1 Introduction

We will review the definition of a Springer fiber and define, for a nilpotent $Y \in \mathfrak{gl}_m$, the Springer fiber at the n-Slodowy slice at Y. For every n and every nilpotent $Y \in \mathfrak{gl}_m$, we will find the irreducible components of the Springer fiber at the n-Slodowy slice at Y. Finally, we will use our results about Springer fibers at n-Slodowy slices to find the irreducible components of some other variety (which probably needs a name), and show that they all have the same dimension.

2 Springer fibers

Let $G \subseteq \mathrm{GL}_m(\mathbb{C})$ be a connected semisimple Lie group, and let $\mathfrak{g} \subseteq \mathfrak{gl}_m(\mathbb{C})$ be its Lie algebra. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the subset consisting of nilpotent elements. Let $\mathcal{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. Let $\widetilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$. Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the projection onto the second coordinate. For $n \in \mathcal{N}$, we call $\pi^{-1}(n)$ the *Springer fiber at n*.

We mention some results about Springer fibers, which we will use later in this paper.

Theorem 2.1. *(Needs citation!) The irreducible components of the Springer fiber at $J(\mu)$ are in bijection with the standard Young tableaus of shape μ. Further, the irreducible components are equidimensional, of dimension $\sum_{i \neq j} \min(\mu_i, \mu_j) = \sum_i (i-1)\mu_i$.*

3 Slodowy slice

A basis for $\mathfrak{sl}_2(\mathbb{C})$ is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra $\mathfrak{g}$, and a homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ sending (e', h', f') to (e, h, f), we say that (e, h, f) is an $\mathfrak{sl}_2$*-triple*. Observe that e, f must be nilpotent, and h must be Cartan (???). If $\mathfrak{g}$ is semisimple, then given any nilpotent $e \in \mathfrak{g}$, the Jacobson-Morozov theorem [1, 3.7.1] says that there exist $h, f \in \mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple.

Given (e,h,f), we define the *Slodowy slice at e* as $\mathcal{S}_e := e + \ker \operatorname{ad}_f$. By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent $e \in \mathfrak{g}$, when $\mathfrak{g}$ is semisimple.

4 Finding $\mathfrak{sl}_2$-triples (E,H,F) in $\mathfrak{gl}_{m+n}$ with a particular E.

Let

$$e = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \in \mathfrak{gl}_n,$$

and let $E = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{gl}_{m+n}$. We will show that there is exactly one $\mathfrak{sl}_2$-triple (E,H,F), and we will find what it looks like. First we solve the case $m=0$ (so $E=e$), and then we use this to solve the case of arbitrary m.

Lemma 4.1. *There is exactly one way to choose $h, f \in \mathfrak{gl}_n$ so that (e,h,f) is an $\mathfrak{sl}_2$-triple.*

Proof. Note that $[h',e'] = 2e'$, and $[e',f'] = h'$, and $[h',f'] = -2f'$. Thus e,h,f must obey the same relations. In particular, $he - eh = 2e$. The matrix eh is h shifted down one, and he is h shifted left one. Thus, $2e_{ij} = (he-eh)_{ij} = h_{i,j+1} - h_{i-1,j}$. We can use this to show that $h_{ij} = 0$ when $i \neq j$. Then we can use it to show that $h_{ii} = h_{i-1,i-1} + 2$, so that $h_{ii} = h_{11} + 2(i-1)$.

Similarly, from $[e,f] = h$ we get that $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$. We can use this to show that $f_{ij} = 0$ when $j \neq i+1$. Then we can use it to show that $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$, that $f_{1,2} = -h_{1,1}$, and that $f_{n-1,n} = h_{n,n}$. From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^{n} h_{ii} \implies$$

$$\sum_i h_{ii} = 0.$$

Remark: this is just the statement that $h \in \mathfrak{sl}_n$; in other words, we will see that every choice of $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_{m+n}$ is also an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_{m+n}$. This shows that $h_{11} = n-1, h_{22} = n-3, ..., h_{nn} = 1-n$. So we have determined h; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for f in terms of h to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & \\ & 0 & (1-n)+(3-n) & & \\ & & \ddots & \ddots & \\ & & & 0 & (1-n)+\cdots+(n-1) \\ & & & & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1(1-n) & & & \\ & 0 & 2(2-n) & & \\ & & 0 & (n-2)(-2) & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

□

Lemma 4.2. *There is exactly one way to choose $H, F \in \mathfrak{gl}_{m+n}$ so that (E, H, F) is an $\mathfrak{sl}_2$-triple.*

Proof. Suppose we have H, F so that (E, H, F) is an $\mathfrak{sl}_2$-triple. Writing $F =: \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & f \end{pmatrix}$, and similarly for H, we have

$$2E = [H, E] = \begin{pmatrix} 0 & H_{12}e \\ 0 & he \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ eH_{21} & eh \end{pmatrix},$$

$$H = [E, F] = \begin{pmatrix} 0 & 0 \\ eF_{21} & ef \end{pmatrix} - \begin{pmatrix} 0 & F_{12}e \\ 0 & fe \end{pmatrix}.$$

From these we observe that (e, h, f) must also be an $\mathfrak{sl}_2$-triple, so h, f must be as in Lemma 4.1. We also see that $H_{11} = 0$. Recalling that left multiplication by e is a down-shift, and right multiplication is a left-shift, we see that H_{12} is all zeroes except for the leftmost column, and H_{21} is all zeroes except for the bottom row. Then our final constraint is that

$$-2F = [H, F] =$$

$$\begin{pmatrix} H_{12}F_{21} & H_{12}F_{22} \\ H_{21}F_{11} + H_{22}F_{21} & H_{21}F_{12} + H_{22}F_{22} \end{pmatrix} - \begin{pmatrix} F_{12}H_{21} & F_{11}H_{12} + F_{12}H_{22} \\ F_{22}H_{21} & F_{21}H_{12} + F_{22}H_{22} \end{pmatrix}.$$

