

# 1 Introduction

We will review the definition of a Springer fiber and define, for a nilpotent  $Y \in \mathfrak{gl}_m$ , the Springer fiber at the  $n$ -Slodowy slice at  $Y$ . For every  $n$  and every nilpotent  $Y \in \mathfrak{gl}_m$ , we will find the irreducible components of the Springer fiber at the  $n$ -Slodowy slice at  $Y$ . Finally, we will use our results about Springer fibers at  $n$ -Slodowy slices to find the irreducible components of some other variety (which probably needs a name), and show that they all have the same dimension.

## 2 Springer fibers

Let  $G \subseteq \mathrm{GL}_m(\mathbb{C})$  be a connected semisimple Lie group, and let  $\mathfrak{g} \subseteq \mathfrak{gl}_m(\mathbb{C})$  be its Lie algebra. Let  $\mathcal{N} \subseteq \mathfrak{g}$  be the subset consisting of nilpotent elements. Let  $\mathcal{B}$  be the variety of Borel subalgebras of  $\mathfrak{g}$ . Let  $\tilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$ . Let  $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  be the projection onto the second coordinate. For  $n \in \mathcal{N}$ , we call  $\pi^{-1}(n)$  the *Springer fiber at  $n$* .

We mention some results about Springer fibers, which we will use later in this paper. TODO: mention them.

## 3 Slodowy slice

A basis for  $\mathfrak{sl}_2(\mathbb{C})$  is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra  $\mathfrak{g}$ , and a homomorphism  $\phi : \mathfrak{sl}_2 \rightarrow \mathfrak{g}$  sending  $(e', h', f')$  to  $(e, h, f)$ , we say that  $(e, h, f)$  is an  *$\mathfrak{sl}_2$ -triple*. Observe that  $e, f$  must be nilpotent, and  $h$  must be Cartan (???). If  $\mathfrak{g}$  is semisimple, then given any nilpotent  $e \in \mathfrak{g}$ , the Jacobson-Morozov theorem [1, 3.7.1] says that there exist  $h, f \in \mathfrak{g}$  such that  $(e, h, f)$  is an  $\mathfrak{sl}_2$ -triple.

Given  $(e, h, f)$ , we define the *Slodowy slice at  $e$*  as  $\mathcal{S}_e := e + \ker \mathrm{ad}_f$ . By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent  $e \in \mathfrak{g}$ , when  $\mathfrak{g}$  is semisimple.

## 4 Finding $\mathfrak{sl}_2$ -triples $(E, H, F)$ in $\mathfrak{gl}_{m+n}$ with a particular $E$ .

Let

$$e = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & 0 \\ & & & 1 & 0 \end{pmatrix} \in \mathfrak{gl}_n,$$

and let  $E = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{gl}_{m+n}$ . We will show that there is exactly one  $\mathfrak{sl}_2$ -triple  $(E, H, F)$ , and we will find what it looks like. First we solve the case  $m = 0$  (so  $E = e$ ), and then we use this to solve the case of arbitrary  $m$ .

**Lemma 4.1.** *There is exactly one way to choose  $h, f \in \mathfrak{gl}_n$  so that  $(e, h, f)$  is an  $\mathfrak{sl}_2$ -triple.*

*Proof.* Note that  $[h', e'] = 2e'$ , and  $[e', f'] = h'$ , and  $[h', f'] = -2f'$ . Thus  $e, h, f$  must obey the same relations. In particular,  $he - eh = 2e$ . The matrix  $eh$  is  $h$  shifted down one, and  $he$  is  $h$  shifted left one. Thus,  $2e_{ij} = (he - eh)_{ij} = h_{i,j+1} - h_{i-1,j}$ . We can use this to show that  $h_{ij} = 0$  when  $i \neq j$ . Then we can use it to show that  $h_{ii} = h_{i-1,i-1} + 2$ , so that  $h_{ii} = h_{11} + 2(i-1)$ .

Similarly, from  $[e, f] = h$  we get that  $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$ . We can use this to show that  $f_{ij} = 0$  when  $j \neq i+1$ . Then we can use it to show that  $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$ , that  $f_{1,2} = -h_{1,1}$ , and that  $f_{n-1,n} = h_{n,n}$ . From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^n h_{ii} \implies \sum_i h_{ii} = 0.$$

Remark: this is just the statement that  $h \in \mathfrak{sl}_n$ ; in other words, we will see that every choice of  $\mathfrak{sl}_2$ -triple in  $\mathfrak{gl}_{m+n}$  is also an  $\mathfrak{sl}_2$ -triple in  $\mathfrak{sl}_{m+n}$ . This shows that  $h_{11} = n-1, h_{22} = n-3, \dots, h_{nn} = 1-n$ . So we have determined

$h$ ; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for  $f$  in terms of  $h$  to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & \\ & 0 & (1-n) + (3-n) & & \\ & & \ddots & \ddots & \\ & & & 0 & (1-n) + \cdots + (n-1) \\ & & & & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1(1-n) & & & \\ & 0 & 2(2-n) & & \\ & & 0 & (n-2)(-2) & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

□

**Lemma 4.2.** *There is exactly one way to choose  $H, F \in \mathfrak{gl}_{m+n}$  so that  $(E, H, F)$  is an  $\mathfrak{sl}_2$ -triple.*

*Proof.* Suppose we have  $H, F$  so that  $(E, H, F)$  is an  $\mathfrak{sl}_2$ -triple. Writing  $F =: \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & f \end{pmatrix}$ , and similarly for  $H$ , we have

$$2E = [H, E] = \begin{pmatrix} 0 & H_{12}e \\ 0 & he \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ eH_{21} & eh \end{pmatrix},$$

$$H = [E, F] = \begin{pmatrix} 0 & 0 \\ eF_{21} & ef \end{pmatrix} - \begin{pmatrix} 0 & F_{12}e \\ 0 & fe \end{pmatrix}.$$

