1 Introduction

We will review the definition of a Springer fiber and define, for a nilpotent $Y \in \mathfrak{gl}_m$, the Springer fiber at the n-Slodowy slice at Y. For every n and every nilpotent $Y \in \mathfrak{gl}_m$, we will find the irreducible components of the Springer fiber at the n-Slodowy slice at Y. Finally, we will use our results about Springer fibers at n-Slodowy slices to find the irreducible components of some other variety (which probably needs a name), and show that they all have the same dimension.

2 Springer fibers

Let $G \subseteq \mathrm{GL}_m(\mathbb{C})$ be a connected semisimple Lie group, and let $\mathfrak{g} \subseteq \mathfrak{gl}_m(\mathbb{C})$ be its Lie algebra. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the subset consisting of nilpotent elements. Let $\mathcal{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. Let $\widetilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$. Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the projection onto the second coordinate. For $n \in \mathcal{N}$, we call $\pi^{-1}(n)$ the *Springer fiber at* n.

We mention some results about Springer fibers, which we will use later in this paper. TODO: mention them.

3 Slodowy slice

A basis for $\mathfrak{sl}_2(\mathbb{C})$ is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra $\mathfrak{g}$, and a homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ sending (e', h', f') to (e, h, f), we say that (e, h, f) is an $\mathfrak{sl}_2$-*triple*. Observe that e, f must be nilpotent, and h must be Cartan (???). If $\mathfrak{g}$ is semisimple, then given any nilpotent $e \in \mathfrak{g}$, the Jacobson-Morozov theorem [1, 3.7.1] says that there exist $h, f \in \mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple.

Given (e, h, f), we define the *Slodowy slice at* e as $\mathcal{S}_e := e + \ker \mathrm{ad}_f$. By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent $e \in \mathfrak{g}$, when $\mathfrak{g}$ is semisimple.

4 Finding $\mathfrak{sl}_2$-triples (E, H, F) in $\mathfrak{gl}_{m+n}$ with a particular E.

Let

$$e = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \in \mathfrak{gl}_n,$$

and let $E = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{gl}_{m+n}$. We will show that there is exactly one $\mathfrak{sl}_2$-triple (E, H, F), and we will find what it looks like. First we solve the case $m = 0$ (so $E = e$), and then we use this to solve the case of arbitrary m.

Lemma 4.1. *There is exactly one way to choose $h, f \in \mathfrak{gl}_n$ so that (e, h, f) is an $\mathfrak{sl}_2$-triple.*

Proof. Note that $[h', e'] = 2e'$, and $[e', f'] = h'$, and $[h', f'] = -2f'$. Thus e, h, f must obey the same relations. In particular, $he - eh = 2e$. The matrix eh is h shifted down one, and he is h shifted left one. Thus, $2e_{ij} = (he - eh)_{ij} = h_{i,j+1} - h_{i-1,j}$. We can use this to show that $h_{ij} = 0$ when $i \neq j$. Then we can use it to show that $h_{ii} = h_{i-1,i-1} + 2$, so that $h_{ii} = h_{11} + 2(i-1)$.

Similarly, from $[e, f] = h$ we get that $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$. We can use this to show that $f_{ij} = 0$ when $j \neq i + 1$. Then we can use it to show that $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$, that $f_{1,2} = -h_{1,1}$, and that $f_{n-1,n} = h_{n,n}$. From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^{n} h_{ii} \implies$$

$$\sum_i h_{ii} = 0.$$

Remark: this is just the statement that $h \in \mathfrak{sl}_n$; in other words, we will see that every choice of $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_{m+n}$ is also an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_{m+n}$. This shows that $h_{11} = n - 1, h_{22} = n - 3, ..., h_{nn} = 1 - n$. So we have determined

h; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for f in terms of h to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & \\ & 0 & (1-n)+(3-n) & & \\ & & \ddots & \ddots & \\ & & & 0 & (1-n)+\cdots+(n-1) \\ & & & & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1(1-n) & & & \\ & 0 & 2(2-n) & & \\ & & 0 & (n-2)(-2) & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

□

Lemma 4.2. *There is exactly one way to choose $H, F \in \mathfrak{gl}_{m+n}$ so that (E, H, F) is an $\mathfrak{sl}_2$-triple.*

Proof. Suppose we have H, F so that (E, H, F) is an $\mathfrak{sl}_2$-triple. Writing $F =: \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & f \end{pmatrix}$, and similarly for H, we have

$$2E = [H, E] = \begin{pmatrix} 0 & H_{12}e \\ 0 & he \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ eH_{21} & eh \end{pmatrix},$$

$$H = [E, F] = \begin{pmatrix} 0 & 0 \\ eF_{21} & ef \end{pmatrix} - \begin{pmatrix} 0 & F_{12}e \\ 0 & fe \end{pmatrix}.$$

From these we observe that (e, h, f) must also be an $\mathfrak{sl}_2$-triple, so h, f must be as in Lemma 4.1. We also see that $H_{11} = 0$. Recalling that left multiplication by e is a down-shift, and right multiplication is a left-shift, we see that H_{12}

is all zeroes except for the leftmost column, and H_{21} is all zeroes except for the bottom row. Then our final constraint is that

$$-2F = [H, F] =$$

$$\begin{pmatrix} H_{12}F_{21} & H_{12}F_{22} \\ H_{21}F_{11} + H_{22}F_{21} & H_{21}F_{12} + H_{22}F_{22} \end{pmatrix} - \begin{pmatrix} F_{12}H_{21} & F_{11}H_{12} + F_{12}H_{22} \\ F_{22}H_{21} & F_{21}H_{12} + F_{22}H_{22} \end{pmatrix}.$$

