

1 Introduction

We will review the definition of a Springer fiber and define, for a nilpotent $Y \in \mathfrak{gl}_m$, the Springer fiber at the n -Slodowy slice at Y . For every n and every nilpotent $Y \in \mathfrak{gl}_m$, we will find the irreducible components of the Springer fiber at the n -Slodowy slice at Y . Finally, we will use our results about Springer fibers at n -Slodowy slices to find the irreducible components of some other variety (which probably needs a name), and show that they all have the same dimension.

2 Springer fibers

Let $G \subseteq \mathrm{GL}_m(\mathbb{C})$ be a connected semisimple Lie group, and let $\mathfrak{g} \subseteq \mathfrak{gl}_m(\mathbb{C})$ be its Lie algebra. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the subset consisting of nilpotent elements. Let \mathcal{B} be the variety of Borel subalgebras of \mathfrak{g} . Let $\tilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$. Let $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ be the projection onto the second coordinate. For $n \in \mathcal{N}$, we call $\pi^{-1}(n)$ the *Springer fiber at n* .

We mention some results about Springer fibers, which we will use later in this paper. TODO: mention them.

3 Slodowy slice

A basis for $\mathfrak{sl}_2(\mathbb{C})$ is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra \mathfrak{g} , and a homomorphism $\phi : \mathfrak{sl}_2 \rightarrow \mathfrak{g}$ sending (e', h', f') to (e, h, f) , we say that (e, h, f) is an *\mathfrak{sl}_2 -triple*. Observe that e, f must be nilpotent, and h must be Cartan (???). If \mathfrak{g} is semisimple, then given any nilpotent $e \in \mathfrak{g}$, the Jacobson-Morozov theorem [1, 3.7.1] says that there exist $h, f \in \mathfrak{g}$ such that (e, h, f) is an \mathfrak{sl}_2 -triple.

Given (e, h, f) , we define the *Slodowy slice at e* as $\mathcal{S}_e := e + \ker \mathrm{ad}_f$. By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent $e \in \mathfrak{g}$, when \mathfrak{g} is semisimple.

4 Finding \mathfrak{sl}_2 -triples (E, H, F) in \mathfrak{gl}_{m+n} with a particular E .

Let

$$e = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \in \mathfrak{gl}_n,$$

and let $E = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{gl}_{m+n}$. We will show that there is exactly one \mathfrak{sl}_2 -triple (E, H, F) , and we will find what it looks like. First we solve the case $m = 0$ (so $E = e$), and then we use this to solve the case of arbitrary m .

Lemma 4.1. *There is exactly one way to choose $h, f \in \mathfrak{gl}_n$ so that (e, h, f) is an \mathfrak{sl}_2 -triple.*

Proof. Note that $[h', e'] = 2e'$, and $[e', f'] = h'$, and $[h', f'] = -2f'$. Thus e, h, f must obey the same relations. In particular, $he - eh = 2e$. The matrix eh is h shifted down one, and he is h shifted left one. Thus, $2e_{ij} = (he - eh)_{ij} = h_{i,j+1} - h_{i-1,j}$. We can use this to show that $h_{ij} = 0$ when $i \neq j$. Then we can use it to show that $h_{ii} = h_{i-1,i-1} + 2$, so that $h_{ii} = h_{11} + 2(i-1)$.

Similarly, from $[e, f] = h$ we get that $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$. We can use this to show that $f_{ij} = 0$ when $j \neq i+1$. Then we can use it to show that $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$, that $f_{1,2} = -h_{1,1}$, and that $f_{n-1,n} = h_{n,n}$. From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^n h_{ii} \implies \sum_i h_{ii} = 0.$$

Remark: this is just the statement that $h \in \mathfrak{sl}_n$; in other words, we will see that every choice of \mathfrak{sl}_2 -triple in \mathfrak{gl}_{m+n} is also an \mathfrak{sl}_2 -triple in \mathfrak{sl}_{m+n} . This shows that $h_{11} = n-1, h_{22} = n-3, \dots, h_{nn} = 1-n$. So we have determined

h ; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for f in terms of h to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & \\ & 0 & (1-n) + (3-n) & & \\ & & \ddots & \ddots & \\ & & & 0 & (1-n) + \cdots + (n-1) \\ & & & & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1(1-n) & & & \\ & 0 & 2(2-n) & & \\ & & 0 & (n-2)(-2) & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

□

Lemma 4.2. *There is exactly one way to choose $H, F \in \mathfrak{gl}_{m+n}$ so that (E, H, F) is an \mathfrak{sl}_2 -triple.*

Proof. Suppose we have H, F so that (E, H, F) is an \mathfrak{sl}_2 -triple. Writing $F =: \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & f \end{pmatrix}$, and similarly for H , we have

$$2E = [H, E] = \begin{pmatrix} 0 & H_{12}e \\ 0 & he \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ eH_{21} & eh \end{pmatrix},$$

$$H = [E, F] = \begin{pmatrix} 0 & 0 \\ eF_{21} & ef \end{pmatrix} - \begin{pmatrix} 0 & F_{12}e \\ 0 & fe \end{pmatrix}.$$

From these we observe that (e, h, f) must also be an \mathfrak{sl}_2 -triple, so h, f must be as in Lemma 4.1. We also see that $H_{11} = 0$. Recalling that left multiplication by e is a down-shift, and right multiplication is a left-shift, we see that H_{12}

is all zeroes except for the leftmost column, and H_{21} is all zeroes except for the bottom row. Then our final constraint is that

$$-2F = [H, F] =$$

$$\begin{pmatrix} H_{12}F_{21} & H_{12}F_{22} \\ H_{21}F_{11} + H_{22}F_{21} & H_{21}F_{12} + H_{22}F_{22} \end{pmatrix} - \begin{pmatrix} F_{12}H_{21} & F_{11}H_{12} + F_{12}H_{22} \\ F_{22}H_{21} & F_{21}H_{12} + F_{22}H_{22} \end{pmatrix}.$$

Now $H_{12} = F_{12}e$, and $H_{21} = eF_{21}$, from the equation $H = [E, F]$. Substituting in the equation above then,

$$\begin{aligned} -2F &= \begin{pmatrix} F_{12}eF_{21} & F_{12}ef \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} F_{12}eF_{21} & F_{11}F_{12}e + F_{12}h \\ feF_{21} & F_{21}F_{12}e + fh \end{pmatrix} = \\ &= \begin{pmatrix} 0 & F_{12}(fe + h) \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e + F_{12}h \\ (ef - h)F_{21} & F_{21}F_{12}e + fh \end{pmatrix} = \\ &= \begin{pmatrix} 0 & F_{12}fe \\ eF_{21}F_{11} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e \\ efF_{21} & F_{21}F_{12}e + fh \end{pmatrix}. \end{aligned}$$

Now we see that $F_{11} = 0$, and consequently that $F_{12} = F_{21} = 0$ as well. This shows that $H_{12} = H_{21} = 0$. We conclude that H and F just have h and f in their bottom-right corners, respectively. \square

5 Finding the Slodowy slices with the same E

First we find $\ker \text{ad}_f$. We have $(fX)_{ij} = i(n-i)A_{i+1,j}$, and $(Xf)_{ij} = (j-1)(n-(j-1))A_{i,j-1}$. So, for all $i, j \in \{1, \dots, n\}$, we have

$$i(n-i)A_{i+1,j} = (j-1)(n-(j-1))A_{i,j-1}.$$

Taking $j = 1$, we find that $A_{i,1} = 0$ for $i \geq 2$. Then, taking $j > 1$, we find that for $i, j \in \{1, \dots, n-1\}$,

$$A_{i+1,j+1} = \frac{(j-1)(n-(j-1))}{i(n-i)}A_{ij}.$$

So, $\ker \text{ad}_f$ is the set of matrices which are upper triangular and satisfy the above condition.

To find the Slodowy slice associated to the previous \mathfrak{sl}_2 -triple (E, H, F) , we just need to find $\ker \operatorname{ad}_F$. We have

$$[F, X] = \begin{pmatrix} 0 & 0 \\ fX_{21} & fX_{22} \end{pmatrix} - \begin{pmatrix} 0 & X_{12}f \\ 0 & X_{22}f \end{pmatrix}.$$

Thus, X_{21} must be all zeroes except for the first row, and X_{12} must be all zeroes except for the last column, and $X_{22} \in \ker \operatorname{ad}_f$. There is no restriction on X_{11} . This describes $\ker \operatorname{ad}_F$.

For $X \in \mathcal{S}_{m,n}$, define $u(X) := X_{11}$.

5.1 Finding $\tilde{\mathcal{N}}_{m,n}$

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m$ be the nilpotent elements. Let $\mathcal{S}'_{m,n}$ be the set of $X \in \mathcal{S}_{m,n}$ such that both X and $u(X)$ are nilpotent. Let $\tilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n}$. Define $\pi_{m,n} : \tilde{\mathcal{N}}_{m,n} \rightarrow \mathcal{N}_m$ by $(\mathfrak{b}, X) \mapsto X_{11}$. For $Y \in \mathfrak{gl}_m$, we call $\pi_{m,n}^{-1}(Y)$ the *Springer fiber at the n -Slodowy slice at Y* .

Lemma 5.1. *Let J be a jordan block with zeroes along the diagonal, and let A be upper triangular and nonzero. Then $J + A$ is not nilpotent.*

Proof. It is straightforward to show by induction that if $v_i = 0$ for $i < j$, and $v_j \neq 0$, then $((J + A)^k v_j)_{j+k} = v_j$. Let i be such that $Ae_i \neq 0$. Then $(J + A)^{i-1}e_1$ has nonzero e_i -component. Then $(J + A)^i e_1$ has some nonzero $e_{i'}$ -component for some $i' \leq i$. Then $(J + A)^{i+(n-i')}e_1$ has some nonzero e_n -component. And $i + (n - i') \geq n$, so we're done. \square

Lemma 5.2. *Let $X \in \mathfrak{gl}_m$, and let*

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ d_1 & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ & d_2 & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ & & \ddots & \cdots & \vdots & \vdots \\ & & & d_{n-2} & y_{n-1,n-1} & y_{n-1,n} \\ & & & & d_{n-1} & y_{nn} \end{pmatrix} \in \mathfrak{gl}_n.$$

For any $a, b \in \mathbb{C}^m$,

$$\det \left(\begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & Y \end{array} \right) =$$

$$\det X \det Y + \left(\prod_i d_i \right) \det \left(\begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & 0 \end{array} \right)$$

Proof. By induction on n . In the case $n = 1$, expanding along the last row (taking the usual interpretation of the empty product) gives the desired result.

Now suppose $n > 1$. Expanding along the last row, we get

$$d_{n-1} \det \left(\begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & Y_{n,n-1} \end{array} \right) - y_{nn} \det \left(\begin{array}{c|c} X & \begin{array}{c} | \\ | \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & Y_{n,n} \end{array} \right).$$

Using our inductive hypothesis for the first determiniant, and using that $\det \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} = \det A_{11} \det A_{22}$ for the second, the expression becomes

$$d_{n-1} \left(\det X \det Y_{n,n-1} + \left(\prod_{i \leq n-2} d_i \right) \det \left(\begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & 0 \end{array} \right) \right) - y_{nn} \det X \det Y_{nn} =$$

$$(d_{n-1} Y_{n,n-1} - y_{nn} \det Y_{nn}) \det X + \left(\prod_i d_i \right) \det \left(\begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \\ a \\ - \end{array} & 0 \end{array} \right) =$$

$$\det Y \det X + \left(\prod_i d_i \right) \det \left(\begin{array}{ccc|c} & & & b \\ & & & | \\ X & & & | \\ \hline - & a & - & 0 \end{array} \right).$$

□

Corollary 5.3. *If X is nilpotent, and*

$$\left(\begin{array}{ccc|c} & & & b \\ & & & | \\ X & & & | \\ \hline - & a & - & Y \end{array} \right)$$

is nilpotent as well, then Y is nilpotent (TODO: and that other determinant is zero).

Proof. By the previous lemma, the characteristic polynomial of the big matrix is

$$g_X(\lambda)g_Y(\lambda) + f(\lambda),$$

where $g_X(\lambda) = \lambda^m$ is the characteristic polynomial of X , and $f(\lambda)$ is some polynomial of degree at most $m - 1$. □

Now, taking the previous corollary and the first lemma together, we see that

$$\mathcal{S}'_{m,n} = \left\{ \left(\begin{array}{ccc|cccc} & & & & & \\ & & & & & b \\ & & & & & | \\ X & & & & & | \\ \hline - & a & - & 0 & & \\ & & & 1 & 0 & \\ & & & & \ddots & \ddots \\ & & & & & 1 & 0 \end{array} \right) : a, b \in \mathbb{C}^m, X \in \mathfrak{gl}_m \text{ is nilpotent} \right\}.$$

6 Simplifying the definition of a Springer fiber at a Slodowy slice

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m(\mathbb{C})$ be the subset consisting of nilpotent elements. For $X \in \mathcal{N}_m$ and $a, b \in \mathbb{C}^m$, let

$$A_{X,a,b} = \left(\begin{array}{c|ccc} X & & & & b \\ \hline - & a & - & & \\ \hline & & 0 & & \\ & & 1 & 0 & \\ & & & 1 & \ddots \\ & & & & \ddots & 0 \\ & & & & & 1 & 0 \end{array} \right) \in \mathfrak{gl}_{m+n}(\mathbb{C}).$$

By the definition given in the previous section, we have

$$\tilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n},$$

and the Springer fiber at the n -Slodowy slice at a nilpotent $X \in \mathfrak{gl}_m$ is

$$\begin{aligned} \pi_{m,n}^{-1}(X) &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \in \mathcal{S}'_{m,n}\} \\ &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \text{ nilpotent}\} \\ &\cong \{(\mathfrak{b}, a, b) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \text{ nilpotent}\}. \end{aligned}$$

We will make one last simplification to this by using a correspondence between Borel subalgebras and complete flags. TODO: fix the next few paragraphs, they are out of context.

We have $\mathcal{M} = \{AHA^{-1} : A \in \text{GL}_{m+n}(\mathbb{C})\}$, where $H \subseteq \mathfrak{gl}_{m+n}(\mathbb{C})$ is the set of upper triangular matrices. We say a map $X : \mathfrak{gl}_{m+n} \rightarrow \mathfrak{gl}_{m+n}$ preserves a flag $V_0 \subseteq \cdots \subseteq V_{m+n}$ if $XV_i \subseteq V_i$ for each i . Let $E_0 \subseteq \cdots \subseteq E_{m+n}$ be the standard flag of \mathbb{C}^{m+n} . Since H is the set of X which preserve E ,

$$\mathcal{M} = \{\{X : \forall i. X(AE_i) \subseteq AE_i\} : A \in \text{GL}_n(\mathbb{C})\}.$$

So, for $X \in \mathcal{N}$,

$$\pi^{-1}(X) \cong \{(V, a, b) : \forall i. A_{X,a,b}V_i \subseteq V_i\}.$$

In this section we will find the irreducible components of $\pi^{-1}(X)$. Since $\pi^{-1}(X) \cong \pi^{-1}(AXA^{-1})$ for any invertible A , we will assume that X is in Jordan normal form.

Let λ be the shape of X , and let $(e_{ij})_{i \leq r, j \leq \lambda_i}$ be a Jordan basis for X , with $Xe_{ij} = e_{i,j-1}$. Let f_1, \dots, f_n be the standard basis for \mathbb{C}^n , with $A_{X,a,b}f_i = f_{i+1}$.

7 Finding a centralizer

For nilpotent X in Jordan form and a in ‘normalized’ form (i.e., with at most one nonzero element, which is a one), we will find the centralizer of $A_{X,a,b}$ in

$$\{(A, I) : A \in \mathfrak{gl}_m\} \subseteq \mathfrak{gl}_{m+n}.$$

Note that an element of the form (A, I) commutes with $A_{X,a,b}$ if and only if A commutes with X , and $aA = a$, and $Ab = b$. So, we just have to find

$$\{A \in \mathfrak{gl}_m : AX = XA, aA = a, Ab = b\}.$$

We begin by finding

$$\{A \in \mathfrak{gl}_m : AX = XA\}.$$

We write the shape of X as $\lambda = (\lambda_1, \dots, \lambda_k)$, so that

$$X = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix}.$$

Then we write A as the block matrix

$$A =: \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix},$$

where A_{ij} is a block of size $\lambda_i \times \lambda_j$. We have

$$XA = \begin{pmatrix} J_{\lambda_1} A_{11} & \cdots & J_{\lambda_1} A_{1k} \\ \vdots & & \vdots \\ J_{\lambda_k} A_{k1} & \cdots & J_{\lambda_k} A_{kk} \end{pmatrix}, \text{ and } AX = \begin{pmatrix} A_{11} J_{\lambda_1} & \cdots & A_{1k} J_{\lambda_k} \\ \vdots & & \vdots \\ A_{k1} J_{\lambda_1} & \cdots & A_{kk} J_{\lambda_k} \end{pmatrix}.$$

So, the constraint that $XA = AX$ is simply saying that

$$\forall i, j. J_{\lambda_i} A_{ij} = A_{ij} J_{\lambda_j}.$$

It's easy to see that left multiplication by J_{λ_i} is just a down-shift by one, and right multiplication by J_{λ_j} is a left-shift by one. This means that A_{ij} is constrained to be a “lower-left-Toeplitz” matrix. A Toeplitz matrix is one which is constant along diagonal bands: $\forall ijk. x_{ij} = x_{i+k, j+k}$. A lower-left-Toeplitz matrix is one which is all zeroes except for the bottom-left corner; that is, the bands start in the bottom-left corner, and continue until hitting the band which includes the lower-right corner or the band which includes the upper-left corner (whichever comes first). In other words, an $n \times m$ lower-left-Toeplitz matrix is one which satisfies $x_{ij} = 0$ for all i, j with $i < j$ and all i, j with $i - j < m - n$. In yet other words, a matrix is lower-left-Toeplitz if it is Toeplitz, satisfies $x_{1j} = 0$ for $j \neq 1$, and satisfies $x_{in} = 0$ for $i \neq m$. Anyway, it is easy to see that the lower-left-Toeplitz matrices are exactly those matrices A such that left-shifting A by one has the same result as down-shifting by one.

So, we have found the set

$$C_1 := \{A \in \mathfrak{gl}_m : AX = XA\}.$$

It is just the set of $A = [A_{ij}]_{ij}$, where each A_{ij} is lower-left-Toeplitz.

Let v_{ij} be the leftmost column of A_{ij} , so that

$$A_{ij} = \begin{pmatrix} v_{ij} \vee 0 & \cdots & v_{ij} \vee [\lambda_j - 1] \end{pmatrix}.$$

Since A_{ij} is lower-left-Toeplitz, v_{ij} is of the form

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$ can be chosen freely. Now we determine which matrices of this form satisfy $aA = a$. For simplicity, we will instead find the matrices A such that $aA = 0$ (so that $a(A + I) = a$). (Note that $I \in C_1$, and C_1 is closed under addition, so this is really the same thing.)

In the case that $a = 0$, clearly every A works. Otherwise, let i_0, j_0 be such that $a_{i_0, j_0} = 1$. Now, clearly the constraint that $aA = 0$ is just saying

that the (i_0, j_0) th row of A must be zero. That is, for each j the j_0 th row of A_{i_0j} must be zero. This just requires that for each j , we must have

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$ can be chosen freely. So, we have now found the set

$$C_2 := \{A \in \mathfrak{gl}_m : AX = XA, aA = a\}.$$

It is the matrices of the form $I + A$, where A_{ij} is a block matrix of size $\lambda_i \times \lambda_j$ with

$$A_{ij} = T(v_{ij}) = T \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$ in the case $i \neq i_0$ and $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$ if $i = i_0$.

Now we are only left with finding the matrices $A \in C_2$ satisfying $Ab = b$. This just means finding the matrices A , of the form given above, which satisfy $Ab = 0$. We may restate this as the condition that for every i ,

$$(A_{i1} \ \cdots \ A_{in})b = 0.$$

Taking the transpose,

$$b^\top \begin{pmatrix} A_{i1}^\top \\ \vdots \\ A_{in}^\top \end{pmatrix} = 0.$$

Now, the rightmost column of A_{ij}^\top is v_{ij} upside-down (rows reversed). Write u_{ij} to refer to v_{ij} upside-down, and u'_{ij} to refer to v'_{ij} upside-down. Then the second-rightmost column is $u_{ij} \frown 1$, and so on until the first column is $u_{ij} \frown [\lambda_i - 1]$. So, we can reformulate this condition by requiring that for all l ,

$$b^\top \begin{pmatrix} u_{i1} \frown l \\ \vdots \\ u_{in} \frown l \end{pmatrix} = 0.$$

Writing

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix},$$

clearly this is equivalent to requiring that for all l ,

$$(b_1^\top \gg l \quad \cdots \quad b_k^\top \gg l) \begin{pmatrix} u_{i1} \\ \vdots \\ u_{ik} \end{pmatrix} = 0.$$

Then writing $b_j = \begin{pmatrix} b'_j \\ \vdots \end{pmatrix}$, where b'_j has the same dimension as u_{ij} , we can rewrite this constraint as

$$(b_1'^\top \gg l \quad \cdots \quad b_k'^\top \gg l) \begin{pmatrix} u'_{i1} \\ \vdots \\ u'_{ik} \end{pmatrix} = 0.$$

Finally, requiring the above equation for all l is the same thing as requiring

$$\begin{pmatrix} b_1'^\top \gg 0 & \cdots & b_k'^\top \gg 0 \\ \vdots & & \vdots \\ b_1'^\top \gg [\lambda_i - 1] & \cdots & b_k'^\top \gg [\lambda_i - 1] \end{pmatrix} \begin{pmatrix} u'_{i1} \\ \vdots \\ u'_{ik} \end{pmatrix} = 0.$$

This is the most that we will simplify the constraint. Recall that the u'_{ij} can be chosen arbitrarily, so the u_{ij} can be chosen arbitrarily as well. That is, the set $C_3 := \{A \in \mathfrak{gl}_m : AX = XA, aA = a, Ab = b\}$ is the set of $I + A$ for any A that can be recovered from u'_{ij} meeting the above condition.

Now we find the dimension of C_3 . Since the u'_{ij} can be chosen arbitrarily, apart from the condition above, we just need to add up their dimensions and subtract the rank of the matrix constructed of the b 's.

8 A necessary condition for $A_{X,a,b}$ to be nilpotent

Suppose $A := A_{X,a,b}$ is nilpotent. Define the *height* of a vector $v \in \mathbb{C}^{m+n}$ as the smallest $k \geq 0$ such that $A_{X,a,b}^k v = 0$. Clearly, for $i \leq j$, we have that the height of f_i is geq the height of f_j . Thus, for each $k \geq 0$, we must have $A_{X,a,b}^k f_n \in \langle e_{ij} \rangle_{ij}$. We have $Af_n = \sum_{ij} b_{ij} e_{ij}$. Then $A^2 f_n = \sum_{ij} b_{ij} (e_{i,j-1} +$

$a_{ij}f_1$), so we must have $\sum_{ij} b_{ij}a_{ij} = 0$. Similarly, since $A^3f_n \in \langle e_{ij} \rangle$, we see that $\sum_{ij} b_{ij}a_{i,j-1} = 0$. Continuing in this way, we obtain that for all $k \geq 0$,

$$\sum_{ij} b_{ij}a_{i,j-k} = 0. \quad (1)$$

In fact this is in fact a sufficient condition for $A_{X,a,b}$ to be nilpotent, as can be seen from the characteristic polynomial of $A_{X,a,b}$. However, we will not do this calculation here, instead just showing that it is sufficient by giving a Jordan basis.

9 A Jordan basis for $A_{X,a,b}$

For illustration, we will begin with the case that X is a single Jordan block. Then we will move to the general case.

9.1 When X is a Jordan block

Suppose X is a single Jordan block of size m . In this case we just write the basis of \mathbb{C}^m as e_1, \dots, e_m , and the above condition on a and b simply becomes

$$\forall k \geq 0. \sum_{i=k+1}^m b_i a_{i-k} = 0. \quad (2)$$

This condition can be simplified even more.

Lemma 9.1. *The condition (1) holds iff there exist nonnegative m_1, m_2, m_3 , a_1, \dots, a_{m_3} , and b_1, \dots, b_{m_3} satisfying the following conditions.*

- $m = m_1 + m_2 + m_3$
- $a = (0, 0, \dots, 0, a_1, \dots, a_{m_3})$
- $b = (b_1, \dots, b_{m_1}, 0, 0, \dots, 0)$
- If $m_1 \neq 0$, then $a_{m_1} \neq 0$
- If $m_3 \neq 0$, then $b_1 \neq 0$

Proof. If a and b are of this form, then for every $k \geq 0$ we have $b_i a_{i-k} = 0$, so clearly the condition holds.

Now suppose (1) holds. If a or b is zero, this is trivial. Otherwise, let $m_1 = \max\{i : a_i \neq 0\}$, and let $m_3 = \min\{i : b_i \neq 0\}$. We just need to show that $m_1 < m_3$. For contradiction, suppose $m_1 - m_3$ is nonnegative. Then by (1), $0 = \sum_i b_i a_{i-(m_1-m_3)} = b_1 a_{m_1}$, contradicting that b_1 and a_{m_1} are nonzero. \square

9.2 A ‘normalization’ fact about Jordan bases

Let V be a finite-dimensional vector space, $A : V \rightarrow V$ a nilpotent operator, and $f : V \rightarrow \mathbb{C}$ a linear map. Let $(e_{ij} : i \leq m, j \leq \lambda_i)$ be a Jordan basis for A .

Lemma 9.2. *There is a Jordan basis e_{ij} for A such that there is at most one pair (i, j) with $f(e_{ij}) \neq 0$.*

Proof. For any change of basis $P : V \rightarrow V$ commuting with A , we obtain a new Jordan basis $(P(e_{ij}) : i \leq m, j \leq \lambda_i)$. For any such P , define

$$S_P = \sum_i \begin{cases} -1, & \forall j. f \circ P(e_{ij}) = 0 \\ \lambda_i - \min\{j : f \circ P(e_{ij}) \neq 0\}, & \text{otherwise} \end{cases}.$$

Let P be any operator, among all invertible operators commuting with A , which minimizes S_P . Write $e'_{i,j} := P(e_{i,j})$.

Suppose for contradiction that there are two distinct i 's (and some j 's) with $f(e'_{ij}) \neq 0$. Then we can take e'_{i_1, j_1} and e'_{i_2, j_2} , where for $k \in \{1, 2\}$ we have $f(e'_{i_k, j_k}) \neq 0$, and $\forall j < j_k. f(e'_{i_k, j}) = 0$. Wlog, assume $\lambda_{i_1} - j_1 \leq \lambda_{i_2} - j_2$. Then, we can define $Q : V \rightarrow V$ by

- $Q(e'_{i_1, \lambda_1}) := e'_{i_1, \lambda_1} - \frac{f(e'_{i_1, j_1})}{f(e'_{i_2, j_2})} e'_{i_2, j_2 + (\lambda_{i_1} - j_1)}$
- For $j < \lambda_1$, $Q(e'_{i_1, j}) := A^{\lambda_{i_1} - j} Q(e'_{i_1, \lambda_{i_1}})$
- For $i \neq i_1$, $Q(e'_{ij}) = e'_{ij}$.

Clearly Q is invertible, and it commutes with A . Further, I claim that $S_{QP} < S_P$. It suffices to show that $\forall j \leq j_1. f \circ Q(e'_{i_1, j}) = 0$. We have

$$f \circ Q(e'_{i_1, j}) = f \left(A^{\lambda_{i_1} - j} \left(e'_{i_1, \lambda_1} - \frac{f(e'_{i_1, j_1})}{f(e'_{i_2, j_2})} e'_{i_2, j_2 + (\lambda_{i_1} - j_1)} \right) \right) =$$

$$f\left(e'_{i_1,j} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})}e'_{i_2,j_2+(j-j_1)}\right) = f(e'_{i_1,j}) - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})}e'_{i_2,j_2+(j-j_1)}.$$

Clearly (by design), this expression is zero when $j = j_1$. And for $j < j_1$, we have $f(e'_{i_1,j}) = f(e'_{i_2,j_2+(j-j_1)}) = 0$, so it is zero then as well. Thus we see that $S_{QP} < S_P$, contradicting that S_P is minimal. So there must be at most one i such that there exists j such that $f(e'_{ij}) \neq 0$.

If there is no such i , we have found the desired basis. So, suppose there is such an i_0 . Let $U = \langle e'_{i_0,j} \rangle$. Simply write $e_j := e'_{i_0,j}$. Let $j_0 = \min\{j : f(e_j) \neq 0\}$. We just have to change basis to zero out e_j for $j \neq j_0$. Let $P_1(e_{\lambda_{i_0}}) := e_{\lambda_{i_0}} - \frac{f(e_{j_0+1})}{f(e_{j_0})}e_{\lambda_{i_0}-1}$, and notice that $f \circ P_1(e_j) = 0$ for all $j \leq j_0 + 1$ except for j_0 . Then we define $P_2(e_{\lambda_{i_0}}) := P_1(e_{\lambda_{i_0}}) - \frac{f \circ P_1(e_{j_0+2})}{f \circ P_1(e_{j_0})}e_{\lambda_{i_0}-2}$, and notice that $f \circ P_2(e_j) = 0$ for all $j \leq j_0 + 2$ except for j_0 . Eventually we get $P_{\lambda_{i_0}-j_0}$, and by applying this to the e_j 's we obtain a basis of U , in which there is exactly one j with $f \circ P_{\lambda_{i_0}-j_0}(e_j) \neq 0$. \square

9.3 The general case

In the case where X was a single Jordan block, it was helpful to have the condition that all the nonzero a 's were to the right of all the nonzero b 's. This allowed some tricks with right-shifting. In the general case, the parameters a and b are not necessarily in such a helpful form. We will see, however, that we can change basis of \mathbb{C}^m to put them in a nice form.

Recall our nilpotent operator $A_{X,a,b}$ on $\mathbb{C}^m \times \mathbb{C}^n$. We can define a map $\mathbb{C}^m \rightarrow \mathbb{C}$ by $x \mapsto x \cdot a$. Then, by the previous lemma, we can choose a Jordan basis e_{ij} of \mathbb{C}^m so that at most one (i, j) has $f(e_{ij}) \neq 0$. This implies that there are some P, k, b' such that $PA_{X,a,b}P^{-1} = A_{X,1_k,b'}$. So, without loss of generality, we may consider this case, where $a = 1_k$.

Write our basis of \mathbb{C}^m as e_{ij} and our basis of \mathbb{C}^n as f_j . Let i_0, r be such that $a = e_{i_0, \lambda_{i_0}-r+1}$. Let $b' = A^{n+r}e_{i_0, \lambda_{i_0}} = b + e_{i_0, \lambda_{i_0}-r-n}$. Let b'_{ij} be the projection of b' onto e_{ij} . For each i , let $p_i = \max\{j : b'_{ij} \neq 0\}$. Then for each $i \neq i_0$, let $q_i = \lambda_i - p_i$. Let $q_{i_0} = \lambda_{i_0} - r - p_{i_0}$. Normally we index e_{ij} so that λ_i is nonincreasing with i . Now, it will be more convenient to assume our indices are such that q_i is nonincreasing, so we will do that.

We have two cases, which will closely resemble the cases we had when X was a single Jordan block.

9.3.1 First case: $n + \max_i p_i \geq q_{i_0} + p_{i_0}$

We list the chains in (roughly) decreasing order. First, for each i such that $q_i \geq n + r$, we take the chain of length $p_i + q_i$ beginning with $e_{i, p_i + q_i}$.

Now we handle $e_{i_0, \lambda_{i_0}}$. Let $P = \max_{i: q_i < n+r} p_i$. Set

$$v = e_{i_0, \lambda_{i_0}} - (A^{r+n+P} e_{i_0, \lambda_{i_0}} \gg r + n + P).$$

Note that $A^{r+n} e_{i_0, \lambda_{i_0}} = b'$, so $A^{r+n+P} e_{i_0, \lambda_{i_0}} = A^P b' = b' \ll P$. Also note that if we shift b' left P times, we zero out all the rows i where $q_i < n + r$. This ensures that the operation of shifting $b' \ll P$ right $r + n + P$ times is invertible by applying A , and thus shifting left, $r + n + P$ times. So, we take the chain of length $r + n + P$ beginning with v . How do we know this is linearly independent from what came before? Something to do with the condition on n ...

Now, we handle the i with $q_i < n + r$. If $p_i \leq \max_{k>i} p_k$, then we take the chain beginning with $e_{i, p_i + q_i}$ of length $p_i + q_i$. Otherwise, we set

$$v_i = A^{n+r-q_i} e_{i_0, \lambda_{i_0}} - \sum_{k=1}^{p_i} b'_{i,k} e_{i,k+q_i}.$$

Note that $A^{q_i} v_i$ is just b' with row i zeroed out. Then, $A^{q_i + \max_{k>i} p_k} v_i$ will have, in addition, rows k zeroed out, for all $k > i$. This ensures that shifting $A^{q_i + \max_{k>i} p_k} v_i$ right $q_i + \max_{k>i} p_k$ times will be inverted by applying A , and thus shifting left, $q_i + \max_{k>i} p_k$ times. So, we take the chain beginning with $v_i - (A^{q_i + \max_{k>i} p_k} v_i \gg q_i + \max_{k>i} p_i)$, which has length $q_i + \max_{k>i} p_k$.

TODO: where did I use the fact that $n + \max_i p_i \geq q_{i_0} + p_{i_0}$?

9.3.2 Second case: $n + \max_i p_i \leq q_{i_0} + p_{i_0}$

Again we list the chains in (roughly) decreasing order. Note that, since $n + \max_i p_i \leq q_{i_0} + p_{i_0}$, in particular we have $n \leq q_{i_0}$, and $n \leq q_i$ for all $i \leq i_0$. Now, for all $i \neq i_0$ with $q_i \geq n$, take the chain of length $p_i + q_i$ starting with $e_{i, p_i + q_i}$. Then, take the chain starting with $e_{i_0, \lambda_{i_0}}$, of length λ_{i_0} .

Let $i_1 = \min\{i : n > q_i\}$. Let $P = \max_{i \geq i_1} p_i$. Then take the chain beginning with $f_1 - (A^{n+P} f_1 \gg n + P)$, of length $n + P$.

Finally, we handle $i \geq i_1$. (In this case $q_i < n$.) If $p_i \leq \max_{k>i} p_k$, then take the chain beginning with $e_{i, p_i + q_i}$ of length $p_i + q_i$. Otherwise, set

$$v_i = A^{n-q_i} f_1 - \sum_{k=1}^{p_i} b'_{i,k} e_{i, q_i + k}.$$

Note that $A^{q_i}v_i$ is just b' with row i zeroed out. Now, we take the chain beginning with $v_i - (A^{q_i + \max_{k>i} p_k} v_i \gg q_i + \max_{k>i} p_k)$, of length $q_i + \max_{k>i} p_k$.

TODO: where did I use the fact that $n + \max_i p_i \leq q_{i_0} + p_{i_0}$.

9.4 When X is all zeroes

In this case we have $A(y, z) = (z_n b, (a \cdot y, z_1, \dots, z_{n-1}))$, and the condition becomes

$$\sum_i b_i a_i = 0.$$

In this case we will be able to write down explicitly the irreducible components of $F := \{(V, a, b) : \forall i. A_{X,a,b} V_i \subseteq V_i\} \cong \pi^{-1}(X)$. For any nonnegative $\delta_0, \delta_1, \dots, \delta_n, \delta_{n+1}$ summing to m , define the corresponding sequence $i_0 = \delta_0$, $i_n = \delta_{n+1} + i_n$, and for $j \in \{1, \dots, n\}$, $i_j = i_{j-1} + 1 + \delta_j$. Let E be the span of the e_i 's, and let $E' = \{(x, 0) \in E : x \cdot a = 0\}$, where the dot is the m -dimensional dot product. Then define F_δ as the set of $(V, a, b) \in F$ such that

- $b \in V_{i_0} \subseteq E'$
- for all $j \in \{1, \dots, n\}$, we have $f_j \in V_{i_j} \subseteq E' + \langle f_1, \dots, f_j \rangle$

I claim that the F_δ 's are the irreducible components of F . To begin, I show that their union is F .

Lemma 9.3. *Let $(V, a, b) \in \mathcal{F}$. Write $f_0 = b$, and $F = \langle f_0, f_1, \dots, f_n \rangle$. For each i , either $F \subseteq V_i$, or else there exists j such that $e_j \notin V_i$, but $\langle e_0, \dots, e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_1, \dots, e_j \rangle$.*

Proof. For $i = 0$, we may take $j = 0$. Now assume the statement holds for i , and we will prove it for $i + 1$. If $F \subseteq V_i$, then $F \subseteq V_{i+1}$, and we are done.

So, suppose there is j such that $e_j \notin V_i$, but $\langle e_0, \dots, e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_0, \dots, e_j \rangle$. We have two cases: either $V_{i+1} = V_i + \langle e_j \rangle$, or not.

- If so, then either $j = n$, in which case $F \subseteq V_{i+1}$, or else $j \neq n$, in which case $e_{j+1} \notin V_{i+1}$, but $\langle e_0, \dots, e_j \rangle \subseteq V_{i+1} \subseteq E' + \langle e_0, \dots, e_{j+1} \rangle$.
- If not, then $e_j \notin V_{i+1}$. Let v_{i+1} be so that $V_{i+1} = V_i + \langle v_{i+1} \rangle$. I just need to show that $v_{i+1} \in Y + \langle e_1, \dots, e_j \rangle$. It suffices to show that $v_{i+1}^\top e_k = 0$

for $k > j$. And to do this, it suffices to show that $(Av_{i+1})^\top e_{k-1} = 0$ for $k > j$.

Note that $Av_{i+1} \in V_i \cap A\mathbb{C}^{2n} \subseteq \langle e_1, \dots, e_j \rangle$. So, for $k > j + 1$ it is clear that $(Av_{i+1})^\top e_{k-1} = 0$. Now, suppose for contradiction that $(Av_{i+1})^\top e_j \neq 0$. Then Av_{i+1} is linearly independent of e_1, \dots, e_{j+1} . Since $Av_{i+1} \in \langle e_1, \dots, e_j \rangle$, it follows that $e_j \in \langle e_1, \dots, e_{j-1}, Av_{i+1} \rangle \subseteq V_i$, a contradiction.

□

Corollary 9.4. $\mathcal{F} = \bigcup_\delta \mathcal{F}_\delta$

Proof. Let $(V, a, b) \in F$. Let $i_0 = \min\{i : b \in V_i\}$, and let $i' = \max\{i : V_i \subseteq E'\}$. We want that $b \in V_{i_0} \subseteq E'$, so we want that $i_0 \leq i'$. For contradiction, suppose $i' < i_0$.

□

Lemma 9.5. Each \mathcal{F}_δ is a closed subvariety of \mathcal{F} .

Proof.

□

Lemma 9.6. Each \mathcal{F}_δ is irreducible of dimension m .

Proof. Let \mathcal{E} be the variety of partial flags of E of shape $(\delta_0, \dots, \delta_{n+1})$. Define $g : \mathcal{F}_\delta \rightarrow \mathcal{E}$ by

$$(V, a, b) \mapsto 0 \subseteq V_{i_0} \cap E \subseteq V_{i_1} \cap E \subseteq \dots \subseteq V_{i_{n+1}} \cap E = E.$$

It is clear that g is surjective, as the definition of \mathcal{F}_δ places no restriction on the intersections $V_i \cap E$. Now let's look more closely at the fibers $g^{-1}(U)$.

First let's find the flags V such that there exist a, b with $g(V, a, b) = U$. We see that, for instance, $V_{i_0} \cap E = U_1$. In fact $V_{i_0} \subseteq E$, so $V_{i_0} = U_1$. But we are free to choose the vector spaces between 0 and V_{i_0} however we wish, so we get some degrees of freedom like \mathcal{F}_{δ_0} , the complete flag variety on $\mathbb{C}^{\delta_0} \cong V_{i_0}/0$. Similarly, for every $j = 1, \dots, n + 1$, we can choose the vector spaces between $V_{i_{j-1}}$ and V_{i_j} arbitrarily, so we get degrees of freedom like $\mathcal{F}_{\delta_{i_j}}$, the complete flag variety on $\mathbb{C}^{\delta_{i_j}} \cong V_{i_j}/V_{i_{j-1}}$. Finally, to meet the constraint of (V, a, b) being in \mathcal{F}_δ , we can choose any $b \in U_1$ and any a such that $\bar{a} \in U_n^\perp$. Thus we get an isomorphism

$$g^{-1}(U) \cong \mathcal{F}_{\delta_0} \times \dots \times \mathcal{F}_{\delta_{n+1}} \times \mathbb{C}^{\delta_0} \times \mathbb{C}^{\delta_{n+1}}.$$

□

Theorem 9.7. The \mathcal{F}_δ 's are the irreducible components of \mathcal{F} .

9.5 In the case that X is a Jordan block

This case seems harder to work with explicitly than the case that X is zero, so our strategy is to reuse

10 A different variety

Define $R = \{(X, \mathfrak{b}_1, \mathfrak{b}_2) : X \in \mathfrak{b}_1 \wedge u(X) \in \mathfrak{b}_2\} \subseteq \mathcal{S}'_{m,n} \times \mathcal{B}_{m+n} \times \mathcal{B}_m$. We can obtain a subvariety of R by requiring that $u(X)$ is in some fixed similarity class. (TODO: why is this a subvariety? Is it? Is this even the right way of explaining the significance of the m^2 ?) We expect that each of these subvarieties is an irreducible component of dimension m^2 . We will verify these things using our previous computations of springer fibers.

10.1 TODO

- Why are SOn flags what they are.

References

- [1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.