

# Irreducible Components and Dimension of the Springer Fiber of a Hook-Type Slodowy Slice

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## Abstract

Let  $n \geq 1$ , and let  $e$  be a regular nilpotent element of  $\mathfrak{gl}_n$ . Let  $m \geq 0$ , and consider the Slodowy slice  $S$  at the element  $(0, e) \in \mathfrak{gl}_m \times \mathfrak{gl}_n \subseteq \mathfrak{gl}_{m+n}$ . We define the *Springer fiber of  $S$*  at a nilpotent  $X \in \mathfrak{gl}_m$  as consisting of the following data: an element  $Y \in S$  which projects to  $X$  coordinate-wise, along with an element of the usual Springer fiber at  $(X, Y) \in \mathfrak{gl}_m \times \mathfrak{gl}_{m+n}$ . The main result of this paper is finding the (equidimensional) irreducible components of a Springer fiber of  $S$ . To do this, we use a well-known result classifying the irreducible components of the usual Springer fiber at a nilpotent element of  $\mathfrak{gl}_m$ . We then use our result to find the (equidimensional) irreducible components of the variety consisting of pairs  $(X, \mathcal{F})$ , where  $X \in \mathfrak{gl}_m$  is nilpotent and  $\mathcal{F}$  is an element of the Springer fiber of  $S$  at  $X$ . We conjecture that there is a correspondence, akin to the geometric RSK correspondence, between irreducible components of this last variety and certain pairs of standard Young tableaux.

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# 1 Introduction

Let  $m \geq 0, n \geq 1$ , and define  $\mathfrak{g} = \mathfrak{gl}_m \times \mathfrak{gl}_{m+n}$ . Let  $\mathcal{N}$  be the nilpotent cone in  $\mathfrak{g}$ , and let  $\tilde{\mathcal{N}} \rightarrow \mathcal{N}$  be the Springer resolution.

Let  $(e, h, f)$  be a principal  $\mathfrak{sl}_2$ -triple in  $\mathfrak{gl}_n$ . Let  $(E, H, F)$  be the  $\mathfrak{sl}_2$ -triple in  $\mathfrak{gl}_{m+n}$  which is the image of  $(e, h, f)$  via the embedding  $\mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_m \times \mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_{m+n}$ . Let  $S$  be the Slodowy slice  $E + \mathfrak{z}_{\mathfrak{gl}_{m+n}}(F)$ . We have a map  $\mathfrak{gl}_{m+n} \rightarrow \mathfrak{gl}_m$  given by coordinate projection. Restricting to  $S$ , we obtain a projection  $p' : S \rightarrow \mathfrak{gl}_m$ . (It turns out that  $p'$  is surjective.) Then, define  $p : S \rightarrow \mathfrak{g}$  by  $x \mapsto (p'(x), x)$ . Taking the map  $p : S \rightarrow \mathfrak{g}$  and the Springer resolution  $\tilde{\mathcal{N}} \rightarrow \mathfrak{g}$ , we obtain a fibred product  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ . The variety  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$  is also discussed in [2][3].

In this paper we study  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ . As a step towards studying  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ , we let  $\mathcal{N}_m$  be the nilpotent cone in  $\mathfrak{gl}_m$ , and we consider the map  $\pi : S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \rightarrow \mathcal{N}_m$  given by taking the map to  $\mathfrak{g}$  and then projecting to  $\mathfrak{gl}_m$ . Given  $X \in \mathcal{N}_m$ , we call  $\pi^{-1}(X)$  the *n-Slodowy-slice Springer fiber at X*.

The main result of this paper is a classification of the irreducible components of an  $n$ -Slodowy-slice Springer fiber. Then we use this result to determine the irreducible components of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ .

## 1.1 Future Work

We describe a possible extension of the result of this paper, as well as an application to representation theory.

The *Steinberg variety* of a Lie algebra  $\mathfrak{h}$  is  $\text{St}_{\mathfrak{h}} = \tilde{\mathcal{N}}_{\mathfrak{h}} \times_{\mathfrak{h}} \tilde{\mathcal{N}}_{\mathfrak{h}}$ , where the map  $\tilde{\mathcal{N}}_{\mathfrak{h}} \rightarrow \mathfrak{h}$  is the Springer resolution. There is a bijective correspondence between the Weyl group of  $\mathfrak{gl}_m$  and the set  $\text{Irr}(\text{St}_{\mathfrak{gl}_m})$  of irreducible components of  $\text{St}_{\mathfrak{gl}_m}$  (see [1][3.6] for details). The Weyl group of  $\mathfrak{gl}_m$  is the symmetric group  $S_m$ .

There is a bijection between the set of irreducible components of  $\text{St}_{\mathfrak{gl}_m}$ —which we denote by  $\text{Irr}(\text{St}_{\mathfrak{gl}_m})$ —and the set

$$T_m = \{(t_1, t_2) : \lambda \vdash m; t_1, t_2 \in \text{SYT}(\lambda)\}.$$

(We write  $\text{SYT}(\lambda)$  to denote the set of standard Young tableaux of shape  $\lambda$ .) One way to obtain the bijection is as follows. First take the aforementioned correspondence between  $\text{St}_{\mathfrak{gl}_m}$  and the Weyl group  $S_m$ . Then take the Robinson-Schensted-Knuth (RSK) correspondence, which gives a bijection between  $S_m$  and  $T_m$ . Putting these together, we get a correspondence  $\text{St}_{\mathfrak{gl}_m} \leftrightarrow T_m$ . A more direct and illuminating explanation of the correspondence  $\text{St}_{\mathfrak{gl}_m} \leftrightarrow T_m$  is given by the geometric RSK correspondence, as described in [3].

An element of  $\text{St}_{\mathfrak{gl}_m}$  is a triple  $(X, \mathfrak{b}_1, \mathfrak{b}_2)$ , where  $X \in \mathcal{N}_m$ , and  $\mathfrak{b}_1, \mathfrak{b}_2$  are elements of the Springer fiber at  $X$ . When the Jordan form of  $X$  has shape  $\lambda$ , the geometric RSK correspondence sends this triple to a pair  $(t_1, t_2)$ , where both are of shape  $\lambda$ .

Somewhat similarly, we will show that  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$  is a subset of the set of tuples  $(X, Y, \mathfrak{b}_1, \mathfrak{b}_2)$ , where  $X \in \mathcal{N}_m, Y \in \mathcal{N}_{m+n}$ , and  $\mathfrak{b}_1$  (resp.  $\mathfrak{b}_2$ ) is an element of the Springer fiber at  $X$  (resp.  $Y$ ).

For any given relation  $R \subseteq \{\lambda : \lambda \vdash m\} \times \{\mu : \mu \vdash (m+n)\}$ , we can define

$$T_{m,n} = \{(t_1, t_2) : \lambda \vdash m; \mu \vdash (m+n); t_1 \in \text{SYT}(\lambda); t_2 \in \text{SYT}(\mu); R(\lambda, \mu)\}.$$

We conjecture that, for some appropriate choice of the relation  $R$ , there is a correspondence, analogous to  $\text{St}_{\mathfrak{gl}_m} \leftrightarrow T_m$ , between  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$  and  $T_{m,n}$ . A natural extension of this project would explore this correspondence.

Given an element of  $\text{Irr}(\text{St}_{\mathfrak{gl}_m})$ , we can first map it to the corresponding  $(t_1, t_2) \in T_m$ , and then obtain an element of  $\text{End}(\mathbb{C}[\text{SYT}(\lambda)])$  by taking the map which sends  $t_1 \mapsto t_2$  and  $t' \mapsto 0$  for  $t' \in \text{SYT}(\lambda) \setminus \{t_1\}$ . In this way, we obtain a  $\mathbb{C}$ -algebra isomorphism

$$\mathbb{C}[\text{Irr}(\text{St}_{\mathfrak{gl}_m})] \cong \bigoplus_{\lambda \vdash m} \text{End}(\mathbb{C}[\text{SYT}(\lambda)]).$$

This is an instance of the fact that for any finite group  $G$ , we have  $\mathbb{C}[G] \cong \bigoplus_{V \in \text{IrrRep}(G)} \text{End}(V)$ .

If we had a correspondence  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \leftrightarrow T_{m,n}$ , this could give an analogous isomorphism

$$\mathbb{C}[\text{Irr}(S \times_{\mathfrak{g}} \tilde{\mathcal{N}})] \cong \bigoplus_{(\lambda, \mu) \in R} \text{Hom}(\mathbb{C}[\text{SYT}(\lambda)], \mathbb{C}[\text{SYT}(\mu)]),$$

which would be an isomorphism of  $\mathbb{C}[S_m]$ - or  $\mathbb{C}[S_{m+n}]$ -modules rather than  $\mathbb{C}$ -algebras. (To give the RHS the structure of a  $\mathbb{C}[S_m]$ -module, we just view  $\mathbb{C}[\text{SYT}(\lambda)]$  as a representation of  $S_m$  in the standard way.)

## 1.2 Overview

Section 2 reviews some preliminary material. Section 3 finds the unique (up to similarity) principal  $\mathfrak{sl}_2$ -triple in  $\mathfrak{gl}_n$ . Section 4 embeds this  $\mathfrak{sl}_2$ -triple into

$\mathfrak{gl}_{m+n}$  as described above. We compute the Slodowy slice and end up with a nice description of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ .

Section 5 discusses how to reduce the problem of finding the irreducible components of an  $n$ -Slodowy-slice Springer fiber at  $X \in \mathcal{N}_m$  to an easier problem. Section 6 solves the easier problem. Section 7 finds the irreducible components of an  $n$ -Slodowy-slice Springer fiber. Section 8 applies the results of Section 7 to find the irreducible components of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ . Finally, Section 9 proves some linear algebra lemmas that were used in the paper.

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## 2 Preliminary Definitions and Facts

### 2.1 Conventions and Notations

We write  $\mathrm{GL}_m, \mathfrak{sl}_m$  to denote  $\mathrm{GL}_m(\mathbb{C}), \mathfrak{sl}_m(\mathbb{C})$ , and so on. The notation  $\mathfrak{z}_{\mathfrak{h}}(X)$  denotes the centralizer of  $X$  in the Lie algebra  $\mathfrak{h}$ .

By  $J_m$  we refer to the nilpotent  $m \times m$  Jordan block (which has ones *below* the diagonal). Given a partition  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ , we write  $J(\lambda)$  to denote the block matrix

$$\begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_\ell} \end{pmatrix}.$$

A partition is always indexed in nonincreasing order, even if it is written down differently when defined. For example, if  $\mu = (4, 6, 3)$ , then  $(\mu_1, \mu_2, \mu_3) = (6, 4, 3)$ .

Throughout the paper, when a nilpotent operator  $X : \mathbb{C}^m \rightarrow \mathbb{C}^m$  of shape  $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash m$  is in the context, we write  $(e_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq \lambda_i}$  to denote a Jordan basis of  $\mathbb{C}^m$  with the property that for all  $i, j$  we have  $Xe_{ij} = e_{i,j-1}$ . (By convention,  $e_{ij} := 0$  for  $j < 1$ , so that condition makes sense.) Also, we write  $f_n, \dots, f_1$  to denote the basis of  $\mathbb{C}^n$  with the property that  $J_n f_i = f_{i-1}$ .

## 2.2 Springer Fibers

Let  $\mathfrak{h}$  be a Lie algebra, let  $\mathcal{N}_{\mathfrak{h}} \subseteq \mathfrak{h}$  be the nilpotent cone in  $\mathfrak{h}$ , and let  $\mathcal{B}_{\mathfrak{h}}$  be the variety of Borel subalgebras of  $\mathfrak{h}$ . Let  $\tilde{\mathcal{N}}_{\mathfrak{h}} = \{(\mathfrak{b}, n) \in \mathcal{B}_{\mathfrak{h}} \times \mathcal{N}_{\mathfrak{h}} : n \in \mathfrak{b}\}$ . Let  $\pi_{\mathfrak{h}} : \tilde{\mathcal{N}}_{\mathfrak{h}} \rightarrow \mathcal{N}_{\mathfrak{h}}$  be the projection onto the second coordinate. We call  $\pi_{\mathfrak{h}}$  the *Springer resolution*. For  $n \in \mathcal{N}_{\mathfrak{h}}$ , we call  $\pi_{\mathfrak{h}}^{-1}(n)$  the *Springer fiber at  $n$* .

