1.1 - Propositions and logical operations

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EXERCISE 1.1.1: Identifying propositions.

Determine whether each of the following sentences is a proposition. If the sentence is a proposition, then write its negation.

(a) Have a nice day. Not a proposition.

(b) The soup is cold. Proposition. Negation is: "The soup is not cold."

(c) The patient has diabetes. Proposition. Negation is: "The patient does not have diabetes."

EXERCISE 1.1.2: Expressing English sentences using logical notation.

Express each English statement using logical operations ∨, ∧, ¬ and the propositional variables t, n, and m defined below. The use of the word "or" means inclusive or.

t: The patient took the medication.
n: The patient had nausea.
m: The patient had migraines.

(a) The patient had nausea and migraines. $n \land m$

(b) The patient took the medication, but still had migraines. $t \land m$

(c) The patient had nausea or migraines. $n \lor m$

EXERCISE 1.1.3: Applying logical operations.

Assume the propositions p, q, r, and s have the following truth values:

p is false
q is true
r is false
s is true

What are the truth values for the following compound propositions?

(a) $\neg p$ $\neg(\text{false}) = \text{true}$

(b) $p \lor r$ $(\text{false}) \lor (\text{false}) = \text{False}$

(c) $q \land s$ $(\text{true}) \land (\text{true}) = \text{true}$

1.2 - Evaluating compound propositions

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EXERCISE 1.2.1: Truth values for compound English sentences.

Determine whether the following propositions are true or false:

(a) 5 is an odd number and 3 is a negative number. $T \land F = F$

(b) 5 is an odd number or 3 is a negative number. $T \lor F = T$

(c) 8 is an odd number or 4 is not an odd number. $F \lor T = T$

EXERCISE 1.2.2: Translating English statements into logic.

Express each statement in logic using the variables:

p: It is windy.
q: It is cold.
r: It is raining.

(a) It is windy and cold. $p \land q$

(b) It is windy but not cold. $p \land \neg q$

(c) It is not true that it is windy or cold. $\neg (p \lor q)$

EXERCISE 1.2.3: Truth values for compound propositions.

The propositional variables, p, q, and s have the following truth assignments: p = T, q = T, s = F. Give the truth value for each proposition.

(a) $p \lor \neg q$ $T \lor \neg T = T \lor F = T$

(b) $(p \land q) \lor s$ $(T \land T) \lor F = T \lor F = T$

(c) $p \land (q \lor s)$ $T \land (T \lor F) = T \land T = T$

EXERCISE | 1.2.4: Writing truth tables.

Write a truth table for each expression.

(a) $\neg p \oplus q$

(b) $\neg(p \vee q)$

Reminder: exclusive or $\oplus$ means we can have one or the other but not both

(a)

p	¬p	q	¬p ⊕ q
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T

← unique to $\oplus$

(b)

p	q	p ∨ q	¬(p ∨ q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

1.3 - Conditional statements

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EXERCISE 1.3.1: Truth values for conditional statements in English.

Which of the following conditional statements are true and why?

(a) If February has 30 days, then 7 is an odd number. $F \to T = T$

(b) If January has 31 days, then 7 is an even number. $T \to F = F$

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

EXERCISE 1.3.2: The inverse, converse, and contrapositive of conditional sentences in English.

Give the inverse, contrapositive, and converse for each of the following statements:

(a) If she finished her homework, then she went to the party.

(b) If he trained for the race, then he finished the race.

(a) Inverse: "If she didn't finish her homework, then she didn't go to the party."
Contrapositive: "If she didn't go to the party, then she didn't finish her homework."
Converse: "If she went to the party, then she finished her homework."

(b) Inverse: "If he didn't train for the race, then he didn't finish the race."
Contrapositive: "If he didn't finish the race, then he didn't train for the race."
Converse: "If he finished the race, then he trained for the race."

EXERCISE 1.3.3: Truth values for the inverse, contrapositive, and converse of a conditional statement.

State the inverse, contrapositive, and converse of each conditional statement. Then indicate whether the inverse, contrapositive, and converse are true.

(a) If 3 is a prime number then 5 is an even number.

(b) If 7 < 5, then 5 < 3.

(a) Inverse: "If 3 is not a prime number then 5 is not an even number."
Contrapositive: "If 5 is not an even number then 3 is not a prime number."
Converse: "If 5 is an even number then 3 is a prime number."

Truth values:

Inverse: $F \to T = T$

Contrapositive: $T \to F = F$

Converse: $F \to T = T$

Inverse: $F \to T = 1$
Contrapositive: $T \to F = F$
Converse: $F \to T = T$

(b) Inverse: "If $7 \geq 5$, then $5 \geq 3$"
Contrapositive: "If $5 \geq 3$, then $7 \geq 5$"
Converse: "If $5 < 3$, then $7 < 5$"

Truth values:
Inverse: $T \to T = T$
Contrapositive: $T \to T = T$
Converse: $F \to F = T$

EXERCISE 1.3.4: Truth tables for logical expressions with conditional operations.

