Discrete Fourier Transform (DFT) and Fast Fourier Transform Algorithm (FFT)

Textbook Reading:

- Section A.6 (Appendix, Section 6) Complex Numbers, pp. 454-460.
- Section 7.5 The Discrete Fourier Transform, pp. 295-302.

Fast Fourier Transform: Applications

- Optics, acoustics, quantum physics, telecommunications, control systems, signal processing, speech recognition, data compression, image processing.
- DVD, JPEG, MP3, MRI, CAT scan.
- Numerical solutions to Poisson's equation used in physics simulations.
- The FFT is one of the truly great computational developments of this [20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. *-Charles van Loan*

Fast Fourier Transform: Brief History

- Cooley-Tukey (1965). Idea came from application of monitoring nuclear tests in Soviet Union and tracking submarines. Independently rediscovered C.F. Gauss divide-and-conquer approach, popularized use FFT.
- C.F. Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

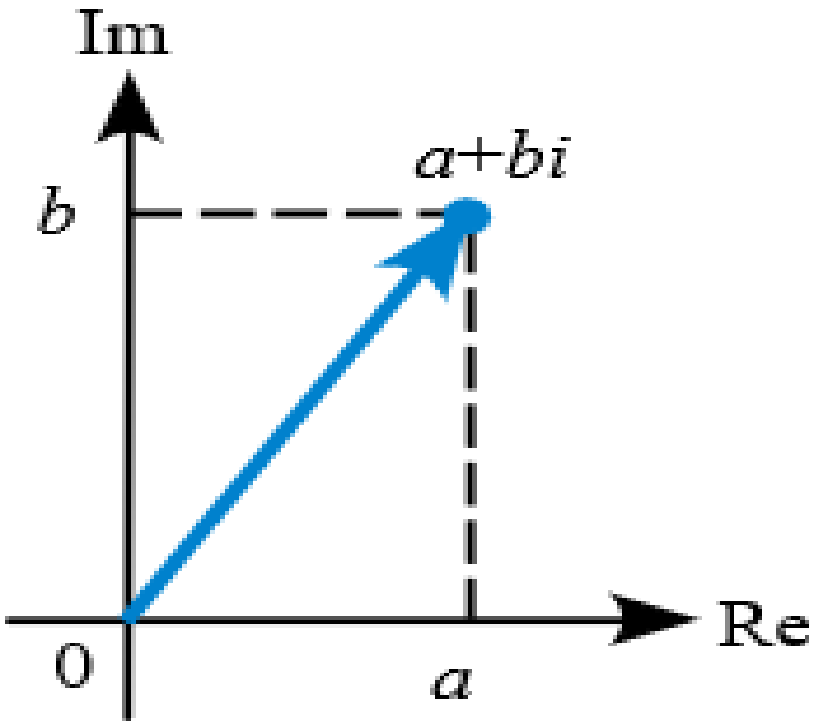
- Importance of FFT not fully realized until advent of fast digital computers, and is still growing.
- There is speculation that FFT was responsible for saving the world during Cuban Missile Crisis 1963. FFT quickly distinguished a small earthquake in Cuba from Soviet missile launch.

Complex Numbers

Mathematicians wanted to solve simple equations such as $x^2 + 1 = 0$, which had no solutions in the real numbers. Thus, the so-called complex numbers of the form $a + ib$ were introduced, where $i = \sqrt{-1}$ stood for a number whose square was –1.

a is called the ***real part*** *and b* the imaginary part of the complex number *a* + *ib.*

Complex Plane



x-axis is real axis and *y*-axis is imaginary axis

Modulus

The *modulus of* $z = a + ib$, denoted by $|z|$, is defined to be the distance from (a, b) to the origin $(0, 0)$, that is,

