Example Stochastic Reserving

Roger Hayne

9/6/2019

```r
library(mvtnorm)
library(MASS)
library(abind)
library(stochasticreserver)
```

Initialize Triangle

Input (B0) is a development array of cumulative averages with a the exposures (claims) used in the denominator appended as the last column. Assumption is for the same development increments as exposure increments and that all development lags with no development have # been removed. Data elements that are not available are indicated as such. This should work (but not tested for) just about any subset of an upper triangular data matrix.
Another requirement of this code is that the matrix contain no columns that are all zero.

```r
B0 = matrix(c(670.25868,1480.24821,1938.53579,2466.25469,2837.84888,3003.52391,
            3055.38674,3132.93838,3141.18638,3159.72524,
            767.98833,1592.50266,2463.79447,3019.71976,3374.72689,3553.61387,3602.27898,
            3627.28386,3645.5656,NA,
            740.57952,1615.79681,2345.85028,2910.52511,3201.5226,3417.71335,3506.58672,
            3529.00243,NA,NA,
            862.11956,1754.90405,2534.77727,3270.85361,3739.88962,4003.00219,4125.30694,
            NA,NA,NA,
            840.94172,1859.02531,2804.54535,3445.34665,3950.47098,4185.95298,NA,NA,NA,NA,
            848.00496,2052.922,3076.13789,3861.03111,4351.57694,NA,NA,NA,NA,NA,
            901.77403,1927.88718,3003.58919,3881.41744,NA,NA,NA,NA,NA,NA,
            935.19866,2103.97736,3181.75054,NA,NA,NA,NA,NA,NA,NA,
            759.32467,1584.91057,NA,NA,NA,NA,NA,NA,NA,NA,
            723.30282,NA,NA,NA,NA,NA,NA,NA,NA,NA),10,10,byrow = TRUE)
dnom = c(39161.,38672.4628,41801.048,42263.2794,41480.8768,40214.3872,43598.5056,
       42118.324,43479.4248,49492.4106)

# Identify model to be used
#   Berquist for the Berquist-Sherman Incremental Severity
#   CapeCod for the Cape Cod
#   Hoerl for the Generalized Hoerl Curve Model with trend
#   Wright for the Generalized Hoerl Curve with individual accident year levels
#   Chain for the Chain Ladder model
model = "Berquist"
# Toggle graphs off if desired
graphs = TRUE

# Toggle simulations off if desired
simulation = TRUE
```

```r
# Set tau to have columns with entries 1 through 10
tau = t(array((1:10), c(10, 10)))
```

```r
# Calculate incremental average matrix
A0 = cbind(B0[, 1], (B0[, (2:10)] + 0 * B0[, (1:9)]) -
             (B0[, (1:9)] + 0 * B0[, (2:10)]))

# Generate a matrix to reflect exposure count in the variance structure
logd = log(matrix(dnom, 10, 10))

# Set up matrix of rows and columns, makes later calculations simpler
rowNum = row(A0)
colNum = col(A0)

# msk is a mask matrix of allowable data, upper triangular assuming same
# development increments as exposure increments, msn picks off the first
# forecast diagonal, msd picks off the to date diagonal
msk = (10 - rowNum) >= colNum - 1
msn = (10 - rowNum) == colNum - 2
msd = (10 - rowNum) == colNum - 1

# Amount paid to date
ptd = rowSums(B0 * msd, na.rm = TRUE)
```

START OF MODEL SPECIFIC CODE

```r
  if (model == "Berquist") {
    model_lst <- berquist(tau, B0, ptd, msk)
  } else if (model == "CapeCod") {
    model_lst <- capecod(tau, B0, ptd, msk)
  } else if (model == "Hoerl") {
    model_lst <- hoerl(tau, B0, ptd, msk)
  } else if (model == "Wright") {
    model_lst <- wright(tau, B0, ptd, msk)
  } else if (model == "Chain") {
    model_lst <- chain(tau, B0, ptd, msk)
  }
g.obj <- model_lst$g.obj
g.grad <- model_lst$g.grad
g.hess <- model_lst$g.hess
a0 <- model_lst$a0
```

