Floating Point

CS2011: Introduction to Computer Systems
Lecture 5 (**2.4**)

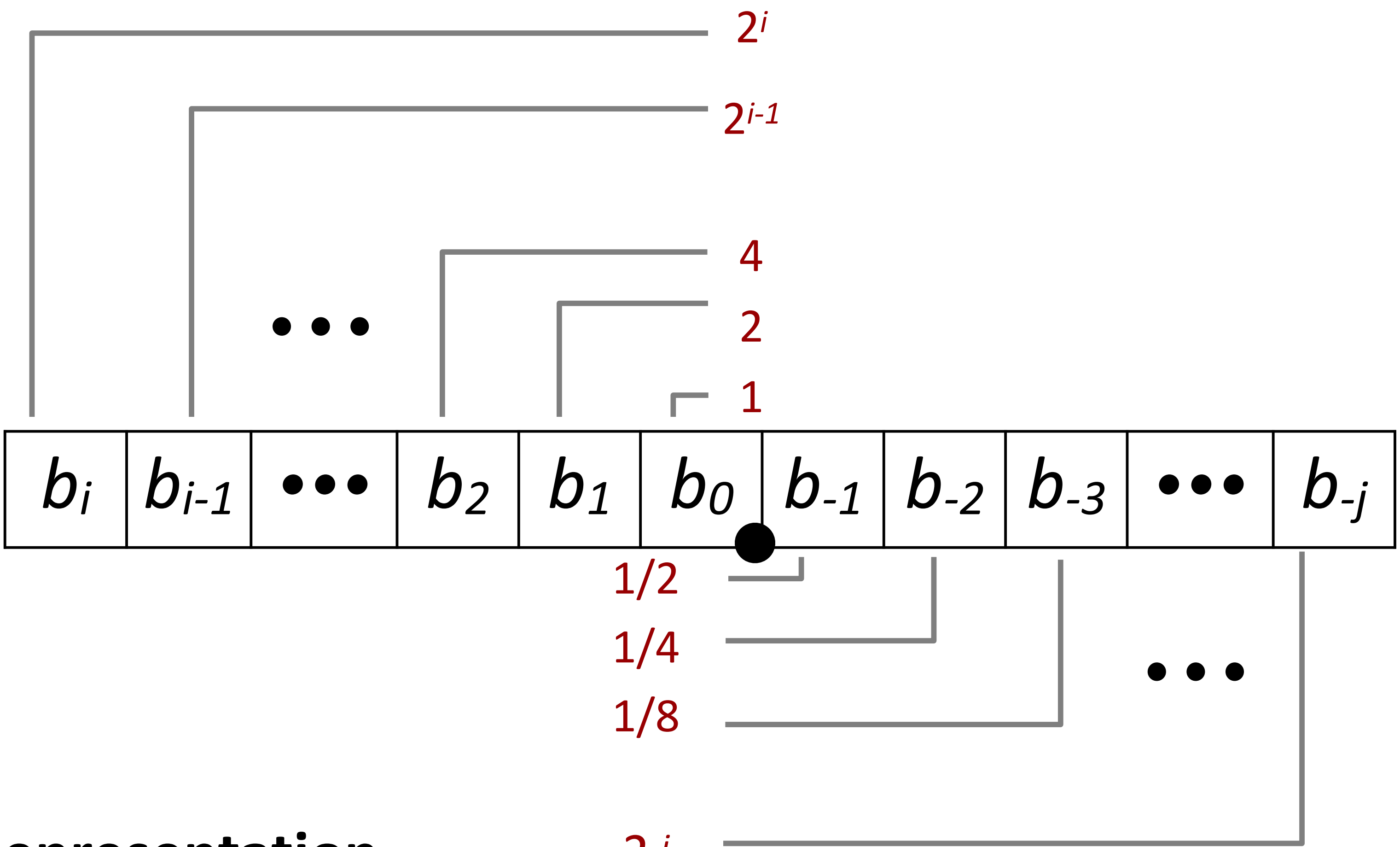
Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101_2?