## 2.3 Springer Fibers in $\mathfrak{gl}_m$

Let  $\mathcal{N}_m$  be the nilpotent cone in  $\mathfrak{gl}_m$ , and let  $\mathcal{B}_m$  be the variety of Borel subalgebras of  $\mathfrak{gl}_m$ . Let  $\mathfrak{h} \subseteq \mathfrak{gl}_m$  be the subalgebra of upper triangular matrices. The variety of Borel subalgebras of  $\mathfrak{gl}_m$  is  $\mathcal{B}_m = \{g\mathfrak{h}g^{-1} : g \in \mathrm{GL}_m\}$ . Thus, the Springer fiber at  $X \in \mathcal{N}_m$  is

$$\mathcal{F}_X = \{g\mathfrak{h}g^{-1} : g \in \mathrm{GL}_m; X \in g\mathfrak{h}g^{-1}\}.$$

**Definition 2.1.** A flag  $V_{\bullet}$  of  $\mathbb{C}^m$  is a sequence of subspaces

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = \mathbb{C}^m,$$

where  $\dim V_i = i$ .

We say that  $X \in \mathfrak{gl}_m$  preserves a flag  $V_{\bullet}$  if for each  $i$  we have  $XV_i \subseteq V_i$ . For  $X \in \mathfrak{gl}_m$ , we have  $[X \in \mathcal{N}_m, \text{ and } X \text{ preserves } V_{\bullet}]$  if and only if [for all  $i$ ,  $XV_{i+1} \subseteq V_i$ ].

One flag is the *standard flag*  $E_{\bullet}$ , where  $E_i$  is generated by the first  $i$  standard basis vectors. The subalgebra  $\mathfrak{h}$  is exactly the subset of  $\mathfrak{gl}_m$  which preserves  $E_{\bullet}$ . So, we can think of  $\mathcal{B}_m$  as the variety of flags of  $\mathbb{C}^m$ , via the correspondence

$$g\mathfrak{h}g^{-1} \leftrightarrow g \cdot E_{\bullet}.$$

( $\mathrm{GL}_m$  acts on the set of flags via  $(g \cdot V_{\bullet})_j := gV_j$ .) Note that  $X$  preserves  $g \cdot E_{\bullet}$  if and only if  $X \in g\mathfrak{h}g^{-1}$ . Thus, we may write the Springer fiber at  $X$  in terms of flags, as

$$\mathcal{F}_X = \{g \cdot E_{\bullet} : g \in \mathrm{GL}_m; X \in g\mathfrak{h}g^{-1}\} = \{V_{\bullet} : \forall i. XV_{i+1} \subseteq V_i\}.$$

**Theorem 2.2** ([4, 2.1]). *The irreducible components of the Springer fiber at  $J(\mu)$  are in bijection with the standard Young tableaux of shape  $\mu$ . The irreducible components are equidimensional, of dimension  $\sum_{i < j} \min(\mu_i, \mu_j) = \sum_i (i-1)\mu_i$ .*

## 2.4 Slodowy Slices

A basis for  $\mathfrak{sl}_2$  is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra  $\mathfrak{h}$  and a homomorphism  $\phi : \mathfrak{sl}_2 \rightarrow \mathfrak{h}$  sending  $(e', h', f')$  to  $(e, h, f)$ , we say that  $(e, h, f)$  is an  $\mathfrak{sl}_2$ -triple. If  $\mathfrak{h}$  is semisimple, then given any nilpotent  $e \in \mathfrak{h}$ , the Jacobson-Morozov theorem [1, 3.7.1] says that there exist  $h, f \in \mathfrak{h}$  such that  $(e, h, f)$  is an  $\mathfrak{sl}_2$ -triple.

Given  $(e, h, f)$ , we define the *Slodowy slice at  $e$*  as  $S_e = e + \mathfrak{z}_{\mathfrak{h}}(f)$ . By the Jacobson-Morozov theorem, if  $\mathfrak{h}$  is semisimple, then we can always find a Slodowy slice at any nilpotent  $e \in \mathfrak{h}$ . In particular, we can find a Slodowy slice at any nilpotent  $e \in \mathfrak{gl}_n$ , since  $\mathfrak{sl}_n \subseteq \mathfrak{gl}_n$  is semisimple and contains every nilpotent element of  $\mathfrak{gl}_n$ .

## 3 Principal $\mathfrak{sl}_2$ -triples in $\mathfrak{gl}_n$

The unique nilpotent regular element of  $\mathfrak{gl}_n$  (up to similarity) is  $J_n$ . In this section we show that, in fact, there is a unique (up to similarity) principal  $\mathfrak{sl}_2$ -triple in  $\mathfrak{gl}_n$ . Let  $e = J_n$ .

**Lemma 3.1.** *There is exactly one way to choose  $h, f \in \mathfrak{gl}_n$  so that  $(e, h, f)$  is an  $\mathfrak{sl}_2$ -triple.*

*Proof.* Note that  $[h', e'] = 2e'$ , and  $[e', f'] = h'$ , and  $[h', f'] = -2f'$ . Thus  $e, h, f$  must obey the same relations. In particular,  $he - eh = 2e$ . The matrix  $eh$  is  $h$  shifted down one, and  $he$  is  $h$  shifted left one. So,  $[h, e] = 2e$  implies that  $h_{ij} = 0$ , except when  $i = j$  or  $(i, j) = (n, 1)$ . It also implies that  $h_{ii} = h_{i-1, i-1} + 2$ , and hence  $h_{ii} = h_{11} + 2(i - 1)$ .

Similarly,  $ef - fe = h$ ; this implies that  $f_{ij} = 0$ , except when  $j = i + 1$  or  $(i, j) = (n, 1)$ . Also,  $f_{i, i+1} = f_{i+1, i+2} + h_{i+1, i+1}$ , and  $f_{1, 2} = -h_{1, 1}$ , and

$f_{n-1,n} = h_{n,n}$ . Putting these equations together,

$$\begin{aligned} h_{nn} + h_{11} &= \\ f_{n-1,n} - f_{1,2} &= \\ \sum_{i=1}^{n-2} (f_{i+1,i+2} - f_{i,i+1}) &= \\ - \sum_{i=1}^{n-2} h_{i+1,i+1}. \end{aligned}$$

That is,  $h \in \mathfrak{sl}_n$ . This shows that  $h_{11} = 1 - n, h_{22} = 3 - n, \dots, h_{nn} = n - 1$ .

Now that we have mostly determined  $h$  and  $f$ , looking at  $[e, f] = h$  and  $[h, f] = -2f$  shows that  $h_{n,1} = f_{n,1} = 0$ . So we have determined  $h$ ; it is

$$h = \begin{pmatrix} 1-n & & & & \\ & 3-n & & & \\ & & \ddots & & \\ & & & n-3 & \\ & & & & n-1 \end{pmatrix}.$$

Now, we can use our expression for  $f$  in terms of  $h$  to obtain

$$\begin{aligned} f &= \begin{pmatrix} 0 & n-1 & & & \\ & 0 & (n-1) + (n-3) & & \\ & & \ddots & \ddots & \\ & & & 0 & (n-1) + \dots + (3-n) \\ & & & & 0 \end{pmatrix} = \\ &\begin{pmatrix} 0 & (1)(n-1) & & & \\ & 0 & (2)(n-2) & & \\ & & \ddots & \ddots & \\ & & & 0 & (-2)(n-2) \\ & & & & 0 & (-1)(n-1) \\ & & & & & 0 \end{pmatrix}. \end{aligned}$$

□



## 4 Finding $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$

### 4.1 Finding the Slodowy Slice $S$

In the previous section we computed the principal  $\mathfrak{sl}_2$ -triple  $(e, h, f)$  in  $\mathfrak{gl}_n$ . Embedding this into  $\mathfrak{gl}_{m+n}$  as described previously, we obtain

$$(E, H, F) = \left( \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} \right).$$

**Lemma 4.1.**  $\mathfrak{z}_{\mathfrak{gl}_n}(f)$  is the set of upper-triangular  $X \in \mathfrak{gl}_n$  such that for all  $i, j \in \{1, \dots, n-1\}$ ,

$$X_{i+1,j+1} = \frac{j(n-j)}{i(n-i)} X_{ij}.$$

*Proof.* Let  $X \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$ . Looking at the definition of  $f$  from the previous section (and zero-padding the matrices), we see that  $(fX)_{ij} = i(n-i)X_{i+1,j}$ , and  $(Xf)_{ij} = (j-1)(n-(j-1))X_{i,j-1}$ . So, for any  $i, j \in \{1, \dots, n\}$ ,

$$i(n-i)X_{i+1,j} = (j-1)(n-(j-1))X_{i,j-1}.$$

Taking  $j = 1$ , the condition above says that for all  $i \geq 2$  we have  $X_{i,1} = 0$ . Taking  $j > 1$  and  $i < n$ , we get that for all  $i, j \in \{1, \dots, n-1\}$ ,

$$X_{i+1,j+1} = \frac{j(n-j)}{i(n-i)} X_{ij}.$$

So, every  $X \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$  is upper triangular and satisfies the condition above. Conversely, it is clear that for such  $X$  we have  $fX = Xf$ .  $\square$

**Lemma 4.2.**

$$\mathfrak{z}_{\mathfrak{gl}_{m+n}}(F) = \left\{ \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - & a & - \end{array} & Y \end{array} \right) : X \in \mathfrak{gl}_m; Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f); a, b \in \mathbb{C}^m \right\}.$$

*Proof.* Let  $Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \in \mathfrak{z}_{m+n}(F)$ . We have

$$\begin{pmatrix} 0 & Z_{12}f \\ 0 & Z_{22}f \end{pmatrix} = ZF = FZ = \begin{pmatrix} 0 & 0 \\ fZ_{21} & fZ_{22} \end{pmatrix}.$$

There is no restriction on  $Z_{11}$ . The condition  $Z_{12}f = 0$  means that all but the last column of  $Z_{12}$  must be zero, and the condition  $0 = fZ_{21}$  means that all but the first row of  $Z_{21}$  must be zero. Finally, the condition  $Z_{22}f = fZ_{22}$  means that  $Z_{22} \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$ .  $\square$

Taken together, the previous lemmas provide a nice characterization of the Slodowy slice  $S = E + \mathfrak{z}_{\mathfrak{gl}_{m+n}}(F)$ .

**Corollary 4.3.**

$$S = \left\{ \left( \begin{array}{ccc|c} & & & b \\ & X & & | \\ \hline - & a & - & | \\ & & & e + Y \end{array} \right) : a, b \in \mathbb{C}^m, X \in \mathfrak{gl}_m, Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f) \right\}.$$

## 4.2 A description of $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$

Recall the map  $S \rightarrow \mathfrak{g}$  given by  $Z \mapsto (p'(Z), Z)$ , where  $p' : \mathfrak{gl}_{m+n} \rightarrow \mathfrak{gl}_m$  is coordinate projection taking a matrix to its upper-left corner. We also have the Springer resolution  $\tilde{\mathcal{N}} \rightarrow \mathfrak{g}$ . From these two maps we define the fibred product  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ .

To obtain an explicit description of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ , we begin by finding the image of the projection  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \rightarrow S$ . Since the image of the Springer resolution is  $\mathcal{N} = \mathcal{N}_m \times \mathcal{N}_{m+n}$ , the image of the projection  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \rightarrow S$  is simply  $S' := \{Z \in S : p'(Z) \in \mathcal{N}_m, Z \in \mathcal{N}_{m+n}\}$ .