Negative Loglikelihood Function to be Minimized

Note that the general form of the model has parameters in addition to those in the loss model, namely the power for the variance and the constant of proprtionality that varies by column. So if the original model has k parameters with 10 columns of data, the total objective function has k+11 parameters

```r
l.obj = function(a, A) {
  npar = length(a) - 2
  e = g.obj(a[1:npar])
  v = exp(-outer(logd[, 1], rep(a[npar + 1], 10), "-")) * (e ^ 2) ^ a[npar +
                                                                        2]
  t1 = log(2 * pi * v) / 2
```

```
  t2 = (A - e) ^ 2 / (2 * v)
  sum(t1 + t2, na.rm = TRUE)
}
# Gradient of the objective function
l.grad = function(a, A) {
  npar = length(a) - 2
  p = a[npar + 2]
  Av = aperm(array(A, c(10, 10, npar)), c(3, 1, 2))
  e = g.obj(a[1:npar])
  ev = aperm(array(e, c(10, 10, npar)), c(3, 1, 2))
  v = exp(-outer(logd[, 1], rep(a[npar + 1], 10), "-")) * (e ^ 2) ^ p
  vv = aperm(array(v, c(10, 10, npar)), c(3, 1, 2))
  dt = rowSums(g.grad(a[1:npar]) * ((p / ev) + (ev - Av) / vv - p * (Av -
                                                                       ev) ^ 2 / (vv * ev)),
               na.rm = TRUE,
               dims = 1)
  yy = 1 - (A - e) ^ 2 / v
  dk = sum(yy / 2, na.rm = TRUE)
  dp = sum(yy * log(e ^ 2) / 2, na.rm = TRUE)
  c(dt, dk, dp)
}
```

Hessian of the objective function

- e is the expectated value matrix
- v is the matrix of variances
- A, e, v all have shape c(10,10)
- The variables _v are copies of the originals to shape c(npar,10,10), paralleling the gradient of g.
- The variables _m are copies of the originals to shape c(npar,npar,10,10), paralleling the hessian of g

```
l.hess = function(a, A) {
  npar = length(a) - 2
  p = a[npar + 2]
  Av = aperm(array(A, c(10, 10, npar)), c(3, 1, 2))
  Am = aperm(array(A, c(10, 10, npar, npar)), c(3, 4, 1, 2))
  e = g.obj(a[1:npar])
  ev = aperm(array(e, c(10, 10, npar)), c(3, 1, 2))
  em = aperm(array(e, c(10, 10, npar, npar)), c(3, 4, 1, 2))
  v = exp(-outer(logd[, 1], rep(a[npar + 1], 10), "-")) * (e ^ 2) ^ p
  vv = aperm(array(v, c(10, 10, npar)), c(3, 1, 2))
  vm = aperm(array(v, c(10, 10, npar, npar)), c(3, 4, 1, 2))
  g1 = g.grad(a[1:npar])
  gg = aperm(array(g1, c(npar, 10, 10, npar)), c(4, 1, 2, 3))
  gg = gg * aperm(gg, c(2, 1, 3, 4))
  gh = g.hess(a[1:npar])
  dtt = rowSums(
    gh * (p / em + (em - Am) / vm - p * (Am - em) ^ 2 / (vm * em)) +
      gg * (
        1 / vm + 4 * p * (Am - em) / (vm * em) + p * (2 * p + 1) * (Am - em) ^ 2 /
          (vm * em ^ 2) - p / em ^ 2
      ),
    dims = 2,
    na.rm = TRUE
```

```
  )
  dkt = rowSums((g1 * (Av - ev) + p * g1 * (Av - ev) ^ 2 / ev) / vv, na.rm = TRUE)
  dtp = rowSums(g1 * (1 / ev + (
    log(ev ^ 2) * (Av - ev) + (p * log(ev ^ 2) - 1) * (Av - ev) ^ 2 / ev
  ) / vv),
  na.rm = TRUE)
  dkk = sum((A - e) ^ 2 / (2 * v), na.rm = TRUE)
  dpk = sum(log(e ^ 2) * (A - e) ^ 2 / (2 * v), na.rm = TRUE)
  dpp = sum(log(e ^ 2) ^ 2 * (A - e) ^ 2 / (2 * v), na.rm = TRUE)
  m1 = rbind(array(dkt), c(dtp))
  rbind(cbind(dtt, t(m1)), cbind(m1, rbind(cbind(dkk, c(
    dpk
  )), c(dpk, dpp))))
}
```

End of funciton specificaitons now on to the minimization

Minimization

Get starting values for kappa and p parameters, default 10 and 1

```
ttt = c(10, 1)
```

For starting values use fitted objective function and assume variance for a cell is estimated by the square of the difference between actual and expected averages. Note since log(0) is -Inf we need to go through some machinations to prep the y values for the fit

```
E = g.obj(a0)
yyy = (A0 - E)^2
yyy = logd + log(((yyy != 0) * yyy) - (yyy == 0))
sss = na.omit(data.frame(x = c(log(E^2)), y = c(yyy)))
ttt = array(coef(lm(sss$y ~ sss$x)))[1:2]
a0 = c(a0, ttt)

set.seed(1) # to check reproducibility with original code
max = list(iter.max = 10000, eval.max = 10000)
```

Actual minimization

```
mle = nlminb(a0, l.obj, gradient = l.grad, hessian = l.hess,
             scale = abs(1 / (2 * ((a0 * (a0 != 0)) + (1 * (a0 == 0))))),
             A = A0, control = max)
```

Model statistics

- **mean** and **var** are model fitted values
- **stres** is the standardized residuals

```
npar = length(a0) - 2
p = mle$par[npar + 2]
mean = g.obj(mle$par[1:npar])
```

```
var = exp(-outer(logd[, 1], rep(mle$par[npar + 1], 10), "-")) * (mean ^
                                                                  2) ^ p
stres = (A0 - mean) / sqrt(var)
g1 = g.grad(mle$par[1:npar])
gg = aperm(array(g1, c(npar, 10, 10, npar)), c(4, 1, 2, 3))
gg = gg * aperm(gg, c(2, 1, 3, 4))
meanv = aperm(array(mean, c(10, 10, npar)), c(3, 1, 2))
meanm = aperm(array(mean, c(10, 10, npar, npar)), c(3, 4, 1, 2))
varm = aperm(array(var, c(10, 10, npar, npar)), c(3, 4, 1, 2))
```

Masks to screen out NA entries in original input matrix

```
s = 0 * A0
sv = aperm(array(s, c(10, 10, npar)), c(3, 1, 2))
sm = aperm(array(s, c(10, 10, npar, npar)), c(3, 4, 1, 2))
```

Calculate the information matrix

- Using second derivatives of the log likelihood function Second with respect to theta parameters

```
tt = rowSums(sm + gg * (1 / varm + 2 * p ^ 2 / (meanm ^ 2)), dims = 2, na.rm = TRUE)
```

Second with respect to theta and kappa

```
kt = p * rowSums(sv + g1 / meanv, na.rm = TRUE)
```

Second with respect to p and theta

```
tp = p * rowSums(sv + g1 * log(meanv ^ 2) / meanv, na.rm = TRUE)
```

Second with respect to kappa

```
kk = (1 / 2) * sum(1 + s, na.rm = TRUE)
```

Second with respect to p and kappa

```
pk = (1 / 2) * sum(s + log(mean ^ 2), na.rm = TRUE)
```

Second with respect to p

```
pp = (1 / 2) * sum(s + log(mean ^ 2) ^ 2, na.rm = TRUE)
```

Create information matrix in blocks

```
m1 = rbind(array(kt), c(tp))
inf = rbind(cbind(tt, t(m1)), cbind(m1, rbind(c(kk, pk), c(pk, pp))))
```

Variance-covariance matrix for parameters, inverse of information matrix

```
vcov = solve(inf)
```

Simulation

Initialize simulation array to keep simulation results

```
sim = matrix(0, 0, 11)
smn = matrix(0, 0, 11)
spm = matrix(0, 0, npar + 2)
```

Simulation for distribution of future amounts

Want 10,000 simulations, but exceeds R capacity, so do in batches of 5,000

```
nsim = 5000
smsk = aperm(array(c(msk), c(10, 10, nsim)), c(3, 1, 2))
smsn = aperm(array(c(msn), c(10, 10, nsim)), c(3, 1, 2))

if (simulation) {
  for (i in 1:5) {
    # Randomly generate parameters from multivariate normal
    spar = rmvnorm(nsim, mle$par, vcov)

    # Arrays to calculate simulated means
    esim = g.obj(spar)

    # Arrays to calculate simulated variances
    ksim = exp(aperm(outer(array(
      spar[, c(npar + 1)], c(nsim, 10)
    ), log(dnom), "-"), c(1, 3, 2)))
    psim = array(spar[, npar + 2], c(nsim, 10, 10))
    vsim = ksim * (esim ^ 2) ^ psim

    # Randomly simulate future averages
    temp = array(rnorm(nsim * 10 * 10, c(esim), sqrt(c(vsim))), c(nsim, 10, 10))

    # Combine to total by exposure period and in aggregate
    # notice separate array with name ending in "n" to capture
    # forecast for next accounting period
    sdnm = t(matrix(dnom, 10, nsim))
    fore = sdnm * rowSums(temp * !smsk, dims = 2)
    forn = sdnm * rowSums(temp * smsn, dims = 2)

    # Cumulate and return for another 5,000
    sim = rbind(sim, cbind(fore, rowSums(fore)))
    smn = rbind(smn, cbind(forn, rowSums(forn)))
    spm = rbind(spm, spar)
  }
}
```

Print Results

```r
model
```

```
## [1] "Berquist"
```

```r
summary(sim)
```

```
##        V1           V2                  V3                  V4
##  Min.   :0   Min.   :-1452523   Min.   :-1460032   Min.   :-1416929
##  1st Qu.:0   1st Qu.:  302558   1st Qu.:  811550   1st Qu.: 2807320
##  Median :0   Median :  597817   Median : 1228080   Median : 3544006
##  Mean   :0   Mean   :  642440   Mean   : 1252318   Mean   : 3558347
##  3rd Qu.:0   3rd Qu.:  932306   3rd Qu.: 1673686   3rd Qu.: 4290686
##  Max.   :0   Max.   : 3508046   Max.   : 4460241   Max.   : 8808302
##        V5                  V6                  V7
##  Min.   :   -6553   Min.   : 5560629   Min.   :20506185
##  1st Qu.: 6227569   1st Qu.:15141382   1st Qu.:37021474
##  Median : 7336299   Median :17015966   Median :40212065
##  Mean   : 7343317   Mean   :17027739   Mean   :40274082
##  3rd Qu.: 8459679   3rd Qu.:18881648   3rd Qu.:43484551
##  Max.   :15788019   Max.   :29283048   Max.   :63449354
##        V8                   V9                 V10
##  Min.   : 42237266   Min.   : 84207233   Min.   :155320285
##  1st Qu.: 69227330   1st Qu.:119093974   1st Qu.:198804197
##  Median : 74008098   Median :126196817   Median :209482103
##  Mean   : 74154814   Mean   :126342190   Mean   :209675354
##  3rd Qu.: 78916652   3rd Qu.:133428500   3rd Qu.:220366856
##  Max.   :110991079   Max.   :173619116   Max.   :279848807
##       V11
##  Min.   :366516154
##  1st Qu.:460324346
##  Median :479599631
##  Mean   :480270601
##  3rd Qu.:499535658
##  Max.   :598174718
```

```r
summary(smn)
```

```
##        V1           V2                  V3                  V4
##  Min.   :0   Min.   :-1452523   Min.   :-1138446   Min.   :-1476281
##  1st Qu.:0   1st Qu.:  302558   1st Qu.:  272247   1st Qu.: 1626501
##  Median :0   Median :  597817   Median :  505270   Median : 2200960
##  Mean   :0   Mean   :  642440   Mean   :  527492   Mean   : 2230028
##  3rd Qu.:0   3rd Qu.:  932306   3rd Qu.:  758791   3rd Qu.: 2800206
##  Max.   :0   Max.   : 3508046   Max.   : 2627158   Max.   : 6670925
##        V5                  V6                  V7
##  Min.   :-1172470   Min.   : 1270164   Min.   : 5239797
##  1st Qu.: 2880155   1st Qu.: 8086341   1st Qu.:18510808
##  Median : 3663228   Median : 9515524   Median :20882468
##  Mean   : 3691096   Mean   : 9561369   Mean   :20957144
##  3rd Qu.: 4467433   3rd Qu.:11003597   3rd Qu.:23334596
##  Max.   : 9452336   Max.   :19788052   Max.   :38041131
##        V8                  V9                 V10
##  Min.   :11507452   Min.   :23972497   Min.   : 23306626
```

```
##  1st Qu.:30127968   1st Qu.:42126743   1st Qu.: 54269334
##  Median :33338663   Median :46211701   Median : 59039646
##  Mean   :33428879   Mean   :46263910   Mean   : 59065905
##  3rd Qu.:36694937   3rd Qu.:50318506   3rd Qu.: 63803067
##  Max.   :56230971   Max.   :74988812   Max.   :103912858
##       V11
##  Min.   :118324354
##  1st Qu.:167789490
##  Median :176256113
##  Mean   :176368264
##  3rd Qu.:184790407
##  Max.   :231225457
```

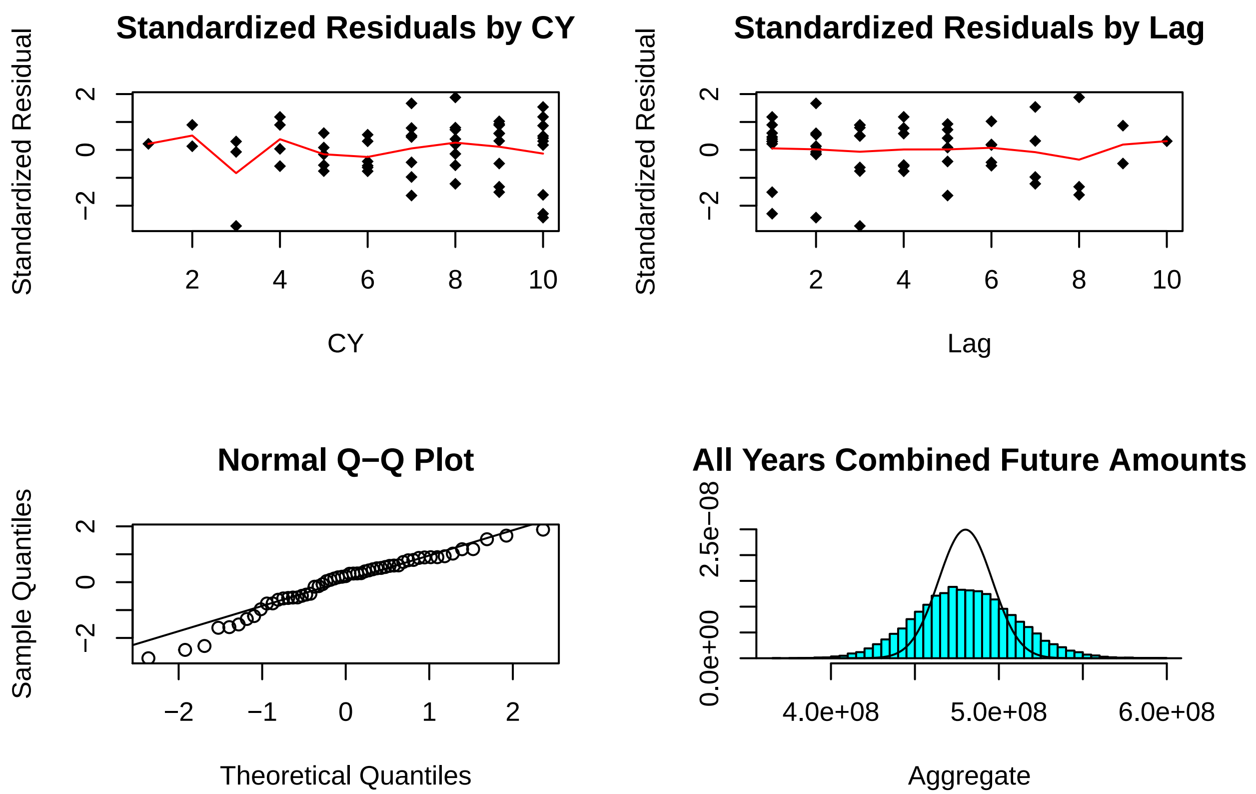
Plots

```
# Scatter plots of residuals & Distribution of Forecasts
if (graphs) {
  #x11(title = model_description(model))

  # Prep data for lines for averages in scatter plots of standardized residuals
  ttt = array(cbind(c(rowNum + colNum - 1), c(stres)), c(length(c(stres)), 2, 19))
  sss = t(array((1:19), c(19, length(c(
    stres
  )))))

  # Plotting
  par(mfrow = c(2, 2))
  plot(
    na.omit(cbind(c(rowNum + colNum - 1), c(stres))),
    main = "Standardized Residuals by CY",
    xlab = "CY",
    ylab = "Standardized Residual",
    pch = 18
  )
  lines(na.omit(list(
    x = (1:19),
    y = colSums(ttt[, 2, ] *
                  (ttt[, 1, ] == sss), na.rm = TRUE) /
      colSums((ttt[, 1, ] == sss) +
                0 *
                ttt[, 2, ], na.rm = TRUE)
  )), col = "red")
  plot(
    na.omit(cbind(c(colNum), c(stres))),
    main = "Standardized Residuals by Lag",
    xlab = "Lag",
    ylab = "Standardized Residual",
    pch = 18
  )
  lines(na.omit(list(
    x = colNum[1, ],
    y = colSums(stres, na.rm = TRUE) /
      colSums(1 + 0 * stres, na.rm = TRUE)
```

```
  )), col = "red")
  qqnorm(c(stres))
  qqline(c(stres))
  if (simulation) {
    proc = list(x = (density(sim[, 11]))$x,
                y = dnorm((density(sim[, 11]))$x,
                          sum(matrix(c(
                            dnom
                          ), 10, 10) * mean * !msk),
                          sqrt(sum(
                            matrix(c(dnom), 10, 10) ^ 2 * var * !msk
                          ))))
    truehist(sim[, 11],
             ymax = max(proc$y),
             main = "All Years Combined Future Amounts",
             xlab = "Aggregate")
    lines(proc)
  }
}
```



Summary From Simulation

Summary of mean, standard deviation, and 90% confidence interval from simulation, similar for one-period forecast

```
sumr = matrix(0, 0, 4)
sumn = matrix(0, 0, 4)

for (i in 1:11) {
  sumr = rbind(sumr, c(mean(sim[, i]), sd(sim[, i]), quantile(sim[, i], c(.05, .95))))
```

```r
  sumn = rbind(sumn, c(mean(smn[, i]), sd(smn[, i]), quantile(smn[, i], c(.05, .95))))
}
sumr
```

```
##                                           5%       95%
##   [1,]         0.0         0.0         0.00         0
##   [2,]    642440.5    481028.8    -44194.36   1501201
##   [3,]   1252318.1    648086.8    234536.28   2350542
##   [4,]   3558346.6   1140237.1   1736139.47   5459264
##   [5,]   7343316.8   1685230.7   4618772.27  10135134
##   [6,]  17027739.1   2812746.4  12455686.23  21698143
##   [7,]  40274081.9   4840499.9  32377990.92  48358166
##   [8,]  74154813.9   7300843.4  62309918.99  86306505
##   [9,] 126342190.5 10639635.7 109032027.02 144116409
## [10,] 209675353.5 15905776.4 183850667.27 236026995
## [11,] 480270600.9 29296749.7 432905077.06 529557186
```

```r
sumn
```

```
##                                           5%       95%
##   [1,]         0.0         0.0         0.00         0
##   [2,]    642440.5    481028.8    -44194.36   1501201
##   [3,]    527492.0    376363.7    -37645.58   1173939
##   [4,]   2230028.1    900647.6    790310.82   3758650
##   [5,]   3691095.9   1201650.8   1761276.09   5701639
##   [6,]   9561369.2   2183081.0   6055215.06  13228795
##   [7,]  20957144.4   3610684.4  15092490.22  26953460
##   [8,]  33428879.0   4931632.4  25433851.64  41583251
##   [9,]  46263909.9   6136563.8  36357479.01  56377102
## [10,]  59065904.7   7160890.5  47441520.52  70937231
## [11,] 176368263.7 12708050.9 155788022.10 197482374
```