Fractional Binary Numbers



Representation

- Bits to right of “binary point” represent fractional powers of 2
- Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

- **Value** **Representation**

Value		Representation	
5 3/4	= 23/4	`101.11`$_2$	= 4 + 1 + 1/2 + 1/4
2 7/8	= 23/8	`10.111`$_2$	= 2 + 1/2 + 1/4 + 1/8
1 7/16	= 23/16	`1.0111`$_2$	= 1 + 1/4 + 1/8 + 1/16

- **Observations**
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

Limitation #1

- Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations
- Value Representation
  - 1/3 `0.0101010101[01]`$..._2$
  - 1/5 `0.001100110011[0011]`$..._2$
  - 1/10 `0.0001100110011[0011]`$..._2$

Limitation #2

- Just one setting of binary point within the *w* bits
  - Limited range of numbers (very small values? very large?)

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IEEE Floating Point

IEEE Standard 754

- Established in 1985 as uniform standard for floating point arithmetic
  - Before that, many idiosyncratic formats
- Supported by all major CPUs
- Some CPUs don't implement IEEE 754 in full
  e.g., early GPUs, Cell BE processor

Driven by numerical concerns

- Nice standards for rounding, overflow, underflow
- Hard to make fast in hardware
  - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

Example:
$15213_{10} = (-1)^0 \times 1.1101101101101_2 \times 2^{13}$

- **Numerical Form:**

$$(-1)^s \, M \; 2^E$$

  - **Sign bit *s*** determines whether number is negative or positive
  - **Significand/Mantissa *M*** normally a fractional value in range [1.0,2.0].
  - **Exponent *E*** weights value by power of two

- **Encoding**
  - MSB S is sign bit ***s***
  - **exp** field encodes ***E*** (but is not equal to E)
  - **frac** field encodes ***M*** (but is not equal to M)

