Mathematics Analysis and Approaches
Higher Level Internal Assessment

Exploring the critical strike chance system in the video game League of Legends

Number of Pages: 22

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Introduction

Randomness plays a role in nearly every game. My favourite video game, League of Legends, is no exception. In League of Legends, players can purchase in-game items that boost their character's attributes. One such attribute is “critical strike chance”, which is the probability that an attack will “critically strike”, dealing extra damage.

Online players have criticized the game’s critical strike system due to its randomness since getting a few lucky critical strikes in a row can significantly influence the outcome of a fight, especially in the game's early stages where characters have low critical strike chances. For example, I have seen videos of a player getting four critical strikes in a row with only a 10% critical strike chance. However, in my many years of playing the game, I have encountered suspiciously few instances of lucky streaks, so I was curious as to whether the game’s critical strike system somehow minimized the occurrence of such streaks.

This investigation aims to explore how critical strike chance works in League of Legends and examine how the game’s system minimizes the impact of randomness for players using probability concepts such as expected value and variance.

Background

This investigation aims to analyze the randomness associated with critical strike chance, specifically, with the length of streaks of critical strikes. The number of critical strikes can be modelled as a discrete random variable since its value must be an integer. The formula for a discrete random variable is as follows:

$$E(X) = \sum xP(X = x)$$

A measure of dispersion is a statistical term that describes how varied a certain variable is. In this investigation, variance will be used to measure the dispersion of the variable. The formula for the variance of a discrete random variable is as follows: $Var(X) = E(X^2) - E(X)^2$. In this investigation, I will explore two different critical strike systems and compare their variances.

This investigation also investigates probability. The most basic formula for theoretical probability taught in this course is $P(A) = \frac{n(A)}{n(U)}$, where $P(A)$ represents the probability of an event A occurring, $n(A)$ represents the number of outcomes in which A occurs and $n(U)$ represents the total number of outcomes in the sample space.

Analysis

Straightforward system

I will begin by analyzing what an expected critical strike chance system would look like. A straightforward critical strike chance system would be as follows: if a player has a critical strike chance p, each attack is independent and has a probability p critically striking.

Let X be the number of critical strikes a player gets in a row with a critical strike chance p.

The probability that the player gets 0 critical strikes in a row would be $1 - p$.

The probability that the player gets 1 critical strike in a row would be $p(1 - p)$.

The probability that the player gets 2 critical strikes in a row would be $p^2(1 - p)$.

The probability that the player gets x critical strikes in a row would be $p^x(1 - p)$.

Since the number of critical strikes must be an integer, it is a discrete random variable and I can apply the formula for its expected value as shown below:

$$E(X) = \sum xP(X = x)$$

$$E(X) = \sum_{x=0}^{\infty} xp^x(1 - p)$$

$$E(X) = (1 - p)\sum_{x=0}^{\infty} xp^x$$

This is not a known summation, but it very closely resembles the known summation of an infinite geometric series:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

The difference is that the general term in summation for $E(X)$ of is multiplied by x, the index of summation. If I would have to slightly alter the summation of the infinite geometric series to apply it to the summation for $E(X)$. I recognized that n in the geometric series summation corresponded to x in the $E(X)$ summation. Thus, I needed to introduce a coefficient n into the general term of the summation for an infinite geometric series. Seeing n in the exponent, I thought of using the power rule, differentiating both sides of the equation with respect to r to bring down the n term.

$$\frac{d}{dr}\sum_{n=0}^{\infty} r^n = \frac{d}{dr}\frac{1}{1-r}$$

Since the left side is a summation, differentiating it is equivalent to differentiating each of the individual terms.

$$\sum_{n=0}^{\infty} nr^{n-1} = \frac{1}{(1-r)^2}$$

Multiplying both sides by r yields:

$$\sum_{n=0}^{\infty} nr^{n} = \frac{r}{(1-r)^2}$$

Substituting $p = r$ and $x = n$,

$$E(X) = (1 - p)\sum_{x=0}^{\infty} xp^{x}$$

$$E(X) = (1 - p)\frac{p}{(1-p)^2}$$

$$E(X) = \frac{p}{1-p}$$

Now I will find the variance of X, $Var(X)$.

