6.4 - Composition of relations

Saturday, June 17, 2023 4:59 PM

$$A \circ B = \{(x,y) \mid \exists z\,((x,z) \in B \;\&\; (z,y) \in A)\}$$

$$B \circ A = \{(x,y) \mid \exists z\,((x,z) \in A \;\&\; (z,y) \in B)\}$$

EXERCISE 6.4.1: Composition of relations expressed as a set of pairs.

Here are two relations defined on the set {a, b, c, d}:

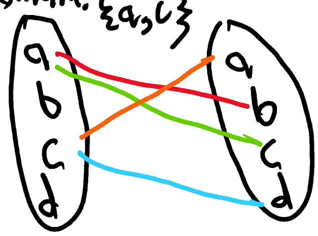
S = { (a, b), (a, c), (c, d), (c, a) }
R = { (b, c), (c, b), (a, d), (d, b) }

Write each relation as a set of ordered pairs.

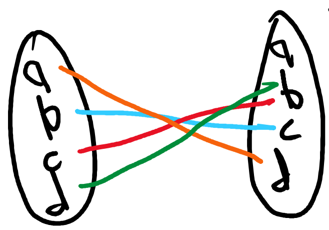
(a) S o R $\{(b,a), (b,d)\}$

(b) R o S $\{(a,c), (a,b), (c,d), (c,b)\}$

(a) domain: {a,c} S



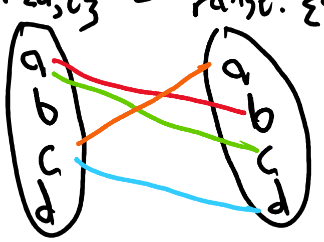
R range: {b,c,d}



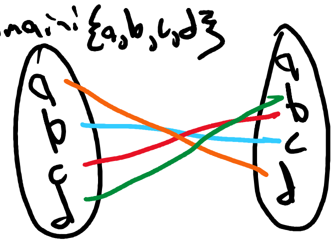
Domain of S & range of R share {c}, {c} maps to {b} in R & {a} & {d} in S.

(b)

domain: {a,c} S range: {a,b,c,d}



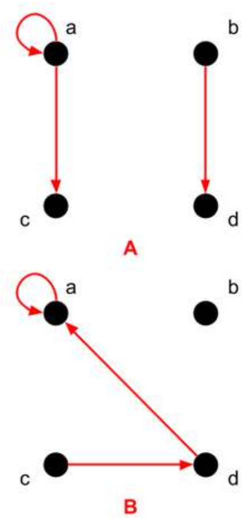
R domain: {a,b,c,d}



Range of S & Domain of R share $\{a,b,c,d\}$. However, they're limited by the domain of S, $\{a,c\}$.

EXERCISE | 6.4.3: Composition of relations and arrow diagrams.

The arrow diagrams for relations A and B are shown below. Both relations have the domain {a, b, c, d}.



$A = \{(a,a), (a,c), (b,d)\}$

$B = \{(a,a), (c,d), (d,a)\}$

(a) Draw the arrow diagram for $B \circ A$.

(b) Draw the arrow diagram for $A \circ B$.

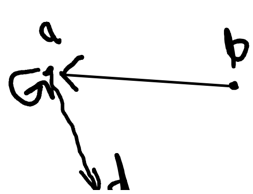
Feedback?

· match
· value

(a) $A = \{(a,a), (a,c), (b,d)\}$

$B = \{(a,a), (c,d), (d,a)\}$

$B \circ A = \{(a,a), (a,d), (b,a)\}$

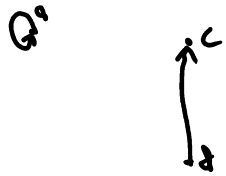


$B(x,z)$; $A(z,y)$

(b) $A = \{(a,a)\ (a,c), (b,d)\}$

$B = \{(a,a), (c,d), (d,a)\}$

$A \circ B = \{(a,a), (d,c)\}$

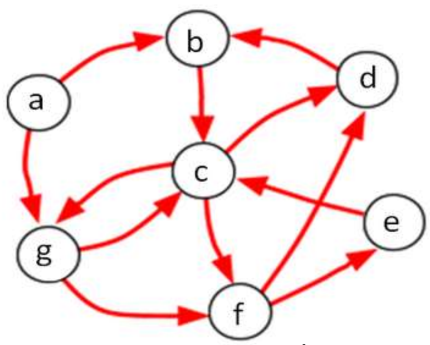


6.5 - Graph powers and the transitive closure

Saturday, June 17, 2023 6:53 PM

EXERCISE 6.5.1: Edges of graph powers.