Now $H_{12} = F_{12}e$, and $H_{21} = eF_{21}$, from the equation $H = [E, F]$. Substituting in the equation above then,

$$-2F = \begin{pmatrix} F_{12}eF_{21} & F_{12}ef \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} F_{12}eF_{21} & F_{11}F_{12}e + F_{12}h \\ feF_{21} & F_{21}F_{12}e + fh \end{pmatrix} =$$

$$\begin{pmatrix} 0 & F_{12}(fe + h) \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e + F_{12}h \\ (ef - h)F_{21} & F_{21}F_{12}e + fh \end{pmatrix} =$$

$$\begin{pmatrix} 0 & F_{12}fe \\ eF_{21}F_{11} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e \\ efF_{21} & F_{21}F_{12}e + fh \end{pmatrix}.$$

Now we see that $F_{11} = 0$, and consequently that $F_{12} = F_{21} = 0$ as well. This shows that $H_{12} = H_{21} = 0$. We conclude that H and F just have h and f in their bottom-right corners, respectively. □

5 Finding the Slodowy slices with the same E

First we find $\ker \mathrm{ad}_f$. We have $(fX)_{ij} = i(n-i)A_{i+1,j}$, and $(Xf)_{ij} = (j-1)(n-(j-1))A_{i,j-1}$. So, for all $i, j \in \{1, ..., n\}$, we have

$$i(n-i)A_{i+1,j} = (j-1)(n-(j-1))A_{i,j-1}.$$

Taking $j = 1$, we find that $A_{i,1} = 0$ for $i \geq 2$. Then, taking $j > 1$, we find that for $i, j \in \{1, ..., n-1\}$,

$$A_{i+1,j+1} = \frac{(j-1)(n-(j-1))}{i(n-i)} A_{ij}.$$

So, $\ker \mathrm{ad}_f$ is the set of matrices which are upper triangular and satisfy the above condition.

To find the Slodowy slice associated to the previous $\mathfrak{sl}_2$-triple (E, H, F), we just need to find $\ker \operatorname{ad}_F$. We have

$$[F, X] = \begin{pmatrix} 0 & 0 \\ fX_{21} & fX_{22} \end{pmatrix} - \begin{pmatrix} 0 & X_{12}f \\ 0 & X_{22}f \end{pmatrix}.$$

Thus, X_{21} must be all zeroes except for the first row, and X_{12} must be all zeroes except for the last column, and $X_{22} \in \ker \operatorname{ad}_f$. There is no restriction on X_{11}. This describes $\ker \operatorname{ad}_F$.

For $X \in \mathcal{S}_{m,n}$, define $u(X) := X_{11}$.

5.1 Finding $\widetilde{\mathcal{N}}_{m,n}$

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m$ be the nilpotent elements. Let $\mathcal{S}'_{m,n}$ be the set of $X \in \mathcal{S}_{m,n}$ such that both X and $u(X)$ are nilpotent. Let $\widetilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n}$. Define $\pi_{m,n} : \widetilde{\mathcal{N}}_{m,n} \to \mathcal{N}_m$ by $(\mathfrak{b}, X) \mapsto X_{11}$. For $Y \in \mathfrak{gl}_m$, we call $\pi_{m,n}^{-1}(Y)$ the *Springer fiber at the n-Slodowy slice at Y.*

Lemma 5.1. *Let J be a jordan block with zeroes along the diagonal, and let A be upper triangular and nonzero. Then $J + A$ is not nilpotent.*

Proof. It is straightforward to show by induction that if $v_i = 0$ for $i < j$, and $v_j \neq 0$, then $((J + A)^k v_j)_{j+k} = v_j$. Let i be such that $Ae_i \neq 0$. Then $(J + A)^{i-1}e_1$ has nonzero e_i-component. Then $(J + A)^i e_1$ has some nonzero $e_{i'}$-component for some $i' \leq i$. Then $(J + A)^{i+(n-i')}e_1$ has some nonzero e_n-component. And $i + (n - i') \geq n$, so we're done. $\square$

Lemma 5.2. *Let $X \in \mathfrak{gl}_m$, and let*

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ d_1 & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ & d_2 & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ & & \ddots & \cdots & \vdots & \vdots \\ & & & d_{n-2} & y_{n-1,n-1} & y_{n-1,n} \\ & & & & d_{n-1} & y_{nn} \end{pmatrix} \in \mathfrak{gl}_n .$$

For any $a, b \in \mathbb{C}^m$,

$$\det \left(\begin{array}{ccc|cc} & X & & & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline - & a & - & & \\ & & & Y & \end{array} \right) =$$

$$\det X \det Y + \left(\prod_i d_i \right) \det \left(\begin{array}{ccc|c} & X & & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline - & a & - & 0 \end{array} \right)$$

Proof. By induction on n. In the case $n = 1$, expanding along the last row (taking the usual interpretation of the empty product) gives the desired result.

Now suppose $n > 1$. Expanding along the last row, we get

$$d_{n-1} \det \left(\begin{array}{ccc|cc} & X & & & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline - & a & - & & \\ & & & Y_{n,n-1} & \end{array} \right) - y_{nn} \det \left(\begin{array}{ccc|cc} & X & & & \\ \hline - & a & - & & \\ & & & & Y_{n,n} \end{array} \right).$$

Using our inductive hypothesis for the first determiniant, and using that $\det \left(\begin{array}{c|c} A_{11} & 0 \\ A_{21} & A_{22} \end{array} \right) = \det A_{11} \det A_{22}$ for the second, the expression becomes

$$d_{n-1} \left(\det X \det Y_{n,n-1} + \left(\prod_{i \le n-2} d_i \right) \det \left(\begin{array}{ccc|c} & X & & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline - & a & - & 0 \end{array} \right) \right) - y_{nn} \det X \det Y_{nn} =$$

$$(d_{n-1} Y_{n,n-1} - y_{nn} \det Y_{nn}) \det X + \left(\prod_i d_i \right) \det \left(\begin{array}{ccc|c} & X & & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline - & a & - & 0 \end{array} \right) =$$

$$\det Y \det X + \left(\prod_i d_i\right) \det \left(\begin{array}{ccc|c} & X & & \vert \\ & & & b \\ & & & \vert \\ \hline — & a & — & 0 \end{array}\right).$$

□

Corollary 5.3. *If X is nilpotent, and*

$$\left(\begin{array}{ccc|ccc} & & & & & \vert \\ & X & & & & b \\ & & & & & \vert \\ \hline — & a & — & & & \\ & & & & Y & \\ & & & & & \end{array}\right)$$

is nilpotent as well, then Y is nilpotent (TODO: and that other determinant is zero).