**Lemma 4.4.**

$$S' = \left\{ A_{X,a,b} := \left( \begin{array}{ccc|c} & & & b \\ & X & & | \\ \hline - & a & - & | \\ & & & e \end{array} \right) \in \mathcal{N}_{m+n} : a, b \in \mathbb{C}^m, X \in \mathcal{N}_m \right\}.$$

*Proof.* Since every element of  $\mathfrak{z}_{\mathfrak{gl}_n}(f)$  is upper triangular by Lemma 4.1, Corollary 9.10 says that if a matrix  $Z \in S$  of the form given in Corollary 4.3 is nilpotent, and the upper-left block  $p'(Z) = X$  is nilpotent as well, then  $e + Y$  must be nilpotent. Finally, Lemma 9.8 says that if  $e + Y$  is nilpotent for  $Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$ , then  $Y = 0$ .

Thus every element of  $S'$  must simply have  $e$  in its bottom-right block. So, every element of  $S'$  is of the desired form.  $\square$

This is not a fully explicit characterization of  $S'$ , since we don't say which choices of  $X$  and  $a, b \in \mathbb{C}^m$  lead to  $A_{X,a,b}$  being nilpotent. We could use Lemma 9.9 to find a necessary and sufficient condition on  $X, a, b$ ; however, the above description of  $S'$  will be good enough for our purposes.

**Corollary 4.5.**

$$\begin{aligned} S \times_{\mathfrak{g}} \tilde{\mathcal{N}} &= \\ &= \{(X, A_{X,a,b}), ((X, A_{X,a,b}), \mathfrak{b})\} \in S \times \tilde{\mathcal{N}} \cong \\ &= \{(A_{X,a,b}, \mathfrak{b}) : X \in \mathfrak{gl}_m; a, b \in \mathbb{C}^m; \mathfrak{b} \in \mathcal{B}; (X, A_{X,a,b}) \in \mathfrak{b} \cap \mathcal{N}\}. \end{aligned}$$

Define  $\pi : S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \rightarrow \mathcal{N}_m$  by  $(A_{X,a,b}, \mathfrak{b}) \mapsto X$ . We define the *n-Slodowsky-slice Springer fiber* at  $X \in \mathcal{N}_m$  to be the fiber  $\pi^{-1}(X)$ . Because  $\mathcal{B} = \mathcal{B}_m \times \mathcal{B}_{m+n}$ ,

$$\pi^{-1}(X) \cong \{(A_{X,a,b}, \mathfrak{b}_{m+n}) : a, b \in \mathbb{C}^m; A_{X,a,b} \in \mathfrak{b}_{m+n} \cap \mathcal{N}_{m+n}\} \times \{\mathfrak{b}_m : X \in \mathfrak{b}_m\}.$$

The right factor of the product is simply the usual Springer fiber  $\mathcal{F}_X$ .

Let

$$\mathcal{P}_X = \{(A_{X,a,b}, V_{\bullet}) : \forall i. A_{X,a,b} V_i \subseteq V_{i-1}\}.$$

By the correspondence between Springer fibers in  $\mathcal{B}_{m+n}$  and flags of  $\mathbb{C}^{m+n}$ , the variety  $\mathcal{P}_X$  is isomorphic to the left factor of  $\pi^{-1}(X)$ .

In the next few sections, we will find the irreducible components of  $\mathcal{P}_X$ . Then we will use this result, along with the result about the usual Springer fiber  $\mathcal{F}_X$ , to find the irreducible components of  $\pi^{-1}(X)$  and then of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ .

## 5 Strategy for Finding Components of $\mathcal{P}_{J(\lambda)}$

Let  $X = J(\lambda) \in \mathfrak{gl}_m$ , where  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ . In this section we write  $\mathcal{P} := \mathcal{P}_X$  and  $A_{a,b} := A_{X,a,b}$ . Let  $(e_{ij})_{1 \leq i \leq k, 1 \leq j \leq \lambda_i}$  be the standard basis for  $\mathbb{C}^m$ .

Let  $(f_j)_{1 \leq j \leq k}$  be the standard basis for  $\mathbb{C}^n$ . We index the bases such that  $Xe_{ij} = e_{i,j-1}$ , and  $J_n f_j = f_{j-1}$ .

For  $1 \leq w \leq \ell$  and  $0 \leq r \leq \lambda_w$  (note that we allow  $r = 0$ ; recall  $e_{i0} = 0$ ), define  $\mathcal{P}_{w,r}$  as the set of  $(A_{a,b}, V_\bullet) \in \mathcal{P}$  such that there exist  $P \in \text{GL}_m$  and  $b' \in \mathbb{C}^n$  such that  $(P^{-1}, I_n)A_{a,b}(P, I_n) = A_{e_{wr}, b'}$ .

**Lemma 5.1.**  $\mathcal{P} = \bigcup_{1 \leq w \leq k, 0 \leq r \leq \lambda_w} \mathcal{P}_{wr}$ .

*Proof.* For any  $a \in \mathbb{C}^m$ , define the linear map  $\phi_a : \mathbb{C}^{m+n} \rightarrow \mathbb{C}$  by  $e_{ij} \mapsto a_{ij}$  and  $f_j \mapsto 0$ . Note that  $A_{a,b}$  is the unique linear map  $\mathbb{C}^{m+n} \rightarrow \mathbb{C}^{m+n}$  that sends  $e_{ij}$  to  $e_{i,j-1} + \phi_a(e_{ij})f_n$ , sends  $f_{i+1}$  to  $f_i$ , and sends  $f_1$  to  $(b, 0)$ .

Let  $(A_{a,b}, V_\bullet) \in \mathcal{P}$ . Since  $X$  is nilpotent, Theorem 9.7 says that there is a change of basis  $P : \mathbb{C}^m \rightarrow \mathbb{C}^m$  such that the vectors  $Pe_{ij}$  form a Jordan basis for  $X$ , and for all but one pair  $(i, j)$  we have  $\phi_a(Pe_{ij}) = 0$ .

Since  $(Pe_{ij})_{ij}$  is a Jordan basis, we have  $(P^{-1}, I_n)A_{a,b}(P, I_n)$  for some  $a', b' \in \mathbb{C}^m$ . All that is left is to show that  $a'$  is of the form  $e_{wr}$  for some  $w$  and  $r$ . Indeed, this results from the fact that  $\phi_{a'}(e_{ij}) = \phi_a(Pe_{ij}) = 0$  for all but one pair  $(i, j)$ .  $\square$

**Lemma 5.2.**  $\mathcal{P}_{w_1 r_1} = \mathcal{P}_{w_2 r_2}$  exactly when either  $r_1 = r_2 = 0$ , or  $(\lambda_{w_1}, r_1) = (\lambda_{w_2}, r_2)$ .

*Proof.* If  $r_1 = r_2 = 0$ , we have  $e_{wr_1} = e_{wr_2} = 0$ , so by definition,  $\mathcal{P}_{w_1 r_1} = \mathcal{P}_{w_2 r_2}$ . And if  $(\lambda_{w_1}, r_1) = (\lambda_{w_2}, r_2)$ , then any matrix of the form  $A_{e_{w_1 r_1} b'}$  can be transformed to a matrix of the form  $A_{e_{w_2 r_2} b'}$  by making the change of basis that swaps  $e_{w_1 j}$  with  $e_{w_2 j}$ .

Conversely, suppose  $\mathcal{P}_{w_1 r_1} = \mathcal{P}_{w_2 r_2}$ . Clearly either  $r_1 = r_2 = 0$  or  $r_1 \neq 0$  and  $r_2 \neq 0$ . In the first case, we're done.

Otherwise, let  $k \geq 0$  be minimal such that there exist  $u, v \in \mathbb{C}^m$  with  $A_{e_{w_1 r_1}, 0}(u, 0) = (v, 0) + f_n$  and  $A_{e_{w_1 r_2}, 0}^k(v, 0) = 0$ . Obviously  $k = r_1 - 1$ . Since (by hypothesis)  $A_{e_{w_1 r_1}, 0}$  and  $A_{e_{w_2 r_2}, 0}$  differ only by a change of basis of  $\mathbb{C}^m$ , the same reasoning shows that  $k = r_2 - 1$ . Hence  $r_1 = r_2$ . Now we only need to show that  $\lambda_{w_1} = \lambda_{w_2}$ .

Let  $k' \geq 0$  be maximal such that there exist  $u, v \in \mathbb{C}^m$  with  $A_{e_{w_1 r_1}, 0}^{k'}(u, 0) = (v, 0) + f_n$ . Obviously  $k' = \lambda_{w_1} - r_1 + 1$ . Since (like  $k$ )  $k'$  is independent of  $\mathbb{C}^m$ -basis, we get  $\lambda_{w_1} - r_1 + 1 = k' = \lambda_{w_2} - r_2 + 1$ , and therefore  $\lambda_{w_1} = \lambda_{w_2}$ .  $\square$

**Lemma 5.3.** When  $\mathcal{P}_{w_1 r_1} \neq \mathcal{P}_{w_2 r_2}$ , we have  $\mathcal{P}_{w_1 r_1} \cap \mathcal{P}_{w_2 r_2} = \emptyset$ .

*Proof.* It suffices to remark that  $\{(x, y) : \exists w, r. x, y \in \mathcal{P}_{wr}\}$  is an equivalence relation. Reflexivity is the statement of Lemma 5.1, and symmetry and transitivity are obvious from the definition of  $\mathcal{P}_{wr}$ .  $\square$

Now, fix any  $w$  and  $r$ . We will find the irreducible components of  $\mathcal{P}_{w,r}$ . These will all happen to be equidimensional (with dimensions independent of  $w$  and  $r$ ), so their closures in  $\mathcal{P}$  will be irreducible components of  $\mathcal{P}$ .

Let

$$G = \{P \in \mathrm{GL}_m : PXP^{-1} = X\},$$

and

$$G_{wr} = \{A \in G : e_{wr}A = e_{wr}\}.$$

Also, define

$$\mathcal{Q}_{wr} = \{(A_{e_{wr},b}, V_\bullet) \in \mathcal{P}_{wr}\}.$$

Let  $G$  act on  $\mathcal{P}_{wr}$  by

$$P \cdot (A_{a,b}, V_\bullet) := ((P, I_n)A_{a,b}(P, I_n)^{-1}, (P, I_n)V_\bullet) = (A_{aP^{-1}, Pb}, (P, I_n)V_\bullet).$$

For any  $x \in \mathcal{Q}_{wr}$ , we have  $G_{wr} = \{g \in G : g \cdot x \in \mathcal{Q}_{wr}\}$ . So, by restriction of  $G$  to  $G_{wr}$  and  $\mathcal{P}_{wr}$  to  $\mathcal{Q}_{wr}$ , we obtain an action of  $G_{wr}$  on  $\mathcal{Q}_{wr}$ .

Consider the map  $\varphi_{wr} : \mathcal{Q}_{wr} \times G \rightarrow \mathcal{P}_{wr}$  defined by

$$(x, P) \mapsto P \cdot x.$$

Letting  $G_{wr}$  act on  $G$  by  $g \cdot h := hg^{-1}$ , we obtain an action of  $G_{wr}$  on  $\mathcal{Q}_{wr} \times G$ .

**Lemma 5.4.** *The map  $\varphi_{wr}$  is a principal  $G_{wr}$ -bundle.*

*Proof.* We need to show that  $G_{wr}$  acts freely and transitively on the fibers of  $\varphi_{wr}$ . It is obvious that  $G_{wr}$  acts freely on  $\mathcal{Q}_{wr} \times G$ ; it is enough to note that it acts freely on  $G$ . Now we check that it acts transitively.

