8.3 - Summations

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EXERCISE 8.3.1: Evaluating summations.

Evaluate the following summations.

(a) $\sum_{k=-1}^{4} k^2$. $(-1)^2+(0)^2+(1)^2+(2)^2+(3)^2+(4)^2 = 1+0+1+4+9+16=31$

(b) $\sum_{k=0}^{4} 2^k$. $2^0+2^1+2^2+2^3+2^4 = 1+2+4+8+16=31$

EXERCISE 8.3.2: Expressing sums using summation notation.

Express the following sums using summation notation.

(a) $(-2)^5 + (-1)^5 + \cdots + 7^5$ $\sum_{k=-2}^{7} [k^5]$

(b) $(-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5$ $\sum_{k=-2}^{5} [k]$

8.4 - Mathematical induction

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EXERCISE 8.4.1: Components of an inductive proof.

Define P(n) to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that P(3) is true.

(b) Express P(k).

(a)

Left-hand side (LHS)

$P(3) = 1^2 + 2^2 + 3^2 = 1+4+9 = 14$

Right-hand side (RHS)

$P(3) = \frac{3(3+1)(2\cdot 3+1)}{6} = \frac{3(4)(7)}{6} = \frac{84}{6} = 14$

$\therefore$ b/c RHS = LHS, P(3) is true

(b)

$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

EXERCISE 8.4.2: Proving identities by induction.

Prove each of the following statements using mathematical induction.

(a) Prove that for any positive integer n, $\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$

Base case: $n=1$

LHS: $1^2 = 1$

RHS: $\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{1(2)}{2}\right)^2 = (1)^2 = 1$

$\therefore$ b/c LHS = RHS, $\sum_{j=1}^{n} j^3 = \left(\frac{1(1+1)}{2}\right)^2$.

Inductive step: Suppose that for positive integer k, $\sum_{j=1}^{k} j^3 = \left(\frac{k(k+1)}{2}\right)^2$.

Then we will show that $\sum_{j=1}^{k+1} j^3 = \left(\frac{k(k+1)(k+2)}{2}\right)^2$.

Starting with the left side of the equation to be proven:

$$\sum_{j=1}^{k+1} j^3 = \sum_{j=1}^{k} j^3 + (k+1)^3 \quad \text{(by separating out last term)}$$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \quad \text{(by the inductive hypothesis)}$$

$$= \frac{k^2(k+1)^2}{4} + \frac{(k+1)^3}{1}$$

$$= (k+1)^2\left[\frac{k^2}{4} + \frac{(k+1)}{1}\right]$$

$$= (k+1)^2\left[\frac{k^2 + 4(k+1)}{4}\right]$$

$$= (k+1)^2\left[\frac{k^2 + 4k + 4}{4}\right]$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2 \quad \text{(by algebra)}$$

$$\therefore \sum_{j=1}^{k+1} j^3 = \left(\frac{k(k+1)(k+2)}{2}\right)^2$$

EXERCISE 8.4.3: Proving inequalities by induction.

Prove each of the following statements using mathematical induction.

(a) Prove that for $n \geq 2, 3^n > 2^n + n^2$.

Base case: $n=2$

$3^2 = 9 > 2^2 + (2)^2 = 4+4 = 8$

$\therefore$ for $n=2$, $3^n > 2^n + n^2$

Inductive step: we will show that for any integer $k \geq 2$, if $3^k > 2^k + k^2$, then $3^{k+1} > 2^{k+1} + (k+1)^2$.

Starting with the left side of the equation to be proven:

$3^{k+1} = 3 \cdot 3^k$ (by algebra)

$> 3 \cdot (2^k + k^2)$ (by inductive hypothesis)

$= 3 \cdot 2^k + 3k^2$

$= 2 \cdot 2^k + 2^k + k^2 + 2k^2$

$= 2^{k+1} + 2^k + k^2 + 2k^2$ (by algebra)

$\geq 2^{k+1} + 1 + k^2 + 2k$ $(k \geq 1, 2^k \geq 1, k^2 \geq k)$

$\geq 2^{k+1} + 1 + k^2 + 2k \qquad (k \geq 1, 2^k \geq 1, k^2 \geq k)$

$= 2^{k+1} + (k+1)^2 \qquad$ (by algebra)

$\therefore 3^{k+1} > 2^{k+1} + (k+1)^2$

8.5 - More inductive proofs

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EXERCISE 8.5.1: Proving divisibility results by induction.

Prove each of the following statements using mathematical induction.

(a) Prove that for any positive integer n, 4 evenly divides 3^{2n}-1.

