1 Introduction

Let $\mathfrak{g} = \mathfrak{gl}_m \times \mathfrak{gl}_{m+n}$, with $n > 0$. Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N} \hookrightarrow \mathfrak{g}$ be the Springer resolution.

Let (e, h, f) be the principal $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n$. We have an embedding $\mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_m \times \mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_{m+n}$. Let (E, H, F) be the $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_{m+n}$ which is the image of (e, h, f) via this embedding. Let S be the Slodowy slice $F + \mathfrak{z}_{\mathfrak{gl}_m}(E)$. We have a map $\mathfrak{gl}_{m+n} \to \mathfrak{gl}_m$ given by coordinate projection. Restricting to S, we obtain $f_1 : S \to \mathfrak{gl}_m$. (It turns out that this map is surjective.) Then, we obtain $p_1 : S \to \mathfrak{g}$ by $x \mapsto (x, f_1(x))$. Taking the map $p_1 : S \to \mathfrak{g}$ and the Springer resolution $\widetilde{\mathcal{N}} \to \mathfrak{g}$, we obtain a fibred product $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$.

In this paper we study $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$. As a step towards studying $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$, we consider the map $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}} \to \mathfrak{gl}_m$ given by taking the map to $\mathfrak{g}$ and then projecting to $\mathfrak{gl}_m$. The main section of this paper will study the fibers of this map. Given $X \in \mathcal{N}_m \subseteq \mathfrak{gl}_m$, we call the fiber at X the *n-Slodowy-slice Springer fiber at X*.

In section 2, we review some preliminary material. In section 3, we find the unique (up to similarity) principal $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n$. In section 4, we embed this $\mathfrak{sl}_2$-triple into $\mathfrak{gl}_{m+n}$ as described above. We compute the Slodowy slice, and we end up with a nice description of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$.

In section 5, we discuss how to reduce the problem of finding the irreducible components of an n-Slodowy-slice Springer fiber at $X \in \mathcal{N}_m$ to an easier problem. In section 6 we solve the easier problem. In section 7 we find the irreducible components of an n-Slodowy-slice Springer fiber. In section 8 we apply the results of section 7 to find the irreducible components of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$. In section 9 we discuss an interesting direction for future work. Then in section 10 we prove some linear algebra facts that were used in the paper.

2 Preliminary Definitions and Facts

2.1 Conventions and Notations

We write $\mathrm{GL}_m, \mathfrak{sl}_m$ to denote $\mathrm{GL}_m(\mathbb{C}), \mathfrak{sl}_m(\mathbb{C})$, and so on. By J_m we refer to the nilpotent $m \times m$ Jordan block (which, by convention, has ones *below* the diagonal). Given a partition $\lambda = (\lambda_1, ..., \lambda_k)$, we write $J(\lambda)$ to denote the block matrix

$$\begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix}.$$

2.2 Springer fibers

Let $\mathfrak{g}$ be a Lie algebra. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the subset consisting of nilpotent elements. Let $\mathcal{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. Let $\widetilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$. Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the projection onto the second coordinate. We call this the *Springer resolution*. For $n \in \mathcal{N}$, we call $\pi^{-1}(n)$ the *Springer fiber at n*.

2.3 Springer fibers in $\mathfrak{gl}_m$

Now we let $\mathcal{N}_m$ be the set of nilpotent elements in $\mathfrak{gl}_m$, and $\mathcal{B}_m$ the variety of Borel subalgebras of $\mathfrak{gl}_m$. Let $H \subseteq \mathfrak{gl}_m$ be the subalgebra of upper triangular matrices. The variety of Borel subalgebras of $\mathfrak{gl}_m$ is $\mathcal{B} = \{gHg^{-1} : g \in \mathrm{GL}_m\}$. Thus, the Springer fiber at $X \in \mathcal{N}$ is

$$S_X = \{gHg^{-1} : X \in gHg^{-1}\}.$$

Definition 2.1. *A* flag (V_i) *of* $\mathbb{C}^m$ *is a sequence of subspaces*

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = \mathbb{C}^m,$$

where $\dim V_i = i$*.*

We say that $X \in \mathfrak{gl}_m$ *preserves* a flag (V_i) if $\forall i.\ XV_i \subseteq V_i$. Note that $X \in \mathcal{N}$ preserves a flag (V_i) if and only if $\forall i.\ XV_i \subseteq V_{i-1}$.

The simplest flag is the *standard flag* (E_i), where $E_i := \langle e_1, ..., e_i \rangle$. Note that the group H is exactly the subset of $\mathfrak{gl}_m$ which preserves E_i.

We can think of $\mathcal{B}$ as the set of flags of $\mathbb{C}^m$, via the correspondence

$$gHg^{-1} \leftrightarrow (gE_i).$$

Note that X preserves (gE_i) if and only if $X \in gHg^{-1}$. Thus, we may write the Springer fiber at $X \in \mathcal{N}$ in terms of flags, as

$$S_X = \{(gE_i) : X \in gHg^{-1}\} = \{(V_i) : \forall i.\ XV_i \subseteq V_{i-1}\}.$$

Theorem 2.2. *(Needs citation!) The irreducible components of the Springer fiber at $J(\mu)$ are in bijection with the standard Young tableaus of shape μ. Further, the irreducible components are equidimensional, of dimension $\sum_{i\neq j}\min(\mu_i,\mu_j)=\sum_i(i-1)\mu_i$.*

2.4 Slodowy Slices

A basis for $\mathfrak{sl}_2$ is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra $\mathfrak{g}$, and a homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ sending (e', h', f') to (e, h, f), we say that (e, h, f) is an $\mathfrak{sl}_2$-*triple*. Observe that e, f must be nilpotent, and h must be Cartan (TODO: what?) If $\mathfrak{g}$ is semisimple, then given any nilpotent $e \in \mathfrak{g}$, the Jacobson-Morozov theorem [1, 3.7.1] says that there exist $h, f \in \mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple.

Given (e, h, f), we define the *Slodowy slice at* e as $\mathcal{S}_e := e + \ker \operatorname{ad}_f$. By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent $e \in \mathfrak{g}$, when $\mathfrak{g}$ is semisimple.

3 Principal $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n$

The unique nilpotent regular element of $\mathfrak{gl}_n$ (up to similarity) is J_n. In this section we use this to find that there is a unique principal $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n$. Write $e = J_n$.

Lemma 3.1. *There is exactly one way to choose $h, f \in \mathfrak{gl}_n$ so that (e, h, f) is an $\mathfrak{sl}_2$-triple.*

Proof. Note that $[h', e'] = 2e'$, and $[e', f'] = h'$, and $[h', f'] = -2f'$. Thus e, h, f must obey the same relations. In particular, $he - eh = 2e$. The matrix eh is h shifted down one, and he is h shifted left one. Thus, $2e_{ij} = (he-eh)_{ij} = h_{i,j+1} - h_{i-1,j}$. We can use this to show that $h_{ij} = 0$ when $i \neq j$. Then we can use it to show that $h_{ii} = h_{i-1,i-1} + 2$, so that $h_{ii} = h_{11} + 2(i-1)$.