Let  $y \in \mathcal{P}_{wr}$ . By definition of  $\mathcal{P}_{wr}$ , there is  $P_y \in G$  with  $P_y \cdot y \in \mathcal{Q}_{wr}$ .

$$\begin{aligned} \varphi_{wr}^{-1}(y) &= \{(x, P) : P \cdot x = y\} = \\ &= \{(P^{-1}y, P) : P^{-1}y \in \mathcal{Q}_{wr}\} = \\ &= \{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \cdot (P_y \cdot y) \in \mathcal{Q}_{wr}\}. \end{aligned}$$

Since  $G_{wr} = \{g \in G : g \cdot (P_y \cdot y) \in \mathcal{Q}_{wr}\}$ , the expression above becomes

$$\{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \in G_{wr}\}.$$

Setting  $Q := P^{-1}P_y^{-1}$ , and observing that  $P = P_y^{-1}Q^{-1}$ , the above becomes

$$\begin{aligned} \{(Q \cdot (P_y \cdot y), P_y^{-1}Q^{-1}) : Q \in G_{wr}\} = \\ \{Q \cdot (P_y \cdot y, P_y^{-1}) : Q \in G_{wr}\}. \end{aligned}$$

Thus the fibers are exactly the  $G_{wr}$ -orbits; or in other words,  $G_{wr}$  acts transitively on the fibers, as desired.  $\square$

Next, we will find the irreducible components  $X \subseteq \mathcal{Q}_{wr}$ , and we then argue that the irreducible components of  $\mathcal{P}_{wr}$  are of the form  $\varphi_{wr}(X \times G)$ .

Actually,  $\mathcal{Q}_{wr}$  is unnecessarily difficult to think about; it is easiest in the case  $r = 0$ . So, we change basis to make  $r = 0$ . Let  $R(r) := \lambda_w$  if  $r = 0$ , or  $r - 1$  otherwise. The point is that  $R(r) = \max\{i \in \{0, \dots, \lambda_w\} : \forall j \leq i. (e_{wr})_{wj} = 0\}$ .

Let  $m' = m + R(r) - \lambda_w$  and  $n' = n + \lambda_w - R(r)$ . Also, let  $\lambda' = (\lambda_1, \dots, \lambda_{w-1}, R(r), \lambda_{w+1}, \dots, \lambda_\ell)$ . Let  $X' = J(\lambda')$ . Let  $\mathcal{Q}' = \{(A'_{X',0,b}, V_\bullet) : \forall i. A'_{X',0,b}V_{i+1} \subseteq V_i\}$ , where

$$A'_{X',a,b} := \left( \begin{array}{c|c} X' & \begin{smallmatrix} | \\ b \\ | \end{smallmatrix} \\ \hline \begin{smallmatrix} - & a & - \end{smallmatrix} & J_{n'} \end{array} \right) \in \mathfrak{gl}_{m'+n'}.$$

**Lemma 5.5.**  $\mathcal{Q}_{wr} \cong \mathcal{Q}'$ .

*Proof.* Let  $k_i = \lambda_i$  for  $i \neq w$ , and let  $k_w = R(r)$ . Let  $(e'_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq k_i}$  be the standard Jordan basis for  $X'$ . Let  $f'_1, \dots, f'_{n'}$  be a basis for  $\mathbb{C}^{n'}$  such that  $A_{X',0,b}f'_{i+1} = f'_i$ , and  $A_{X',0,b}f_1 = (b, 0)$ . Define the linear map  $Q_{wr} : \mathbb{C}^{m'+n'} \rightarrow \mathbb{C}^{m+n}$  by:

- For all  $i$  and  $j$ ,  $e'_{ij} \mapsto e_{ij}$ .
- For  $j \in \{1, \dots, \lambda_w - R(r)\}$ ,  $f'_{n+j} \mapsto e_{w,R(r)+j}$ .
- For  $j \in \{1, \dots, n\}$ ,  $f'_j \mapsto f_j + e_{w,R(r)-n+j}$ .

It is straightforward to check that for any  $b$  such that  $A_{X,e_{wr},b}$  is nilpotent, there exists  $b'$  such that  $Q_{wr}^{-1}A_{X,e_{wr},b}Q_{wr} = A_{X',0,b'}$ . Similarly, for any  $b'$ , there exists  $b$  such that  $Q_{wr}A_{X',0,b'}Q_{wr}^{-1} = A_{X,e_{wr},b}$ .

Thus change of basis by  $Q_{wr}$  maps  $\mathcal{Q}_{wr}$  to  $\mathcal{Q}'$ , and the inverse map, change of basis by  $Q_{wr}^{-1}$ , maps  $\mathcal{Q}'$  to  $\mathcal{Q}_{wr}$ .  $\square$

The next section finds the irreducible components of  $\mathcal{Q}'$ . To make the notation nicer, we will refer to it as  $\mathcal{Q}$ , and refer to  $m', n'$  as  $m, n$  (and so on), throughout the next section.

## 6 The Components of $\mathcal{Q}$

### 6.1 Setup

Let  $X = J(\lambda) \in \mathfrak{gl}_m$ . Let  $(e_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq \lambda_i}$  be the standard (Jordan) basis for  $\mathbb{C}^m$ . Let

$$\mathcal{Q} = \{(A_{0,b}, V_\bullet) : \forall i. A_{0,b}V_{i+1} \subseteq V_i\}.$$

In this section we find the irreducible components of  $\mathcal{Q}$ .

We write  $b_{ij}$  to denote the projection of  $b \in \mathbb{C}^m$  onto  $e_{ij}$ . For each row  $i$ , let  $p_i(b) = \max\{j : b_{ij} \neq 0\}$  (by convention, the max of the empty set is zero). Then set  $q_i(b) = \lambda_i - p_i(b)$ . When it is clear enough from context where the  $b$  is coming from, we will just write  $p_i$  and  $q_i$  instead of  $p_i(b)$  and  $q_i(b)$ .

Let  $I = \{i_1 < \dots < i_s\} \subseteq \{1, \dots, \ell\}$ , and let  $(\rho_i)_{i \in I}$  be any function  $I \rightarrow \mathbb{N}_{>0}$  such that (1)  $\rho_i \leq \lambda_i$ , (2)  $\rho_i$  is decreasing with  $i$ , (3)  $\lambda_i - \rho_i$  is decreasing with  $i$ , and (4)  $\rho_i < n$ . For notational convenience—although we assign meaning to neither  $i_0$  nor  $i_{s+1}$ —we define  $q_{i_0} := n$ , and  $p_{i_{s+1}} := 0$ . Then, we define  $B_{I,(\rho_i)}$  as the set of  $b \in \mathbb{C}^m$  satisfying the following conditions.

- For all  $k \in \{1, \dots, s\}$ ,  $p_i = \rho_i$ .
- For all  $k \in \{0, \dots, s\}$ ,  $p_{i_{k+1}} = \max_{i: q_i < q_{i_k}} p_i$ .

For any  $b \in B_{I,(\rho_i)}$  we have  $p_{i_1} > \dots > p_{i_s} > p_{i_{s+1}}$ , and also  $q_{i_0} > q_{i_1} > \dots > q_{i_s}$ .

**Lemma 6.1.**  $\mathbb{C}^m = \bigcup_{I,(\rho_i)} B_{I,(\rho_i)}$ , where  $I$  ranges over all subsets of  $\{1, \dots, \ell\}$ , and  $(\rho_i)$  ranges over all maps  $I \rightarrow \mathbb{N}_{>0}$  satisfying conditions (1),(2),(3),(4). Further, none of the  $B_{I,(\rho_i)}$  is contained in the union of the others.

*Proof.* Let  $b \in \mathbb{C}^m$ . If  $\{i : q_{i_0} > q_i\}$  is the empty set, then stop. Otherwise, take any  $i_1 \in \arg \max_{i: q_{i_0} > q_i} p_i$ , and set  $\rho_{i_1} := p_{i_1}$ . If  $\{i : q_{i_1} > q_i\} = \emptyset$ ,

then stop. Otherwise, take any  $i_2 \in \arg \max_{i: q_{i_1} > q_i} p_i$ , and set  $\rho_{i_2} := p_{i_2}$ . Continuing on in this way, eventually we reach an  $s$  where  $\{i : q_{i_s} > q_i\} = \emptyset$ . Then we set  $I = \{i_1, \dots, i_s\}$ . Note that  $I, (\rho_i)$  satisfy conditions (1)–(4), and furthermore  $b \in B_{I,(\rho_i)}$ .

Now, we check that no  $B_{I,(\rho_i)}$  is contained in the union of the others. Fix  $I$  and  $(\rho_i)$ . Take any  $b \in B_{I,(\rho_i)}$  with  $p_i = \rho_i$  for  $i \in I$  and  $p_i = 0$  for  $i \notin I$ . It is clear that for any  $I'$  and  $(\rho'_i)$ , if  $I' \neq I$  or  $(\rho'_i) \neq (\rho_i)$ , then  $b \notin B_{I',(\rho'_i)}$ .  $\square$

Let  $\mathcal{Q}_{I,(\rho_i)} = \{(A_{0,b}, V_\bullet) \in \mathcal{Q} : b \in B_{I,(\rho_i)}\}$ . We will show that there is some partition  $\mu$  such that  $\mathcal{Q}_{I,(\rho_i)} \cong B_{I,(\rho_i)} \times \mathcal{F}_{J(\mu)}$ . Then we will use Theorem 2.2 to find the irreducible components of  $\mathcal{Q}_{I,(\rho_i)}$ , and the closures of these will be the irreducible components of  $\mathcal{Q}$ .

## 6.2 Study of $B_{I,(\rho_i)}$

Fix any  $I$  and  $(\rho_i)$  satisfying the conditions (1)–(4) of Lemma 6.1. As before, we write  $\{i_1 < \dots < i_s\} := I$ , and  $q_{i_0} := n$ , and  $p_{i_{s+1}} := 0$ .

First, we provide an alternative characterization of  $B_{I,(\rho_i)}$ .

**Lemma 6.2.**  *$B_{I,(\rho_i)}$  is the set of  $b \in \mathbb{C}^m$  satisfying the following conditions.*

- For all  $k \in \{1, \dots, s\}$ ,  $p_{i_k} = \rho_{i_k}$ .
- For all  $i \notin I$ ,
  - For all  $k \in \{0, \dots, s\}$  such that  $\lambda_i > q_{i_k} + p_{i_{k+1}}$ , we have  $q_{i_k} \leq q_i$ .
  - For all  $k \in \{1, \dots, s+1\}$  such that  $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$ , we have  $p_i \leq p_{i_k}$ .

*Proof.* First we show that every element of  $B_{I,(\rho_i)}$  satisfies those conditions. Let  $b \in B_{I,(\rho_i)}$ . It is clear that for each  $k$  we have  $p_{i_k} = \rho_{i_k}$ .

Let  $i \notin I$  and  $k \in \{0, \dots, s\}$  such that  $\lambda_i > q_{i_k} + p_{i_{k+1}}$ . Suppose for contradiction that  $q_i < q_{i_k}$ . Then  $p_i \leq \max_{j: q_j < q_{i_k}} p_j = p_{i_{k+1}}$ . Then  $\lambda_i = p_i + q_i < q_{i_k} + p_{i_{k+1}}$ , a contradiction. So we must have  $q_{i_k} \leq q_i$ , as desired.

Now let  $i \notin I$  and  $k \in \{1, \dots, s+1\}$  such that  $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$ . Suppose for contradiction that  $p_i > p_{i_k}$ . Then, putting this together with the first inequality, we obtain  $\lambda_i - p_i < q_{i_{k-1}} + p_{i_k} - p_{i_k}$ ; that is,  $q_i < q_{i_{k-1}}$ . Consequently,  $p_i \leq \max_{j: q_j < q_{i_{k-1}}} p_j = p_{i_k}$ , as desired.