$$Var(X) = E(X^2) - E(X)^2$$

$$Var(X) = \sum x^2 P(X = x) - (\frac{p}{1-p})^2$$

$$Var(X) = \sum_{x=0}^{\infty} x^2 p^x (1 - p) - (\frac{p}{1-p})^2$$

Again, the left term is not a known summation, but a similar strategy of differentiating the known formula for a summation of an infinite geometric series can be applied. Since I am computing $E(X^2)$, there is an x^2 term instead of x. This suggests that I will have to take the derivative twice to obtain the squared term. I will begin with this intermediate result obtained in the calculation of $E(X)$.

$$\sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}$$

I will replace n and r with x and p respectively to match the variables in the summation for $E(X^2)$.

$$\sum_{x=0}^{\infty} xp^x = \frac{p}{(1-p)^2}$$

Again, I will differentiate both sides of the equation with respect to p using the power rule to bring down the exponent x. The right side of the equation can be differentiated using the quotient rule.

$$\frac{d}{dp} \sum_{x=0}^{\infty} xp^x = \frac{d}{dp} \frac{p}{(1-p)^2}$$

$$\sum_{x=0}^{\infty} x^2 p^{x-1} = \frac{(1-p)^2 + 2(1-p)p}{(1-p)^4}$$

$$\sum_{x=0}^{\infty} x^2 p^{x-1} = \frac{1+p}{(1-p)^3}$$

Multiplying both sides by p to restore the exponent of p yields:

$$\sum_{x=0}^{\infty} x^2 p^x = \frac{p+p^2}{(1-p)^3}$$

Multiplying both sides by $p - 1$ yields:

$$\sum_{x=0}^{\infty} x^2 p^x (p - 1) = E(X^2) = \frac{p+p^2}{(1-p)^2}$$

$$Var(X) = E(X^2) - E(X)^2$$

$$Var(X) = \frac{p+p^2}{(1-p)^2} - \frac{p^2}{(1-p)^2}$$

$$Var(X) = \frac{p}{(1-p)^2}$$

Now, the expected value and variance of X, the number of critical strikes in a row, can be determined for any critical strike chance p using the formulas below.

$$E(X) = \frac{p}{1-p}$$

$$Var(X) = \frac{p}{(1-p)^2}$$

However, I would like to examine the average variance across all critical strike chances p, which is the expected value of the variance. In League of Legends, critical strike chance can be any real number from 0 to 1, so I recognized that $Var(X)$ would be a continuous random variable, and that I would need to use integration rather than a summation. Using the formula for the average value of a function over the interval $[a, b]$: $average = \frac{1}{b-a}\int_a^b f(x)dx$,

$$E(V) = \int_0^1 \frac{p}{(1-p)^2} dp)$$

I started by considering the indefinite integral first since I find it simpler not to avoid having to change the bounds in case I were to use substitution. I started by expanding the denominator to see if it would lead anywhere.

$$E(V) = \int \frac{p}{1-2p+p^2} dp$$

Seeing a polynomial of order 2 on the denominator and a polynomial of order 1, I instinctively thought of using u-substitution with u as the denominator, since its derivative would be in the same order as the numerator. However, upon further examination, letting $u = 1 - 2p + p^2$ would result in $\frac{du}{dp} = 2p - 2$. It would not be possible to introduce the $- 2$ into the expression. I then returned to the factored form of the integral.

$$E(V) = \int \frac{p}{(1-p)^2} dp$$

One reason that expanding the binomial did not provide any help is that it is in the denominator. If the numerator and denominator somehow switched places, expanding each term in the expanded binomial could be divided by p and the resulting integral would be trivial. This thinking led me to the substitution $u = 1 - p$. This simple substitution solves the issue of having a binomial on the bottom, by replacing it with a single variable u.

$$u = p - 1$$

$$\frac{du}{dp} = 1$$

$$E(V) = \int \frac{p}{u^2} du$$

The equation for u can be rearranged to $p = u + 1$. Substituting this value for p in the integral yields:

$$E(V) = \int \frac{u+1}{u^2} du$$

$$E(V) = \int \frac{1}{u} + \frac{1}{u^2} du$$

$$E(V) = ln(u) - \frac{1}{u} + C$$

$$E(V) = ln(|p - 1|) - \frac{1}{p-1} + C$$

While u-substitution was a simple and elegant way to solve this integral, I also recognized that integration by partial fractions could have been used due to the factored denominator. I could have represented $\frac{p}{(p-1)^2}$ as the sum of $\frac{A}{p-1} + \frac{B}{(p-1)^2}$ where A and B are constants to be determined and integrated into that sum instead.

