1 Introduction

We will review the definition of a Springer fiber and define, for a nilpotent $Y \in \mathfrak{gl}_m$, the Springer fiber at the n-Slodowy slice at Y. For every n and every nilpotent $Y \in \mathfrak{gl}_m$, we will find the irreducible components of the Springer fiber at the n-Slodowy slice at Y. Finally, we will use our results about Springer fibers at n-Slodowy slices to find the irreducible components of some other variety (which probably needs a name), and show that they all have the same dimension.

2 Springer fibers

Let $G \subseteq \mathrm{GL}_m(\mathbb{C})$ be a connected semisimple Lie group, and let $\mathfrak{g} \subseteq \mathfrak{gl}_m(\mathbb{C})$ be its Lie algebra. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the subset consisting of nilpotent elements. Let $\mathcal{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. Let $\widetilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$. Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the projection onto the second coordinate. For $n \in \mathcal{N}$, we call $\pi^{-1}(n)$ the *Springer fiber at n*.

We mention some results about Springer fibers, which we will use later in this paper. TODO: mention them.

3 Slodowy slice

A basis for $\mathfrak{sl}_2(\mathbb{C})$ is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra $\mathfrak{g}$, and a homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ sending (e', h', f') to (e, h, f), we say that (e, h, f) is an $\mathfrak{sl}_2$*-triple*. Observe that e, f must be nilpotent, and h must be Cartan (???). If $\mathfrak{g}$ is semisimple, then given any nilpotent $e \in \mathfrak{g}$, the Jacobson-Morozov theorem [1, 3.7.1] says that there exist $h, f \in \mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple.

Given (e, h, f), we define the *Slodowy slice at e* as $\mathcal{S}_e := e + \ker \mathrm{ad}_f$. By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent $e \in \mathfrak{g}$, when $\mathfrak{g}$ is semisimple.

3.1 Finding an $\mathfrak{sl}_2$-triple in a simple case

Let

$$e = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \in \mathfrak{sl}_n,$$

where there are $n-1$ ones. We will find the Slodowy slice at e.

As a first step, we will find an $\mathfrak{sl}_2$-triple with e as its first element. Note that $[h', e'] = 2e'$, and $[e', f'] = h'$, and $[h', f'] = -2f'$. Thus e, h, f must obey the same relations. In particular, $he - eh = 2e$. The matrix eh is h shifted down one, and he is h shifted left one. Thus, $2e_{ij} = (he - eh)_{ij} = h_{i,j+1} - h_{i-1,j}$. We can use this to show that $h_{ij} = 0$ when $i \neq j$. Then we can use it to show that $h_{ii} = h_{i-1,i-1} + 2$, so that $h_{ii} = h_{11} + 2(i-1)$.

Similarly, from $[e, f] = h$ we get that $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$. We can use this to show that $f_{ij} = 0$ when $j \neq i+1$. Then we can use it to show that $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$, that $f_{1,2} = -h_{1,1}$, and that $f_{n-1,n} = h_{n,n}$. From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^{n} h_{ii} \implies$$

$$\sum_i h_{ii} = 0.$$

Oops, I didn't actually need to prove that, since $h \in \mathfrak{sl}_n$... it's nice to know that it's the unique solution in $\mathfrak{gl}_n$ though? This shows that $h_{11} = n-1, h_{22} = n-3, ..., h_{nn} = 1-n$. So we have determined h; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for f in terms of h to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & & \\ & 0 & (1-n)+(3-n) & & & \\ & & \ddots & \ddots & & \\ & & & 0 & (1-n)+\cdots+(n-1) \\ & & & & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1(1-n) & & & & \\ & 0 & 2(2-n) & & & \\ & & 0 & (n-2)(-2) & & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

3.2 Finding an $\mathfrak{sl}_2$-triple in a slightly less simple case

Now, let

$$E = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{sl}_{m+n},$$

with e as in the previous subsection. We will find H, F so that (E, H, F) is an $\mathfrak{sl}_2$-triple. Writing $F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & f \end{pmatrix}$, and similarly for H, we have

$$2E = [H, E] = \begin{pmatrix} 0 & H_{12}e \\ 0 & he \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ eH_{21} & eh \end{pmatrix},$$

$$H = [E, F] = \begin{pmatrix} 0 & 0 \\ eF_{21} & ef \end{pmatrix} - \begin{pmatrix} 0 & F_{12}e \\ 0 & fe \end{pmatrix}.$$