Similarly, from $[e, f] = h$ we get that $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$. We can use this to show that $f_{ij} = 0$ when $j \neq i + 1$. Then we can use it to show that $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$, that $f_{1,2} = -h_{1,1}$, and that $f_{n-1,n} = h_{n,n}$. From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^{n} h_{ii} \implies$$

$$\sum_i h_{ii} = 0.$$

Remark: this is just the statement that $h \in \mathfrak{sl}_n$; in other words, we will see that every choice of $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_{m+n}$ is also an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_{m+n}$. This shows that $h_{11} = n-1, h_{22} = n-3, ..., h_{nn} = 1-n$. So we have determined h; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for f in terms of h to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & \\ & 0 & (1-n)+(3-n) & & \\ & & \ddots & \ddots & \\ & & & 0 & (1-n)+\cdots+(n-1) \\ & & & & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1(1-n) & & & \\ & 0 & 2(2-n) & & \\ & & 0 & (n-2)(-2) & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

□

4 Finding $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$

4.1 Finding the Slodowy slice S

In the previous section we computed the principal $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{gl}_n$. Embedding this into $\mathfrak{gl}_{m+n}$ as described previously, we obtain

$$(E, H, F) = \left(\begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} \right).$$

Lemma 4.1. *$\mathfrak{z}_{\mathfrak{gl}_n}(f)$ is the set of upper-triangular $X \in \mathfrak{gl}_n$ such that for all i, j,*

$$X_{i+1,j+1} = \frac{j(n-j)}{i(n-i)} X_{ij}.$$

Proof. Let $X \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$. Looking at the definition of f from the previous section (and zero-padding the matrices), we see that $(fX)_{ij} = i(n-i)X_{i+1,j}$, and $(Xf)_{ij} = (j-1)(n-(j-1))X_{i,j-1}$. So,

$$\forall i, j \in \{1, ..., n\}.\ i(n-i)X_{i+1,j} = (j-1)(n-(j-1))X_{i,j-1}.$$

Taking $j = 1$, the condition above tells us that $\forall i \geq 2.\ X_{i,1} = 0$. Taking $j > 1$ and $i < n$, we obtain that

$$\forall i, j \in \{1, ..., n-1\}.\ X_{i+1,j+1} = \frac{j(n-j)}{i(n-i)} X_{ij}.$$

So, every $X \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$ is upper triangular and satisfies the condition above. Conversely, it is clear that for such X we have $fX = Xf$. $\square$

Lemma 4.2.

$$\mathfrak{z}_{\mathfrak{gl}_{m+n}}(F) = \left\{ \left(\begin{array}{c|c} X & \begin{matrix} | \\ b \\ | \end{matrix} \\ \hline \begin{matrix} — & a & — \end{matrix} & \\ & Y \end{array} \right) : X \in \mathfrak{gl}_m; Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f); a, b \in \mathbb{C}^m \right\}.$$

Proof. Let $Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \in \mathfrak{z}_{m+n}(F)$. We have

$$\begin{pmatrix} 0 & Z_{12}f \\ 0 & Z_{22}f \end{pmatrix} = ZF = FZ = \begin{pmatrix} 0 & 0 \\ fZ_{21} & fZ_{22} \end{pmatrix}.$$

We see that there is no restriction on Z_{11}. The condition $Z_{12}f = 0$ means that all but the last column of Z_{12} must be zero, and the condition $0 = fZ_{21}$ means that all but the first row of Z_{21} must be zero. And the condition $Z_{22}f = fZ_{22}$ means that $Z_{22} \in \mathfrak{z}_{\mathfrak{gl}_{m+n}}(f)$. □

These lemmas provide an explicit characterization of the Slodowy slice $S = E + \mathfrak{z}_{\mathfrak{gl}_{m+n}}(F)$.

Corollary 4.3.

$$S = \left\{ \left(\begin{array}{c|c} X & \begin{matrix} & | \\ & b \\ & | \end{matrix} \\ \hline \begin{matrix} - & a & - \\ & & \end{matrix} & e + Y \end{array} \right) : a, b \in \mathbb{C}^m, X \in \mathfrak{gl}_m, Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f) \right\}.$$

4.2 A description of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$

Recall that we have the map $S \to \mathfrak{g}$ given by $Z \mapsto (p(Z), Z)$, where $p : \mathfrak{gl}_{m+n} \to \mathfrak{gl}_m$ is the coordinate projection. We also have the Springer resolution $\widetilde{\mathcal{N}} \to \mathfrak{g}$. And from these two maps we define the fibered product $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$.

To obtain an explicit description of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$, we will begin by finding the image of the projection $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}} \to S$. As the image of the Springer resolution is $\mathcal{N} = \mathcal{N}_m \times \mathcal{N}_{m+n}$, the image of the projection $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}} \to S$ is simply $S' = \{Z \in S : p(Z) \in \mathcal{N}_m, Z \in \mathcal{N}_{m+n}\}$.

Lemma 4.4.

$$S' = \left\{ A_{X,a,b} := \left(\begin{array}{ccc|cccc} & & & & & & | \\ & X & & & & & b \\ & & & & & & | \\ \hline — & a & — & 0 & & & \\ & & & 1 & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & 0 \end{array}\right) \in \mathcal{N}_{m+n} : a, b \in \mathbb{C}^m, X \in \mathcal{N}_m \right\}.$$

Proof. As every element of $\mathfrak{z}_{\mathfrak{gl}_n}(f)$ is upper triangular, Corollary 10.9 tells us that if a matrix of the form given in Corollary 4.3 is nilpotent, and the upper-left block X is nilpotent as well, then $e + Y$ must be nilpotent. Then, again using that every element $Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$ is upper triangular, Lemma 10.7 tells us that if $e + Y$ is nilpotent, then $Y = 0$.

Hence every element of S' must simply have e in its bottom-right block. We see that every element of S' is of the desired form. $\square$

This is not a fully explicit characterization of S', since we don't say which choices of X and $a, b \in \mathbb{C}^m$ lead to $A_{X,a,b}$ being nilpotent. We could use Lemma 10.8 to find a necessary and sufficient condition on X, a, b; however, the description above of S' will be good enough for our purposes.

5 Simplifying the definition of a Springer fiber at a Slodowy slice

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m(\mathbb{C})$ be the subset consisting of nilpotent elements. For $X \in \mathcal{N}_m$ and $a, b \in \mathbb{C}^m$, let

$$A_{X,a,b} = \left(\begin{array}{ccc|ccccc} & & & & & & & | \\ & X & & & & & & b \\ & & & & & & & | \\ \hline — & a & — & 0 & & & & \\ & & & 1 & 0 & & & \\ & & & & 1 & \ddots & & \\ & & & & & \ddots & 0 & \\ & & & & & & 1 & 0 \end{array}\right) \in \mathfrak{gl}_{m+n}(\mathbb{C}).$$

By the definition given in the previous section, we have

$$\widetilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n},$$

and the Springer fiber at the n-Slodowy slice at a nilpotent $X \in \mathfrak{gl}_m$ is

$$\begin{aligned}\pi_{m,n}^{-1}(X) &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \in \mathcal{S}'_{m,n}\} \\ &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b}\text{nilpotent}\} \\ &\cong \{(\mathfrak{b}, a, b) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \text{ nilpotent}\}.\end{aligned}$$

Using our correspondence between $\mathcal{B}_{m+n}$ and complete flags of $\mathbb{C}^{m+n}$, we obtain

$$\pi_{m,n}^{-1}(X) \cong \{(V, a, b) : \forall i.\ A_{X,a,b}V_i \subseteq V_{i-1}\}.$$

6 Strategy and setup for finding the irreducible components of a Springer fiber at a Slodowy slice

Fix any $X \in \mathcal{N}$. As we have fixed X, we now write $A_{a,b} := A_{X,a,b}$. Let $(e_{ij})_{1\leq i\leq k, 1\leq j\leq \lambda_i}$ be a Jordan basis for X. For convenience we define $e_{i0} := 0$; now we may express the fact that (e_{ij}) is a Jordan basis by writing $\forall i. X e_{ij} = e_{i,j-1}$.