From these we observe that  $(e, h, f)$  must also be an  $\mathfrak{sl}_2$ -triple, so  $h, f$  must be as in Lemma 4.1. We also see that  $H_{11} = 0$ . Recalling that left multiplication by  $e$  is a down-shift, and right multiplication is a left-shift, we see that  $H_{12}$

is all zeroes except for the leftmost column, and  $H_{21}$  is all zeroes except for the bottom row. Then our final constraint is that

$$-2F = [H, F] =$$

$$\begin{pmatrix} H_{12}F_{21} & H_{12}F_{22} \\ H_{21}F_{11} + H_{22}F_{21} & H_{21}F_{12} + H_{22}F_{22} \end{pmatrix} - \begin{pmatrix} F_{12}H_{21} & F_{11}H_{12} + F_{12}H_{22} \\ F_{22}H_{21} & F_{21}H_{12} + F_{22}H_{22} \end{pmatrix}.$$

Now  $H_{12} = F_{12}e$ , and  $H_{21} = eF_{21}$ , from the equation  $H = [E, F]$ . Substituting in the equation above then,

$$\begin{aligned} -2F &= \begin{pmatrix} F_{12}eF_{21} & F_{12}ef \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} F_{12}eF_{21} & F_{11}F_{12}e + F_{12}h \\ feF_{21} & F_{21}F_{12}e + fh \end{pmatrix} = \\ &= \begin{pmatrix} 0 & F_{12}(fe + h) \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e + F_{12}h \\ (ef - h)F_{21} & F_{21}F_{12}e + fh \end{pmatrix} = \\ &= \begin{pmatrix} 0 & F_{12}fe \\ eF_{21}F_{11} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e \\ efF_{21} & F_{21}F_{12}e + fh \end{pmatrix}. \end{aligned}$$

Now we see that  $F_{11} = 0$ , and consequently that  $F_{12} = F_{21} = 0$  as well. This shows that  $H_{12} = H_{21} = 0$ . We conclude that  $H$  and  $F$  just have  $h$  and  $f$  in their bottom-right corners, respectively.  $\square$

## 5 Finding the Slodowy slices with the same $E$

First we find  $\ker \text{ad}_f$ . We have  $(fX)_{ij} = i(n-i)A_{i+1,j}$ , and  $(Xf)_{ij} = (j-1)(n-(j-1))A_{i,j-1}$ . So, for all  $i, j \in \{1, \dots, n\}$ , we have

$$i(n-i)A_{i+1,j} = (j-1)(n-(j-1))A_{i,j-1}.$$

Taking  $j = 1$ , we find that  $A_{i,1} = 0$  for  $i \geq 2$ . Then, taking  $j > 1$ , we find that for  $i, j \in \{1, \dots, n-1\}$ ,

$$A_{i+1,j+1} = \frac{(j-1)(n-(j-1))}{i(n-i)}A_{ij}.$$

So,  $\ker \text{ad}_f$  is the set of matrices which are upper triangular and satisfy the above condition.

To find the Slodowy slice associated to the previous  $\mathfrak{sl}_2$ -triple  $(E, H, F)$ , we just need to find  $\ker \operatorname{ad}_F$ . We have

$$[F, X] = \begin{pmatrix} 0 & 0 \\ fX_{21} & fX_{22} \end{pmatrix} - \begin{pmatrix} 0 & X_{12}f \\ 0 & X_{22}f \end{pmatrix}.$$

Thus,  $X_{21}$  must be all zeroes except for the first row, and  $X_{12}$  must be all zeroes except for the last column, and  $X_{22} \in \ker \operatorname{ad}_f$ . There is no restriction on  $X_{11}$ . This describes  $\ker \operatorname{ad}_F$ .

For  $X \in \mathcal{S}_{m,n}$ , define  $u(X) := X_{11}$ .

### 5.1 Finding $\tilde{\mathcal{N}}_{m,n}$

Let  $\mathcal{N}_m \subseteq \mathfrak{gl}_m$  be the nilpotent elements. Let  $\mathcal{S}'_{m,n}$  be the set of  $X \in \mathcal{S}_{m,n}$  such that both  $X$  and  $u(X)$  are nilpotent. Let  $\tilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n}$ . Define  $\pi_{m,n} : \tilde{\mathcal{N}}_{m,n} \rightarrow \mathcal{N}_m$  by  $(\mathfrak{b}, X) \mapsto X_{11}$ . For  $Y \in \mathfrak{gl}_m$ , we call  $\pi_{m,n}^{-1}(Y)$  the *Springer fiber at the  $n$ -Slodowy slice at  $Y$* .

**Lemma 5.1.** *Let  $J$  be a jordan block with zeroes along the diagonal, and let  $A$  be upper triangular and nonzero. Then  $J + A$  is not nilpotent.*

*Proof.* It is straightforward to show by induction that if  $v_i = 0$  for  $i < j$ , and  $v_j \neq 0$ , then  $((J + A)^k v_j)_{j+k} = v_j$ . Let  $i$  be such that  $Ae_i \neq 0$ . Then  $(J + A)^{i-1}e_1$  has nonzero  $e_i$ -component. Then  $(J + A)^i e_1$  has some nonzero  $e_{i'}$ -component for some  $i' \leq i$ . Then  $(J + A)^{i+(n-i')}e_1$  has some nonzero  $e_n$ -component. And  $i + (n - i') \geq n$ , so we're done.  $\square$

**Lemma 5.2.** *Let  $X \in \mathfrak{gl}_m$ , and let*

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ d_1 & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ & d_2 & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ & & \ddots & \cdots & \vdots & \vdots \\ & & & d_{n-2} & y_{n-1,n-1} & y_{n-1,n} \\ & & & & d_{n-1} & y_{nn} \end{pmatrix} \in \mathfrak{gl}_n.$$

For any  $a, b \in \mathbb{C}^m$ ,

$$\det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - & a & - \end{array} & Y \end{array} \right) =$$

$$\det X \det Y + \left( \prod_i d_i \right) \det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - & a & - \end{array} & 0 \end{array} \right)$$

*Proof.* By induction on  $n$ . In the case  $n = 1$ , expanding along the last row (taking the usual interpretation of the empty product) gives the desired result.