However, in my experience, u-substitution is typically more efficient, which is why I considered it before using partial fractions. Returning to my definite integral, I noticed that substituting the values to evaluate the area resulted in a value that was not computable.

$$E(V) = (ln(|p - 1|) - \frac{1}{p-1})|_0^1$$

$$E(V) = ln(|1 - 1|) - \frac{1}{1-1} - (ln(1 - 0) - \frac{1}{0-1})$$

$$E(V) = ln(0) - \frac{1}{0} + 1$$

The left bound of the integral provided no problems, the issue was with the higher bound when p approached 1 since it resulted in $ln(0) - \frac{1}{0}$, which is undefined. I decided to graph my function for variance to see what was going on.

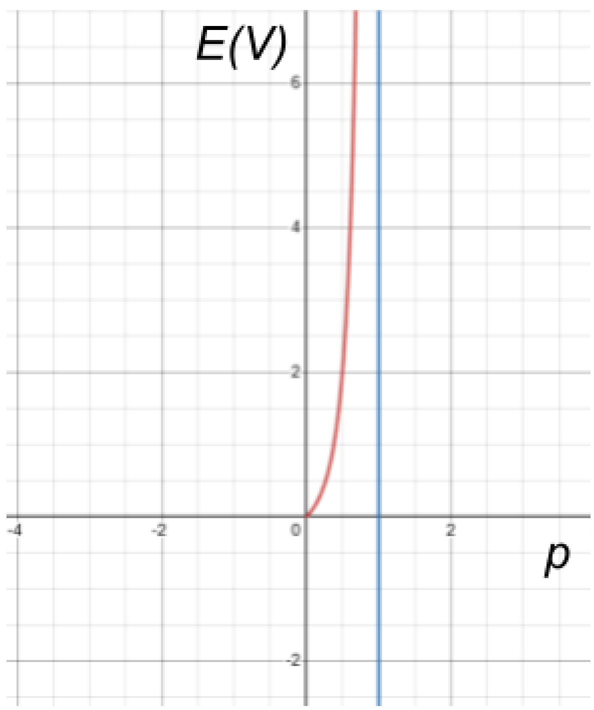

Figure 1: Graph of $E(V)$ against p

As previously mentioned, there were no errors with the left bound, but the right bound approaches infinity which results in the integral diverging. I thought back to what the function was representing: the variance of X, the expected number of critical strikes in a row. The left side of the bound made sense, since at 0 critical strike chance, the number of critical strikes in a row is 0 every time. As for the right side, as the critical strike chance approaches 1, making long streaks of critical strikes in a row becomes more likely, and the value of X becomes unbounded, leading to extremely high variance. The current integral for variance provided little insight into the average variance a player experienced, so I needed to change it somehow. I instead decided on limiting the bounds of the integral from $[0, 1]$ to $[0, 0.5]$, since the original inspiration of this investigation was to investigate variance at lower critical strike chances. Thus, the average variance could be calculated using:

$$E(V) = \frac{1}{0.5}(ln(|p - 1|) - \frac{1}{p-1})\Big|_0^{\frac{1}{2}}$$

$$E(V) = 2(ln(|\frac{1}{2} - 1|) - \frac{1}{\frac{1}{2}-1} - (ln(|0 - 1|) + \frac{1}{0-1}))$$

$$E(V) = 2(ln(\frac{1}{2}) + 2 - (ln(1) - 1))$$

$$E(V) = 2 - 2ln(2) \approx 0.6137$$

Altered system

I will now examine how an altered critical strike system works and compare it to the straightforward system analyzed in the previous section. The exact critical strike algorithm used in League of Legends is not publicly available, but according to the official Wiki-page, it works something like this: "*The probability of a critical strike dynamically updates based on how many attacks have critically striked … where multiple successful critical strikes will cause the probability to progressively decrease for future attempts*".