Give a truth table for each expression.

(a) $(\neg p \wedge q) \to p$

(b) $(p \to q) \to (q \to p)$

(a)

p	¬p	q	¬p ∧ q	(¬p ∧ q) → p
T	F	T	F	T
T	F	F	F	T
F	T	T	T	F
F	T	F	F	T

(b)

p	q	p → q	q → p	(p → q) → (q → p)
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

EXERCISE 1.3.5: Expressing conditional statements in English using logic.

Define the following propositions:

c: I will return to college.
j: I will get a job.

Translate the following English sentences into logical expressions using the definitions above:

(a) Not getting a job is a sufficient condition for me to return to college.

(b) If I return to college, then I won't get a job.

(a) $\neg j \to c$

(b) $c \to \neg j$

EXERCISE 1.3.7: Expressing conditional statements in English using logic.

Define the following propositions:

s: a person is a senior
y: a person is at least 17 years of age
p: a person is allowed to park in the school parking lot

Express each of the following English sentences with a logical expression:

(a) A person is allowed to park in the school parking lot only if they are a senior and at least seventeen years of age.

(b) A person can park in the school parking lot if they are a senior or at least seventeen years of age.

(a) $p \leftrightarrow (s \wedge y)$

(b) $(s \vee y) \rightarrow p$

EXERCISE 1.3.10: Determining if a truth value of a compound expression is known given a partial truth assignment.

The variable p is true, q is false, and the truth value for variable r is unknown. Indicate whether the truth value of each logical expression is true, false, or unknown.

(a) $p \rightarrow (q \wedge r)$

(b) $(p \vee r) \rightarrow r$

(a) $T \rightarrow (F \wedge r) = T \rightarrow F = F$

(b) $(T \vee r) \rightarrow r = T \rightarrow r$, unknown

Consider: $\begin{cases} T \rightarrow r = T & \text{if } r = T \\ T \rightarrow r = F & \text{if } r = F \end{cases}$

1.4 - Logical equivalence

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EXERCISE 1.4.1: Proving tautologies and contradictions.

Show whether each logical expression is a tautology, contradiction or neither.

(a) (p ∨ q) ∨ (q → p) tautology

(b) (p → q) ↔ (p ∧ ¬q) contradiction

(a)

p	q	p ∨ q	q → p	(p ∨ q) ∨ (q → p)
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(b)

p	q	p → q	¬q	(p ∧ ¬q)	(p → q) ↔ (p ∧ ¬q)
T	T	T	F	F	F
T	F	F	T	T	F
F	T	T	F	F	F
F	F	T	T	F	F

The biconditional operation

If p and q are propositions, the proposition "p if and only if q" is expressed with the ***biconditional operation*** and is denoted p ↔ q. The proposition p ↔ q is true when p and q have the same truth value and is false when p and q have different truth values.

Alternative ways of expressing p ↔ q in English include "p is necessary and sufficient for q" or "if p then q, and conversely". The term ***iff*** is an abbreviation of the expression "if and only if", as in "p iff q". The truth table for p ↔ q is given below:

Table 1.3.4: Truth table for the biconditional operation.

opposite of "xor"

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Feedback?

EXERCISE | 1.4.2: Truth tables to prove logical equivalence.

Use truth tables to show that the following pairs of expressions are logically equivalent.

(a) $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$

(b) $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$

logically equivalent

(a)

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

p	q	p → q	q → p	(p → q) ∧ (q → p)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(b)

logically equivalent

p	q	p ↔ q	¬(p ↔ q)
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

p	q	¬p	¬p ↔ q
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	F

EXERCISE | 1.4.3: Proving two logical expressions are not logically equivalent.

Prove that the following pairs of expressions are not logically equivalent.

(a) $p \to q$ and $q \to p$

(b) $\neg p \to q$ and $\neg p \vee q$

not logically equivalent

(a)

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

p	q	q → p
T	T	T
T	F	T
F	T	F
F	F	T

(b)

not logically equivalent

p	q	¬p	¬p → q
T	T	F	T
T	F	T	T
F	T	F	T
F	F	T	F

p	q	¬p	¬p ∨ q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

1.5 - Laws of propositional logic

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Table 1.5.1: Laws of propositional logic.

Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	$p \vee F \equiv p$	$p \wedge T \equiv p$
Domination laws:	$p \wedge F \equiv F$	$p \vee T \equiv T$
Double negation law:	$\neg\neg p \equiv p$	
Complement laws:	$p \wedge \neg p \equiv F$ $\neg T \equiv F$	$p \vee \neg p \equiv T$ $\neg F \equiv T$
De Morgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional identities:	$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

EXERCISE 1.5.1: Label the steps in a proof of logical equivalence.

Below are several proofs showing that two logical expressions are logically equivalent. Label the steps in each proof with the law used to obtain each proposition from the previous proposition. The first line in the proof does not have a label.

