Ch4 - Continuous Random Variables and Probability Distributions (Part II)

Monday, June 5, 2023 6:16 PM

1. [–/10 Points] DETAILS DEVORESTAT9 4.3.028.MI.S. MY NOTES ASK YOUR

Let Z be a standard normal random variable and calculate the following probabilities, drawing pictures wherever appropriate. (Round your answers to four decimal places.)

USE SALT

(a) $P(0 \le Z \le 2.61)$

$P(0 \le Z \le 2.61) = \Phi(2.61) - \Phi(0) = 0.4955$

(b) $P(0 \le Z \le 2)$

$P(0 \le Z \le 2) = \Phi(2) - \Phi(0) = 0.4772$

(c) $P(-2.70 \le Z \le 0)$

$P(-2.70 \le Z \le 0) = \Phi(0) - \Phi(-2.70) = 0.4965$

(d) $P(-2.70 \le Z \le 2.70)$

$P(-2.70 \le Z \le 2.70) = \Phi(2.70) - \Phi(-2.70) = 0.9931$

(e) $P(Z \le 1.04)$

$P(Z \le 1.04) = \Phi(1.04) = 0.8508$

(f) $P(-1.65 \le Z)$

$P(-1.65 \le Z) = P(Z \ge -1.65) = 1 - P(Z \le -1.65) = 1 - \Phi(-1.65) = 0.9505$

(g) $P(-1.70 \le Z \le 2.00)$

$P(-1.70 \le Z \le 2.00) = \Phi(2.00) - \Phi(-1.70) = 0.9327$

(h) $P(1.04 \le Z \le 2.50)$

$P(1.04 \le Z \le 2.50) = \Phi(2.50) - \Phi(1.04) = 0.1430$

(i) $P(1.70 \le Z)$

$P(1.70 \le Z) = P(Z \ge 1.70) = 1 - P(Z \le 1.70) = 1 - \Phi(1.70) = 0.0446$

(j) $P(|Z| \le 2.50)$

$P(|Z| \le 2.50) = P(-2.50 \le Z \le 2.50) = \Phi(2.50) - \Phi(-2.50) = 0.9876$

$|Z|$ cannot be neg.

Definition

The normal distribution with parameter values $\mu = 0$ and $\sigma = 1$ is called the **standard normal distribution**. A random variable having a standard normal distribution is called a **standard normal random variable** and will be denoted by Z. The pdf of Z is

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty$$

The graph of $f(z; 0, 1)$ is called the *standard normal* (or z) curve. Its inflection points are at 1 and -1. The cdf of Z is $P(Z \le z) = \int_{-\infty}^{z} f(y; 0, 1)\, dy$, which we will denote by $\Phi(z)$.

2. [–/3 Points] DETAILS DEVORESTAT9 4.3.031.S. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Determine z_α for the following of α. (Round your answers to two decimal places.)

USE SALT Using invnorm

(a) $\alpha = 0.0079$

$z_{1-\alpha} = z_{1-0.0079} = z_{0.9921} = 2.41$

(b) $\alpha = 0.09$

$z_{1-\alpha} = z_{1-0.09} = z_{0.91} = 1.34$

(c) $\alpha = 0.693$

$z_{1-\alpha} = z_{1-0.693} = z_{0.307} = -0.50$

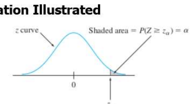
You may need to use the appropriate table in the Appendix of Tables to answer this question.

Need Help? Read It

Notation

z_α will denote the value on the z axis for which α of the area under the z curve lies to the right of z_α. (See Figure 4.19.)

Figure 4.19 z_α **Notation Illustrated**



For example, $z_{.10}$ captures upper-tail area .10, and $z_{.01}$ captures upper-tail area .01.

Since α of the area under the z curve lies to the right of z_α, $1 - \alpha$ of the area lies to its left. Thus z_α *is the* $100(1-\alpha)$*th percentile of the standard normal distribution.* By symmetry the area under the standard normal curve to the left of $-z_\alpha$ is also α. The z_α's are usually referred to as **z critical values**. Table 4.1 lists the most useful z percentiles and z_α values.

3. [–/5 Points] DETAILS **DEVORESTAT9 4.3.032.MI.S.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Suppose the force acting on a column that helps to support a building is a normally distributed random variable X with mean value 11.0 kips and standard deviation 1.50 kips. Compute the following probabilities by standardizing and then using a standard normal curve table from the Appendix Tables or SALT. (Round your answers to four decimal places.)

