Ch5 - Joint Probability Distributions and Random Samples (Part I)

Thursday, May 18, 2023 9:44 PM

Topics:

1. [–/12 Points] DETAILS DEVORESTAT9 5.1.001.MI. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let X denote the number of hoses being used on the self-service island at a particular time, and let Y denote the number of hoses on the full-service island in use at that time. The joint pmf of X and Y appears in the accompanying tabulation.

		y		
$p(x, y)$		0	1	2
	0	0.10	0.05	0.02
x	1	0.06	0.20	0.08
	2	0.06	0.14	0.29

(a) What is $P(X = 1 \text{ and } Y = 1)$?

$P(X = 1 \text{ and } Y = 1) =$ 0.20

(b) Compute $P(X \le 1 \text{ and } Y \le 1)$.

$P(X \le 1 \text{ and } Y \le 1) =$ $0.20 + 0.05 + 0.06 + 0.10 = 0.41$

(c) Give a word description of the event $\{X \ne 0 \text{ and } Y \ne 0\}$.

- ○ At most one hose is in use at both islands.
- ● At least one hose is in use at both islands.
- ○ One hose is in use on both islands.
- ○ One hose is in use on one island.

Compute the probability of this event.

$P(X \ne 0 \text{ and } Y \ne 0) =$ $0.20 + 0.08 + 0.14 + 0.29 = 0.71$

Definition

Let X and Y be two discrete rv's defined on the sample space $\mathscr{S}$ of an experiment. The **joint probability mass function** $p(x, y)$ is defined for each pair of numbers (x, y) by

$$p(x, y) = P(X = x \text{ and } Y = y)$$

It must be the case that $p(x, y) \ge 0$ and $\sum_x \sum_y p(x, y) = 1$.

Now let A be any particular set consisting of pairs of (x, y) values (e.g., $A = \{(x, y)\colon x + y = 5\}$ or $\{(x, y)\colon \max(x, y) \le 3\}$). Then the probability $P[(X, Y) \in A]$ that the random pair (X, Y) lies in the set A is obtained by summing the joint pmf over pairs in A:

$$P[(X, Y) \in A] = \sum_{(x,\, y)} \sum_{\in A} p(x, y)$$

(d) Compute the marginal pmf of X.

x	0	1	2	
$p_X(x)$	0.17 +	0.34 +	0.49	= 1
	$0.10+0.05+0.02$	$0.06+0.20+0.08$	$0.06+0.14+0.29$	

Compute the marginal pmf of Y.

y	0	1	2	
$p_Y(y)$	0.22 +	0.39 +	0.39	= 1
	$0.10+0.06+0.06$	$0.05+0.20+0.14$	$0.02+0.08+0.29$	

Using $p_X(x)$, what is $P(X \le 1)$?

$P(X \le 1) =$ $P(X \le 1) = 0.34 + 0.17 = 0.51$

(e) Are X and Y independent rv's? Explain.

- ● X and Y are not independent because $P(x,y) \ne p_X(x) \cdot p_Y(y)$. $P(0,0) \ne P_X(0) \cdot P_Y(0) \Rightarrow 0.10 \ne 0.17 \cdot 0.22 = 0.0374$
- ○ X and Y are independent because $P(x,y) \ne p_X(x) \cdot p_Y(y)$.
- ○ X and Y are not independent because $P(x,y) = p_X(x) \cdot p_Y(y)$.
- ○ X and Y are independent because $P(x,y) = p_X(x) \cdot p_Y(y)$.

Definition

The **marginal probability mass function of X**, denoted by $p_X(x)$, is given by

$$p_X(x) = \sum_{y:\, p(x,\, y) > 0} p(x, y) \qquad \text{for each possible value } x$$

Similarly, the **marginal probability mass function of Y** is

$$p_Y(y) = \sum_{x:\, p(x,\, y) > 0} p(x, y) \qquad \text{for each possible value } y.$$

2. [–/6 Points] DETAILS DEVORESTAT9 5.1.003. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

A certain market has both an express checkout line and a superexpress checkout line. Let X_1 denote the number of customers in line at the express checkout at a particular time of day, and let X_2 denote the number of customers in line at the superexpress checkout at the same time. Suppose the joint pmf of X_1 and X_2 is as given in the accompanying table.

		x_2			
		0	1	2	3
	0	0.08	0.07	0.04	0.00
	1	0.05	0.15	0.04	0.04
x_1	2	0.05	0.03	0.10	0.06
	3	0.01	0.03	0.04	0.07
	4	0.00	0.01	0.05	0.08

(a) What is $P(X_1 = 1, X_2 = 1)$, that is, the probability that there is exactly one customer in each line?