The diagram below shows a directed graph G.



(a) Is (a, b) in G^2? No, there are no paths from a to b of length 2.

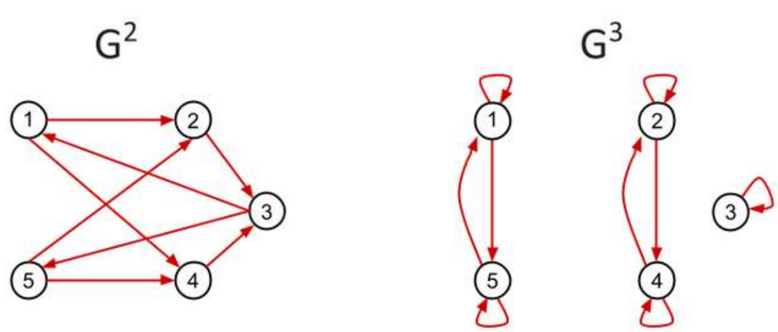
(b) Is (b, e) in G^3? Yes, path b→c→f→e has length 3.

(c) Is (g, g) in G^3? No, there are no paths from g to g of length 3.

(d) Is (g, g) in G^4? Yes, path g→f→e→c→g has length 4.

EXERCISE 6.5.4: Inferring facts about a graph from its graph powers.

The drawing below shows G^2 and G^3 for a graph G.



Use the information provided in G^2 and G^3 to answer the following questions about G.

(a) Is there a walk of length 3 from vertex 4 to vertex 5 in G? No, they're closed off from one another.

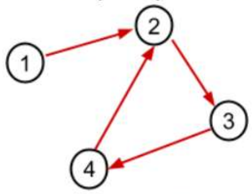
(b) Is there any closed walk of length 2 in G? Yes, there are several closed walks of length 2.

6.6 - Matrix multiplication and graph powers

Saturday, June 17, 2023 7:15 PM

EXERCISE 6.6.1: Adjacency matrices for graph powers and the transitive closure via matrix multiplication.

(a) Give the adjacency matrix for the graph below.



$\{(1,2), (2,3), (3,4), (4,2)\}$

4×4 matrix

Then use matrix multiplication to find the adjacency matrices for G^2, G^3, G^4, and G^+.

Feedback?

$$G^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$G^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$G^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G^4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$G^+ = G^1 \cup G^2 \cup G^3 \cup G^4$$

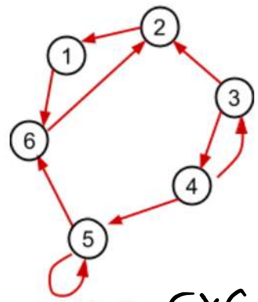
sum matrices:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

EXERCISE 6.6.2: Interpreting the adjacency matrix for a graph power.

A graph G is pictured below:

{(1,6), (2,1), (3,2), (3,4), (4,3), (4,5), (5,5), (5,6), (6,2)}



(a) Give the adjacency matrix for G. 6x6 matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

EXERCISE 6.6.3: Inferring information about a graph from matrices of graph powers.

A directed graph G has 5 vertices, numbered 1 through 5. The 5x5 matrix A is the adjacency matrix for G. The matrices A^2 and A^3 are given below.

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Use the information given to answer the questions about the graph G.

(a) Which vertices can reach vertex 2 by a walk of length 3? 2, 4, 5.