From these we observe that (e, h, f) must also be an $\mathfrak{sl}_2$-triple, so h, f must be as in the previous subsection. We also see that $H_{11} = 0$. Recalling that left multiplication by e is a down-shift, and right multiplication is a left-shift, we see that H_{12} is all zeroes except for the leftmost column, and H_{21} is all zeroes except for the bottom row. Then our final constraint is that

$$-2F = [H, F] =$$

$$\begin{pmatrix} H_{12}F_{21} & H_{12}F_{22} \\ H_{21}F_{11} + H_{22}F_{21} & H_{21}F_{12} + H_{22}F_{22} \end{pmatrix} - \begin{pmatrix} F_{12}H_{21} & F_{11}H_{12} + F_{12}H_{22} \\ F_{22}H_{21} & F_{21}H_{12} + F_{22}H_{22} \end{pmatrix}.$$

Now $H_{12} = F_{12}e$, and $H_{21} = eF_{21}$, from the equation $H = [E, F]$. Substituting in the equation above then,

$$-2F = \begin{pmatrix} F_{12}eF_{21} & F_{12}ef \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} F_{12}eF_{21} & F_{11}F_{12}e + F_{12}h \\ feF_{21} & F_{21}F_{12}e + fh \end{pmatrix} =$$

$$\begin{pmatrix} 0 & F_{12}(fe + h) \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e + F_{12}h \\ (ef - h)F_{21} & F_{21}F_{12}e + fh \end{pmatrix} =$$

$$\begin{pmatrix} 0 & F_{12}fe \\ eF_{21}F_{11} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e \\ efF_{21} & F_{21}F_{12}e + fh \end{pmatrix}.$$

Now we see that $F_{11} = 0$, and consequently that $F_{12} = F_{21} = 0$ as well. This shows that $H_{12} = H_{21} = 0$. We conclude that H and F just have h and f in their bottom-right corners, respectively.

3.3 Finding the previous Slodowy slices

First we find $\ker \mathrm{ad}_f$. We have $(fX)_{ij} = i(n-i)A_{i+1,j}$, and $(Xf)_{ij} = (j-1)(n-(j-1))A_{i,j-1}$. So, for all $i, j \in \{1, ..., n\}$, we have

$$i(n-i)A_{i+1,j} = (j-1)(n-(j-1))A_{i,j-1}.$$

Taking $j = 1$, we find that $A_{i,1} = 0$ for $i \geq 2$. Then, taking $j > 1$, we find that for $i, j \in \{1, ..., n-1\}$,

$$A_{i+1,j+1} = \frac{(j-1)(n-(j-1))}{i(n-i)} A_{ij}.$$

So, $\ker \mathrm{ad}_f$ is the set of matrices which are upper triangular and satisfy the above condition.

To find the Slodowy slice associated to the previous $\mathfrak{sl}_2$-triple (E, H, F), we just need to find $\ker \mathrm{ad}_F$. We have

$$[F, X] = \begin{pmatrix} 0 & 0 \\ fX_{21} & fX_{22} \end{pmatrix} - \begin{pmatrix} 0 & X_{12}f \\ 0 & X_{22}f \end{pmatrix}.$$

Thus, X_{21} must be all zeroes except for the first row, and X_{12} must be all zeroes except for the last column, and $X_{22} \in \ker \mathrm{ad}_f$. There is no restriction on X_{11}. This describes $\ker \mathrm{ad}_F$.

For $X \in \mathcal{S}_{m,n}$, define $u(X) := X_{11}$.

3.4 Finding $\widetilde{\mathcal{N}}_{m,n}$

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m$ be the nilpotent elements. Let $\mathcal{S}'_{m,n}$ be the set of $X \in \mathcal{S}_{m,n}$ such that both X and $u(X)$ are nilpotent. Let $\widetilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n}$. Define $\pi_{m,n} : \widetilde{\mathcal{N}}_{m,n} \to \mathcal{N}_m$ by $(\mathfrak{b}, X) \mapsto X_{11}$. For $Y \in \mathfrak{gl}_m$, we call $\pi_{m,n}^{-1}(Y)$ the *n-dim-Slodowy-slice Springer fiber* at Y.

