1 Definition of one thing

Let $G \subseteq \mathrm{GL}_m(\mathbb{C})$ be a connected semisimple Lie group, and let $\mathfrak{g} \subseteq \mathfrak{gl}_m(\mathbb{C})$ be its Lie algebra. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the subset consisting of nilpotent elements. Let $\mathcal{F}l$ be the variety of Borel subalgebras of $\mathfrak{g}$. Let $\widetilde{\mathcal{N}} = \{(\mathfrak{m}, n) : n \in \mathfrak{m}\} \subseteq \mathcal{M} \times \mathcal{N}$. Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the projection onto the second coordinate. For $n \in \mathcal{N}$, we call $\pi^{-1}(n)$ the *Springer fiber at* n.

2 Slodowy slice

A basis for $\mathfrak{sl}_2(\mathbb{C})$ is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra $\mathfrak{g}$, and a homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ sending (e', h', f') to (e, h, f), we say that (e, h, f) is an $\mathfrak{sl}_2$*-triple*. Observe that e, f must be nilpotent, and h must be Cartan (???). If $\mathfrak{g}$ is semisimple, then given any nilpotent $e \in \mathfrak{g}$, the Jacobson-Morozov theorem [1, 3.7.1] says that there exist $h, f \in \mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple.

Given (e, h, f), we define the *Slodowy slice at* e as $\mathcal{S}_e := e + \ker \mathrm{ad}_f$. By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent $e \in \mathfrak{g}$, when $\mathfrak{g}$ is semisimple.

Let

$$e = \left(\begin{array}{c|ccccc} & & & & & \\ \hline & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & 0 & \\ & & & & 1 & 0 \end{array}\right) \in \mathfrak{sl}_{m+n},$$

where there are n ones. We will find the Slodowy slice at e.

As a first step, we will find an $\mathfrak{sl}_2$ triple with e as its first element. Note that $[h', e'] = 2e'$, and $[e', f'] = h'$, and $[h', f'] = -2f'$. Thus e, h, f must obey the same relations. In particular, $he - eh = 2e$. eh is the last n rows of h, shifted down one, and he is the last $n - 1$ columns of h, shifted left one. Somehow (TODO: fill in) we conclude that f and h are something.

3 Definition of springer fiber at slodowy slice

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m(\mathbb{C})$ be the subset consisting of nilpotent elements. For $X \in \mathcal{N}$ and $a, b \in \mathbb{C}^m$, let

$$A_{X,a,b} = \left(\begin{array}{ccc|ccccc} & & & & & & & | \\ & X & & & & & & b \\ & & & & & & & | \\ \hline — & a & — & 0 & & & & \\ & & & 1 & 0 & & & \\ & & & & 1 & \ddots & & \\ & & & & & \ddots & 0 & \\ & & & & & & 1 & 0 \end{array}\right) \in \mathfrak{gl}_{m+n}(\mathbb{C}).$$

Let $\mathcal{M}$ be the variety of Borel subalgebras of $\mathfrak{gl}_{m+n}(\mathbb{C})$. Let

$$\widetilde{\mathcal{N}} = \{(\mathfrak{m}, a, b, X) : A_{X,a,b} \in \mathfrak{m}\} \subseteq \mathcal{M} \times \mathbb{C}^m \times \mathbb{C}^m \times \mathcal{N}.$$

Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the projection onto the last coordinate. For $X \in \mathcal{N}$, we call $\pi^{-1}(X)$ the *other Springer fiber at* X.

4 Finding the Springer fibers of section 2

We have $\mathcal{M} = \{AHA^{-1} : A \in \mathrm{GL}_{m+n}(\mathbb{C})\}$, where $H \subseteq \mathfrak{gl}_{m+n}(\mathbb{C})$ is the set of upper triangular matrices. We say a map $X : \mathfrak{gl}_{m+n} \to \mathfrak{gl}_{m+n}$ preserves a flag $V_0 \subseteq \cdots \subseteq V_{m+n}$ if $XV_i \subseteq V_i$ for each i. Let $E_0 \subseteq \cdots \subseteq E_{m+n}$ be the standard flag of $\mathbb{C}^{m+n}$. Since H is the set of X which preserve E,

$$\mathcal{M} = \{\{X : \forall i.\ X(AE_i) \subseteq AE_i\} : A \in \mathrm{GL}_n(\mathbb{C})\}.$$

So, for $X \in \mathcal{N}$,

$$\pi^{-1}(X) \cong \{(V, a, b) : \forall i.\ A_{X,a,b}V_i \subseteq V_i\}.$$

In this section we will find the irreducible components of $\pi^{-1}(X)$. Since $\pi^{-1}(X) \cong \pi^{-1}(AXA^{-1})$ for any invertible A, we will assume that X is in Jordan normal form.

Let λ be the shape of X, and let $(e_{ij})_{i \leq r, j \leq \lambda_i}$ be a Jordan basis for X, with $Xe_{ij} = e_{i,j-1}$. Let $f_1, ..., f_n$ be the standard basis for $\mathbb{C}^n$, with $A_{X,a,b}f_i = f_{i+1}$.