$P(X_1 = 1, X_2 = 1) =$ 0.15

(b) What is $P(X_1 = X_2)$, that is, the probability that the numbers of customers in the two lines are identical?

$P(X_1 = X_2) =$ $P(X_1=0, X_2=0) + P(X_1=1, X_2=1) + P(X_1=2, X_2=2) + P(X_1=3, X_2=3) = 0.08 + 0.15 + 0.10 + 0.97 = 0.40$

(c) Let A denote the event that there are at least two more customers in one line than in the other line. Express A in terms of X_1 and X_2.

- ○ $A = \{X_1 \ge 2 + X_2 \cup X_2 \le 2 + X_1\}$
- ○ $A = \{X_1 \le 2 + X_2 \cup X_2 \le 2 + X_1\}$
- ● $A = \{X_1 \ge 2 + X_2 \cup X_2 \ge 2 + X_1\}$
- ○ $A = \{X_1 \le 2 + X_2 \cup X_2 \ge 2 + X_1\}$

$x_1 \ge 2 + x_2$: $2 \ge 2+0$, $3 \ge 2+0$, $4 \ge 2+0$

$x_1 \ge 2 + x_2$: $3 \ge 2+1$, $4 \ge 2+1$

$x_1 \ge 2 + x_2$: $4 \ge 2+2$

$x_2 \ge 2 + x_1$: $2 \ge 2+0$, $3 \ge 2+0$

$x_2 \ge 2 + x_1$: $3 \ge 2+1$

Calculate the probability of this event.

$P(A) =$ $P(A) = P(2,0) + P(3,0) + P(4,0) + P(3,1) + P(4,1) + P(4,2) + P(0,2) + P(0,3) + P(1,3)$

$= 0.05 + 0.01 + 0.00 + 0.03 + 0.01 + 0.05 + 0.04 + 0.00 + 0.04 = 0.23$

(d) What is the probability that the total number of customers in the two lines is exactly four? At least four?

$P(\text{exactly four}) =$ $P(1,3) + P(2,2) + P(3,1) + P(4,0) = 0.04 + 0.10 + 0.03 + 0.00 = 0.17$ (when $x_1 + x_2 = 4$)

$P(\text{at least four}) =$ $P(\text{exactly four}) + P(2,3) + P(3,2) + P(3,3) + P(4,1) + P(4,2) + P(4,3)$

(when $x_1 + x_2 \ge 4$)

$= 0.17 + 0.06 + 0.04 + 0.07 + 0.01 + 0.05 + 0.08$

$= 0.48$

Definition

Two random variables X and Y are said to be **independent** if for every pair of x and y values

$$p(x, y) = p_X(x) \cdot p_Y(y) \qquad \text{when } X \text{ and } Y \text{ are discrete} \qquad \textbf{(5.1)}$$

or

$$f(x, y) = f_X(x) \cdot f_Y(y) \qquad \text{when } X \text{ and } Y \text{ are continuous}$$

If (5.1) is not satisfied for all (x, y), then X and Y are said to be **dependent**.

3. [–/5 Points] DETAILS DEVORESTAT9 5.1.009. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 28 psi. Suppose the actual air pressure in each tire is a random variable—X for the right tire and Y for the left tire, with joint pdf

$$f(x,y) = \begin{cases} K(x^2 + y^2) & 20 \le x \le 30,\ 20 \le y \le 30 \\ 0 & \text{otherwise} \end{cases}$$

use: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

same calculation twice on both sides of +

(a) What is the value of K? (Enter your answer as a fraction.)