In this section we find the irreducible components of

$$\pi^{-1}(X) \cong V := \{(A_{a,b}, U) : \forall i.\ A_{a,b}U_{i+1} \subseteq U_i\}.$$

For $1 \leq w \leq k$ and $0 \leq r \leq \lambda_w$ (note that we allow $r = 0$), define

$$V_{w,r} := \{(A_{a,b}, U) \in V : \exists P \in \mathrm{GL}_m\ .\ \exists b'.\ (P^{-1}, I_n)A_{a,b}(P, I_n) = A_{e_{wr},b'}\}.$$

Lemma 6.1. *$V = \bigcup_{1\leq w\leq k, 0\leq r\leq \lambda_w} V_{w,r}$. Further, $V_{w_1r_1} = V_{w_2r_2}$ exactly when either $r_1 = r_2 = 0$, or $\lambda_{w_1} = \lambda_{w_2} \wedge r_1 = r_2$. When $V_{w_1r_1} \neq V_{w_2r_2}$, we have $V_{w_1r_1} \cap V_{w_2r_2} = \emptyset$.*

Proof. TODO □

Now, fix any w and r. We will find the irreducible components of $V_{w,r}$. These will all happen to be equidimensional (with dimensions independent of w and r), so their closures in V will be the irreducible components of V.

Let
$$G := \{P \in \mathrm{GL}_m : P^{-1}XP = X\},$$
and
$$G_{wr} = \{A \in G : e_{wr}A = e_{wr}\}.$$

Now, define
$$U_{wr} = \{(A_{e_{wr},b}, U) \in V_{wr}\}.$$
Let G act on V_{wr} by
$$P \cdot (A_{e_{wr},b}, U) := ((P, I_n)A_{e_{wr},b}(P, I_n)^{-1}, (P, I_n)U) = (A_{e_{wr}P^{-1},Pb}, (P, I_n)U).$$

Consider the map $\varphi : U_{wr} \times G \to V_{wr}$ defined by
$$(x, P) \mapsto P \cdot x.$$
By restriction of G to G_{wr} and V_{wr} to U_{wr}, we obtain an action of G_{wr} on U_{wr}. Then, letting G_{wr} act on G by $g \cdot h := hg^{-1}$, we obtain an action of G_{wr} on $U_{wr} \times G$.

Lemma 6.2. *As an algebraic variety, G is irreducible.*

Proof. It's just $\mathbb{C}^{something}$, since its blocks are the lower-left-toeplitz matrices. □

Lemma 6.3. *The map φ is a principal G_{wr}-bundle.*

Proof. We need to show that G_{wr} acts freely and transitively on the fibers of φ. It is obvious that G_{wr} acts freely on $U_{wr} \times G$; it is enough to note that it acts freely on G.

Let $y \in V_{wr}$. By definition of V_{wr}, there is $P_y \in G$ with $P_y \cdot y \in U_{wr}$. We have
$$\begin{aligned}\varphi^{-1}(y) = \{(x, P) : P \cdot x = y\} = \\ \{(P^{-1}y, P) : P^{-1}y \in U_{wr}\} = \\ \{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \cdot (P_y \cdot y) \in U_{wr}\}.\end{aligned}$$

For any $x \in U_{wr}$ (in particular for $x = P_y \cdot y$), we have $G_{wr} = \{g \in G : g \cdot x \in U_{wr}\}$. Therefore, the fiber above becomes

$$\{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \in G_{wr}\}.$$

Setting $Q := P^{-1}P_y^{-1}$, so that $P = P_y^{-1}Q^{-1}$, the above becomes

$$\{(Q \cdot (P_y \cdot y), P_y^{-1}Q^{-1}) : Q \in G_{wr}\} = \\ \{Q \cdot (P_y \cdot y, P_y^{-1}) : Q \in G_{wr}\}.$$

We see that the fibers are exactly the G_{wr}-orbits; or in other words, G_{wr} acts transitively on the fibers, as desired. $\square$

Our strategy is to find the irreducible components $X \subseteq U_{wr}$, and we will then argue that the irreducible components of V_{wr} are of the form $\varphi(X \times G)$. So, we will now find the irreducible components of U_{wr}.

Actually this will be unnecessarily difficult to think about; it is easiest in the case $r = 0$. So, we will change basis to make $r = 0$. Let $\lambda' = (\lambda_1, ..., \lambda_{w-1}, \lambda_w - r, \lambda_{w+1}, ..., \lambda_k)$. Let X' be in Jordan normal form with shape λ'. Let $U'_{wr} = \{(A_{X',0,b}, U) : \forall i. A_{X',0,b} U_{i+1} \subseteq U_i\}$. Let e'_{ij} be a Jordan basis for X'. Let $f_1, ..., f_n$ be a Jordan basis for the restriction of $A_{X,0,b}$ to ... Set $m' := m - r$, and $n' := n + r$. Let $f'_1, ..., f'_{n'}$ blah. Define the linear map $Q_{wr} : \mathbb{C}^{m'+n'} \to \mathbb{C}^{m+n}$ by:

- For all i, $e'_{ij} \mapsto e_{ij}$.
- For $j = 1, ..., r$, $f'_{n+j} \mapsto e_{w,(\lambda_w - r)+j}$.
- For $j = 1, ..., n$, $f'_j \mapsto f_j + e_{w,(\lambda_w - r)-n+j}$.

Observe that conjugation by Q_{wr} maps U_{wr} to U'_{wr}, and conjugation by Q_{wr}^{-1} maps U'_{wr} to U_{wr}. We conclude that $U_{wr} \cong U'_{wr}$; so to find the irreducible components of U_{wr} we just need to find the irreducible components of U'_{wr}. To clear the context, which is rather cluttered by now, and to avoid writing primes everywhere, we move to a new section.

7 Finding the irreducible components of U_0

7.1 Setup

Let X be nilpotent with Jordan basis $(e'_{ij})_{1\le i\le k, 1\le j\le\lambda'_i}$. Let $V = \{(A_{a,b}, U) : \forall i.\ A_{a,b}U_{i+1} \subseteq U\}$. Let

$$U_0 = \{(A_{0,b}, U) \in V\}.$$

We will find the irreducible components of U_0.

We write b_{ij} to denote the projection of $b \in \mathbb{C}^m$ onto e_{ij}. For each row i, let $p_i(b) = \max\{j : b_{ij} \neq 0\}$ (the maximum of the empty set is zero). Then set $q_i(b) = \lambda_i - p_i(b)$. When it is clear enough from context, we will just write p_i and q_i instead of $p_i(b)$ and $q_i(b)$.

Let $I = \{i_1 < \cdots < i_r\} \subseteq \{1, ..., k\}$, and let $(\rho_i)_{i\in I}$ be any map $I \to \mathbb{N}_{>0}$ such that (1) $\rho_i \le \lambda_i$, (2) ρ_i is decreasing with i, (3) $\lambda_i - \rho_i$ is decreasing with i, and (4) $\rho_i < n$. For notational convenience (although we assign meaning to neither i_0 nor i_{r+1}), we define $q_{i_0} := n$, and $p_{i_{r+1}} := 0$. Then, we define $B_{I,(\rho_i)}$ as the set of $b \in \mathbb{C}^m$ satisfying the following conditions.

- For all $k \in \{1, ..., r\}$, $p_i = \rho_i$.
- For all $k \in \{0, ..., r\}$, $p_{i_{k+1}} = \max_{i:q_i<q_{i_k}} p_i$.

Note that for any $b \in B_{I,(\rho_i)}$ we have $p_{i_1} > \cdots > p_{i_r} > p_{i_{r+1}}$, and also $q_{i_0} > q_{i_1} > \cdots > q_{i_r}$.

