1 Introduction

We will review the definition of a Springer fiber and define, for a nilpotent $Y \in \mathfrak{gl}_m$, the Springer fiber at the n-Slodowy slice at Y. For every n and every nilpotent $Y \in \mathfrak{gl}_m$, we will find the irreducible components of the Springer fiber at the n-Slodowy slice at Y. Finally, we will use our results about Springer fibers at n-Slodowy slices to find the irreducible components of some other variety (which probably needs a name), and show that they all have the same dimension.

2 Preliminary Definitions and Facts

2.1 Conventions and Notations

We write $\mathrm{GL}_m, \mathfrak{sl}_m$ to denote $\mathrm{GL}_m(\mathbb{C}), \mathfrak{sl}_m(\mathbb{C})$, and so on. By J_m we refer to the nilpotent $m \times m$ Jordan block (which, by convention, has ones *below* the diagonal). Given a partition $\lambda = (\lambda_1, ..., \lambda_k)$, we write $J(\lambda)$ to denote the block matrix

$$\begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix}.$$

2.2 Springer fibers

Let $\mathfrak{g}$ be a Lie algebra. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the subset consisting of nilpotent elements. Let $\mathcal{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. Let $\widetilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$. Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the projection onto the second coordinate. We call this the *Springer resolution*. For $n \in \mathcal{N}$, we call $\pi^{-1}(n)$ the *Springer fiber at n*.

2.3 Springer fibers in $\mathfrak{gl}_m$

Now we let $\mathcal{N}_m$ be the set of nilpotent elements in $\mathfrak{gl}_m$, and $\mathcal{B}_m$ the variety of Borel subalgebras of $\mathfrak{gl}_m$. Let $H \subseteq \mathfrak{gl}_m$ be the subalgebra of upper triangular matrices. The variety of Borel subalgebras of $\mathfrak{gl}_m$ is $\mathcal{B} = \{gHg^{-1} : g \in \mathrm{GL}_m\}$. Thus, the Springer fiber at $X \in \mathcal{N}$ is

$$S_X = \{gHg^{-1} : X \in gHg^{-1}\}.$$

Definition 2.1. *A* flag (V_i) *of* $\mathbb{C}^m$ *is a sequence of subspaces*

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = \mathbb{C}^m,$$

where $\dim V_i = i$.

We say that $X \in \mathfrak{gl}_m$ *preserves* a flag (V_i) if $\forall i.\ XV_i \subseteq V_i$. Note that $X \in \mathcal{N}$ preserves a flag (V_i) if and only if $\forall i.\ XV_i \subseteq V_{i-1}$.

The simplest flag is the *standard flag* (E_i), where $E_i := \langle e_1, ..., e_i \rangle$. Note that the group H is exactly the subset of $\mathfrak{gl}_m$ which preserves E_i.

We can think of $\mathcal{B}$ as the set of flags of $\mathbb{C}^m$, via the correspondence

$$gHg^{-1} \leftrightarrow (gE_i).$$

Note that X preserves (gE_i) if and only if $X \in gHg^{-1}$. Thus, we may write the Springer fiber at $X \in \mathcal{N}$ in terms of flags, as

$$S_X = \{(gE_i) : X \in gHg^{-1}\} = \{(V_i) : \forall i.\ XV_i \subseteq V_{i-1}\}.$$

Theorem 2.2. *(Needs citation!) The irreducible components of the Springer fiber at* $J(\mu)$ *are in bijection with the standard Young tableaus of shape* μ*. Further, the irreducible components are equidimensional, of dimension* $\sum_{i \neq j} \min(\mu_i, \mu_j) = \sum_i (i-1)\mu_i$.

2.4 Slodowy Slices

A basis for $\mathfrak{sl}_2$ is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra $\mathfrak{g}$, and a homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ sending (e', h', f') to (e, h, f), we say that (e, h, f) is an $\mathfrak{sl}_2$*-triple*. Observe that e, f must be nilpotent, and h must be Cartan (TODO: what?) If $\mathfrak{g}$ is semisimple, then given any nilpotent $e \in \mathfrak{g}$, the Jacobson-Morozov theorem [1, 3.7.1] says that there exist $h, f \in \mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple.

Given (e, h, f), we define the *Slodowy slice at* e as $\mathcal{S}_e := e + \ker \operatorname{ad}_f$. By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent $e \in \mathfrak{g}$, when $\mathfrak{g}$ is semisimple.

3 Linear Algebra Facts

In this section we prove linear algebra facts that will be useful later. A reader uninterested in their verification may wish to read the theorem statements and skip ahead to the next section.

3.1 The centralizer of a nilpotent matrix

Definition 3.1. *A matrix Y is* Toeplitz *if it is constant along bands parallel to the main diagonal. That is, $\forall i, j, k.\ Y_{ij} = Y_{i+k,j+k}$.*

Definition 3.2. *An $m \times n$ matrix Y is* lower-left Toeplitz *if it is Toeplitz and, in addition, we have $y_{n-i,j-1} = 0$ whenever $i + j \geq \min(m, n)$.*

That is, Y is lower-left Toeplitz if it is Toeplitz, and the only nonzero entries are those with Manhattan distance less than $\min(m, n)$ from the entry in the bottom-left corner. In yet other words, all but the leftmost (equivalently, bottommost) $\min(m, n)$ diagonal bands are zero.

Lemma 3.3. *Let $\lambda = (\lambda_1, ..., \lambda_k)$ be a partition of m. The centralizer of $J(\lambda)$ in $\mathfrak{gl}_m$ is the subalgebra consisting of matrices*

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix},$$

where each M_{ij} is a $\lambda_i \times \lambda_j$ matrix, such that each M_{ij} is lower-left Toeplitz.

Proof. Let

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix}.$$

We need to show that $J(\lambda)M = MJ(\lambda)$ if and only if each M_{ij} is lower-left Toeplitz.

We have

$$J(\lambda)M = \begin{pmatrix} J_{\lambda_1} M_{11} & \cdots & J_{\lambda_1} M_{1k} \\ \vdots & & \vdots \\ J_{\lambda_k} M_{k1} & \cdots & J_{\lambda_k} M_{kk} \end{pmatrix}, \text{ and } MJ(\lambda) = \begin{pmatrix} M_{11} J_{\lambda_1} & \cdots & M_{1k} J_{\lambda_k} \\ \vdots & & \vdots \\ M_{k1} J_{\lambda_1} & \cdots & M_{kk} J_{\lambda_k} \end{pmatrix}.$$

So, we have $J(\lambda)M = MJ(\lambda)$ if and only if $\forall i,j.\ J_{\lambda_i}M_{ij} = M_{ij}J_{\lambda_j}$. Multiplying on the left by J_{λ_i} just shifts each row down by one, and multiplying on the right by J_{λ_j} shifts each column left by one. The matrices for which left-shifting gives the same result as down-shifting are exactly the lower-left Toeplitz matrices. $\square$

For nilpotent X in Jordan form and a in 'normalized' form (i.e., with at most one nonzero element, which is a one), we will find the centralizer of $A_{X,a,b}$ in

$$\{(A,I) : A \in \mathfrak{gl}_m\} \subseteq \mathfrak{gl}_{m+n}\,.$$

Note that an element of the form (A,I) commutes with $A_{X,a,b}$ if and only if A commutes with X, and $aA = a$, and $Ab = b$. So, we just have to find

$$\{A \in \mathfrak{gl}_m : AX = XA, aA = a, Ab = b\}.$$

It is just the set of $A = [A_{ij}]_{ij}$, where each A_{ij} is lower-left-Toeplitz.