s	exp	frac

Precision options

- **Single precision: 32 bits**

  ≈ 7 decimal digits, $10^{\pm 38}$

s	exp	frac
1	8-bits	23-bits

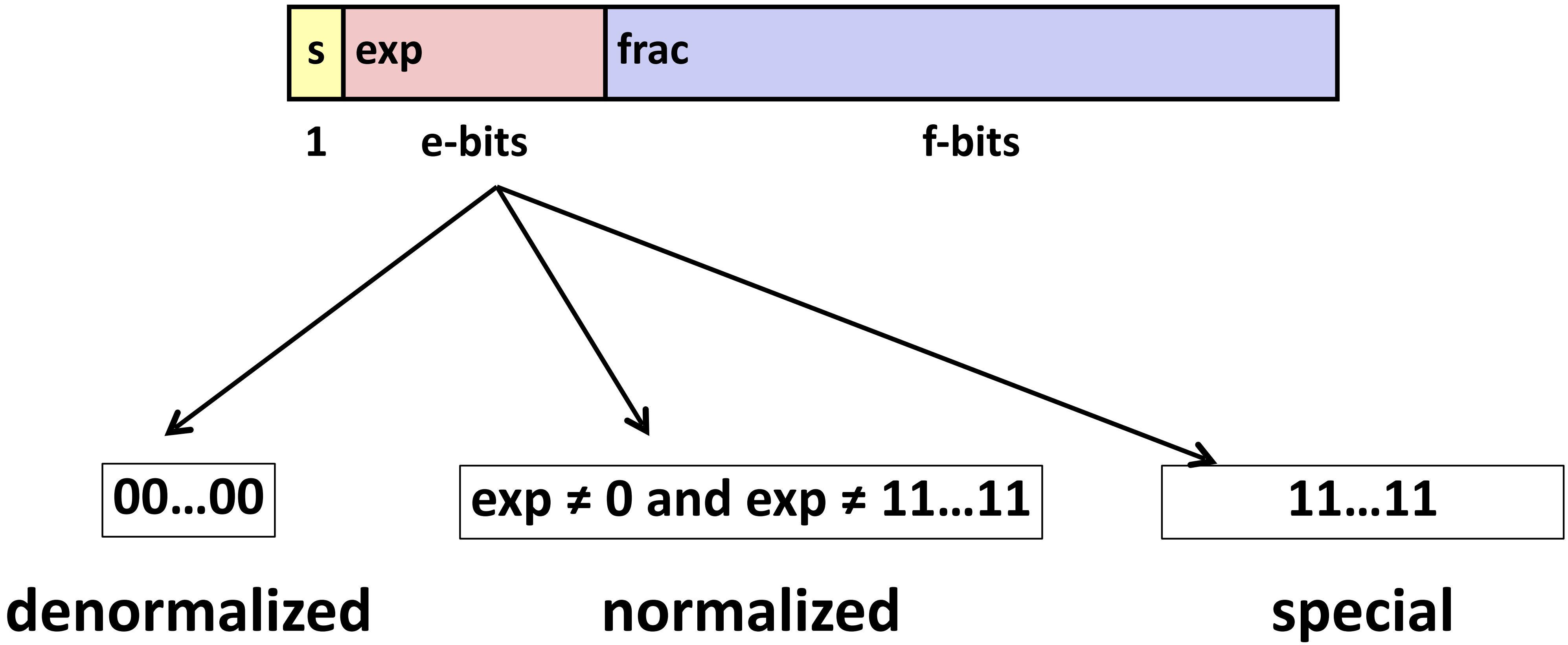
- **Double precision: 64 bits**

  ≈ 16 decimal digits, $10^{\pm 308}$

s	exp	frac
1	11-bits	52-bits

- **Other formats: half precision, quad precision**

Three “kinds” of floating point numbers



"Normalized" Values

$v = (-1)^s\, M\, 2^E$

- **When: `exp` ≠ 000…0 and `exp` ≠ 111…1 (2^8 - 1 = 255 in single precision and 2^{11} - 1 = 2047 in double precision)**
- **Exponent E coded as a *biased* value: *E* = `exp` – *Bias***
  - *exp*: unsigned value of exp field
  - *Bias* = **2^{k-1} - 1**, where *k* is number of exponent bits
    - **Single precision**: k = 8, Bias = 127 (**`exp`**: 1…254, **E: -126…127**)
    - **Double precision**: k = 11, Bias = 1023 (**`exp`**: 1…2046, **E: -1022…1023**)
- **Significand M coded with implied leading 1: *M* = 1.xxx…x$_2$**
  - xxx…x: bits of frac field *f* (0 <= *f* < 1)……. ***M* = 1 + *f***
  - Minimum when **`frac`**=000…0 (**M = 1.0**)
  - Maximum when **`frac`**=111…1 (**M = 2.0 – ε**)
  - *Get extra leading bit for **"free"***

Normalized Encoding Example

$$v = (-1)^s \, M \, 2^E$$
$$E = \mathbf{exp} - Bias$$

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  M = $1.\underline{1101101101101}_2$

  `frac` = $\underline{1101101101101}0000000000_2$

- **Exponent**

  E = 13

  $Bias$ = 127

  `exp` = 140 = 10001100_2

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

Denormalized Values

$$v = (-1)^s \, M \, 2^E$$
$$E = 1 - Bias$$

- **Helps us represent value 0 since normalized format always has M >= 1 and represent numbers that are very close to 0.0**
- **Condition:** exp = 000…0
- **Significand coded with implied leading 0: $M = 0.xxx\ldots x_2$**
  - `xxx…x`: bits of `frac` f ……… `M` = f
- **Exponent value: $E = 1 - \text{Bias}$ (instead of `exp` – *Bias*)** (**why?**)