$$|z| = \sqrt{a^2 + b^2}$$

Euler's Formula

$$\cos(x) + i \sin(x) = e^{ix}$$

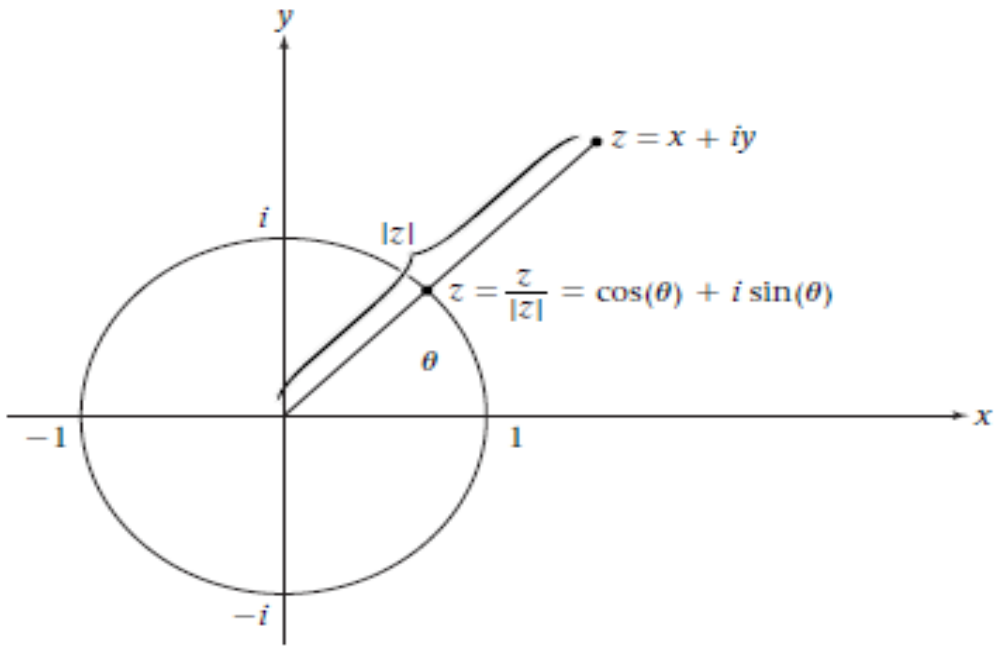
Fun corollary relating five most important numbers in mathematics 0, 1, π, e, i. It has been called the most beautiful equation in mathematics.

$$e^{i\pi} + 1 = 0$$

Plotting Complex Numbers

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$

where $r = |z| = \sqrt{x^2 + y^2}$



x-axis in real axis and *y*-axis is imaginary axis

n^{th} roots of unity

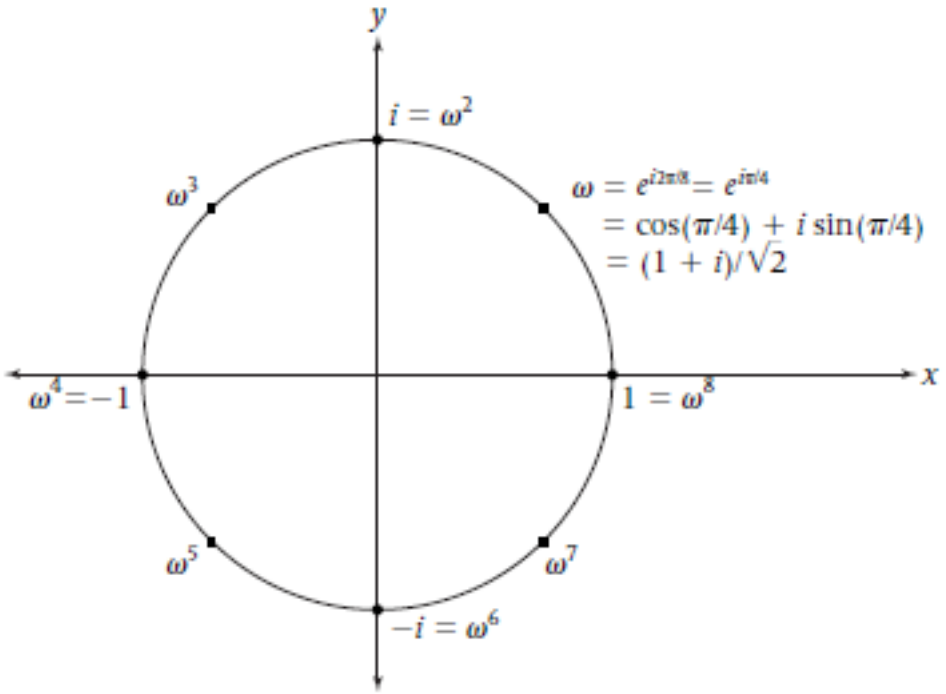
Def. An ***n*th root of unity** is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$ *called a* ***primitive*** *root because it generates all the other roots by taking powers.*

Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{2\pi i})^k$

$= (\cos(2\pi) + i \sin(2\pi))^k = (1 + 0)^k = 1.$

Eight 8th Roots of Unity



Definition of Discrete Fourier Transform (DFT)

Given a polynomial $P(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$

$DFT_\omega P(x)$ maps $P(x)$ to the polynomial

$$Q(x) = b_0 + b_1 x + \ldots + b_{n-1} x^{n-1}$$

where

$$b_j = P(\omega^j)$$

and ω is **primitive** n^{th} root of unity, i.e., an n^{th} root of unity whose powers generate all other n^{th} roots of unity.