Proof. By the previous lemma, the characteristic polynomial of the big matrix is

$$g_X(\lambda) g_Y(\lambda) + f(\lambda),$$

where $g_X(\lambda) = \lambda^m$ is the characteristic polynomial of X, and $f(\lambda)$ is some polynomial of degree at most $m - 1$. □

Now, taking the previous corolloary and the first lemma together, we see that

$$\mathcal{S}'_{m,n} = \left\{ \left(\begin{array}{ccc|cccc} & & & & & & \vert \\ & X & & & & & b \\ & & & & & & \vert \\ \hline — & a & — & 0 & & & \\ & & & 1 & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & 0 \end{array}\right) : a, b \in \mathbb{C}^m, X \in \mathfrak{gl}_m \text{ is nilpotent} \right\}.$$

6 Simplifying the definition of a Springer fiber at a Slodowy slice

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m(\mathbb{C})$ be the subset consisting of nilpotent elements. For $X \in \mathcal{N}_m$ and $a, b \in \mathbb{C}^m$, let

$$A_{X,a,b} = \left(\begin{array}{c|ccccc} & & & & & | \\ X & & & & & b \\ & & & & & | \\ \hline - \; a \; - & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & 0 & \\ & & & & 1 & 0 \end{array}\right) \in \mathfrak{gl}_{m+n}(\mathbb{C}).$$

By the definition given in the previous section, we have

$$\widetilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n},$$

and the Springer fiber at the n-Slodowy slice at a nilpotent $X \in \mathfrak{gl}_m$ is

$$\begin{aligned} \pi_{m,n}^{-1}(X) &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \in \mathcal{S}'_{m,n}\} \\ &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b}\text{nilpotent}\} \\ &\cong \{(\mathfrak{b}, a, b) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \text{ nilpotent}\}. \end{aligned}$$

We will make one last simplification to this by using a correspondence between Borel subalgebras and complete flags. TODO: fix the next few paragraphs, they are out of context.

We have $\mathcal{M} = \{AHA^{-1} : A \in \mathrm{GL}_{m+n}(\mathbb{C})\}$, where $H \subseteq \mathfrak{gl}_{m+n}(\mathbb{C})$ is the set of upper triangular matrices. We say a map $X : \mathfrak{gl}_{m+n} \to \mathfrak{gl}_{m+n}$ preserves a flag $V_0 \subseteq \cdots \subseteq V_{m+n}$ if $XV_i \subseteq V_i$ for each i. Let $E_0 \subseteq \cdots \subseteq E_{m+n}$ be the standard flag of $\mathbb{C}^{m+n}$. Since H is the set of X which preserve E,

$$\mathcal{M} = \{\{X : \forall i.\ X(AE_i) \subseteq AE_i\} : A \in \mathrm{GL}_n(\mathbb{C})\}.$$

So, for $X \in \mathcal{N}$,

$$\pi^{-1}(X) \cong \{(V, a, b) : \forall i.\ A_{X,a,b}V_i \subseteq V_i\}.$$

In this section we will find the irreducible components of $\pi^{-1}(X)$. Since $\pi^{-1}(X) \cong \pi^{-1}(AXA^{-1})$ for any invertible A, we will assume that X is in Jordan normal form.

Let λ be the shape of X, and let $(e_{ij})_{i\leq r, j\leq \lambda_i}$ be a Jordan basis for X, with $Xe_{ij} = e_{i,j-1}$. Let $f_1, ..., f_n$ be the standard basis for $\mathbb{C}^n$, with $A_{X,a,b}f_i = f_{i+1}$.

7 Finding a centralizer

For nilpotent X in Jordan form and a in 'normalized' form (i.e., with at most one nonzero element, which is a one), we will find the centralizer of $A_{X,a,b}$ in

$$\{(A, I) : A \in \mathfrak{gl}_m\} \subseteq \mathfrak{gl}_{m+n}\,.$$

Note that an element of the form (A, I) commutes with $A_{X,a,b}$ if and only if A commutes with X, and $aA = a$, and $Ab = b$. So, we just have to find

$$\{A \in \mathfrak{gl}_m : AX = XA, aA = a, Ab = b\}.$$

We begin by finding

$$\{A \in \mathfrak{gl}_m : AX = XA\}.$$

We write the shape of X as $\lambda = (\lambda_1, ..., \lambda_k)$, so that

$$X = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix}.$$

Then we write A as the block matrix

$$A =: \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix},$$

where A_{ij} is a block of size $\lambda_i \times \lambda_j$. We have

$$XA = \begin{pmatrix} J_{\lambda_1}A_{11} & \cdots & J_{\lambda_1}A_{1k} \\ \vdots & & \vdots \\ J_{\lambda_k}A_{k1} & \cdots & J_{\lambda_k}A_{kk} \end{pmatrix}, \text{ and } AX = \begin{pmatrix} A_{11}J_{\lambda_1} & \cdots & A_{1k}J_{\lambda_k} \\ \vdots & & \vdots \\ A_{k1}J_{\lambda_1} & \cdots & A_{kk}J_{\lambda_k} \end{pmatrix}.$$