(a)

$(p \rightarrow q) \wedge (q \vee p)$	
$(\neg p \vee q) \wedge (q \vee p)$	Conditional identity
$(q \vee \neg p) \wedge (q \vee p)$	Commutative law
$q \vee (\neg p \wedge p)$	Distributive law
$q \vee (p \wedge \neg p)$	Commutative law
$q \vee F$	Complement law
q	Identity law

(b)

$(\neg p \vee q) \rightarrow (p \wedge q)$	
$\neg(\neg p \vee q) \vee (p \wedge q)$	Conditional identity
$(\neg\neg p \wedge \neg q) \vee (p \wedge q)$	De Morgan's law
$(p \wedge \neg q) \vee (p \wedge q)$	Double negation law
$p \wedge (\neg q \vee q)$	Distributive law
$p \wedge (q \vee \neg q)$	Commutative law
$p \wedge T$	Complement law
p	Identity law

(c)

$r \vee (\neg r \rightarrow p)$	
$r \vee (\neg\neg r \vee p)$	Conditional identity

(c)

$r \vee (\neg r \rightarrow p)$	
$r \vee (\neg\neg r \vee p)$	Conditional identity
$r \vee (r \vee p)$	Double negation law
$(r \vee r) \vee p$	Associative law
$r \vee p$	Idempotent law

EXERCISE 1.5.2: Using the laws of logic to prove logical equivalence.

Use the laws of propositional logic to prove the following:

(a) $\neg p \rightarrow \neg q \equiv q \rightarrow p$

(b) $p \wedge (\neg p \rightarrow q) \equiv p$

(c) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

(a) $\neg p \rightarrow \neg q$

$\neg\neg p \vee \neg q$ Conditional identity

$p \vee \neg q$ Double negation law

$\neg q \vee p$ Commutative law

$q \rightarrow p$ Conditional identity

$\therefore \neg p \rightarrow \neg q \equiv q \rightarrow p$

(b) $p \wedge (\neg p \rightarrow q)$

$p \wedge (\neg\neg p \vee q)$ Conditional identity

$p \wedge (p \vee q)$ Double negation law

p Absorption law

$\therefore p \wedge (\neg p \rightarrow q) \equiv p$

(c) $(p \rightarrow q) \wedge (p \rightarrow r)$

$(\neg p \vee q) \wedge (\neg p \vee r)$ Conditional identity

$\neg p \vee (q \wedge r)$ Distributive law

$p \rightarrow (q \wedge r)$ Conditional identity

$\therefore (p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

EXERCISE | 1.5.3: Using the laws of logic to prove tautologies.

Use the laws of propositional logic to prove that each statement is a tautology.

(a) $(p \wedge q) \rightarrow (p \vee r)$

(b) $p \rightarrow (r \rightarrow p)$

(a) $(p \wedge q) \rightarrow (p \vee r)$

$\neg(p \wedge q) \vee (p \vee r)$ Conditional identity

$(\neg p \vee \neg q) \vee (p \vee r)$ De Morgan's law

$((\neg p \vee \neg q) \vee p) \vee r$ Associative law

$((\neg q \vee \neg p) \vee p) \vee r$ commutative law

$(\neg q \vee (\neg p \vee p)) \vee r$ Associative law

$(\neg q \vee T) \vee r$ Complement law

$T \vee r$ Domination law

$$T$$ Domination law

$\therefore (p \wedge q) \to (p \vee r)$ is a tautology

(b)

$$p \to (r \to p)$$

$$p \to (\neg r \vee p)$$ Conditional identity

$$\neg p \vee (\neg r \vee p)$$ Conditional identity

$$\neg p \vee (p \vee \neg r)$$ Commutative law

$$(\neg p \vee p) \vee \neg r$$ Associative law

$$(p \vee \neg p) \vee \neg r$$ Commutative law

$$T \vee \neg r$$ Complement law

T Domination law

1.6 - Predicates and quantifiers

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EXERCISE 1.6.1: Which expressions with predicates are propositions?

Predicates P, T and E are defined below. The domain is the set of all positive integers.

$P(x)$: x is even
$T(x, y)$: $2^x = y$
$E(x, y, z)$: $x^y = z$

A proposition if no variables

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(a) P(3) 3 is even. This is a proposition. Its truth value is false.

(b) ¬P(3) 3 is not even. This is a proposition. Its truth value is true.

(c) T(5, 32) $2^5 = 32$. This is a proposition. Its truth value is true.

EXERCISE 1.6.2: Truth values for quantified statements about integers.

In this problem, the domain is the set of all integers. Which statements are true? If an existential statement is true, give an example. If a universal statement is false, give a counterexample.

(a) $\exists x\,(x + x = 1)$ The statement is false. $x+x = 2x = 1 \Rightarrow x = 1/2 \notin \mathbb{Z}$

(b) $\exists x\,(x + 2 = 1)$ The statement is true for $x=-1$. Ex. $-1+2=1$

(c) $\forall x\,(x^2 - x \neq 1)$ The statement is true.