USE SALT

$\mu = 11.0 \quad \sigma = 1.50$

(a) $P(X \le 11)$

$P(X \le 11) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = P\left(Z \le \frac{11-11.0}{1.50}\right) = P(Z \le 0) = \Phi(0) = 0.5090$

(b) $P(X \le 12.5)$

$P(X \le 12.5) = P\left(Z \le \frac{a-\mu}{\sigma}\right) = P\left(Z \le \frac{12.5-11.0}{1.50}\right) = P(Z \le 1) = \Phi(1) = 0.8413$

(c) $P(X \ge 3.5)$

$P(X \ge 3.5) = 1 - P(X \le 3.5) = 1 - P\left(Z \le \frac{a-\mu}{\sigma}\right) = 1 - P\left(Z \le \frac{3.5-11.0}{1.50}\right) = 1 - P(Z \le -5) = 1 - \Phi(-5) = 0.9999$

(d) $P(9 \le X \le 14)$

$P(9 \le X \le 14) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right) = P\left(\frac{9-11.0}{1.50} \le Z \le \frac{14-11.0}{1.50}\right) = P(-1.33 \le Z \le 2.00) = \Phi(2.00) - \Phi(-1.33) = 0.8855$

(e) $P(|X - 11| \le 2)$

$P(|X-11| \le 2) = P(-2 \le X-11 \le 2) = P(9 \le X \le 13) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right) = P\left(\frac{9-11.0}{1.50} \le Z \le \frac{13-11.0}{1.50}\right) = P(-1.33 \le Z \le 1.33) = \Phi(1.33) - \Phi(-1.33) = 0.8165$

Need Help? Read It Watch It Master It

Proposition

If X has a normal distribution with mean μ and standard deviation σ, then

$$Z = \frac{X-\mu}{\sigma}$$

has a standard normal distribution. Thus

$$\begin{aligned} P(a \le X \le b) &= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \\ P(X \le a) &= \Phi\left(\frac{a-\mu}{\sigma}\right) \qquad P(X \ge b) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right) \end{aligned}$$

4. [–/5 Points] DETAILS **DEVORESTAT9 4.3.036.S.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Spray drift is a constant concern for pesticide applicators and agricultural producers. The inverse relationship between droplet size and drift potential is well known. The paper "Effects of 2,4-D Formulation and Quinclorac on Spray Droplet Size and Deposition"† investigated the effects of herbicide formulation on spray atomization. A figure in a paper suggested the normal distribution with mean 1050 μm and standard deviation 150 μm was a reasonable model for droplet size for water (the "control treatment") sprayed through a 760 ml/min nozzle.

USE SALT

$\mu = 1{,}050 \quad \sigma = 150$

(a) What is the probability that the size of a single droplet is less than 1410 μm? At least 975 μm? (Round your answers to four decimal places.)

less than 1410 μm $\frac{x-\mu}{\sigma} = \frac{1{,}410-1{,}050}{150} = 2.40$ $\quad P(X < 1{,}410) = P(Z < 2.40) = \Phi(2.40) = 0.9918$

at least 975 μm $\frac{x-\mu}{\sigma} = \frac{975-1{,}050}{150} = -0.50$ $\quad P(X \ge 975) = 1 - P(X \le 975) = 1 - P(Z \le -0.50) = 1 - \Phi(-0.50) = 0.6915$

(b) What is the probability that the size of a single droplet is between 975 and 1410 μm? (Round your answer to four decimal places.)

$P(975 \le X \le 1{,}410) = P(-0.50 \le Z \le 2.40) = \Phi(2.40) - \Phi(-0.50) = 0.6834$

(c) How would you characterize the smallest 2% of all droplets? (Round your answer to two decimal places.)

The smallest 2% of droplets are those smaller than ____ μm in size. $Z = \frac{X-\mu}{\sigma} \Rightarrow -2.05 = \frac{x-1{,}050}{150} \Rightarrow x = 742.50$

$\alpha = 0.02 \quad Z_{1-\alpha} = Z_{1-0.02} = Z_{0.98} = 2.05$ (smallest 2% so $Z = -2.05$)

(d) If the sizes of five independently selected droplets are measured, what is the probability that at least one exceeds 1410 μm? (Round your answer to four decimal places.)