So, the constraint that $XA = AX$ is simply saying that

$$\forall i, j.\ J_{\lambda_i} A_{ij} = A_{ij} J_{\lambda_j}.$$

It's easy to see that left multiplication by J_{λ_i} is just a down-shift by one, and right multiplication by J_{λ_j} is a left-shift by one. This means that A_{ij} is constrained to be a "lower-left-Toeplitz" matrix. A Toeplitz matrix is one which is constant along diagonal bands: $\forall ijk.\ x_{ij} = x_{i+k,j+k}$. A lower-left-Toeplitz matrix is one which is all zeroes except for the bottom-left corner; that is, the bands start in the bottom-left corner, and continue until hitting the band which includes the lower-right corner or the band which includes the upper-left corner (whichever comes first). In other words, an $n \times m$ lower-left-Toeplitz matrix is one which satisfies $x_{ij} = 0$ for all i, j with $i < j$ and all i, j with $i - j < m - n$. In yet other words, a matrix is lower-left-Toeplitz if it is Toeplitz, satisfies $x_{1j} = 0$ for $j \neq 1$, and satisfies $x_{in} = 0$ for $i \neq m$. Anyway, it is easy to see that the lower-left-Toeplitz matrices are exactly those matrices A such that left-shifting A by one has the same result as down-shifting by one.

So, we have found the set

$$C_1 := \{A \in \mathfrak{gl}_m : AX = XA\}.$$

It is just the set of $A = [A_{ij}]_{ij}$, where each A_{ij} is lower-left-Toeplitz.

Let v_{ij} be the leftmost column of A_{ij}, so that

$$A_{ij} = \begin{pmatrix} v_{ij} \Vee 0 & \cdots & v_{ij} \Vee [\lambda_j - 1] \end{pmatrix}.$$

Since A_{ij} is lower-left-Toeplitz, v_{ij} is of the form

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$ can be chosen freely. Now we determine which matrices of this form satisfy $aA = a$. For simplicity, we will instead find the matrices A such that $aA = 0$ (so that $a(A + I) = a$). (Note that $I \in C_1$, and C_1 is closed under addition, so this is really the same thing.)

In the case that $a = 0$, clearly every A works. Otherwise, let i_0, j_0 be such that $a_{i_0, j_0} = 1$. Now, clearly the constraint that $aA = 0$ is just saying that the (i_0, j_0)th row of A must be zero. That is, for each j the j_0th row of $A_{i_0 j}$ must be zero. This just requires that for each j, we must have

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$ can be chosen freely. So, we have now found the set

$$C_2 := \{A \in \mathfrak{gl}_m : AX = XA, aA = a\}.$$

It is the matrices of the form $I + A$, where A_{ij} is a block matrix of size $\lambda_i \times \lambda_j$ with

$$A_{ij} = T(v_{ij}) = T\begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$ in the case $i \neq i_0$ and $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$ if $i = i_0$.

Now we are only left with finding the matrices $A \in C_2$ satisfying $Ab = b$. This just means finding the matrices A, of the form given above, which satisfy $Ab = 0$. We may restate this as the condition that for every i,

$$\begin{pmatrix} A_{i1} & \cdots & A_{in} \end{pmatrix} b = 0.$$

Taking the transpose,

$$b^\top \begin{pmatrix} A_{i1}^\top \\ \vdots \\ A_{in}^\top \end{pmatrix} = 0.$$

Now, the rightmost column of $A_{ij}^\top$ is v_{ij} upside-down (rows reversed). Write u_{ij} to refer to v_{ij} upside-down, and u'_{ij} to refer to v'_{ij} upside-down. Then the second-rightmost column is $u_{ij} \Uparrow 1$, and so on until the first column is $u_{ij} \Uparrow [\lambda_i - 1]$. So, we can reformulate this condition by requiring that for all l,

$$b^\top \begin{pmatrix} u_{i1} \Uparrow l \\ \vdots \\ u_{in} \Uparrow l \end{pmatrix} = 0.$$

Writing

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix},$$

clearly this is equivalent to requiring that for all l,

$$\begin{pmatrix} b_1^\top \gg l & \cdots & b_k^\top \gg l \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{ik} \end{pmatrix} = 0.$$

Then writing $b_j = \begin{pmatrix} b_j' \\ \vdots \end{pmatrix}$, where b_j' has the same dimension as u_{ij}, we can rewrite this constraint as

$$\begin{pmatrix} b_1'^\top \gg l & \cdots & b_k'^\top \gg l \end{pmatrix} \begin{pmatrix} u_{i1}' \\ \vdots \\ u_{ik}' \end{pmatrix} = 0.$$

Finally, requiring the above equation for all l is the same thing as requiring

$$\begin{pmatrix} b_1'^\top \gg 0 & \cdots & b_k'^\top \gg 0 \\ \vdots & & \vdots \\ b_1'^\top \gg [\lambda_i - 1] & \cdots & b_k'^\top \gg [\lambda_i - 1] \end{pmatrix} \begin{pmatrix} u_{i1}' \\ \vdots \\ u_{ik}' \end{pmatrix} = 0.$$

This is the most that we will simplify the constraint. Recall that the v_{ij}' can be chosen arbitrarily, so the u_{ij}' can be chosen arbitrarily as well. That is, the set $C_3 := \{A \in \mathfrak{gl}_m : AX = XA, aA = a, Ab = b\}$ is the set of $I + A$ for any A that can be recovered from u_{ij}' meeting the above condition.

