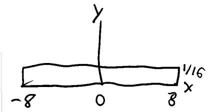
Ch4 - Continuous Random Variables and Probability Distributions (Part I)

Thursday, May 18, 2023 9:28 AM



1. [–/4 Points] DETAILS DEVORESTAT9 4.1.002.MI. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Suppose the reaction temperature X (in °C) in a certain chemical process has a uniform distribution with $A = -8$ and $B = 8$.

$f(x) = \begin{cases} \frac{1}{B-A} & \text{for } -8 \le x \le 8 \\ 0 & \text{for otherwise} \end{cases} = \frac{1}{8-(-8)} = \frac{1}{16}$

(a) Compute $P(X < 0)$.

$P(x<0) = P(-8 \le x \le 0) = \int_a^b f(x)dx = \int_{-8}^{0} \frac{1}{16}dx = \frac{1}{16}x \Big|_{-8}^{0} = \frac{0}{16} - \left(-\frac{8}{16}\right) = \frac{8}{16} = \frac{1}{2} = 0.50$

(b) Compute $P(-4 < X < 4)$.

$P(-4 \le x \le 4) = \int_a^b f(x)dx = \int_{-4}^{4} \frac{1}{16}dx = \frac{1}{16}x \Big|_{-4}^{4} = \frac{4}{16} - \left(-\frac{4}{16}\right) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2} = 0.50$

(c) Compute $P(-5 \le X \le 6)$. (Round your answer to two decimal places.)

$P(-5 \le x \le 6) = \int_a^b f(x)dx = \int_{-5}^{6} \frac{1}{16}dx = \frac{1}{16}x \Big|_{-5}^{6} = \frac{6}{16} - \left(-\frac{5}{16}\right) = \frac{6}{16} + \frac{5}{16} = \frac{11}{16} = 0.69$

(d) For k satisfying $-8 < k < k + 4 < 8$, compute $P(k < X < k + 4)$. (Round your answer to two decimal places.)

$P(k < x < k+4) = \int_a^b f(x)dx = \int_k^{k+4} \frac{1}{16}dx = \frac{1}{16}x \Big|_k^{k+4} = \frac{k+4}{16} - \frac{k}{16} = \frac{(k+4)-k}{16} = \frac{4}{16} = \frac{1}{4} = 0.25$

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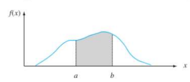
Definition

Let X be a continuous rv. Then a **probability distribution** or **probability density function** (pdf) of X is a function $f(x)$ such that for any two numbers a and b with $a \le b$,

$$P(a \le X \le b) = \int_a^b f(x)dx$$

That is, the probability that X takes on a value in the interval $[a, b]$ is the area above this interval and under the graph of the density function, as illustrated in Figure 4.2. The graph of $f(x)$ is often referred to as the *density curve*.

Figure 4.2 $P(a \le X \le b) =$ the area under the density curve between a



Definition

A continuous rv X is said to have a **uniform distribution** on the interval $[A, B]$ if the pdf of X is

$$f(x; A, B) = \begin{cases} \frac{1}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

2. [1/4 Points] DETAILS PREVIOUS ANSWERS DEVORESTAT9 4.1.003.

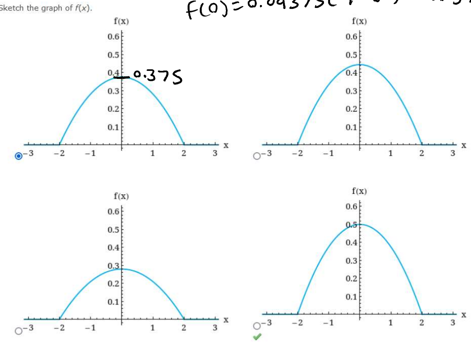
The error involved in making a certain measurement is a continuous rv X with the following pdf.

$f(x) = \begin{cases} 0.09375(4 - x^2) & -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

try $x = 0$

$f(0) = 0.09375(4 - 0^2) = 0.375$

(a) Sketch the graph of $f(x)$.