$P(X > 1{,}410) = 1 - P(X \le 1{,}410) = 1 - P\left(Z \le \frac{a-\mu}{\sigma}\right) = 1 - P\left(Z \le \frac{1{,}410-1{,}050}{150}\right) = 1 - P(Z \le 2.40) = 0.0082$

You may need to use the appropriate table in the Appendix of Tables to answer this question. $n = 5,\ p = 0.0082$ Binomial dist: $P(Y=1) = b(y; n, p) = b(1; 5, 0.0082) = 0.0403$

Need Help? Read It Watch It

5. [–/4 Points] DETAILS **DEVORESTAT9 4.4.059.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Let X = the time between two successive arrivals at the drive-up window of a local bank. If X has an exponential distribution with $\lambda = 1$, (which is identical to a standard gamma distribution with $\alpha = 1$), compute the following. (If necessary, round your answer to three decimal places.)

(a) The expected time between two successive arrivals

$\mu = \frac{1}{\lambda} = \frac{1}{1} = 1$

(b) The standard deviation of the time between successive arrivals

$\sigma = \mu = 1$

(c) $P(X \le 2)$

$P(X \le 2) = F(x; \lambda) = F(2; 1) = 1 - e^{-(1)(2)} = 1 - e^{-2} = 0.865$

(d) $P(1 \le X \le 4)$

$P(1 \le X \le 4) = P(X=4) - P(X=1) = (1 - e^{-(1)(4)}) - (1 - e^{-(1)(1)}) = (1 - e^{-4}) - (1 - e^{-1}) = 0.350$

You may need to use the appropriate table in the Appendix of Tables

Need Help? Read It

Definition

X is said to have an **exponential distribution** with (scale) parameter λ ($\lambda > 0$) if the pdf of X is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{(4.5)}$$

Some sources write the exponential pdf in the form $(1/\beta)e^{-x/\beta}$, so that $\beta = 1/\lambda$. The expected value of an exponentially distributed random variable X is

$$\mu = E(X) = \int_0^\infty x\lambda e^{-\lambda x}\,dx$$

Obtaining this expected value necessitates doing an integration by parts. The variance of X can be computed using the fact that $V(X) = E(X^2) - [E(X)]^2$. The determination of $E(X^2)$ requires integrating by parts twice in succession. The results of these integrations are as follows:

$$\mu = \frac{1}{\lambda} \qquad \sigma^2 = \frac{1}{\lambda^2}$$

Both the mean and standard deviation of the exponential distribution equal $1/\lambda$. Graphs of several exponential pdf's are illustrated in Figure 4.26.

The exponential pdf is easily integrated to obtain the cdf.

$$F(x;\lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

6. [–/5 Points] DETAILS **DEVORESTAT9 4.4.060.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Let X denote the distance (m) that an animal moves from its birth site to the first territorial vacancy it encounters. Suppose that for banner-tailed kangaroo rats, X has an exponential distribution with parameter $\lambda = 0.01367$.

(a) What is the probability that the distance is at most 100 m? At most 200 m? Between 100 and 200 m? (Round your answers to four decimal places.)

at most 100 m $P(X \le 100) = 1 - e^{-\lambda x} = 1 - e^{-(0.01367)(100)} = 0.7451$

at most 200 m $P(X \le 200) = 1 - e^{-\lambda x} = 1 - e^{-(0.01367)(200)} = 0.9350$

between 100 and 200 m $P(100 \le X \le 200) = (1 - e^{-(0.01367)(200)}) - (1 - e^{-(0.01367)(100)}) = 0.1899$

(b) What is the probability that distance exceeds the mean distance by more than 2 standard deviations? (Round your answer to four decimal places.)

$M = \sigma = \frac{1}{\lambda} = \frac{1}{0.01367} = 73.1529$ $\quad P(X > M + 2\sigma) = 1 - P(X < M + 2\sigma) = 1 - P(X < 73.1529 + 2(73.1529)) = 1 - P(X < 219.4587)$

$= 1 - (1 - e^{-(0.01367)(219.4587)})$

$= 0.0498$

(c) What is the value of the median distance? (Round your answer to two decimal places.)

m

$e^{-\lambda M} = 0.5$

$-\lambda M = \ln(0.5)$

$M = \frac{\ln(0.5)}{-\lambda} = \frac{\ln(0.5)}{-0.01367} = 50.71$

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7. [–/6 Points] DETAILS **DEVORESTAT9 4.4.069.** MY NOTES ASK YOUR TEACHER

A system consists of five identical components connected in series as shown:

— 1 — 2 — 3 — 4 — 5 —

As soon as one components fails, the entire system will fail. Suppose each component has a lifetime that is exponentially distributed with $\lambda = 0.01$ and that components fail independently of one another. Define events A_i = {ith component lasts at least t hours}, $i = 1, \ldots, 5$, so that the A_is are independent events. Let X = the time at which the system fails—that is, the shortest (minimum) lifetime among the five components.