Now we find the dimension of C_3. Since the u_{ij}' can be chosen arbitrarily, apart from the condition above, we just need to add up their dimensions and subtract the rank of the matrix constructed of the b's. The dimension of u_{ij}' is $\min(\lambda_i, \lambda_j)$ in the case that $i \neq i_0$, and $\min(\lambda_i - j_0, \lambda_j)$ when $i = i_0$. Write d_{ij} for this dimension. For each i, let q_i be the length of the longest prefix of b_i which consists of zeroes. Then if $A_{X,a,b}$ is nilpotent, we must have $q_{i_0} \geq j_0$. So we may as well set $q_{i_0} := q_{i_0} - j_0$.

Fix an i. I claim the rank of the matrix of b's is $h := \max_j [d_{ij} - q_j]$, or zero if this quantity is always negative. First, to see that the rank is at least this big, note that for any $j \in \arg\max_j [d_{ij} - q_j]$, the rank of the jth "column block" of the matrix of b's is equal to h. Then, to see that the rank is at most this big, note that every row after the first h rows is just zeroes. So, the total degrees of freedom for the u_{ij} is

$$\sum_j d_{ij} - \max(0, \max_j [d_{ij} - q_j]).$$

Then summing over all the i, we get the dimension of C_3:

$$\sum_i \left[\sum_j d_{ij} - \max(0, \max_j [d_{ij} - q_j]) \right] = \sum_{ij} d_{ij} - \sum_i \max(0, \max_j [d_{ij} - q_j]).$$

When $i \neq i_0$, then $d_{ii} = \lambda_i \geq q_i$, and therefore $\max(0, \max_j [d_{ij} - q_j]) = \max_j [d_{ij} - q_j]$. When $i = i_0$, then $d_{ii} = \lambda_i - j_0 \geq q_{i_0}$. So, we can simplify the above expression:

$$\sum_{ij} d_{ij} - \sum_i \max_j [d_{ij} - q_j].$$

Note that $\max_j [d_{ij} - q_j] = \max_j [\min(p_i + q_i, p_j + q_j) - q_j]$. This thing being maximized equals $\min(p_i + q_i - q_j, p_j)$. It would be really helpful if I could simplify this to p_j or something. I don't think so, though.

8 A necessary condition for $A_{X,a,b}$ to be nilpotent

Suppose $A := A_{X,a,b}$ is nilpotent. Define the *height* of a vector $v \in \mathbb{C}^{m+n}$ as the smallest $k \geq 0$ such that $A^k_{X,a,b} v = 0$. Clearly, for $i \leq j$, we have that the height of f_i is geq the height of f_j. Thus, for each $k \geq 0$, we must have $A^k_{X,a,b} f_n \in \langle e_{ij} \rangle_{ij}$. We have $A f_n = \sum_{ij} b_{ij} e_{ij}$. Then $A^2 f_n = \sum_{ij} b_{ij} (e_{i,j-1} + a_{ij} f_1)$, so we must have $\sum_{ij} b_{ij} a_{ij} = 0$. Similarly, since $A^3 f_n \in \langle e_{ij} \rangle$, we see that $\sum_{ij} b_{ij} a_{i,j-1} = 0$. Continuing in this way, we obtain that for all $k \geq 0$,

$$\sum_{ij} b_{ij} a_{i,j-k} = 0. \tag{1}$$

In fact this is in fact a sufficient condition for $A_{X,a,b}$ to be nilpotent, as can be seen from the characteristic polynomial of $A_{X,a,b}$. However, we will not do this calculation here, instead just showing that it is sufficient by giving a Jordan basis.

9 A Jordan basis for $A_{X,a,b}$

For illustration, we will begin with the case that X is a single Jordan block. Then we will move to the general case.

9.1 When X is a Jordan block

Suppose X is a single Jordan block of size m. In this case we just write the basis of $\mathbb{C}^m$ as $e_1, ..., e_m$, and the above condition on a and b simply becomes

$$\forall k \geq 0. \sum_{i=k+1}^{m} b_i a_{i-k} = 0. \tag{2}$$

This condition can be simplified even more.

Lemma 9.1. *The condition (1) holds iff there exist nonnegative m_1, m_2, m_3, $a_1, ..., a_{m_3}$, and $b_1, ..., b_{m_3}$ satisfying the following conditions.*

- $m = m_1 + m_2 + m_3$
- $a = (0, 0, ..., 0, a_1, ..., a_{m_3})$
- $b = (b_1, ..., b_{m_1}, 0, 0, ..., 0)$
- *If $m_1 \neq 0$, then $a_{m_1} \neq 0$*
- *If $m_3 \neq 0$, then $b_1 \neq 0$*

Proof. If a and b are of this form, then for every $k \geq 0$ we have $b_i a_{i-k} = 0$, so clearly the condition holds.

Now suppose (1) holds. If a or b is zero, this is trivial. Otherwise, let $m_1 = \max\{i : a_i \neq 0\}$, and let $m_3 = \min\{i : b_i \neq 0\}$. We just need to show that $m_1 < m_3$. For contradiction, suppose $m_1 - m_3$ is nonnegative. Then by (1), $0 = \sum_i b_i a_{i-(m_1-m_3)} = b_1 a_{m_1}$, contradicting that b_1 and a_{m_1} are nonzero. □

9.2 A 'normalization' fact about Jordan bases

Let V be a finite-dimensional vector space, $A : V \to V$ a nilpotent operator, and $f : V \to \mathbb{C}$ a linear map. Let $(e_{ij} : i \leq m, j \leq \lambda_i)$ be a Jordan basis for A.

Lemma 9.2. *There is a Jordan basis e_{ij} for A such that there is at most one pair (i, j) with $f(e_{ij}) \neq 0$.*

Proof. For any change of basis $P : V \to V$ commuting with A, we obtain a new Jordan basis $(P(e_{ij}) : i \le m, j \le \lambda_i)$. For any such P, define

$$S_P = \sum_i \begin{cases} -1, & \forall j.\ f \circ P(e_{ij}) = 0 \\ \lambda_i - \min\{j : f \circ P(e_{ij}) \neq 0\}, & \text{otherwise} \end{cases}.$$

Let P be any operator, among all invertible operators commuting with A, which minimizes S_P. Write $e'_{i,j} := P(e_{i,j})$.