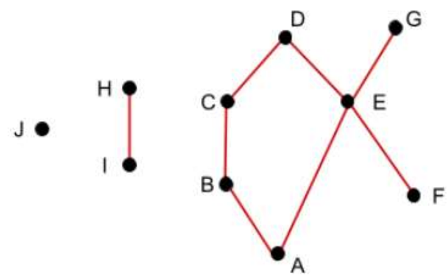
(b) What is the out-degree of vertex 4 in the transitive closure of G? 5, we can reach all verticies from 4.

6.7 - Partial orders

Saturday, June 17, 2023 7:57 PM

EXERCISE 6.7.1: Interpreting Hasse diagrams.

The drawing below shows a Hasse diagram for a partial order on the set {A, B, C, D, E, F, G, H, I, J}



(a) What are the minimal elements of the partial order? J, I, A, F which are not linked downwards.

(b) What are the maximal elements of the partial order? J, H, D, G which are not linked upwards.

(c) Which of the following pairs are comparable? (A, D), (G, F), (D, B), (H, I) are comparable.
(A, D), (J, F), (B, E), (G, F), (D, B), (C, F), (H, I), (C, E)

Feedback?

6.8 - Strict orders and directed acyclic graphs

Saturday, June 17, 2023 8:06 PM

EXERCISE 6.8.1: Identifying partial, strict, and total orders.

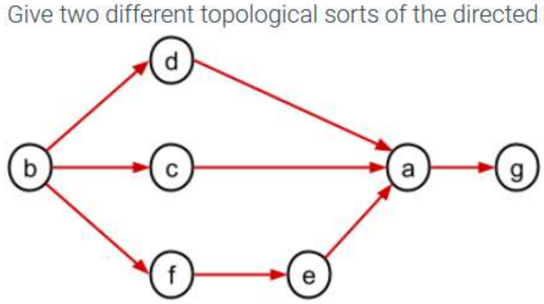
For each relation, indicate whether the relation is a partial order, a strict order, or neither. If the relation is a partial or strict order, indicate whether the relation is also a total order. Justify your answers.

(a) The domain is the set of all words in the English language (as defined by, say, Webster's dictionary). Word x is related to word y if x appears before y in alphabetical order. Assume that each word appears exactly once in the dictionary.

There is a strict & total order. There is a strict order b/c the set of all words in the English language are transitive & anti-reflexive. The set is anti-reflexive b/c no word can appear before itself with only one instance. The set is transitive b/c if x appears before y, then a word after y appears after x too. There is a total order b/c the set has every pair of elements comparable by being alphabetized.

EXERCISE 6.8.2: Topological sort of a DAG.

(a) Give two different topological sorts of the directed acyclic graph shown below.



1.) b→d→c→f→e→a→g

2.) b→c→f→e→d→a→g

6.9 - Equivalence relations

Saturday, June 17, 2023 9:08 PM

EXERCISE 6.9.2: Equivalence classes for remainders of integer division.

(a) The domain of the equivalence relation D is the set S:

$$S = \{7, 2, 13, 44, 56, 34, 99, 31, 4, 17\}$$

For any x, y ∈ S, xDy if x has the same remainder as y when divided by 4. Show the partition of S defined by the equivalence classes of D.

Feedback?

$x \in S/4$	Remainder
7	3
2	2
13	1
44	0
56	0
34	2
99	3
31	3
4	0
17	1

Remainder 0, $A = \{44, 56, 4\}$

Remainder 1, $B = \{13, 17\}$

Remainder 2, $C = \{2, 34\}$

Remainder 3, $E = \{7, 99, 31\}$

B/c $A \subseteq S$, $B \subseteq S$, $C \subseteq S$, $E \subseteq S$;

$A \neq \{\emptyset\}$, $B \neq \{\emptyset\}$, $C \neq \{\emptyset\}$, $E \neq \{\emptyset\}$;

$A \cap B \cap C \cap E = \{\emptyset\}$; $A \cup B \cup C \cup E = S$

then A, B, C, E are disjoint sets of S

B/c of the property $A \cup B \cup C \cup E = S$,

the sets are equivalent classes of D