(b) Compute $P(X > 0)$.

$P(x > 0) = P(0 \le x \le 2) = \int_a^b f(x)dx = \int_0^2 0.09375(4) - 0.09375x^2 = 0.50$

(c) Compute $P(-1 < X < 1)$. (Enter your answer to four decimal places.)

$P(-1 < x < 1) = \int_a^b f(x)dx = \int_{-1}^{1} 0.09375(4) - 0.09375x^2 dx = 0.6875$

(d) Compute $P(X < -1.2$ or $X > 1.2)$. (Round your answer to four decimal places.)

$P(x < -1.2) = P(-2 < x < -1.2) = \int_{-2}^{-1.2} 0.09375(4) - 0.09375x^2 dx = 0.104$

$P(x > 1.2) = P(1.2 < x < 2) = \int_{1.2}^{2} 0.09375(4) - 0.09375x^2 dx = 0.104$

$P(x < -1.2 \cup x > 1.2) = P(-2 < x < -1.2) + P(1.2 < x < 2) = 0.104 + 0.104 = 0.208$

(two-tailed distribution)

3. [–/8 Points] DETAILS **DEVORESTAT9 4.1.004.**

Let X denote the vibratory stress (psi) on a wind turbine blade at a particular wind speed in a wind tunnel. Use the Rayleigh distribution, with pdf

$$f(x;\theta) = \begin{cases} \dfrac{x}{\theta^2} \cdot e^{-x^2/(2\theta^2)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$u = \dfrac{x^2}{2\theta^2}$; $du = 2x \cdot \dfrac{1}{2\theta^2} dx = \dfrac{x}{\theta^2} dx \Rightarrow dx = \dfrac{\theta^2}{x} du$

as a model for the X distribution.

(a) Verify that $f(x; \theta)$ is a legitimate pdf.

$$\int_0^{\infty} \frac{x}{\theta^2} \cdot e^{-x^2/(2\theta^2)}\, dx = \Big[\quad\Big]_0^{\infty} = 0 - (\quad) = $$

$\int_0^{\infty} \frac{x}{\theta^2} \cdot e^{-\frac{x^2}{2\theta^2}} dx = \int_0^{\infty} \frac{x}{\theta^2} \cdot e^{-u} \cdot \frac{\theta^2}{x} du = \int_0^{\infty} e^{-u} = -e^{-u}\Big|_0^{\infty} = -e^{-\frac{x^2}{2\theta^2}}$

$= -\frac{1}{\infty} - \left(-\frac{1}{e^0}\right) = 0 - (-1) = 1$

(b) Suppose θ = 112. What is the probability that X is at most 200? Less than 200? At least 200? (Round your answer to four decimal places.) $f(x;\theta=112) = \frac{x}{112^2} \cdot e^{-\frac{x^2}{2(112)^2}}$

at most 200 $P(X \le 200) = P(0 \le x \le 200) = \int_a^b f(x) dx = \int_0^{200} \frac{x}{112^2} \cdot e^{-\frac{x^2}{2(112)^2}} dx = 0.7970$

less than 200 $P(X < 200) = P(0 \le x \le 200) = 0.7970$

at least 200 $P(x \ge 200) = 1 - P(x < 200) = 1 - 0.7970 = 0.2030$

(c) What is the probability that X is between 100 and 200 (again assuming θ = 112)? (Round your answer to four decimal places.)