$K =$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{20}^{39}\int_{20}^{30} k(x^2+y^2)\,dx\,dy = k\cdot\int_{20}^{30}\int_{20}^{30} x^2\,dy\,dx + k\cdot\int_{20}^{30}\int_{20}^{30} y^2\,dx\,dy = k\cdot\int_{20}^{30} x^2 \left(y\Big|_{20}^{30}\right)dx + k\cdot\int_{20}^{30} y^2 \left(x\Big|_{20}^{30}\right)dy = 10k\cdot\frac{x^3}{3}\Big|_{20}^{30} + 10k\cdot\frac{y^3}{3}\Big|_{20}^{30} = 2\cdot 10k\left[\frac{19,000}{3}\right] = k\cdot\frac{380,000}{3} = 1 \Rightarrow k = \frac{3}{380,000}$$

(b) What is the probability that both tires are underfilled? (Round your answer to four decimal places.)

$$P(X<28 \cup Y<28) = \int_{20}^{28}\int_{20}^{28} \frac{3}{380,000}(x^2+y^2)\,dx\,dy = \frac{3}{380,000}\cdot\int_{20}^{28}\int_{20}^{28} x^2\,dy\,dx + \frac{3}{380,000}\cdot\int_{20}^{28}\int_{20}^{28} y^2\,dx\,dy = \frac{3\cdot(28-20)}{380,000}\left[\int_{20}^{28} x^2\,dx + \int_{20}^{28} y^2\,dy\right] = 0.5875$$

(c) What is the probability that the difference in air pressure between the two tires is at most 2 psi? (Round your answer to four decimal places.)

$$P(|X-Y| \le 2) = \int_{20}^{22}\int_{20}^{x+2} \frac{3}{380,000}(x^2+y^2)\,dy\,dx + \int_{22}^{28}\int_{x-2}^{x+2} \frac{3}{380,000}(x^2+y^2)\,dy\,dx + \int_{28}^{30}\int_{x-2}^{30} \frac{3}{380,000}(x^2+y^2)\,dy\,dx = 0.3593$$

(d) Determine the (marginal) distribution of air pressure in the right tire alone.

for $20 \le x \le 30$

$$f_X(x) = \int_{20}^{30} \frac{3}{380,000}(x^2+y^2)\,dy = \frac{3}{380,000}\left[\int_{20}^{30} x^2\,dy + \int_{20}^{30} y^2\,dy\right] = \frac{3}{380,000}\left[10x^2 + \frac{y^3}{3}\Big|_{20}^{30}\right] = \frac{3}{380,000}\left[10x^2 + \left(\frac{30^3}{3} - \frac{20^3}{3}\right)\right]$$

(e) Are X and Y independent rv's?

- ○ Yes, $f(x,y) = f_X(x)\cdot f_Y(y)$, so X and Y are independent.
- ○ Yes, $f(x,y) \ne f_X(x)\cdot f_Y(y)$, so X and Y are independent.
- ○ No, $f(x,y) = f_X(x)\cdot f_Y(y)$, so X and Y are not independent.
- ● No, $f(x,y) \ne f_X(x)\cdot f_Y(y)$, so X and Y are not independent. The eq'n for $f_X(x)$ & $f_Y(y)$ are not close to $f(x,y)$

Need Help? Read It

Definition

Let X and Y be continuous rv's. A **joint probability density function** $f(x,y)$ for these two variables is a function satisfying $f(x,y) \ge 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$. Then for any two-dimensional set A

$$P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy$$

In particular, if A is the two-dimensional rectangle $\{(x,y): a \le x \le b, c \le y \le d\}$, then

$$P[(X,Y) \in A] = P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

4. [–/4 Points] DETAILS DEVORESTAT9 5.1.010. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Annie and Alvie have agreed to meet between 5:00 P.M. and 6:00 P.M. for dinner at a local health-food restaurant. Let X = Annie's arrival time and Y = Alvie's arrival time. Suppose X and Y are independent with each uniformly distributed on the interval [5, 6].

(a) What is the joint pdf of X and Y?

$$f(x,y) = \begin{cases} 1 & 5 \le x \le 6,\ 5 \le y \le 6 \\ 0 & \text{otherwise} \end{cases}$$

b/c $f(x) = \begin{cases} \frac{1}{6-5} = 1 & 5 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$ & $f(y) = \begin{cases} 1 & 5 \le y \le 6 \\ 0 & \text{otherwise} \end{cases}$, ind. $\therefore f(x,y) = f(x)\cdot f(y)$

b/c x & y are identically distributed

(b) What is the probability that they both arrive between 5:15 and 5:45?