EXERCISE 1.6.3: Translating mathematical statements in English into logical expressions.

Consider the following statements in English. Write a logical expression with the same meaning. The domain is the set of all real numbers.

(a) There is a number whose cube is equal to 2. $\exists x\,(x^3 = 2)$

(b) The square of every number is at least 0. $\forall x\,(x^2 \geq 0)$

EXERCISE 1.6.4: Truth values for quantified statements for a given set of predicates.

The domain for this problem is a set $\{a, b, c, d\}$. The table below shows the value of three predicates for each of the elements in the domain. For example, $Q(b)$ is false because the truth value in row b, column Q is F.

	P	Q	R
a	T	T	F
b	T	F	F
c	T	F	F
d	T	F	F

Which statements are true? Justify your answer.

(a) $\forall x\, P(x)$ This statement is true for every value of x b/c P(x) is a tautology (all columns true).

(b) $\exists x\, P(x)$ This statement is true b/c there exists a value of x for which P(x) is true.

1.7 - Quantified statements

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EXERCISE 1.7.1: Determining whether a quantified statement about the integers is true.

Predicates P and Q are defined below. The domain is the set of all positive integers.

P(x): x is prime
Q(x): x is a perfect square (i.e., $x = y^2$, for some integer y)

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(a) $\exists x\, Q(x)$ This statement is a proposition. Its truth value is true.

(b) $\forall x\, Q(x) \land \neg P(x)$ This statement is not a proposition b/c x is a free variable in P(x), consider: $\forall(Q(x)) \land \neg P(x)$

(c) $\forall x\, Q(x) \lor P(3)$ This statement is a proposition. Its truth value is true.

For all x, x is a perfect square OR 3 is prime

$F \vee T = T$

EXERCISE 1.7.2: Translating quantified statements from English to logic, part 1.

In the following question, the domain is a set of students at a university. Define the following predicates:

E(x): x is enrolled in the class
T(x): x took the test

Translate the following English statements into a logical expression with the same meaning.

(a) Someone who is enrolled in the class took the test. $\exists x\, (E(x) \land T(x))$

(b) All students enrolled in the class took the test. $\forall x\, (E(x) \rightarrow T(x))$

(c) Everyone who took the test is enrolled in the class. $\forall x\, (T(x) \rightarrow E(x))$

EXERCISE 1.7.3: Translating quantified statements in English into logic, part 2.

In the following question, the domain is the set of employees at a company. One of the employees is named Sam. Define the following predicates:

T(x): x is a member of the executive team
B(x): x received a large bonus

Translate the following English statements into a logical expression with the same meaning.

(a) Someone did not get a large bonus. $\exists x\ \neg B(x)$

(b) Everyone got a large bonus. $\forall x\ B(x)$

EXERCISE 1.7.4: Translating quantified statements from English to logic, part 3.

In the following question, the domain is a set of employees who work at a company. Ingrid is one of the employees at the company. Define the following predicates:

S(x): x was sick yesterday
W(x): x went to work yesterday
V(x): x was on vacation yesterday

Translate the following English statements into a logical expression with the same meaning.

(a) At least one person was sick yesterday. $\exists x\ S(x)$

(b) Everyone was well and went to work yesterday. $\forall x\ (\neg S(x) \land W(x))$

EXERCISE 1.7.5: Translating quantified statements from English to logic, part 4.

A student club holds a meeting. The predicate M(x) denotes whether person x came to the meeting on time. The predicate O(x) refers to whether person x is an officer of the club. The predicate D(x) indicates whether person x has paid his or her club dues. The domain is the set of all members of the club. Give a logical expression that is equivalent to each English statement.

(a) Someone is not an officer. $\exists x\ \neg O(x)$

(b) All the officers came on time to the meeting. $\forall x\ (O(x) \rightarrow M(x))$

EXERCISE 1.7.7: Determining whether a quantified logical statement is true and translating into English, part 1.

The domain is a group working on a project at a company. Define the following predicates.

- D(x): x missed the deadline.
- N(x): x is a new employee.

Consider a situation in which there are five people in the group. The following table gives values for the predicates D and N for each member of the group. For example, Bert did not miss the deadline because the truth value in the row labeled Bert and the column labeled D(x) is F.

Using these values, determine whether each quantified expression evaluates to true or false. Then translate the statement into English.

Name	D(x)	N(x)
Sam	T	F
Beth	T	T
Melanie	F	T
Al	T	T
Bert	F	T

(a) $\forall x\, (D(x) \vee N(x))$

(b) $\forall x\, ((x \neq \text{Sam}) \rightarrow N(x))$

(a) All x evaluate to true.
For any person in the group, they have either missed a deadline or are a new employee.
(b) True.
For any person in the group, if they are not Sam, then they are a new employee.

EXERCISE 1.7.8: Determining whether a quantified logical statement is true and translating into English, part 2.