$P(100 \le x \le 200) = \int_a^b f(x) dx = \int_{100}^{200} \frac{x}{112^2} \cdot e^{-\frac{x^2}{2(112)^2}} dx = 0.4682$

(d) Give an expression for $P(X \le x)$. or $P(x \ge X)$

$P(X \le x) =$ $1 - P(x < X) = 1 - e^{-\frac{x^2}{2\theta^2}}$

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where $e^{-\frac{x^2}{2\theta^2}}$ is the integral simplified

4. [–/5 Points] DETAILS **DEVORESTAT9 4.1.005.** MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

A college professor never finishes his lecture before the end of the hour and always finishes his lectures within 2 min after the hour. Let X = the time that elapses between the end of the hour and the end of the lecture and suppose the pdf of X is as follows.

$$f(x) = \begin{cases} kx^2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \Rightarrow \begin{cases} (0.375)x^2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

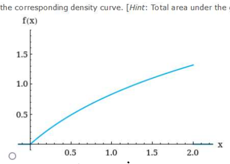
(a) Find the value of k. (Enter your answer to three decimal places.)

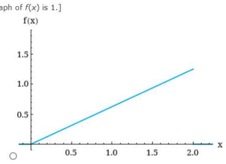
$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^2 kx^2 dx = 1 \Rightarrow k \cdot \int_0^2 x^2 dx = 1 \Rightarrow k \cdot \frac{x^3}{3}\Big|_0^2 = 1 \Rightarrow k\left[\frac{2^3}{3} - \frac{0^3}{3}\right] = 1 \Rightarrow k \cdot \frac{8}{3} = 1 \Rightarrow k = \frac{3}{8} = 0.375$

For $f(x)$ to be a legitimate pdf, it must satisfy the following two conditions:

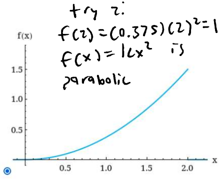
1. $f(x) \ge 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)\,dx =$ area under the entire graph of $f(x)$
$= 1$

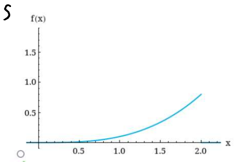
Draw the corresponding density curve. [*Hint*: Total area under the graph of $f(x)$ is 1.]





try 2:
$f(2) = (0.375)(2)^2 = 1.5$
$f(x) = kx^2$ is parabolic





(b) What is the probability that the lecture ends within 1 min of the end of the hour? (Enter your answer to three decimal places.)

$P(X \le 1) = P(0 \le x \le 1) = \int_a^b f(x) dx = \int_0^1 (0.375)x^2 dx = 0.125$

(c) What is the probability that the lecture continues beyond the hour for between 45 and 105 sec? (Round your answer to four decimal places.)

$P\left(\frac{45}{60} \le x \le \frac{105}{60}\right) = P(0.75 \le x \le 1.75) = \int_a^b f(x) dx = \int_{0.75}^{1.75} (0.375)x^2 dx = 0.6172$

(d) What is the probability that the lecture continues for at least 105 sec beyond the end of the hour? (Round your answer to four decimal places.)

$P\left(x \ge \frac{105}{60}\right) = P\left(\frac{105}{60} \le x \le 2\right) = P(1.75 \le x \le 2) = \int_{1.75}^{2} (0.375)x^2 dx = 0.3301$

5. [–/9 Points] DETAILS **DEVORESTAT9 4.2.011.MI.**

Let X denote the amount of time a book on two-hour reserve is actually checked out, and suppose the cdf is the following.

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^2}{16} & 0 \le x < 4 \\ 1 & 4 \le x \end{cases}$$

Use the cdf to obtain the following. (If necessary, round your answer to four decimal places.)

(a) Calculate $P(X \le 2)$. $P(X \le x) = F(x)$

$P(X \le 2) = F(2) = \frac{2^2}{16} = \frac{4}{16} = \frac{1}{4} = 0.25$

(b) Calculate $P(1.5 \le X \le 2)$. $P(a \le X \le b) = F(b) - F(a)$

$P(1.5 \le X \le 2) = F(2) - F(1.5) = \frac{2^2}{16} - \frac{(1.5)^2}{16} = 0.1094$

(c) Calculate $P(X > 2.5)$. $P(X > a) = 1 - F(a)$

$P(X > 2.5) = 1 - F(2.5) = 1 - \frac{(2.5)^2}{16} = 0.6094$

(d) What is the median checkout duration $\tilde{\mu}$? [solve $0.5 = F(\tilde{\mu})$].