Let v_{ij} be the leftmost column of A_{ij}, so that

$$A_{ij} = \begin{pmatrix} v_{ij} \curlyveedownarrow 0 & \cdots & v_{ij} \curlyveedownarrow [\lambda_j - 1] \end{pmatrix}.$$

Since A_{ij} is lower-left-Toeplitz, v_{ij} is of the form

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i,\lambda_j)}$ can be chosen freely. Now we determine which matrices of this form satisfy $aA = a$. For simplicity, we will instead find the matrices A such that $aA = 0$ (so that $a(A+I) = a$). (Note that $I \in C_1$, and C_1 is closed under addition, so this is really the same thing.)

In the case that $a = 0$, clearly every A works. Otherwise, let i_0, j_0 be such that $a_{i_0,j_0} = 1$. Now, clearly the constraint that $aA = 0$ is just saying that the (i_0, j_0)th row of A must be zero. That is, for each j the j_0th row of $A_{i_0 j}$ must be zero. This just requires that for each j, we must have

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$ can be chosen freely. So, we have now found the set

$$C_2 := \{A \in \mathfrak{gl}_m : AX = XA, aA = a\}.$$

It is the matrices of the form $I+A$, where A_{ij} is a block matrix of size $\lambda_i \times \lambda_j$ with

$$A_{ij} = T(v_{ij}) = T\begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$ in the case $i \neq i_0$ and $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$ if $i = i_0$.

3.2 A 'normalization' fact about Jordan bases

Lemma 3.4. *For any finite-dimensional V, nilpotent $A : V \to V$, and linear $f : V \to \mathbb{C}$, there is a Jordan basis e_{ij} for A such that there is at most one i such that there exists j such that $f(e_{ij}) \neq 0$.*

Proof. For any Jordan basis $(e_{ij})_{ij}$ of A, define

$$S((e_{ij})_{1\le i\le k, 1\le j\le\lambda_i}) := \sum_i \begin{cases} -1, & \forall j.\ f(e_{ij}) = 0 \\ \lambda_i - \min\{j : f(e_{ij}) \neq 0\}, & \text{otherwise} \end{cases}.$$

We proceed by induction on the measure S. That is, let $(e_{ij})_{ij}$ be a Jordan basis for A. Our inductive hypothesis is that if there exists a Jordan basis $(e'_{ij})_{ij}$ with $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$, then we get the desired conclusion.

Now, we have two cases. In the first case, $(e_{ij})_{ij}$ already satisfies the desired property. In this case we are done. In the other case, there exist i_1, j_1, i_2, j_2 with $i_1 \neq i_2$, and $f(e_{i_1 j_1}) \neq 0$, and $f(e_{i_2 j_2}) \neq 0$. We let j_1, j_2 be minimal with this property, so that $\forall j < j_1.\ f(e_{i_1 j}) = 0$, and $\forall j < j_2.\ f(e_{i_2 j}) = 0$. Wlog, we assume that $\lambda_{i_1} - j_1 \le \lambda_{i_2} - j_2$.

By our inductive hypothesis, all we need to do is find a Jordan basis $(e'_{ij})_{ij}$ with $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$. This is what we do. Define e'_{ij} as follows.

- $e'_{i_1,\lambda_1} := e_{i_1,\lambda_1} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} e_{i_2, j_2 + (\lambda_{i_1} - j_1)}$
- For $j < \lambda_1$, $e'_{i_1,j} := A^{\lambda_{i_1} - j} e'_{i_1,\lambda_{i_1}}$
- For $i \neq i_1$, $e'_{ij} := e_{ij}$.

Clearly this is a Jordan basis for A. Further, I claim that $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$. It suffices to show that $\forall j \le j_1.\ f(e_{i_1,j}) = 0$. We have

$$f(e'_{i_1,j}) = f\left(A^{\lambda_{i_1} - j}\left(e_{i_1,\lambda_1} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} e_{i_2, j_2 + (\lambda_{i_1} - j_1)}\right)\right) =$$

$$f\left(e_{i_1,j} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} e_{i_2,j_2+(j-j_1)}\right) = f(e_{i_1,j}) - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} f(e_{i_2,j_2+(j-j_1)}).$$

Clearly (by design), this expression is zero when $j = j_1$. And for $j < j_1$, we have $f(e_{i_1,j}) = f(e_{i_2,j_2+(j-j_1)}) = 0$, so it is zero then as well. Hence the measure S of this new basis is smaller, as desired. $\square$

Lemma 3.5. *For any n and linear $f : \mathbb{C}^n \to \mathbb{C}$, there is a Jordan basis e_j for J_n such that there is at most one j with $f(e_j) \neq 0$.*

Proof. Let e_j be a Jordan basis for J_n. If $\{j : f(e_j) \neq 0\}$ is the empty set, we are done. Otherwise, let $j_0 = \min\{j : f(e_j) \neq 0\}$. For any Jordan basis f_j with $j_0 = \min\{j : f(e_j) \neq 0\}$, define

$$S((f_j)_j) := \begin{cases} -1, & \{j > j_0 : f(e_j) \neq 0\} = \emptyset \\ n - \min\{j > j_0 : f(e_j) \neq 0\}, & \text{otherwise} \end{cases}.$$

We proceed by induction on S. That is, let $(e_j)_j$ be a Jordan basis for J_n with $j_0 = \min\{j : f(e_j) \neq 0\}$. Our inductive hypothesis is that if there exists a Jordan basis $(e'_j)_j$ with $j_0 = \min\{j : f(e'_j) \neq 0\}$ and $S((e'_j)_j) < S((e_j)_j)$, then the conclusion holds.

We have two cases: either $(e_j)_j$ satisfies the desired property, or not. If not, then let $j_1 = \min\{j > j_0 : f(e_j) \neq 0\}$, and define a new Jordan basis e'_j as follows.