Lemma 3.1. *Let J be a jordan block with zeroes along the diagonal, and let A be upper triangular and nonzero. Then $J + A$ is not nilpotent.*

Proof. It is straightforward to show by induction that if $v_i = 0$ for $i < j$, and $v_j \neq 0$, then $((J + A)^k v_j)_{j+k} = v_j$. Let i be such that $Ae_i \neq 0$. Then $(J + A)^{i-1} e_1$ has nonzero e_i-component. Then $(J + A)^i e_1$ has some nonzero $e_{i'}$-component for some $i' \leq i$. Then $(J + A)^{i+(n-i')} e_1$ has some nonzero e_n-component. And $i + (n - i') \geq n$, so we're done. □

Lemma 3.2. *Let $X \in \mathfrak{gl}_m$, and let*

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ d_1 & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ & d_2 & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ & & \ddots & \cdots & \vdots & \vdots \\ & & & d_{n-2} & y_{n-1,n-1} & y_{n-1,n} \\ & & & & d_{n-1} & y_{nn} \end{pmatrix} \in \mathfrak{gl}_n .$$

For any $a, b \in \mathbb{C}^m$,

$$\det \left(\begin{array}{c|c} X & \begin{matrix} | \\ b \\ | \end{matrix} \\ \hline \begin{matrix} — & a & — \end{matrix} & \\ & Y \end{array} \right) =$$

$$\det X \det Y + \left(\prod_i d_i \right) \det \left(\begin{array}{c|c} X & \begin{matrix} | \\ b \\ | \end{matrix} \\ \hline \begin{matrix} — & a & — \end{matrix} & 0 \end{array} \right)$$

Proof. By induction on n. In the case $n = 1$, expanding along the last row (taking the usual interpretation of the empty product) gives the desired result.

Now suppose $n > 1$. Expanding along the last row, we get

$$d_{n-1} \det \left(\begin{array}{c|c} X & \begin{matrix} & \vert \\ & b \\ & \vert \end{matrix} \\ \hline \text{—}\ \ a\ \ \text{—} & \\ & Y_{n,n-1} \end{array}\right) - y_{nn} \det \left(\begin{array}{c|c} X & \\ \hline \text{—}\ \ a\ \ \text{—} & \\ & Y_{n,n} \end{array}\right).$$

Using our inductive hypothesis for the first determiniant, and using that $\det \left(\begin{array}{c|c} A_{11} & 0 \\ A_{21} & A_{22} \end{array}\right) = \det A_{11} \det A_{22}$ for the second, the expression becomes

$$d_{n-1} \left(\det X \det Y_{n,n-1} + \left(\prod_{i \leq n-2} d_i \right) \det \left(\begin{array}{c|c} X & \begin{matrix} \vert \\ b \\ \vert \end{matrix} \\ \hline \text{—}\ \ a\ \ \text{—} & 0 \end{array}\right) \right) - y_{nn} \det X \det Y_{nn} =$$

$$(d_{n-1} Y_{n,n-1} - y_{nn} \det Y_{nn}) \det X + \left(\prod_i d_i \right) \det \left(\begin{array}{c|c} X & \begin{matrix} \vert \\ b \\ \vert \end{matrix} \\ \hline \text{—}\ \ a\ \ \text{—} & 0 \end{array}\right) =$$

$$\det Y \det X + \left(\prod_i d_i \right) \det \left(\begin{array}{c|c} X & \begin{matrix} \vert \\ b \\ \vert \end{matrix} \\ \hline \text{—}\ \ a\ \ \text{—} & 0 \end{array}\right).$$

$\square$

Corollary 3.3. *If X is nilpotent, and*

$$\left(\begin{array}{c|c} X & \begin{matrix} & \vert \\ & b \\ & \vert \end{matrix} \\ \hline \text{—}\ \ a\ \ \text{—} & \\ & Y \end{array}\right)$$

is nilpotent as well, then Y is nilpotent (TODO: and that other determinant is zero).

Proof. By the previous lemma, the characteristic polynomial of the big matrix is

$$g_X(\lambda)g_Y(\lambda) + f(\lambda),$$

where $g_X(\lambda) = \lambda^m$ is the characteristic polynomial of X, and $f(\lambda)$ is some polynomial of degree at most $m-1$. $\square$

Now, taking the previous corolloary and the first lemma together, we see that

$$\mathcal{S}'_{m,n} = \left\{ \left(\begin{array}{ccc|cccc} & & & & & & | \\ & X & & & & & b \\ & & & & & & | \\ \hline — & a & — & 0 & & & \\ & & & 1 & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & 0 \end{array}\right) : a, b \in \mathbb{C}^m, X \in \mathfrak{gl}_m \text{ is nilpotent} \right\}.$$