$0.5 = F(\tilde{\mu}) \Rightarrow 0.5 = \frac{\tilde{\mu}^2}{16} \Rightarrow \tilde{\mu}^2 = 8 \Rightarrow \tilde{\mu} = \sqrt{8} = 2.8284$

(e) Obtain the density function $f(x)$.

$$f(x) = F'(x) = \begin{cases} \quad & 0 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$$

$F'(x) = \frac{d}{dx}\left[\frac{x^2}{16}\right] = \frac{2x}{16} = \frac{x}{8}$

with $f(x) = F'(x)$ as pdf

(f) Calculate $E(X)$.

$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_0^4 x \cdot \frac{x}{8} dx = \int_0^4 \frac{x^2}{8} = 2.6667$

(g) Calculate $V(X)$ and σ_X.

$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx = \int_0^4 x^2 \cdot \frac{x}{8} dx = \int_0^4 \frac{x^3}{8} = 8$

$V(X)$ $V(X) = E(X^2) - [E(X)]^2 = 8 - (2.6667)^2 = 0.8889$

σ_X $\sigma_X = \sqrt{V(X)} = \sqrt{0.8889} = 0.9428$

(h) If the borrower is charged an amount $h(X) = X^2$ when checkout duration is X, compute the expected charge $E[h(X)]$.

$\mu_{h(x)} = E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f(x) dx = \int_0^4 x^2 \cdot \frac{x}{8} = \int_0^4 \frac{x^3}{8} = 8$

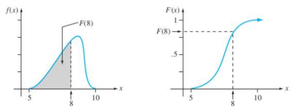
Definition

The **cumulative distribution function** $F(x)$ for a continuous rv X is defined for every number x by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\,dy$$

For each x, $F(x)$ is the area under the density curve to the left of x. This is illustrated in Figure 4.5, where $F(x)$ increases smoothly as x increases.

Figure 4.5 **A pdf and associated cdf**



Proposition

Let X be a continuous rv with pdf $f(x)$ and cdf $F(x)$. Then for any number a,

$$P(X > a) = 1 - F(a)$$

and for any two numbers a and b with $a < b$,

$$P(a \le X \le b) = F(b) - F(a)$$

Definition

The **median** of a continuous distribution, denoted by $\tilde{\mu}$, is the 50th percentile, so $\tilde{\mu}$ satisfies $.5 = F(\tilde{\mu})$. That is, half the area under the density curve is to the left of $\tilde{\mu}$ and half is to the right of $\tilde{\mu}$.

Proposition

If X is a continuous rv with pdf $f(x)$ and cdf $F(x)$, then at every x at which the derivative $F'(x)$ exists, $F'(x) = f(x)$.

Definition

The **expected** or **mean value** of a continuous rv X with pdf $f(x)$ is

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

Definition

The **variance** of a continuous random variable X with pdf $f(x)$ and mean value μ is

$$\sigma_X^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)dx = E[(X-\mu)^2]$$

The **standard deviation** (SD) of X is $\sigma_X = \sqrt{V(X)}$.

The variance and standard deviation give quantitative measures of how much spread there is in the distribution or population of x values. Again σ is roughly the size of a typical deviation from μ. Computation of σ^2 is facilitated by using the same shortcut formula employed in the discrete case.

Proposition

$$V(X) = E(X^2) - [E(X)]^2$$

Proposition

If X is a continuous rv with pdf $f(x)$ and $h(X)$ is any function of X, then

$$E[h(X)] = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x) \cdot f(x)\,dx$$

6. [–/7 Points] DETAILS DEVORESTAT9 4.2.013.MI. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

"Time headway" in traffic flow is the elapsed time between the time that one car finishes passing a fixed point and the instant that the next car begins to pass that point. Let X = the time headway for two randomly chosen consecutive cars on a freeway during a period of heavy flow (sec). Suppose that in a particular traffic environment, the distribution of time headway has the following form.