$$P(5.25 \le x \le 5.75,\ 5.25 \le y \le 5.75) = \left[P(5.25 \le x \le 5.75)\right]^2 = \left[\int_{5.25}^{5.75} f(x)\,dx\right]^2 = \left[\int_{5.25}^{5.75} (1)\,dx\right]^2 = [0.5]^2 = 0.25$$

(c) If the first one to arrive will wait only 15 min before leaving to eat elsewhere, what is the probability that they have dinner at the health-food restaurant? [*Hint*: The event of interest is $A = \{(x, y): |x - y| \le 1/4\}$.] (Round your answer to three decimal places.)

$$P(|x-y| \le 0.25) = P(y-0.25 \le x \le y+0.25) = \int_5^6\int_{y-0.25}^{y+0.25} f(x,y)\,dx\,dy = \int_5^{5.25}\int_{y-0.25}^{y+0.25} f(x,y)\,dx\,dy + \int_{5.25}^{5.75}\int_{y-0.25}^{y+0.25} f(x,y)\,dx\,dy + \int_{5.75}^{6}\int_{y-0.25}^{y+0.25} f(x,y)\,dx\,dy$$

$$= \int_5^{5.25}\int_5^{y+0.25} (1)\,dx\,dy + \int_{5.25}^{5.75}\int_{y-0.25}^{y+0.25} (1)\,dx\,dy + \int_{5.75}^{6}\int_{y-0.25}^{6} (1)\,dx\,dy$$

$$= 0.438$$

Need Help? Read It

5. [–/2 Points] DETAILS DEVORESTAT9 5.2.022.MI. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

An instructor has given a short quiz consisting of two parts. For a randomly selected student, let X = the number of points earned on the first part and Y = the number of points earned on the second part. Suppose that the joint pmf of X and Y is given in the accompanying table.

	$p(x, y)$	y = 0	5	10	15
x	0	0.03	0.06	0.02	0.10
	5	0.04	0.16	0.20	0.10
	10	0.01	0.15	0.12	0.01

$P(x+y) = 0 \Rightarrow P(0,0) = 0.03$
$P(x+y) = 5 \Rightarrow P(0,5) + P(5,0) = 0.06 + 0.04 = 0.10$
$P(x+y) = 10 \Rightarrow P(0,10) + P(10,0) + P(5,5) = 0.02 + 0.01 + 0.16 = 0.19$
$P(x+y) = 15 \Rightarrow P(0,15) + P(5,10) + P(10,5) = 0.10 + 0.20 + 0.15 = 0.45$
$P(x+y) = 20 \Rightarrow P(5,15) + P(10,10) = 0.10 + 0.12 = 0.22$
$P(x+y) = 25 \Rightarrow P(10,15) = 0.01$

(a) If the score recorded in the grade book is the total number of points earned on the two parts, what is the expected recorded score $E(X + Y)$? (Enter your answer to one decimal place.)

$E(x+y) = [(0)(0.03)] + [(5)(0.10)] + [(10)(0.19)] + [(15)(0.45)] + [(20)(0.22)] + [(25)(0.01)] = 13.8$
$E(x+y) = \sum\sum (x+y)\cdot P(x,y) = E(x) + E(y)$

(b) If the maximum of the two scores is recorded, what is the expected recorded score? (Enter your answer to two decimal places.)

$\sum\sum \max(x,y)\cdot P(x,y) = \max(0,0)\cdot(0.03) + \max(0,5)\cdot(0.06) + \max(0,10)\cdot(0.02) + \ldots + \max(10,15)\cdot(0.01)$
$= (0)(0.03) + (5)(0.06) + (10)(0.02) + \ldots + (15)(0.01)$
$= 9.45$

Proposition

Let X and Y be jointly distributed rv's with pmf $p(x,y)$ or pdf $f(x,y)$ according to whether the variables are discrete or continuous. Then the expected value of a function $h(X,Y)$, denoted by $E[h(X,Y)]$ or $\mu_{h(X,Y)}$, is given by

$$E[h(X,Y)] = \begin{cases} \sum_x \sum_y h(x,y)\cdot p(x,y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\cdot f(x,y)\,dx\,dy & \text{if } X \text{ and } Y \text{ are continuous} \end{cases}$$

6. [–/1 Points] DETAILS DEVORESTAT9 5.2.029.