<u>Base case:</u> $n=1$

$3^{2(1)}-1 = 3^2-1 = 9-1 = 8$

Since 8 evenly divides 4, the theorem holds for the case $n=1$.

<u>Inductive step:</u> Suppose that for positive integer k, 4 evenly divides $3^{2k}-1$. Then we will show that 4 evenly divides $3^{2(k+1)}-1$.

By the inductive hypothesis, 4 evenly divides $3^{2k}-1$, which means that $3^{2k}-1=4m$ for some integer m. By adding 1 to both sides of the equation $3^{2k}-1=4m$, we get $3^{2k}=4m+1$ which is an equivalent statement of the inductive hypothesis.

We must show that $3^{2(k+1)}-1$ can be expressed as 4 times an integer.

can be expressed as 4 times an integer.

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1$$

$$= 3^2 \cdot 3^{2k} - 1$$

$$= 9 \cdot 3^{2k} - 1 \quad \text{(by algebra)}$$

$$= 9(4m+1) - 1 \quad \text{(by the inductive hypothesis)}$$

$$= 9 \cdot 4m + 9 - 1$$

$$= 9 \cdot 4m + 8$$

$$= 4(9m+2)$$

Since m is an integer, $(9m+2)$ is also an integer. Therefore $3^{2(k+1)} - 1$ is equal to 4 times an integer which means that $3^{2(k+1)} - 1$ is divisible by 4.

EXERCISE | 8.5.3: Proving explicit formulas for recurrence relations by induction.

Prove each of the following statements using mathematical induction.

(a) Define the sequence $\{c_n\}$ as follows:

- $c_0 = 5$
- $c_n = (c_{n-1})^2$ for $n \geq 1$

Prove that for $n \geq 0$, $c_n = 5^{2^n}$.

Note that in the explicit formula for c_n, the exponent of 5 is 2^n.

<u>Base case:</u> $n = 0$

we must show that $c_0 = 5^{2^0} = 5^1 = 5$. Since $5^{2^0} = 5^1 = 5$ & the initial condition states that $c_0 = 5$, the theorem holds for $n=0$.

Inductive case: we will show that for every $k \geq 0$, if $c_k = 5^{2^k}$ is true, then $c_{k+1} = 5^{2^{k+1}}$. The recurrence relation is equivalent to the statement $c_{k+1} = (c_k)^2$ for $k \geq 0$.

$$c_{k+1} = (c_k)^2 \quad \text{(by definition)}$$
$$= (5^{2^k})^2 \quad \text{(by inductive hypothesis)}$$
$$= 5^{2^k \cdot 2}$$
$$= 5^{2^{k+1}}$$

$\therefore \; c_{k+1} = 5^{2^{k+1}}$

8.6 - Strong induction and well-ordering

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EXERCISE | 8.6.2: Proofs by strong induction - explicit formulas for recurrence relations.

Prove each of the following statements using strong induction.

(a) The Fibonacci sequence is defined as follows:

- $f_0 = 0$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$

Prove that for $n \geq 0$,

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

<u>Base cases:</u> $n=0$

$$f_0 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right]$$

$$= \frac{1}{\sqrt{5}}\left[1-1\right]$$

$$= \frac{1}{\sqrt{5}}\left[0\right]$$ (by algebra)

$$= 0$$

$\therefore$ b/c $f_0 = 0$, proved

$n=1$

$$f_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}}{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\sqrt{5}\right]$$

$$= 1$$

$\therefore$ b/c $f_1 = 1$, proved

Note:

let $\phi = \frac{\sqrt{5}+1}{2}$

$\phi^{-1} = \frac{1}{\phi} = \frac{2}{\sqrt{5}+1} = \frac{2}{\sqrt{5}+1}\frac{(-\sqrt{5}-1)}{(-\sqrt{5}-1)} = \frac{\sqrt{5}-1}{2}$

$-\phi^{-1} = -\frac{\sqrt{5}-1}{2} = \frac{1-\sqrt{5}}{2}$

Inductive hypothesis: suppose $n=k$,

$$f_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] = \frac{1}{\sqrt{5}}\left[\phi^k - \left(-\frac{1}{\phi}\right)^k\right]$$

$$f_{k-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right] = \frac{1}{\sqrt{5}}\left[\phi^{k-1} - \left(-\frac{1}{\phi}\right)^{k-1}\right] = \frac{1}{\sqrt{5}}\left[\phi^k\cdot\frac{1}{\phi} - \left(-\frac{1}{\phi}\right)^k\cdot(-\phi)\right]$$