Suppose for contradiction that there are two distinct i's (and some j's) with $f(e'_{ij}) \neq 0$. Then we can take e'_{i_1,j_1} and e'_{i_2,j_2}, where for $k \in \{1,2\}$ we have $f(e'_{i_k,j_k}) \neq 0$, and $\forall j < j_k.\ f(e'_{i_k,j}) = 0$. Wlog, assume $\lambda_{i_1} - j_1 \le \lambda_{i_2} - j_2$. Then, we can define $Q : V \to V$ by

- $Q(e'_{i_1,\lambda_1}) := e'_{i_1,\lambda_1} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(\lambda_{i_1}-j_1)}$
- For $j < \lambda_1$, $Q(e'_{i_1,j}) := A^{\lambda_{i_1}-j} Q(e'_{i_1,\lambda_{i_1}})$
- For $i \neq i_1$, $Q(e'_{ij}) = e'_{ij}$.

Clearly Q is invertible, and it commutes with A. Further, I claim that $S_{QP} < S_P$. It suffices to show that $\forall j \le j_1.\ f \circ Q(e'_{i_1,j}) = 0$. We have

$$f \circ Q(e'_{i_1,j}) = f\left(A^{\lambda_{i_1}-j}\left(e'_{i_1,\lambda_1} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(\lambda_{i_1}-j_1)}\right)\right) =$$

$$f\left(e'_{i_1,j} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(j-j_1)}\right) = f(e'_{i_1,j}) - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(j-j_1)}.$$

Clearly (by design), this expression is zero when $j = j_1$. And for $j < j_1$, we have $f(e'_{i_1,j}) = f(e'_{i_2,j_2+(j_2-j_1)}) = 0$, so it is zero then as well. Thus we see that $S_{QP} < S_P$, contradicting that S_P is minimal. So there must be at most one i such that there exists j such that $f(e'_{ij}) \neq 0$.

If there is no such i, we have found the desired basis. So, suppose there is such an i_0. Let $U = \langle e'_{i_0,j} \rangle$. Simply write $e_j := e'_{i_0,j}$. Let $j_0 = \min\{j : f(e_j) \neq 0\}$. We just have to change basis to zero out e_j for $j \neq j_0$. Let $P_1(e_{\lambda_{i_0}}) := e_{\lambda_{i_0}} - \frac{f(e_{j_0+1})}{f(e_{j_0})} e_{\lambda_{i_0}-1}$, and notice that $f \circ P_1(e_j) = 0$ for all $j \le j_0 + 1$ except for j_0. Then we define $P_2(e_{\lambda_{i_0}}) := P_1(e_{\lambda_{i_0}}) - \frac{f \circ P_1(e_{j_0+2})}{f \circ P_1(e_{j_0})} e_{\lambda_{i_0}-2}$, and notice that $f \circ P_2(e_j) = 0$ for all $j \le j_0 + 2$ except for j_0. Eventually we get $P_{\lambda_{i_0}-j_0}$, and by applying this to the e_j's we obtain a basis of U, in which there is exactly one j with $f \circ P_{\lambda_{i_0}-j_0}(e_j) \neq 0$. $\square$

9.3 The general case

In the case where X was a single Jordan block, it was helpful to have the condition that all the nonzero a's were to the right of all the nonzero b's. This allowed some tricks with right-shifting. In the general case, the parameters a and b are not necessarily in such a helpful form. We will see, however, that we can change basis of $\mathbb{C}^m$ to put them in a nice form.

Recall our nilpotent operator $A_{X,a,b}$ on $\mathbb{C}^m \times \mathbb{C}^n$. We can define a map $\mathbb{C}^m \to \mathbb{C}$ by $x \mapsto x \cdot a$. Then, by the previous lemma, we can choose a Jordan basis e_{ij} of $\mathbb{C}^m$ so that at most one (i,j) has $f(e_{ij}) \neq 0$. This implies that there are some P, k, b' such that $PA_{X,a,b}P^{-1} = A_{X,1_k,b'}$. So, without loss of generality, we may consider this case, where $a = 1_k$.

Write our basis of $\mathbb{C}^m$ as e_{ij} and our basis of $\mathbb{C}^n$ as f_j. Let i_0, r be such that $a = e_{i_0,\lambda_{i_0}-r+1}$. Let $b' = A^{n+r}e_{i_0,\lambda_{i_0}} = b + e_{i_0,\lambda_{i_0}-r-n}$. Let b'_{ij} be the projection of b' onto e_{ij}. For each i, let $p_i = \max\{j : b'_{ij} \neq 0\}$. Then for each $i \neq i_0$, let $q_i = \lambda_i - p_i$. Let $q_{i_0} = \lambda_{i_0} - r - p_{i_0}$. Normally we index e_{ij} so that λ_i is nonincreasing with i. Now, it will be more convenient to assume our indices are such that q_i is nonincreasing, so we will do that.

We have two cases, which will closely resemble the cases we had when X was a single Jordan block.

9.3.1 First case: $n + \max_i p_i \geq q_{i_0} + p_{i_0}$

We list the chains in (roughly) decreasing order. First, for each i such that $q_i \geq n + r$, we take the chain of length $p_i + q_i$ beginning with e_{i,p_i+q_i}.