Now suppose  $n > 1$ . Expanding along the last row, we get

$$d_{n-1} \det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - & a & - \end{array} & Y_{n,n-1} \end{array} \right) - y_{nn} \det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ \\ | \end{array} \\ \hline \begin{array}{c} - & a & - \end{array} & Y_{n,n} \end{array} \right).$$

Using our inductive hypothesis for the first determiniant, and using that  $\det \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} = \det A_{11} \det A_{22}$  for the second, the expression becomes

$$d_{n-1} \left( \det X \det Y_{n,n-1} + \left( \prod_{i \leq n-2} d_i \right) \det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - & a & - \end{array} & 0 \end{array} \right) \right) - y_{nn} \det X \det Y_{nn} =$$

$$(d_{n-1} Y_{n,n-1} - y_{nn} \det Y_{nn}) \det X + \left( \prod_i d_i \right) \det \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - & a & - \end{array} & 0 \end{array} \right) =$$

$$\det Y \det X + \left( \prod_i d_i \right) \det \left( \begin{array}{ccc|c} & & & b \\ & & & | \\ & & & | \\ \hline - & a & - & 0 \end{array} \right).$$

□

**Corollary 5.3.** *If  $X$  is nilpotent, and*

$$\left( \begin{array}{ccc|c} & & & b \\ & & & | \\ & & & | \\ \hline - & a & - & \\ & & & Y \end{array} \right)$$

*is nilpotent as well, then  $Y$  is nilpotent (TODO: and that other determinant is zero).*

*Proof.* By the previous lemma, the characteristic polynomial of the big matrix is

$$g_X(\lambda)g_Y(\lambda) + f(\lambda),$$

where  $g_X(\lambda) = \lambda^m$  is the characteristic polynomial of  $X$ , and  $f(\lambda)$  is some polynomial of degree at most  $m - 1$ . □

Now, taking the previous corollary and the first lemma together, we see that

$$\mathcal{S}'_{m,n} = \left\{ \left( \begin{array}{ccc|cccc} & & & & & & b \\ & & & & & & | \\ & & & & & & | \\ \hline - & a & - & 0 & & & \\ & & & 1 & 0 & & \\ & & & & & \ddots & \\ & & & & & & \ddots & \\ & & & & & & & 1 & 0 \end{array} \right) : a, b \in \mathbb{C}^m, X \in \mathfrak{gl}_m \text{ is nilpotent} \right\}.$$

## 6 Simplifying the definition of a Springer fiber at a Slodowy slice

Let  $\mathcal{N}_m \subseteq \mathfrak{gl}_m(\mathbb{C})$  be the subset consisting of nilpotent elements. For  $X \in \mathcal{N}_m$  and  $a, b \in \mathbb{C}^m$ , let

$$A_{X,a,b} = \left( \begin{array}{c|ccc} X & & & & b \\ \hline - & a & - & & \\ \hline & & 0 & & \\ & & 1 & 0 & \\ & & & 1 & \ddots \\ & & & & \ddots & 0 \\ & & & & & 1 & 0 \end{array} \right) \in \mathfrak{gl}_{m+n}(\mathbb{C}).$$

By the definition given in the previous section, we have

$$\tilde{\mathcal{N}}_{m,n} = \{(\mathfrak{b}, X) : X \in \mathfrak{b}\} \subseteq \mathcal{B}_{m+n} \times \mathcal{S}'_{m,n},$$

and the Springer fiber at the  $n$ -Slodowy slice at a nilpotent  $X \in \mathfrak{gl}_m$  is

$$\begin{aligned} \pi_{m,n}^{-1}(X) &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \in \mathcal{S}'_{m,n}\} \\ &= \{(\mathfrak{b}, A_{X,a,b}) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \text{ nilpotent}\} \\ &\cong \{(\mathfrak{b}, a, b) : A_{X,a,b} \in \mathfrak{b}, A_{X,a,b} \text{ nilpotent}\}. \end{aligned}$$

We will make one last simplification to this by using a correspondence between Borel subalgebras and complete flags. TODO: fix the next few paragraphs, they are out of context.

We have  $\mathcal{M} = \{AHA^{-1} : A \in \text{GL}_{m+n}(\mathbb{C})\}$ , where  $H \subseteq \mathfrak{gl}_{m+n}(\mathbb{C})$  is the set of upper triangular matrices. We say a map  $X : \mathfrak{gl}_{m+n} \rightarrow \mathfrak{gl}_{m+n}$  preserves a flag  $V_0 \subseteq \cdots \subseteq V_{m+n}$  if  $XV_i \subseteq V_i$  for each  $i$ . Let  $E_0 \subseteq \cdots \subseteq E_{m+n}$  be the standard flag of  $\mathbb{C}^{m+n}$ . Since  $H$  is the set of  $X$  which preserve  $E$ ,

$$\mathcal{M} = \{\{X : \forall i. X(AE_i) \subseteq AE_i\} : A \in \text{GL}_n(\mathbb{C})\}.$$

So, for  $X \in \mathcal{N}$ ,

$$\pi^{-1}(X) \cong \{(V, a, b) : \forall i. A_{X,a,b}V_i \subseteq V_i\}.$$



In this section we will find the irreducible components of  $\pi^{-1}(X)$ . Since  $\pi^{-1}(X) \cong \pi^{-1}(AXA^{-1})$  for any invertible  $A$ , we will assume that  $X$  is in Jordan normal form.

Let  $\lambda$  be the shape of  $X$ , and let  $(e_{ij})_{i \leq r, j \leq \lambda_i}$  be a Jordan basis for  $X$ , with  $Xe_{ij} = e_{i,j-1}$ . Let  $f_1, \dots, f_n$  be the standard basis for  $\mathbb{C}^n$ , with  $A_{X,a,b}f_i = f_{i+1}$ .