  Normalized = *Smallest number* is 1.0 x 2^{-126}

  Denormalized = 0.1111111111…. x 2^{-126} (slightly smaller than 1 and approaching 0)
- **Cases**
  - `exp` = 000…0, `frac` = 000…0
    - Represents zero value 0.0
    - Note distinct values: +0.0 and –0.0 (why?) — does not account for sign bit - the two zeros are considered different in some ways and same in others under IEEE floating-point format
  - `exp` = 000…0, `frac` ≠ 000…0
    - Numbers closest to 0.0
    - Equispaced (spaced evenly near 0.0)
    - *Gradual underflow* property

Special Values

- **Condition: `exp` = `111…1`**

- **Case: `exp` = `111…1`, `frac` = `000…0`**
  - **Represents value ∞ (infinity)**
  - Operation that overflows
  - Both positive (+∞) and negative (-∞)
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞
- **Case: `exp` = `111…1`, `frac` ≠ `000…0`**
  - **Not-a-Number (NaN)**
  - When result of an operation is not real number of infinity
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0 …. Imaginary number

C float Decoding Example

$$v = (-1)^s \, M \, 2^E$$
$$E = \text{exp} - \textit{Bias}$$

Bias = $2^{k-1} - 1 = 127$

float: `0xC0A00000`

binary: ____ ____ ____ ____ ____ ____ ____ ____

E =

S =

M =

v = $(-1)^s$ M 2^E =

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example #1

$$v = (-1)^s M\, 2^E$$
$$E = \text{exp} - Bias$$

float: 0xC0A00000

binary: 1100 0000 1010 0000 0000 0000 0000 0000

1	1000 0001	010 0000 0000 0000 0000 0000
1	8-bits	23-bits

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
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7	7	0111
8	8	1000
9	9	1001
A	10	1010
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C	12	1100
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E	14	1110
F	15	1111

E =

S =

M = 1 .

$v = (-1)^s M\, 2^E =$

C float Decoding Example #1

$$v = (-1)^s M\, 2^E$$
$$E = \text{exp} - Bias$$

$Bias = 2^{k-1} - 1 = 127$

float: 0xC0A00000

binary: 1100 0000 1010 0000 0000 0000 0000 0000

1	1000 0001	010 0000 0000 0000 0000 0000
1	8-bits	23-bits

Hex	Decimal	Binary
0	0	0000
1	1	0001
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8	8	1000
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A	10	1010
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F	15	1111

E = exp – Bias = 129 – 127 = 2 (decimal)

S = 1 -> negative number

M = 1.010 0000 0000 0000 0000 0000

= 1 + 1/4 = 1.25

$v = (-1)^s M\, 2^E = (-1)^1 * 1.25 * 2^2 = -5$

C float Decoding Example #2

$$v = (-1)^s M\, 2^E$$
$$E = 1 - Bias$$

float: 0x001C0000

binary: 0000 0000 0001 1100 0000 0000 0000 0000

0	0000 0000	001 1100 0000 0000 0000 0000
1	8-bits	23-bits

E =

S =

M = 0 .

v = $(-1)^s$ M 2^E =

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example #2

$$v = (-1)^s \, M \, 2^E$$
$$E = 1 - Bias$$

$Bias = 2^{k-1} - 1 = 127$

float: 0x001C0000

binary: 0000 0000 0001 1100 0000 0000 0000 0000

0	0000 0000	001 1100 0000 0000 0000 0000
1	8-bits	23-bits

E = 1 – Bias = 1 – 127 = –126 (decimal)