Now we handle $e_{i_0,\lambda_{i_0}}$. Let $P = \max_{i:q_i<n+r} p_i$. Set

$$v = e_{i_0,\lambda_{i_0}} - (A^{r+n+P}e_{i_0,\lambda_{i_0}} >> r + n + P).$$

Note that $A^{r+n}e_{i_0,\lambda_{i_0}} = b'$, so $A^{r+n+P}e_{i_0,\lambda_{i_0}} = A^P b' = b' << P$. Also note that if we shift b' left P times, we zero out all the rows i where $q_i < n + r$. This ensures that the operation of shifting $b' << P$ right $r + n + P$ times is invertible by applying A, and thus shifting left, $r + n + P$ times. So, we take the chain of length $r + n + P$ beginning with v.

Now, it is clear that $e_{i_0,p_{i_0}+q_{i_0}+i}$ is in the span of the chains we have listed so far for each $i \geq 1$. It is also clear that $b' \ll k$ is in the span, for each $k \geq 0$.

Now, we handle the i with $q_i < n + r$. We take as an inductive hypothesis that for all $i' < i$ and all j, we have $e_{i'j}$ in the span of the chains we have already listed.

If $p_i \leq \max_{k>i} p_k$, then we take the chain beginning with e_{i,p_i+q_i} of length $p_i + q_i$. Clearly we then have $e_{i'j}$ in the span of the chains we have listed, for $i' \leq i$.

Otherwise, we set

$$v_i = A^{n+r-q_i} e_{i_0,\lambda_{i_0}} - \sum_{k=1}^{p_i} b'_{i,k} e_{i,k+q_i}.$$

Note that $A^{q_i} v_i$ is just b' with row i zeroed out. Then, $A^{q_i+\max_{k>i} p_k} v_i$ will have, in addition, rows k zeroed out, for all $k > i$. This ensures that shifting $A^{q_i+\max_{k>i} p_k} v_i$ right $q_i + \max_{k>i} p_k$ times will be inverted by applying A, and thus shifting left, $q_i + \max_{k>i} p_k$ times. So, we take the chain beginning with $v_i - (A^{q_i+\max_{k>i} p_k} v_i >> q_i + \max_{k>i} p_i)$, which has length $q_i + \max_{k>i} p_k$.

Now, we want to show that for each j, e_{ij} is in the span of the chains we've listed so far. Since $A^{q_i+\max_{k>i} p_k} v_i$ has rows k zeroed out for $k \geq i$, our inductive hypothesis tells us that $A^{q_i+\max_{k>i} p_k} v_i \gg q_i + \max_{k>i} p_k$ is in the span of the chains already listed. So it suffices to show that the closure of $\langle v_i \rangle$ under action by A contains the span of e_{ij}. And $A^{n+r-q_i} e_{i_0,\lambda_{i_0}}$ is also in the span of chains already listed, so it suffices to show that the closure of $\langle \sum_k b'_{ik} e_{i,k+q_i} \rangle$ under action by A contains $\langle e_{ij} \rangle$. And this just follows from the fact that b'_{i,p_i} is nonzero.

Now, I've shown that my purported Jordan basis has a large enough span; to show that it is indeed a Jordan basis I just need to count the number of vectors and show that we obtain $m + n$. TODO: do this.

9.3.2 Second case: $n + \max_i p_i \leq q_{i_0} + p_{i_0}$

Again we list the chains in (roughly) decreasing order. Note that, since $n + \max_i p_i \leq q_{i_0} + p_{i_0}$, in particular we have $n \leq q_{i_0}$, and $n \leq q_i$ for all $i \leq i_0$. Now, for all $i \neq i_0$ with $q_i \geq n$, take the chain of length $p_i + q_i$ starting with e_{i,p_i+q_i}. Then, take the chain starting with $e_{i_0,\lambda_{i_0}}$, of length λ_{i_0}.

Let $i_1 = \min\{i : n > q_i\}$. Let $P = \max_{i \geq i_1} p_i$. Then take the chain beginning with $f_1 - (A^{n+p} f_1 >> n + p)$, of length $n + P$.

Finally, we handle $i \geq i_1$. (In this case $q_i < n$.) If $p_i \leq \max_{k>i} p_k$, then take the chain beginning with e_{i,p_i+q_i} of length $p_i + q_i$. Otherwise, set

$$v_i = A^{n-q_i} f_1 - \sum_{k=1}^{p_i} b'_{i,k} e_{i,q_i+k}.$$

Note that $A^{q_i} v_i$ is just b' with row i zeroed out. Now, we take the chain beginning with $v_i - (A^{q_i + \max_{k>i} p_k} v_i >> q_i + \max_{k>i} p_k)$, of length $q_i + \max_{k>i} p_k$.

TODO: where did I use the fact that $n + \max_i p_i \leq q_{i_0} + p_{i_0}$.

9.4 When X is all zeroes

In this case we have $A(y, z) = (z_n b, (a \cdot y, z_1, ..., z_{n-1}))$, and the condition becomes

$$\sum_i b_i a_i = 0.$$

In this case we will be able to write down explicitly the irreducible components of $F := \{(V, a, b) : \forall i.\ A_{X,a,b} V_i \subseteq V_i\} \cong \pi^{-1}(X)$. For any nonnegative $\delta_0, \delta_1, ..., \delta_n, \delta_{n+1}$ summing to m, define the corresponding sequence $i_0 = \delta_0$, $i_n = \delta_{n+1} + i_n$, and for $j \in \{1, ..., n\}$, $i_j = i_{j-1} + 1 + \delta_j$. Let E be the span of the e_i's, and let $E' = \{(x, 0) \in E : x \cdot a = 0\}$, where the dot is the m-dimensional dot product. Then define F_δ as the set of $(V, a, b) \in F$ such that

- $b \in V_{i_0} \subseteq E'$
- for all $j \in \{1, ..., n\}$, we have $f_j \in V_{i_j} \subseteq E' + \langle f_1, ..., f_j \rangle$

I claim that the F_δ's are the irreducible components of F. To begin, I show that their union is F.

