2.1 - Mathematical definitions

Wednesday, May 24, 2023 7:40 PM

EXERCISE 2.1.1: Even and odd integers.

Indicate whether each integer n is even or odd. If n is even, show that n equals 2k, for some integer k. If n is odd, show that n equals 2k+1, for some integer k.

(a) n = -1

(b) n = -101

(a) $n=-1$ is odd

$n=2k+1$

$-1=2k+1$

$-2=2k$

$-1=k$

$\therefore$ since $n=2(-1)+1=-2+1=-1$, $n=-1$ is odd

(b) $n=-101$ is odd

$n=2k+1$

$-101=2k+1$

$-102=2k$

$-51=k$

$\therefore$ since $n=2(-51)+1=-102+1=-101$, $n=-101$ is odd

EXERCISE 2.1.3: Showing a number is rational.

Show that each number n is rational by showing that n is equal to the ratio of two integers, where the denominator is non-zero.

(a) n = .25

(b) n = -5

(a) $r=\frac{x}{y}=\frac{25}{100}=\frac{1}{4}=0.25$ where $x,y\in\mathbb{Z}$ & $y\neq 0$

(b) $r=\frac{x}{y}=\frac{-5}{1}=-5$ where $x,y\in\mathbb{Z}$ & $y\neq 0$

EXERCISE 2.1.5: Prime and composite numbers.

Indicate whether each number n is prime, composite or neither. If n is composite give a divisor of n that is less than n and greater than 1.

(a) n = 0

(b) n = 1

(a) $0 \ngtr 1$

$n > 1$ for both prime & composite numbers,

$\therefore$ $n=0$ is neither prime nor composite

(b) $1 \ngtr 1$

$n > 1$ for both prime & composite numbers,

$\therefore$ $n=1$ is neither prime nor composite

2.2 - Introduction to proofs

Wednesday, May 24, 2023 8:44 PM

EXERCISE | 2.2.1: Methods of proof.

Determine whether each statement is true or false. Provide a justification for each answer.

(a) Showing that a statement holds for a few cases is sufficient to prove a universal statement.

(b) Providing one example when the statement holds is sufficient to prove an existential statement.

(a) False, we would need to prove statement expressed as predicate $P(x)$ is always true. $\forall x P(x)$. We cannot prove the statement for every instance to infinity.

Example: all colors are red. This is true for red but not green.

(b) True. This is largely the definition of the existential modifier that a single true statement evaluates to true.

Example: at most one computer has windows 10, if a computer lab has many computers but only one with windows 10, then the statement is true.

EXERCISE 2.2.2: Proof by exhaustion.

Prove each statement using a proof by exhaustion.

(a) For every integer n such that $0 \leq n < 3$, $(n+1)^2 > n^3$.

$n=0$ $\quad (0+1)^2 > (0)^3$

$(1)^2 > (0)^3$

$\therefore 1 > 0$ (true)

$n=1$ $\quad (1+1)^2 > (1)^3$

$(2)^2 > 1$

$\therefore 4 > 1$ (true)

$n=2$ $\quad (2+1)^2 > (2)^3$

$(3)^2 > 8$

$\therefore 9 > 8$ (true)

$\therefore$ statement holds true with proof by exhaustion

EXERCISE 2.2.3: Find a counterexample.

Find a counterexample to show that each of the statements is false.

(b) If n is an integer and n^2 is divisible by 4, then n is divisible by 4.

$n, r \in \mathbb{Z}$ $\quad \frac{n^2}{4} = r \rightarrow \frac{n}{4} = r$

Counterexample: $n = 6$

$$\frac{(6)^2}{4} = \frac{36}{4} = 9 \in \mathbb{Z} \text{ (true)}$$

$$\frac{6}{4} \notin \mathbb{Z} \quad \text{(false)}$$

$T \rightarrow F = F$

EXERCISE | 2.2.5: Proving existential statements.

Prove each existential statement given below.

(a) There are positive integers x and y such that $\frac{1}{x} + \frac{1}{y}$ is an integer.

Let $x = 1, y = 1$ where $x, y \in \mathbb{N}$

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{1} + \frac{1}{1} = 1 + 1 = 2$$

$\therefore$ there exists some positive integers x & y s.t. $\frac{1}{x} + \frac{1}{y} = r \in \mathbb{Z}$ (true)

2.3 - Best practices and common errors in proofs

Wednesday, May 24, 2023 9:33 PM

EXERCISE 2.3.1: Fill in the words to form a complete proof.

Use the given equations in a complete proof of each theorem. Your proof should be expressed in complete English sentences.

(a) **Theorem:** If a, b, and c are integers such that $a^3|b$ and $b^2|c$, then $a^6|c$.

