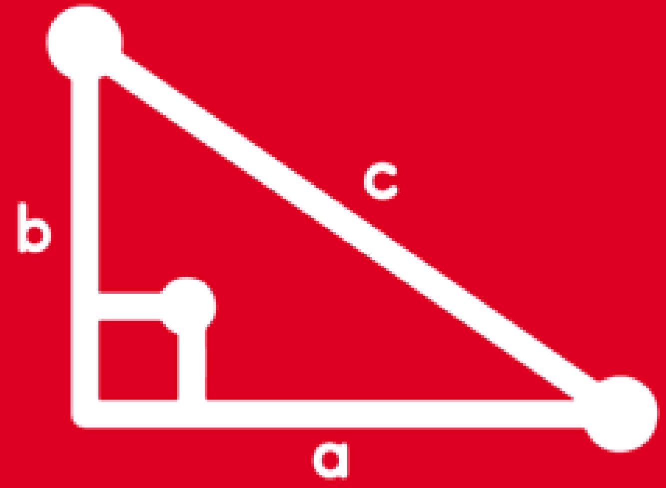
TRIGONOMETRY

Chapter 19



5th
SECONDARY

TRANSFORMACIONES TRIGONOMÉTRICAS I

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En el siglo XVI, aparecieron en Europa una serie de identidades conocidas como las *reglas de prostaféresis*; en la actualidad son conocidas como las identidades de **Transformaciones Trigonométricas**, las cuales convierten una suma y diferencia de senos y cosenos a un producto y viceversa.

Para deducir estas identidades se usan las *identidades del ángulo compuesto*:

$$\operatorname{sen}(x+y)=\operatorname{sen}x.\cos y+\cos x.\operatorname{sen}y \quad \text{... (1)}$$

$$\operatorname{sen}(x-y)=\operatorname{sen}x.\cos y-\cos x.\operatorname{sen}y \quad \text{... (2)}$$

Sumando **(1)** y **(2)**:

$$\operatorname{sen}(x+y)+\operatorname{sen}(x-y)=2\operatorname{sen}x.\cos y \quad \text{... (*)}$$

Hacemos un cambio de variable:

$$\text{Sea } \begin{cases} x+y=A \\ x-y=B \end{cases}$$

$$\Rightarrow \quad x=\frac{A+B}{2} \quad ; \quad y=\frac{A-B}{2}$$

Reemplazando en **(*)** , se obtiene:

$$\operatorname{sen}A+\operatorname{sen}B=2\operatorname{sen}\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

TRANSFORMACIONES TRIGONOMÉTRICAS

1er caso: De suma y diferencia de senos y cosenos a producto

$$\operatorname{sen}A+\operatorname{sen}B=2\operatorname{sen}\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\operatorname{sen}A-\operatorname{sen}B=2\cos\left(\frac{A+B}{2}\right)\operatorname{sen}\left(\frac{A-B}{2}\right)$$

$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A-\cos B=-2\operatorname{sen}\left(\frac{A+B}{2}\right)\operatorname{sen}\left(\frac{A-B}{2}\right)$$

Ejemplos:

- $\operatorname{sen}3x+\operatorname{sen}x=2\operatorname{sen}\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$

$\Rightarrow$ $\operatorname{sen}3x+\operatorname{sen}x=2\operatorname{sen}2x\cos x$

- $\cos 80^\circ+\cos 40^\circ=2\cos\left(\frac{80^\circ+40^\circ}{2}\right)\cos\left(\frac{80^\circ-40^\circ}{2}\right)$

$\Rightarrow$ $\cos 80^\circ+\cos 40^\circ=2\underbrace{\cos 60^\circ}_{1/2}\cos 20^\circ$

$\Rightarrow$ $\cos 80^\circ+\cos 40^\circ=\cos 20^\circ$

TRANSFORMACIONES TRIGONOMÉTRICAS

2do caso: De producto de senos y cosenos a suma y diferencia

$$2\,\text{sen}\alpha\cos\beta = \text{sen}(\alpha+\beta)+\text{sen}(\alpha-\beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$$

$$2\,\text{sen}\alpha\,\text{sen}\beta = \cos(\alpha-\beta)-\cos(\alpha+\beta)$$

Observación:

Si al aplicar las transformaciones trigonométricas obtenemos ángulos negativos, debes usar :

$\text{sen}(-x) = -\text{sen}x$ $\cos(-x) = \cos x$

Ejemplos:

- $2\,\text{sen}3x\cos x = \text{sen}(3x+x)+\text{sen}(3x-x)$

$\Rightarrow$ $2\,\text{sen}3x\cos x = \text{sen}4x+\text{sen}2x$

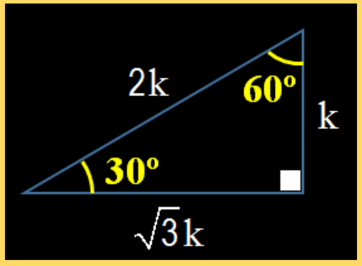
- $2\cos 20^\circ\cos 10^\circ = \cos(20^\circ+10^\circ)+\cos(20^\circ-10^\circ)$