To replicate this algorithm, I will try an approach where the critical strike chance decreases by a percent amount after each consecutive critical strike but resets upon a non-critical strike. However, it is necessary to adjust the initial probability to ensure that the overall critical strike probability across all attacks remains equal to the probability displayed to the player. Specifically, the probability of landing a critical strike on the first attack must be set higher than the displayed value. This compensates for the subsequent decreases so that when averaged over many attacks, the effective critical strike chance aligns with the actual critical strike chance.

Let p be the displayed critical strike chance.
Let p_0 be the probability of a critical strike on the first attack.
Let r be the percentage decrease for consecutive critical strikes.

The probability of a critical strike after k successful critical strikes is given by $p_k = p_0 r^k$. To begin, I needed to determine a new expression for $E(X)$, the expected number of consecutive critical strikes. I returned to the formula for the expected value of a discrete random variable X.

$$E(X) = \sum xP(X = x)$$

I determined $P(X = x)$ of the altered system in the same way I did for the straightforward system. For the number of consecutive critical strikes to be equal to x, there must be x successful critical strikes followed by 1 unsuccessful attempt.

$$P(X = x) = p_0 \cdot p_1 \cdot \ldots \cdot p_{x-2} \cdot p_{x-1} \cdot (1 - p_x)$$

$$P(X = x) = (\prod_{x=0}^{x-1} p_x)(1 - p_0 r^x)$$

$$P(X = x) = (\prod_{x=0}^{x-1} p_0 r^x)(1 - p_0 r^x)$$

The exponents of r in the product $\prod_{x=0}^{x-1} p_0 r^x$, form an arithmetic sequence starting with 0 and ending with $x - 1$. This sum can be expressed as $x(x - 1)/2$. Using exponent rules, the product can be rewritten as shown below:

$$\prod_{x=0}^{x-1} p_0 r^x = p_0^x r^{x(x-1)/2}$$

Thus,

$$P(X = x) = p_0^x r^{x(x-1)/2}(1 - p_0 r^x)$$

and

$$E(X) = \sum_{x=0}^{\infty} x p_0^x r^{x(x-1)/2}(1 - p_0 r^x)$$

I then tried to determine a closed-form expression for $E(X)$. However, this summation was much more complex than the one for the straightforward system. It did not resemble any known summations that I had learned. Furthermore, the quadratic term in the exponent of $r^{x(x-1)/2}$ would require a very specific known summation to produce through the same method of differentiation used for the straightforward system. After researching, I learned that certain advanced methods such as *generating functions* could be applied to evaluate $E(X)$ more rigorously. However, these approaches are beyond the scope of this course.

I then decided to look towards approximating the function $E(X)$. I recognized that each term in my summation was a polynomial with p_0 as the variable, multiplied by some coefficient in terms of r. I then made the connection between my series and a Maclaurin series. Unlike a Maclaurin series whose coefficients are determined from the derivatives of a known function at $x = 0$, my coefficients come from $P(X = x)$. However, both series are expressed as sums of powers of a variable. Additionally, similar to a Maclaurin series which is centred at $x = 0$, my series is centred around a probability p_0, which must be in the range of $[0,\ 1]$. I remembered that the Maclaurin series problems I worked on in class only ever asked to determine the first three or four terms of the series. This is because for a small x, $(|x| < 1)$, each successive power x^n becomes significantly smaller. This means that the contribution of higher power terms in a Maclaurin series diminishes rapidly. Since my series deals with small values, namely $0 \leq p_0 \leq 1$, this same property of higher power terms contributing less to the accuracy of estimation could be applied.

Using this information, I decided to approximate $E(X)$ by taking the first four nonzero terms of the summation. While this does result in a loss of accuracy, the below graph shows that the two functions are approximately equal, especially in the interval $[0,\ 0.5]$, the focus of this investigation. The below graph used a value of 0.9 for r.