In the following question, the domain is a set of male patients in a clinical study. Define the following predicates:

P(x): x was given the placebo
D(x): x was given the medication
A(x): x had fainting spells
M(x): x had migraines

Suppose that there are five patients who participated in the study. The table below shows the names of the patients and the truth value for each patient and each predicate:

Name	P(x)	D(x)	A(x)	M(x)
Frodo	T	F	F	T
Gandalf	F	T	F	F
Gimli	F	T	T	F
Aragorn	T	F	T	T
Bilbo	T	T	F	F

For each of the following quantified statements, indicate whether the statement is a proposition. If the statement is a proposition, give its truth value and translate the expression into English.

(a) $\exists x\, (M(x) \wedge D(x))$

(b) $\exists x\, M(x) \wedge \exists x\, D(x)$

(a) The statement is a proposition.
Its truth value is false.
There exists at least one male patient who had migraines & was given the medication.

(b) The statement is a proposition.
Its truth value is true. Consider: $(\exists x\,(M(x))) \wedge (\exists x\,(D(x))) = T \wedge T = T$.
There exists at least one male patient who had migraines & there exists at least one male patient who was given the medication.

1.8 - De Morgan's law for quantified statements

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EXERCISE | 1.8.1: Applying De Morgan's law for quantified statements to logical expressions.

Apply De Morgan's law to each expression to obtain an equivalent expression in which each negation sign applies directly to a predicate. For example, $\exists x\,(\neg P(x) \vee \neg Q(x))$ is an acceptable final answer, but not $\neg\exists x\, P(x)$ or $\exists x\, \neg(P(x) \wedge Q(x))$.

(a) $\neg\exists x\, P(x)$

(b) $\neg\exists x\, (P(x) \vee Q(x))$

(a) $\neg \exists x\, P(x)$

$\forall x\, \neg P(x)$ De Morgan's law for quantified statements

$\therefore \neg \exists x P(x) \equiv \forall x \neg P(x)$

(b) $\neg \exists x\, (P(x) \vee Q(x))$

$\forall x\, \neg (P(x) \vee Q(x))$ De Morgan's law for quantified statements

$\forall x\, (\neg P(x) \wedge Q(x))$ De Morgan's law

$\therefore \neg \exists x (P(x) \vee Q(x)) \equiv \forall (\neg P(x) \wedge Q(x))$

EXERCISE | 1.8.2: Applying De Morgan's law for quantified statements to English statements.

In the following question, the domain is a set of male patients in a clinical study. Define the following predicates:

P(x): x was given the placebo
D(x): x was given the medication
M(x): x had migraines

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.

Sample question: Some patient was given the placebo and the medication.

- $\exists x\, (P(x) \wedge D(x))$
- Negation: $\neg\exists x\, (P(x) \wedge D(x))$
- Applying De Morgan's law: $\forall x\, (\neg P(x) \vee \neg D(x))$
- English: Every patient was either not given the placebo or not given the medication (or both).

(a) Every patient was given the medication.

(b) Every patient was given the medication or the placebo or both. inclusive or

(a) $\forall x\, D(x)$

Negation: $\neg \forall x\, D(x)$

$\exists x \neg D(x)$ De Morgan's law for quantified statements

At least one male patient was not given the medication.

(b) $\forall x (D(x) \vee P(x))$

Negation: $\neg \forall x (D(x) \vee P(x))$

$\exists x \neg (D(x) \vee P(x))$ De Morgan's law for quantified statements

$\exists x (\neg D(x) \wedge \neg P(x))$ De Morgan's law

At least one male patient was not given the medication & not given the placebo.

EXERCISE | 1.8.4: Using De Morgan's law for quantified statements to prove logical equivalence.

Use De Morgan's law for quantified statements and the laws of propositional logic to show the following equivalences:

(a) $\neg\forall x (P(x) \wedge \neg Q(x)) \equiv \exists x (\neg P(x) \vee Q(x))$

(b) $\neg\forall x (\neg P(x) \rightarrow Q(x)) \equiv \exists x (\neg P(x) \wedge \neg Q(x))$

(a) $\neg \forall x (P(x) \wedge \neg Q(x))$

$\exists x \neg (P(x) \wedge \neg Q(x))$ De Morgan's law for quantified statements

$\exists x (\neg P(x) \vee \neg\neg Q(x))$ De Morgan's law

$\exists x (\neg P(x) \vee Q(x))$ Double negation law

$\therefore \neg \forall x (P(x) \wedge \neg Q(x)) \equiv \exists x (\neg P(x) \vee Q(x))$

(b) $\neg \forall x (\neg P(x) \rightarrow Q(x))$

$\exists x \neg (\neg P(x) \rightarrow Q(x))$ De Morgan's law for quantified statements

$\exists x \neg (\neg\neg P(x) \vee Q(x))$ Conditional identity

$\exists x \neg (P(x) \vee Q(x))$ Double negation law

$\exists x (\neg P(x) \wedge \neg Q(x))$ De Morgan's law

$\therefore \neg \forall x (\neg P(x) \rightarrow Q(x)) \equiv \exists x (\neg P(x) \wedge \neg Q(x))$

1.9 - Nested quantifiers

Tuesday, May 23, 2023 1:27 PM

EXERCISE 1.9.1: Which logical expressions with nested quantifiers are propositions?