(a) The event $\{X \ge t\}$ is equivalent to what event involving $A_1, \ldots, A_5$?

- ● $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$
- ○ $A_1 \cap A_2 \cup A_3 \cap A_4 \cup A_5$
- ○ $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$
- ○ $A_1 \cup A_2 \cap A_3 \cup A_4 \cap A_5$

(b) Using the independence of the A_i's, compute $P(X \ge t)$.

$P(X \ge t) =$ $P(X \ge t) = 1 - P(X \le t) = [1 - F(A_i)]^5 = \left[1 - \int_0^t \lambda e^{-\lambda t}\,dt\right]^5 = [1 - (1 - e^{-\lambda t})]^5 = [1 - 1 + e^{-\lambda t}]^5 = e^{-5\lambda t} = e^{-5(0.01)t} = e^{-0.05t}$

Obtain $F(t) = P(X \le t)$.

$F(t) =$ $P(X \le t) = 1 - e^{-0.05t}$

Obtain the pdf of X.

$f(t) =$ $f(t) = \frac{d}{dt}[F(t)] = \frac{d}{dt}[1 - e^{-0.05t}] = 0.05e^{-0.05t}$

What type of distribution does X have?

- ○ X is a gamma distribution with parameters $\alpha = 0$ and $\beta = 1$.
- ● X is an exponential distribution with $\lambda = 0.05$.
- ○ X is a gamma distribution with parameters $\alpha = 1$ and $\beta = 0.05$.
- ○ X is an exponential distribution with $\lambda = 1$.

(c) Suppose there are n components, each having exponential lifetime with parameter λ. What type of distribution does X have?

- ○ X is a gamma distribution with parameters $\alpha = 1$ and $\beta = 1/\lambda$.
- ○ X is an exponential distribution with parameter $\lambda = e$.
- ○ X is a gamma distribution with parameters $\alpha = \lambda$ and $\beta = n$.
- ● X is an exponential distribution with parameter $n\lambda$. $P(X \ge t) = 1 - P(X \le t) = [1 - F(A_i)]^n = e^{-n\lambda t}$

parameters: n & λ

Definition

For $\alpha > 0$, the **gamma function** $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx \qquad \textbf{(4.6)}$$

The most important properties of the gamma function are the following:

1. For any $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1)\cdot\Gamma(\alpha-1)$ [via integration by parts]
2. For any positive integer, n, $\Gamma(n) = (n-1)!$
3. $\Gamma(1/2) = \sqrt{\pi}$

Now let

$$f(x;\alpha) = \begin{cases} \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{(4.7)}$$

8. [–/4 Points] DETAILS **DEVORESTAT9 4.5.072.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

$\Gamma(\alpha)$ $\alpha > 1$ then $(\alpha-1)\Gamma(\alpha-1)$

The lifetime X (in hundreds of hours) of a certain type of vacuum tube has a Weibull distribution with parameters $\alpha = 2$ and $\beta = 3$.† Compute the following. (Round your answers to three decimal places.)

(a) $E(X)$ and $V(X)$

$E(X) =$

$V(X) =$

$E(x) = \beta\cdot\Gamma\left(1+\frac{1}{\alpha}\right) = 3\cdot\Gamma\left(1+\frac{1}{2}\right) = 3\cdot\Gamma\left(\frac{3}{2}\right)$ $(\alpha > 1) \Rightarrow 3\cdot\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right) = \frac{3}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}(\sqrt{\pi}) = 2.659$

$V(X) = \beta^2\left(\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right) = (3)^2\left(\Gamma\left(1+\frac{2}{2}\right) - \left[\Gamma\left(1+\frac{1}{2}\right)\right]^2\right) = 9\left(\Gamma(2) - \left[\Gamma\left(\frac{3}{2}\right)\right]^2\right) = 9\left((2-1)\Gamma(2-1) - \left[(3/2-1)\Gamma(3/2-1)\right]^2\right) = 9\left(\Gamma(1) - \left[\frac{1}{2}\Gamma(1/2)\right]^2\right) = 9\left((1-1)! - \left[\frac{1}{2}\sqrt{\pi}\right]^2\right) = 9\left(1 - \frac{\pi}{4}\right) = 1.931$