Let $U_{I,(\rho_i)} := \{(A_{0,b}, U) \in V : b \in B_{I,(\rho_i)}\}$. The idea here is to show that blah relation holds between U and B.

Lemma 7.1. *$\mathbb{C}^m = \bigcup_{I,(\rho_i)} B_{I,(\rho_i)}$, where I ranges over all subsets of $\{1, ..., k\}$, and (ρ_i) ranges over all maps $I \to \mathbb{N}_{>0}$ satisfying the conditions (1),(2),(3),(4). Further, none of the $B_{I,(\rho_i)}$ is contained in the union of the others.*

Proof. Let $b \in \mathbb{C}^m$. If $\{i : q_{i_0} > q_i\}$ is the empty set, then stop. Otherwise, take any $i_1 \in \arg\max_{i:q_{i_0}>q_i} p_i$, and set $\rho_{i_1} := p_{i_1}$. If $\{i : q_{i_1} > q_i\} = \emptyset$, then stop. Otherwise, take any $i_2 \in \arg\max_{i:q_{i_1}>q_i} p_i$, and set $\rho_{i_2} := p_{i_2}$. Continuing on in this way, eventually we reach a point where $\{i : q_{i_k} > q_i\} = \emptyset$. Then we set $I = \{i_1, ..., i_k\}$. Note that $I, (\rho_i)$ satisfy conditions (1)–(4), and furthermore $b \in B_{I,(\rho_i)}$.

Now, we show that no $B_{I,(\rho_i)}$ is contained in the union of the others. Indeed, fix I and (ρ_i). Take any $b \in B_{I,(\rho_i)}$ with $p_i = \rho_i$ for $i \in I$ and $p_i = 0$ for $i \notin I$. It is clear that $b \notin B_{I',(\rho'_i)}$ whenever $I' \neq I$ or $(\rho'_i) \neq (\rho_i)$. $\square$

7.2 A study of $B_{I,(\rho_i)}$

Fix any I and (ρ_i) satisfying the conditions (1)–(4). As before, we write $\{i_1 < \cdots < i_r\} := I$, and $q_{i_0} := n$, and $p_{i_{r+1}} := 0$.

First, we provide an alternative characterization of $B_{I,(\rho_i)}$.

Lemma 7.2. *$B_{I,(\rho_i)}$ is the set of $b \in \mathbb{C}^m$ satisfying the following conditions.*

- *For all $k \in \{1, ..., r\}$, $p_{i_k} = \rho_{i_k}$.*
- *For all $i \notin I$,*
  - *For all $k \in \{0, ..., r\}$ such that $\lambda_i > q_{i_k} + p_{i_{k+1}}$, we have $q_{i_k} \leq q_i$.*
  - *For all $k \in \{1, ..., r+1\}$ such that $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$, we have $p_i \leq p_{i_k}$.*

Proof. First we show that every element of $B_{I,(\rho_i)}$ satisfies those conditions. Let $b \in B_{I,(\rho_i)}$. It is clear that $\forall k.\ p_{i_k} = \rho_{i_k}$.

Take $i \notin I$ and $k \in \{0, ..., r\}$ such that $\lambda_i > q_{i_k} + p_{i_{k+1}}$. Suppose for contradiction that $q_i < q_{i_k}$. Then $p_i \leq \max_{j: q_j < q_{i_k}} p_j = p_{i_{k+1}}$. Then $\lambda_i = p_i + q_i < q_{i_k} + p_{i_{k+1}}$, a contradiction. So we must have $q_{i_k} \leq q_i$, as desired.

Now take $i \notin I$ and $k \in \{1, ..., r+1\}$ such that $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$. Suppose for contradiction that $p_i > p_{i_k}$. Then, putting this together with the first inequality, $\lambda_i - p_i < q_{i_{k-1}} + p_{i_k} - p_{i_k}$; that is, $q_i < q_{i_{k-1}}$. Consequently, $p_i \leq \max_{j: q_j < q_{i_{k-1}}} p_j = p_{i_k}$, as desired.

Now we have shown that every element of $B_{I,(\rho_i)}$ satisfies the conditions of the lemma, and we proceed to the converse. Let $b \in \mathbb{C}^m$ satisfy the conditions. Let $k \in \{0, ..., r\}$. We need to show that $\max_{j: q_j < q_{i_k}} p_j = p_{i_{k+1}}$. Given the conditions (1)–(4) on ρ_i, it suffices to show that for each $i \notin I$ with $q_i > q_{i_k}$, we have $p_i \leq p_{i_{k+1}}$. Indeed, given $i \notin I$ with $q_i > q_{i_k}$, we cannot have $\lambda_i > q_{i_k} + p_{i_{k+1}}$, as that would imply that $q_{i_k} \leq q_i$. Hence we must have $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$, and consequently $p_i \leq p_{i_k}$. $\square$

Corollary 7.3. *$B_{I,(\rho_i)}$ is the set of $b \in \mathbb{C}^m$ satisfying the following conditions.*

- *For all $k \in \{1, ..., r\}$, $p_{i_k} = \rho_{i_k}$.*
- *For $i \notin I$,*
  - *If $\lambda_i \geq q_{i_0} + p_{i_1}$, then $p_i \leq \lambda_i - q_{i_0}$.*
  - *If there is $k \in \{1, ..., r\}$ with $q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}$, then $p_i \leq \min(p_{i_k}, \lambda_i - q_{i_k})$.*

– *If* $q_{i_r} + p_{i_{r+1}} > \lambda_i$, *then* $p_i \leq p_{i_{r+1}}$.

Proof. Both q_{i_k} and p_{i_k} are decreasing as k increases, so this follows directly from Lemma 7.2. (Note that $p_i \leq \lambda_i - q_{i_k}$ iff $q_{i_k} \leq q_i$.) $\square$

Corollary 7.4.

$$B_{I,(\rho_i)} \cong \prod_{k=1}^{r} (\mathbb{C}^{\rho_{i_k}-1} \times (\mathbb{C} \backslash \{0\})) \times \prod_{i:\lambda_i \geq q_{i_0}+p_{i_1}} \mathbb{C}^{\lambda_i - q_{i_0}} \times \prod_{k=1}^{r} \prod_{\{i \notin I: q_{i_{k-1}}+p_{i_k} > \lambda_i \geq q_{i_k}+p_{i_{k+1}}\}} \mathbb{C}^{\min(p_{i_k}, \lambda_i - q_{i_k})}$$

Proof. Here I use the notation $x \times y = (x, y)$, and so on. The isomorphism sends $b \in B_{I,(\rho_i)}$ to

$$\prod_{k=1}^{r} (b_{i_k 1}, ..., b_{i_k \rho_{i_k}}) \times \prod_{i:\lambda_i \geq q_{i_0}+p_{i_1}} (b_{i1}, ..., b_{i,\lambda_i - q_{i_0}}) \times$$

$$\prod_{k=1}^{r} \prod_{\{i \notin I: q_{i_{k-1}}+p_{i_k} > \lambda_i \geq q_{i_k}+p_{i_{k+1}}\}} (b_{i1}, ..., b_{i,\min(p_{i_k}, \lambda_i - q_{i_k})}).$$

Corollary 7.3 tells us that this is an isomorphism. $\square$

Corollary 7.5.

$$\dim B_{I,(\rho_i)} = \sum_{i:\lambda_i \geq q_{i_0}+\rho_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{r} \sum_{\{i: q_{i_{k-1}}+p_{i_k} > \lambda_i \geq q_{i_k}+p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}).$$

Proof. Immediate from Corollary 7.4. $\square$

7.3 A study of $U_{I,(\rho_i)}$

Fix any I and (ρ_i) satisfying the conditions of Lemma 8.1. We will find the irreducible components of $U_{I,(\rho_i)}$, and show that their closures are in fact irreducible components of U_0.