The table below shows the value of a predicate M(x, y) for every possible combination of values of the variables x and y. The domain for x and y is {1, 2, 3}. The row number indicates the value for x and the column number indicates the value for y. For example M(1, 2) = F because the value in row 1, column 2, is F.

M		y 1	2	3
x	1	T	F	T
	2	T	F	T
	3	T	T	F

Indicate whether each of the logical expressions is a proposition. If so, indicate whether the proposition is true or false.

(a) M(1, 1) Is a proposition. Truth value is true,

(b) ∀y M(x, y) Is not a proposition (x is a free variable).

(c) ∃x M(x, 3) Is a proposition. Truth value is true.

EXERCISE 1.9.2: Truth values for statements with nested quantifiers - small finite domain.

The tables below show the values of predicates P(x, y), Q(x, y), and S(x, y) for every possible combination of values of the variables x and y. The row number indicates the value for x and the column number indicates the value for y. The domain for x and y is {1, 2, 3}.

P	1	2	3
1	T	F	T
2	T	F	T
3	T	T	F

Q	1	2	3
1	F	F	F
2	T	T	T
3	T	F	F

S	1	2	3
1	F	F	F
2	F	F	F
3	F	F	F

Indicate whether each of the quantified statements is true or false.

(a) ∃x ∀y P(x, y) False.

(b) ∃x ∀y Q(x, y) True. (@ x=2)

(c) ∃y ∀x P(x, y) True. (@ y=1)

EXERCISE 1.9.3: Truth values for mathematical expressions with nested quantifiers.

Determine the truth value of each expression below. The domain is the set of all real numbers.

(a) ∀x∃y (xy > 0) False. (if y=0, there exists no x s.t. xy>0)

(b) ∃x∀y (xy = 0) True. (if x=0, all values of y result in xy=0)

EXERCISE 1.9.4: De Morgan's law and nested quantifiers.

Write the negation of each of the following logical expressions so that all negations immediately precede predicates. In some cases, it may be necessary to apply one or more laws of propositional logic.

(a) $\forall x\ \exists y\ \exists z\ P(y, x, z)$

(b) $\forall x\ \exists y\ (P(x, y) \land Q(x, y))$

(c) $\exists x\ \forall y\ (P(x, y) \to Q(x, y))$

(a) Negation: $\neg(\forall x\ \exists y\ \exists z\ P(x,y,z))$

$\exists x\ \neg\exists y\ \neg\exists z\ P(x,y,z)$ De Morgan's law

$\exists x\ \forall y\ \forall z\ \neg P(x,y,z)$ De Morgan's law

(b) Negation: $\neg(\forall x\ \exists y\ (P(x,y) \land Q(x,y)))$

$\exists x\ \neg\exists y\ (P(x,y) \land Q(x,y))$ De Morgan's law

$\exists x\ \forall y\ \neg(P(x,y) \land Q(x,y))$ De Morgan's law

$\exists x\ \forall y\ (\neg P(x,y) \lor \neg Q(x,y))$ De Morgan's law

(c) Negation: $\neg(\exists x\ \forall y\ (P(x,y) \to Q(x,y)))$

$\neg\exists x\ \exists y\ (P(x,y) \to Q(x,y))$ De Morgan's law

$\forall x\ \exists y\ \neg(P(x,y) \to Q(x,y))$ De Morgan's law

$\forall x\ \exists y\ \neg(\neg P(x,y) \lor Q(x,y))$ Conditional identity

$\forall x\ \exists y\ (\neg\neg P(x,y) \land \neg Q(x,y))$ De Morgan's law

$\forall x\ \exists y\ (P(x,y) \land \neg Q(x,y))$ Double negation law

1.10 - More nested quantified statements

Tuesday, May 23, 2023 3:18 PM

EXERCISE 1.10.1: Truth values for expressions with nested quantifiers.

The domain for this problem is a group of three people who are working on a project. To make notation easier, the people are numbered 1, 2, 3. The predicate M(x, y) indicates whether x has sent an email to y, so M(2, 3) is read "Person 2 has sent an email to person 3." The table below shows the value of the predicate M(x,y) for each (x,y) pair. The truth value in row x and column y gives the truth value for M(x,y).

M	1	2	3
1	T	T	T
2	T	F	T
3	T	T	F

Indicate whether the quantified statement is true or false. Justify your answer.

(a) $\forall x\ \forall y\ M(x,y)$ False (only true if M(x,y) is a T for all x & y), counterexample: M(2,2)=F.