4.1 A necessary condition for $A_{X,a,b}$ to be nilpotent

Suppose $A := A_{X,a,b}$ is nilpotent. Define the *height* of a vector $v \in \mathbb{C}^{m+n}$ as the smallest $k \geq 0$ such that $A^k_{X,a,b}v = 0$. Clearly, for $i \leq j$, we have that the height of f_i is geq the height of f_j. Thus, for each $k \geq 0$, we must have $A^k_{X,a,b}f_n \in \langle e_{ij}\rangle_{ij}$. We have $Af_n = \sum_{ij} b_{ij}e_{ij}$. Then $A^2 f_n = \sum_{ij} b_{ij}(e_{i,j-1} + a_{ij}f_1)$, so we must have $\sum_{ij} b_{ij}a_{ij} = 0$. Similarly, since $A^3 f_n \in \langle e_{ij}\rangle$, we see that $\sum_{ij} b_{ij}a_{i,j-1} = 0$. Continuing in this way, we obtain that for all $k \geq 0$,

$$\sum_{ij} b_{ij}a_{i,j-k} = 0. \tag{1}$$

In fact this is in fact a sufficient condition for $A_{X,a,b}$ to be nilpotent, as can be seen from the characteristic polynomial of $A_{X,a,b}$. However, we will not do this calculation here, instead just showing that it is sufficient by giving a Jordan basis.

4.2 When X is all zeroes

In this case we have $A(y,z) = (z_n b, (a \cdot y, z_1, ..., z_{n-1}))$, and the condition becomes

$$\sum_i b_i a_i = 0.$$

In this case we will be able to write down explicitly the irreducible components of $F := \{(V,a,b) : \forall i.\ A_{X,a,b}V_i \subseteq V_i\} \cong \pi^{-1}(X)$. For any nonnegative $\delta_0, \delta_1, ..., \delta_n, \delta_{n+1}$ summing to m, define the corresponding sequence $i_0 = \delta_0$, $i_n = \delta_{n+1} + i_n$, and for $j \in \{1, ..., n\}$, $i_j = i_{j-1} + 1 + \delta_j$. Let E be the span of the e_i's, and let $E' = \{(x,0) \in E : x \cdot a = 0\}$, where the dot is the m-dimensional dot product. Then define F_δ as the set of $(V,a,b) \in F$ such that

- $b \in V_{i_0} \subseteq E'$
- for all $j \in \{1, ..., n\}$, we have $f_j \in V_{i_j} \subseteq E' + \langle f_1, ..., f_j\rangle$

I claim that the F_δ's are the irreducible components of F. To begin, I show that their union is F.

Lemma 4.1. *Let $(V,a,b) \in \mathcal{F}$. Write $f_0 = b$, and $F = \langle f_0, f_1, ..., f_n\rangle$. For each i, either $F \subseteq V_i$, or else there exists j such that $e_j \notin V_i$, but $\langle e_0, ..., e_{j-1}\rangle \subseteq V_i \subseteq E' + \langle e_1, ..., e_j\rangle$.*

Proof. For $i = 0$, we may take $j = 0$. Now assume the statement holds for i, and we will prove it for $i + 1$. If $F \subseteq V_i$, then $F \subseteq V_{i+1}$, and we are done.

So, suppose there is j such that $e_j \notin V_i$, but $\langle e_0, ..., e_{j-1}\rangle \subseteq V_i \subseteq E' + \langle e_0, ..., e_j\rangle$. We have two cases: either $V_{i+1} = V_i + \langle e_j\rangle$, or not.

- If so, then either $j = n$, in which case $F \subseteq V_{i+1}$, or else $j \neq n$, in which case $e_{j+1} \notin V_{i+1}$, but $\langle e_0, ..., e_j\rangle \subseteq V_{i+1} \subseteq E' + \langle e_0, ..., e_{j+1}\rangle$.

- If not, then $e_j \notin V_{i+1}$. Let v_{i+1} be so that $V_{i+1} = V_i + \langle v_{i+1}\rangle$. I just need to show that $v_{i+1} \in Y + \langle e_1, ..., e_j\rangle$. It suffices to show that $v_{i+1}^\top e_k = 0$ for $k > j$. And to do this, it suffices to show that $(Av_{i+1})^\top e_{k-1} = 0$ for $k > j$.

  Note that $Av_{i+1} \in V_i \cap A\mathbb{C}^{2n} \subseteq \langle e_1, ..., e_j\rangle$. So, for $k > j + 1$ it is clear that $(Av_{i+1})^\top e_{k-1} = 0$. Now, suppose for contradiction that $(Av_{i+1})^\top e_j \neq 0$. Then Av_{i+1} is linearly independent of $e_1, ..., e_{j+1}$. Since $Av_{i+1} \in \langle e_1, ..., e_j\rangle$, it follows that $e_j \in \langle e_1, ..., e_{j-1}, Av_{i+1}\rangle \subseteq V_i$, a contradiction.