Example

$$P(x) = 5x^3 + 3x + 10 \leftrightarrow [10, 3, 0, 5]$$

$$P(1) = 18$$

$$P(i) = 10 - 2i$$

$$P(-1) = 2$$

$$P(-i) = 10 + 2i$$

$$Q(x) = DFT\omega\, P(x)$$

$$= (10 + 2i)x^3 + 2x^2 + (10 - 2i)x + 18$$

$$\leftrightarrow [18, 10 - 2i, 2, 10 + 2i]$$

PSN. Compute $DFT\omega\ P(x)$ for

$P(x) = x^3 + 3x^2 + 2x - 1$ and $\omega = -i$

and give associated coefficient arrays.

Matrix Formulation for Discrete Fourier Transform

Using coefficient arrays and matrix notions

$[a_0, a_1, \ldots, a_{n-1}]$ mapped to $[b_0, b_1, \ldots, b_{n-1}]$ by

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

DFT matrix $F\omega$

Straightforward Solution

Use Horner's rule to evaluate $P(x)$ at each of the n n^{th} roots of unity, i.e., evaluate $P(x)$ at each x for

$$x = \omega^k = e^{2\pi ik/n} = \cos(2\pi k/n) + i \sin(2\pi k/n)), \; k = 0,1, \ldots, n.$$

Since Horner's Rule involves n multiplications for each of these n evaluations, computing $DFT_\omega P(x)$ in this naïve way has worst-case complexity

$$W(n) = n^2.$$

We can do much better using *FFT.*

Roots of Unity

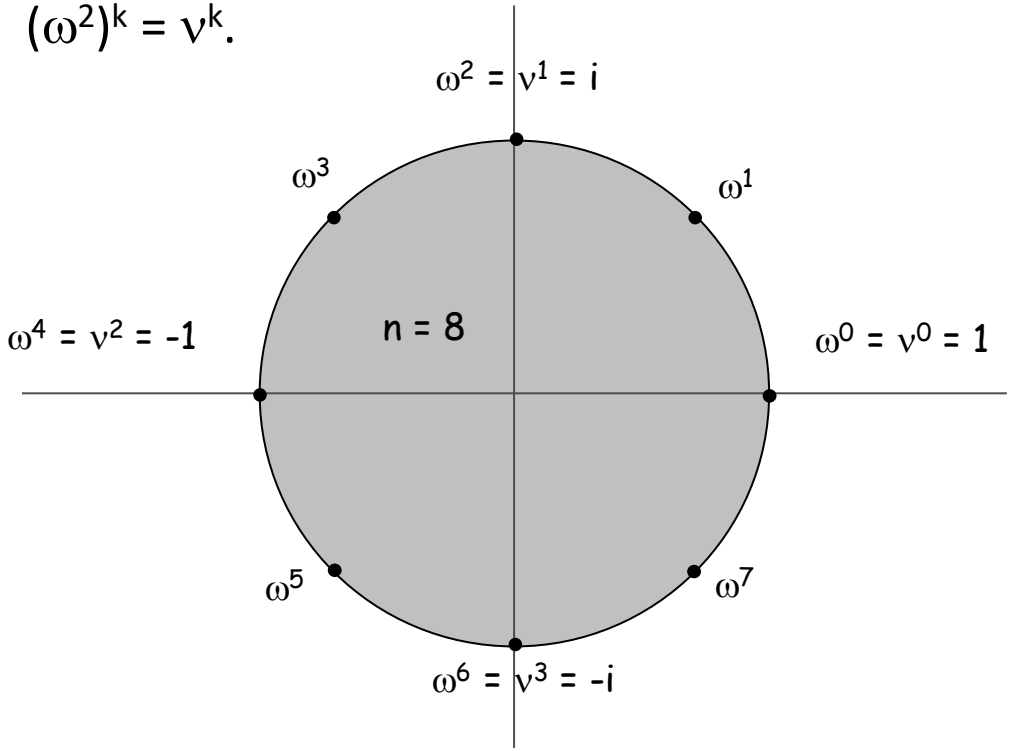
Def. An n^{th} root of unity is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.

Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{2\pi i})^k = (\cos(2\pi) + \sin(2\pi))^k = (1 + 0)^k = 1$.

Fact. The $(\frac{1}{2}n)^{th}$ roots of unity are: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$ where $\nu = e^{4\pi i / n}$.

Fact. $\omega^2 = \nu$ and $(\omega^2)^k = \nu^k$.



Fast Fourier Transform Design Strategy

Goal. Evaluate a degree $n - 1$ polynomial $P(x) = a_0 + \ldots + a_{n-1}\, x^{n-1}$ at its n^{th} roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

Divide. Break polynomial up into even and odd powers.

$$E(x) = a_0 + a_2 x + a_4 x^2 + \ldots + a_{n/2-2}\, x^{(n-1)/2}.$$

$$D(x) = a_1 + a_3 x + a_5 x^2 + \ldots + a_{n/2-1}\, x^{(n-1)/2}.$$

$$P(x) = E(x^2) + x\, D(x^2).$$

Conquer. Evaluate degree $E(x)$ and $D(x)$ at the $(\frac{1}{2}n)^{\text{th}}$ roots of unity:

$$\nu^0, \nu^1, \ldots, \nu^{n/2-1}.$$

Combine.

$$P(\omega^k) = E(\nu^k) + \omega^k\, D(\nu^k), \quad 0 \le k < n/2$$

$$P(\omega^{k+n/2}) = E(\nu^k) - \omega^k\, D(\nu^k), \quad 0 \le k < n/2 \qquad (\omega^{k+n/2} = -\omega^k)$$

Pseudocode for recursive version of FFT

procedure *FFTRec*(a[0:n – 1], n, ω, b[0:n – 1]) **recursive**

Input: a[0:n – 1] (an array of coefficients of the polynomial $P(x) = a_{n-1}x^{n-1} + \ldots + a_1x + a_0$)

n (a power of two) //$n = 2^k$

ω (a primitive n^{th} root of unity)

Output: b[0:n – 1] (an array of values $b[j] = P(\omega^j,\ j = 0, \ldots, n-1)$

if $n = 1$ **then**

$b[0] \leftarrow a[0]$

else

//divide into even-indexed and odd-indexed coefficients

for $j \leftarrow 0$ **to** $n/2 - 1$ **do**

$Even[j] \leftarrow a[2*j]$ //$[a_0, a_2, \ldots, a_{n-2}]$

$Odd[j] \leftarrow a[2*j + 1]$ //$[a_1, a_3, \ldots, a_{n-1}]$

endfor

FFTRec($Even$[0:$n/2 - 1$],$n/2$, ω^2,e[0:$n/2 - 1$])

FFTRec(Odd[0:$n/2 - 1$],$n/2$, ω^2,d[0:$n/2 - 1$])

for $j \leftarrow 0$ to $n/2 - 1$ **do**

$b[j] \leftarrow e[j] + \omega^j * d[j]$

$b[j + n/2] \leftarrow e[j] - \omega^j * d[j]$

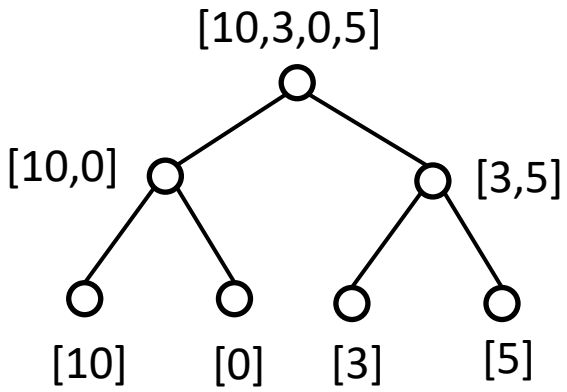
endfor

endif

end *FFTRec*

Action of FFT for example

$P(x) = 5x^3 + 3x + 10 \leftrightarrow [10, 3, 0, 5]$, $\omega = i$



$n = 1$, $\omega = 1$: [10] [0] [3] [5]

$n = 2$, $\omega = -1$: [10 + 0, 10 – 0] [3 + 5] [3 – 5]

= [10, 10] = [8, -2]

$n = 4$, $\omega = i$: $[10 + 8, 10 - 2i, 10 - 8, 10 + 2i] = [18, 10 - 2i, 2, 10 + 2i]$

$$Q(x) = (10 + 2i)x^3 + 2x^2 + (10 - 2i)x + 18$$

PSN. Show action of FFT in computing $DFT\omega\ P(x)$ for

$P(x) = x^3 + 3x^2 + 2x - 1$ and $\omega = -i$

and give associated coefficient arrays.