## 7 Finding a centralizer

For nilpotent  $X$  in Jordan form and  $a$  in ‘normalized’ form (i.e., with at most one nonzero element, which is a one), we will find the centralizer of  $A_{X,a,b}$  in

$$\{(A, I) : A \in \mathfrak{gl}_m\} \subseteq \mathfrak{gl}_{m+n}.$$

Note that an element of the form  $(A, I)$  commutes with  $A_{X,a,b}$  if and only if  $A$  commutes with  $X$ , and  $aA = a$ , and  $Ab = b$ . So, we just have to find

$$\{A \in \mathfrak{gl}_m : AX = XA, aA = a, Ab = b\}.$$

We begin by finding

$$\{A \in \mathfrak{gl}_m : AX = XA\}.$$

We write the shape of  $X$  as  $\lambda = (\lambda_1, \dots, \lambda_k)$ , so that

$$X = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix}.$$

Then we write  $A$  as the block matrix

$$A =: \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix},$$

where  $A_{ij}$  is a block of size  $\lambda_i \times \lambda_j$ . We have

$$XA = \begin{pmatrix} J_{\lambda_1} A_{11} & \cdots & J_{\lambda_1} A_{1k} \\ \vdots & & \vdots \\ J_{\lambda_k} A_{k1} & \cdots & J_{\lambda_k} A_{kk} \end{pmatrix}, \text{ and } AX = \begin{pmatrix} A_{11} J_{\lambda_1} & \cdots & A_{1k} J_{\lambda_k} \\ \vdots & & \vdots \\ A_{k1} J_{\lambda_1} & \cdots & A_{kk} J_{\lambda_k} \end{pmatrix}.$$

So, the constraint that  $XA = AX$  is simply saying that

$$\forall i, j. J_{\lambda_i} A_{ij} = A_{ij} J_{\lambda_j}.$$

It's easy to see that left multiplication by  $J_{\lambda_i}$  is just a down-shift by one, and right multiplication by  $J_{\lambda_j}$  is a left-shift by one. This means that  $A_{ij}$  is constrained to be a “lower-left-Toeplitz” matrix. A Toeplitz matrix is one which is constant along diagonal bands:  $\forall i, j, k. x_{ij} = x_{i+k, j+k}$ . A lower-left-Toeplitz matrix is one which is all zeroes except for the bottom-left corner; that is, the bands start in the bottom-left corner, and continue until hitting the band which includes the lower-right corner or the band which includes the upper-left corner (whichever comes first). In other words, an  $n \times m$  lower-left-Toeplitz matrix is one which satisfies  $x_{ij} = 0$  for all  $i, j$  with  $i < j$  and all  $i, j$  with  $i - j < m - n$ . In yet other words, a matrix is lower-left-Toeplitz if it is Toeplitz, satisfies  $x_{1j} = 0$  for  $j \neq 1$ , and satisfies  $x_{in} = 0$  for  $i \neq m$ . Anyway, it is easy to see that the lower-left-Toeplitz matrices are exactly those matrices  $A$  such that left-shifting  $A$  by one has the same result as down-shifting by one.

So, we have found the set

$$C_1 := \{A \in \mathfrak{gl}_m : AX = XA\}.$$

It is just the set of  $A = [A_{ij}]_{ij}$ , where each  $A_{ij}$  is lower-left-Toeplitz.

Let  $v_{ij}$  be the leftmost column of  $A_{ij}$ , so that

$$A_{ij} = \begin{pmatrix} v_{ij} \vee 0 & \cdots & v_{ij} \vee [\lambda_j - 1] \end{pmatrix}.$$

Since  $A_{ij}$  is lower-left-Toeplitz,  $v_{ij}$  is of the form

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where  $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$  can be chosen freely. Now we determine which matrices of this form satisfy  $aA = a$ . For simplicity, we will instead find the matrices  $A$  such that  $aA = 0$  (so that  $a(A + I) = a$ ). (Note that  $I \in C_1$ , and  $C_1$  is closed under addition, so this is really the same thing.)

In the case that  $a = 0$ , clearly every  $A$  works. Otherwise, let  $i_0, j_0$  be such that  $a_{i_0, j_0} = 1$ . Now, clearly the constraint that  $aA = 0$  is just saying

that the  $(i_0, j_0)$ th row of  $A$  must be zero. That is, for each  $j$  the  $j_0$ th row of  $A_{i_0j}$  must be zero. This just requires that for each  $j$ , we must have

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where  $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$  can be chosen freely. So, we have now found the set

$$C_2 := \{A \in \mathfrak{gl}_m : AX = XA, aA = a\}.$$

It is the matrices of the form  $I + A$ , where  $A_{ij}$  is a block matrix of size  $\lambda_i \times \lambda_j$  with

$$A_{ij} = T(v_{ij}) = T \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where  $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$  in the case  $i \neq i_0$  and  $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$  if  $i = i_0$ .

Now we are only left with finding the matrices  $A \in C_2$  satisfying  $Ab = b$ . This just means finding the matrices  $A$ , of the form given above, which satisfy  $Ab = 0$ . We may restate this as the condition that for every  $i$ ,

$$(A_{i1} \ \cdots \ A_{in})b = 0.$$

Taking the transpose,

$$b^\top \begin{pmatrix} A_{i1}^\top \\ \vdots \\ A_{in}^\top \end{pmatrix} = 0.$$

Now, the rightmost column of  $A_{ij}^\top$  is  $v_{ij}$  upside-down (rows reversed). Write  $u_{ij}$  to refer to  $v_{ij}$  upside-down, and  $u'_{ij}$  to refer to  $v'_{ij}$  upside-down. Then the second-rightmost column is  $u_{ij} \hat{\wedge} 1$ , and so on until the first column is  $u_{ij} \hat{\wedge} [\lambda_i - 1]$ . So, we can reformulate this condition by requiring that for all  $l$ ,

$$b^\top \begin{pmatrix} u_{i1} \hat{\wedge} l \\ \vdots \\ u_{in} \hat{\wedge} l \end{pmatrix} = 0.$$