The joint and marginal pdf's of X = amount of almonds and Y = amount of cashews are

$$f(x, y) = \begin{cases} 24xy & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} 12x(1 - x)^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow f_Y(y) = \begin{cases} 12y(1-y)^2 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

with $f_Y(y)$ obtained by replacing x by y in $f_X(x)$. It is easily verified that $\mu_X = \mu_Y = \frac{2}{5}$, and $E(XY) = \frac{2}{15}$. Thus $\text{Cov}(X, Y) = -\frac{2}{75}$.

$V(x) = \int_{-\infty}^{\infty} (x-\mu_x)^2\cdot f(x)\,dx = \int_0^1 (x-\frac{2}{5})^2\cdot[12x(1-x)^2]\,dx = 0.04 = V(y)$ (b/c $f(x)$ & $f(y)$ along with μ_x & μ_y are identical)

$\sigma_x = \sqrt{V(x)} = \sqrt{0.04} = 0.20 = \sigma_y$

Compute the correlation coefficient ρ for X and Y.

$\rho =$

$$\rho_{x,y} = \frac{\text{Cov}(x,y)}{\sigma_x\cdot\sigma_y} = \frac{(-2/75)}{(0.20)(0.20)} = -\frac{2}{3}$$

Covariance

When two random variables X and Y are not independent, it is frequently of interest to assess how strongly they are related to one another.

Definition

The **covariance** between two rv's X and Y is

$$\text{Cov}(X,Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

Covariance

When two random variables X and Y are not independent, it is frequently of interest to assess how strongly they are related to one another.

Definition

The **covariance** between two rv's X and Y is

$$\text{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= \begin{cases} \sum_x \sum_y (x-\mu_X)(y-\mu_Y)p(x,y) & X, Y \text{ discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f(x,y)\,dx\,dy & X, Y \text{ continuous} \end{cases}$$

The following shortcut formula for $\text{Cov}(X,Y)$ simplifies the computations.

Proposition

$$\text{Cov}(X,Y) = E(XY) - \mu_X \cdot \mu_Y$$

According to this formula, no intermediate subtractions are necessary; only at the end of the computation is $\mu_X \cdot \mu_Y$ subtracted from $E(XY)$. The proof involves expanding $(X-\mu_X)(Y-\mu_Y)$ and then carrying the summation or integration through to each individual term.

Correlation

Definition

The **correlation coefficient** of X and Y, denoted by $\text{Corr}(X,Y)$, $\rho_{X,Y}$, or just ρ, is defined by

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

7. [–/2 Points] DETAILS DEVORESTAT9 5.2.030. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

An instructor has given a short quiz consisting of two parts. For a randomly selected student, let X = the number of points earned on the first part and Y = the number of points earned on the second part. Suppose that the joint pmf of X and Y is given in the accompanying table.

$p(x, y)$		y = 0	5	10	15
x	0	0.03	0.06	0.02	0.10
	5	0.04	0.14	0.20	0.10
	10	0.01	0.15	0.14	0.01

$E(X) = \sum p_X(x)\cdot x = (0.03+0.06+0.02+0.10)(0) + (0.04+0.14+0.20+0.10)(5) + (0.01+0.15+0.14+0.01)(10) = 5.5$

$E(Y) = \sum p_Y(y)\cdot y = (0.03+0.04+0.01)(0) + (0.06+0.14+0.15)(5) + (0.02+0.20+0.14)(10) + (0.10+0.10+0.01)(15) = 8.5$

$E(XY) = \sum\sum (x\cdot y)\cdot p(x,y) = (0\cdot 0)(0.03) + (0\cdot 5)(0.06) + \dots + (10\cdot 15)(0.01) = 44$

$E(X^2) = \sum p_X(x)\cdot x^2 = (0.03+0.06+0.02+0.10)(0)^2 + (0.04+0.14+0.20+0.10)(5)^2 + (0.01+0.15+0.14+0.01)(10)^2 = 43$

$E(Y^2) = \sum p_Y(y)\cdot y^2 = (0.03+0.04+0.01)(0)^2 + (0.06+0.14+0.15)(5)^2 + (0.02+0.20+0.14)(10)^2 + (0.10+0.10+0.01)(15)^2 = 92$

(a) Compute the covariance for X and Y. (Round your answer to two decimal places.)