I claim that $A_{0,b}$ has the same shape for every $(A_{0,b}, U) \in U_{I,(\rho_i)}$. Call this shape μ. By finding a algebraic map taking b to a Jordan basis for $A_{0,b}$, we will put $U_{I,(\rho_i)}$ in isomorphism with the product (choice of b) $\times$ (Springer fiber at $J(\mu)$). Then we will use our result about the usual Springer fiber at $J(\mu)$ to find the irreducible components of $U_{I,(\rho_i)}$.

So, now we find a Jordan basis for $A_{0,b}$.

Lemma 7.6. *The following vectors give a Jordan basis for $A_{0,b}$. (For convenience, we write $A := A_{0,b}$ in this lemma and proof.)*

- *For $i \notin I$, the chain of length $p_i + q_i$ beginning with e_{i,p_i+q_i}*
- *The chain of length $n + p_{i_1}$ beginning with $f_n - (A^{n+p_{i_1}} f_n \texttt{>>}\ n + p_{i_1})$*
- *For $k \in \{1, ..., r\}$, the chain of length $q_{i_k} + p_{i_{k+1}}$ beginning with $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \texttt{>>}\ q_{i_k} + p_{i_{k+1}})$, where $v_{i_k} := A^{n-q_{i_k}} f_n - \sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i,l+q_{i_k}}$*

Proof. There are three things to check: (1) the chains are no longer than their claimed lengths, (2) the claimed lengths sum to $m + n$, and (3) the span of the chains is $\mathbb{C}^{m+n}$.

Proof of (1). It is obvious that a chain beginning with e_{i,p_i+q_i} has length $p_i + q_i$.

Now consider the chain beginning with $f_n - (A^{n+p_{i_1}} f_n \texttt{>>}\ n + p_{i_1})$. Note that $A^n f_n = b$, so $A^{n+p_{i_1}} f_n = A^{p_{i_1}} b = b \texttt{<<} p_{i_1}$. By shifting b left p_{i_1} times, we zero out all the rows i where $q_i < n$. This ensures that the operation of shifting $b \texttt{<<} p_{i_1}$ right $n + p_{i_1}$ times is invertible by shifting left $n + p_{i_1}$ times. That is,

$$\begin{aligned} A^{n+p_{i_1}} f_n &= \\ b \texttt{<<} p_{i_1} &= \\ ((b \texttt{<<} p_{i_1})\ \texttt{>>}\ n + p_{i_1}) \texttt{<<} n + p_{i_1} &= \\ A^{n+p_{i_1}} (A^{n+p_{i_1}} f_n\ \texttt{>>}\ n + p_{i_1}). \end{aligned}$$

This shows that the chain has length at most $n + p_{i_1}$, as desired.

Now, let $k \in \{1, ..., r\}$. We have $v_{i_k} = A^{n-q_{i_k}} f_n - \sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i_k,l+q_{i_k}}$. First we note that $q_{i_k} < n$ by the definition of $U_{I,(\rho_i)}$, so the definition of v_{i_k} makes sense. We consider the chain beginning with $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \texttt{>>} q_{i_k} + p_{i_{k+1}})$. Note that $A^{q_{i_k}} v_{i_k}$ is just b with row i_k zeroed out. For brevity, we write $b_{i_k} := A^{q_{i_k}} v_{i_k}$. Note that $b_{i_k} \texttt{<<} p_{i_{k+1}}$ has rows l zeroed out, for all l with $q_l < q_{i_k}$. This ensures that shifting $b_{i_k} \texttt{<<} p_{i_{k+1}}$ right $q_{i_k} + p_{i_{k+1}}$ times can be inverted by shifting left $q_{i_k} + p_{i_{k+1}}$ times. That is,

$$\begin{aligned} A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} &= \\ b_{i_k} \texttt{<<} p_{i_{k+1}} &= \\ ((b_{i_k} \texttt{<<} p_{i_{k+1}})\ \texttt{>>}\ q_{i_k} + p_{i_{k+1}}) \texttt{<<} q_{i_k} + p_{i_{k+1}} &= \\ A^{q_{i_k}+p_{i_{k+1}}} (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k}\ \texttt{>>}\ q_{i_k} + p_{i_{k+1}}). \end{aligned}$$

This shows that the chain has length at most $q_{i_k} + p_{i_{k+1}}$, as desired. □

Proof of (2). The sum of the lengths is

$$\sum_{i \notin I} (p_i + q_i) + (n + P) + \sum_{i \in I} (q_i + P_i).$$

Writing $I = \{i_1 < \cdots < i_{|I|}\}$, we note that $P = p_{i_1}$, that $P_{i_{|I|}} = 0$, and that for $l < |I|$ we have $P_{i_l} = p_{i_{l+1}}$. Thus, the sum above is

$$\sum_{i \notin I} (p_i + q_i) + (n + p_{i_1}) + \sum_{l=1}^{|I|} (q_{i_l} + p_{i_{l+1}}) =$$

$$\sum_{i \notin I} (p_i + q_i) + n + \sum_{l=1}^{|I|} (q_{i_l} + p_{i_l}) = n + \sum_{i} (q_i + p_i) = m + n.$$

□

Proof of (3). Let W be the span of the chains listed. We need to show that $W = \mathbb{C}^{m+n}$. Because every $i \in I$ satisfies $q_i < n$, clearly $\langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$.

I claim that $f_n \in W$ as well. To see this, we consider the chain beginning with $f_n - (A^{n+P} f_n >> n + P)$. As explained in the proof of (1), we have $A^{n+P} f_n \in \langle e_{ij} \rangle_{i,j:q_i \geq n}$. Consequently, $(A^{n+P} f_n >> n + P) \in \langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$. Because $f_n - (A^{n+P} f_n >> n + P) \in W$, this implies that $f_n \in W$.

Because $AW \subseteq W$ (obvious, since W is the span of chains), the fact that $f_n \in W$ implies that $f_i \in W$ for each i, and also $b << l \in W$ for each $l \geq 0$.

Now we are left with showing that $\langle e_{ij} \rangle_{i,j:q_i < n} \subseteq W$. This is obvious for $i \notin I$. For $i \in I$, we do it inductively. Fix $i \in I$, and suppose we have already shown that for all $i' \in I$ with $q_{i'} > q_i$, we have $\langle e_{ij} \rangle_{j \text{ arbitrary}} \subseteq W$. We will show that $\langle e_{ij} \rangle_{j \text{ arbitrary}} \subseteq W$.

To see this, we consider the chain beginning with $v_i - (A^{q_i+P_i} v_i >> q_i + P_i)$. (Recall $v_i = A^{n-q_i} f_n - \sum_{l=1}^{p_i} b_{il} e_{i,l+q_i}$.) Because $AW \subseteq W$, and $b_{i,p_i} \neq 0$ (by definition of p_i), it suffices to show that $\sum_{l=1}^{p_i} b_{il} e_{i,k+q_i} \in W$. As explained in the proof of (1), we have $A^{q_i+P_i} v_i \in \langle e_{lj} \rangle_{q_l \geq q_i}$. And since $A^{q_i+P_i} v_i$ has row i zeroed out, in fact $A^{q_i+P_i} v_i \in \langle e_{lj} \rangle_{l:l \neq i \wedge q_l \geq q_i}$. Hence, $A^{q_i+P_i} v_i >> q_i + P_i \in \langle e_{lj} \rangle_{l:l \neq i \wedge q_l \geq q_i}$. By our inductive hypothesis, $\langle e_{lj} \rangle_{l:l \neq i \wedge q_l \geq q_i} \subseteq W$, and consequently $A^{q_i+P_i} v_i >> q_i + P_i \in W$. Since we know $v_i - (A^{q_i+P_i} v_i >> q_i + P_i) \in W$, this implies that $v_i \in W$. Because $A^{n-q_i} f_n \in W$, this then implies that $\sum_{l=1}^{p_i} b_{il} e_{i,l+q_i} \in W$, as desired. □

We proved (1), (2), (3), so we are done. $\square$

Let $\mu(I,(\rho_i))$ be the shape of the Jordan basis given in the previous lemma. Let X_μ be the Springer fiber at $J(\mu)$, and let $(X_{\mu,\alpha})_{\alpha\in\mathrm{SYT}(\mu)}$ be the irreducible components.