4 Definition of springer fiber at slodowy slice

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m(\mathbb{C})$ be the subset consisting of nilpotent elements. For $X \in \mathcal{N}$ and $a, b \in \mathbb{C}^m$, let

$$A_{X,a,b} = \left(\begin{array}{ccc|ccccc} & & & & & & & | \\ & X & & & & & & b \\ & & & & & & & | \\ \hline — & a & — & 0 & & & & \\ & & & 1 & 0 & & & \\ & & & & 1 & \ddots & & \\ & & & & & \ddots & 0 & \\ & & & & & & 1 & 0 \end{array}\right) \in \mathfrak{gl}_{m+n}(\mathbb{C}).$$

Let $\mathcal{M}$ be the variety of Borel subalgebras of $\mathfrak{gl}_{m+n}(\mathbb{C})$. Let

$$\widetilde{\mathcal{N}} = \{(\mathfrak{m}, a, b, X) : A_{X,a,b} \in \mathfrak{m}\} \subseteq \mathcal{M} \times \mathbb{C}^m \times \mathbb{C}^m \times \mathcal{N}.$$

Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the projection onto the last coordinate. For $X \in \mathcal{N}$, we call $\pi^{-1}(X)$ the *other Springer fiber at X*.

5 A necessary condition for $A_{X,a,b}$ to be nilpotent

Suppose $A := A_{X,a,b}$ is nilpotent. Define the *height* of a vector $v \in \mathbb{C}^{m+n}$ as the smallest $k \geq 0$ such that $A^k_{X,a,b}v = 0$. Clearly, for $i \leq j$, we have that the height of f_i is geq the height of f_j. Thus, for each $k \geq 0$, we must have $A^k_{X,a,b}f_n \in \langle e_{ij} \rangle_{ij}$. We have $Af_n = \sum_{ij} b_{ij}e_{ij}$. Then $A^2 f_n = \sum_{ij} b_{ij}(e_{i,j-1} + a_{ij}f_1)$, so we must have $\sum_{ij} b_{ij}a_{ij} = 0$. Similarly, since $A^3 f_n \in \langle e_{ij} \rangle$, we see that $\sum_{ij} b_{ij}a_{i,j-1} = 0$. Continuing in this way, we obtain that for all $k \geq 0$,

$$\sum_{ij} b_{ij}a_{i,j-k} = 0. \tag{1}$$

In fact this is in fact a sufficient condition for $A_{X,a,b}$ to be nilpotent, as can be seen from the characteristic polynomial of $A_{X,a,b}$. However, we will not do this calculation here, instead just showing that it is sufficient by giving a Jordan basis.

6 A Jordan basis for $A_{X,a,b}$

For illustration, we will begin with the case that X is a single Jordan block. Then we will move to the general case.

6.1 When X is a Jordan block

Suppose X is a single Jordan block of size m. In this case we just write the basis of $\mathbb{C}^m$ as $e_1, ..., e_m$, and the above condition on a and b simply becomes

$$\forall k \geq 0. \sum_{i=k+1}^{m} b_i a_{i-k} = 0. \tag{2}$$

This condition can be simplified even more.

Lemma 6.1. *The condition (1) holds iff there exist nonnegative m_1, m_2, m_3, $a_1, ..., a_{m_3}$, and $b_1, ..., b_{m_3}$ satisfying the following conditions.*

- $m = m_1 + m_2 + m_3$
- $a = (0, 0, ..., 0, a_1, ..., a_{m_3})$

- $b = (b_1, ..., b_{m_1}, 0, 0, ..., 0)$
- *If* $m_1 \neq 0$, *then* $a_{m_1} \neq 0$
- *If* $m_3 \neq 0$, *then* $b_1 \neq 0$

Proof. If a and b are of this form, then for every $k \geq 0$ we have $b_i a_{i-k} = 0$, so clearly the condition holds.

Now suppose (1) holds. If a or b is zero, this is trivial. Otherwise, let $m_1 = \max\{i : a_i \neq 0\}$, and let $m_3 = \min\{i : b_i \neq 0\}$. We just need to show that $m_1 < m_3$. For contradiction, suppose $m_1 - m_3$ is nonnegative. Then by (1), $0 = \sum_i b_i a_{i-(m_1-m_3)} = b_1 a_{m_1}$, contradicting that b_1 and a_{m_1} are nonzero. □

6.2 A 'normalization' fact about Jordan bases

Let V be a finite-dimensional vector space, $A : V \to V$ a nilpotent operator, and $f : V \to \mathbb{C}$ a linear map. Let $(e_{ij} : i \leq m, j \leq \lambda_i)$ be a Jordan basis for A.

Lemma 6.2. *There is a Jordan basis* e_{ij} *for* A *such that there is at most one pair* (i, j) *with* $f(e_{ij}) \neq 0$.