$$f(x) = \begin{cases} \frac{k}{x^{10}} & x > 1 \\ 0 & x \le 1 \end{cases} \Rightarrow \begin{cases} \frac{9}{x^{10}} & x > 1 \\ 0 & x \le 1 \end{cases}$$

(a) Determine the value of k for which $f(x)$ is a legitimate pdf.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_1^{\infty} \frac{k}{x^{10}}\,dx = 1 \Rightarrow k\cdot\int_1^{\infty} x^{-10}\,dx = 1 \Rightarrow k\cdot\frac{x^{-9}}{-9}\Big|_1^{\infty} = 1 \Rightarrow -\frac{k}{9}\left[\frac{1}{\infty^9} - \frac{1}{1^9}\right] = 1 \Rightarrow -\frac{k}{9}[-1] = 1 \Rightarrow \frac{k}{9} = 1 \Rightarrow k = 9$$

(b) Obtain the cumulative distribution function.

$$F(x) = \begin{cases} \boxed{} & x > 1 \\ 0 & x \le 1 \end{cases}$$

$$\overset{cdf}{F(x)} = \int_{-\infty}^{x} \overset{pdf}{f(y)}\,dy = \int_1^{x} \frac{9}{y^{10}}\,dy = 9\cdot\int_1^{x} y^{-10}\,dy = 9\cdot\frac{y^{-9}}{-9}\Big|_1^{x} = -\left[\frac{1}{x^9} - \frac{1}{1^9}\right] = 1 - \frac{1}{x^9}$$

(c) Use the cdf from (b) to determine the probability that headway exceeds 2 sec. (Round your answer to four decimal places.)

$$P(x > 2) = 1 - F(2) = 1 - \left(1 - \frac{1}{2^9}\right) = 0.0020$$

Use the cdf from (b) to determine the probability that headway is between 2 and 3 sec. (Round your answer to four decimal places.)

$$P(2 \le x \le 3) = F(3) - F(2) = \left(1 - \frac{1}{3^9}\right) - \left(1 - \frac{1}{2^9}\right) = 0.0019$$

(d) Obtain the mean value of headway and the standard deviation of headway. (Round your standard deviation to three decimal places.)

mean

standard deviation

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x\cdot f(x)\,dx = \int_1^{\infty} x\cdot\frac{9}{x^{10}}\,dx = 9\cdot\int_1^{\infty} x^{-9} = 9\cdot\frac{x^{-8}}{-8}\Big|_1^{\infty} = -\frac{9}{8}\left[\frac{1}{\infty^8} - \frac{1}{(1)^8}\right] = -\frac{9}{8}[-1] = \frac{9}{8} = 1.125$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\cdot f(x)\,dx = \int_1^{\infty} x^2\cdot\frac{9}{x^{10}}\,dx = 9\cdot\int_1^{\infty} x^{-8}\,dx = 9\cdot\frac{x^{-7}}{-7}\Big|_1^{\infty} = -\frac{9}{7}\left[\frac{1}{\infty^7} - \frac{1}{(1)^7}\right] = \frac{9}{7} = 1.2857$$

$$\sigma_X = \sqrt{V(X)} = \sqrt{0.0201} = 0.1417 \quad V(X) = E(X^2) - [E(X)]^2 = 1.2857 - (1.125)^2 = 0.0201$$

(e) What is the probability that headway is within 1 standard deviation of the mean value? (Round your answer to three decimal places.)

$$P(\mu_X - \sigma_X \le X \le \mu_X + \sigma_X) = P(1.125 - 0.1417 \le X \le 1.125 + 0.1417) = P(0.9833 \le X \le 1.2667) = F(1.2667) - F(0.9833)$$

$$= \left(1 - \frac{1}{(1.2667)^9}\right) - 0$$

$$= 0.881$$

Reminder;

$$F(x) = \begin{cases} 1 - \frac{1}{x^9} & x < 1 \\ 0 & x \ge 1 \end{cases}$$

so $F(0.9833) = 0$

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