Cov(X, Y) =

$\text{Cov}(X,Y) = E(XY) - E(X)\cdot E(Y) = 44 - (5.5)(8.5) = -2.75$

(b) Compute ρ for X and Y. (Round your answer to two decimal places.)

ρ =

$V(X) = E(X^2) - [E(X)]^2 = 43 - (5.5)^2 = 12.75$

$V(Y) = E(Y^2) - [E(Y)]^2 = 92 - (8.5)^2 = 19.75$

$\sigma_X = \sqrt{V(X)} = \sqrt{12.75} = 3.5707$

$\sigma_Y = \sqrt{V(Y)} = \sqrt{19.75} = 4.4441$

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X\cdot\sigma_Y} = \frac{-2.75}{(3.5707)(4.441)} = -0.17$$

8. [–/2 Points] DETAILS DEVORESTAT9 5.2.031. MY NOTES ASK YOUR TEACHER PRACTICE ANOTHER

Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a random variable—X for the right tire and Y for the left tire, with joint pdf given below.

$$f(x,y) = \begin{cases} K(x^2+y^2) & 22 \le x \le 32,\ 22 \le y \le 32 \\ 0 & \text{otherwise} \end{cases}$$

$E(X) = E(Y)$ & $\sigma_X = \sigma_Y$ $f(x) = f(y)$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_{22}^{32}\int_{22}^{32} K(x^2+y^2)\,dx\,dy = 1 \Rightarrow K = \frac{3}{442{,}400}$

$f_Y(y) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{22}^{32} K(x^2+y^2)\,dy = K\cdot\left[x^2\cdot\left(y\Big|_{22}^{32}\right) + \frac{y^3}{3}\Big|_{22}^{32}\right] = K\left[10x^2 + \frac{22{,}120}{3}\right] = 10Kx^2 + \frac{22{,}120}{3}\cdot\frac{3}{442{,}400} = 10Kx^2 + \frac{1}{20} = 10Kx^2 + 0.05$

(a) Compute the covariance between X and Y. (Round your answer to four decimal places.)

Cov(X, Y) =

$E(X) = E(Y) = \int_{-\infty}^{\infty} x\cdot f(x)\,dx = \int_{22}^{32} x\cdot(10Kx^2+0.05)\,dx = 27.3052$

$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\cdot f(x,y)\,dx\,dy = \int_{22}^{32}\int_{22}^{32} xy\,(K(x^2+y^2))\,dx\,dy = 745.4783$

$\text{Cov}(X,Y) = E(X,Y) - E(X)\cdot E(Y) = 745.4783 - (27.3052)(27.3052) = -0.0931$

(b) Compute the correlation coefficient ρ for this X and Y. (Round your answer to four decimal places.)

ρ =

$E(X^2) = E(Y^2) = \int_{-\infty}^{\infty} x^2\cdot f(x)\,dx = \int_{22}^{32} x^2(10Kx^2+0.05)\,dx = 753.8493$

$V(X) = V(Y) = E(X^2) - [E(X)]^2 = 753.8493 - (27.3052)^2 = 8.2754$

$\sigma_X = \sigma_Y = \sqrt{V(X)} = \sqrt{8.2754} = 2.8767$

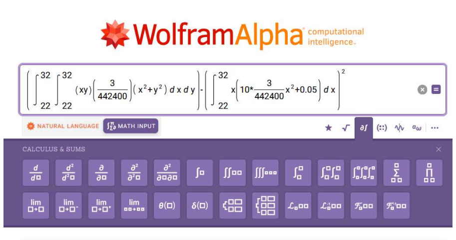
$\rho = \frac{\text{Cov}(X,Y)}{\sigma_X\cdot\sigma_Y} = \frac{-0.0931}{(2.8767)(2.8767)} = -0.0113$

Definition

The **marginal probability density functions** of X and Y, denoted by $f_X(x)$ and $f_Y(y)$, respectively, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad \text{for } -\infty < x < \infty$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \quad \text{for } -\infty < y < \infty$$

The exact covariance:



Input

$$\int_{22}^{32}\int_{22}^{32}(x\,y)\times\frac{3}{442400}(x^2+y^2)\,dx\,dy - \left(\int_{22}^{32} x\left(10\times\frac{3}{442400}x^2+0.05\right)dx\right)^2$$

Result

−0.0931188