Proof. For any change of basis $P : V \to V$ commuting with A, we obtain a new Jordan basis $(P(e_{ij}) : i \leq m, j \leq \lambda_i)$. For any such P, define

$$S_P = \sum_i \begin{cases} -1, & \forall j.\ f \circ P(e_{ij}) = 0 \\ \lambda_i - \min\{j : f \circ P(e_{ij}) \neq 0\}, & \text{otherwise} \end{cases}.$$

Let P be any operator, among all invertible operators commuting with A, which minimizes S_P. Write $e'_{i,j} := P(e_{i,j})$.

Suppose for contradiction that there are two distinct i's (and some j's) with $f(e'_{ij}) \neq 0$. Then we can take e'_{i_1,j_1} and e'_{i_2,j_2}, where for $k \in \{1, 2\}$ we have $f(e'_{i_k,j_k}) \neq 0$, and $\forall j < j_k.\ f(e'_{i_k,j}) = 0$. Wlog, assume $\lambda_{i_1} - j_1 \leq \lambda_{i_2} - j_2$. Then, we can define $Q : V \to V$ by

- $Q(e'_{i_1,\lambda_1}) := e'_{i_1,\lambda_1} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(\lambda_{i_1}-j_1)}$
- For $j < \lambda_1$, $Q(e'_{i_1,j}) := A^{\lambda_{i_1}-j} Q(e'_{i_1,\lambda_{i_1}})$

- For $i \neq i_1$, $Q(e'_{ij}) = e'_{ij}$.

Clearly Q is invertible, and it commutes with A. Further, I claim that $S_{QP} < S_P$. It suffices to show that $\forall j \leq j_1.\ f \circ Q(e'_{i_1,j}) = 0$. We have

$$f \circ Q(e'_{i_1,j}) = f\left(A^{\lambda_{i_1}-j}\left(e'_{i_1,\lambda_1} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})}e'_{i_2,j_2+(\lambda_{i_1}-j_1)}\right)\right) =$$

$$f\left(e'_{i_1,j} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})}e'_{i_2,j_2+(j-j_1)}\right) = f(e'_{i_1,j}) - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})}e'_{i_2,j_2+(j-j_1)}.$$

Clearly (by design), this expression is zero when $j = j_1$. And for $j < j_1$, we have $f(e'_{i_1,j}) = f(e'_{i_2,j_2+(j_2-j_1)}) = 0$, so it is zero then as well. Thus we see that $S_{QP} < S_P$, contradicting that S_P is minimal. So there must be at most one i such that there exists j such that $f(e'_{ij}) \neq 0$.

If there is no such i, we have found the desired basis. So, suppose there is such an i_0. Let $U = \langle e'_{i_0,j}\rangle$. Simply write $e_j := e'_{i_0,j}$. Let $j_0 = \min\{j : f(e_j) \neq 0\}$. We just have to change basis to zero out e_j for $j \neq j_0$. Let $P_1(e_{\lambda_{i_0}}) := e_{\lambda_{i_0}} - \frac{f(e_{j_0+1})}{f(e_{j_0})}e_{\lambda_{i_0}-1}$, and notice that $f \circ P_1(e_j) = 0$ for all $j \leq j_0+1$ except for j_0. Then we define $P_2(e_{\lambda_{i_0}}) := P_1(e_{\lambda_{i_0}}) - \frac{f \circ P_1(e_{j_0+2})}{f \circ P_1(e_{j_0})}e_{\lambda_{i_0}-2}$, and notice that $f \circ P_2(e_j) = 0$ for all $j \leq j_0 + 2$ except for j_0. Eventually we get $P_{\lambda_{i_0}-j_0}$, and by applying this to the e_j's we obtain a basis of U, in which there is exactly one j with $f \circ P_{\lambda_{i_0}-j_0}(e_j) \neq 0$. $\square$

6.3 The general case

In the case where X was a single Jordan block, it was helpful to have the condition that all the nonzero a's were to the right of all the nonzero b's. This allowed some tricks with right-shifting. In the general case, the parameters a and b are not necessarily in such a helpful form. We will see, however, that we can change basis of $\mathbb{C}^m$ to put them in a nice form.

Recall our nilpotent operator $A_{X,a,b}$ on $\mathbb{C}^m \times \mathbb{C}^n$. We can define a map $\mathbb{C}^m \to \mathbb{C}$ by $x \mapsto x \cdot a$. Then, by the previous lemma, we can choose a Jordan basis e_{ij} of $\mathbb{C}^m$ so that at most one (i,j) has $f(e_{ij}) \neq 0$. This implies that there are some P, k, b' such that $PA_{X,a,b}P^{-1} = A_{X,1_k,b'}$. So, without loss of generality, we may consider this case, where $a = 1_k$.