S = 0 -> positive number

M = 0.001 1100 0000 0000 0000 0000

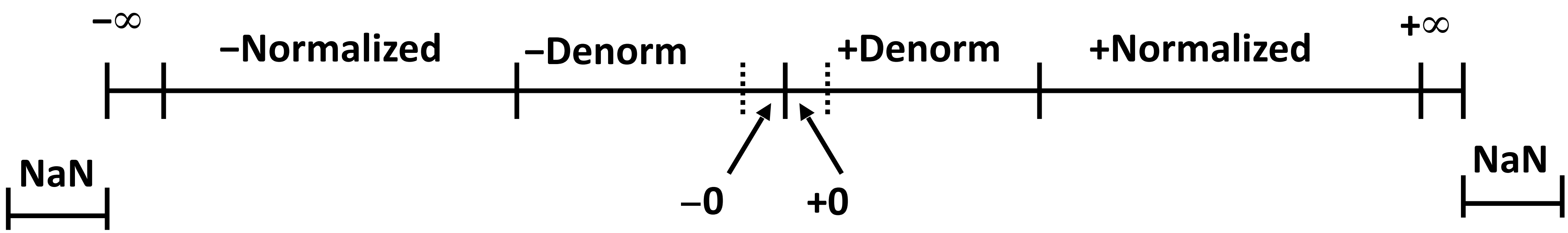
= 1/8 + 1/16 + 1/32 = 7/32 = $7 * 2^{-5}$

$v = (-1)^s \, M \, 2^E = (-1)^0 * 7*2^{-5} * 2^{-126} = 7*2^{-131}$

$\approx 2.571393892 \times 10^{-39}$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Visualization: Floating Point Encodings



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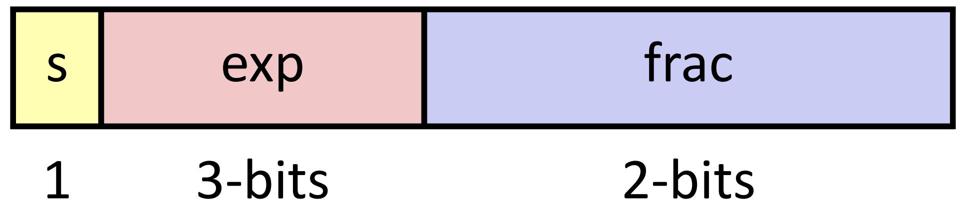
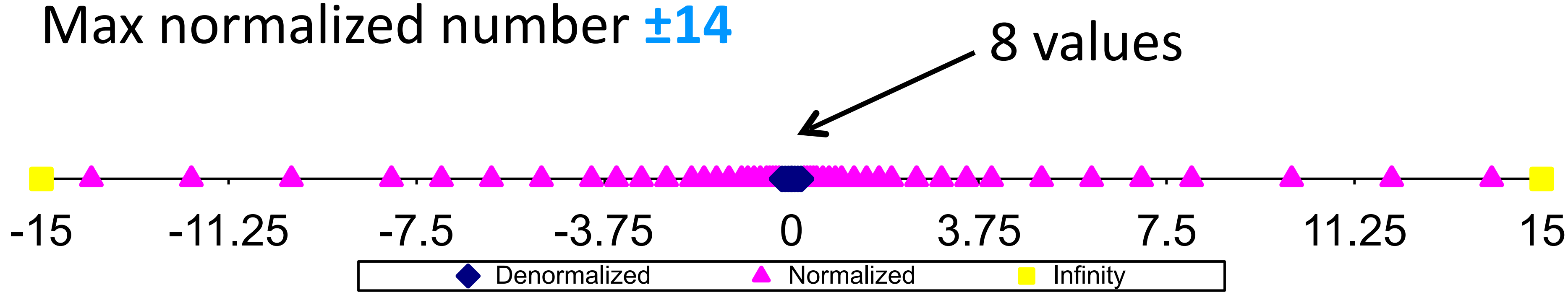
Distribution of Values

- **6-bit IEEE-like hypothetical format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$