Writing

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix},$$

clearly this is equivalent to requiring that for all  $l$ ,

$$(b_1^\top \gg l \quad \cdots \quad b_k^\top \gg l) \begin{pmatrix} u_{i1} \\ \vdots \\ u_{ik} \end{pmatrix} = 0.$$

Then writing  $b_j = \begin{pmatrix} b'_j \\ \vdots \end{pmatrix}$ , where  $b'_j$  has the same dimension as  $u_{ij}$ , we can rewrite this constraint as

$$(b_1'^\top \gg l \quad \cdots \quad b_k'^\top \gg l) \begin{pmatrix} u'_{i1} \\ \vdots \\ u'_{ik} \end{pmatrix} = 0.$$

Finally, requiring the above equation for all  $l$  is the same thing as requiring

$$\begin{pmatrix} b_1'^\top \gg 0 & \cdots & b_k'^\top \gg 0 \\ \vdots & & \vdots \\ b_1'^\top \gg [\lambda_i - 1] & \cdots & b_k'^\top \gg [\lambda_i - 1] \end{pmatrix} \begin{pmatrix} u'_{i1} \\ \vdots \\ u'_{ik} \end{pmatrix} = 0.$$

This is the most that we will simplify the constraint. Recall that the  $v'_{ij}$  can be chosen arbitrarily, so the  $u'_{ij}$  can be chosen arbitrarily as well. That is, the set  $C_3 := \{A \in \mathfrak{gl}_m : AX = XA, aA = a, Ab = b\}$  is the set of  $I + A$  for any  $A$  that can be recovered from  $u'_{ij}$  meeting the above condition.

Now we find the dimension of  $C_3$ . Since the  $u'_{ij}$  can be chosen arbitrarily, apart from the condition above, we just need to add up their dimensions and subtract the rank of the matrix constructed of the  $b$ 's. The dimension of  $u'_{ij}$  is  $\min(\lambda_i, \lambda_j)$  in the case that  $i \neq i_0$ , and  $\min(\lambda_i - j_0, \lambda_j)$  when  $i = i_0$ . Write  $d_{ij}$  for this dimension. For each  $i$ , let  $q_i$  be the length of the longest prefix of  $b_i$  which consists of zeroes. Then if  $A_{X,a,b}$  is nilpotent, we must have  $q_{i_0} \geq j_0$ . So we may as well set  $q_{i_0} := q_{i_0} - j_0$ .

Fix an  $i$ . I claim the rank of the matrix of  $b$ 's is  $h := \max_j [d_{ij} - q_j]$ , or zero if this quantity is always negative. First, to see that the rank is at least this big, note that for any  $j \in \arg \max_j [d_{ij} - q_j]$ , the rank of the  $j$ th ‘‘column block’’ of the matrix of  $b$ 's is equal to  $h$ . Then, to see that the rank is at most this big, note that every row after the first  $h$  rows is just zeroes. So, the total degrees of freedom for the  $u_{ij}$  is

$$\sum_j d_{ij} - \max(0, \max_j [d_{ij} - q_j]).$$

Then summing over all the  $i$ , we get the dimension of  $C_3$ :

$$\sum_i \left[ \sum_j d_{ij} - \max(0, \max_j [d_{ij} - q_j]) \right] = \sum_{ij} d_{ij} - \sum_i \max(0, \max_j [d_{ij} - q_j]).$$

When  $i \neq i_0$ , then  $d_{ii} = \lambda_i \geq q_i$ , and therefore  $\max(0, \max_j [d_{ij} - q_j]) = \max_j [d_{ij} - q_j]$ . When  $i = i_0$ , then  $d_{ii} = \lambda_i - j_0 \geq q_{i_0}$ . So, we can simplify the above expression:

$$\sum_{ij} d_{ij} - \sum_i \max_j [d_{ij} - q_j].$$

Note that  $\max_j [d_{ij} - q_j] = \max_j [\min(p_i + q_i, p_j + q_j) - q_j]$ . This thing being maximized equals  $\min(p_i + q_i - q_j, p_j)$ . It would be really helpful if I could simplify this to  $p_j$  or something. I don't think so, though.

## 8 A necessary condition for $A_{X,a,b}$ to be nilpotent

Suppose  $A := A_{X,a,b}$  is nilpotent. Define the *height* of a vector  $v \in \mathbb{C}^{m+n}$  as the smallest  $k \geq 0$  such that  $A_{X,a,b}^k v = 0$ . Clearly, for  $i \leq j$ , we have that the height of  $f_i$  is geq the height of  $f_j$ . Thus, for each  $k \geq 0$ , we must have  $A_{X,a,b}^k f_n \in \langle e_{ij} \rangle_{ij}$ . We have  $A f_n = \sum_{ij} b_{ij} e_{ij}$ . Then  $A^2 f_n = \sum_{ij} b_{ij} (e_{i,j-1} + a_{ij} f_1)$ , so we must have  $\sum_{ij} b_{ij} a_{ij} = 0$ . Similarly, since  $A^3 f_n \in \langle e_{ij} \rangle$ , we see that  $\sum_{ij} b_{ij} a_{i,j-1} = 0$ . Continuing in this way, we obtain that for all  $k \geq 0$ ,

$$\sum_{ij} b_{ij} a_{i,j-k} = 0. \tag{1}$$

In fact this is in fact a sufficient condition for  $A_{X,a,b}$  to be nilpotent, as can be seen from the characteristic polynomial of  $A_{X,a,b}$ . However, we will not do this calculation here, instead just showing that it is sufficient by giving a Jordan basis.

## 9 A Jordan basis for $A_{X,a,b}$

For illustration, we will begin with the case that  $X$  is a single Jordan block. Then we will move to the general case.

