Stable Matchings

- Stable matchings have many important applications to economics, assignment of medical students to hospitals, assignment of organs to donors.
- Gale-Shapley designed famous algorithm for find a stable perfect matching
- Lloyd Shapley and Alan Roth won the Nobel Prize for their work on stable matchings.

Stable Matching Problem

- Complete bipartite graph $G = (X,Y)$ where $|X| = |Y| = n$.
- For example, X and Y represent n boys and n girls.
- Each boy has a strict, complete preference ordering over the girls, and vice versa

Example Preference Profiles

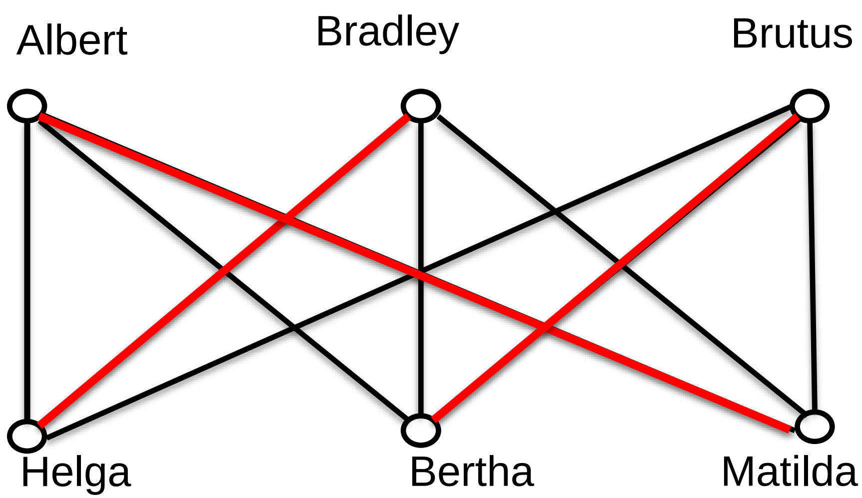
  

Albert	Diane	Emily	Bertha
Bradley	Emily	Diane	Bertha
Brutus	Diane	Emily	Bertha

Diane	Bradley	Albert	Brutus
Emily	Albert	Bradley	Brutus
Bertha	Albert	Bradley	Brutus

Matching

- A **matching** M is a set of **independent edges**, i.e., a set of edges xy such that
  - ➢ Each boy $x \in X$ appears in at most one edge of M.
  - ➢ Each girl $g \in G$ appears in at most one edge of M.
- A matching M is **perfect** if $|M| = |X| = |Y| = n$.



Unstable Pair

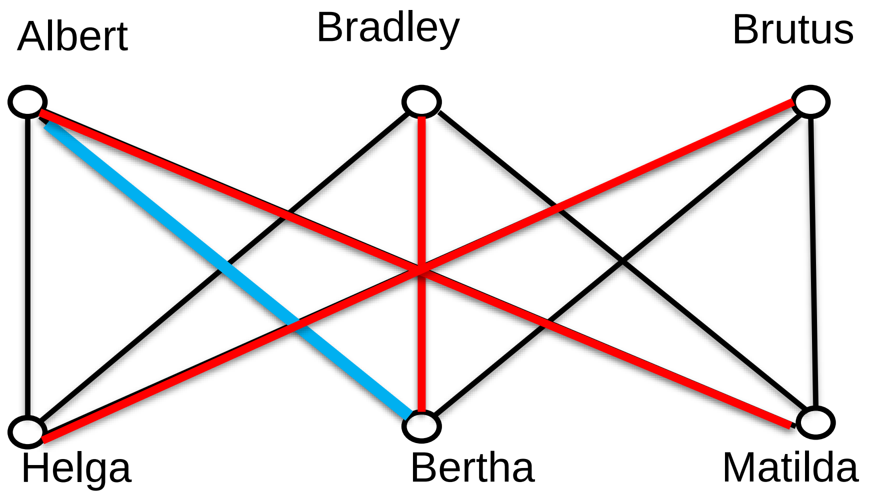
Given a perfect matching M, a boy b and girl g form an **unstable pair** if both:

b prefers g to the girl he is matched with.

g prefers b to the boy she is matched with.