Lemma 9.3. *Let $(V, a, b) \in \mathcal{F}$. Write $f_0 = b$, and $F = \langle f_0, f_1, ..., f_n \rangle$. For each i, either $F \subseteq V_i$, or else there exists j such that $e_j \notin V_i$, but $\langle e_0, ..., e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_1, ..., e_j \rangle$.*

Proof. For $i = 0$, we may take $j = 0$. Now assume the statement holds for i, and we will prove it for $i + 1$. If $F \subseteq V_i$, then $F \subseteq V_{i+1}$, and we are done.

So, suppose there is j such that $e_j \notin V_i$, but $\langle e_0, ..., e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_0, ..., e_j \rangle$. We have two cases: either $V_{i+1} = V_i + \langle e_j \rangle$, or not.

- If so, then either $j = n$, in which case $F \subseteq V_{i+1}$, or else $j \neq n$, in which case $e_{j+1} \notin V_{i+1}$, but $\langle e_0, ..., e_j \rangle \subseteq V_{i+1} \subseteq E' + \langle e_0, ..., e_{j+1} \rangle$.
- If not, then $e_j \notin V_{i+1}$. Let v_{i+1} be so that $V_{i+1} = V_i + \langle v_{i+1} \rangle$. I just need to show that $v_{i+1} \in Y + \langle e_1, ..., e_j \rangle$. It suffices to show that $v_{i+1}^\top e_k = 0$

for $k > j$. And to do this, it suffices to show that $(Av_{i+1})^\top e_{k-1} = 0$ for $k > j$.

Note that $Av_{i+1} \in V_i \cap A\mathbb{C}^{2n} \subseteq \langle e_1, ..., e_j \rangle$. So, for $k > j + 1$ it is clear that $(Av_{i+1})^\top e_{k-1} = 0$. Now, suppose for contradiction that $(Av_{i+1})^\top e_j \neq 0$. Then Av_{i+1} is linearly independent of $e_1, ..., e_{j+1}$. Since $Av_{i+1} \in \langle e_1, ..., e_j \rangle$, it follows that $e_j \in \langle e_1, ..., e_{j-1}, Av_{i+1} \rangle \subseteq V_i$, a contradiction.

□

Corollary 9.4. $\mathcal{F} = \bigcup_\delta \mathcal{F}_\delta$

Proof. Let $(V, a, b) \in F$. Let $i_0 = \min\{i : b \in V_i\}$, and let $i' = \max\{i : V_i \subseteq E'\}$. We want that $b \in V_{i_0} \subseteq E'$, so we want that $i_0 \leq i'$. For contradiction, suppose $i' < i_0$.

□

Lemma 9.5. *Each $\mathcal{F}_\delta$ is a closed subvariety of $\mathcal{F}$.*

Proof. □

Lemma 9.6. *Each $\mathcal{F}_\delta$ is irreducible of dimension m.*

Proof. Let $\mathcal{E}$ be the variety of partial flags of E of shape $(\delta_0, ..., \delta_{n+1})$. Define $g : \mathcal{F}_\delta \to \mathcal{E}$ by

$$(V, a, b) \mapsto 0 \subseteq V_{i_0} \cap E \subseteq V_{i_1} \cap E \subseteq \cdots \subseteq V_{i_{n+1}} \cap E = E.$$

It is clear that g is surjective, as the definition of $\mathcal{F}_\delta$ places no restriction on the intersections $V_i \cap E$. Now let's look more closely at the fibers $g^{-1}(U)$.

First let's find the flags V such that there exist a, b with $g(V, a, b) = U$. We see that, for instance, $V_{i_0} \cap E = U_1$. In fact $V_{i_0} \subseteq E$, so $V_{i_0} = U_1$. But we are free to choose the vector spaces between 0 and V_{i_0} however we wish, so we get some degrees of freedom like $\mathcal{F}_{\delta_0}$, the complete flag variety on $\mathbb{C}^{\delta_0} \cong V_{i_0}/0$. Similarly, for every $j = 1, ..., n + 1$, we can choose the vector spaces between $V_{i_{j-1}}$ and V_{i_j} arbitrarily, so we get degrees of freedom like $\mathcal{F}_{\delta_{i_j}}$, the complete flag variety on $\mathbb{C}^{\delta_{i_j}} \cong V_{i_j}/V_{i_{j-1}}$. Finally, to meet the constraint of (V, a, b) being in $\mathcal{F}_\delta$, we can choose any $b \in U_1$ and any a such that $\overline{a} \in U_n^\perp$. Thus we get an isomorphism

$$g^{-1}(U) \cong \mathcal{F}_{\delta_0} \times \cdots \times \mathcal{F}_{\delta_{n+1}} \times \mathbb{C}^{\delta_0} \times \mathbb{C}^{\delta_{n+1}}.$$

□

Theorem 9.7. *The $\mathcal{F}_\delta$'s are the irreducible components of $\mathcal{F}$.*

9.5 In the case that X is a Jordan block

This case seems harder to work with explicitly than the case that X is zero, so our strategy is to reuse

10 A different variety

Define $R = \{(X, \mathfrak{b}_1, \mathfrak{b}_2) : X \in \mathfrak{b}_1 \wedge u(X) \in \mathfrak{b}_2\} \subseteq \mathcal{S}'_{m,n} \times \mathcal{B}_{m+n} \times \mathcal{B}_m$. We can obtain a subvariety of R by requiring that $u(X)$ is in some fixed similarity class. (TODO: why is this a subvariety? Is it? Is this even the right way of explaining the significance of the m^2?) We expect that each of these subvarieties is an irreducible component of dimension m^2. We will verify these things using our prvious computations of springer fibers.

10.1 TODO

- Why are SOn flags what they are.

References

[1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.