Given a zero-indexed list $L=[L_0,...,L_{l-1}]$, we define $\gamma(L)=\sum_i iL_i$. We are interested in this thing because the dimension of X_μ is $\gamma([\mu_1,...,\mu_l])$.

Lemma 7.7.

$$\gamma([\mu_1,...,\mu_l])=\gamma([0,\lambda_1,...,\lambda_k])-$$

$$\left[\sum_{i:\lambda_i\geq q_{i_0}+\rho_{i_1}}(\lambda_i-q_{i_0})+\sum_{k=1}^{r}\sum_{\{i:q_{i_{k-1}}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_{k+1}}\}}\min(p_{i_k},\lambda_i-q_{i_k})\right].$$

Proof. Let $L=[q_{i_0},\lambda_1,...,\lambda_k]$. Note that $\mu=[...,q_{i_0}+p_{i_1},...,q_{i_1}+p_{i_2},...,...,q_{i_r}+p_{i_{r+1}},...]$. Let L' be the result of taking μ and, for each x, replacing one occurence of $q_{i_x}+p_{i_{x+1}}$ by $q_{i_x}+p_{i_x}$; that is, $L'=[...,q_{i_0},...,q_{i_1}+p_{i_1},...,q_{i_2}+p_{i_2},...,q_{i_r}+p_{i_r}]$. Note that we get from μ to L' by just moving each p_{i_k} to the right by $1+\#\{i\notin I:q_{i_{k-1}}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_{k+1}}\}$ slots. So,

$$\gamma(L')-\gamma(\mu)=\sum_{k=1}^{r}p_{i_k}(1+\#\{i\notin I:q_{i_{k-1}}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_{k+1}}\})=$$

$$\sum_{k=1}^{r}p_{i_k}\cdot\#\{i:q_{i_{k-1}}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_{k+1}}\}.$$

Now we consider how to get from L' to L. First we shift q_{i_0} to the left by $\#\{i\notin I:\lambda_i\geq q_{i_0}+p_{i_1}\}$ slots. Then we leave q_{i_0} in place and sort the rest of the list. This entails shifting each $q_{i_x}+p_{i_x}$ to the left by $\#\{i\notin I:q_{i_x}+p_{i_x}>\lambda_i\geq q_{i_x}+p_{i_{x+1}}\}$ slots. Shifting $q_{i_x}+p_{i_x}$ to the left one slot, by swapping it with λ_i, changes the value of γ by $\lambda_i-(q_{i_x}+p_{i_x})$. To go from L' to L, we can just make these swaps repeatedly. So,

$$\gamma(L)-\gamma(L')=$$

$$\sum_{i\notin I:\lambda_i\geq q_{i_0}+p_{i_1}}(\lambda_i-q_{i_0})+\sum_{k=1}^{s}\sum_{\{i\notin I:q_{i_k}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_{k+1}}\}}(\lambda_i-(q_{i_k}+p_{i_k})).$$

Now,

$$
\begin{aligned}
&\gamma(L)-\gamma(\mu) = \\
&[\gamma(L)-\gamma(L')]+[\gamma(L')-\gamma(\mu)] = \\
&\left[\sum_{i:\lambda_i \geq q_{i_0}+p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{s} \sum_{\{i:q_{i_k}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_{k+1}}\}} (\lambda_i - (q_{i_k}+p_{i_k}))\right] + \\
&\left[\sum_{k=1}^{s} \sum_{\{i:q_{i_{k-1}}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_{k+1}}\}} p_{i_k}\right] = \\
&\sum_{i:\lambda_i \geq q_{i_0}+p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{s} \sum_{\{i:q_{i_{k-1}}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_k}\}} p_{i_k} + \\
&\sum_{k=1}^{s} \sum_{i:q_{i_k}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_{k+1}}} (\lambda_i - q_{i_k}) = \\
&\sum_{i:\lambda_i \geq q_{i_0}+p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{s} \sum_{\{i:q_{i_{k-1}}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}).
\end{aligned}
$$

□

Lemma 7.8. $U_{I,(\rho_i)} \cong B_{I,(\rho_i)} \times X_\mu$.

Proof. For $b \in B_{I,(\rho_i)}$, let P_b be the change-of-basis matrix, with columns given by the Jordan basis of the previous lemma, so that $J(\mu) = P_b^{-1} A_{0,b} P_b$. From looking at the Jordan basis of the previous lemma (the basis can be expressed in terms of $A, \lambda, e_{ij}, f_i, I, \rho_i, b$), it is clear that the map $P : B_{I,(\rho_i)} \to \mathrm{GL}_{m+n}$, given by $b \mapsto P_b$, is algebraic.

Now, we remark that the Springer fiber at $A_{0,b}$ is simply $\{P_b U : U \in X_\mu\}$. This gives us the isomorphism $B_{I,(\rho_i)} \times X_\mu \to U_{I,(\rho_i)}$.

$$(b, U) \mapsto (P_b J(\mu) P_b^{-1}, P_b U),$$

with inverse

$$(A_{0,b}, U) \mapsto (b, P_b^{-1} U).$$

□

Corollary 7.9. *The* $U_{I,(\rho_i)}$ *is irreducible, and its dimension is* $\gamma([0, \lambda_1, ..., \lambda_k])$.

Proof. Irreducibility follows from Lemma 7.8, since $B_{I,(\rho_i)}$ is irreducible (by Corollary 7.4), as is X_μ. To get the dimension, we add the dimension of $B_{I,(\rho_i)}$ to the dimension of X_μ. The dimension of X_μ is □

7.4 Conclusion

At last we have reached the base of our recursive procedure, and we begin to propagate upwards. We write $(X_{\mu,\alpha})_\alpha$ to denote the irreducible components of X_μ.

Lemma 7.10. *The irreducible components of $U_{I,(\rho_i)}$ are exactly the subvarieties*

$$U_{I,(\rho_i),\alpha} := \{(P_b J(\mu) P_b^{-1}, P_b U) : b \in B_{I,(\rho_i)}, U \in X_{\mu,\alpha}\}.$$

These are distinct. There are blah of them, and each has dimension blah.

Proof. We know from Corollary 7.4 that $B_{I,(\rho_i)}$ is irreducible. So, the components of $B_{I,(\rho_i)} \times X_\mu$ are obviously $B_{I,(\rho_i)} \times X_{\mu,\alpha}$. Their dimensions are blah, and there are... of them. taking image under the isomorphism gives the desired result. □

Lemma 7.11. *The irreducible components of U_0 are exactly the $U_{I,(\rho_i),\alpha}$. They are distinct. There are blah of them, each of dimension blah.*

8 Finding the irreducible components of a springer fiber at a slodowy slice

Apply things from the previous two sections, and conclude.