$\Rightarrow$ $2\cos 20^\circ\cos 10^\circ = \underbrace{\cos 30^\circ}+\cos 10^\circ$

$\Rightarrow$ $2\cos 20^\circ\cos 10^\circ = \dfrac{\sqrt{3}}{2}+\cos 10^\circ$

1. Reduzca: $E = \dfrac{\text{sen}40° + \text{sen}20°}{\cos 40° + \cos 20°}$

Recordar:



RESOLUCIÓN

Recordar:

$$\text{sen}A + \text{sen}B = 2\text{sen}\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$$

$$E = \frac{\overbrace{\text{sen}40° + \text{sen}20°}^{\cancel{2}\text{sen}30°\cancel{\cos 10°}}}{\underbrace{\cos 40° + \cos 20°}_{\cancel{2}\cos 30°\cancel{\cos 10°}}}$$

$$E = \frac{\text{sen}30°}{\cos 30°}$$

$$E = \tan 30°$$

$$E = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$\therefore \boxed{E = \frac{\sqrt{3}}{3}}$$

2. Halle el valor del ángulo agudo x en:

$$\frac{\operatorname{sen}9x-\operatorname{sen}3x}{\cos 9x+\cos 3x}=\sqrt{3}$$

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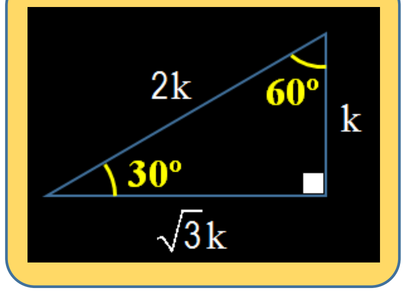


RESOLUCIÓN

Recordar:

$$\operatorname{sen}A-\operatorname{sen}B=2\cos\left(\frac{A+B}{2}\right).\operatorname{sen}\left(\frac{A-B}{2}\right)$$

$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$$

$$\frac{\overbrace{\operatorname{sen}9x-\operatorname{sen}3x}^{\cancel{2}\cancel{\cos 6x}\operatorname{sen}3x}}{\underbrace{\cos 9x+\cos 3x}_{\cancel{2}\cancel{\cos 6x}\cos 3x}}=\sqrt{3}$$

$$\tan 3x=\sqrt{3}$$

$$3x=60^\circ$$

$$\therefore \boxed{x=20^\circ}$$

3. Para $x=\dfrac{\pi}{24}$, calcule: $E=\dfrac{\operatorname{sen}6x+\operatorname{sen}4x+\operatorname{sen}2x}{\cos 6x+\cos 4x+\cos 2x}$

RESOLUCIÓN

Recordar:

$$\operatorname{sen}A+\operatorname{sen}B=2\operatorname{sen}\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$$

$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$$

$$E=\frac{\overbrace{\operatorname{sen}6x+\operatorname{sen}2x}^{2\operatorname{sen}4x\cos 2x}+\operatorname{sen}4x}{\underbrace{\cos 6x+\cos 2x}_{2\cos 4x\cos 2x}+\cos 4x}$$

$$E=\frac{\operatorname{sen}4x(\cancel{2\cos 2x+1})}{\cos 4x(\cancel{2\cos 2x+1})}$$

$$E=\tan 4x$$

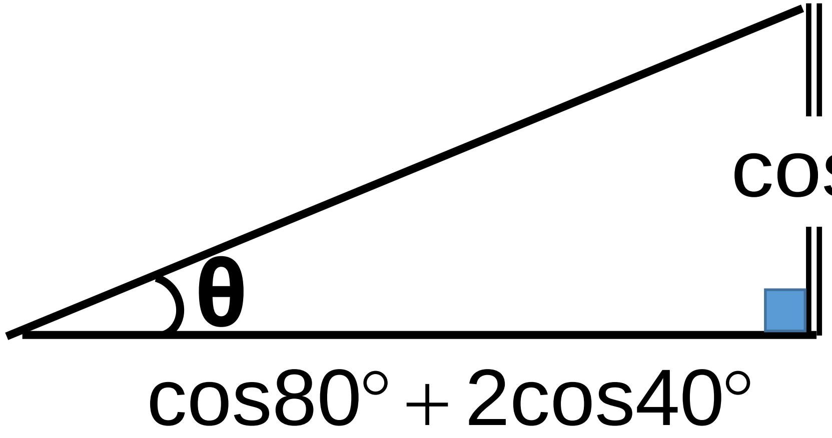
LUEGO:

$$E=\tan\left(4\text{x}\frac{\pi}{24}\right)$$

$$E=\tan\left(\frac{\pi}{6}\right)$$

$$\therefore\ \boxed{E=\frac{\sqrt{3}}{3}}$$

4. Una barra metálica descansa sobre una pared lisa, tal como se muestra en la figura. Halle el valor de θ.



RESOLUCIÓN

Recordar:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$$

$$\cot\theta = \frac{\cos 80° + 2\cos 40°}{\cos 10°} = \frac{\overbrace{\cos 80° + \cos 40°}^{2\cos 60°\cos 20°} + \cos 40°}{\cos 10°}$$

$$\cot\theta = \frac{\cancel{2}\left(\frac{1}{\cancel{2}}\right)\cos 20° + \cos 40°}{\cos 10°} = \frac{\overbrace{\cos 40° + \cos 20°}^{2\cos 30°\cancel{\cos 10°}}}{\cancel{\cos 10°}}$$

$$\cot\theta = 2\left(\frac{\sqrt{3}}{2}\right) \Rightarrow \cot\theta = \sqrt{3}$$

$$\therefore \boxed{\theta = 30°}$$

5.

α

$\alpha = \quad ° \quad ° - \quad °$

Recordar:

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

RESOLUCIÓN

$$\text{sen} \propto = 2\cos 40° \cos 10° - \cos 50°$$

$$\text{sen} \propto = \cancel{\cos 50°} + \cos 30° - \cancel{\cos 50°}$$

$$\text{sen} \propto = \cos 30°$$

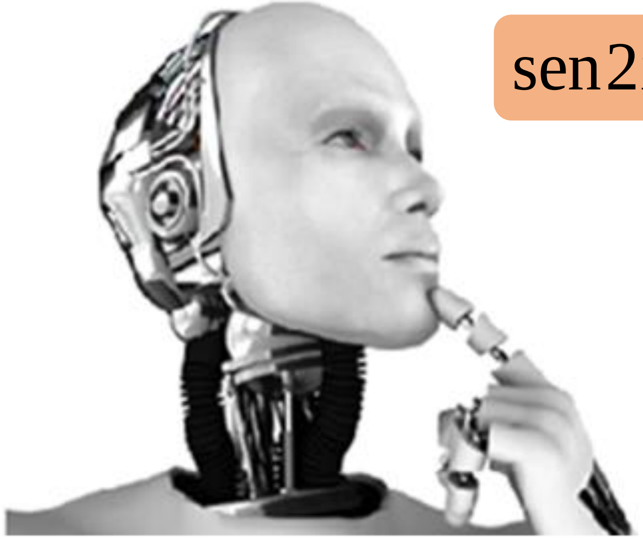
R.T. de ángulos complementarios:

$$\Rightarrow \propto + 30° = 90° \quad \therefore \boxed{\propto = 60°}$$

6.

Recordar:

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$



$$\operatorname{sen}2x = 2\operatorname{sen}x\cos x$$

$$\csc x = \frac{1}{\operatorname{sen}x}$$

RESOLUCIÓN

$$R = \frac{2\cos 4x\cos 3x - \cos 7x}{\operatorname{sen}2x}$$

$$R = \frac{\cancel{\cos 7x} + \cos x - \cancel{\cos 7x}}{\operatorname{sen}2x}$$

$$R = \frac{\cancel{\cos x}}{2\operatorname{sen}x\cancel{\cos x}}$$

$$R = \frac{1}{2\operatorname{sen}x}$$

$$\therefore \boxed{R = \frac{1}{2}\csc x}$$

7.

RESOLUCIÓN

Recordar:

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$\frac{2\,\cos40^\circ\cos20^\circ}{2\,\cos60^\circ} = k + sen70^\circ$$

$$\frac{\cos60^\circ + \cos20^\circ}{\cancel{2}\left(\frac{1}{\cancel{2}}\right)} = k + sen70^\circ$$

$$\frac{1}{2} + \cancel{sen70^\circ} = k + \cancel{sen70^\circ}$$

$$\therefore \boxed{k = \frac{1}{2}}$$

8.

− = −

+

Recordar:

$$2\operatorname{sen}\alpha\operatorname{sen}\beta = \cos(\alpha-\beta)-\cos(\alpha+\beta)$$

RESOLUCIÓN

$$2(\operatorname{sen}7x - \operatorname{sen}x)\operatorname{sen}4x = \cos(Px) - \cos(Qx)$$

$$2\operatorname{sen}7x\operatorname{sen}4x - 2\operatorname{sen}4x\operatorname{sen}x = \cos(Px) - \cos(Qx)$$

$$\cancel{\cos 3x} - \cos 11x - (\cancel{\cos 3x} - \cos 5x) = \cos(Px) - \cos(Qx)$$

$$\cos 5x - \cos 11x = \cos(Px) - \cos(Qx)$$

Comparando: $P = 5$ y $Q = 11$ $\quad\therefore\quad \boxed{P + Q = 16}$