- $e'_n := e_n - \frac{f(e_{j_1})}{f(e_{j_0})} e_{n-(j_1-j_0)}$
- For $j < n$, $e'_j := J_n^{n-j} e'_n$

It is straightforward to check that $j_0 = \min\{j : f(e_j) \neq 0\}$, and that $S((e'_j)_j) \leq S((e_j)_j) - 1$. By our inductive hypothesis, we are done. $\square$

Theorem 3.6. *For any finite-dimensional V, nilpotent $A : V \to V$, and linear $f : V \to \mathbb{C}$, there is a Jordan basis e_{ij} for A such that there is at most one pair (i, j) with $f(e_{ij}) \neq 0$.*

Proof. Lemma 3.4 provides a Jordan basis e_{ij} such that for all $i \neq i_0$ and all j, we have $f(e_{ij}) = 0$. Restricting A to $\langle e_{i_0 j} \rangle_{j \text{ arbitrary}}$ gives a Jordan block, and then applying Lemma 3.5 gives the desired result. $\square$

3.3 A determinant

Lemma 3.7. *Let $X \in \mathfrak{gl}_m$, and let*

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ d_1 & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ & d_2 & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ & & \ddots & \cdots & \vdots & \vdots \\ & & & d_{n-2} & y_{n-1,n-1} & y_{n-1,n} \\ & & & & d_{n-1} & y_{nn} \end{pmatrix} \in \mathfrak{gl}_n .$$

For any $a, b \in \mathbb{C}^m$,

$$\det \left(\begin{array}{c|c} X & \begin{matrix} & | \\ & b \\ & | \end{matrix} \\ \hline \begin{matrix} - & a & - \\ & & \end{matrix} & Y \end{array}\right) =$$

$$\det X \det Y + \left(\prod_i d_i\right) \det \left(\begin{array}{c|c} X & \begin{matrix} | \\ b \\ | \end{matrix} \\ \hline \begin{matrix} - & a & - \end{matrix} & 0 \end{array}\right)$$

Proof. By induction on n. In the case $n = 1$, expanding along the last row (taking the usual interpretation of the empty product) gives the desired result.

Now suppose $n > 1$. Expanding along the last row, we get

$$d_{n-1} \det \left(\begin{array}{c|c} X & \begin{matrix} & | \\ & b \\ & | \end{matrix} \\ \hline \begin{matrix} - & a & - \\ & & \end{matrix} & Y_{n,n-1} \end{array}\right) - y_{nn} \det \left(\begin{array}{c|c} X & \\ \hline \begin{matrix} - & a & - \\ & & \end{matrix} & Y_{n,n} \end{array}\right).$$

Using our inductive hypothesis for the first determiniant, and using that

$\det\left(\begin{array}{c|c} A_{11} & 0 \\ A_{21} & A_{22} \end{array}\right) = \det A_{11} \det A_{22}$ for the second, the expression becomes

$$d_{n-1}\left(\det X \det Y_{n,n-1} + \left(\prod_{i\le n-2} d_i\right)\det\left(\begin{array}{ccc|c} & & & | \\ & X & & b \\ & & & | \\ \hline — & a & — & 0 \end{array}\right)\right) - y_{nn}\det X \det Y_{nn} =$$

$$(d_{n-1}Y_{n,n-1} - y_{nn}\det Y_{nn})\det X + \left(\prod_i d_i\right)\det\left(\begin{array}{ccc|c} & & & | \\ & X & & b \\ & & & | \\ \hline — & a & — & 0 \end{array}\right) =$$

$$\det Y \det X + \left(\prod_i d_i\right)\det\left(\begin{array}{ccc|c} & & & | \\ & X & & b \\ & & & | \\ \hline — & a & — & 0 \end{array}\right).$$

□

Corollary 3.8. *If X is nilpotent, and*

$$\left(\begin{array}{ccc|c} & & & | \\ & X & & b \\ & & & | \\ \hline — & a & — & \\ & & & Y \\ & & & \end{array}\right)$$

is nilpotent as well, then Y is nilpotent. If all the d_i are nonzero, then

$$\left(\begin{array}{ccc|c} & & & | \\ & X & & b \\ & & & | \\ \hline — & a & — & 0 \end{array}\right)$$

is nilpotent as well.

Proof. By the previous lemma, the characteristic polynomial of the big matrix is

$$g_X(\lambda)g_Y(\lambda) + f(\lambda),$$

where $g_X(\lambda) = \lambda^m$ is the characteristic polynomial of X, and $f(\lambda)$ is some polynomial of degree at most $m-1$. □

4 Finding $\mathfrak{sl}_2$-triples (E, H, F) in $\mathfrak{gl}_{m+n}$ with a particular E.

Let

$$e = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \in \mathfrak{gl}_n,$$

and let $E = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{gl}_{m+n}$. We will show that there is exactly one $\mathfrak{sl}_2$-triple (E, H, F), and we will find what it looks like. First we solve the case $m = 0$ (so $E = e$), and then we use this to solve the case of arbitrary m.

Lemma 4.1. *There is exactly one way to choose $h, f \in \mathfrak{gl}_n$ so that (e, h, f) is an $\mathfrak{sl}_2$-triple.*

Proof. Note that $[h', e'] = 2e'$, and $[e', f'] = h'$, and $[h', f'] = -2f'$. Thus e, h, f must obey the same relations. In particular, $he - eh = 2e$. The matrix eh is h shifted down one, and he is h shifted left one. Thus, $2e_{ij} = (he - eh)_{ij} = h_{i,j+1} - h_{i-1,j}$. We can use this to show that $h_{ij} = 0$ when $i \neq j$. Then we can use it to show that $h_{ii} = h_{i-1,i-1} + 2$, so that $h_{ii} = h_{11} + 2(i-1)$.

Similarly, from $[e, f] = h$ we get that $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$. We can use this to show that $f_{ij} = 0$ when $j \neq i + 1$. Then we can use it to show that $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$, that $f_{1,2} = -h_{1,1}$, and that $f_{n-1,n} = h_{n,n}$. From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^{n} h_{ii} \implies$$

$$\sum_i h_{ii} = 0.$$

Remark: this is just the statement that $h \in \mathfrak{sl}_n$; in other words, we will see that every choice of $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_{m+n}$ is also an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_{m+n}$. This shows that $h_{11} = n - 1, h_{22} = n - 3, ..., h_{nn} = 1 - n$. So we have determined

h; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for f in terms of h to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & \\ & 0 & (1-n)+(3-n) & & \\ & & \ddots & \ddots & \\ & & & 0 & (1-n)+\cdots+(n-1) \\ & & & & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1(1-n) & & & \\ & 0 & 2(2-n) & & \\ & & 0 & (n-2)(-2) & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

□

Lemma 4.2. *There is exactly one way to choose $H, F \in \mathfrak{gl}_{m+n}$ so that (E, H, F) is an $\mathfrak{sl}_2$-triple.*

Proof. Suppose we have H, F so that (E, H, F) is an $\mathfrak{sl}_2$-triple. Writing $F =: \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & f \end{pmatrix}$, and similarly for H, we have

$$2E = [H, E] = \begin{pmatrix} 0 & H_{12}e \\ 0 & he \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ eH_{21} & eh \end{pmatrix},$$

$$H = [E, F] = \begin{pmatrix} 0 & 0 \\ eF_{21} & ef \end{pmatrix} - \begin{pmatrix} 0 & F_{12}e \\ 0 & fe \end{pmatrix}.$$

From these we observe that (e, h, f) must also be an $\mathfrak{sl}_2$-triple, so h, f must be as in Lemma 4.1. We also see that $H_{11} = 0$. Recalling that left multiplication by e is a down-shift, and right multiplication is a left-shift, we see that H_{12}

is all zeroes except for the leftmost column, and H_{21} is all zeroes except for the bottom row. Then our final constraint is that

$$-2F = [H, F] =$$

$$\begin{pmatrix} H_{12}F_{21} & H_{12}F_{22} \\ H_{21}F_{11} + H_{22}F_{21} & H_{21}F_{12} + H_{22}F_{22} \end{pmatrix} - \begin{pmatrix} F_{12}H_{21} & F_{11}H_{12} + F_{12}H_{22} \\ F_{22}H_{21} & F_{21}H_{12} + F_{22}H_{22} \end{pmatrix}.$$

Now $H_{12} = F_{12}e$, and $H_{21} = eF_{21}$, from the equation $H = [E, F]$. Substituting in the equation above then,

$$-2F = \begin{pmatrix} F_{12}eF_{21} & F_{12}ef \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} F_{12}eF_{21} & F_{11}F_{12}e + F_{12}h \\ feF_{21} & F_{21}F_{12}e + fh \end{pmatrix} =$$

$$\begin{pmatrix} 0 & F_{12}(fe + h) \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e + F_{12}h \\ (ef - h)F_{21} & F_{21}F_{12}e + fh \end{pmatrix} =$$

$$\begin{pmatrix} 0 & F_{12}fe \\ eF_{21}F_{11} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e \\ efF_{21} & F_{21}F_{12}e + fh \end{pmatrix}.$$

Now we see that $F_{11} = 0$, and consequently that $F_{12} = F_{21} = 0$ as well. This shows that $H_{12} = H_{21} = 0$. We conclude that H and F just have h and f in their bottom-right corners, respectively. $\square$

5 Finding the Slodowy slices with the same E

First we find $\ker \mathrm{ad}_f$. We have $(fX)_{ij} = i(n-i)A_{i+1,j}$, and $(Xf)_{ij} = (j-1)(n-(j-1))A_{i,j-1}$. So, for all $i, j \in \{1, ..., n\}$, we have

$$i(n-i)A_{i+1,j} = (j-1)(n-(j-1))A_{i,j-1}.$$

Taking $j = 1$, we find that $A_{i,1} = 0$ for $i \geq 2$. Then, taking $j > 1$, we find that for $i, j \in \{1, ..., n-1\}$,

$$A_{i+1,j+1} = \frac{(j-1)(n-(j-1))}{i(n-i)} A_{ij}.$$

So, $\ker \mathrm{ad}_f$ is the set of matrices which are upper triangular and satisfy the above condition.

To find the Slodowy slice associated to the previous $\mathfrak{sl}_2$-triple (E, H, F), we just need to find $\ker \operatorname{ad}_F$. We have

$$[F, X] = \begin{pmatrix} 0 & 0 \\ fX_{21} & fX_{22} \end{pmatrix} - \begin{pmatrix} 0 & X_{12}f \\ 0 & X_{22}f \end{pmatrix}.$$

Thus, X_{21} must be all zeroes except for the first row, and X_{12} must be all zeroes except for the last column, and $X_{22} \in \ker \operatorname{ad}_f$. There is no restriction on X_{11}. This describes $\ker \operatorname{ad}_F$.

For $X \in \mathcal{S}_{m,n}$, define $u(X) := X_{11}$.

5.1 Finding $\widetilde{\mathcal{N}}_{m,n}$

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m$ be the nilpotent elements. Let $\mathcal{S}'_{m,n}$ be the set of $X \in \mathcal{S}_{m,n}$ such that both X and $u(X)$ are nilpotent. Let $\widetilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n}$. Define $\pi_{m,n} : \widetilde{\mathcal{N}}_{m,n} \to \mathcal{N}_m$ by $(\mathfrak{b}, X) \mapsto X_{11}$. For $Y \in \mathfrak{gl}_m$, we call $\pi^{-1}_{m,n}(Y)$ the *Springer fiber at the n-Slodowy slice at* Y.

Lemma 5.1. *Let J be a jordan block with zeroes along the diagonal, and let A be upper triangular and nonzero. Then $J + A$ is not nilpotent.*

Proof. It is straightforward to show by induction that if $v_i = 0$ for $i < j$, and $v_j \neq 0$, then $((J + A)^k v_j)_{j+k} = v_j$. Let i be such that $Ae_i \neq 0$. Then $(J + A)^{i-1}e_1$ has nonzero e_i-component. Then $(J + A)^i e_1$ has some nonzero $e_{i'}$-component for some $i' \leq i$. Then $(J + A)^{i+(n-i')}e_1$ has some nonzero e_n-component. And $i + (n - i') \geq n$, so we're done. $\square$

Now, taking the previous corolloary and the first lemma together, we see that

$$\mathcal{S}'_{m,n} = \left\{ \left(\begin{array}{c|cccc} & & & & | \\ X & & & & b \\ & & & & | \\ \hline - \ a \ - & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{array} \right) : a, b \in \mathbb{C}^m, X \in \mathfrak{gl}_m \text{ is nilpotent} \right\}.$$

6 Simplifying the definition of a Springer fiber at a Slodowy slice

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m(\mathbb{C})$ be the subset consisting of nilpotent elements. For $X \in \mathcal{N}_m$ and $a, b \in \mathbb{C}^m$, let

$$A_{X,a,b} = \left(\begin{array}{ccc|ccccc} & & & & & & & | \\ & X & & & & & & b \\ & & & & & & & | \\ \hline — & a & — & 0 & & & & \\ & & & 1 & 0 & & & \\ & & & & 1 & \ddots & & \\ & & & & & \ddots & 0 & \\ & & & & & & 1 & 0 \end{array}\right) \in \mathfrak{gl}_{m+n}(\mathbb{C}).$$

By the definition given in the previous section, we have

$$\widetilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n},$$

and the Springer fiber at the n-Slodowy slice at a nilpotent $X \in \mathfrak{gl}_m$ is

$$\begin{aligned} \pi_{m,n}^{-1}(X) &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \in \mathcal{S}'_{m,n}\} \\ &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b}\text{nilpotent}\} \\ &\cong \{(\mathfrak{b}, a, b) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \text{ nilpotent}\}. \end{aligned}$$

Using our correspondence between $\mathcal{B}_{m+n}$ and complete flags of $\mathbb{C}^{m+n}$, we obtain

$$\pi_{m,n}^{-1}(X) \cong \{(V, a, b) : \forall i.\ A_{X,a,b} V_i \subseteq V_{i-1}\}.$$

7 Strategy and setup for finding the irreducible components of a Springer fiber at a Slodowy slice

Fix any $X \in \mathcal{N}$. As we have fixed X, we now write $A_{a,b} := A_{X,a,b}$. Let $(e_{ij})_{1 \leq i \leq k, 1 \leq j \leq \lambda_i}$ be a Jordan basis for X. For convenience we define $e_{i0} := 0$;

now we may express the fact that (e_{ij}) is a Jordan basis by writing $\forall i. X e_{ij} = e_{i,j-1}$.

In this section we find the irreducible components of

$$\pi^{-1}(X) \cong V := \{(A_{a,b}, U) : \forall i.\ A_{a,b} U_{i+1} \subseteq U_i\}.$$

For $1 \le w \le k$ and $0 \le r \le \lambda_w$ (note that we allow $r = 0$), define

$$V_{w,r} := \{(A_{a,b}, U) \in V : \exists P \in \mathrm{GL}_m\,.\ \exists b'.\ (P^{-1}, I_n) A_{a,b} (P, I_n) = A_{e_{wr}, b'}\}.$$

Lemma 7.1. *$V = \bigcup_{1 \le w \le k, 0 \le r \le \lambda_w} V_{w,r}$. Further, $V_{w_1 r_1} = V_{w_2 r_2}$ exactly when either $r_1 = r_2 = 0$, or $\lambda_{w_1} = \lambda_{w_2} \wedge r_1 = r_2$. When $V_{w_1 r_1} \neq V_{w_2 r_2}$, we have $V_{w_1 r_1} \cap V_{w_2 r_2} = \emptyset$.*

Proof. TODO □

Now, fix any w and r. We will find the irreducible components of $V_{w,r}$. These will all happen to be equidimensional (with dimensions independent of w and r), so their closures in V will be the irreducible components of V.

Let

$$G := \{P \in \mathrm{GL}_m : P^{-1} X P = X\},$$

and

$$G_{wr} = \{A \in G : e_{wr} A = e_{wr}\}.$$

Now, define

$$U_{wr} = \{(A_{e_{wr}, b}, U) \in V_{wr}\}.$$

Let G act on V_{wr} by

$$P \cdot (A_{e_{wr}, b}, U) := ((P, I_n) A_{e_{wr}, b} (P, I_n)^{-1}, (P, I_n) U) = (A_{e_{wr} P^{-1}, Pb}, (P, I_n) U).$$

Consider the map $\varphi : U_{wr} \times G \to V_{wr}$ defined by

$$(x, P) \mapsto P \cdot x.$$

By restriction of G to G_{wr} and V_{wr} to U_{wr}, we obtain an action of G_{wr} on U_{wr}. Then, letting G_{wr} act on G by $g \cdot h := h g^{-1}$, we obtain an action of G_{wr} on $U_{wr} \times G$.

Lemma 7.2. *As an algebraic variety, G is irreducible.*

Proof. It's just $\mathbb{C}^{something}$, since its blocks are the lower-left-toeplitz matrices. □

Lemma 7.3. *The map φ is a principal G_{wr}-bundle.*

Proof. We need to show that G_{wr} acts freely and transitively on the fibers of φ. It is obvious that G_{wr} acts freely on $U_{wr} \times G$; it is enough to note that it acts freely on G.

Let $y \in V_{wr}$. By definition of V_{wr}, there is $P_y \in G$ with $P_y \cdot y \in U_{wr}$. We have

$$\begin{aligned}\varphi^{-1}(y) = \{(x, P) : P \cdot x = y\} = \\ \{(P^{-1}y, P) : P^{-1}y \in U_{wr}\} = \\ \{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \cdot (P_y \cdot y) \in U_{wr}\}.\end{aligned}$$

For any $x \in U_{wr}$ (in particular for $x = P_y \cdot y$), we have $G_{wr} = \{g \in G : g \cdot x \in U_{wr}\}$. Therefore, the fiber above becomes

$$\{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \in G_{wr}\}.$$

Setting $Q := P^{-1}P_y^{-1}$, so that $P = P_y^{-1}Q^{-1}$, the above becomes

$$\begin{aligned}\{(Q \cdot (P_y \cdot y), P_y^{-1}Q^{-1}) : Q \in G_{wr}\} = \\ \{Q \cdot (P_y \cdot y, P_y^{-1}) : Q \in G_{wr}\}.\end{aligned}$$

We see that the fibers are exactly the G_{wr}-orbits; or in other words, G_{wr} acts transitively on the fibers, as desired. □

Our strategy is to find the irreducible components $X \subseteq U_{wr}$, and we will then argue that the irreducible components of V_{wr} are of the form $\varphi(X \times G)$. So, we will now find the irreducible components of U_{wr}.

Actually this will be unnecessarily difficult to think about; it is easiest in the case $r = 0$. So, we will change basis to make $r = 0$. Let $\lambda' = (\lambda_1, ..., \lambda_{w-1}, \lambda_w - r, \lambda_{w+1}, ..., \lambda_k)$. Let X' be in Jordan normal form with shape λ'. Let $U'_{wr} = \{(A_{X',0,b}, U) : \forall i. A_{X',0,b} U_{i+1} \subseteq U_i\}$. Let e'_{ij} be a Jordan basis for X'. Let $f_1, ..., f_n$ be a Jordan basis for the restriction of $A_{X,0,b}$ to ... Set $m' := m - r$, and $n' := n + r$. Let $f'_1, ..., f'_{n'}$ blah. Define the linear map $Q_{wr} : \mathbb{C}^{m'+n'} \to \mathbb{C}^{m+n}$ by:

- For all i, $e'_{ij} \mapsto e_{ij}$.

- For $j = 1, ..., r$, $f'_{n+j} \mapsto e_{w,(\lambda_w - r)+j}$.
- For $j = 1, ..., n$, $f'_j \mapsto f_j + e_{w,(\lambda_w - r)-n+j}$.

Observe that conjugation by Q_{wr} maps U_{wr} to U'_{wr}, and conjugation by Q_{wr}^{-1} maps U'_{wr} to U_{wr}. We conclude that $U_{wr} \cong U'_{wr}$; so to find the irreducible components of U_{wr} we just need to find the irreducible components of U'_{wr}. To clear the context, which is rather cluttered by now, and to avoid writing primes everywhere, we move to a new section.

8 Finding the irreducible components of U_0

8.1 Setup

Let X be nilpotent with Jordan basis $(e'_{ij})_{1 \le i \le k, 1 \le j \le \lambda'_i}$. Let $V = \{(A_{a,b}, U) : \forall i.\ A_{a,b} U_{i+1} \subseteq U\}$. Let

$$U_0 = \{(A_{0,b}, U) \in V\}.$$

We will find the irreducible components of U_0.

We write b_{ij} to denote the projection of $b \in \mathbb{C}^m$ onto e_{ij}. For each row i, let $p_i(b) = \max\{j : b_{ij} \neq 0\}$ (the maximum of the empty set is zero). Then set $q_i(b) = \lambda_i - p_i(b)$. When it is clear enough from context, we will just write p_i and q_i instead of $p_i(b)$ and $q_i(b)$.

Let $I = \{i_1 < \cdots < i_r\} \subseteq \{1, ..., k\}$, and let $(\rho_i)_{i \in I}$ be any map $I \to \mathbb{N}_{>0}$ such that (1) $\rho_i \le \lambda_i$, (2) ρ_i is decreasing with i, (3) $\lambda_i - \rho_i$ is decreasing with i, and (4) $\rho_i < n$. For notational convenience (although we assign meaning to neither i_0 nor i_{r+1}), we define $q_{i_0} := n$, and $p_{i_{r+1}} := 0$. Then, we define $B_{I,(\rho_i)}$ as the set of $b \in \mathbb{C}^m$ satisfying the following conditions.

- For all $k \in \{1, ..., r\}$, $p_i = \rho_i$.
- For all $k \in \{0, ..., r\}$, $p_{i_{k+1}} = \max_{i : q_i < q_{i_k}} p_i$.

Note that for any $b \in B_{I,(\rho_i)}$ we have $p_{i_1} > \cdots > p_{i_r} > p_{i_{r+1}}$, and also $q_{i_0} > q_{i_1} > \cdots > q_{i_r}$.

Let $U_{I,(\rho_i)} := \{(A_{0,b}, U) \in V : b \in B_{I,(\rho_i)}\}$. The idea here is to show that blah relation holds between U and B.