□

Corollary 4.2. $\mathcal{F} = \bigcup_\delta \mathcal{F}_\delta$

Proof. Let $(V, a, b) \in F$. Let $i_0 = \min\{i : b \in V_i\}$, and let $i' = \max\{i : V_i \subseteq E'\}$. We want that $b \in V_{i_0} \subseteq E'$, so we want that $i_0 \leq i'$. For contradiction, suppose $i' < i_0$.

□

Lemma 4.3. *Each $\mathcal{F}_\delta$ is a closed subvariety of $\mathcal{F}$.*

Proof. □

Lemma 4.4.

4.3 When X is a Jordan block

Suppose X is a single Jordan block of size m. In this case we just write the basis of $\mathbb{C}^m$ as $e_1, ..., e_m$, and the above condition on a and b simply becomes

$$\forall k \geq 0. \sum_{i=k+1}^{m} b_i a_{i-k} = 0. \tag{2}$$

This condition can be simplified even more.

Lemma 4.5. *The condition (1) holds iff there exist nonnegative* m_1, m_2, m_3, $a_1, ..., a_{m_3}$*, and* $b_1, ..., b_{m_3}$ *satisfying the following conditions.*

- $m = m_1 + m_2 + m_3$
- $a = (0, 0, ..., 0, a_1, ..., a_{m_3})$
- $b = (b_1, ..., b_{m_1}, 0, 0, ..., 0)$
- *If* $m_1 \neq 0$*, then* $a_{m_1} \neq 0$
- *If* $m_3 \neq 0$*, then* $b_1 \neq 0$

Proof. If a and b are of this form, then for every $k \geq 0$ we have $b_i a_{i-k} = 0$, so clearly the condition holds.

Now suppose (1) holds. If a or b is zero, this is trivial. Otherwise, let $m_1 = \max\{i : a_i \neq 0\}$, and let $m_3 = \min\{i : b_i \neq 0\}$. We just need to show that $m_1 < m_3$. For contradiction, suppose $m_1 - m_3$ is nonnegative. Then by (1), $0 = \sum_i b_i a_{i-(m_1-m_3)} = b_1 a_{m_1}$, contradicting that b_1 and a_{m_1} are nonzero. □

4.4 In the case of general X

In the case where X was a single Jordan block, it was helpful to have the condition that all the nonzero a's were to the right of all the nonzero b's. In the general case, it is not clear that such a thing should be true. And indeed it is not. However, we will see that we can change basis of $\mathbb{C}^m$ so that a similar thing is true.

Let $c^i_{i',j'} \in \mathbb{C}$ be some arbitrary coefficients. Given our Jordan basis e_{ij} of $\mathbb{C}^m$, we can define a new Jordan basis e'_{ij} of $\mathbb{C}^m$ by

$$e'_{i,\lambda_i} = e_{i,\lambda_i} + \sum_{\{(i',j') : i' \leq i, j' \leq \lambda_i\} \setminus \{(i,j)\}} c^i_{i',j'} e_{i',j'}.$$

Then we define

$$e'_{i,j} = X^{\lambda_i - j} e'_{i,\lambda_i} = e_{ij} + \sum_{\{(i',j') : i' \leq i, j' \leq \lambda_i\} \setminus \{(i,j)\}} c^i_{i',j'} e_{i',j'-(\lambda_i - j)}.$$

Let us calculate the new value a' of a in this new basis. That is, we want the values a'_{ij} that satisfy $Ae'_{ij} = e'_{i,j-1} + a'_{ij} f_1$. Looking at the definition of e'_{ij} above, we see that

$$a'_{ij} = a_{ij} + \sum_{\{(i',j') : i' \leq i, j' \leq \lambda_i\} \setminus \{(i,j)\}} c^i_{i',j'} a_{i',j'-(\lambda_i - j)}.$$

Now, for each i, choose the coefficients $c^i_{i',j'}$ to maximize the value $\max\{j : (a'_{i,1}, ..., a'_{i,j}) = 0\}$. In other words, we choose $c^i_{i'j'}$ to zero out the largest possible prefix of $(a_{i,1}, ..., a_{i,\lambda_i})$.

Lemma 4.6. *The condition of Equation* (1) *on a and b holds iff there exist nonnegative p_i, q_i, r_i, $\alpha_{i,j}$, $\beta_{i,j}$ such that for each i, the following conditions are satisfied.*

- $\lambda_i = p_i + q_i + r_i$
- $(a_{i,1}, ..., a_{i,\lambda_i}) = (0, 0, ..., 0, \alpha_{i,1}, ..., \alpha_{i,r_i})$
- $(b_{i,1}, ..., b_{i,\lambda_i}) = (\beta_{i,1}, ..., \beta_{i,p_i}, 0, 0, ..., 0)$
- *If $r_i \neq 0$, then $a_1 \neq 0$*
- *If $p_i \neq 0$, then $b_{p_i} \neq 0$*

Proof. □

4.5 TODO

- Why are SOn flags what they are.

References

[1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.