Key point. An unstable pair $\{b,g\}$ could each improve by joint action.

Example of an Unstable Pair



	1st	2nd	3rd
Albert	Helga	Bertha	Matilda
Bradley	Bertha	Helga	Matilda
Brutus	Helga	Bertha	Matilda

	1st	2nd	3rd
Helga	Bradley	Albert	Brutus
Bertha	Albert	Bradley	Brutus
Matilda	Albert	Bradley	Brutus

Albert-Bertha is an unstable pair, i.e., they both prefer each other to their partner

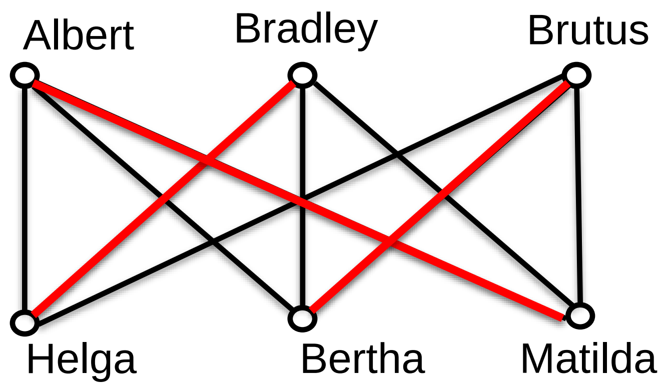
Stable matchings

A **stable matching** is a perfect matching with no unstable pairs.

Stable matching problem.

Given the preference lists of n boys and n girls, find a stable matching.

Example perfect matching shown in complete bipartite graph and in preference tables

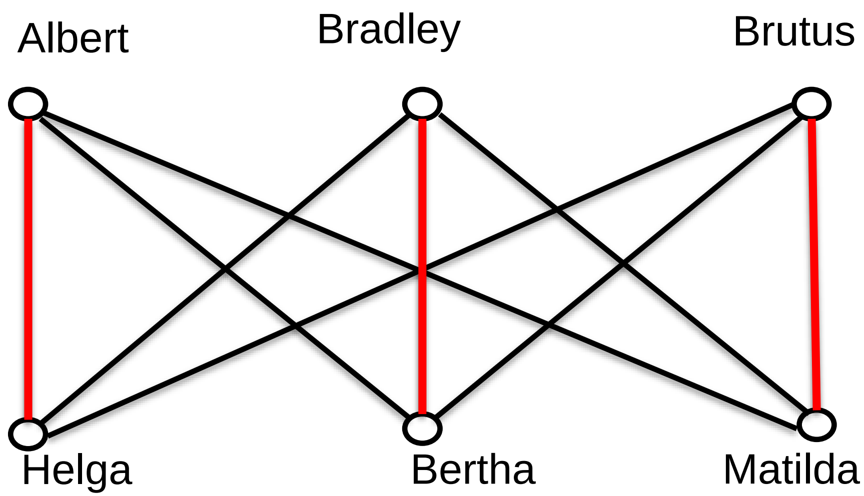


	1st	2nd	3rd
Albert	Helga	Bertha	Matilda
Bradley	Bertha	Helga	Matilda
Brutus	Helga	Bertha	Matilda

	1st	2nd	3rd
Helga	Bradley	Albert	Brutus
Bertha	Albert	Bradley	Brutus
Matilda	Albert	Bradley	Brutus

PSN. Is this perfect matching stable?

Example of a Stable Matching



	1st	2nd	3rd
Albert	Helga	Bertha	Matilda
Bradley	Bertha	Helga	Matilda
Brutus	Helga	Bertha	Matilda

	1st	2nd	3rd
Helga	Bradley	Albert	Brutus
Bertha	Albert	Bradley	Brutus
Matilda	Albert	Bradley	Brutus

An intuitive method that guarantees to find a stable matching

Gale–Shapley (preference lists for boys and girls)

Intialize M to empty matching.

while (some boy b is unmatched and hasn't proposed to every girl)

$g \leftarrow$ first girl on b's list to whom b has not yet proposed

if (g is unmatched)

add edge $\{b,g\}$ to matching M

else if (g prefers b to current partner b')

replace edge $\{b',g\}$ with $\{b,g\}$

else

g rejects b

return stable matching M

Demo of Gale-Shapely Stable Matching Algorithm

Gale-Shapley Demo

Complexity Analysis

Observation 1. Boys propose to girls in decreasing order of preference.