Now $H_{12} = F_{12}e$, and $H_{21} = eF_{21}$, from the equation $H = [E, F]$. Substituting in the equation above then,

$$-2F = \begin{pmatrix} F_{12}eF_{21} & F_{12}ef \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} F_{12}eF_{21} & F_{11}F_{12}e + F_{12}h \\ feF_{21} & F_{21}F_{12}e + fh \end{pmatrix} =$$

$$\begin{pmatrix} 0 & F_{12}(fe + h) \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e + F_{12}h \\ (ef - h)F_{21} & F_{21}F_{12}e + fh \end{pmatrix} =$$

$$\begin{pmatrix} 0 & F_{12}fe \\ eF_{21}F_{11} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e \\ efF_{21} & F_{21}F_{12}e + fh \end{pmatrix}.$$

Now we see that $F_{11} = 0$, and consequently that $F_{12} = F_{21} = 0$ as well. This shows that $H_{12} = H_{21} = 0$. We conclude that H and F just have h and f in their bottom-right corners, respectively. □

5 Finding the Slodowy slices with the same E

First we find $\ker \operatorname{ad}_f$. We have $(fX)_{ij} = i(n-i)A_{i+1,j}$, and $(Xf)_{ij} = (j-1)(n-(j-1))A_{i,j-1}$. So, for all $i, j \in \{1, ..., n\}$, we have

$$i(n-i)A_{i+1,j} = (j-1)(n-(j-1))A_{i,j-1}.$$

Taking $j = 1$, we find that $A_{i,1} = 0$ for $i \geq 2$. Then, taking $j > 1$, we find that for $i, j \in \{1, ..., n-1\}$,

$$A_{i+1,j+1} = \frac{(j-1)(n-(j-1))}{i(n-i)} A_{ij}.$$

So, $\ker \operatorname{ad}_f$ is the set of matrices which are upper triangular and satisfy the above condition.

To find the Slodowy slice associated to the previous $\mathfrak{sl}_2$-triple (E, H, F), we just need to find $\ker \operatorname{ad}_F$. We have

$$[F, X] = \begin{pmatrix} 0 & 0 \\ fX_{21} & fX_{22} \end{pmatrix} - \begin{pmatrix} 0 & X_{12}f \\ 0 & X_{22}f \end{pmatrix}.$$

Thus, X_{21} must be all zeroes except for the first row, and X_{12} must be all zeroes except for the last column, and $X_{22} \in \ker \operatorname{ad}_f$. There is no restriction on X_{11}. This describes $\ker \operatorname{ad}_F$.

For $X \in \mathcal{S}_{m,n}$, define $u(X) := X_{11}$.

5.1 Finding $\widetilde{\mathcal{N}}_{m,n}$

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m$ be the nilpotent elements. Let $\mathcal{S}'_{m,n}$ be the set of $X \in \mathcal{S}_{m,n}$ such that both X and $u(X)$ are nilpotent. Let $\widetilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n}$. Define $\pi_{m,n} : \widetilde{\mathcal{N}}_{m,n} \to \mathcal{N}_m$ by $(\mathfrak{b}, X) \mapsto X_{11}$. For $Y \in \mathfrak{gl}_m$, we call $\pi_{m,n}^{-1}(Y)$ the *Springer fiber at the n-Slodowy slice at Y*.

Lemma 5.1. *Let J be a jordan block with zeroes along the diagonal, and let A be upper triangular and nonzero. Then $J + A$ is not nilpotent.*

Proof. It is straightforward to show by induction that if $v_i = 0$ for $i < j$, and $v_j \neq 0$, then $((J + A)^k v_j)_{j+k} = v_j$. Let i be such that $Ae_i \neq 0$. Then $(J + A)^{i-1} e_1$ has nonzero e_i-component. Then $(J + A)^i e_1$ has some nonzero $e_{i'}$-component for some $i' \leq i$. Then $(J + A)^{i+(n-i')} e_1$ has some nonzero e_n-component. And $i + (n - i') \geq n$, so we're done. $\square$

Lemma 5.2. *Let $X \in \mathfrak{gl}_m$, and let*

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ d_1 & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ & d_2 & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ & & \ddots & \cdots & \vdots & \vdots \\ & & & d_{n-2} & y_{n-1,n-1} & y_{n-1,n} \\ & & & & d_{n-1} & y_{nn} \end{pmatrix} \in \mathfrak{gl}_n .$$

For any $a, b \in \mathbb{C}^m$,

$$\det \left(\begin{array}{ccc|c} & X & & \begin{matrix}|\\ b\\ |\end{matrix} \\ \hline - & a & - & \\ & & & Y \end{array}\right) =$$

$$\det X \det Y + \left(\prod_i d_i\right) \det \left(\begin{array}{ccc|c} & X & & \begin{matrix}|\\ b\\ |\end{matrix} \\ \hline - & a & - & 0 \end{array}\right)$$

Proof. By induction on n. In the case $n = 1$, expanding along the last row (taking the usual interpretation of the empty product) gives the desired result.

Now suppose $n > 1$. Expanding along the last row, we get

$$d_{n-1} \det \left(\begin{array}{ccc|c} & X & & \begin{matrix}|\\ b\\ |\end{matrix} \\ \hline - & a & - & \\ & & & Y_{n,n-1} \end{array}\right) - y_{nn} \det \left(\begin{array}{ccc|c} & X & & \\ \hline - & a & - & \\ & & & Y_{n,n} \end{array}\right).$$

Using our inductive hypothesis for the first determiniant, and using that $\det \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} = \det A_{11} \det A_{22}$ for the second, the expression becomes

$$d_{n-1} \left(\det X \det Y_{n,n-1} + \left(\prod_{i \le n-2} d_i\right) \det \left(\begin{array}{ccc|c} & X & & \begin{matrix}|\\ b\\ |\end{matrix} \\ \hline - & a & - & 0 \end{array}\right)\right) - y_{nn} \det X \det Y_{nn} =$$

$$(d_{n-1} Y_{n,n-1} - y_{nn} \det Y_{nn}) \det X + \left(\prod_i d_i\right) \det \left(\begin{array}{ccc|c} & X & & \begin{matrix}|\\ b\\ |\end{matrix} \\ \hline - & a & - & 0 \end{array}\right) =$$

$$\det Y \det X + \left(\prod_i d_i\right) \det \left(\begin{array}{ccc|c} & X & & | \\ & & & b \\ & & & | \\ \hline — & a & — & 0 \end{array}\right).$$

□

Corollary 5.3. *If X is nilpotent, and*

$$\left(\begin{array}{ccc|cc} & & & & | \\ & X & & & b \\ & & & & | \\ \hline — & a & — & & \\ & & & Y & \\ & & & & \end{array}\right)$$

is nilpotent as well, then Y is nilpotent (TODO: and that other determinant is zero).