(b) $\forall x\ \forall y\ ((x \neq y) \rightarrow M(x,y))$ True, M(2,2) & M(3,3) (when x=y) are the only false values.

(c) $\exists x\ \exists y\ \neg M(x,y)$ True, example: $\neg M(2,2) = \neg F = T$ (complement law)

EXERCISE 1.10.2: Truth values for mathematical statements with nested quantifiers.

The domain for all variables in the expressions below is the set of real numbers. Determine whether each statement is true or false. Justify your answer.

(a) $\forall x\ \exists y\ (x + y = 0)$ True. $x+y=0 \Rightarrow y=-x$ works for existential variable.

(b) $\exists x\ \forall y\ (x + y = 0)$ False. For every x there is at least one y s.t. $x+y \neq 0$.

(c) $\exists x\ \forall y\ (xy = y)$ True. There exists an x (i.e. x=1) s.t. xy=y for all y.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

EXERCISE 1.10.4: Mathematical statements into logical statements with nested quantifiers.

Translate each of the following English statements into logical expressions. The domain is the set of all real numbers.

(a) There are two numbers whose ratio is less than 1. $\exists x \exists y \left(\frac{x}{y} < 1\right)$ where $x, y \in \mathbb{R}$

(b) The reciprocal of every positive number is also positive. $\forall x \left(\frac{1}{x} > 0\right)$ where $x \in \mathbb{R}$

(c) There are two numbers whose sum is equal to their product. $\exists x \exists y (x+y = xy)$ where $x, y \in \mathbb{R}$

1.11 - Logical reasoning

Tuesday, May 23, 2023 4:02 PM

EXERCISE | 1.11.1: Valid and invalid arguments expressed in logical notation.

Indicate whether the argument is valid or invalid. For valid arguments, prove that the argument is valid using a truth table. For invalid arguments, give truth values for the variables showing that the argument is not valid.

(a)

$$\begin{array}{l} p \lor q \\ \underline{p} \\ \therefore q \end{array}$$

equivalent to $((p \lor q) \land (p)) \to q$

(b)

$$\begin{array}{l} p \leftrightarrow q \\ \underline{p \lor q} \\ \therefore p \end{array}$$

equivalent to $((p \leftrightarrow q) \land (p \land q)) \to p$

(c)

$$\begin{array}{l} p \\ \underline{q} \\ \therefore p \leftrightarrow q \end{array}$$

equivalent to $(p \land q) \to (p \leftrightarrow q)$

(a)

Hyp. Conc. Hyp.

p	q	p V q
T	T	T
T	F	T
F	T	T
F	F	F

← Hyp. true, conc. False, argument invalid

(b)

concl. Hyp. Hyp.

p	q	p ↔ q	p V q
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	F

← Hyp. true, concl. true, argument valid

(c)

Hyp. Hyp. Concl.

p	q	p ↔ q
T	T	T
T	F	F

← Hyp. true, concl. true, argument valid

F	T	F
F	F	T

1.12 - Rules of inference with propositions

Tuesday, May 23, 2023 11:10 PM

Table 1.12.1: Rules of inference known to be valid arguments.

Rule of inference	Name
p $\underline{p \rightarrow q}$ $\therefore q$	Modus ponens
$\neg q$ $\underline{p \rightarrow q}$ $\therefore \neg p$	Modus tollens
$\underline{p}$ $\therefore p \lor q$	Addition
$\underline{p \land q}$ $\therefore p$	Simplification
p $\underline{q}$ $\therefore p \land q$	Conjunction
$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore p \rightarrow r$	Hypothetical syllogism
$p \lor q$ $\underline{\neg p}$ $\therefore q$	Disjunctive syllogism
$p \lor q$ $\underline{\neg p \lor r}$ $\therefore q \lor r$	Resolution

EXERCISE | 1.12.2: Proving arguments are valid using rules of inference.

Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

(a)

$$\begin{array}{l} p \to q \\ q \to r \\ \underline{\neg r} \\ \therefore \neg p \end{array}$$

(b)

$$\begin{array}{l} p \to (q \land r) \\ \underline{\neg q} \\ \therefore \neg p \end{array}$$

(c)

$$\begin{array}{l} (p \land q) \to r \\ \neg r \\ \underline{q} \\ \therefore \neg p \end{array}$$

(a) (1) $p \to q$ Hypothesis

(2) $q \to r$ Hypothesis

(3) $p \to r$ Hypothetical syllogism (on (1) & (2))

(4) $\neg r$ Hypothesis

(5) $\neg p$ Modus tollens (in (3) & (4))

$\therefore$ argument is valid

(b) (1) $q \land r$ Hypothesis

(2) q simplification (on (1))

(3) $\neg q$ Hypothesis

(4) $p \to q$ Hypothesis (after simplification in (2))

(5) $\neg p$ Modus tollens (in (4) & (5))

$\therefore$ argument is valid

(c) (1) $p \wedge q$ Hypothesis

(2) p Simplification (in (1))

(3) $\neg r$ Hypothesis

(4) $p \to r$ Hypothesis (after simplification in (2))

(5) $\neg p$ Modus tollens (in (3) & (4))

$\therefore$ argument is valid

1.13 - Rules of inference with quantifiers

Wednesday, May 24, 2023 6:20 PM

Table 1.13.1: Rules of inference for quantified statements.