## 9.1 When $X$ is a Jordan block

Suppose  $X$  is a single Jordan block of size  $m$ . In this case we just write the basis of  $\mathbb{C}^m$  as  $e_1, \dots, e_m$ , and the above condition on  $a$  and  $b$  simply becomes

$$\forall k \geq 0. \sum_{i=k+1}^m b_i a_{i-k} = 0. \quad (2)$$

This condition can be simplified even more.

**Lemma 9.1.** *The condition (1) holds iff there exist nonnegative  $m_1, m_2, m_3$ ,  $a_1, \dots, a_{m_3}$ , and  $b_1, \dots, b_{m_1}$  satisfying the following conditions.*

- $m = m_1 + m_2 + m_3$
- $a = (0, 0, \dots, 0, a_1, \dots, a_{m_3})$
- $b = (b_1, \dots, b_{m_1}, 0, 0, \dots, 0)$
- If  $m_1 \neq 0$ , then  $a_{m_1} \neq 0$
- If  $m_3 \neq 0$ , then  $b_1 \neq 0$

*Proof.* If  $a$  and  $b$  are of this form, then for every  $k \geq 0$  we have  $b_i a_{i-k} = 0$ , so clearly the condition holds.

Now suppose (1) holds. If  $a$  or  $b$  is zero, this is trivial. Otherwise, let  $m_1 = \max\{i : a_i \neq 0\}$ , and let  $m_3 = \min\{i : b_i \neq 0\}$ . We just need to show that  $m_1 < m_3$ . For contradiction, suppose  $m_1 - m_3$  is nonnegative. Then by (1),  $0 = \sum_i b_i a_{i-(m_1-m_3)} = b_1 a_{m_1}$ , contradicting that  $b_1$  and  $a_{m_1}$  are nonzero.  $\square$

## 9.2 A ‘normalization’ fact about Jordan bases

Let  $V$  be a finite-dimensional vector space,  $A : V \rightarrow V$  a nilpotent operator, and  $f : V \rightarrow \mathbb{C}$  a linear map. Let  $(e_{ij} : i \leq m, j \leq \lambda_i)$  be a Jordan basis for  $A$ .

**Lemma 9.2.** *There is a Jordan basis  $e_{ij}$  for  $A$  such that there is at most one pair  $(i, j)$  with  $f(e_{ij}) \neq 0$ .*

*Proof.* For any change of basis  $P : V \rightarrow V$  commuting with  $A$ , we obtain a new Jordan basis  $(P(e_{ij}) : i \leq m, j \leq \lambda_i)$ . For any such  $P$ , define

$$S_P = \sum_i \begin{cases} -1, & \forall j. f \circ P(e_{ij}) = 0 \\ \lambda_i - \min\{j : f \circ P(e_{ij}) \neq 0\}, & \text{otherwise} \end{cases}.$$

Let  $P$  be any operator, among all invertible operators commuting with  $A$ , which minimizes  $S_P$ . Write  $e'_{i,j} := P(e_{i,j})$ .

Suppose for contradiction that there are two distinct  $i$ 's (and some  $j$ 's) with  $f(e'_{ij}) \neq 0$ . Then we can take  $e'_{i_1,j_1}$  and  $e'_{i_2,j_2}$ , where for  $k \in \{1, 2\}$  we have  $f(e'_{i_k,j_k}) \neq 0$ , and  $\forall j < j_k. f(e'_{i_k,j}) = 0$ . Wlog, assume  $\lambda_{i_1} - j_1 \leq \lambda_{i_2} - j_2$ . Then, we can define  $Q : V \rightarrow V$  by

- $Q(e'_{i_1,\lambda_1}) := e'_{i_1,\lambda_1} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(\lambda_{i_1}-j_1)}$
- For  $j < \lambda_1$ ,  $Q(e'_{i_1,j}) := A^{\lambda_{i_1}-j} Q(e'_{i_1,\lambda_1})$
- For  $i \neq i_1$ ,  $Q(e'_{ij}) = e'_{ij}$ .

Clearly  $Q$  is invertible, and it commutes with  $A$ . Further, I claim that  $S_{QP} < S_P$ . It suffices to show that  $\forall j \leq j_1. f \circ Q(e'_{i_1,j}) = 0$ . We have

$$\begin{aligned} f \circ Q(e'_{i_1,j}) &= f \left( A^{\lambda_{i_1}-j} \left( e'_{i_1,\lambda_1} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(\lambda_{i_1}-j_1)} \right) \right) = \\ &= f \left( e'_{i_1,j} - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} e'_{i_2,j_2+(j-j_1)} \right) = f(e'_{i_1,j}) - \frac{f(e'_{i_1,j_1})}{f(e'_{i_2,j_2})} f(e'_{i_2,j_2+(j-j_1)}). \end{aligned}$$

Clearly (by design), this expression is zero when  $j = j_1$ . And for  $j < j_1$ , we have  $f(e'_{i_1,j}) = f(e'_{i_2,j_2+(j-j_1)}) = 0$ , so it is zero then as well. Thus we see that  $S_{QP} < S_P$ , contradicting that  $S_P$  is minimal. So there must be at most one  $i$  such that there exists  $j$  such that  $f(e'_{ij}) \neq 0$ .

If there is no such  $i$ , we have found the desired basis. So, suppose there is such an  $i_0$ . Let  $U = \langle e'_{i_0,j} \rangle$ . Simply write  $e_j := e'_{i_0,j}$ . Let  $j_0 = \min\{j : f(e_j) \neq 0\}$ . We just have to change basis to zero out  $e_j$  for  $j \neq j_0$ . Let  $P_1(e_{\lambda_{i_0}}) := e_{\lambda_{i_0}} - \frac{f(e_{j_0+1})}{f(e_{j_0})} e_{\lambda_{i_0}-1}$ , and notice that  $f \circ P_1(e_j) = 0$  for all  $j \leq j_0+1$  except for  $j_0$ . Then we define  $P_2(e_{\lambda_{i_0}}) := P_1(e_{\lambda_{i_0}}) - \frac{f \circ P_1(e_{j_0+2})}{f \circ P_1(e_{j_0})} e_{\lambda_{i_0}-2}$ , and notice that  $f \circ P_2(e_j) = 0$  for all  $j \leq j_0+2$  except for  $j_0$ . Eventually we get  $P_{\lambda_{i_0}-j_0}$ , and by applying this to the  $e_j$ 's we obtain a basis of  $U$ , in which there is exactly one  $j$  with  $f \circ P_{\lambda_{i_0}-j_0}(e_j) \neq 0$ .  $\square$

### 9.3 The general case

In the case where  $X$  was a single Jordan block, it was helpful to have the condition that all the nonzero  $a$ 's were to the right of all the nonzero  $b$ 's. This allowed some tricks with right-shifting. In the general case, the parameters  $a$  and  $b$  are not necessarily in such a helpful form. We will see, however, that we can change basis of  $\mathbb{C}^m$  to put them in a nice form.