9 The components of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$

Define $R = \{(X, \mathfrak{b}_1, \mathfrak{b}_2) : X \in \mathfrak{b}_1 \wedge u(X) \in \mathfrak{b}_2\} \subseteq \mathcal{S}'_{m,n} \times \mathcal{B}_{m+n} \times \mathcal{B}_m$. We can obtain a subvariety of R by requiring that $u(X)$ is in some fixed similarity class. We expect that each of these subvarieties is an irreducible component of dimension m^2. We will verify these things using our prvious computations of springer fibers.

10 Linear Algebra Facts

In this section we prove linear algebra facts that were used in the paper. They are confined to this section to avoid interrupting the flow of the paper.

10.1 The centralizer of a nilpotent matrix

Definition 10.1. *A matrix Y is* Toeplitz *if it is constant along bands parallel to the main diagonal. That is, $\forall i, j, k$. $Y_{ij} = Y_{i+k,j+k}$.*

Definition 10.2. *An $m \times n$ matrix Y is* lower-left Toeplitz *if it is Toeplitz and, in addition, we have $y_{n-i,j-1} = 0$ whenever $i + j \geq \min(m, n)$.*

That is, Y is lower-left Toeplitz if it is Toeplitz, and the only nonzero entries are those with Manhattan distance less than $\min(m, n)$ from the entry in the bottom-left corner. In yet other words, all but the leftmost (equivalently, bottommost) $\min(m, n)$ diagonal bands are zero.

Lemma 10.3. *Let $\lambda = (\lambda_1, ..., \lambda_k)$ be a partition of m. The centralizer of $J(\lambda)$ in $\mathfrak{gl}_m$ is the subalgebra consisting of matrices*

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix},$$

where each M_{ij} is a $\lambda_i \times \lambda_j$ matrix, such that each M_{ij} is lower-left Toeplitz.

Proof. Let

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix}.$$

We need to show that $J(\lambda)M = MJ(\lambda)$ if and only if each M_{ij} is lower-left Toeplitz.

We have

$$J(\lambda)M = \begin{pmatrix} J_{\lambda_1}M_{11} & \cdots & J_{\lambda_1}M_{1k} \\ \vdots & & \vdots \\ J_{\lambda_k}M_{k1} & \cdots & J_{\lambda_k}M_{kk} \end{pmatrix}, \text{ and } MJ(\lambda) = \begin{pmatrix} M_{11}J_{\lambda_1} & \cdots & M_{1k}J_{\lambda_k} \\ \vdots & & \vdots \\ M_{k1}J_{\lambda_1} & \cdots & M_{kk}J_{\lambda_k} \end{pmatrix}.$$

So, we have $J(\lambda)M = MJ(\lambda)$ if and only if $\forall i,j.\ J_{\lambda_i}M_{ij} = M_{ij}J_{\lambda_j}$. Multiplying on the left by J_{λ_i} just shifts each row down by one, and multiplying on the right by J_{λ_j} shifts each column left by one. The matrices for which left-shifting gives the same result as down-shifting are exactly the lower-left Toeplitz matrices. $\square$

For nilpotent X in Jordan form and a in 'normalized' form (i.e., with at most one nonzero element, which is a one), we will find the centralizer of $A_{X,a,b}$ in

$$\{(A, I) : A \in \mathfrak{gl}_m\} \subseteq \mathfrak{gl}_{m+n}\,.$$

Note that an element of the form (A, I) commutes with $A_{X,a,b}$ if and only if A commutes with X, and $aA = a$, and $Ab = b$. So, we just have to find

$$\{A \in \mathfrak{gl}_m : AX = XA, aA = a, Ab = b\}.$$

It is just the set of $A = [A_{ij}]_{ij}$, where each A_{ij} is lower-left-Toeplitz.

Let v_{ij} be the leftmost column of A_{ij}, so that

$$A_{ij} = \begin{pmatrix} v_{ij} \curlyvee 0 & \cdots & v_{ij} \curlyvee [\lambda_j - 1] \end{pmatrix}.$$

Since A_{ij} is lower-left-Toeplitz, v_{ij} is of the form

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i,\lambda_j)}$ can be chosen freely. Now we determine which matrices of this form satisfy $aA = a$. For simplicity, we will instead find the matrices A such that $aA = 0$ (so that $a(A + I) = a$). (Note that $I \in C_1$, and C_1 is closed under addition, so this is really the same thing.)

In the case that $a = 0$, clearly every A works. Otherwise, let i_0, j_0 be such that $a_{i_0,j_0} = 1$. Now, clearly the constraint that $aA = 0$ is just saying that the (i_0, j_0)th row of A must be zero. That is, for each j the j_0th row of $A_{i_0 j}$ must be zero. This just requires that for each j, we must have

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0,\lambda_j)}$ can be chosen freely. So, we have now found the set

$$C_2 := \{A \in \mathfrak{gl}_m : AX = XA, aA = a\}.$$

It is the matrices of the form $I+A$, where A_{ij} is a block matrix of size $\lambda_i \times \lambda_j$ with
$$A_{ij} = T(v_{ij}) = T\begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$
where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$ in the case $i \neq i_0$ and $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$ if $i = i_0$.

10.2 A 'normalization' fact about Jordan bases

Lemma 10.4. *For any finite-dimensional V, nilpotent $A : V \to V$, and linear $f : V \to \mathbb{C}$, there is a Jordan basis e_{ij} for A such that there is at most one i such that there exists j such that $f(e_{ij}) \neq 0$.*

Proof. For any Jordan basis $(e_{ij})_{ij}$ of A, define
$$S((e_{ij})_{1\le i\le k, 1\le j\le \lambda_i}) := \sum_i \begin{cases} -1, & \forall j.\ f(e_{ij}) = 0 \\ \lambda_i - \min\{j : f(e_{ij}) \neq 0\}, & \text{otherwise} \end{cases}.$$
We proceed by induction on the measure S. That is, let $(e_{ij})_{ij}$ be a Jordan basis for A. Our inductive hypothesis is that if there exists a Jordan basis $(e'_{ij})_{ij}$ with $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$, then we get the desired conclusion.

Now, we have two cases. In the first case, $(e_{ij})_{ij}$ already satisfies the desired property. In this case we are done. In the other case, there exist i_1, j_1, i_2, j_2 with $i_1 \neq i_2$, and $f(e_{i_1 j_1}) \neq 0$, and $f(e_{i_2 j_2}) \neq 0$. We let j_1, j_2 be minimal with this property, so that $\forall j < j_1.\ f(e_{i_1 j}) = 0$, and $\forall j < j_2.\ f(e_{i_2 j}) = 0$. Wlog, we assume that $\lambda_{i_1} - j_1 \le \lambda_{i_2} - j_2$.

By our inductive hypothesis, all we need to do is find a Jordan basis $(e'_{ij})_{ij}$ with $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$. This is what we do. Define e'_{ij} as follows.

- $e'_{i_1,\lambda_1} := e_{i_1,\lambda_1} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} e_{i_2, j_2 + (\lambda_{i_1} - j_1)}$
- For $j < \lambda_1$, $e'_{i_1,j} := A^{\lambda_{i_1} - j} e'_{i_1,\lambda_{i_1}}$
- For $i \neq i_1$, $e'_{ij} := e_{ij}$.