Rule of Inference	Name	Example
c is an element (arbitrary or particular) $\underline{\forall x\ P(x)}$ $\therefore$ P(c)	Universal instantiation	Sam is a student in the class. Every student in the class completed the assignment. Therefore, Sam completed his assignment.
c is an arbitrary element $\underline{P(c)}$ $\therefore \forall x\ P(x)$	Universal generalization	Let c be an arbitrary integer. $c \leq c^2$ Therefore, every integer is less than or equal to its square.
$\underline{\exists x\ P(x)}$ $\therefore$ (c is a particular element) $\wedge$ P(c)	Existential instantiation*	There is an integer that is equal to its square. Therefore, $c^2 = c$, for some integer c.
c is an element (arbitrary or particular) $\underline{P(c)}$ $\therefore \exists x\ P(x)$	Existential generalization	Sam is a particular student in the class. Sam completed the assignment. Therefore, there is a student in the class who completed the assignment.

*Each use of existential instantiation must define a new element with its own name (e.g., "c" or "d").

Note: In existential instantiation, the variable definition is part of the conclusion in the rule. This is because the application of existential instantiation is tied to that particular variable. By contrast, in applying universal instantiation, it is valid to plug in any variable defined earlier in the proof.

EXERCISE | 1.13.1: Proving the validity of arguments with quantified statements.

Prove that the given argument is valid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. Then use the rules of inference to prove that the form is valid.

(b) The domain is the set of people who live in a city. Linda lives in the city.

Linda lives in the city. $L(\text{"Linda"})$
Linda owns a Ferrari. $F(\text{"Linda"})$
Everyone who owns a Ferrari has gotten a speeding ticket. $\forall x\,(F(x) \to S(x))$
$\therefore$ Linda has gotten a speeding ticket. $\therefore S(\text{"Linda"})$

x is a person in the city, F(x) means x owns a ferrari. S(x) means x has a speeding ticket. L(x) means x lives in in the city.

(1) $L(\text{"Linda"})$ Hypothesis

(2) $\forall x(F(x) \to S(x))$ Hypothesis

(3) $F(\text{"Linda"}) \to S(\text{"Linda"})$ Universal instantiation
$\therefore F(x) \to S(x)$ (c in (1) & (2))

(4) $F(\text{"Linda"})$ Hypothesis

(4) F("Linda") Hypothesis

$\therefore F(x)$

(5) S("Linda") Modus ponens

$\therefore S(x)$ (in (3) & (4))

B/c S("Linda") $\equiv$ S(x), argument is valid

EXERCISE | 1.13.3: Show an argument with quantified statements is invalid.

Show that the given argument is invalid by giving values for the predicates P and Q over the domain {a, b}.

(a)

$\forall x\,(P(x) \to Q(x))$
$\exists x\, \neg P(x)$
$\therefore \exists x\, \neg Q(x)$

Assume $P(a) = T$ & $Q(a) = T$
$P(b) = F$ & $Q(b) = T$

$P(a) \to Q(a) = T \to T = T$

$P(b) \to Q(b) = F \to T = T$

$\neg P(b) = \neg F = T$

$\therefore \exists x\, \neg Q(x)$

with $\neg Q(a) = \neg Q(b) = \neg T = F$

$\therefore$ argument is invalid b/c hypotheses T but conclusion F.

EXERCISE | 1.13.4: Determine and prove whether an argument is valid or invalid.

Determine whether each argument is valid. If the argument is valid, give a proof using the laws of logic. If the argument is invalid, give values for the predicates P and Q over the domain {a, b} that demonstrate the argument is invalid.

(a)

$\exists x\,(P(x) \land Q(x))$
$\therefore \exists x\, Q(x) \land \exists x\, P(x)$

(1) $\exists x\,(P(x) \wedge Q(x))$ Hypothesis

(2) (c is a particular element) $\wedge P(c) \wedge Q(c)$ Existential instantiation (in (1))
with new element c declared

(3) $P(c) \wedge Q(c)$ Simplification (in (2))

(4) $P(c)$ Simplification (in (3))

(5) c is a particular element Simplification (in (2))

(6) $\exists x\, P(x)$ Existential generalization (in (4) & (5))

(7) $Q(c)$ Simplification (in (3))

(8) $\exists x\, Q(x)$ Existential generalization (in (5) & (7))

(9) $\exists x\, P(x) \wedge \exists x\, Q(x)$ Conjunction (in (6) & (8))

(10) $\exists x\, Q(x) \wedge \exists x\, P(x)$ Commutative law

$\therefore$ the argument is valid