Write our basis of $\mathbb{C}^m$ as e_{ij} and our basis of $\mathbb{C}^n$ as f_j. Let i_0, r be such that $a = e_{i_0,\lambda_{i_0}-r+1}$. Let $b' = A^{n+r}e_{i_0,\lambda_{i_0}} = b + e_{i_0,\lambda_{i_0}-r-n}$. Let b'_{ij} be the projection of b' onto e_{ij}. For each i, let $p_i = \max\{j : b'_{ij} \neq 0\}$. Then for each $i \neq i_0$, let $q_i = \lambda_i - p_i$. Let $q_{i_0} = \lambda_{i_0} - r - p_{i_0}$. Normally we index e_{ij} so that λ_i is nonincreasing with i. Now, it will be more convenient to assume our indices are such that q_i is nonincreasing, so we will do that.

We have two cases, which will closely resemble the cases we had when X was a single Jordan block.

6.3.1 First case: $n + \max_i p_i \geq q_{i_0} + p_{i_0}$

We list the chains in (roughly) decreasing order. First, for each i such that $q_i \geq n + r$, we take the chain of length $p_i + q_i$ beginning with e_{i,p_i+q_i}.

Now we handle $e_{i_0,\lambda_{i_0}}$. Let $P = \max_{i:q_i<n+r} p_i$. Set

$$v = e_{i_0,\lambda_{i_0}} - (A^{r+n+P} e_{i_0,\lambda_{i_0}} >> r + n + P).$$

Note that $A^{r+n} e_{i_0,\lambda_{i_0}} = b'$, so $A^{r+n+P} e_{i_0,\lambda_{i_0}} = A^P b' = b' << P$. Also note that if we shift b' left P times, we zero out all the rows i where $q_i < n + r$. This ensures that the operation of shifting $b' << P$ right $r + n + P$ times is invertible by applying A, and thus shifting left, $r + n + P$ times. So, we take the chain of length $r + n + P$ beginning with v. How do we know this is linearly independent from what came before? Something to do with the condition on n...

Now, we handle the i with $q_i < n + r$. If $p_i \leq \max_{k>i} p_k$, then we take the chain beginning with e_{i,p_i+q_i} of length $p_i + q_i$. Otherwise, we set

$$v_i = A^{n+r-q_i} e_{i_0,\lambda_{i_0}} - \sum_{k=1}^{p_i} b'_{i,k} e_{i,k+q_i}.$$

Note that $A^{q_i} v_i$ is just b' with row i zeroed out. Then, $A^{q_i + \max_{k>i} p_k} v_i$ will have, in addition, rows k zeroed out, for all $k > i$. This ensures that shifting $A^{q_i + \max_{k>i} p_k} v_i$ right $q_i + \max_{k>i} p_k$ times will be inverted by applying A, and thus shifting left, $q_i + \max_{k>i} p_k$ times. So, we take the chain beginning with $v_i - (A^{q_i + \max_{k>i} p_k} v_i >> q_i + \max_{k>i} p_i)$, which has length $q_i + \max_{k>i} p_k$.

TODO: where did I use the fact that $n + \max_i p_i \geq q_{i_0} + p_{i_0}$?

6.3.2 Second case: $n + \max_i p_i \leq q_{i_0} + p_{i_0}$

Again we list the chains in (roughly) decreasing order. Note that, since $n + \max_i p_i \leq q_{i_0} + p_{i_0}$, in particular we have $n \leq q_{i_0}$, and $n \leq q_i$ for all

$i \leq i_0$. Now, for all $i \neq i_0$ with $q_i \geq n$, take the chain of length $p_i + q_i$ starting with e_{i,p_i+q_i}. Then, take the chain starting with $e_{i_0,\lambda_{i_0}}$, of length λ_{i_0}.

Let $i_1 = \min\{i : n > q_i\}$. Let $P = \max_{i \geq i_1} p_i$. Then take the chain beginning with $f_1 - (A^{n+p} f_1 >> n + p)$, of length $n + P$.

Finally, we handle $i \geq i_1$. (In this case $q_i < n$.) If $p_i \leq \max_{k>i} p_k$, then take the chain beginning with e_{i,p_i+q_i} of length $p_i + q_i$. Otherwise, set

$$v_i = A^{n-q_i} f_1 - \sum_{k=1}^{p_i} b'_{i,k} e_{i,q_i+k}.$$

Note that $A^{q_i} v_i$ is just b' with row i zeroed out. Now, we take the chain beginning with $v_i - (A^{q_i + \max_{k>i} p_k} v_i >> q_i + \max_{k>i} p_k)$, of length $q_i + \max_{k>i} p_k$.

