Computing Time – Time Complexities

Textbook Reading:

- Section 2.5, pp. 32-35
- Subsections 2.6.1 & 2.6.2, pp. 36-38

Analyzing Algorithm Performance

- We measure the computing time or (time) **complexity** of an algorithm as a function of the **input size *n*** to the algorithm.

- For example, when searching or sorting a list, the input size is the number of elements n in the list;

- when evaluating a polynomial, the input size is either the degree of the polynomial or the number of nonzero coefficients in the polynomial;

- when multiplying two square $n \times n$ matrices, the input size is n;

- when testing whether an integer is a prime, the input size is the number of digits of the integer;

- when traversing a tree, the input size is the number of nodes in the tree; and so forth.

Measuring complexity

- In measuring the complexity (computing time) of an algorithm we identify a **basic operation**.
- And **count how many times an algorithm performs this basic operation**.
- Analysis based on a suitably chosen basic operation yields measurements that are proportional to actual run time behavior exhibited when running the algorithm on various computers, so that the analysis is not dependent on a particular computer.

Catch

- Counting the number of basic operations is not well-defined, even restricted to a given input size *n*, because a different number of operations may be performed for different inputs of the same size.
- This is solved by defining best-case, worst-case and average complexities.

Best-Case Complexity

The *best-case complexity* B(n) of an algorithm is the **fewest** basic operations performed over all inputs of **size *n***.

This can be expressed mathematically as follows:

$$B(n) = \min\{\tau(I) \mid I \text{ in } \mathscr{I}_n\}$$

Worst-Case Complexity

The *worst-case complexity* $W(n)$ of an algorithm is the **most** basic operations performed over all inputs of **size** n.

This can be expressed mathematically as follows:

$$W(n) = \max\{\tau(I) \mid I \text{ in } \mathscr{I}_n\}$$

Average Complexity

We define a random variable τ that maps the sample space $\mathcal{I}_n$ of all inputs I of **size *n*** onto the number of basic operations performed by the algorithm for input I. The average complexity $A(n)$ is defined to be the **expected value** of τ, i.e.,

$$A(n) = \mathrm{E}(\tau).$$

Note that the average complexity $A(n)$ **is dependent on the probability distribution** on $\mathcal{I}_n$.

A review of probability theory and random variables is given in Appendix E of the textbook *Algorithms: Foundations and Design Strategies.*

Formula Average Complexity

Let p_i denote the number of basic operations algorithm performs for an input of size n. Then

$$A(n) = \sum_{i=B(n)}^{W(n)} i p_i$$

Linear Search

function *LinearSearch* ($L[0{:}n - 1], X$)

Input: $L[0{:}n - 1]$ (a list of size n), X (a search item)

Output: returns index of first occurrence of X in the list, or -1 if X is not in the list

for $i \leftarrow 0$ **to** $n - 1$ **do**

if ($X = L[i]$) **then**

return(i)

endif

endfor

return(-1)

end *LinearSearch*

Complexity Analysis of *LinearSearch*

The basic operation of *LinearSearch* is the **comparison** of the search element to a list element.   In the input size n is the size of the list.

Best-Case Complexity

Clearly, *LinearSearch* performs only one comparison when the input X is the first element in the list, so that the best-case complexity is

$$B(n) = 1.$$

Worst-case complexity

The most comparisons are performed when X is not in the list, or when X occurs in the last position only. Thus, the worst-case complexity of *LinearSearch* is

$$W(n) = n.$$

Average Complexity of *LinearSearch*

To simplify the discussion of the average behavior of *LinearSearch*, we assume that the search element X is in the list $L[0:n-1]$ and is equally likely to be found in any of the n positions. Note that i comparisons are performed when X is found at position i in the list. Thus, the probability that *LinearSearch* performs i comparisons is given by $p_i = 1/n$.

Substituting these probabilities into the formula for *A*(*n*) yields:

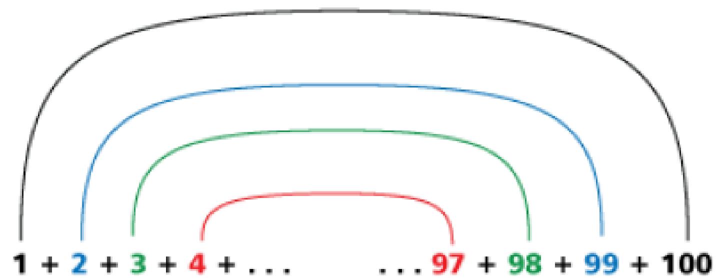
$$A(n) = \sum_{i=B(n)}^{W(n)} ip_i = \sum_{i=1}^{n} i\,\frac{1}{n} = \frac{1}{n}(1 + 2 + \ ... + n).$$

Can we obtain a closed-form formula for

$1 + 2 + \ldots + n$?

Sum of numbers 1 + 2 + … + 100

I love the story of Carl Friedrich Gauss—who, as an elementary student in the late 1700s, amazed his teacher with how quickly he found the sum of the integers from 1 to 100 to be 5,050. Gauss recognized he had fifty pairs of numbers when he added the first and last number in the series, the second and second-last number in the series, and so on. For example: (1 + 100), (2 + 99), (3 + 98), . . . , and each pair has a sum of 101.



50 pairs × 101 (the sum of each pair) = 5,050.

Another way to represent the problem could be to list the integers from 1 to 100 and write the same list in reverse order below the first list.

1 + 2 + 3 + 4 + ... + 97 + 98 + 99 + 100

100 + 99 + 98 + 97 + ... + 4 + 3 + 2 + 1

101 + 101 + 101 + 101 + ... + 101 + 101 + 101 + 101

This gives us 100 addends of 101 for 10,100. Because the list of numbers from 1 to 100 was doubled, we need to divide the total by 2, giving us a sum of 5,050.

Formula for $1 + 2 + \ldots + n$

Let $S = 1 + 2 + \ldots + n.$

$$
\begin{array}{cccccc}
1 & + & n & & = & n + 1 \\
2 & + & n - 1 & & = & n + 1 \\
3 & + & n - 2 & & = & n + 1 \\
 & & & \vdots & & \\
\mathrm{n} & + & 1 & & = & n + 1 \\
\hline
S & + & S & & = & n(n + 1)
\end{array}
$$

Summing columns we obtain

$$2\,S = n(n+1) \text{ or } S = n(n+1)/2$$

Substituting formula for 1 + 2 + … + *n* we obtain:

$$A(n) = \tfrac{1}{n}(1 + 2 + \ \ldots + n) = \tfrac{1}{n}\,\frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Binary Search

Now suppose as a precondition the list is sorted. Then, there is a much faster algorithm for searching called Binary Search.

Pseudocode for Binary Search

function *BinarySearch* (L[0:n – 1],X)
Input: L[0:n – 1] (a sorted array of n list elements)
X (a search item)
Output: returns the index of an occurrence of X in the list, or -1 if X is not in the list

```
    Found ← .false.
    low ← 0
    high ← n – 1
    while .not. Found .and. low ≤ high do
        mid ← ⌊(low + high)/2⌋
        if  X = L[mid] then Found ← .true.
        else  if  X < L[mid] then
                high ← mid – 1
            else
                low ← mid + 1
            endif
        endif
    endwhile
    if Found then
        return(mid)
    else
        return(-1)
    endif
end BinarySearch
```

Recursive Version

function *BinarySearch*(*L*[0:*n* – 1],low,high,*X*)
Input: *L*[0:*n* – 1] (an array of *n* list elements, sorted in increasing order
low, high (nonnegative integers)
X (a search item)
Output: returns the index of an occurrence of *X* in the sublist *L*[0:*n* – 1,low,high] or -1 if *X* is not in the list

if *high* < *low* **then return**(-1) **endif** // empty list

$mid \leftarrow \lfloor (low + high)/2 \rfloor$

if *X* = *L*[*mid*] **then return**(*mid*) **endif**

if *X* < *L*[*mid*] **then**
BinarySearch(*L*[0:*n* – 1],low,mid - 1,*X*)
else
BinarySearch(*L*[0:*n* – 1],mid + 1,high,*X*)
endif

end *BinarySearch*

PSN. Write in C++. Include good commenting.

Best-case complexity

The best-case complexity of *BinarySearch* occurs when X is found in the midpoint position $\lfloor (n-1)/2 \rfloor$ of $L[0:n-1]$ and involves a single operations, i.e.,

$$B(\mathrm{n}) = 1$$

Worst-case complexity

The worst-case complexity is equal to twice the longest string of midpoints (values of *mid*) ever generated by the algorithm for an input X. In particular, if we assume that $n = 2^k - 1$ for some positive integer k, then such a string is generated by searching for $X = L[0]$. We then compare X successively to the midpoints $2^{k-1} - 1$, $2^{k-2} - 1$, . . . , 0, so that this longest string has length k. To express k in terms of n, we note that $n + 1 = 2^k$, so that we have $k = \log_2(n + 1)$. Since to comparisons are involved at each stage

$$W(n) = 2\log_2(n + 1) \approx 2\log_2 n.$$

PSN. Give C++ code for the a recursive version of binary search where no equality check with midpoint is made and analyze your algorithm.

Complexity Analysis

For convenience assume $n = 2^k$. Then

$$B(n) = W(n) = k = \log_2 n.$$

For any probability distribution on the sample space of inputs of size n

$$B(n) \le A(n) \le W(n).$$

It follows that

$$A(n) = \log_2 n.$$

Why did the computer show up at work late?

Answer:

It had a hard drive.