Now we have shown that every element of  $B_{I,(\rho_i)}$  satisfies the conditions of the lemma, and we proceed to the converse. Let  $b \in \mathbb{C}^m$  satisfy the



conditions. Let  $k \in \{0, \dots, s\}$ . We need to show that  $\max_{j: q_j < q_{i_k}} p_j = p_{i_{k+1}}$ . Given the conditions (1)–(4) on  $\rho_i$ , it suffices to show that for each  $i \notin I$  with  $q_i < q_{i_k}$ , we have  $p_i \leq p_{i_{k+1}}$ . Indeed, given  $i \notin I$  with  $q_i < q_{i_k}$ , we cannot have  $\lambda_i > q_{i_k} + p_{i_{k+1}}$ , as that would imply (by hypothesis) that  $q_{i_k} \leq q_i$ . Hence  $\lambda_i \leq q_{i_k} + p_{i_{k+1}} \leq q_{i_{k-1}} + p_{i_k}$ , and consequently (by hypothesis)  $p_i \leq p_{i_k}$ .  $\square$

**Corollary 6.3.**  $B_{I,(\rho_i)}$  is the set of  $b \in \mathbb{C}^m$  satisfying the following conditions.

- For all  $k \in \{1, \dots, s\}$ ,  $p_{i_k} = \rho_{i_k}$ .
- For  $i \notin I$ ,
  - If  $\lambda_i \geq q_{i_0} + p_{i_1}$ , then  $p_i \leq \lambda_i - q_{i_0}$ .
  - If there is  $k \in \{1, \dots, s\}$  with  $q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}$ , then  $p_i \leq \min(p_{i_k}, \lambda_i - q_{i_k})$ .
  - If  $q_{i_s} + p_{i_{s+1}} > \lambda_i$ , then  $p_i \leq p_{i_{s+1}}$ .

*Proof.* Both  $q_{i_k}$  and  $p_{i_k}$  are decreasing as  $k$  increases, so this follows directly from Lemma 6.2. (Here we use that  $p_i \leq \lambda_i - q_{i_k}$  if and only if  $q_{i_k} \leq q_i$ .)  $\square$

**Corollary 6.4.**

$$B_{I,(\rho_i)} \cong \prod_{k=1}^s (\mathbb{C}^{\rho_{i_k}-1} \times (\mathbb{C} \setminus \{0\})) \times \prod_{i: \lambda_i \geq q_{i_0} + p_{i_1}} \mathbb{C}^{\lambda_i - q_{i_0}} \times \prod_{k=1}^s \prod_{\{i \notin I: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} \mathbb{C}^{\min(p_{i_k}, \lambda_i - q_{i_k})}.$$

*Proof.* Here we use the notation  $x \times y = (x, y)$ , and so on. The isomorphism sends  $b \in B_{I,(\rho_i)}$  to

$$\prod_{k=1}^s (b_{i_{k1}}, \dots, b_{i_k \rho_{i_k}}) \times \prod_{i: \lambda_i \geq q_{i_0} + p_{i_1}} (b_{i1}, \dots, b_{i, \lambda_i - q_{i_0}}) \times \prod_{k=1}^s \prod_{\{i \notin I: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} (b_{i1}, \dots, b_{i, \min(p_{i_k}, \lambda_i - q_{i_k})}).$$

Corollary 6.3 says that this is an isomorphism.  $\square$

**Corollary 6.5.**

$$\dim B_{I,(\rho_i)} = \sum_{i: \lambda_i \geq q_{i_0} + \rho_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^s \sum_{\{i: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}).$$

*Proof.* Immediate from Corollary 6.4.  $\square$

### 6.3 Study of $\mathcal{Q}_{I,(\rho_i)}$

Fix any  $I$  and  $(\rho_i)$  satisfying the conditions (1)–(4) of Lemma 6.1. In this subsection we find the irreducible components of  $\mathcal{Q}_{I,(\rho_i)}$ .

We will find a partition  $\mu$ , depending only on  $I$  and  $(\rho_i)$ , such that for every  $b \in B_{I,(\rho_i)}$ , the matrix  $A_{0,b}$  is similar to  $J(\mu)$ . By finding an algebraic map taking  $b \in B_{I,(\rho_i)}$  to a Jordan basis for  $A_{0,b}$ , we will put  $\mathcal{Q}_{I,(\rho_i)}$  in isomorphism with the product  $B_{I,(\rho_i)} \times \mathcal{F}_{J(\mu)}$ .

So, now we find a Jordan basis for  $A_{0,b}$ . As before, we write  $\{i_1 < \dots < i_s\} := I$ , and  $q_{i_0} := n$ , and  $p_{i_{s+1}} := 0$ . We will also write, somewhat abusively,  $b \in \mathbb{C}^{m+n}$  to refer to the vector  $(b, 0) \in \mathbb{C}^m \times \mathbb{C}^n$ .

Given a vector  $v$  such that  $A^k v \neq 0$  but  $A^{k+1} v = 0$ , we refer to the list of vectors  $[v, A_{0,b} v, \dots, A_{0,b}^k v]$  as the *chain starting with  $v$* . Given a vector  $v = \sum_{ij} v_{ij} e_{ij} \in \mathbb{C}^{m+n}$ , we define the left-shift operator  $v \ll k := \sum_{ij} v_{ij} e_{i,j-k}$ . If (and only if)  $v \in \langle e_{ij} \rangle_{i,j: j \leq \lambda_i - k}$ , then we define the right-shift operator  $v \gg k := \sum_{ij} v_{ij} e_{i,j+k}$ .

**Lemma 6.6.** *For any  $b \in B_{I,(\rho_i)}$ , the following vectors give a Jordan basis for  $A_{0,b}$ . (For convenience, we write  $A := A_{0,b}$  in this lemma and proof.)*

- For  $i \notin I$ , the chain of length  $\lambda_i$  beginning with  $e_{i,\lambda_i}$
- The chain of length  $q_{i_0} + p_{i_1}$  beginning with  $f_n - (A^{n+p_{i_1}} f_n \gg n + p_{i_1})$
- For  $k \in \{1, \dots, s\}$ , the chain of length  $q_{i_k} + p_{i_{k+1}}$  beginning with  $v_{i_k} - (A^{q_{i_k} + p_{i_{k+1}}} v_{i_k} \gg q_{i_k} + p_{i_{k+1}})$ , where  $v_{i_k} := A^{n-q_{i_k}} f_n - \sum_{j=1}^{p_{i_k}} b_{i_k j} e_{i,j+q_{i_k}}$

*Proof.* There are three things to check: (1) the chains are no longer than their claimed lengths, (2) the claimed lengths sum to  $m + n$ , and (3) the span of the chains is  $\mathbb{C}^{m+n}$ .

*Proof of (1).* It is obvious that a chain beginning with  $e_{i,\lambda_i}$  has length  $\lambda_i$ .

Now consider the chain beginning with  $f_n - (A^{n+p_{i_1}} f_n \gg n + p_{i_1})$ . Note that  $A^n f_n = b$ , so  $A^{n+p_{i_1}} f_n = A^{p_{i_1}} b = b \ll p_{i_1}$ . By shifting  $b$  left  $p_{i_1}$  times, we zero out all the rows  $i$  where  $q_i < n$ . (More formally speaking, for all  $i$  and  $j$  such that  $q_i < n$ , we have  $(b \ll p_{i_1})_{ij} = 0$ .) This ensures that the operation of shifting  $b \ll p_{i_1}$  right  $n + p_{i_1}$  times is defined, and thus it is invertible by shifting left  $n + p_{i_1}$  times. That is,

$$\begin{aligned} A^{n+p_{i_1}} f_n &= \\ b \ll p_{i_1} &= \\ ((b \ll p_{i_1}) \gg n + p_{i_1}) \ll n + p_{i_1} &= \\ A^{n+p_{i_1}} (A^{n+p_{i_1}} f_n \gg n + p_{i_1}). \end{aligned}$$

This shows that the chain has length at most  $n + p_{i_1}$ , as desired.

Now, let  $k \in \{1, \dots, s\}$ . Since  $b \in B_{I,(\rho_i)}$ , we have  $q_{i_k} < n$ , and thus the definition of  $v_{i_k}$  makes sense—i.e., the exponent of  $A$  is nonnegative. We consider the chain beginning with  $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \gg q_{i_k} + p_{i_{k+1}})$ . Note that  $A^{q_{i_k}} v_{i_k}$  is just  $b$  with row  $i_k$  zeroed out. For brevity, we write  $b_{i_k} := A^{q_{i_k}} v_{i_k}$ . For each  $i$  with  $q_i < q_{i_k}$ , the vector  $b_{i_k} \ll p_{i_{k+1}}$  has row  $i$  zeroed out. This ensures that the operation of shifting  $b_{i_k} \ll p_{i_{k+1}}$  right  $q_{i_k} + p_{i_{k+1}}$  times is defined, and thus it is invertible by shifting left  $q_{i_k} + p_{i_{k+1}}$  times. That is,

$$\begin{aligned} A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} &= \\ b_{i_k} \ll p_{i_{k+1}} &= \\ ((b_{i_k} \ll p_{i_{k+1}}) \gg q_{i_k} + p_{i_{k+1}}) \ll q_{i_k} + p_{i_{k+1}} &= \\ A^{q_{i_k}+p_{i_{k+1}}} (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \gg q_{i_k} + p_{i_{k+1}}). \end{aligned}$$

This shows that the chain has length at most  $q_{i_k} + p_{i_{k+1}}$ , as desired.  $\square$

*Proof of (2).* The sum of the lengths is

$$\sum_{i \notin I} \lambda_i + \sum_{k=0}^s (q_{i_k} + p_{i_{k+1}}) = n + \sum_i (q_i + p_i) = m + n.$$

$\square$

*Proof of (3).* Let  $W$  be the span of the chains listed. We need to show that  $W = \mathbb{C}^{m+n}$ . Because every  $i \in I$  satisfies  $q_i < n$ , clearly  $\langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$ .

We claim that  $f_n \in W$  as well. To see this, consider the chain beginning with  $f_n - (A^{n+p_{i_1}} f_n \gg n + p_{i_1})$ . As explained in the proof of (1), we have  $A^{n+p_{i_1}} f_n \in \langle e_{ij} \rangle_{i,j:q_i \geq n}$ . Consequently,  $(A^{n+p_{i_1}} f_n \gg n + p_{i_1}) \in \langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$ . Because  $f_n - (A^{n+p_{i_1}} f_n \gg n + p_{i_1}) \in W$ , this implies that  $f_n \in W$ .

Because  $AW \subseteq W$  (obvious, since  $W$  is a span of chains), the fact that  $f_n \in W$  implies that  $f_i \in W$  for each  $i$ , and also  $b \ll l \in W$  for each  $l \geq 0$ .

Now we are left with showing that  $\langle e_{ij} \rangle_{i,j:q_i < n} \subseteq W$ . It suffices to show that  $e_{i,\lambda_i} \in W$  for each  $i$  with  $q_i < n$ . This is obvious for  $i \notin I$ . So, we just need to show that  $e_{i_k, \lambda_{i_k}} \in W$  for each  $k \in \{1, \dots, s\}$ . We do this inductively; fix  $k$ , and suppose we have already shown that  $e_{i_{k'}, \lambda_{i_{k'}}} \in W$  for  $k' < k$ . We will show that  $e_{i_k, \lambda_{i_k}} \in W$ .