Recall our nilpotent operator  $A_{X,a,b}$  on  $\mathbb{C}^m \times \mathbb{C}^n$ . We can define a map  $\mathbb{C}^m \rightarrow \mathbb{C}$  by  $x \mapsto x \cdot a$ . Then, by the previous lemma, we can choose a Jordan basis  $e_{ij}$  of  $\mathbb{C}^m$  so that at most one  $(i, j)$  has  $f(e_{ij}) \neq 0$ . This implies that there are some  $P, k, b'$  such that  $PA_{X,a,b}P^{-1} = A_{X,1_k,b'}$ . So, without loss of generality, we may consider this case, where  $a = 1_k$ .

Write our basis of  $\mathbb{C}^m$  as  $e_{ij}$  and our basis of  $\mathbb{C}^n$  as  $f_j$ . Let  $i_0, r$  be such that  $a = e_{i_0, \lambda_{i_0} - r + 1}$ . Let  $b' = A^{n+r}e_{i_0, \lambda_{i_0}} = b + e_{i_0, \lambda_{i_0} - r - n}$ . Let  $b'_{ij}$  be the projection of  $b'$  onto  $e_{ij}$ . For each  $i$ , let  $p_i = \max\{j : b'_{ij} \neq 0\}$ . Then for each  $i \neq i_0$ , let  $q_i = \lambda_i - p_i$ . Let  $q_{i_0} = \lambda_{i_0} - r - p_{i_0}$ . Normally we index  $e_{ij}$  so that  $\lambda_i$  is nonincreasing with  $i$ . Now, it will be more convenient to assume our indices are such that  $q_i$  is nonincreasing, so we will do that.

We have two cases, which will closely resemble the cases we had when  $X$  was a single Jordan block.

#### 9.3.1 First case: $n + \max_i p_i \geq q_{i_0} + p_{i_0}$

We list the chains in (roughly) decreasing order. First, for each  $i$  such that  $q_i \geq n + r$ , we take the chain of length  $p_i + q_i$  beginning with  $e_{i, p_i + q_i}$ .

Now we handle  $e_{i_0, \lambda_{i_0}}$ . Let  $P = \max_{i: q_i < n+r} p_i$ . Set

$$v = e_{i_0, \lambda_{i_0}} - (A^{r+n+P}e_{i_0, \lambda_{i_0}} \gg r + n + P).$$

Note that  $A^{r+n}e_{i_0, \lambda_{i_0}} = b'$ , so  $A^{r+n+P}e_{i_0, \lambda_{i_0}} = A^P b' = b' \ll P$ . Also note that if we shift  $b'$  left  $P$  times, we zero out all the rows  $i$  where  $q_i < n + r$ . This ensures that the operation of shifting  $b' \ll P$  right  $r + n + P$  times is invertible by applying  $A$ , and thus shifting left,  $r + n + P$  times. So, we take the chain of length  $r + n + P$  beginning with  $v$ . How do we know this is linearly independent from what came before? Something to do with the condition on  $n$ ...

Now, we handle the  $i$  with  $q_i < n + r$ . If  $p_i \leq \max_{k>i} p_k$ , then we take



the chain beginning with  $e_{i,p_i+q_i}$  of length  $p_i + q_i$ . Otherwise, we set

$$v_i = A^{n+r-q_i} e_{i_0, \lambda_{i_0}} - \sum_{k=1}^{p_i} b'_{i,k} e_{i,k+q_i}.$$

Note that  $A^{q_i} v_i$  is just  $b'$  with row  $i$  zeroed out. Then,  $A^{q_i + \max_{k>i} p_k} v_i$  will have, in addition, rows  $k$  zeroed out, for all  $k > i$ . This ensures that shifting  $A^{q_i + \max_{k>i} p_k} v_i$  right  $q_i + \max_{k>i} p_k$  times will be inverted by applying  $A$ , and thus shifting left,  $q_i + \max_{k>i} p_k$  times. So, we take the chain beginning with  $v_i - (A^{q_i + \max_{k>i} p_k} v_i >> q_i + \max_{k>i} p_k)$ , which has length  $q_i + \max_{k>i} p_k$ .

TODO: where did I use the fact that  $n + \max_i p_i \geq q_{i_0} + p_{i_0}$ ?

### 9.3.2 Second case: $n + \max_i p_i \leq q_{i_0} + p_{i_0}$

Again we list the chains in (roughly) decreasing order. Note that, since  $n + \max_i p_i \leq q_{i_0} + p_{i_0}$ , in particular we have  $n \leq q_{i_0}$ , and  $n \leq q_i$  for all  $i \leq i_0$ . Now, for all  $i \neq i_0$  with  $q_i \geq n$ , take the chain of length  $p_i + q_i$  starting with  $e_{i,p_i+q_i}$ . Then, take the chain starting with  $e_{i_0, \lambda_{i_0}}$ , of length  $\lambda_{i_0}$ .

Let  $i_1 = \min\{i : n > q_i\}$ . Let  $P = \max_{i \geq i_1} p_i$ . Then take the chain beginning with  $f_1 - (A^{n+p} f_1 >> n + p)$ , of length  $n + P$ .