Lemma 8.1. $\mathbb{C}^m = \bigcup_{I,(\rho_i)} B_{I,(\rho_i)}$, *where I ranges over all subsets of $\{1, ..., k\}$, and (ρ_i) ranges over all maps $I \to \mathbb{N}_{>0}$ satisfying the conditions (1),(2),(3),(4). Further, none of the $B_{I,(\rho_i)}$ is contained in the union of the others.*

Proof. Let $b \in \mathbb{C}^m$. If $\{i : q_{i_0} > q_i\}$ is the empty set, then stop. Otherwise, take any $i_1 \in \arg\max_{i:q_{i_0}>q_i} p_i$, and set $\rho_{i_1} := p_{i_1}$. If $\{i : q_{i_1} > q_i\} = \emptyset$, then stop. Otherwise, take any $i_2 \in \arg\max_{i:q_{i_1}>q_i} p_i$, and set $\rho_{i_2} := p_{i_2}$. Continuing on in this way, eventually we reach a point where $\{i : q_{i_k} > q_i\} = \emptyset$. Then we set $I = \{i_1, ..., i_k\}$. Note that $I, (\rho_i)$ satisfy conditions (1)–(4), and furthermore $b \in B_{I,(\rho_i)}$.

Now, we show that no $B_{I,(\rho_i)}$ is contained in the union of the others. Indeed, fix I and (ρ_i). Take any $b \in B_{I,(\rho_i)}$ with $p_i = \rho_i$ for $i \in I$ and $p_i = 0$ for $i \notin I$. It is clear that $b \notin B_{I',(\rho'_i)}$ whenever $I' \neq I$ or $(\rho'_i) \neq (\rho_i)$. $\square$

8.2 A study of $B_{I,(\rho_i)}$

Fix any I and (ρ_i) satisfying the conditions (1)–(4). As before, we write $\{i_1 < \cdots < i_r\} := I$, and $q_{i_0} := n$, and $p_{i_{r+1}} := 0$.

First, we provide an alternative characterization of $B_{I,(\rho_i)}$.

Lemma 8.2. *$B_{I,(\rho_i)}$ is the set of $b \in \mathbb{C}^m$ satisfying the following conditions.*

- *For all $k \in \{1, ..., r\}$, $p_{i_k} = \rho_{i_k}$.*
- *For all $i \notin I$,*
  - *For all $k \in \{0, ..., r\}$ such that $\lambda_i > q_{i_k} + p_{i_{k+1}}$, we have $q_{i_k} \leq q_i$.*
  - *For all $k \in \{1, ..., r+1\}$ such that $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$, we have $p_i \leq p_{i_k}$.*

Proof. First we show that every element of $B_{I,(\rho_i)}$ satisfies those conditions. Let $b \in B_{I,(\rho_i)}$. It is clear that $\forall k.\ p_{i_k} = \rho_{i_k}$.

Take $i \notin I$ and $k \in \{0, ..., r\}$ such that $\lambda_i > q_{i_k} + p_{i_{k+1}}$. Suppose for contradiction that $q_i < q_{i_k}$. Then $p_i \leq \max_{j:q_j<q_{i_k}} p_j = p_{i_{k+1}}$. Then $\lambda_i = p_i + q_i < q_{i_k} + p_{i_{k+1}}$, a contradiction. So we must have $q_{i_k} \leq q_i$, as desired.

Now take $i \notin I$ and $k \in \{1, ..., r+1\}$ such that $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$. Suppose for contradiction that $p_i > p_{i_k}$. Then, putting this together with the first inequality, $\lambda_i - p_i < q_{i_{k-1}} + p_{i_k} - p_{i_k}$; that is, $q_i < q_{i_{k-1}}$. Consequently, $p_i \leq \max_{j:q_j<q_{i_{k-1}}} p_j = p_{i_k}$, as desired.

Now we have shown that every element of $B_{I,(\rho_i)}$ satisfies the conditions of the lemma, and we proceed to the converse. Let $b \in \mathbb{C}^m$ satisfy the conditions. Let $k \in \{0, ..., r\}$. We need to show that $\max_{j:q_j<q_{i_k}} p_j = p_{i_{k+1}}$. Given the conditions (1)–(4) on ρ_i, it suffices to show that for each $i \notin I$ with $q_i > q_{i_k}$, we have $p_i \leq p_{i_{k+1}}$. Indeed, given $i \notin I$ with $q_i > q_{i_k}$, we cannot have $\lambda_i > q_{i_k} + p_{i_{k+1}}$, as that would imply that $q_{i_k} \leq q_i$. Hence we must have $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$, and consequently $p_i \leq p_{i_k}$. □

Corollary 8.3. *$B_{I,(\rho_i)}$ is the set of $b \in \mathbb{C}^m$ satisfying the following conditions.*

- *For all $k \in \{1, ..., r\}$, $p_{i_k} = \rho_{i_k}$.*
- *For $i \notin I$,*
  - *If $\lambda_i \geq q_{i_0} + p_{i_1}$, then $p_i \leq \lambda_i - q_{i_0}$.*
  - *If there is $k \in \{0, ..., r\}$ with $q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}$, then $p_i \leq \min(p_{i_k}, \lambda_i - q_{i_k})$.*
  - *If $q_{i_r} + p_{i_{r+1}} > \lambda_i$, then $p_i \leq p_{i_{r+1}}$.*

Proof. Both q_{i_k} and p_{i_k} are decreasing as k increases, so this follows directly from Lemma 8.2. (Note that $p_i \leq \lambda_i - q_{i_k}$ iff $q_{i_k} \leq q_i$.) □

Corollary 8.4.

$$B_{I,(\rho_i)} \cong \prod_{k=1}^{r} (\mathbb{C}^{\rho_{i_k}-1} \times (\mathbb{C}\setminus\{0\})) \times \prod_{i:\lambda_i \geq q_{i_0}+p_{i_1}} \mathbb{C}^{\lambda_i - q_{i_0}} \times \prod_{k=0}^{r} \prod_{\{i \notin I: q_{i_{k-1}}+p_{i_k} > \lambda_i \geq q_{i_k}+p_{i_{k+1}}\}} \mathbb{C}^{\min(p_{i_k}, \lambda_i - q_{i_k})}$$

Proof. Here I use the notation $x \times y = (x, y)$, and so on. The isomorphism sends $b \in B_{I,(\rho_i)}$ to

$$\prod_{k=1}^{r} (b_{i_k 1}, ..., b_{i_k \rho_{i_k}}) \times \prod_{i:\lambda_i \geq q_{i_0}+p_{i_1}} (b_{i1}, ..., b_{i,\lambda_i - q_{i_0}}) \times$$

$$\prod_{k=0}^{r} \prod_{\{i \notin I: q_{i_{k-1}}+p_{i_k} > \lambda_i \geq q_{i_k}+p_{i_{k+1}}\}} (b_{i1}, ..., b_{i,\min(p_{i_k}, \lambda_i - q_{i_k})}).$$

Corollary 8.3 tells us that this is an isomorphism. □

Corollary 8.5.

$$\dim B_{I,(\rho_i)} = \sum_{i:\lambda_i \geq q_{i_0}+\rho_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=0}^{r} \sum_{\{i: q_{i_{k-1}}+p_{i_k} > \lambda_i \geq q_{i_k}+p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}).$$

Proof. Immediate from Corollary 8.4. □

8.3 A study of $U_{I,(\rho_i)}$

Fix any I and (ρ_i) satisfying the conditions of Lemma 8.1. We will find the irreducible components of $U_{I,(\rho_i)}$, and show that their closures are in fact irreducible components of U_0.

I claim that $A_{0,b}$ has the same shape for every $(A_{0,b}, U) \in U_{I,(\rho_i)}$. Call this shape μ. By finding a algebraic map taking b to a Jordan basis for $A_{0,b}$, we will put $U_{I,(\rho_i)}$ in isomorphism with the product (choice of b) $\times$ (Springer fiber at $J(\mu)$). Then we will use our result about the usual Springer fiber at $J(\mu)$ to find the irreducible components of $U_{I,(\rho_i)}$.

So, now we find a Jordan basis for $A_{0,b}$.

Lemma 8.6. *The following vectors give a Jordan basis for $A_{0,b}$. (For convenience, we write $A := A_{0,b}$ in this lemma and proof.)*

- *For $i \notin I$, the chain of length $p_i + q_i$ beginning with e_{i,p_i+q_i}*
- *The chain of length $n + p_{i_1}$ beginning with $f_n - (A^{n+p_{i_1}} f_n \texttt{>>}\ n + p_{i_1})$*
- *For $k \in \{1, ..., r\}$, the chain of length $q_{i_k} + p_{i_{k+1}}$ beginning with $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \texttt{>>}\ q_{i_k} + p_{i_{k+1}})$, where $v_{i_k} := A^{n-q_{i_k}} f_n - \sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i,l+q_{i_k}}$*

Proof. There are three things to check: (1) the chains are no longer than their claimed lengths, (2) the claimed lengths sum to $m + n$, and (3) the span of the chains is $\mathbb{C}^{m+n}$.

Proof of (1). It is obvious that a chain beginning with e_{i,p_i+q_i} has length $p_i + q_i$.

Now consider the chain beginning with $f_n - (A^{n+p_{i_1}} f_n \texttt{>>}\ n + p_{i_1})$. Note that $A^n f_n = b$, so $A^{n+p_{i_1}} f_n = A^{p_{i_1}} b = b \texttt{<<} p_{i_1}$. By shifting b left p_{i_1} times, we zero out all the rows i where $q_i < n$. This ensures that the operation of shifting $b \texttt{<<} p_{i_1}$ right $n + p_{i_1}$ times is invertible by shifting left $n + p_{i_1}$ times. That is,

$$\begin{aligned} A^{n+p_{i_1}} f_n &= \\ b \texttt{<<} p_{i_1} &= \\ ((b \texttt{<<} p_{i_1}) \texttt{>>}\ n + p_{i_1}) \texttt{<<} n + p_{i_1} &= \\ A^{n+p_{i_1}} (A^{n+p_{i_1}} f_n \texttt{>>}\ n + p_{i_1}). \end{aligned}$$

This shows that the chain has length at most $n + p_{i_1}$, as desired.

Now, let $k \in \{1, ..., r\}$. We have $v_{i_k} = A^{n-q_{i_k}} f_n - \sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i_k, l+q_{i_k}}$. First we note that $q_{i_k} < n$ by the definition of $U_{I,(\rho_i)}$, so the definition of v_{i_k} makes sense. We consider the chain beginning with $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \gg q_{i_k} + p_{i_{k+1}})$. Note that $A^{q_{i_k}} v_{i_k}$ is just b with row i_k zeroed out. For brevity, we write $b_{i_k} := A^{q_{i_k}} v_{i_k}$. Note that $b_{i_k} \ll p_{i_{k+1}}$ has rows l zeroed out, for all l with $q_l < q_{i_k}$. This ensures that shifting $b_{i_k} \ll p_{i_{k+1}}$ right $q_{i_k} + p_{i_{k+1}}$ times can be inverted by shifting left $q_{i_k} + p_{i_{k+1}}$ times. That is,

$$\begin{aligned} A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} &= \\ b_{i_k} \ll p_{i_{k+1}} &= \\ ((b_{i_k} \ll p_{i_{k+1}}) \gg q_{i_k} + p_{i_{k+1}}) \ll q_{i_k} + p_{i_{k+1}} &= \\ A^{q_{i_k}+p_{i_{k+1}}} (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \gg q_{i_k} + p_{i_{k+1}}). & \end{aligned}$$

This shows that the chain has length at most $q_{i_k} + p_{i_{k+1}}$, as desired. $\square$

Proof of (2). The sum of the lengths is

$$\sum_{i \notin I} (p_i + q_i) + (n + P) + \sum_{i \in I} (q_i + P_i).$$

Writing $I = \{i_1 < \cdots < i_{|I|}\}$, we note that $P = p_{i_1}$, that $P_{i_{|I|}} = 0$, and that for $l < |I|$ we have $P_{i_l} = p_{i_{l+1}}$. Thus, the sum above is

$$\sum_{i \notin I} (p_i + q_i) + (n + p_{i_1}) + \sum_{l=1}^{|I|} (q_{i_l} + p_{i_{l+1}}) =$$

$$\sum_{i \notin I} (p_i + q_i) + n + \sum_{l=1}^{|I|} (q_{i_l} + p_{i_l}) = n + \sum_{i} (q_i + p_i) = m + n.$$

$\square$

Proof of (3). Let W be the span of the chains listed. We need to show that $W = \mathbb{C}^{m+n}$. Because every $i \in I$ satisfies $q_i < n$, clearly $\langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$.

I claim that $f_n \in W$ as well. To see this, we consider the chain beginning with $f_n - (A^{n+P} f_n \gg n + P)$. As explained in the proof of (1), we have $A^{n+P} f_n \in \langle e_{ij} \rangle_{i,j:q_i \geq n}$. Consequently, $(A^{n+P} f_n \gg n + P) \in \langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$. Because $f_n - (A^{n+P} f_n \gg n + P) \in W$, this implies that $f_n \in W$.

Because $AW \subseteq W$ (obvious, since W is the span of chains), the fact that $f_n \in W$ implies that $f_i \in W$ for each i, and also $b \ll l \in W$ for each $l \geq 0$.

Now we are left with showing that $\langle e_{ij}\rangle_{i,j:q_i<n} \subseteq W$. This is obvious for $i \notin I$. For $i \in I$, we do it inductively. Fix $i \in I$, and suppose we have already shown that for all $i' \in I$ with $q_{i'} > q_i$, we have $\langle e_{ij}\rangle_{j \text{ arbitrary}} \subseteq W$. We will show that $\langle e_{ij}\rangle_{j \text{ arbitrary}} \subseteq W$.

To see this, we consider the chain beginning with $v_i - (A^{q_i+P_i} v_i \text{ >> } q_i + P_i)$. (Recall $v_i = A^{n-q_i} f_n - \sum_{l=1}^{p_i} b_{il} e_{i,l+q_i}$.) Because $AW \subseteq W$, and $b_{i,p_i} \neq 0$ (by definition of p_i), it suffices to show that $\sum_{l=1}^{p_i} b_{il} e_{i,k+q_i} \in W$. As explained in the proof of (1), we have $A^{q_i+P_i} v_i \in \langle e_{lj}\rangle_{q_l \geq q_i}$. And since $A^{q_i+P_i} v_i$ has row i zeroed out, in fact $A^{q_i+P_i} v_i \in \langle e_{lj}\rangle_{l:l\neq i \wedge q_l \geq q_i}$. Hence, $A^{q_i+P_i} v_i \text{ >> } q_i + P_i \in \langle e_{lj}\rangle_{l:l\neq i \wedge q_l \geq q_i}$. By our inductive hypothesis, $\langle e_{lj}\rangle_{l:l\neq i \wedge q_l \geq q_i} \subseteq W$, and consequently $A^{q_i+P_i} v_i \text{ >> } q_i + P_i \in W$. Since we know $v_i - (A^{q_i+P_i} v_i \text{ >> } q_i + P_i) \in W$, this implies that $v_i \in W$. Because $A^{n-q_i} f_n \in W$, this then implies that $\sum_{l=1}^{p_i} b_{il} e_{i,l+q_i} \in W$, as desired. $\square$

We proved (1), (2), (3), so we are done. $\square$

Let $\mu(I, (\rho_i))$ be the shape of the Jordan basis given in the previous lemma. Let X_μ be the Springer fiber at $J(\mu)$, and let $(X_{\mu,\alpha})_{\alpha \in \mathrm{SYT}(\mu)}$ be the irreducible components.

Given a zero-indexed list $L = [L_0, ..., L_{l-1}]$, we define $\gamma(L) = \sum_i i L_i$. We will be interested in this thing because the dimension of X_μ is $\gamma([\mu_1, ..., \mu_l])$. Let s be the function taking integer lists to lists which sorts the input in nonincreasing order.

Lemma 8.7. *$\gamma([\mu_1, ..., \mu_l]) = \gamma([0, \lambda_1, ..., \lambda_k]) -$ something.*

Proof. Let $L = [q_{i_0}, \lambda_1, ..., \lambda_k]$. Note that $\mu = [..., q_{i_0} + p_{i_1}, ..., q_{i_1} + p_{i_2}, ..., ..., q_{i_r} + p_{i_{r+1}}, ...]$. Let L' be the result of taking μ and, for each x, replacing one occurence of $q_{i_x} + p_{i_{x+1}}$ by $q_{i_x} + p_{i_x}$; that is, $L' = [..., q_{i_0}, ..., q_{i_1} + p_{i_1}, ..., q_{i_2} + p_{i_2}, ..., q_{i_r} + p_{i_r}]$. Note that we get from μ to L' by just moving each p_{i_k} to the right by $1 + \#\{i \notin I : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}$ slots. So,

$$\gamma(L') - \gamma(\mu) = \sum_{k=1}^{r} p_{i_{k+1}}(1 + \#\{i \notin I : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}) =$$

$$\sum_{k=1}^{r} p_{i_k} \cdot \#\{i : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}.$$

Now we consider how to get from L' to L. First we shift q_{i_0} to the left by $\#\{i \notin I : \lambda_i \geq q_{i_0} + p_{i_1}\}$ slots. Then we leave q_{i_0} in place and sort the rest of the list. This entails shifting each $q_{i_x}+p_{i_x}$ to the left by $\#\{i \notin I : q_{i_x}+p_{i_x} > \lambda_i \geq q_{i_x}+p_{i_{x+1}}\}$ slots. Shifting $q_{i_x}+p_{i_x}$ to the left one slot, by swapping it with λ_i, changes the value of γ by $\lambda_i - (q_{i_x}+p_{i_x})$. To go from L' to $s(L')$, we can just make these swaps repeatedly. So,

$$\gamma(L) - \gamma(L') =$$

$$\sum_{i \notin I : q_{i_0}+p_{i_1} > \lambda_i \geq q_{i_0}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{s} \sum_{\{i \notin I : q_{i_k}+p_{i_{k+1}} > \lambda_i \geq q_{i_k}+p_{i_k}\}} (\lambda_i - (q_{i_k}+p_{i_k})).$$

TODO: the thing above would be cleaner if $p_{i_0} := 0$.

Now,

$$\gamma(L) - \gamma(\mu) =$$
$$[\gamma(L) - \gamma(L')] + [\gamma(L') - \gamma(\mu)] =$$

□

Lemma 8.8. $U_{I,(\rho_i)} \cong B_{I,(\rho_i)} \times X_\mu$.

Proof. For $b \in B_{I,(\rho_i)}$, let P_b be the change-of-basis matrix, with columns given by the Jordan basis of the previous lemma, so that $J(\mu) = P_b^{-1} A_{0,b} P_b$. From looking at the Jordan basis of the previous lemma (the basis can be expressed in terms of $A, \lambda, e_{ij}, f_i, I, \rho_i, b$), it is clear that the map $P : B_{I,(\rho_i)} \to \mathrm{GL}_{m+n}$, given by $b \mapsto P_b$, is algebraic.

Now, we remark that the Springer fiber at $A_{0,b}$ is simply $\{P_b U : U \in X_\mu\}$. This gives us the isomorphism $B_{I,(\rho_i)} \times X_\mu \to U_{I,(\rho_i)}$.

$$(b, U) \mapsto (P_b J(\mu) P_b^{-1}, P_b U),$$

with inverse

$$(A_{0,b}, U) \mapsto (b, P_b^{-1} U).$$

□

8.4 Conclusion

At last we have reached the base of our recursive procedure, and we begin to propagate upwards. We write $(X_{\mu,\alpha})_\alpha$ to denote the irreducible components of X_μ.

Lemma 8.9. *The irreducible components of $U_{I,(\rho_i)}$ are exactly the subvarieties*

$$U_{I,(\rho_i),\alpha} := \{(P_b J(\mu) P_b^{-1}, P_b U) : b \in B_{I,(\rho_i)}, U \in X_{\mu,\alpha}\}.$$

These are distinct. There are blah of them, and each has dimension blah.

Proof. We know from Corollary 8.4 that $B_{I,(\rho_i)}$ is irreducible. So, the components of $B_{I,(\rho_i)} \times X_\mu$ are obviously $B_{I,(\rho_i)} \times X_{\mu,\alpha}$. Their dimensions are blah, and there are... of them. taking image under the isomorphism gives the desired result. $\square$

Lemma 8.10. *The irreducible components of U_0 are exactly the $U_{I,(\rho_i),\alpha}$. They are distinct. There are blah of them, each of dimension blah.*

9 Finding the irreducible components of a springer fiber at a slodowy slice

Apply things from the previous two sections, and conclude.

10 A different variety

Define $R = \{(X, \mathfrak{b}_1, \mathfrak{b}_2) : X \in \mathfrak{b}_1 \wedge u(X) \in \mathfrak{b}_2\} \subseteq \mathcal{S}'_{m,n} \times \mathcal{B}_{m+n} \times \mathcal{B}_m$. We can obtain a subvariety of R by requiring that $u(X)$ is in some fixed similarity class. We expect that each of these subvarieties is an irreducible component of dimension m^2. We will verify these things using our prvious computations of springer fibers.

References

[1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.