Inductive step: then $n=k+1$,

$$f_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] = \frac{1}{\sqrt{5}}\left[\phi^{k+1} - \left(-\frac{1}{\phi}\right)^{k+1}\right]$$

$= f_k + f_{k-1}$ (derived from $f_n = f_{n-1} + f_{n-2}$)

$$= \frac{1}{\sqrt{5}}\left[\phi^k - \left(-\frac{1}{\phi}\right)^k\right] + \frac{1}{\sqrt{5}}\left[\phi^k\cdot\frac{1}{\phi} - \left(-\frac{1}{\phi}\right)^k\cdot(-\phi)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\phi^k - \left(-\frac{1}{\phi}\right)^k + \phi^k\cdot\frac{1}{\phi} - \left(-\frac{1}{\phi}\right)^k\cdot(-\phi)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\phi^k\left(1+\frac{1}{\phi}\right) - \left(-\frac{1}{\phi}\right)^k(1-\phi)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\phi^k\cdot\phi - \left(-\frac{1}{\phi}\right)^k\left(-\frac{1}{\phi}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\phi^{k+1} - \left(-\frac{1}{\phi}\right)^{k+1}\right]$$

$\therefore f_2 = f_0 + f_1 = 2$

hence proved $\forall n (n \geq 0)$

8.8 - Recursive definitions

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EXERCISE 8.8.2: Devising recursive definitions for sets of strings.

Let A = {a, b}.

(a) Give a recursive definition for A^*.

Base case: $\lambda \in A^*$

Recursive rule: if $x \in A^*$ then,

$xa \in A^*$

$xb \in A^*$

Addendum:
Every binary string can be constructed by starting with λ & repeatedly concatenating a or b to the end of the string.

EXERCISE 8.8.4: Recursive definitions for subsets of binary strings.

Give a recursive definition for each subset of the binary strings. A string x should be in the recursively defined set if and only if x has the property described.

(a) The set S consists of all strings with an even number of 1's.

Base cases: 0, 11

Recursive rule: if $0, 11 \in S$ & $x \in S$ then,

$1x1$, $0x$, $0x \in S$

EXERCISE 8.8.6: Recursive definitions for subsets of integers.

(a) A subset S of the integers is defined recursively as follows:

- Base case: $2 \in S$
- Recursive rule: if $k \in S$, then
  - $k + 5 \in S$
  - $k - 5 \in S$

List the elements of S whose absolute value is less than 20.

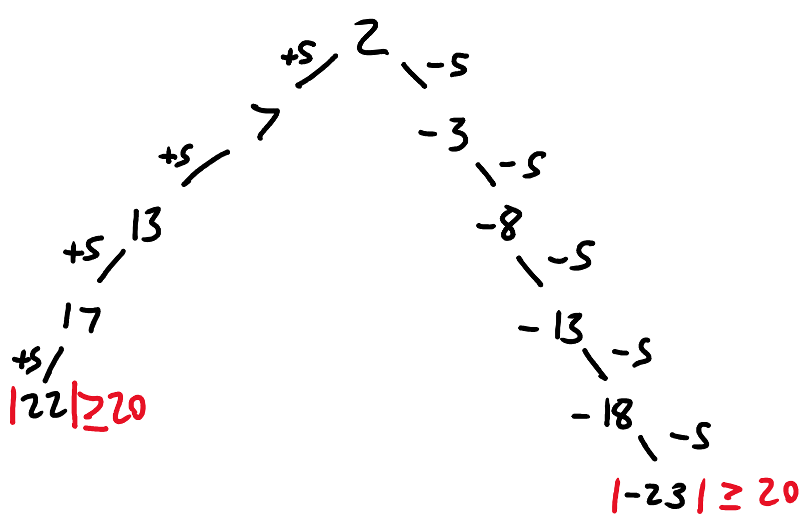
Base case: $2 \in S$

Recursive rule:

$k+5 \in S$ $\qquad$ $k-5 \in S$

$2+5 \in S$ $\qquad$ $2-5 \in S$

$7 \in S$ $\qquad$ $-3 \in S$



∴ the list of elements who's absolute value is less than 20 is:

$\{-18, -13, -8, -3, 2, 7, 12, 17\}$

EXERCISE 8.8.7: Interpreting a recursive definition of a set of strings.