- **Notice how the distribution gets denser toward zero.**

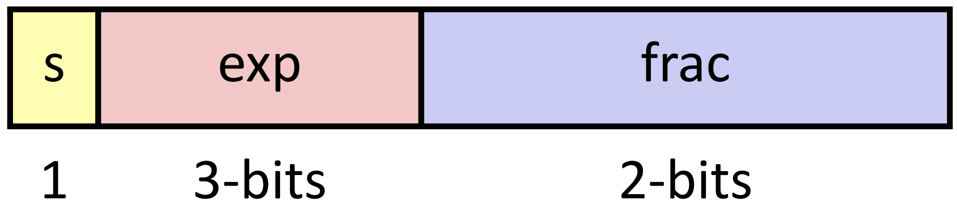
  Max normalized number **±14**

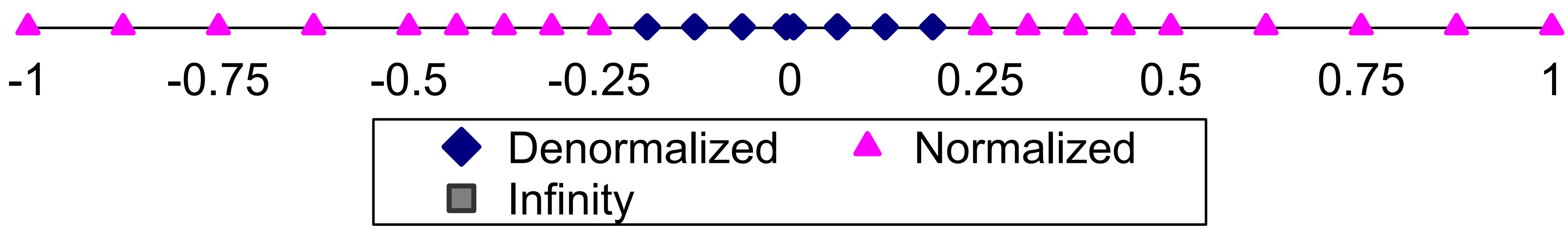


Distribution of Values (close-up view)

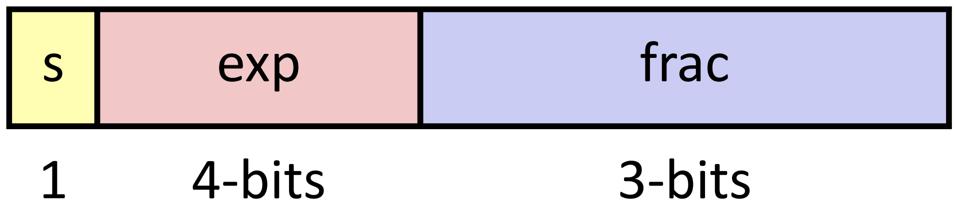
6-bit IEEE-like hypothetical format

- e = 3 exponent bits
- f = 2 fraction bits
- Bias is 3
- Denormalized (equispaced - spaced evenly near 0)
- Normalized (***not*** equispaced)





Tiny Floating Point Example



8-bit Hypothetical Floating Point Representation

- the sign bit is in the most significant bit
- the next four bits are the **`exp`**, with a bias of 7
- the last three bits are the **`frac`**

Same general form as IEEE Format

- normalized, denormalized
- representation of 0, NaN, infinity

Dynamic Range (s=0 only)

$$v = (-1)^s\, M\, 2^E$$

norm: E = `exp` *– Bias*

denorm: E = 1 – Bias

	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	$(-1)^0(0+1/4)*2^{-6}$
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	$(-1)^0(1+1/8)*2^{-6}$
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Note: As floating point numbers increase, bit patterns also show an increase (similar to unsigned int)

Interesting Numbers

`{single,double}`

Description	*exp*	*frac*	*Numeric Value*
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
▪ Single ≈ 1.4×10^{-45}			
▪ Double ≈ 4.9×10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
▪ Single ≈ 1.18×10^{-38}			
▪ Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
▪ Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
▪ Single ≈ 3.4×10^{38}			
▪ Double ≈ 1.8×10^{308}			