Since  $AW \subseteq W$ , and  $b_{i_k, p_{i_k}} \neq 0$  by definition of  $p_{i_k}$ , it suffices to show that  $\sum_{j=1}^{p_{i_k}} b_{i_k, j} e_{i_k, j+q_{i_k}} \in W$ . To see that  $\sum_j b_{i_k, j} e_{i_k, j+q_{i_k}} \in W$ , consider the chain beginning with  $v_{i_k} - (A^{q_{i_k}+p_{i_k+1}} v_{i_k} \gg q_{i_k} + p_{i_k+1})$ . As explained in the proof of (1),  $A^{q_{i_k}+p_{i_k+1}} v_{i_k} \in \langle e_{ij} \rangle_{i,j:q_i \geq q_{i_k}}$ . And since  $A^{q_{i_k}+p_{i_k+1}} v_{i_k}$  has row  $i_k$  zeroed out, in fact  $A^{q_{i_k}+p_{i_k+1}} v_{i_k} \in \langle e_{ij} \rangle_{i,j:i \neq i_k \wedge q_i \geq q_{i_k}}$ . Therefore,  $A^{q_{i_k}+p_{i_k+1}} v_{i_k} \gg q_{i_k} + p_{i_k+1} \in \langle e_{ij} \rangle_{i,j:i \neq i_k \wedge q_i \geq q_{i_k}}$ . By our inductive hypothesis,  $\langle e_{ij} \rangle_{i,j:i \neq i_k \wedge q_i \geq q_{i_k}} \subseteq W$ , and consequently  $A^{q_{i_k}+p_{i_k+1}} v_{i_k} \gg q_{i_k} + p_{i_k+1} \in W$ . Since we know that  $v_{i_k} - (A^{q_{i_k}+p_{i_k+1}} v_{i_k} \gg q_{i_k} + p_{i_k+1}) \in W$ , this implies that  $v_{i_k} \in W$ . Because  $A^{n-q_{i_k}} f_n \in W$ , this implies that  $\sum_j b_{i_k, j} e_{i_k, j+q_{i_k}} \in W$ , as desired.  $\square$

$\square$

Let  $\mu = (\mu_1, \dots, \mu_{\ell+1})$  be the shape of the Jordan basis given in the lemma. Let  $(\mathcal{F}_{\mu, \alpha})_{\alpha \in \text{SYT}(\mu)}$  be the irreducible components of the Springer fiber  $\mathcal{F}_{J(\mu)}$ .

Given a zero-indexed list  $L = [L_0, \dots, L_{\ell-1}]$ , define  $\gamma(L) = \sum_i i L_i$ . We are interested in  $\gamma$  because the dimension of each  $\mathcal{F}_{\mu, \alpha}$  is  $\gamma(\mu) = \gamma([\mu_1, \dots, \mu_{\ell+1}])$ .

**Lemma 6.7.**

$$\gamma(\mu) = \gamma([0, \lambda_1, \dots, \lambda_\ell]) - \left[ \sum_{i: \lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^s \sum_{\{i: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}) \right].$$

*Proof.* We have

$$\mu = [\dots, q_{i_0} + p_{i_1}, \dots, q_{i_1} + p_{i_2}, \dots, \dots, q_{i_s} + p_{i_{s+1}}, \dots].$$

Let  $L'$  be the result of taking  $\mu$  and, for each  $k$ , replacing an occurrence of  $q_{i_k} + p_{i_{k+1}}$  by  $q_{i_k} + p_{i_k}$ ; that is,

$$L' = [\dots, q_{i_0}, \dots, q_{i_1} + p_{i_1}, \dots, q_{i_2} + p_{i_2}, \dots, \dots, q_{i_s} + p_{i_s}, \dots].$$

We can transform  $\mu$  into  $L'$  by just ‘moving’ each  $p_{i_k}$  to the right by  $1 + \#\{i \notin I : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}$  slots. So,

$$\begin{aligned} \gamma(L') - \gamma(\mu) &= \sum_{k=1}^s p_{i_k} (1 + \#\{i \notin I : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}) = \\ &= \sum_{k=1}^s p_{i_k} \cdot \#\{i : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}. \end{aligned}$$

Let  $L = [q_{i_0}, \lambda_1, \dots, \lambda_\ell]$ . Now we consider how to transform  $L'$  into  $L$ . First we shift  $q_{i_0}$  to the left by  $\#\{i \notin I : \lambda_i \geq q_{i_0} + p_{i_1}\}$  slots. Then we leave  $q_{i_0}$  in place and sort the rest of the list. This entails shifting each  $q_{i_k} + p_{i_k}$  to the left by  $\#\{i \notin I : q_{i_k} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}$  slots. Shifting  $q_{i_k} + p_{i_k}$  to the left one slot, by swapping it with  $\lambda_i$ , changes the value of  $\gamma$  by  $\lambda_i - (q_{i_k} + p_{i_k})$ . To go from  $L'$  to  $L$ , we can just make these swaps repeatedly. So,

$$\begin{aligned} \gamma(L) - \gamma(L') &= \\ &= \sum_{i \notin I : \lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^s \sum_{\{i \notin I : q_{i_k} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} (\lambda_i - (q_{i_k} + p_{i_k})). \end{aligned}$$

Now, we put these two results together to get  $\gamma(L) - \gamma(\mu)$ .

$$\begin{aligned}
& \gamma(L) - \gamma(\mu) = \\
& [\gamma(L) - \gamma(L')] + [\gamma(L') - \gamma(\mu)] = \\
& \left[ \sum_{i: \lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^s \sum_{\{i: q_{i_k} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} (\lambda_i - (q_{i_k} + p_{i_k})) \right] + \\
& \left[ \sum_{k=1}^s \sum_{\{i: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} p_{i_k} \right] = \\
& \sum_{i: \lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^s \sum_{\{i: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} p_{i_k} + \\
& \sum_{k=1}^s \sum_{i: q_{i_k} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}} (\lambda_i - q_{i_k}) = \\
& \sum_{i: \lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^s \sum_{\{i: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}).
\end{aligned}$$

Since  $\gamma(L) = \gamma([0, \lambda_1, \dots, \lambda_\ell])$ , the equation above implies the desired result.  $\square$

**Lemma 6.8.**  $\mathcal{Q}_{I,(\rho_i)} \cong B_{I,(\rho_i)} \times \mathcal{F}_{J(\mu)}$ .

*Proof.* For each  $b \in B_{I,(\rho_i)}$ , let  $P_b$  be the change-of-basis matrix, with columns given by the Jordan basis of Lemma 6.6, such that  $J(\mu) = P_b^{-1} A_{0,b} P_b$ . From looking at the Jordan basis, it is clear that the map  $P : B_{I,(\rho_i)} \rightarrow \text{GL}_{m+n}$ , given by  $b \mapsto P_b$ , is algebraic.

The Springer fiber at  $A_{0,b}$  is  $\{P_b V_\bullet : V_\bullet \in \mathcal{F}_{J(\mu)}\}$ . So, there is an isomorphism  $B_{I,(\rho_i)} \times \mathcal{F}_{J(\mu)} \rightarrow \mathcal{Q}_{I,(\rho_i)}$  given by

$$(b, V_\bullet) \mapsto (P_b J(\mu) P_b^{-1}, P_b V_\bullet),$$

with inverse

$$(A_{0,b}, V_\bullet) \mapsto (b, P_b^{-1} V_\bullet).$$

$\square$

**Corollary 6.9.** *For  $\alpha \in \text{SYT}(\mu)$ , let  $\mathcal{Q}_{I,(\rho_i),\alpha}$  be the subvariety of  $\mathcal{Q}_{I,(\rho_i)}$  corresponding to  $B_{I,(\rho_i)} \times \mathcal{F}_{\mu,\alpha}$  via the isomorphism of Lemma 6.8. The subvarieties  $(\mathcal{Q}_{I,(\rho_i),\alpha})_{\alpha \in \text{SYT}(\mu)}$  are the irreducible components of  $\mathcal{Q}_{I,(\rho_i)}$ . Each has dimension  $\gamma([0, \lambda_1, \dots, \lambda_\ell])$ .*

*Proof.* We know that  $B_{I,(\rho_i)}$  is irreducible by Corollary 6.4. Then, the fact that the  $B_{I,(\rho_i)} \times \mathcal{F}_{\mu,\alpha}$  are the irreducible components of  $B_{I,(\rho_i)} \times \mathcal{F}_{J(\mu)}$  just follows from the fact that the  $\mathcal{F}_{\mu,\alpha}$  are the irreducible components of  $\mathcal{F}_{J(\mu)}$ .

To get the dimension, we add the dimension of  $B_{I,(\rho_i)}$  to the dimension of  $\mathcal{F}_{\mu,\alpha}$ . We get the dimension of  $B_{I,(\rho_i)}$  from Corollary 6.5, and we get the dimension of  $\mathcal{F}_{\mu,\alpha}$  from Lemma 6.7. Adding them together, things cancel out and we get  $\gamma([0, \lambda_1, \dots, \lambda_\ell])$ .  $\square$

## 6.4 Conclusion

**Theorem 6.10.** *The irreducible components of  $\mathcal{Q}$  are the closures of the subvarieties  $\mathcal{Q}_{I,(\rho_i),\alpha}$ , as we let  $I, (\rho_i)$  range over all possibilities satisfying the conditions (1)–(4) of Lemma 6.1, and we let  $\alpha \in \text{SYT}(\mu(I, (\rho_i)))$ .*

*Proof.* By Corollary 6.9, the  $\mathcal{Q}_{I,(\rho_i),\alpha}$  are irreducible and equidimensional. Then Lemma 6.1 says that their union is  $\mathcal{Q}$ , and none is contained in the union of the others.  $\square$

## 7 The Components of $\mathcal{P}_{J(\lambda)}$

As before, we write  $X = J(\lambda) \in \mathfrak{gl}_m$ , where  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ . We write  $\mathcal{P} := \mathcal{P}_{J(\lambda)}$ , and  $A_{a,b} := A_{J(\lambda),a,b}$ . And as before,  $(e_{ij})_{ij}$  and  $(f_j)_j$  form the standard basis for  $\mathbb{C}^{m+n}$ .

### 7.1 The Components of $\mathcal{Q}_{wr}$

Fix any  $w$  and  $r$ . Recall from Section 5 the variety  $\mathcal{Q}_{wr} = \{(A_{e_{wr},b}, V_\bullet) \in \mathcal{P}\}$ . Lemma 5.5 says that

$$\mathcal{Q}_{wr} \cong \mathcal{Q}' := \{(A'_{X',0,b}, V_\bullet) : \forall i. A'_{X',0,b} V_{i+1} \subseteq V_i\},$$

where  $X' = J(\lambda')$ , and  $\lambda' = (\lambda_1, \dots, R(r), \dots, \lambda_\ell)$ , and  $m' = m - \lambda_w + R(r)$ , and  $n' = n + \lambda_w - R(r)$ .

Let  $(\mathcal{Q}'_{I,(\rho_i),\alpha})_{I,(\rho_i),\alpha}$  be the irreducible components of  $\mathcal{Q}'$  given by Theorem 6.10. Write  $\mathcal{Q}_{w,r,I,(\rho_i),\alpha}$  to denote the irreducible component of  $\mathcal{Q}_{wr}$  corresponding to  $\mathcal{Q}'_{I,(\rho_i),\alpha}$  via the isomorphism  $\mathcal{Q}_{wr} \cong \mathcal{Q}'$  of Lemma 5.5.

**Theorem 7.1.** *The irreducible components of  $\mathcal{Q}_{wr}$  are the subvarieties  $\mathcal{Q}_{w,r,I,(\rho_i),\alpha}$ . Each has dimension  $\sum_{i \leq j} \min(\lambda'_i, \lambda'_j)$ , where  $\lambda' = (\lambda_1, \dots, \lambda_{w-1}, R(r), \lambda_{w+1}, \dots, \lambda_\ell)$ .*

*Proof.* From the foregoing discussion, it is clear that they are indeed the irreducible components. To calculate the dimension, we refer to Corollary 6.9, which says the dimension is  $\gamma([0, \lambda'_1, \dots, \lambda'_\ell])$ .  $\square$

## 7.2 The Varieties $G_{wr}$ and $G$

Fix  $w, r$ . Recall from Section 5 the groups  $G = \{P \in \mathrm{GL}_m : P^{-1}XP = X\}$  and  $G_{wr} = \{A \in G : e_{wr}A = e_{wr}\}$ .