Proof. By the previous lemma, the characteristic polynomial of the big matrix is

$$g_X(\lambda)g_Y(\lambda) + f(\lambda),$$

where $g_X(\lambda) = \lambda^m$ is the characteristic polynomial of X, and $f(\lambda)$ is some polynomial of degree at most $m-1$. □

Now, taking the previous corolloary and the first lemma together, we see that

$$\mathcal{S}'_{m,n} = \left\{ \left(\begin{array}{ccc|cccc} & & & & & & | \\ & X & & & & & b \\ & & & & & & | \\ \hline — & a & — & 0 & & & \\ & & & 1 & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & 0 \end{array}\right) : a, b \in \mathbb{C}^m, X \in \mathfrak{gl}_m \text{ is nilpotent} \right\}.$$

6 Simplifying the definition of a Springer fiber at a Slodowy slice

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m(\mathbb{C})$ be the subset consisting of nilpotent elements. For $X \in \mathcal{N}_m$ and $a, b \in \mathbb{C}^m$, let

$$A_{X,a,b} = \left(\begin{array}{c|ccccc} & & & & & | \\ X & & & & & b \\ & & & & & | \\ \hline — \; a \; — & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & 0 & \\ & & & & 1 & 0 \end{array}\right) \in \mathfrak{gl}_{m+n}(\mathbb{C}).$$

By the definition given in the previous section, we have

$$\widetilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n},$$

and the Springer fiber at the n-Slodowy slice at a nilpotent $X \in \mathfrak{gl}_m$ is

$$\begin{aligned} \pi_{m,n}^{-1}(X) &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \in \mathcal{S}'_{m,n}\} \\ &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \text{nilpotent}\} \\ &\cong \{(\mathfrak{b}, a, b) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \text{ nilpotent}\}. \end{aligned}$$

We will make one last simplification to this by using a correspondence between Borel subalgebras and complete flags. TODO: fix the next few paragraphs, they are out of context.

We have $\mathcal{M} = \{AHA^{-1} : A \in \mathrm{GL}_{m+n}(\mathbb{C})\}$, where $H \subseteq \mathfrak{gl}_{m+n}(\mathbb{C})$ is the set of upper triangular matrices. We say a map $X : \mathfrak{gl}_{m+n} \to \mathfrak{gl}_{m+n}$ preserves a flag $V_0 \subseteq \cdots \subseteq V_{m+n}$ if $XV_i \subseteq V_i$ for each i. Let $E_0 \subseteq \cdots \subseteq E_{m+n}$ be the standard flag of $\mathbb{C}^{m+n}$. Since H is the set of X which preserve E,

$$\mathcal{M} = \{\{X : \forall i.\ X(AE_i) \subseteq AE_i\} : A \in \mathrm{GL}_n(\mathbb{C})\}.$$

So, for $X \in \mathcal{N}$,

$$\pi^{-1}(X) \cong \{(V, a, b) : \forall i.\ A_{X,a,b} V_i \subseteq V_{i-1}\}.$$

7 Strategy and setup for finding the irreducible components of a Springer fiber at a Slodowy slice

Fix any $X \in \mathcal{N}$. As we have fixed X, we now write $A_{a,b} := A_{X,a,b}$. Let $(e_{ij})_{1 \leq i \leq k, 1 \leq j \leq \lambda_i}$ be a Jordan basis for X. For convenience we define $e_{i0} := 0$; now we may express the fact that (e_{ij}) is a Jordan basis by writing $\forall i. X e_{ij} = e_{i,j-1}$.

In this section we find the irreducible components of

$$\pi^{-1}(X) \cong V := \{(A_{a,b}, U) : \forall i.\ A_{a,b} U_{i+1} \subseteq U_i\}.$$

For $1 \leq w \leq k$ and $0 \leq r \leq \lambda_w$ (note that we allow $r = 0$), define

$$V_{w,r} := \{(A_{a,b}, U) \in V : \exists P \in \mathrm{GL}_m \,.\, \exists b'.\ (P^{-1}, I_n) A_{a,b} (P, I_n) = A_{e_{wr}, b'}\}.$$

Lemma 7.1. *$V = \bigcup_{1 \leq w \leq k, 0 \leq r \leq \lambda_w} V_{w,r}$. Further, $V_{w_1 r_1} = V_{w_2 r_2}$ exactly when either $r_1 = r_2 = 0$, or $\lambda_{w_1} = \lambda_{w_2} \wedge r_1 = r_2$. When $V_{w_1 r_1} \neq V_{w_2 r_2}$, we have $V_{w_1 r_1} \cap V_{w_2 r_2} = \emptyset$.*

Proof. TODO □

Now, fix any w and r. We will find the irreducible components of $V_{w,r}$. These will all happen to be equidimensional (with dimensions independent of w and r), so their closures in V will be the irreducible components of V.

Let

$$G := \{P \in \mathrm{GL}_m : P^{-1} X P = X\},$$

and

$$G_{wr} = \{A \in G : e_{wr} A = e_{wr}\}.$$

Now, define

$$U_{wr} = \{(A_{e_{wr},b}, U) \in V_{wr}\}.$$

Let G act on V_{wr} by

$$P \cdot (A_{e_{wr},b}, U) := ((P, I_n) A_{e_{wr},b} (P, I_n)^{-1}, (P, I_n) U) = (A_{e_{wr} P^{-1}, Pb}, (P, I_n) U).$$

Consider the map $\varphi : U_{wr} \times G \to V_{wr}$ defined by

$$(x, P) \mapsto P \cdot x.$$

By restriction of G to G_{wr} and V_{wr} to U_{wr}, we obtain an action of G_{wr} on U_{wr}. Then, letting G_{wr} act on G by $g \cdot h := hg^{-1}$, we obtain an action of G_{wr} on $U_{wr} \times G$.

Lemma 7.2. *As an algebraic variety, G is irreducible.*

Proof. It's just $\mathbb{C}^{something}$, since its blocks are the lower-left-toeplitz matrices. $\square$

Lemma 7.3. *The map φ is some sort of quotient by the action of G_{wr}.*

Proof. Let $y \in V_{wr}$. By definition of V_{wr}, there is $P_y \in G$ with $P_y \cdot y \in U_{wr}$. We have

$$\varphi^{-1}(y) = \{(x, P) : P \cdot x = y = P_y^{-1} \cdot (P_y \cdot y)\} =$$
$$\{(x, P) : x = (P^{-1}P_y^{-1}) \cdot (P_y \cdot y)\} =$$
$$\{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \in G_{wr}\} =$$
$$(\text{taking } Q := P^{-1}P_y^{-1}, \text{ so } P = (QP_y)^{-1} = P_y^{-1}Q^{-1})$$
$$\{(Q \cdot (P_y \cdot y), P_y^{-1}Q^{-1}) : Q \in G_{wr}\} =$$
$$\{Q \cdot (P_y \cdot y, P_y^{-1}) : Q \in G_{wr}\}.$$

So we see that the fibers are exactly the G_{wr}-orbits.

Now, what kind of quotient is this? All I need is something that allows me to deduce the dimension of the quotient... $\square$

Our strategy is to find the irreducible components $X \subseteq U_{wr}$, and we will then argue that the irreducible components of V_{wr} are of the form $\varphi(X \times G)$. So, we will now find the irreducible components of U_{wr}.

Actually this will be unnecessarily difficult to think about; it is easiest in the case $r = 0$. So, we will change basis to make $r = 0$. Let $\lambda' = (\lambda_1, ..., \lambda_{w-1}, \lambda_w - r, \lambda_{w+1}, ..., \lambda_k)$. Let X' be in Jordan normal form with shape λ'. Let $U'_{wr} = \{(A_{X',0,b}, U) : \forall i. A_{X',0,b}U_{i+1} \subseteq U_i\}$. Let e'_{ij} be a Jordan basis for X'. Let $f_1, ..., f_n$ be a Jordan basis for the restriction of $A_{X,0,b}$ to ... Set $m' := m - r$, and $n' := n + r$. Let $f'_1, ..., f'_{n'}$ blah. Define the linear map $Q_{wr} : \mathbb{C}^{m'+n'} \to \mathbb{C}^{m+n}$ by:

- For all i, $e'_{ij} \mapsto e_{ij}$.
- For $j = 1, ..., r$, $f'_{n+j} \mapsto e_{w,(\lambda_w - r)+j}$.

- For $j = 1, ..., n$, $f'_j \mapsto f_j + e_{w,(\lambda_w - r)-n+j}$.

Observe that conjugation by Q_{wr} maps U_{wr} to U'_{wr}, and conjugation by Q_{wr}^{-1} maps U'_{wr} to U_{wr}. We conclude that $U_{wr} \cong U'_{wr}$; so to find the irreducible components of U_{wr} we just need to find the irreducible components of U'_{wr}. To clear the context, which is rather cluttered by now, and to avoid writing primes everywhere, we move to a new section.

8 Finding the irreducible components of U_0

Let X be nilpotent with Jordan basis $(e'_{ij})_{1 \leq i \leq k, 1 \leq j \leq \lambda'_i}$. Let $V = \{(A_{a,b}, U) : \forall i.\ A_{a,b} U_{i+1} \subseteq U\}$. Let

$$U_0 = \{(A_{0,b}, U) \in V\}.$$

We will find the irreducible components of U_0.

We write b_{ij} to denote the projection of $b \in \mathbb{C}^m$ onto e_{ij}. For each row i, let $p_i(b) = \max\{j : b_{ij} \neq 0\}$ (the maximum of the empty set is zero). Then set $q_i(b) = \lambda_i - p_i(b)$. When it is clear enough from context, we will just write p_i and q_i instead of $p_i(b)$ and $q_i(b)$.

For any $I \subseteq \{1, ..., k\}$ and any values $(\rho_i)_{i \in I}$, define

$$B_{I,(\rho_i)} := \{b \in \mathbb{C}^m :$$

$$\{(p_i, q_i) : i \in I\} = \{(p_i, q_i) : n > q_i \wedge p_i = \max_{j : q_j = q_i} p_j > \max_{j : q_j < q_i} p_j\}$$

$$\wedge \forall i \in I.\ p_i = \rho_i\}.$$

Then let $U_{I,(\rho_i)} = \{(A_{0,b}, U) \in V : b \in B_{I,(\rho_i)}\}$.

Lemma 8.1. $\mathbb{C}^m = \bigcup_{I,(\rho_i)} B_{I,(\rho_i)}$*, where* I *ranges over all subsets of* $\{1, ..., k\}$*, and* (ρ_i) *ranges over all maps* $I \to \mathbb{N}_{>0}$ *such that* ρ_i *and* $\lambda_i - \rho_i$ *are both decreasing with* i*. TODO: something about overlap being small (they're not disjoint). They are distinct. Is that good enough for my purposes? Ah, I think it is good enough if I instead say that the sets of* b*'s are distinct.*

Proof. Let $b \in \mathbb{C}^m$. Let $I' = \{(p_i, q_i) : p_i = n > q_i \wedge \max_{j : q_j = q_i} p_j > \max_{j : q_j < q_i} p_j\}$. Then let $I = \{\max\{i : (p_i, q_i) = (p, q)\} : (p, q) \in I'\}$. Choosing the maximal i is an arbitrary choice; we just have to pick one. For $i \in I$,

$\rho_i := p_i$. All we need to do is show that each ρ_i is positive, and that ρ_i and $\lambda_i - \rho_i$ are decreasing. Then we get that $(A_{0,b}, U) \in U_{I,(\rho_i)}$, and we are done.

Recall that by convention the maximum of the empty set is zero. So, the fact that $p_i > \max_{j>i} p_j$ tells us that $\rho_i = p_i$ is positive.

Let $i, j \in I$ with $i < j$. Since $\lambda_i \geq \lambda_j$, we have either $p_i \geq p_j$ or $q_i \geq q_j$. Further, since $(p_i, q_i) \neq (p_j, q_j)$ (clear from definition of I), either $p_i > p_j$ or $q_i > q_j$. And for $i, j \in I$ we clearly have $p_i > p_j \iff q_i > q_j$. So in either case, both $p_i > p_j$ and $q_i > q_j$, and consequently both $\rho_i > \rho_j$ and $\lambda_i - \rho_i > \lambda_j - \rho_j$. $\square$

Lemma 8.2. *Here we have an alternative characterization of $B_{I,(\rho_i)}$. Namely,*

$$B_{I,(\rho_i)} = \{b \in \mathbb{C}^m : \forall i \in I.\ p_i = \rho_i \wedge \forall i \notin I.\ p_i = \min(p_{i_k}, \lambda_i - q_{i_k}),$$

$$\textit{where } k = ...\}$$

Proof. TODO $\square$

Corollary 8.3.

$$B_{I,(\rho_i)} \cong \prod_{i \in I} \mathbb{C}^{blah} \times \prod_{i \notin I}$$

Fix any I and (ρ_i) satisfying the conditions of Lemma 8.1. We will find the irreducible components of $U_{I,(\rho_i)}$, and show that their closures are in fact irreducible components of U_0.

I claim that $A_{0,b}$ has the same shape for every $(A_{0,b}, U) \in U_{I,(\rho_i)}$. Call this shape μ. By finding a algebraic map taking b to a Jordan basis for $A_{0,b}$, we will put $U_{I,(\rho_i)}$ in isomorphism with the product (choice of b) $\times$ (Springer fiber at $J(\mu)$). Then we will use our result about the usual Springer fiber at $J(\mu)$ to find the irreducible components of $U_{I,(\rho_i)}$. (I should define this somewhere else, but $J(\mu)$ just has the blocks in order of nonincreasing size.)

So, now we find a Jordan basis for $A_{0,b}$.

Lemma 8.4. *Let $P = \max_{i \in I} \rho_i$, and for $i \in I$, let $P_i = \max_{j \in I : j > i} \rho_j$. The following vectors give a Jordan basis for $A_{0,b}$. (For convenience, we write $A := A_{0,b}$ in this lemma and proof.)*

- *For $i \notin I$, the chain of length $p_i + q_i$ beginning with e_{i,p_i+q_i}*
- *The chain of length $n + P$ beginning with $f_n - (A^{n+P} f_n \texttt{>>}\ n + P)$*

- *For $i \in I$, the chain of length $q_i + P_i$ beginning with $v_i - (A^{q_i+P_i} v_i \gg q_i + P_i)$, where $v_i := A^{n-q_i} f_n - \sum_{l=1}^{p_i} b_{il} e_{i,l+q_i}$*

Proof. There are three things to check: (1) the chains are no longer than their claimed lengths, (2) the claimed lengths sum to $m+n$, and (3) the span of the chains is $\mathbb{C}^{m+n}$.

We have defined P and P_i in terms of the ρ_i to make it clear that they only depend on I and the ρ_i. Now we provide more useful characterizations. It is immediate from the definition of $U_{I,(\rho_i)}$ that $P = \max_{l:q_l<n} p_l$, and $P_i = \max_{l:q_l<q_i} p_l$.

Proof of (1). It is obvious that a chain beginning with e_{i,p_i+q_i} has length $p_i + q_i$.

Now consider the chain beginning with $f_n - (A^{n+P} f_n \gg n + P)$. Note that $A^n f_n = b$, so $A^{n+P} f_n = A^P b = b \ll P$. By shifting b left P times, we zero out all the rows i where $q_i < n$. This ensures that the operation of shifting $b \ll P$ right $n+P$ times is invertible by shifting left $n+P$ times. That is,

$$A^{n+P} f_n = b \ll P = ((b \ll P) \gg n+P) \ll n+P = A^{n+P}(A^{n+P} f_n \gg n+P).$$

This shows that the chain has length at most $n+P$, as desired.

Now, let $i \in I$. We have $v_i = A^{n-q_i} f_n - \sum_{l=1}^{p_i} b_{il} e_{i,l+q_i}$. First we note that $q_i < n$ by the definition of $U_{I,(\rho_i)}$, so the definition of v_i makes sense. We consider the chain beginning with $v_i - (A^{q_i+P_i} v_i \gg q_i + P_i)$. Note that $A^{q_i} v_i$ is just b with row i zeroed out. For brevity, we write $b_i := A^{q_i} v_i$. Note that $b_i \ll P_i$ has rows l zeroed out, for all l with $q_l < q_i$. This ensures that shifting $b_i \ll P_i$ right $q_i + P_i$ times can be inverted by shifting left $q_i + P_i$ times. That is,

$$A^{q_i+P_i} v_i = b_i \ll P_i = ((b_i \ll P_i) \gg q_i{+}P_i) \ll q_i{+}P_i = A^{q_i+P_i}(A^{q_i+P_i} v_i \gg q_i{+}P_i).$$

This shows that the chain has length at most $q_i + P_i$, as desired. $\square$

Proof of (2). The sum of the lengths is

$$\sum_{i \notin I} (p_i + q_i) + (n + P) + \sum_{i \in I} (q_i + P_i).$$

Writing $I = \{i_1 < \cdots < i_{|I|}\}$, we note that $P = p_{i_1}$, that $P_{i_{|I|}} = 0$, and that for $l < |I|$ we have $P_{i_l} = p_{i_{l+1}}$. Thus, the sum above is

$$\sum_{i \notin I}(p_i + q_i) + (n + p_{i_1}) + \sum_{l=1}^{|I|}(q_{i_l} + p_{i_{l+1}}) =$$

$$\sum_{i \notin I}(p_i + q_i) + n + \sum_{l=1}^{|I|}(q_{i_l} + p_{i_l}) = n + \sum_i (q_i + p_i) = m + n.$$

□

Proof of (3). Let W be the span of the chains listed. We need to show that $W = \mathbb{C}^{m+n}$. Because every $i \in I$ satisfies $q_i < n$, clearly $\langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$.

I claim that $f_n \in W$ as well. To see this, we consider the chain beginning with $f_n - (A^{n+P} f_n \gg n + P)$. As explained in the proof of (1), we have $A^{n+P} f_n \in \langle e_{ij} \rangle_{i,j:q_i \geq n}$. Consequently, $(A^{n+P} f_n \gg n + P) \in \langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$. Because $f_n - (A^{n+P} f_n \gg n + P) \in W$, this implies that $f_n \in W$.

Because $AW \subseteq W$ (obvious, since W is the span of chains), the fact that $f_n \in W$ implies that $f_i \in W$ for each i, and also $b \ll l \in W$ for each $l \geq 0$.