(b) $P(X \le 3)$

$P(X \le 3) = 1 - e^{-(X/\beta)^\alpha} = 1 - e^{-(3/3)^2} = 0.632$

(c) $P(1 \le X \le 3)$

$P(1 \le X \le 3) = F(3;2,3) - F(1;2,3) = (1 - e^{-(3/3)^2}) - (1 - e^{-(1/3)^2}) = 0.527$

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$F(x;\alpha,\beta)$

Definition

A random variable X is said to have a **Weibull distribution** with shape parameter α and scale parameter β $(\alpha > 0, \beta > 0)$ if the pdf of X is

$$f(x;\alpha,\beta) = \begin{cases} \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha} & x \ge 0 \\ 0 & x < 0 \end{cases} \qquad \textbf{(4.11)}$$

The cdf of a Weibull rv having parameters α and β is

$$F(x;\alpha,\beta) = \begin{cases} 0 & x < 0 \\ 1 - e^{-(x/\beta)^\alpha} & x \ge 0 \end{cases} \qquad \textbf{(4.12)}$$

Integrating to obtain $E(X)$ and $E(X^2)$ yields

$$\mu = \beta\Gamma\left(1+\frac{1}{\alpha}\right) \qquad \sigma^2 = \beta^2\left\{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right\}$$

The computation of μ and σ^2 thus necessitates using the gamma function.

9. [–/5 Points] DETAILS **DEVORESTAT9 4.5.073.MI.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

The Weibull distribution is widely used in statistical problems relating to aging of solid insulating materials subjected to aging and stress. Use this distribution as a model for time (in hours) to failure of solid insulating specimens subjected to AC voltage. The values of the parameters depend on the voltage and temperature; suppose $\alpha = 2.8$ and $\beta = 180$.

(a) What is the probability that a specimen's lifetime is at most 250? Less than 250? More than 300? (Round your answers to five decimal places.)

at most 250 $P(X \le 250) = F(x;\alpha,\beta) = F(250;2.8,180) = 1 - e^{-(x/\beta)^\alpha} = 1 - e^{-(250/180)^{2.8}} = 0.91864$

less than 250 Same as above, $P(x \le 250)$

more than 300 $P(X > 300) = 1 - P(x \le 300) = 1 - F(300;2.8,180) = 1 - (1 - e^{-(x/\beta)^\alpha}) = e^{-(x/\beta)^\alpha} = e^{-(300/180)^{2.8}} = 0.01530$

(b) What is the probability that a specimen's lifetime is between 100 and 250? (Round your answer to four decimal places.)

$P(100 \le X \le 250) = F(250;2.8,180) - F(100;2.8,180) = (1 - e^{-(250/180)^{2.8}}) - (1 - e^{-(100/180)^{2.8}}) = 0.7432$

(c) What value (in hr) is such that exactly 50% of all specimens have lifetimes exceeding that value? (Round your answer to three decimal places.)

hr

$1 - e^{-(x/\beta)^\alpha} = 0.5$

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$-e^{-(x/\beta)^\alpha} = 0.5 - 1$

$e^{-(x/\beta)^\alpha} = 1 - 0.5$

$e^{-(x/\beta)^\alpha} = 0.5$

$-\left(\frac{x}{\beta}\right)^\alpha = \ln(0.5)$

$\left(\frac{x}{\beta}\right)^\alpha = -\ln(0.5)$

$\frac{x}{\beta} = \sqrt[\alpha]{-\ln(0.5)}$

$x = \beta\cdot\sqrt[\alpha]{-\ln(0.5)}$

$x = 180\sqrt[2.8]{-\ln(0.5)}$

$x = 157.915$

10. [–/4 Points] DETAILS **DEVORESTAT9 4.5.076.S.** MY NOTES | ASK YOUR TEACHER | PRACTICE ANOTHER

An article suggests the lognormal distribution as a model for SO_2 concentration above a certain forest. Suppose the parameter values are $\mu = 2.1$ and $\sigma = 0.7$.