Complexity Analysis

Theorem. *FFT* evaluates a polynomial of size n at each of the n^{th} roots of unity in time $O(n \log n)$.

Recurrence Relation for Worst-Case Complexity. Assume n is a power of 2.

$$W(n) = 2W(n/2) + n, \quad W(1) = 0$$

This is the same recurrence relation as Mergesort! Thus,

$$W(n) = \Theta(n \log n).$$

Inverse DFT

A very useful and important property of *DFT* is that its inverse can be obtained as just another *DFT*.

Theorem. $DFT_{\omega}^{-1}Q(x) = \frac{1}{n} DFT_{\omega^{-1}} Q(x)$

What this means is that

if $Q(x) = DFT_{\omega} P(x)$ then $P(x) = \frac{1}{n} DFT_{\omega^{-1}} Q(x)$

where $\omega^{-1} = 1/\omega$, *i.e.,* if ω is the primitive root $e^{2\pi i / n} = cos(2\pi / n) + i\, sin(2\pi / n)$ then ω^{-1} is the primitive root $e^{-2\pi i / n} = cos(2\pi / n) + i\, sin(2\pi / n)$.

DFT matrices evaluated at ω and ω^{-1}

$F_\omega =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \cdots & \omega^{n-1} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & \cdots & \omega^{2(n-1)} \\ 1 & \omega^{3} & \omega^{6} & \omega^{9} & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}$$

$F_{\omega^{-1}} =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

Inverse FFT: Proof of Correctness

Claim. F_ω and $\frac{1}{n} F_{\omega^{-1}}$ are inverse matrices, i.e., the their product is the identity matrix.

This follows from the fact that

$$\sum_{j=0}^{n-1} \omega^{kj} = \begin{cases} n & \text{if } k \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases}$$

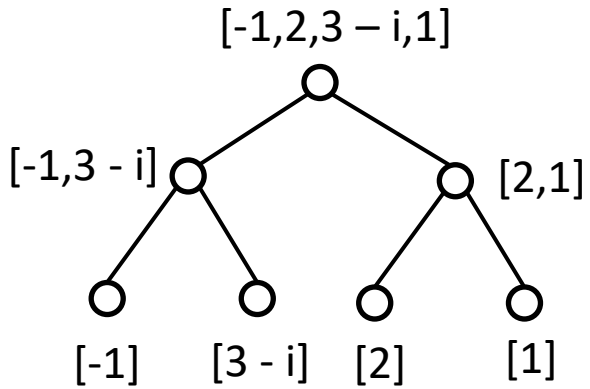
Example of action of FFT to compute inverse of DFT

Compute $DFT_{\omega}{}^{-1}Q(x)$ for

$Q(x) = x^3 + (3 - i)x^2 + x - 2$ at $\omega = i$

Solution

$Q(x) = x^3 + (3 - i)x^2 + 2x - 1 \leftrightarrow [-1\ 2, 3\ -i, 1]$



$n = 1$, $\omega = 1$: [-1] [3 – i] [2] [1]

$n = 2$, $\omega = -1$: [-1 + 3 - i, -1 – (3 – i)] [2 + 1] [2 – 1]

= [2 – i, -4 + i] = [3, 1]

$n = 4$, $\omega = -i$: [(2 – i) + 3, -4 + i + 1(-i), (2 – i) – 3, -4 + i – 1(-i)]

= [5 – i, -4, -1 – i, -4 + 2i]

$DFT_\omega{}^{-1}Q(x) \leftrightarrow \frac{1}{4}[5 - i, -4, -1 - i, -4 + 2i] \leftrightarrow \frac{-2+i}{2}x^3 + \frac{-1-i}{4}x^2 - x + \frac{5-i}{4}$

Application to Ordering Bagels in Case of Shortages