Observation 2. Once a girl is matched, the girl never becomes unmatched; only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of the while loop.

Proof. Each time through the while loop, a boy proposes to a new girl. Thus, there are at most n^2 possible proposals, i.e., $W(n) \in O(n^2)$.

Complexity Analysis, cont'd

There are examples where boys make $n(n-1)+1$ proposals. Thus, $W(n) \in \Omega(n^2)$. Since $W(n) \in O(n^2)$ and $W(n) \in \Omega(n^2)$, we have

$$W(n) \in \Theta(n^2).$$

	1st	2nd	3rd	4th	5th
A	V	W	X	Y	Z
B	W	X	Y	V	Z
C	X	Y	V	W	Z
D	Y	V	W	X	Z
E	V	W	X	Y	Z

	1st	2nd	3rd	4th	5th
V	B	C	D	E	A
W	C	D	E	A	B
X	D	E	A	B	C
Y	E	A	B	C	D
Z	A	B	C	D	E

Proof of correctness:
Gale–Shapley outputs a perfect matching

Claim. Gale–Shapley outputs a matching.

Proof.

Boy proposes only if unmatched. $\Rightarrow$ matched to ≤ 1 girl

Girl keeps only best boy. $\Rightarrow$ matched to ≤ 1 boy

Claim. In Gale–Shapley matching, all boys get matched.

Proof by contradiction. Suppose that some boy b is unmatched upon termination of Gale–Shapley algorithm. Then some girl g is unmatched upon termination. Then g was never been proposed to. But, b proposes to every girl, since b ends up unmatched. In particular, b proposed to g, a contradiction

Claim. In Gale–Shapley matching, all girls get matched.

Proof. By previous claim, all n boys get matched. Thus, all n students get matched.

Proof of correctness: Stability

Theorem. In Gale–Shapley matching M, there are no unstable pairs.

Proof. Consider any pair $\{b,g\}$ that is not in M.

Case 1:

b never proposed to g.

$\Rightarrow$ b prefers its Gale–Shapley partner to g.

$\Rightarrow$ $\{b,g\}$ is not unstable.

boys propose in decreasing order of preference

Case 2:

b proposed to g.

$\Rightarrow$ g rejected b (either right away or later)

$\Rightarrow$ g prefers Gale–Shapley partner to b.

$\Rightarrow$ $\{b,g\}$ is not unstable.

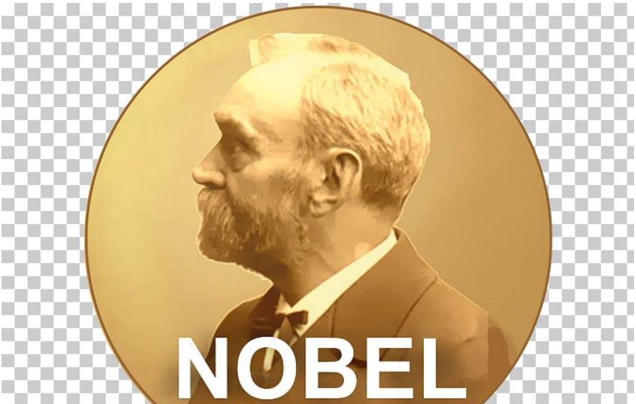
girls only trade up

In either case, the pair $\{b,g\}$ is not unstable.

Nobel Prize

https://www.nature.com/news/a-nobel-for-the-art-of-matchmaking-1.11607

[popular-economicsciences2012.pdf (nobelprize.org)]()



What prize did the person receive who invented the first knock-knock joke?

The No-bell Prize