$b = ka^3$
$c = jb^2$
$c = jb^2 = j(ka^3)^2 = (jk^2)a^6$

Let a, b, c be integers. There exist some integers k, j such that $b = ka^3$ & $c = jb^2$. We can substitute $b = ka^2$ into $c = j(ka^3)^2$ to obtain $c = jk^2a^6$. If we let $jk^2 = u$, then we obtain $c = ua^6$. Thus, we can prove that if $a^3|b$ & $b^2|c$, then $c|a^6$ given $a, b, c \in \mathbb{Z}$.

EXERCISE 2.3.2: Find the mistake in the proof - integer division.

Theorem: If w, x, y, and z are integers where w divides x and y divides z, then wy divides xz.

For each "proof" of the theorem, explain where the proof uses invalid reasoning or skips essential steps.

(a) **Proof.**
Let w, x, y, z be integers such that w divides x and y divides z. Since, by assumption, w divides x, then $x = kw$ for some integer k and $w \neq 0$. Since, by assumption, y divides z, then $z = ky$ for some integer k and $y \neq 0$. Plug in the expression kw for x and ky for z in the expression xz to get

$$xz = (kw)(ky) = (k^2)(wy)$$

Since k is an integer, then k^2 is also an integer. Since $w \neq 0$ and $y \neq 0$, then $wy \neq 0$. Since xz equals wy times an integer and $wy \neq 0$, then wy divides xz. ■

(b) **Proof.**
Let w, x, y, and z be integers such that w divides x and y divides z. Since, by assumption, w divides x, then $x = kw$ for some integer k and $w \neq 0$. Since, by assumption, y divides z, then $z = jy$ for some integer j and $y \neq 0$. Since $w \neq 0$ and $y \neq 0$, then $wy \neq 0$. Let m be an integer such that $xz = m \cdot wy$. Since xz equals wy times an integer and $wy \neq 0$, then wy divides xz. ■

(a) This proof is invalid b/c we assume some integer k exists to satisfy both some integer k exists to satisfy both conditions $x=kw$ & $z=ky$. It would make more sense to permit an additional variable.

(b) The proof is invalid b/c it should consider $xz = m \cdot wy$ as its own product variable. There is no proof that some integer m exists.

EXERCISE 2.3.3: Find the mistake in the proof - odd and even numbers.

Theorem: If n and m are odd integers, then $n^2 + m^2$ is even

For each "proof" of the theorem, explain where the proof uses invalid reasoning or skips essential steps.

(a) **Proof.**
m = 7 is odd because 7 = 2·3+1. n = 9 is odd because 9 = 2·4 + 1.

$$7^2 + 9^2 = 49 + 81 = 130 = 2 \cdot 65$$

Since $7^2 + 9^2$ is equal to 2 times an integer, $7^2 + 9^2$ is even. Therefore the theorem is true. ■

(b) **Proof.**
Let n and m be odd integers. Since n is an odd integer, then n = 2k+1. Since m is an odd integer, then m = 2j+1. Plugging in 2k+1 for n and 2j+1 for m into the expression $n^2 + m^2$ gives

$$n^2 + m^2 = (2k+1)^2 + (2j+1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 2(2k^2 + 2k + 2j^2 + 2j + 1)$$

Since k and j are integers, $2k^2 + 2k + 2j^2 + 2j + 1$ is also an integer. Since $n^2 + m^2$ is equal to two times an integer, then $n^2 + m^2$ is an even integer. ■

(a) This specific example holds true but does not universally prove n^2+m^2 is always even, only that 7^2+9^2 is even.

(b) Because $2k^2+2k+2j^2+j+1$ where $k, j \in \mathbb{Z}$ is an integer. n^2+m^2 being two times some integer implies the final sum will be an even integer.

Although the proof skips the step of proving that double any integer is an even integer $-2k$.

2.4 - Writing direct proofs

Thursday, May 25, 2023 10:01 AM

EXERCISE 2.4.1: Proving statements about odd and even integers with direct proofs.

Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as $2k+1$, where k is an integer. An even integer is an integer that can be expressed as $2k$, where k is an integer.

Prove each of the following statements using a direct proof.

(b) The sum of two odd integers is an even integer.

$k, j \in \mathbb{Z}$

$(2k+1)+(2j+1) = 2k+2j+2 = 2(k+j+1)$

$(k+j+1) \in \mathbb{Z}$

$\therefore (2k+1)+(2j+1)$ is an even integer

EXERCISE 2.4.2: Proving statements about rational numbers with direct proofs.

Prove each of the following statements using a direct proof.

(b) The quotient of a rational number and a non-zero rational number is a rational number.

$p, q \in \mathbb{Q}$ & $q \neq 0$

$\exists a, b \in \mathbb{Z}$ s.t. $p = \frac{a}{b}$, $b \neq 0$

$\exists c, d \in \mathbb{Z}$ s.t. $q = \frac{c}{d}$, $d \neq 0$ (b/c $q \neq 0$, then $c \neq 0$)

$\frac{p}{q} = p \cdot \frac{1}{q} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

$ad, bc \in \mathbb{Z}$ (product of integers)

$\therefore \frac{?}{y} \in \mathbb{Q}$

EXERCISE 2.4.4: Showing a statement is true or false by direct proof or counterexample.

Determine whether the statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample.

(a) If x and y are even integers, then $x + y$ is an even integer.

(b) If $x + y$ is an even integer, then x and y are both even integers.

(a) $x, y \in \mathbb{Z}$; $k, j \in \mathbb{Z}$

$x+y = 2k+2j = 2(k+j)$

$k+j \in \mathbb{Z}$

$\therefore$ $x+y$ is an even integer

True.

(b) False.

odd + odd = even

Counterexample: $3+1=4$

2.5 - Proof by contrapositive

Thursday, May 25, 2023 11:33 AM

EXERCISE 2.5.1: Proof by contrapositive of statements about odd and even integers.

Prove each statement by contrapositive

(a) For every integer n, if n^2 is odd, then n is odd.

$\forall n \in \mathbb{Z}$

contrapositive:

suppose n is even ($\neg$ conclusion)

then $n = 2k$, $k \in \mathbb{Z}$

$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$

$2k^2$ is an even integer

$\therefore n^2$ is an even integer ($\neg$ hypothesis)

EXERCISE 2.5.2: Proof by contrapositive of statements about integer division.

Prove each statement by contrapositive

(a) If x and y are integers such that $3 \nmid xy$ then $3 \nmid x$.

$x, y \in \mathbb{Z}$ s.t. $3 \mid xy$ then $3 \mid x$ ($\neg$ conclusion)

$x = 3k$, $k \in \mathbb{Z}$ (derived from $3 \mid x$)

$xy = 3ky$

$xy = 3(k_y)$

$\therefore\ 3 \mid xy$ ($\neg$ hypothesis)

2.6 - Proof by contradiction

Thursday, May 25, 2023 12:12 PM

EXERCISE 2.6.1: Rational and irrational numbers.

You can use the fact that $\sqrt{2}$ is irrational to answer the questions below. You can also use other facts proven within this exercise.

(b) Prove that $2 - \sqrt{2}$ is irrational.

Assume $(2-\sqrt{2}) \in \mathbb{Q}$

$$(2-\sqrt{2}) = \frac{p}{q} ; \quad p, q \in \mathbb{Z}$$

$$2 - \sqrt{2} - \frac{p}{q} = 0$$

$$\frac{2}{1} - \frac{p}{q} = \sqrt{2}$$

$$\frac{2q - p}{q} = \sqrt{2}$$

$\therefore \sqrt{2} \in \mathbb{Q}$ b/c $q, p \in \mathbb{Z}$

This contradicts the fact that $\sqrt{2} \in \mathbb{I}$ (irrational)

So in our proof by contradiction,

$2 - \sqrt{2} \in \mathbb{I}$

$2-\sqrt{2} \in \mathbb{I}$

2.7 - Proof by cases

Thursday, May 25, 2023 12:58 PM

EXERCISE 2.7.1: Proofs by cases - statements about numbers.

Prove each statement.

(a) For every real number x, $x^2 \geq 0$.

(b) For every integer n, $n^2 \geq n$.

(a) Case 1: $x \geq 0$

$(x \geq 0) \wedge (x \in \mathbb{R})$

$\therefore (x^2 \geq 0) \wedge (x^2 \in \mathbb{R})$

Case 2: $x < 0$

$(x < 0) \wedge (x \in \mathbb{R})$

$\therefore (x^2 \geq 0) \wedge (x^2 \in \mathbb{R})$

from cases 1 & 2:

$\forall x \in \mathbb{R}, x^2 \geq 0$

(b) $\forall n \in \mathbb{Z}$

Case 1: $n = 0$

$n^2 \geq n$

$(0)^2 \geq (0)$

$0 \geq 0$ True

Case 2: $n = 1$

$n^2 \geq n$

$(1)^2 \geq (1)$

$1 \geq 1$

Case 3: $n = 2$

$n^2 \geq n$

$$(2)^2 \geq 2$$

$$4 \geq 2$$

$$\forall (n > 1),\ n^2 \geq n \Rightarrow (n^2 - n) = n(n-1)$$

$$\therefore\ n^2 \geq n,\ n \in \mathbb{Z}$$