**Lemma 7.2.**  *$G$  is irreducible and has dimension  $\sum_{ij} \min(\lambda_i, \lambda_j)$ .*

*Proof.* The closure of  $G$  in  $\mathfrak{gl}_m$  is  $\mathfrak{z}_{\mathfrak{gl}_m}(X)$ . Lemma 9.3 says that  $\mathfrak{z}_{\mathfrak{gl}_m}(X)$  is isomorphic to  $\mathbb{C}^{\sum_{ij} \min(\lambda_i, \lambda_j)}$ .  $\square$

**Lemma 7.3.**  *$G_{wr}$  is irreducible and has dimension  $\sum_{ij} \min(\lambda_i, \lambda'_j)$ , where  $\lambda' = (\lambda_1, \dots, \lambda_{w-1}, R(r), \lambda_{w+1}, \dots, \lambda_\ell)$ .*

*Proof.* The closure of  $G_{wr}$  in  $\mathfrak{gl}_m$  is  $V = \{Y \in \mathfrak{z}_{\mathfrak{gl}_m}(X) : e_{wr}Y = e_{wr}\}$ . In the case that  $r = 0$ , the constraint that  $e_{wr}Y = e_{wr}$  is no constraint at all, so we have  $G_{wr} = G$ , and the result follows from Lemma 7.2.

In the case that  $r > 0$ , observe that  $V = \{Y + I : Y \in \mathfrak{z}_{\mathfrak{gl}_m}(X), e_{wr}Y = 0\}$ . The constraint  $e_{wr}Y = 0$  just says that a certain row of  $Y$ , the row with index  $\sum_{i \leq w} \lambda_i - R(r)$ , must be all zeroes. So, the set of  $Y$  such that  $Y + I \in V$  is the set described by Corollary 9.4. Hence  $V \cong \mathbb{C}^{\sum_{ij} \min(\lambda_i, \lambda'_j)}$ .  $\square$

## 7.3 The Components of $\mathcal{P}_{wr}$

Recall from section 5 the subvarieties  $\mathcal{P}_{wr} \subseteq \mathcal{P}$ . From Lemma 5.4 we have the principal  $G_{wr}$ -bundle  $\varphi_{wr} : \mathcal{Q}_{wr} \times G \rightarrow \mathcal{P}_{wr}$ .

**Theorem 7.4.** *Every irreducible component of  $\mathcal{P}_{wr}$  is the closure of some subvariety of the form  $\varphi_{wr}(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G)$ , and the closure of each subvariety of this form is an irreducible component. Each has dimension  $\sum_{i \leq j} \min(\lambda_i, \lambda_j)$ .*



**Remark 7.5.** *Theorem 7.4 does not say “the irreducible components of  $\mathcal{P}_{wr}$  are the closures of the subvarieties  $\varphi_{wr}(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G)$ ”, as that would seem to suggest some claim about distinctness. Further analysis is required to determine which ones are distinct.*

*Proof.* Together, Theorem 6.10 and Lemma 7.2 tell us that the  $\mathcal{Q}_{w,r,I,(\rho_i),\alpha}$  are irreducible, and their union is  $\mathcal{Q}_{wr}$ . Hence their images are irreducible, and the surjectivity of  $\varphi_{wr}$  implies that their union of their images is  $\mathcal{P}_{wr}$ .

Now, to verify that each image is an irreducible component, we need only verify that they are equidimensional. We use the result of Lemma 5.4, namely that  $\varphi_{wr}$  is a principal  $G_{wr}$ -bundle.

Let  $V_{w,r,I,(\rho_i),\alpha}$  be the closure of  $\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G$  in  $\mathcal{Q}_{wr} \times G$  under action by  $G_{wr}$ . Clearly  $\varphi_{wr}(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G) = \varphi_{wr}(V_{w,r,I,(\rho_i),\alpha})$ . So, we have only to compute the dimension of  $\varphi(V_{w,r,I,(\rho_i),\alpha})$ .

This is easy, because  $\varphi_{wr}$  (and, consequently, the restriction of  $\varphi_{wr}$  to  $V_{w,r,I,(\rho_i),\alpha}$ ) is a principal  $G_{wr}$ -bundle, and principal bundles play nicely with dimensions. That is, we can conclude that

$$\dim \varphi_{wr}(V_{w,r,I,(\rho_i),\alpha}) = \dim V_{w,r,I,(\rho_i),\alpha} - \dim G_{wr}.$$

Since  $\dim(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G) = \dim(\mathcal{Q}_{wr} \times G)$  by Theorem 7.1, we know that  $\dim V_{w,r,I,(\rho_i),\alpha} = \dim(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G) = \dim \mathcal{Q}_{w,r,I,(\rho_i),\alpha} + \dim G$ . By the equation above then, we get

$$\dim \varphi_{wr}(V_{w,r,I,(\rho_i),\alpha}) = \dim \mathcal{Q}_{w,r,I,(\rho_i),\alpha} + \dim G - \dim G_{wr}.$$

We already calculated those dimensions in Theorem 7.1, Lemma 7.2, and Lemma 7.3 respectively. Referring to those, we get

$$\dim \varphi_{wr}(V_{w,r,I,(\rho_i),\alpha}) = \sum_{i \leq j} \min(\lambda'_i, \lambda'_j) + \sum_{ij} \min(\lambda_i, \lambda_j) - \sum_{ij} \min(\lambda_i, \lambda'_j). \quad (1)$$

Now,

$$\sum_{i \leq j} \min(\lambda'_i, \lambda'_j) = \sum_{i \leq j} \min(\lambda_i, \lambda_j) - \sum_j \min(\lambda_w, \lambda_j) + \sum_j \min(\lambda_w - r, \lambda_j),$$

and

$$\sum_{ij} \min(\lambda_i, \lambda'_j) = \sum_{ij} \min(\lambda_i, \lambda_j) - \sum_i \min(\lambda_i, \lambda_w) + \sum_i \min(\lambda_i, \lambda_w - r).$$

Substituting these into the RHS of Equation (1), things cancel out, and we get

$$\sum_{i \leq j} \min(\lambda_i, \lambda_j) + \sum_{ij} \min(\lambda_i, \lambda_j) - \sum_{ij} \min(\lambda_i, \lambda_j) = \sum_{i \leq j} \min(\lambda_i, \lambda_j).$$

□

## 7.4 The Components of $\mathcal{P}$

**Corollary 7.6.** *Every irreducible component of  $\mathcal{P}$  is the closure of some subvariety of the form  $\varphi_{wr}(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G)$ , and the closure of each subvariety of this form is an irreducible component. Each has dimension  $\sum_{i \leq j} \min(\lambda_i, \lambda_j)$ .*

*Proof.* Follows directly from Theorem 7.4. □

## 8 The Components of $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$

Recall from Corollary 4.5 that

$$S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \cong \{(A_{X,a,b}, U_{\bullet}, V_{\bullet}) : \forall i. XU_{i+1} \subseteq U_i; \forall i. A_{X,a,b}V_{i+1} \subseteq V_i\}.$$

We defined  $\pi : S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \rightarrow \mathcal{N}_m$  by  $(A_{X,a,b}, U_{\bullet}, V_{\bullet}) \mapsto X$ , and called  $\pi^{-1}(X)$  the  $n$ -Slodowy-slice Springer fiber at  $X$ .

**Theorem 8.1.**  $\pi^{-1}(J(\lambda)) \cong \mathcal{P}_{J(\lambda)} \times X_{\lambda}$ . *Each irreducible component of  $\pi^{-1}(J(\lambda))$  has dimension  $\sum_{ij} \min(\lambda_i, \lambda_j)$ .*

*Proof.* The isomorphism sends  $(A_{X,a,b}, U_{\bullet}, V_{\bullet})$  to  $((A_{X,a,b}, U_{\bullet}), V_{\bullet})$ . The irreducible components of  $\mathcal{P}_{J(\lambda)}$  are given by Corollary 7.6. Those of  $X_{\lambda}$  are given by Theorem 2.2. Adding the dimensions gives

$$\sum_{i \leq j} \min(\lambda_i, \lambda_j) + \sum_{i < j} \min(\lambda_i, \lambda_j) = \sum_{ij} \min(\lambda_i, \lambda_j).$$

□

Let  $\lambda$  be any partition of  $m$ . Let  $\mathrm{GL}_m \times \{I_n\}$  act on  $\pi^{-1}(J(\lambda))$  by conjugation; that is,

$$(P, I_n) \cdot (A_{X,a,b}, U_{\bullet}, V_{\bullet}) :=$$

$$((P, I_n)A_{X,a,b}(P, I_n)^{-1}, (P, I_n)U_\bullet, (P, I_n)V_\bullet).$$

Let

$$K = \{(g, I_n) \in \mathrm{GL}_m \times \{I_n\} : gXg^{-1} = X\}.$$

Define  $\phi_\lambda : \pi^{-1}(J(\lambda)) \times (\mathrm{GL}_m \times \{I_n\}) \rightarrow S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$  by  $\phi_\lambda(x, g) = g \cdot x$ .

**Lemma 8.2.** *For each partition  $\lambda$  of  $m$ , the map  $\phi_\lambda$  is a principal  $K$ -bundle.*

*Proof.* Analogous to Lemma 5.4.  $\square$

Let  $(C_{\lambda,\beta})_\beta$  be the irreducible components of  $\pi^{-1}(J(\lambda))$ . These are described by Theorem 8.1.

**Theorem 8.3.** *Every irreducible component of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$  is the closure of some subvariety of the form  $\phi_\lambda(C_{\lambda,\beta} \times (\mathrm{GL}_m \times \{I_n\}))$ , and the closure of each subvariety of this form is an irreducible component. Each has dimension  $m^2$ .*

*Proof.* This is analogous to Theorem 7.4; the only difference is the dimension calculation. This time,

$$\begin{aligned} \dim \phi_\lambda(C_{\lambda,\beta} \times (\mathrm{GL}_m \times \{I_n\})) &= \\ \dim C_{\lambda,\beta} + \dim \mathrm{GL}_m - \dim K &= \\ \sum_{ij} \min(\lambda_i, \lambda_j) + m^2 - \sum_{ij} \min(\lambda_i, \lambda_j) &= \\ m^2. \end{aligned}$$

The dimension of  $C_{\lambda,\beta}$  comes from Theorem 7.4; the dimension of  $\mathrm{GL}_m$  is obvious; and the dimension of  $K$  comes from Lemma 9.3.  $\square$

## 9 Linear Algebra Facts

In this section we prove linear algebra facts that were used earlier. They are confined to this section to avoid interrupting the rest of the paper.

### 9.1 The Centralizer of a Nilpotent Matrix

**Definition 9.1.** *A matrix  $Y$  is Toeplitz if it is constant along bands parallel to the main diagonal. That is,  $\forall i, j, k. Y_{ij} = Y_{i+k, j+k}$ .*

**Definition 9.2.** An  $m \times n$  matrix  $Y$  is lower-left Toeplitz if it is Toeplitz and, in addition, we have  $y_{n-i,j-1} = 0$  whenever  $i + j \geq \min(m, n)$ .

That is,  $Y$  is lower-left Toeplitz if it is Toeplitz, and the only nonzero entries are those with Manhattan distance less than  $\min(m, n)$  from the entry in the bottom-left corner. In yet other words, all but the leftmost (equivalently, bottommost)  $\min(m, n)$  diagonal bands are zero.

**Lemma 9.3.** Let  $\lambda = (\lambda_1, \dots, \lambda_k)$  be a partition of  $m$ . The centralizer of  $J(\lambda)$  in  $\mathfrak{gl}_m$  is the subalgebra consisting of matrices

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix},$$

where each  $M_{ij}$  is a  $\lambda_i \times \lambda_j$  matrix, such that each  $M_{ij}$  is lower-left Toeplitz.

*Proof.* Let

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix}.$$

We need to show that  $J(\lambda)M = MJ(\lambda)$  if and only if each  $M_{ij}$  is lower-left Toeplitz.