Now we are left with showing that $\langle e_{ij} \rangle_{i,j:q_i < n} \subseteq W$. This is obvious for $i \notin I$. For $i \in I$, we do it inductively. Fix $i \in I$, and suppose we have already shown that for all $i' \in I$ with $q_{i'} > q_i$, we have $\langle e_{ij} \rangle_{j \text{ arbitrary}} \subseteq W$. We will show that $\langle e_{ij} \rangle_{j \text{ arbitrary}} \subseteq W$.

To see this, we consider the chain beginning with $v_i - (A^{q_i+P_i} v_i \gg q_i + P_i)$. (Recall $v_i = A^{n-q_i} f_n - \sum_{l=1}^{p_i} b_{il} e_{i,l+q_i}$.) Because $AW \subseteq W$, and $b_{i,p_i} \neq 0$ (by definition of p_i), it suffices to show that $\sum_{l=1}^{p_i} b_{il} e_{i,k+q_i} \in W$. As explained in the proof of (1), we have $A^{q_i+P_i} v_i \in \langle e_{lj} \rangle_{q_l \geq q_i}$. And since $A^{q_i+P_i} v_i$ has row i zeroed out, in fact $A^{q_i+P_i} v_i \in \langle e_{lj} \rangle_{l:l \neq i \wedge q_l \geq q_i}$. Hence, $A^{q_i+P_i} v_i \gg q_i + P_i \in \langle e_{lj} \rangle_{l:l \neq i \wedge q_l \geq q_i}$. By our inductive hypothesis, $\langle e_{lj} \rangle_{l:l \neq i \wedge q_l \geq q_i} \subseteq W$, and consequently $A^{q_i+P_i} v_i \gg q_i + P_i \in W$. Since we know $v_i - (A^{q_i+P_i} v_i \gg q_i + P_i) \in W$, this implies that $v_i \in W$. Because $A^{n-q_i} f_n \in W$, this then implies that $\sum_{l=1}^{p_i} b_{il} e_{i,l+q_i} \in W$, as desired. □

We proved (1), (2), (3), so we are done. □

Let μ be the shape of the Jordan basis given in the previous lemma. Let X_μ be the Springer fiber at $J(\mu)$.

Lemma 8.5. $U_{I,(\rho_i)} \cong B_{I,(\rho_i)} \times X_\mu$.

Proof. For $b \in B_{I,(\rho_i)}$, let P_b be the change-of-basis matrix, with columns given by the Jordan basis of the previous lemma, so that $J(\mu) = P_b^{-1} A_{0,b} P_b$. From looking at the Jordan basis of the previous lemma (the basis can be expressed in terms of $A, \lambda, e_{ij}, f_i, I, \rho_i, b$), it is clear that the map $P : B_{I,(\rho_i)} \to \mathrm{GL}_{m+n}$, given by $b \mapsto P_b$, is algebraic.

Now, we remark that the Springer fiber at $A_{0,b}$ is simply $\{P_b U : U \in X_\mu\}$. This gives us the isomorphism $B_{I,(\rho_i)} \times X_\mu \to U_{I,(\rho_i)}$.

$$(b, U) \mapsto (P_b J(\mu) P_b^{-1}, P_b U),$$

with inverse

$$(A_{0,b}, U) \mapsto (b, P_b^{-1} U).$$

□

At last we have reached the base of our recursive procedure, and we begin to propagate upwards.

9 Finding the irreducible components of a springer fiber at a slodowy slice

Apply things from the previous two sections, and conclude.

10 Finding a centralizer

For nilpotent X in Jordan form and a in 'normalized' form (i.e., with at most one nonzero element, which is a one), we will find the centralizer of $A_{X,a,b}$ in

$$\{(A, I) : A \in \mathfrak{gl}_m\} \subseteq \mathfrak{gl}_{m+n}\,.$$

Note that an element of the form (A, I) commutes with $A_{X,a,b}$ if and only if A commutes with X, and $aA = a$, and $Ab = b$. So, we just have to find

$$\{A \in \mathfrak{gl}_m : AX = XA, aA = a, Ab = b\}.$$

We begin by finding

$$\{A \in \mathfrak{gl}_m : AX = XA\}.$$

We write the shape of X as $\lambda = (\lambda_1, ..., \lambda_k)$, so that

$$X = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix}.$$

Then we write A as the block matrix

$$A =: \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix},$$

where A_{ij} is a block of size $\lambda_i \times \lambda_j$. We have

$$XA = \begin{pmatrix} J_{\lambda_1} A_{11} & \cdots & J_{\lambda_1} A_{1k} \\ \vdots & & \vdots \\ J_{\lambda_k} A_{k1} & \cdots & J_{\lambda_k} A_{kk} \end{pmatrix}, \text{ and } AX = \begin{pmatrix} A_{11} J_{\lambda_1} & \cdots & A_{1k} J_{\lambda_k} \\ \vdots & & \vdots \\ A_{k1} J_{\lambda_1} & \cdots & A_{kk} J_{\lambda_k} \end{pmatrix}.$$

So, the constraint that $XA = AX$ is simply saying that

$$\forall i, j.\ J_{\lambda_i} A_{ij} = A_{ij} J_{\lambda_j}.$$

It's easy to see that left multiplication by J_{λ_i} is just a down-shift by one, and right multiplication by J_{λ_j} is a left-shift by one. This means that A_{ij} is constrained to be a "lower-left-Toeplitz" matrix. A Toeplitz matrix is one which is constant along diagonal bands: $\forall ijk.\ x_{ij} = x_{i+k,j+k}$. A lower-left-Toeplitz matrix is one which is all zeroes except for the bottom-left corner; that is, the bands start in the bottom-left corner, and continue until hitting the band which includes the lower-right corner or the band which includes the upper-left corner (whichever comes first). In other words, an $n \times m$ lower-left-Toeplitz matrix is one which satisfies $x_{ij} = 0$ for all i, j with $i < j$ and all i, j with $i - j < m - n$. In yet other words, a matrix is lower-left-Toeplitz if it is Toeplitz, satisfies $x_{1j} = 0$ for $j \neq 1$, and satisfies $x_{in} = 0$ for $i \neq m$. Anyway, it is easy to see that the lower-left-Toeplitz matrices are exactly those matrices A such that left-shifting A by one has the same result as down-shifting by one.