Special Properties of the IEEE Encoding

- **FP Zero Same as Integer Zero**
  - All bits = 0
- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield? The answer is complicated.
  - **Otherwise OK**
    - **Denorm vs. normalized**
    - **Normalized vs. infinity**

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Floating Point Operations: Basic Idea

- $x +_f y = \text{Round}(x + y)$

- $x \times_f y = \text{Round}(x \times y)$

- **Basic idea**
  - First compute exact result (say $x +_f y$ or $x \times_f y$)
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

Rounding Modes (illustrate with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
Towards zero	$1 ↓	$1 ↓	$1 ↓	$2 ↓	–$1 ↑
Round down (−∞)	$1 ↓	$1 ↓	$1 ↓	$2 ↓	–$2 ↓
Round up (+∞)	$2 ↑	$2 ↑	$2 ↑	$3 ↑	–$1 ↑
Nearest Even* (default)	$1 ↓	$2 ↑	$2 ↑	$2 ↓	–$2 ↓

*Round to nearest, but if half-way in-between then round to nearest even

Closer Look at Round-To-Even

Default Rounding Mode

- **Hard** to get any other kind **without dropping into assembly**
  - C99 has support for rounding mode management
- All others are statistically biased
  - Sum of set of positive numbers will consistently be over- or under-estimated

Applying to Other Decimal Places / Bit Positions

- When exactly halfway between two possible values
  - Round so that least significant digit is even
- E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

Binary Fractional Numbers

- "Even" when least significant bit is `0`
- "Half way" when bits to right of rounding position = `100`…$_2$

Examples

- Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

Rounding In Hardware

1.BBGRXXX

Round to nearest 1/8

Guard bit: LSB of result

Sticky bit: OR of remaining bits

Round bit: 1st bit removed

Round up conditions

- Round = 1, Sticky = 1 → > 0.5
- Guard = 1, Round = 1, Sticky = 0 → Round to even

Fraction	*GRS*	*Incr?*	*Rounded*
1.0000000	000	N	1.000
1.1010000	100	N	1.101
1.0001000	010	N	1.000
1.0011000	110	Y	1.010
1.0001010	011	Y	1.001
1.1111100	111	Y	10.000

FP Multiplication

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ \text{x}\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign *s*: $s1$ ^ $s2$
  - Significand *M*: $M1$ x $M2$
  - Exponent *E*: $E1 + E2$
- **Fixing**
  - If $M \geq 2$, shift *M* right, increment *E*
  - If *E* out of range, overflow
  - Round *M* to fit `frac` precision
- **Implementation**
  - Biggest chore is multiplying significands

Suppose a frac f of 3 bits

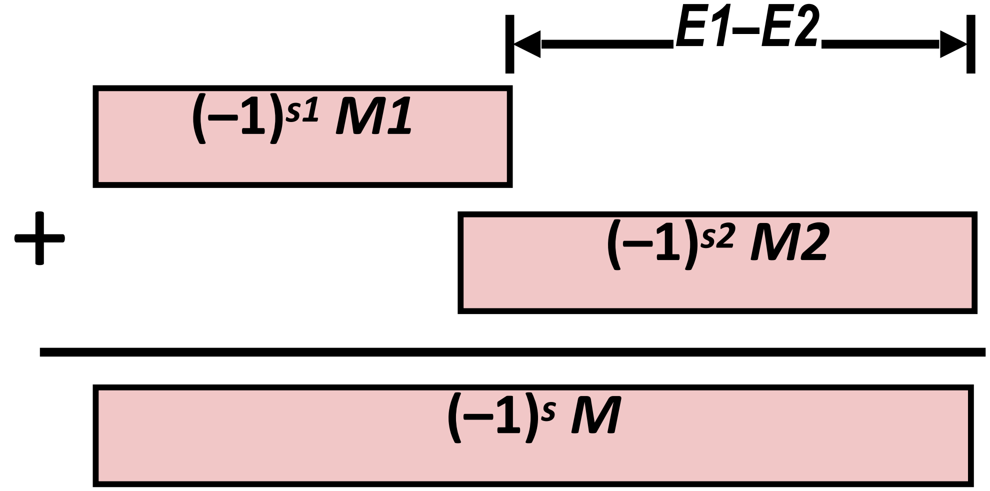
4 bit significand: `1.010*2`^{2} x `1.110*2`^{3} = `10.0011*2`^{5}