TODO: where did I use the fact that $n + \max_i p_i \leq q_{i_0} + p_{i_0}$.

7 Finding the Springer fibers of section 2

We have $\mathcal{M} = \{AHA^{-1} : A \in \mathrm{GL}_{m+n}(\mathbb{C})\}$, where $H \subseteq \mathfrak{gl}_{m+n}(\mathbb{C})$ is the set of upper triangular matrices. We say a map $X : \mathfrak{gl}_{m+n} \to \mathfrak{gl}_{m+n}$ preserves a flag $V_0 \subseteq \cdots \subseteq V_{m+n}$ if $XV_i \subseteq V_i$ for each i. Let $E_0 \subseteq \cdots \subseteq E_{m+n}$ be the standard flag of $\mathbb{C}^{m+n}$. Since H is the set of X which preserve E,

$$\mathcal{M} = \{\{X : \forall i.\ X(AE_i) \subseteq AE_i\} : A \in \mathrm{GL}_n(\mathbb{C})\}.$$

So, for $X \in \mathcal{N}$,

$$\pi^{-1}(X) \cong \{(V, a, b) : \forall i.\ A_{X,a,b} V_i \subseteq V_i\}.$$

In this section we will find the irreducible components of $\pi^{-1}(X)$. Since $\pi^{-1}(X) \cong \pi^{-1}(AXA^{-1})$ for any invertible A, we will assume that X is in Jordan normal form.

Let λ be the shape of X, and let $(e_{ij})_{i \leq r, j \leq \lambda_i}$ be a Jordan basis for X, with $Xe_{ij} = e_{i,j-1}$. Let $f_1, ..., f_n$ be the standard basis for $\mathbb{C}^n$, with $A_{X,a,b} f_i = f_{i+1}$.

7.1 When X is all zeroes

In this case we have $A(y, z) = (z_n b, (a \cdot y, z_1, ..., z_{n-1}))$, and the condition becomes

$$\sum_i b_i a_i = 0.$$

In this case we will be able to write down explicitly the irreducible components of $F := \{(V, a, b) : \forall i.\ A_{X,a,b} V_i \subseteq V_i\} \cong \pi^{-1}(X)$. For any nonnegative $\delta_0, \delta_1, ..., \delta_n, \delta_{n+1}$ summing to m, define the corresponding sequence $i_0 = \delta_0$, $i_n = \delta_{n+1} + i_n$, and for $j \in \{1, ..., n\}$, $i_j = i_{j-1} + 1 + \delta_j$. Let E be the span of the e_i's, and let $E' = \{(x, 0) \in E : x \cdot a = 0\}$, where the dot is the m-dimensional dot product. Then define F_δ as the set of $(V, a, b) \in F$ such that

- $b \in V_{i_0} \subseteq E'$
- for all $j \in \{1, ..., n\}$, we have $f_j \in V_{i_j} \subseteq E' + \langle f_1, ..., f_j \rangle$

I claim that the F_δ's are the irreducible components of F. To begin, I show that their union is F.

Lemma 7.1. *Let $(V, a, b) \in \mathcal{F}$. Write $f_0 = b$, and $F = \langle f_0, f_1, ..., f_n \rangle$. For each i, either $F \subseteq V_i$, or else there exists j such that $e_j \notin V_i$, but $\langle e_0, ..., e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_1, ..., e_j \rangle$.*

Proof. For $i = 0$, we may take $j = 0$. Now assume the statement holds for i, and we will prove it for $i + 1$. If $F \subseteq V_i$, then $F \subseteq V_{i+1}$, and we are done.

So, suppose there is j such that $e_j \notin V_i$, but $\langle e_0, ..., e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_0, ..., e_j \rangle$. We have two cases: either $V_{i+1} = V_i + \langle e_j \rangle$, or not.

- If so, then either $j = n$, in which case $F \subseteq V_{i+1}$, or else $j \neq n$, in which case $e_{j+1} \notin V_{i+1}$, but $\langle e_0, ..., e_j \rangle \subseteq V_{i+1} \subseteq E' + \langle e_0, ..., e_{j+1} \rangle$.
- If not, then $e_j \notin V_{i+1}$. Let v_{i+1} be so that $V_{i+1} = V_i + \langle v_{i+1} \rangle$. I just need to show that $v_{i+1} \in Y + \langle e_1, ..., e_j \rangle$. It suffices to show that $v_{i+1}^\top e_k = 0$ for $k > j$. And to do this, it suffices to show that $(Av_{i+1})^\top e_{k-1} = 0$ for $k > j$.