So, we have found the set

$$C_1 := \{A \in \mathfrak{gl}_m : AX = XA\}.$$

It is just the set of $A = [A_{ij}]_{ij}$, where each A_{ij} is lower-left-Toeplitz.

Let v_{ij} be the leftmost column of A_{ij}, so that

$$A_{ij} = \begin{pmatrix} v_{ij} \curlyvee 0 & \cdots & v_{ij} \curlyvee [\lambda_j - 1] \end{pmatrix}.$$

Since A_{ij} is lower-left-Toeplitz, v_{ij} is of the form

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$ can be chosen freely. Now we determine which matrices of this form satisfy $aA = a$. For simplicity, we will instead find the matrices A such that $aA = 0$ (so that $a(A + I) = a$). (Note that $I \in C_1$, and C_1 is closed under addition, so this is really the same thing.)

In the case that $a = 0$, clearly every A works. Otherwise, let i_0, j_0 be such that $a_{i_0,j_0} = 1$. Now, clearly the constraint that $aA = 0$ is just saying that the (i_0, j_0)th row of A must be zero. That is, for each j the j_0th row of $A_{i_0 j}$ must be zero. This just requires that for each j, we must have

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$ can be chosen freely. So, we have now found the set

$$C_2 := \{A \in \mathfrak{gl}_m : AX = XA, aA = a\}.$$

It is the matrices of the form $I + A$, where A_{ij} is a block matrix of size $\lambda_i \times \lambda_j$ with

$$A_{ij} = T(v_{ij}) = T \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$ in the case $i \neq i_0$ and $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$ if $i = i_0$.

11 A Jordan basis for $A_{X,a,b}$

11.1 A 'normalization' fact about Jordan bases

Let V be a finite-dimensional vector space, $A : V \to V$ a nilpotent operator, and $f : V \to \mathbb{C}$ a linear map. Let $(e_{ij} : i \leq m, j \leq \lambda_i)$ be a Jordan basis for A.

Lemma 11.1. *There is a Jordan basis e_{ij} for A such that there is at most one pair (i,j) with $f(e_{ij}) \neq 0$.*

Proof. For any change of basis $P : V \to V$ commuting with A, we obtain a new Jordan basis $(P(e_{ij}) : i \leq m, j \leq \lambda_i)$. For any such P, define

$$S_P = \sum_i \begin{cases} -1, & \forall j.\ f \circ P(e_{ij}) = 0 \\ \lambda_i - \min\{j : f \circ P(e_{ij}) \neq 0\}, & \text{otherwise} \end{cases}.$$

Let P be any operator, among all invertible operators commuting with A, which minimizes S_P. Write $e'_{i,j} := P(e_{i,j})$.

Suppose for contradiction that there are two distinct i's (and some j's) with $f(e'_{ij}) \neq 0$. Then we can take e'_{i_1,j_1} and e'_{i_2,j_2}, where for $k \in \{1,2\}$ we have $f(e'_{i_k,j_k}) \neq 0$, and $\forall j < j_k.\ f(e'_{i_k,j}) = 0$. Wlog, assume $\lambda_{i_1} - j_1 \leq \lambda_{i_2} - j_2$. Then, we can define $Q : V \to V$ by

- $Q(e'_{i_1,\lambda_1}) := e'_{i_1,\lambda_1} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(\lambda_{i_1}-j_1)}$
- For $j < \lambda_1$, $Q(e'_{i_1,j}) := A^{\lambda_{i_1}-j} Q(e'_{i_1,\lambda_{i_1}})$
- For $i \neq i_1$, $Q(e'_{ij}) = e'_{ij}$.

Clearly Q is invertible, and it commutes with A. Further, I claim that $S_{QP} < S_P$. It suffices to show that $\forall j \leq j_1.\ f \circ Q(e'_{i_1,j}) = 0$. We have

$$f \circ Q(e'_{i_1,j}) = f\left(A^{\lambda_{i_1}-j}\left(e'_{i_1,\lambda_1} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(\lambda_{i_1}-j_1)}\right)\right) =$$

$$f\left(e'_{i_1,j} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(j-j_1)}\right) = f(e'_{i_1,j}) - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(j-j_1)}.$$

Clearly (by design), this expression is zero when $j = j_1$. And for $j < j_1$, we have $f(e'_{i_1,j}) = f(e'_{i_2,j_2+(j_2-j_1)}) = 0$, so it is zero then as well. Thus we see that $S_{QP} < S_P$, contradicting that S_P is minimal. So there must be at most one i such that there exists j such that $f(e'_{ij}) \neq 0$.

If there is no such i, we have found the desired basis. So, suppose there is such an i_0. Let $U = \langle e'_{i_0,j} \rangle$. Simply write $e_j := e'_{i_0,j}$. Let $j_0 = \min\{j : f(e_j) \neq 0\}$. We just have to change basis to zero out e_j for $j \neq j_0$. Let $P_1(e_{\lambda_{i_0}}) := e_{\lambda_{i_0}} - \frac{f(e_{j_0+1})}{f(e_{j_0})} e_{\lambda_{i_0}-1}$, and notice that $f \circ P_1(e_j) = 0$ for all $j \leq j_0+1$

except for j_0. Then we define $P_2(e_{\lambda_{i_0}}) := P_1(e_{\lambda_{i_0}}) - \frac{f \circ P_1(e_{j_0+2})}{f \circ P_1(e_{j_0})} e_{\lambda_{i_0}-2}$, and notice that $f \circ P_2(e_j) = 0$ for all $j \le j_0 + 2$ except for j_0. Eventually we get $P_{\lambda_{i_0}-j_0}$, and by applying this to the e_j's we obtain a basis of U, in which there is exactly one j with $f \circ P_{\lambda_{i_0}-j_0}(e_j) \neq 0$. $\square$

11.2 When X is all zeroes

In this case we have $A(y, z) = (z_n b, (a \cdot y, z_1, ..., z_{n-1}))$, and the condition becomes

$$\sum_i b_i a_i = 0.$$

In this case we will be able to write down explicitly the irreducible components of $F := \{(V, a, b) : \forall i.\ A_{X,a,b} V_i \subseteq V_i\} \cong \pi^{-1}(X)$. For any nonnegative $\delta_0, \delta_1, ..., \delta_n, \delta_{n+1}$ summing to m, define the corresponding sequence $i_0 = \delta_0$, $i_n = \delta_{n+1} + i_n$, and for $j \in \{1, ..., n\}$, $i_j = i_{j-1} + 1 + \delta_j$. Let E be the span of the e_i's, and let $E' = \{(x, 0) \in E : x \cdot a = 0\}$, where the dot is the m-dimensional dot product. Then define F_δ as the set of $(V, a, b) \in F$ such that

- $b \in V_{i_0} \subseteq E'$
- for all $j \in \{1, ..., n\}$, we have $f_j \in V_{i_j} \subseteq E' + \langle f_1, ..., f_j \rangle$

I claim that the F_δ's are the irreducible components of F. To begin, I show that their union is F.

Lemma 11.2. *Let $(V, a, b) \in \mathcal{F}$. Write $f_0 = b$, and $F = \langle f_0, f_1, ..., f_n \rangle$. For each i, either $F \subseteq V_i$, or else there exists j such that $e_j \notin V_i$, but $\langle e_0, ..., e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_1, ..., e_j \rangle$.*

Proof. For $i = 0$, we may take $j = 0$. Now assume the statement holds for i, and we will prove it for $i + 1$. If $F \subseteq V_i$, then $F \subseteq V_{i+1}$, and we are done.

So, suppose there is j such that $e_j \notin V_i$, but $\langle e_0, ..., e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_0, ..., e_j \rangle$. We have two cases: either $V_{i+1} = V_i + \langle e_j \rangle$, or not.

- If so, then either $j = n$, in which case $F \subseteq V_{i+1}$, or else $j \neq n$, in which case $e_{j+1} \notin V_{i+1}$, but $\langle e_0, ..., e_j \rangle \subseteq V_{i+1} \subseteq E' + \langle e_0, ..., e_{j+1} \rangle$.
- If not, then $e_j \notin V_{i+1}$. Let v_{i+1} be so that $V_{i+1} = V_i + \langle v_{i+1} \rangle$. I just need to show that $v_{i+1} \in Y + \langle e_1, ..., e_j \rangle$. It suffices to show that $v_{i+1}^\top e_k = 0$

for $k > j$. And to do this, it suffices to show that $(Av_{i+1})^\top e_{k-1} = 0$ for $k > j$.

Note that $Av_{i+1} \in V_i \cap A\mathbb{C}^{2n} \subseteq \langle e_1, ..., e_j \rangle$. So, for $k > j+1$ it is clear that $(Av_{i+1})^\top e_{k-1} = 0$. Now, suppose for contradiction that $(Av_{i+1})^\top e_j \neq 0$. Then Av_{i+1} is linearly independent of $e_1, ..., e_{j+1}$. Since $Av_{i+1} \in \langle e_1, ..., e_j \rangle$, it follows that $e_j \in \langle e_1, ..., e_{j-1}, Av_{i+1} \rangle \subseteq V_i$, a contradiction.

□

Corollary 11.3. $\mathcal{F} = \bigcup_\delta \mathcal{F}_\delta$

Proof. Let $(V, a, b) \in F$. Let $i_0 = \min\{i : b \in V_i\}$, and let $i' = \max\{i : V_i \subseteq E'\}$. We want that $b \in V_{i_0} \subseteq E'$, so we want that $i_0 \leq i'$. For contradiction, suppose $i' < i_0$.

□

Lemma 11.4. *Each $\mathcal{F}_\delta$ is a closed subvariety of $\mathcal{F}$.*

Proof. □

Lemma 11.5. *Each $\mathcal{F}_\delta$ is irreducible of dimension m.*

Proof. Let $\mathcal{E}$ be the variety of partial flags of E of shape $(\delta_0, ..., \delta_{n+1})$. Define $g : \mathcal{F}_\delta \to \mathcal{E}$ by

$$(V, a, b) \mapsto 0 \subseteq V_{i_0} \cap E \subseteq V_{i_1} \cap E \subseteq \cdots \subseteq V_{i_{n+1}} \cap E = E.$$

It is clear that g is surjective, as the definition of $\mathcal{F}_\delta$ places no restriction on the intersections $V_i \cap E$. Now let's look more closely at the fibers $g^{-1}(U)$.

First let's find the flags V such that there exist a, b with $g(V, a, b) = U$. We see that, for instance, $V_{i_0} \cap E = U_1$. In fact $V_{i_0} \subseteq E$, so $V_{i_0} = U_1$. But we are free to choose the vector spaces between 0 and V_{i_0} however we wish, so we get some degrees of freedom like $\mathcal{F}_{\delta_0}$, the complete flag variety on $\mathbb{C}^{\delta_0} \cong V_{i_0}/0$. Similarly, for every $j = 1, ..., n+1$, we can choose the vector spaces between $V_{i_{j-1}}$ and V_{i_j} arbitrarily, so we get degrees of freedom like $\mathcal{F}_{\delta_{i_j}}$, the complete flag variety on $\mathbb{C}^{\delta_{i_j}} \cong V_{i_j}/V_{i_{j-1}}$. Finally, to meet the constraint of (V, a, b) being in $\mathcal{F}_\delta$, we can choose any $b \in U_1$ and any a such that $\overline{a} \in U_n^\perp$. Thus we get an isomorphism

$$g^{-1}(U) \cong \mathcal{F}_{\delta_0} \times \cdots \times \mathcal{F}_{\delta_{n+1}} \times \mathbb{C}^{\delta_0} \times \mathbb{C}^{\delta_{n+1}}.$$

□

Theorem 11.6. *The $\mathcal{F}_\delta$'s are the irreducible components of $\mathcal{F}$.*

11.3 In the case that X is a Jordan block

This case seems harder to work with explicitly than the case that X is zero, so our strategy is to reuse

12 A different variety

Define $R = \{(X, \mathfrak{b}_1, \mathfrak{b}_2) : X \in \mathfrak{b}_1 \wedge u(X) \in \mathfrak{b}_2\} \subseteq \mathcal{S}'_{m,n} \times \mathcal{B}_{m+n} \times \mathcal{B}_m$. We can obtain a subvariety of R by requiring that $u(X)$ is in some fixed similarity class. (TODO: why is this a subvariety? Is it? Is this even the right way of explaining the significance of the m^2?) We expect that each of these subvarieties is an irreducible component of dimension m^2. We will verify these things using our prvious computations of springer fibers.

12.1 TODO

- Why are SOn flags what they are.

References

[1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.