= `1.00011*2`^{6} = `1.001*2`^{6}

Floating Point Addition

- $(-1)^{s1}\,M1\ \ 2^{E1} \;+\; (-1)^{s2}\,M2\ \ 2^{E2}$
  - Assume $E1 > E2$
- **Exact Result:** $(-1)^{s}\,M\ \ 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: $E1$
- **Fixing**
  - If $M \geq 2$, shift M right, increment E
  - if $M < 1$, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision

Get binary points lined up



Suppose a frac f of 3 bits

$$1.010 * 2^{2} + 1.110 * 2^{3} = (0.1010 + 1.1100) * 2^{3}$$
$$= 10.0110 * 2^{3} = 1.00110 * 2^{4} = 1.010 * 2^{4}$$

Mathematical Properties of FP Add

Compare to those of Abelian Group

- Closed under addition? ***Yes***
  - Sum of two FP will always be FP, but may generate infinity or NaN
- Commutative? ***Yes***
- Associative? ***No***
  - Overflow and inexactness of rounding
  - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
- 0 is additive identity? ***Yes***
- Every element has additive inverse? ***Almost***
  - Yes, except for infinities & NaNs

Monotonicity

- $a \geq b \Rightarrow a+c \geq b+c$? ***Almost***
  - Except for infinities & NaNs

Mathematical Properties of FP Mult

Compare to Commutative Ring

- Closed under multiplication? **Yes**
  - Multiplication of two FP will always be FP, but may generate infinity or NaN
- Multiplication Commutative? **Yes**
- Multiplication is Associative? **No**
  - Possibility of overflow, inexactness of rounding
  - Ex: `(1e20*1e20)*1e-20= inf`, `1e20*(1e20*1e-20)= 1e20`
- 1 is multiplicative identity? **Yes**
- Multiplication distributes over addition? **No**
  - Possibility of overflow, inexactness of rounding
  - `1e20*(1e20-1e20)= 0.0`, `1e20*1e20 – 1e20*1e20 = NaN`

Monotonicity

Almost

- $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$?
  - Except for infinities & NaNs

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Floating Point in C

- **C Guarantees Two Levels**
  - `float` single precision
  - `double` double precision

- **Conversions/Casting**
  - Casting between `int`, `float`, and `double` changes bit representation
  - `double`/`float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - `int` → `double`
    - Exact conversion, as long as `int` has ≤ 53 bit word size
  - `int` → `float`
    - Will *lose precision* and round according to rounding mode

Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither **d** nor **f** is NaN

- `x == (int)(float) x` ✗
- `x == (int)(double) x` ✓
- `f == (float)(double) f` ✓
- `d == (double)(float) d` ✗
- `f == -(-f);` ✓
- `2/3 == 2/3.0` ✗
- `d < 0.0` $\Rightarrow$ `((d*2) < 0.0)` ✓
- `d > f` $\Rightarrow$ `-f > -d` ✓
- `d * d >= 0.0` ✓
- `(d+f)-d == f` ✗

Summary

- **IEEE Floating Point has clear mathematical properties**
- **Represents numbers of form M x 2^E**
- **One can reason about operations independent of implementation**
  - As if computed with perfect precision and then rounded
- **Not the same as real arithmetic**
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers