Ch3 - Discrete Random Variables and Probability Distributions (Part II)

Sunday, June 4, 2023   3:24 PM

1. [–/4 Points] DETAILS **DEVORESTAT9 3.4.046.MI.**

Compute the following binomial probabilities directly from the formula for $b(x; n, p)$. (Round your answers to three decimal places.)

(a) $b(4; 8, 0.3)$

$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{8}{4}(0.3)^4(1-0.3)^{8-4} = 0.136$

(b) $b(6; 8, 0.65)$

$b(6; 8, 0.65) = \binom{8}{6}(0.65)^6(1-0.65)^{8-6} = 0.259$

(c) $P(3 \le X \le 5)$ when $n = 7$ and $p = 0.55$

$P(3 \le X \le 5) = b(3; 7, 0.55) + b(4; 7, 0.55) + b(5; 7, 0.55) = 0.239 + 0.292 + 0.214 = 0.745$

(d) $P(1 \le X)$ when $n = 9$ and $p = 0.15$

$P(1 \le X) = 1 - P(X=0) = 1 - b(0; 9, 0.15) = 1 - 0.232 = 0.768$

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Notation

Because the pmf of a binomial rv X depends on the two parameters n and p, we denote the pmf by $b(x; n, p)$.

Theorem

$$b(x; n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

Notation

For $X \sim \text{Bin}(n, p)$, the cdf will be denoted by

$$B(x; n, p) = P(X \le x) = \sum_{y=0}^{x} b(y; n, p) \qquad x = 0, 1, \ldots, n$$

2. [–/4 Points] DETAILS **DEVORESTAT9 3.4.050.S.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

A particular telephone number is used to receive both voice calls and fax messages. Suppose that 25% of the incoming calls involve fax messages, and consider a sample of 25 incoming calls. (Round your answers to three decimal places.)

$n = 25$
$p = 0.25$

USE SALT

(a) What is the probability that at most 8 of the calls involve a fax message?

$P(X \le 8) = \sum_{y=0}^{8} b(y; 25, 0.25) = B(8; 25, 0.25) = 0.851$

(b) What is the probability that exactly 8 of the calls involve a fax message?

$P(X = 8) = b(8; 25, 0.25) = 0.124$

(c) What is the probability that at least 8 of the calls involve a fax message?

$P(X \ge 8) = 1 - P(X < 8) = 1 - P(X \le 7) = 1 - B(7; 25, 0.25) = 1 - 0.727 = 0.273$

(d) What is the probability that more than 8 of the calls involve a fax message?

$P(X > 8) = 1 - P(X \le 8) = 1 - B(8; 25, 0.25) = 1 - 0.851 = 0.149$

You may need to use the appropriate table in the Appendix of Tables to answer this question.

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3. [–/1 Points] DETAILS **DEVORESTAT9 3.4.055.S.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Twenty percent of all telephones of a certain type are submitted for service while under warranty. Of these, 60% can be repaired, whereas the other 40% must be replaced with new units. If a company purchases ten of these telephones, what is the probability that exactly two will end up being replaced under warranty? (Round your answer to three decimal places.)

$n = 10$ $p = \%$ need servicing $\cdot \%$ need replaced $= (0.20)(0.40) = 0.08$ $x = 2$

USE SALT

$P(X=2) = b(2; 10, 0.08) = 0.148$

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4. [–/1 Points] DETAILS **DEVORESTAT9 3.4.060.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

A toll bridge charges $1.00 for passenger cars and $2.50 for other vehicles. Suppose that during daytime hours, 70% of all vehicles are passenger cars. If 25 vehicles cross the bridge during a particular daytime period, what is the resulting expected toll revenue? [*Hint:* Let X = the number of passenger cars; then the toll revenue $h(X)$ is a linear function of X.]

$

$n=25 \quad p=0.7 \quad h(X)=1.00X+2.50(25-X)=1.00X+62.5-2.50X=62.5-1.50X$

$E(h(X))=E(62.5-1.5X)=E(62.5)-E(1.5X)=62.5-1.5E(X)=62.5-1.5(17.5)=36.25$

$E(X)=np=(25)(0.7)=17.5$

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Proposition

If $X \sim \text{Bin}(n,p)$, then $E(X)=np$, $V(X)=np(1-p)=npq$, and $\sigma_X=\sqrt{npq}$ (where $q=1-p$).

5. [–/7 Points] DETAILS **DEVORESTAT9 3.5.070.MI.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

An instructor who taught two sections of engineering statistics last term, the first with 25 students and the second with 40, decided to assign a term project. After all projects had been turned in, the instructor randomly ordered them before grading. Consider the first 15 graded projects. $N=25+40=65 \quad n=15$

(a) What is the probability that exactly 10 of these are from the second section? (Round your answer to four decimal places.) $M=40$

$$P(X=10)=\frac{\binom{M}{x}\cdot\binom{N-M}{n-x}}{\binom{N}{n}}=\frac{\binom{40}{10}\cdot\binom{65-40}{15-10}}{\binom{65}{15}}=0.2172$$

(b) What is the probability that at least 10 of these are from the second section? (Round your answer to four decimal places.) using $H(x;n,M,N)$

$P(X\geq 10)=1-P(X<10)=1-P(X\leq 9)=1-H(9;15,40,65)=1-0.5594=0.4406$

$M=40$ (second section)

(c) What is the probability that at least 10 of these are from the same section? (Round your answer to four decimal places.)

$P(X\geq 10)=1-P(X\leq 9)=1-H(9;15,25,65)=1-0.9874=0.0126$ so $P(X\geq 10)\big|_{M=40}+P(X\geq 10)\big|_{M=25}=0.4406+0.0126=0.4532$

$M=25$ (first section)

(d) What are the mean value and standard deviation of the number among these 15 that are from the second section? (Round your mean to the nearest whole number and your standard deviation to three decimal places.)

mean projects

standard deviation projects

$E(X)=n\cdot\frac{M}{N}=15\cdot\frac{40}{65}\approx 9$ $\quad \sigma=\sqrt{V(X)}=\sqrt{2.774}=1.665$

$V(X)=\left(\frac{N-n}{N-1}\right)\cdot n\cdot\frac{M}{N}\cdot\left(1-\frac{M}{N}\right)=\left(\frac{65-15}{65-1}\right)\cdot 15\cdot\frac{40}{65}\cdot\left(1-\frac{40}{65}\right)=2.774$

(e) What are the mean value and standard deviation of the number of projects not among these first 15 that are from the second section? (Round your mean to the nearest whole number and your standard deviation to three decimal places.) $h(X)=40-X$

mean projects

standard deviation projects

$E(h(X))=E(40-X)=E(40)-E(X)=40-E(X)=40-9=31$

$V(h(X))=V(40-X)=V(X)$ $\therefore$ $V(X)$ & σ are the same as above

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Proposition

If X is the number of S's in a completely random sample of size n drawn from a population consisting of M S's and $(N-M)$ F's, then the probability distribution of X, called the **hypergeometric distribution**, is given by

$$P(X=x)=h(x;n,M,N)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \qquad \textbf{(3.15)}$$

let $H(x;n,M,N)$ be the cdf

for x an integer satisfying $\max(0, n-N+M)\leq x\leq \min(n,M)$.

Proposition

The mean and variance of the hypergeometric rv X having pmf $h(x;n,M,N)$ are

$$E(X)=n\cdot\frac{M}{N} \qquad V(X)=\left(\frac{N-n}{N-1}\right)\cdot n\cdot\frac{M}{N}\cdot\left(1-\frac{M}{N}\right)$$

6. [–/2 Points] DETAILS **DEVORESTAT9 3.5.072.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

A personnel director interviewing 9 senior engineers for six job openings has scheduled seven interviews for the first day and two for the second day of interviewing. Assume that the candidates are interviewed in a random order.

$N=9$ $n=7$

(a) What is the probability that x of the top six candidates are interviewed on the first day? $M=6$

- ○ $h(N;\ 7, 9, 6)$
- ● $h(x;\ 7, 6, 9)$
- ○ $h(N;\ 2, 6, 9)$
- ○ $h(x;\ 2, 9, 6)$

$h(x; n, M, N) = h(x; 7, 6, 9)$

(b) How many of the top six candidates can be expected to be interviewed on the first day? (Round your answer to two decimal places.)

[ ] candidates

$E(x) = n \cdot \frac{M}{N} = 7 \cdot \frac{6}{9} = 4.67$

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7. [–/7 Points] DETAILS **DEVORESTAT9 3.5.076.**

A family decides to have children until it has three children of the same gender. Assuming $P(B) = P(G) = 0.5$, what is the pmf of X = the number of children in the family?

x	0	1	2	3	4	5	6
$p(x)$	0	0	0	0.25	0.375	0.375	0

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Until the family has three children of the same gender, all feasible combinations of boys and girls are:

$$\{BBB, GGG, BBGB, BGBB, GBBB, GGBG, GBGG, BGGG, BBGGB, BGBGB, BGGBB, GBGBB, GBBGB, GGBBB, BBGGG, BGBGG, BGGBG, GGBBG, GBGBG, GBBGG\}$$

Then we see that there are two combinations that result in three children, six combinations that result in four children, and twelve combinations that result in five children.

Let X be the family's total number of children. Add up the probability for each conceivable combination:

$P(BBB) = (0.5)(0.5)(0.5) = 0.125$

$$\begin{aligned} P(X=3) &= P(BBB) + P(GGG) \\ &= 0.125 + 0.125 \\ &= 0.25 \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} P(X=4) &= P(BBGB) + \ldots + P(BGGG) \\ &= 6P(BBGB) \\ &= 6(0.0625) \\ &= 0.375 \\ &= \frac{3}{8} \end{aligned}$$

$$\begin{aligned} P(X=5) &= P(BBGGB) + \ldots + P(GBBGG) \\ &= 12P(BBGGB) \\ &= 12(0.03125) \\ &= 0.375 \\ &= \frac{3}{8} \end{aligned}$$

8. [–/3 Points] DETAILS **DEVORESTAT9 3.5.078.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

For a certain river, suppose the drought length Y is the number of consecutive time intervals in which the water supply remains below a critical value y_0 (a deficit), preceded by and followed by periods in which the supply exceeds this critical value (a surplus). An article proposes a geometric distribution with $p = 0.376$ for this random variable. (Round your answers to three decimal places.)

(a) What is the probability that a drought lasts exactly 3 intervals? At most 3 intervals?

exactly 3 intervals [ ] $P(x=3) = nb(3;1,0.376) = (1-0.376)^3(0.376) = 0.091$

at most 3 intervals [ ] $P(X \le 3) = \text{geocdf}(4;1,0.376) = 0.848$

(b) What is the probability that the length of a drought exceeds its mean value by at least one standard deviation?

[ ] $E(X) = \frac{(1-p)}{p} = \frac{1-0.376}{0.376} = 1.660$ $V(X)$ $\frac{1-p}{p^2} = \frac{1-0.376}{(0.376)^2} = 4.414$ $\sigma = \sqrt{V(X)} = \sqrt{4.414} = 2.101$

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$P(X \ge \mu+\sigma) = P(X \ge 1.660+2.101) = P(X \ge 3.761) = 1 - P(X \le 3.761) = 1 - P(X \le 3) = 1 - \text{geocdf}(4; 0.376) = 0.152$

In the special case $r = 1$, the pmf is

$$nb(x;1,p) = (1-p)^x p \qquad x = 0, 1, 2, \ldots \qquad \textbf{(3.17)}$$

In Example 3.12, we derived the pmf for the number of trials necessary to obtain the first S, and the pmf there is similar to Expression (3.17). Both $X = \text{number of}F\text{'s}$ and $Y = \text{number of trials}(=1+X)$ are referred to in the literature as **geometric random variables**, and the pmf in Expression (3.17) is called the **geometric distribution**.

Proposition

If X is a negative binomial rv with pmf $nb(x; r, p)$, then

$$E(X) = \frac{r(1-p)}{p} \qquad V(X) = \frac{r(1-p)}{p^2}$$

9. [–/5 Points] DETAILS **DEVORESTAT9 3.6.080.MI.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Let X be the number of material anomalies occurring in a particular region of an aircraft gas-turbine disk. The article "Methodology for Probabilistic Life Prediction of Multiple-Anomaly Materials"† proposes a Poisson distribution for X. Suppose that $\mu = 4$. (Round your answers to three decimal places.)

(a) Compute both $P(X \le 4)$ and $P(X < 4)$.

$P(X \le 4) =$ $P(X \le 4) = P_{cdf}(4; 4) = 0.629$

$P(X < 4) =$ $P(x < 4) = P(x \le 3) = P_{cdf}(3; 4) = 0.433$

(b) Compute $P(4 \le X \le 7)$.

$P(4 \le x \le 7) = \sum_{x=4}^{7} \left[\frac{e^{-4} \cdot 4^x}{x!}\right] = 0.515$

(c) Compute $P(7 \le X)$.

$P(7 \le X) = P(x \ge 7) = 1 - P(x \le 6) = 0.111$

(d) What is the probability that the number of anomalies does not exceed the mean value by more than one standard deviation?

$E(x) = V(x) = \mu = 4$

$\sigma = \sqrt{V(x)} = \sqrt{4} = 2$

$P(x \ngtr \mu + \sigma) = P(x \le \mu + \sigma) = P(x \le 4+2) = P(x \le 6) = P_{cdf}(6; 4) = 0.889$

You may need to use the appropriate table in the Appendix of Tables to answer this question.

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Definition

A discrete random variable X is said to have a **Poisson distribution** with parameter μ ($\mu > 0$) if the pmf of X is

$$p(x; \mu) = \frac{e^{-\mu} \cdot \mu^x}{x!} \qquad x = 0, 1, 2, 3, \ldots$$

Proposition

If X has a Poisson distribution with parameter μ, then $E(X) = V(X) = \mu$.

10. [–/3 Points] DETAILS **DEVORESTAT9 3.6.082.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Consider writing onto a computer disk and then sending it through a certifier that counts the number of missing pulses. Suppose this number X has a Poisson distribution with parameter $\mu = 0.5$. (Round your answers to three decimal places.)

(a) What is the probability that a disk has exactly one missing pulse?

$P(X=1) = p(1; 0.5) = 0.303$

(b) What is the probability that a disk has at least two missing pulses?

$P(x \ge 2) = 1 - P(x \le 1) = 1 - P_{cdf}(1; 0.5) = 0.090$

(c) If two disks are independently selected, what is the probability that neither contains a missing pulse?

$P(x=0 \cap y=0) = P(x=0) \cdot P(x=0) = [p(0; 0.5)]^2 = 0.368$

You may need to use the appropriate table in the Appendix of Tables to answer this question.

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11. [–/3 Points] DETAILS **DEVORESTAT9 3.6.089.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

An article suggests that a Poisson process can be used to represent the occurrence of structural loads over time. Suppose the mean time between occurrences of loads is 0.4 year.

(a) How many loads can be expected to occur during a 4-year period?

loads

11. [–/3 Points] DETAILS DEVORESTAT9 3.6.089. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

An article suggests that a Poisson process can be used to represent the occurrence of structural loads over time. Suppose the mean time between occurrences of loads is 0.4 year.

(a) How many loads can be expected to occur during a 4-year period?

[ ] loads $\text{rate} = \alpha = \frac{1}{t} = \frac{1}{0.4} = 2.5$ $\quad \mu = \alpha \cdot t = (2.5)(4) = 10$

(b) What is the probability that more than thirteen loads occur during a 4-year period? (Round your answer to three decimal places.)

[ ] $P(X > 13) = 1 - P(X \le 13) = 1 - P_{cdf}(13; 10) = 0.136$

(c) How long must a time period be so that the probability of no loads occurring during that period is at most 0.2? (Round your answer to four decimal places.)

[ ] yr $P(X=0) \le 0.2 \Rightarrow \frac{e^{-\alpha t}(\alpha t)^x}{x!} \le 0.2 \Rightarrow \frac{e^{-2.5t}(2.5t)^0}{0!} \le 0.2 \Rightarrow e^{-2.5t} \le 0.2 \Rightarrow -2.5t \le \ln(0.2) \Rightarrow t \ge \frac{\ln(0.2)}{-2.5}$

$\Rightarrow t \ge 0.6438$

You may need to use the appropriate table in the Appendix of Tables to answer this question.

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Proposition

$P_k(t) = e^{-\alpha t} \cdot (\alpha t)^k / k!$, so that the number of events during a time interval of length t is a Poisson rv with parameter $\mu = \alpha t$. The expected number of events during any such time interval is then αt, so the expected number during a unit interval of time is α.

The occurrence of events over time as described is called a *Poisson process;* the parameter α specifies the *rate* for the process.

12. [–/3 Points] DETAILS DEVORESTAT9 3.6.092. MY NOTES ASK YOUR TEACHER

Automobiles arrive at a vehicle equipment inspection station according to a Poisson process with rate $\alpha = 8$ per hour. Suppose that with probability 0.5 an arriving vehicle will have no equipment violations.

(a) What is the probability that exactly eight arrive during the hour and all eight have no violations? (Round your answer to four decimal places.)

[ ] $t=1,\ \alpha = 8$ so $\mu = \alpha \cdot t = 8 \cdot 1 = 8$ $\quad P(X=8 \cap \text{no violations}) = P(X=8) \cdot P_{\text{no violations}}(X=8) = p(8;8) \cdot (0.5)^8 = 0.0005$

(b) For any fixed $y \ge 8$, what is the probability that y arrive during the hour, of which eight have no violations?

[ ] $P(y \text{ arrive} \cap y \ge 8 \text{ no violations}) = \frac{e^{-8} \cdot 8^y}{y!} \cdot \frac{y!}{8!(y-8)!} \cdot (0.5)^y = \frac{e^{-8} 8^y (0.5)^y}{8!(y-8)!}$

(c) What is the probability that eight "no-violation" cars arrive during the next hour? [*Hint:* Sum the probabilities in part (b) from $y = 8$ to ∞.] (Round your answer to three decimal places.)

[ ] $\sum_{y=8}^{\infty} \frac{e^{-8} 8^y (0.5)^y}{8!(y-8)!} = 0.030$

You may need to use the appropriate table in the Appendix of Tables to answer this question.

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