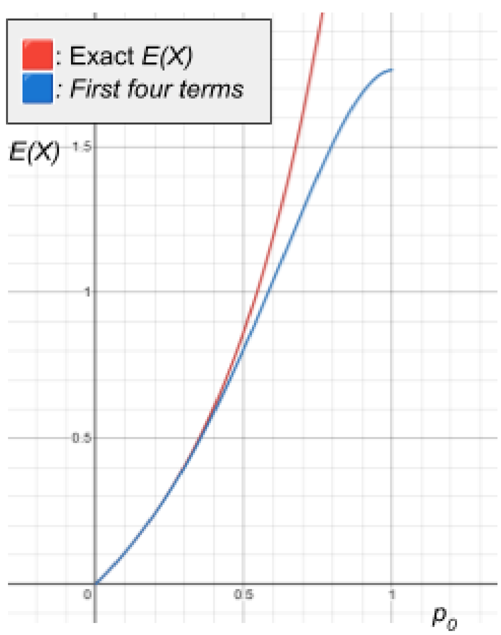

Figure 2: Graph of exact $E(X)$ and approximated $E(X)$ against p_0

Thus,

$$E(X) \approx \sum_{x=0}^{4} x p_0^x r^{x(x-1)/2}(1 - p_0 r^x)$$

$$E(X) \approx \sum_{x=0}^{4} x p_0^x r^{x(x-1)/2} - x p_0^x r^{x(x-1)/2} p_0 r^x$$

$$E(X) \approx \sum_{x=0}^{4} x p_0^x r^{x(x-1)/2} - x p_0^{x+1} r^{x(x+1)/2}$$

Now, I could simply evaluate the first few indices x of the summation. Note that the $x = 0$ term is equal to 0, so we start at $x = 1$.

Let the first term ($x = 1$) be T_1.

$$T_1 = 1 \cdot p_0^1 r^{1(1-1)/2} - 1 \cdot p_0^{1+1} r^{1(1+1)/2}$$

$$T_1 = 1 \cdot p_0^1 r^0 - 1 \cdot p_0^2 r^1$$

$$T_1 = p_0 - p_0^2 r$$

Substituting $x = 2, 3, 4$ yields:

$$T_2 = 2p_0^2 r - 2p_0^3 r^3$$

$$T_3 = 3p_0^3 r^3 - 3p_0^4 r^6$$

$$T_4 = 4p_0^4 r^6 - 4p_0^5 r^{10}$$

$$E(X) \approx T_1 + T_2 + T_3 + T_4 \approx p_0 - p_0^2 r + 2p_0^2 r - 2p_0^3 r^3 + 3p_0^3 r^3 - 3p_0^4 r^6 + 4p_0^4 r^6 - 4p_0^5 r^{10}$$

$$E(X) \approx p_0 + p_0^2 r + p_0^3 r^3 + p_0^4 r^6 - 4p_0^5 r^{10}$$

The variance can be determined using the formula $Var(X) = E(X^2) - E(X)^2$.

$$Var(X) = E(X^2) - E(X)^2$$

$$Var(X) \approx \sum_{x=0}^{\infty} x p_0^x r^{x(x-1)/2} - x p_0^{x+1} r^{x(x+1)/2} - (p_0 + p_0^2 r + p_0^3 r^3 + p_0^4 r^6 - 4p_0^5 r^{10})^2$$

The summation for $E(X^2)$, like the summation for $E(X)$ is not expressible in a closed form due to similar reasons. I will again estimate it using the first four nonzero terms.

$$T_1 = p_0 - p_0^2 r$$

$$T_2 = 4p_0^2 r - 4p_0^3 r^3$$

$$T_3 = 9p_0^3 r^3 - 9p_0^4 r^6$$

$$T_4 = 16p_0^4 r^6 - 16p_0^5 r^{10}$$

$$E(X^2) \approx p_0 - p_0^2 r + 4p_0^2 r - 4p_0^3 r^3 + 9p_0^3 r^3 - 9p_0^4 r^6 + 16p_0^4 r^6 - 16p_0^5 r^{10}$$

$$E(X^2) \approx p_0 + 3p_0^2 r + 5p_0^3 r^3 + 7p_0^4 r^6 - 16p_0^5 r^{10}$$

The expansion of $E(X)^2$ is very long, so I will truncate it to only include up to the p_0^5 term. As previously mentioned, terms with higher exponents become less and less significant since $|p_0| \leq 1$, so truncating it up to the p_0^5 term does not significantly reduce accuracy.

$$E(X^2) \approx p_0^2 + 2p_0^3 r + p_0^4(r^2 + r^3) + 2p_0^5(r^4 + r^6)$$

Finally,

$$Var(X) \approx p_0 + p_0^2(3r - 1) + p_0^3(5r^3 - 2r) + p_0^4(7r^6 - r^2 - r^3) - 2p_0^5 r^{10}(8r^{10} + r^4 + r^6)$$

While this expression for $Var(X)$ is not exact, it can be seen in the diagram below that the graph of the exact $Var(X)$ is nearly indistinguishable from the approximated $Var(X)$ over the interval $[0, 0.5]$. The graph below used a value of 0.9 for r.