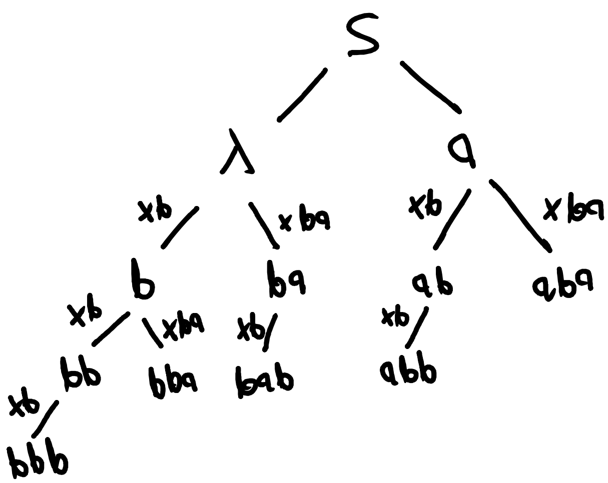
(a) The recursive definition given below defines a set S of strings over the alphabet {a, b}:

- Base case: $\lambda \in S$ and $a \in S$
- Recursive rule: if $x \in S$ then,
  - $xb \in S$
  - $xba \in S$

List all the strings of length at most 3 in S.

Feedback?

$\{\lambda, a, b, ba, ab, aba, bb, bba, bab, abb, bbb\}$



Base cases

8.9 - Structural induction

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EXERCISE 8.9.2: Proving facts about recursively defined sets of strings.

(a) Consider a set of strings defined recursively as follows:

- Base case: a ∈ S
- Recursive rule: if x ∈ S then,
  - xb ∈ S (Rule 1)
  - xa ∈ S (Rule 2)

Prove that every string in S begins with the character a.

$S = \{a, aa, ab, aaa, aab, aba, abb, \ldots\}$

$a(a+b)^* \in S$

$\therefore$ every string in S begins with character a

8.10 - Recursive algorithms

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EXERCISE | 8.10.1: Recursively computing sums of cubes.

(a) Give a recursive algorithm to compute the sum of the cubes of the first n positive integers. The input to the algorithm is a positive integer n. The output is $\sum_{j=1}^{n} j^3$. The algorithm should be recursive, it should not compute the sum using a closed form expression or an iterative loop.

Feedback?

SumCube(n)

Input: a positive integer n.

output: sum of the cubes of the first n positive integers.

If (n==1), Return(1)

Else,

Return((n*n*n) + SumCube(n-1))

EXERCISE | 8.10.2: Recursively computing the sum of the first n positive odd integers.

(a) Give a recursive algorithm which takes as input a positive integer n and returns the sum of the first n positive odd integers.

Feedback?

SumOdds(n)

Input: a positive integer n.

Output: the sum of the first n positive odd integers.

```
If (n==1), Return( 1 )

Else,
Return((2*n -1)+ SumOdds(n-1))
```

8.11 - Induction and recursive algorithms

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EXERCISE | 8.11.1: Recursively computing sums of cubes, cont.

(a) Use induction to prove that the algorithm to compute the sum of the cubes of the first n positive integers (shown below) returns the correct value for every positive integer input.

```
SumCube(n)

Input: A positive integer n.
Output: 1^3 + 2^3 + ... + n^3.

If (n = 1), Return( 1 )
s := SumCube(n - 1) // The recursive call

Return( n^3 + s )
```

Feedback?

Base case: $n=1$

$T(1)=1$

our recursive relation $T(n)$ can be found by Return(n^3+s) where

$n^3+s = n^3 + \text{SumCube}(n-1) = n^3 + T(n-1)$

So, $T(n) = n^3 + T(n-1)$

Inductive step: Suppose we have $T(k)$ for positive integer k where we represent our recursive relation with $T(k) = k^3 + T(k-1)$, then we must prove,

$T(k+1) = (k+1)^3 + T(k) = 0^3 + 1^3 + \ldots + (k+1)^3$

$T(0) = 0^3 = 0$ for $k=0$

$T(k+1) = (k+1)^3 + T(k)$

$= (k+1)^3 + k^3 + T(k-1)$

$= (k+1)^3 + k^3 + (k-1)^3 + T(k-2)$

$= (k+1)^3 + k^3 + (k-1)^3 + T(k-2)$

$= (k+1)^3 + k^3 + (k-1)^3 + (k-2)^3 + T(k-3)$ (by seperating out terms)

$= (k+1)^3 + k^3 + (k-1)^3 + (k-2)^3 + \ldots + (k-(k-1))^3 + (k-k)^3$ (by expanding sum)

$= (k+1)^3 + k^3 + (k-1)^3 + (k-2)^3 + \ldots + 1^3 + 0^3$ (by algebra)

$= 0^3 + 1^3 + \ldots + (k+1)^3$ (rewritten)