Clearly this is a Jordan basis for A. Further, I claim that $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$. It suffices to show that $\forall j \le j_1.\ f(e_{i_1,j}) = 0$. We have
$$f(e'_{i_1,j}) = f\left(A^{\lambda_{i_1} - j}\left(e_{i_1,\lambda_1} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} e_{i_2,j_2+(\lambda_{i_1}-j_1)}\right)\right) =$$

$$f\left(e_{i_1,j} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} e_{i_2,j_2+(j-j_1)}\right) = f(e_{i_1,j}) - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} f(e_{i_2,j_2+(j-j_1)}).$$

Clearly (by design), this expression is zero when $j = j_1$. And for $j < j_1$, we have $f(e_{i_1,j}) = f(e_{i_2,j_2+(j-j_1)}) = 0$, so it is zero then as well. Hence the measure S of this new basis is smaller, as desired. $\square$

Lemma 10.5. *For any n and linear $f : \mathbb{C}^n \to \mathbb{C}$, there is a Jordan basis e_j for J_n such that there is at most one j with $f(e_j) \neq 0$.*

Proof. Let e_j be a Jordan basis for J_n. If $\{j : f(e_j) \neq 0\}$ is the empty set, we are done. Otherwise, let $j_0 = \min\{j : f(e_j) \neq 0\}$. For any Jordan basis f_j with $j_0 = \min\{j : f(e_j) \neq 0\}$, define

$$S((f_j)_j) := \begin{cases} -1, & \{j > j_0 : f(e_j) \neq 0\} = \emptyset \\ n - \min\{j > j_0 : f(e_j) \neq 0\}, & \text{otherwise} \end{cases}.$$

We proceed by induction on S. That is, let $(e_j)_j$ be a Jordan basis for J_n with $j_0 = \min\{j : f(e_j) \neq 0\}$. Our inductive hypothesis is that if there exists a Jordan basis $(e'_j)_j$ with $j_0 = \min\{j : f(e'_j) \neq 0\}$ and $S((e'_j)_j) < S((e_j)_j)$, then the conclusion holds.

We have two cases: either $(e_j)_j$ satisfies the desired property, or not. If not, then let $j_1 = \min\{j > j_0 : f(e_j) \neq 0\}$, and define a new Jordan basis e'_j as follows.

- $e'_n := e_n - \frac{f(e_{j_1})}{f(e_{j_0})} e_{n-(j_1-j_0)}$
- For $j < n$, $e'_j := J_n^{n-j} e'_n$

It is straightforward to check that $j_0 = \min\{j : f(e'_j) \neq 0\}$, and that $S((e'_j)_j) \leq S((e_j)_j) - 1$. By our inductive hypothesis, we are done. $\square$

Theorem 10.6. *For any finite-dimensional V, nilpotent $A : V \to V$, and linear $f : V \to \mathbb{C}$, there is a Jordan basis e_{ij} for A such that there is at most one pair (i, j) with $f(e_{ij}) \neq 0$.*

Proof. Lemma 10.4 provides a Jordan basis e_{ij} such that for all $i \neq i_0$ and all j, we have $f(e_{ij}) = 0$. Restricting A to $\langle e_{i_0 j}\rangle_{j \text{ arbitrary}}$ gives a Jordan block, and then applying Lemma 10.5 gives the desired result. $\square$

10.3 Nilpotency Lemmas

Lemma 10.7. *Let $X \in \mathfrak{gl}_n$ be upper triangular. Then $J_n + X$ is nilpotent if and only if $X = 0$.*

Proof. Clearly if $X = 0$, then $J_n + X$ is nilpotent. Inversely, suppose $X \neq 0$. Let $e_1, ..., e_n$ be the standard basis, with $J_n e_i = e_{i+1}$. Let $i_1 = \max\{i : Xe_i \neq 0\}$. As X is upper triangular, we have $Xe_{i_1} = v + ae_{i_2}$, with $a \in \mathbb{C} \setminus \{0\}$, $i_2 \leq i_1$, and $v \in \langle e_1, ..., e_{i_1 - 1}\rangle$.

Now, $(J_n + X)^{i_1} e_1 = e_{i_1+1} + v + ae_{i_2}$. Then, $(J_n + X)^{i_1 + (n-i_1)} e_1 = 0 + (J_n + X)^{n-i_1}(v + ae_{i_2})$. Clearly $(J_n + X)^{n-i_1}(v + ae_{i_2}) = v' + ae_{i_2+n-i_1}$, with $v' \in \langle e_1, ..., e_{i_2+n-i_1-1}\rangle$. Now, since $i_2 \leq i_1$, we have $i_2 + n - i_1 \leq n$, and therefore $(J_n + X)^n e_1 \neq 0$. It follows that $J_n + X$ is not nilpotent. $\square$

Lemma 10.8. *Let $X \in \mathfrak{gl}_m$, and let*

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ d_1 & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ & d_2 & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ & & \ddots & \cdots & \vdots & \vdots \\ & & & d_{n-2} & y_{n-1,n-1} & y_{n-1,n} \\ & & & & d_{n-1} & y_{nn} \end{pmatrix} \in \mathfrak{gl}_n \,.$$

For any $a, b \in \mathbb{C}^m$,

$$\det \left(\begin{array}{c|c} X & \begin{matrix} & | \\ & b \\ & | \end{matrix} \\ \hline \begin{matrix} - & a & - \\ & & \end{matrix} & Y \end{array}\right) =$$

$$\det X \det Y + \left(\prod_i d_i\right) \det \left(\begin{array}{c|c} X & \begin{matrix} | \\ b \\ | \end{matrix} \\ \hline - \; a \; - & 0 \end{array}\right)$$

Proof. By induction on n. In the case $n = 1$, expanding along the last row (taking the usual interpretation of the empty product) gives the desired result.

Now suppose $n > 1$. Expanding along the last row, we get

$$d_{n-1} \det \left(\begin{array}{ccc|c} & X & & \vert \\ & & & b \\ & & & \vert \\ \hline — & a & — & \\ & & & Y_{n,n-1} \\ & & & \end{array}\right) - y_{nn} \det \left(\begin{array}{ccc|c} & X & & \\ & & & \\ \hline — & a & — & \\ & & & Y_{n,n} \\ & & & \end{array}\right).$$

Using our inductive hypothesis for the first determiniant, and using that $\det \left(\begin{array}{c|c} A_{11} & 0 \\ A_{21} & A_{22} \end{array}\right) = \det A_{11} \det A_{22}$ for the second, the expression becomes

$$d_{n-1} \left(\det X \det Y_{n,n-1} + \left(\prod_{i \le n-2} d_i \right) \det \left(\begin{array}{ccc|c} & X & & \vert \\ & & & b \\ & & & \vert \\ \hline — & a & — & 0 \end{array}\right) \right) - y_{nn} \det X \det Y_{nn} =$$

$$(d_{n-1} Y_{n,n-1} - y_{nn} \det Y_{nn}) \det X + \left(\prod_i d_i \right) \det \left(\begin{array}{ccc|c} & X & & \vert \\ & & & b \\ & & & \vert \\ \hline — & a & — & 0 \end{array}\right) =$$

$$\det Y \det X + \left(\prod_i d_i \right) \det \left(\begin{array}{ccc|c} & X & & \vert \\ & & & b \\ & & & \vert \\ \hline — & a & — & 0 \end{array}\right).$$

$\square$

Corollary 10.9. *If X is nilpotent, and*

$$\left(\begin{array}{ccc|c} & X & & \vert \\ & & & b \\ & & & \vert \\ \hline — & a & — & \\ & & & Y \\ & & & \end{array}\right)$$

is nilpotent as well, then Y is nilpotent.

Proof. By the previous lemma, the characteristic polynomial of the big matrix is

$$g_X(\lambda)g_Y(\lambda) + f(\lambda),$$

where $g_X(\lambda) = \lambda^m$ is the characteristic polynomial of X, and $f(\lambda)$ is some polynomial of degree at most $m - 1$. $\square$

References

[1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.