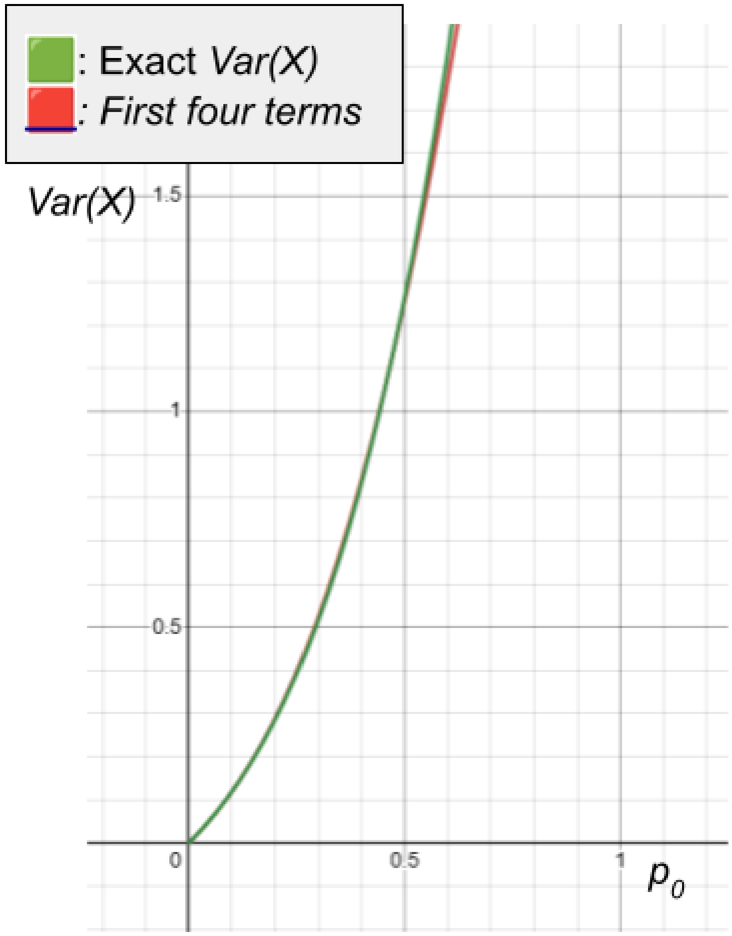

Figure 3: Graph of exact $Var(X)$ and approximated $Var(X)$ against p_0

Next, I needed to determine an expression for p_0 in terms of p and r. As mentioned previously, the probability of a critical strike over all attacks for the straightforward system should equal the probability of a critical strike over all attacks for the altered system. The average probability of a critical strike for the straightforward system is the constant p. Determining the average probability for the altered system was not as simple. I decided to consult the most basic definition of theoretical probability:

$$P(A) = \frac{n(A)}{n(U)}$$

In the context of this investigation, the event A is an attack being a critical strike and the sample set U is all attacks. I recalled that my random variable X was the average number of a streak of consecutive critical strikes. Thinking in terms of streaks of critical strikes, each critical strike is part of a streak, and each streak ends with a non-critical strike. Thus, I could set $n(A) = E(X)$ and $n(U) = E(X) + 1$, yielding: $P(A) = \frac{E(X)}{E(X)+1}$