We have

$$J(\lambda)M = \begin{pmatrix} J_{\lambda_1}M_{11} & \cdots & J_{\lambda_1}M_{1k} \\ \vdots & & \vdots \\ J_{\lambda_k}M_{k1} & \cdots & J_{\lambda_k}M_{kk} \end{pmatrix}, \text{ and } MJ(\lambda) = \begin{pmatrix} M_{11}J_{\lambda_1} & \cdots & M_{1k}J_{\lambda_k} \\ \vdots & & \vdots \\ M_{k1}J_{\lambda_1} & \cdots & M_{kk}J_{\lambda_k} \end{pmatrix}.$$

So, we have  $J(\lambda)M = MJ(\lambda)$  if and only if  $\forall i, j. J_{\lambda_i}M_{ij} = M_{ij}J_{\lambda_j}$ . Multiplying on the left by  $J_{\lambda_i}$  just shifts each row down by one, and multiplying on the right by  $J_{\lambda_j}$  shifts each column left by one. The matrices for which left-shifting gives the same result as down-shifting are exactly the lower-left Toeplitz matrices.  $\square$

**Corollary 9.4.** Let  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  be a partition of  $m$ . Let  $w \in \{1, \dots, \ell\}$  and  $R \in \{0, \dots, \lambda_w - 1\}$ . Let  $i = [\sum_{j \leq w} \lambda_j] - R$ . The set of  $M \in \mathfrak{gl}_m(J(\lambda))$  such that the  $i$ th row of  $M$  is equal to zero is the set of matrices

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1\ell} \\ \vdots & & \vdots \\ M_{\ell 1} & \cdots & M_{\ell\ell} \end{pmatrix},$$

where each  $M_{ij}$  is a  $\lambda_i \times \lambda_j$  matrix, such that:

- For  $i \neq w$ ,  $M_{ij}$  is lower-left Toeplitz
- Each  $M_{wj}$  is of the form

$$M_{wj} = \begin{pmatrix} 0 \\ M'_{wj} \end{pmatrix},$$

where  $M'_{wj}$  is a  $R \times \lambda_j$  lower-left Toeplitz matrix.

## 9.2 A ‘Standardization’ Fact about Jordan Bases

**Lemma 9.5.** *For any finite-dimensional  $V$ , nilpotent  $A : V \rightarrow V$ , and linear  $f : V \rightarrow \mathbb{C}$ , there is a Jordan basis  $e_{ij}$  for  $A$  such that there is at most one  $i$  such that there exists  $j$  such that  $f(e_{ij}) \neq 0$ .*

*Proof.* For any Jordan basis  $(e_{ij})_{ij}$  of  $A$ , define

$$S((e_{ij})_{1 \leq i \leq k, 1 \leq j \leq \lambda_i}) := \sum_i \begin{cases} -1, & \forall j. f(e_{ij}) = 0 \\ \lambda_i - \min\{j : f(e_{ij}) \neq 0\}, & \text{otherwise} \end{cases}.$$

We proceed by induction on the measure  $S$ . That is, let  $(e_{ij})_{ij}$  be a Jordan basis for  $A$ . Our inductive hypothesis is that if there exists a Jordan basis  $(e'_{ij})_{ij}$  with  $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$ , then we get the desired conclusion.

Now, we have two cases. In the first case,  $(e_{ij})_{ij}$  already satisfies the desired property. In this case we are done. In the other case, there exist  $i_1, j_1, i_2, j_2$  with  $i_1 \neq i_2$ , and  $f(e_{i_1 j_1}) \neq 0$ , and  $f(e_{i_2 j_2}) \neq 0$ . We let  $j_1, j_2$  be minimal with this property, so that  $\forall j < j_1. f(e_{i_1 j}) = 0$ , and  $\forall j < j_2. f(e_{i_2 j}) = 0$ . Wlog, we assume that  $\lambda_{i_1} - j_1 \leq \lambda_{i_2} - j_2$ .

By our inductive hypothesis, all we need to do is find a Jordan basis  $(e'_{ij})_{ij}$  with  $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$ . This is what we do. Define  $e'_{ij}$  as follows.

- $e'_{i_1, \lambda_1} := e_{i_1, \lambda_1} - \frac{f(e_{i_1, j_1})}{f(e_{i_2, j_2})} e_{i_2, j_2 + (\lambda_{i_1} - j_1)}$
- For  $j < \lambda_1$ ,  $e'_{i_1, j} := A^{\lambda_{i_1} - j} e'_{i_1, \lambda_1}$
- For  $i \neq i_1$ ,  $e'_{ij} := e_{ij}$ .

Clearly this is a Jordan basis for  $A$ . Further, we claim that  $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$ . It suffices to show that  $\forall j \leq j_1, f(e_{i_1,j}) = 0$ . We have

$$\begin{aligned} f(e'_{i_1,j}) &= f\left(A^{\lambda_{i_1}-j}\left(e_{i_1,\lambda_1} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})}e_{i_2,j_2+(\lambda_{i_1}-j_1)}\right)\right) = \\ &= f\left(e_{i_1,j} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})}e_{i_2,j_2+(j-j_1)}\right) = f(e_{i_1,j}) - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})}f(e_{i_2,j_2+(j-j_1)}). \end{aligned}$$

Clearly (by design), this expression is zero when  $j = j_1$ . And for  $j < j_1$ , we have  $f(e_{i_1,j}) = f(e_{i_2,j_2+(j-j_1)}) = 0$ , so it is zero then as well. Hence the measure  $S$  of this new basis is smaller, as desired.  $\square$

**Lemma 9.6.** *For any  $n$  and linear  $f : \mathbb{C}^n \rightarrow \mathbb{C}$ , there is a Jordan basis  $e_j$  for  $J_n$  such that there is at most one  $j$  with  $f(e_j) \neq 0$ .*

*Proof.* Let  $e_j$  be a Jordan basis for  $J_n$ . If  $\{j : f(e_j) \neq 0\}$  is the empty set, we are done. Otherwise, let  $j_0 = \min\{j : f(e_j) \neq 0\}$ . For any Jordan basis  $f_j$  with  $j_0 = \min\{j : f(e_j) \neq 0\}$ , define

$$S((f_j)_j) := \begin{cases} -1, & \{j > j_0 : f(e_j) \neq 0\} = \emptyset \\ n - \min\{j > j_0 : f(e_j) \neq 0\}, & \text{otherwise} \end{cases}.$$

We proceed by induction on  $S$ . That is, let  $(e_j)_j$  be a Jordan basis for  $J_n$  with  $j_0 = \min\{j : f(e_j) \neq 0\}$ . Our inductive hypothesis is that if there exists a Jordan basis  $(e'_j)_j$  with  $j_0 = \min\{j : f(e'_j) \neq 0\}$  and  $S((e'_j)_j) < S((e_j)_j)$ , then the conclusion holds.

We have two cases: either  $(e_j)_j$  satisfies the desired property, or not. If not, then let  $j_1 = \min\{j > j_0 : f(e_j) \neq 0\}$ , and define a new Jordan basis  $e'_j$  as follows.

- $e'_n := e_n - \frac{f(e_{j_1})}{f(e_{j_0})}e_{n-(j_1-j_0)}$
- For  $j < n$ ,  $e'_j := J_n^{n-j}e'_n$

It is straightforward to check that  $j_0 = \min\{j : f(e'_j) \neq 0\}$ , and that  $S((e'_j)_j) \leq S((e_j)_j) - 1$ . By our inductive hypothesis, we are done.  $\square$

**Theorem 9.7.** *For any finite-dimensional  $V$ , nilpotent  $A : V \rightarrow V$ , and linear  $f : V \rightarrow \mathbb{C}$ , there is a Jordan basis  $e_{ij}$  for  $A$  such that there is at most one pair  $(i, j)$  with  $f(e_{ij}) \neq 0$ .*

*Proof.* Lemma 9.5 provides a Jordan basis  $e_{ij}$  such that for all  $i \neq i_0$  and all  $j$ , we have  $f(e_{ij}) = 0$ . Restricting  $A$  to  $\langle e_{i_0j} \rangle_j$  arbitrary gives a Jordan block, and then applying Lemma 9.6 gives the desired result.  $\square$

### 9.3 Nilpotency Lemmas

**Lemma 9.8.** *Let  $X \in \mathfrak{gl}_n$  be upper triangular. Then  $J_n + X$  is nilpotent if and only if  $X = 0$ .*

*Proof.* Clearly if  $X = 0$ , then  $J_n + X$  is nilpotent. Inversely, suppose  $X \neq 0$ . Let  $e_1, \dots, e_n$  be the standard basis, with  $J_n e_i = e_{i+1}$ . Let  $i_1 = \max\{i : X e_i \neq 0\}$ . As  $X$  is upper triangular, we have  $X e_{i_1} = v + a e_{i_2}$ , with  $a \in \mathbb{C} \setminus \{0\}$ ,  $i_2 \leq i_1$ , and  $v \in \langle e_1, \dots, e_{i_1-1} \rangle$ .

Now,  $(J_n + X)^{i_1} e_1 = e_{i_1+1} + v + a e_{i_2}$ . Then,  $(J_n + X)^{i_1+(n-i_1)} e_1 = 0 + (J_n + X)^{n-i_1}(v + a e_{i_2})$ . Clearly  $(J_n + X)^{n-i_1}(v + a e_{i_2}) = v' + a e_{i_2+n-i_1}$ , with  $v' \in \langle e_1, \dots, e_{i_2+n-i_1-1} \rangle$ . Now, since  $i_2 \leq i_1$ , we have  $i_2 + n - i_1 \leq n$ , and therefore  $(J_n + X)^n e_1 \neq 0$ . It follows that  $J_n + X$  is not nilpotent.  $\square$

**Lemma 9.9.** *Let  $X \in \mathfrak{gl}_m$ , and let*

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ d_1 & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ & d_2 & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ & & \ddots & \cdots & \vdots & \vdots \\ & & & d_{n-2} & y_{n-1,n-1} & y_{n-1,n} \\ & & & & d_{n-1} & y_{nn} \end{pmatrix} \in \mathfrak{gl}_n.$$

For any  $a, b \in \mathbb{C}^m$ ,

$$\det \left( \begin{array}{ccc|c} X & & & b \\ \hline - & a & - & \end{array} \right) = \det X \det Y + \left( \prod_i d_i \right) \det \left( \begin{array}{ccc|c} X & & & b \\ \hline - & a & - & 0 \end{array} \right)$$

*Proof.* By induction on  $n$ . In the case  $n = 1$ , expanding along the last row (taking the usual interpretation of the empty product) gives the desired result.

Now suppose  $n > 1$ . Expanding along the last row, we get

$$d_{n-1} \det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & Y_{n,n-1} \end{array} \right) - y_{nn} \det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ | \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & Y_{n,n} \end{array} \right).$$

Using our inductive hypothesis for the first determiniant, and using that  $\det \left( \begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array} \right) = \det A_{11} \det A_{22}$  for the second, the expression becomes

$$d_{n-1} \left( \det X \det Y_{n,n-1} + \left( \prod_{i \leq n-2} d_i \right) \det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & 0 \end{array} \right) \right) - y_{nn} \det X \det Y_{nn} =$$

$$(d_{n-1} Y_{n,n-1} - y_{nn} \det Y_{nn}) \det X + \left( \prod_i d_i \right) \det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & 0 \end{array} \right) =$$

$$\det Y \det X + \left( \prod_i d_i \right) \det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & 0 \end{array} \right).$$

□

**Corollary 9.10.** *If  $X$  is nilpotent, and*

$$\left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & Y \end{array} \right)$$

*is nilpotent as well, then  $Y$  is nilpotent.*



*Proof.* By the previous lemma, the characteristic polynomial of the big matrix is

$$g_X(\lambda)g_Y(\lambda) + f(\lambda),$$

where  $g_X(\lambda) = \lambda^m$  is the characteristic polynomial of  $X$ , and  $f(\lambda)$  is some polynomial of degree at most  $m - 1$ .  $\square$

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