USE SALT

(a) What are the mean value and standard deviation of concentration? (Round your answers to three decimal places.)

mean

$E(X) = e^{\mu+\sigma^2/2} = e^{2.1+(0.7)^2/2} = 10.433$

standard deviation

$V(X) = e^{2\mu+\sigma^2}\cdot(e^{\sigma^2}-1) = e^{2(2.1)+(0.7)^2}\cdot(e^{(0.7)^2}-1) = 68.830$ $\quad S.D. = \sqrt{V(X)} = \sqrt{68.830} = 8.296$

(b) What is the probability that concentration is at most 10? Between 5 and 10? (Round your answers to four decimal places.)

at most 10

$P(X \le 10) = P(\ln(X) \le \ln(10)) = P\left(Z \le \frac{\ln(x)-\mu}{\sigma}\right) = P\left(Z \le \frac{\ln(10)-2.1}{0.7}\right) = P(Z \le 0.29) = \Phi(0.29) = 0.6139$

between 5 and 10

$P(5 \le X \le 10) = P(\ln(5) \le \ln(X) \le \ln(10)) = P\left(\frac{\ln(a)-\mu}{\sigma} \le Z \le \frac{\ln(b)-\mu}{\sigma}\right) = P\left(\frac{\ln(5)-2.1}{0.7} \le Z \le \frac{\ln(10)-2.1}{0.7}\right) = P(-0.70 \le Z \le 0.29) = \Phi(0.29) - \Phi(-0.70) = 0.3721$

You may need to use the appropriate table in the Appendix of Tables to answer this question.

Need Help? Read It Watch It

Definition

A nonnegative rv X is said to have a **lognormal distribution** if the rv $Y = \ln(X)$ has a normal distribution. The resulting pdf of a lognormal rv when $\ln(X)$ is normally distributed with parameters μ and σ is

$$f(x;\mu,\sigma) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} e^{-[\ln(x)-\mu]^2/(2\sigma^2)} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

Be careful here; the parameters μ and σ are not the mean and standard deviation of X but of $\ln(X)$. The mean and variance of X can be shown to be

$$E(X) = e^{\mu+\sigma^2/2} \qquad V(X) = e^{2\mu+\sigma^2}\cdot(e^{\sigma^2}-1)$$

Because $\ln(X)$ has a normal distribution, the cdf of X can be expressed in terms of the cdf $\Phi(z)$ of a standard normal rv Z.

$$\begin{aligned} F(x;\mu,\sigma) &= P(X \le x) = P[\ln(X) \le \ln(x)] \\ &= P\left(Z \le \frac{\ln(x)-\mu}{\sigma}\right) = \Phi\left(\frac{\ln(x)-\mu}{\sigma}\right) \qquad x \ge 0 \end{aligned} \tag{4.13}$$

11. [–/5 Points] DETAILS **DEVORESTAT9 4.5.084.** MY NOTES | ASK YOUR TEACHER | PRACTICE ANOTHER

$A=0,\ B=1$

Suppose the proportion X of surface area in a randomly selected quadrat that is covered by a certain plant has a standard beta distribution with $\alpha = 5$ and $\beta = 2$.

(a) Compute $E(X)$ and $V(X)$. (Round your answers to four decimal places.)

$E(X) =$

$E(X) = A + (B-A)\cdot\frac{\alpha}{\alpha+\beta} = 0 + (1-0)\cdot\frac{5}{5+2} = 0.7143$

$V(X) =$

$V(X) = \frac{(B-A)^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{(1-0)^2(5)(2)}{(5+2)^2(5+2+1)} = 0.0255$

(b) Compute $P(X \le 0.7)$. (Round your answer to four decimal places.)

Excel: beta.dist(x, α, β, cum.) = beta.dist(0.7, 5, 2, TRUE) = 0.4202

(c) Compute $P(0.7 \le X \le 0.9)$. (Round your answer to four decimal places.)

Excel: beta.dist(0.9, 5, 2, TRUE) − beta.dist(0.7, 5, 2, TRUE) = 0.4656

(d) What is the expected proportion of the sampling region not covered by the plant? (Round your answer to four decimal places.)

$1 - E(X) = 1 - 0.7143 = 0.2857$

Need Help? Read It

Definition

A random variable X is said to have a **beta distribution** with parameters α, β (both positive), A, and B if the pdf of X is

$$f(x;\alpha,\beta,A,B) = \begin{cases} \frac{1}{B-A}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\left(\frac{x-A}{B-A}\right)^{\alpha-1}\left(\frac{B-x}{B-A}\right)^{\beta-1} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

The case $A = 0$, $B = 1$ gives the **standard beta distribution**.

Figure 4.32 illustrates several standard beta pdf's. Graphs of the general pdf are similar, except they are shifted and then stretched or compressed to fit over $[A, B]$. Unless α and β are integers, integration of the pdf to calculate probabilities is difficult. Either a table of the incomplete beta function or appropriate software should be used. The mean and variance of X are

$$\mu = A + (B-A)\cdot\frac{\alpha}{\alpha+\beta} \qquad \sigma^2 = \frac{(B-A)^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$