Since this theoretical probability is equal to p,

$$p = \frac{E(X)}{E(X)+1}$$

$$p = \frac{p_0 + p_0^2 r + p_0^3 r^3 + p_0^4 r^6 - 4p_0^5 r^{10}}{p_0 + p_0^2 r + p_0^3 r^3 + p_0^4 r^6 - 4p_0^5 r^{10} + 1}$$

$$p(p_0 + p_0^2 r + p_0^3 r^3 + p_0^4 r^6 - 4p_0^5 r^{10} + 1) = p_0 + p_0^2 r + p_0^3 r^3 + p_0^4 r^6 - 4p_0^5 r^{10}$$

$$p(p_0 + p_0^2 r + p_0^3 r^3 + p_0^4 r^6 - 4p_0^5 r^{10}) + p = p_0 + p_0^2 r + p_0^3 r^3 + p_0^4 r^6 - 4p_0^5 r^{10}$$

$$(p - 1)(p_0 + p_0^2 r + p_0^3 r^3 + p_0^4 r^6 - 4p_0^5 r^{10}) + p = 0$$

Ideally, I would solve this equation for p_0 in terms of p and r. Then I could rewrite $Var(X)$ in terms of p and r and calculate the average variance of the altered system. However, this is a fifth-degree polynomial in p_0, making any algebraic solutions near impossible. Since an analytical solution was not an option, I used computer software to determine numerical solutions to the equation. For example, given $p = 0.3$ and $r = 0.9$, $p_0 \approx 0.3153$. Since p_0 could not be determined analytically, I was not able to determine the average variance analytically either. So again, I used computer software to determine the numerical value. I decided to set $r = 0.9$. The program looped through values of p from 0 to 0.5 using a step-size of 0.00001. For each value of p, the value of p_0 was determined and the variance was calculated. Finally, the program determined the average variance of the altered system to be roughly 0.5314.

Conclusion

Overall, this investigation successfully explored the statistics and probability concepts such as expected value and variance and their applications to the critical strike chance system in the video game League of Legends. I began by deriving exact expressions for $E(X)$ and $Var(X)$ of a straightforward probability system. I then tried to determine expressions for $E(X)$ and $Var(X)$ of the altered system but was not able to derive closed-form expressions. Instead, I truncated the summations for $E(X)$ and $Var(X)$ to produce simplified estimations. The results showed that the altered system had a lower average variance (0.5314) than the straightforward system (0.6137). While the many estimations used throughout this investigation limited the accuracy of my results, I am confident in the relative average variances of the two systems—on average, the altered system has less variance than the straightforward system.

Further investigations could examine different values of r and their effect on the average variance. Another possible extension is the use of a probability density function for the critical strike chances. The addition of a probability density function would better reflect average gameplay. Finally, I could look into exact solutions rather than estimations. However, this is beyond the scope of the course. In conclusion, this investigation provided insight into the application of statistics and probability in minimizing variance in video games.

References

"Critical Strike." *League of Legends Wiki*, wiki.leagueoflegends.com/en-us/Critical_strike#Critical_strike_chance. Accessed 26 Feb. 2025.

Appendices

Appendix A: Python Program used to calculate the values of p_0 and variance for the altered system

```
import numpy as np
from scipy.optimize import fsolve

# Parameters
r = 0.9
p_values = np.arange(0, 0.500, 0.00001)  # p from 0 to 0.5 (step = 0.001)
variances = []

# Define the equation to solve for p0 given p and r
def equation(p0, p, r):
    return (p - 1) * (p0 + p0**2 * r + p0**3 * r**3 + p0**4 * r**6 - 4 * p0**5
* r**10) + p

def calc_var1(p):
    return p/((1-p)*(1-p))
def calc_var2(p):
    p0_initial_guess = p / (1 - p * (1 - r)) if p != 1/(1 - r) else 0.5

    # Solve for p0 numerically
    p0_solution = fsolve(equation, p0_initial_guess, args=(p, r))[0]

    # Compute variance using the given formula
    term1 = p0_solution
    term2 = p0_solution**2 * (3 * r - 1)
    term3 = p0_solution**3 * (5 * r**3 - 2 * r)
    term4 = p0_solution**4 * (7 * r**6 - r**2 - r**3)
    term5 = -2 * p0_solution**5 * r**10 * (8 * r**10 + r**4 + r**6)
    return term1 + term2 + term3 + term4 + term5

var1 = []
for p in p_values:
    var1.append(calc_var1(p))
```

```
    variances.append(calc_var2(p))

# Calculate average variance
average_variance = np.mean(variances)

print(f"Average Variance (r = 0.9, p ∈ [0, 0.5]): {average_variance:.6f}")
print(np.mean(var1))
```