  Note that $Av_{i+1} \in V_i \cap A\mathbb{C}^{2n} \subseteq \langle e_1, ..., e_j \rangle$. So, for $k > j + 1$ it is clear that $(Av_{i+1})^\top e_{k-1} = 0$. Now, suppose for contradiction that $(Av_{i+1})^\top e_j \neq 0$. Then Av_{i+1} is linearly independent of $e_1, ..., e_{j+1}$. Since $Av_{i+1} \in \langle e_1, ..., e_j \rangle$, it follows that $e_j \in \langle e_1, ..., e_{j-1}, Av_{i+1} \rangle \subseteq V_i$, a contradiction.

□

Corollary 7.2. $\mathcal{F} = \bigcup_\delta \mathcal{F}_\delta$

Proof. Let $(V, a, b) \in F$. Let $i_0 = \min\{i : b \in V_i\}$, and let $i' = \max\{i : V_i \subseteq E'\}$. We want that $b \in V_{i_0} \subseteq E'$, so we want that $i_0 \leq i'$. For contradiction, suppose $i' < i_0$.

$\square$

Lemma 7.3. *Each $\mathcal{F}_\delta$ is a closed subvariety of $\mathcal{F}$.*

Proof. $\square$

Lemma 7.4. *Each $\mathcal{F}_\delta$ is irreducible of dimension m.*

Proof. Let $\mathcal{E}$ be the variety of partial flags of E of shape $(\delta_0, ..., \delta_{n+1})$. Define $g : \mathcal{F}_\delta \to \mathcal{E}$ by

$$(V, a, b) \mapsto 0 \subseteq V_{i_0} \cap E \subseteq V_{i_1} \cap E \subseteq \cdots \subseteq V_{i_{n+1}} \cap E = E.$$

It is clear that g is surjective, as the definition of $\mathcal{F}_\delta$ places no restriction on the intersections $V_i \cap E$. Now let's look more closely at the fibers $g^{-1}(U)$.

First let's find the flags V such that there exist a, b with $g(V, a, b) = U$. We see that, for instance, $V_{i_0} \cap E = U_1$. In fact $V_{i_0} \subseteq E$, so $V_{i_0} = U_1$. But we are free to choose the vector spaces between 0 and V_{i_0} however we wish, so we get some degrees of freedom like $\mathcal{F}_{\delta_0}$, the complete flag variety on $\mathbb{C}^{\delta_0} \cong V_{i_0}/0$. Similarly, for every $j = 1, ..., n+1$, we can choose the vector spaces between $V_{i_{j-1}}$ and V_{i_j} arbitrarily, so we get degrees of freedom like $\mathcal{F}_{\delta_{i_j}}$, the complete flag variety on $\mathbb{C}^{\delta_{i_j}} \cong V_{i_j}/V_{i_{j-1}}$. Finally, to meet the constraint of (V, a, b) being in $\mathcal{F}_\delta$, we can choose any $b \in U_1$ and any a such that $\overline{a} \in U_n^\perp$. Thus we get an isomorphism

$$g^{-1}(U) \cong \mathcal{F}_{\delta_0} \times \cdots \times \mathcal{F}_{\delta_{n+1}} \times \mathbb{C}^{\delta_0} \times \mathbb{C}^{\delta_{n+1}}.$$

$\square$

Theorem 7.5. *The $\mathcal{F}_\delta$'s are the irreducible components of $\mathcal{F}$.*

7.2 In the case that X is a Jordan block

This case seems harder to work with explicitly than the case that X is zero, so our strategy is to reuse

8 A different variety

Define $R = \{(X, \mathfrak{b}_1, \mathfrak{b}_2) : X \in \mathfrak{b}_1 \wedge u(X) \in \mathfrak{b}_2\} \subseteq \mathcal{S}'_{m,n} \times \mathcal{B}_{m+n} \times \mathcal{B}_m$. We can obtain a subvariety of R by requiring that $u(X)$ is in some fixed similarity class. (TODO: why is this a subvariety? Is it? Is this even the right way of explaining the significance of the m^2?) We expect that each of these subvarieties is an irreducible component of dimension m^2. We will verify these things using our prvious computations of springer fibers.

8.1 TODO

- Why are SOn flags what they are.

References

[1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.