Finally, we handle  $i \geq i_1$ . (In this case  $q_i < n$ .) If  $p_i \leq \max_{k>i} p_k$ , then take the chain beginning with  $e_{i,p_i+q_i}$  of length  $p_i + q_i$ . Otherwise, set

$$v_i = A^{n-q_i} f_1 - \sum_{k=1}^{p_i} b'_{i,k} e_{i,q_i+k}.$$

Note that  $A^{q_i} v_i$  is just  $b'$  with row  $i$  zeroed out. Now, we take the chain beginning with  $v_i - (A^{q_i + \max_{k>i} p_k} v_i >> q_i + \max_{k>i} p_k)$ , of length  $q_i + \max_{k>i} p_k$ .

TODO: where did I use the fact that  $n + \max_i p_i \leq q_{i_0} + p_{i_0}$ ?

## 9.4 When $X$ is all zeroes

In this case we have  $A(y, z) = (z_n b, (a \cdot y, z_1, \dots, z_{n-1}))$ , and the condition becomes

$$\sum_i b_i a_i = 0.$$

In this case we will be able to write down explicitly the irreducible components of  $F := \{(V, a, b) : \forall i. A_{X,a,b} V_i \subseteq V_i\} \cong \pi^{-1}(X)$ . For any nonnegative

$\delta_0, \delta_1, \dots, \delta_n, \delta_{n+1}$  summing to  $m$ , define the corresponding sequence  $i_0 = \delta_0$ ,  $i_n = \delta_{n+1} + i_n$ , and for  $j \in \{1, \dots, n\}$ ,  $i_j = i_{j-1} + 1 + \delta_j$ . Let  $E$  be the span of the  $e_i$ 's, and let  $E' = \{(x, 0) \in E : x \cdot a = 0\}$ , where the dot is the  $m$ -dimensional dot product. Then define  $F_\delta$  as the set of  $(V, a, b) \in F$  such that

- $b \in V_{i_0} \subseteq E'$
- for all  $j \in \{1, \dots, n\}$ , we have  $f_j \in V_{i_j} \subseteq E' + \langle f_1, \dots, f_j \rangle$

I claim that the  $F_\delta$ 's are the irreducible components of  $F$ . To begin, I show that their union is  $F$ .

**Lemma 9.3.** *Let  $(V, a, b) \in \mathcal{F}$ . Write  $f_0 = b$ , and  $F = \langle f_0, f_1, \dots, f_n \rangle$ . For each  $i$ , either  $F \subseteq V_i$ , or else there exists  $j$  such that  $e_j \notin V_i$ , but  $\langle e_0, \dots, e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_1, \dots, e_j \rangle$ .*

*Proof.* For  $i = 0$ , we may take  $j = 0$ . Now assume the statement holds for  $i$ , and we will prove it for  $i + 1$ . If  $F \subseteq V_i$ , then  $F \subseteq V_{i+1}$ , and we are done.

So, suppose there is  $j$  such that  $e_j \notin V_i$ , but  $\langle e_0, \dots, e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_0, \dots, e_j \rangle$ . We have two cases: either  $V_{i+1} = V_i + \langle e_j \rangle$ , or not.

- If so, then either  $j = n$ , in which case  $F \subseteq V_{i+1}$ , or else  $j \neq n$ , in which case  $e_{j+1} \notin V_{i+1}$ , but  $\langle e_0, \dots, e_j \rangle \subseteq V_{i+1} \subseteq E' + \langle e_0, \dots, e_{j+1} \rangle$ .
- If not, then  $e_j \notin V_{i+1}$ . Let  $v_{i+1}$  be so that  $V_{i+1} = V_i + \langle v_{i+1} \rangle$ . I just need to show that  $v_{i+1} \in Y + \langle e_1, \dots, e_j \rangle$ . It suffices to show that  $v_{i+1}^\top e_k = 0$  for  $k > j$ . And to do this, it suffices to show that  $(Av_{i+1})^\top e_{k-1} = 0$  for  $k > j$ .

Note that  $Av_{i+1} \in V_i \cap A\mathbb{C}^{2n} \subseteq \langle e_1, \dots, e_j \rangle$ . So, for  $k > j + 1$  it is clear that  $(Av_{i+1})^\top e_{k-1} = 0$ . Now, suppose for contradiction that  $(Av_{i+1})^\top e_j \neq 0$ . Then  $Av_{i+1}$  is linearly independent of  $e_1, \dots, e_{j+1}$ . Since  $Av_{i+1} \in \langle e_1, \dots, e_j \rangle$ , it follows that  $e_j \in \langle e_1, \dots, e_{j-1}, Av_{i+1} \rangle \subseteq V_i$ , a contradiction.

□

**Corollary 9.4.**  $\mathcal{F} = \bigcup_\delta \mathcal{F}_\delta$

*Proof.* Let  $(V, a, b) \in F$ . Let  $i_0 = \min\{i : b \in V_i\}$ , and let  $i' = \max\{i : V_i \subseteq E'\}$ . We want that  $b \in V_{i_0} \subseteq E'$ , so we want that  $i_0 \leq i'$ . For contradiction, suppose  $i' < i_0$ .

□

**Lemma 9.5.** *Each  $\mathcal{F}_\delta$  is a closed subvariety of  $\mathcal{F}$ .*

*Proof.* □

**Lemma 9.6.** *Each  $\mathcal{F}_\delta$  is irreducible of dimension  $m$ .*

*Proof.* Let  $\mathcal{E}$  be the variety of partial flags of  $E$  of shape  $(\delta_0, \dots, \delta_{n+1})$ . Define  $g : \mathcal{F}_\delta \rightarrow \mathcal{E}$  by

$$(V, a, b) \mapsto 0 \subseteq V_{i_0} \cap E \subseteq V_{i_1} \cap E \subseteq \dots \subseteq V_{i_{n+1}} \cap E = E.$$

It is clear that  $g$  is surjective, as the definition of  $\mathcal{F}_\delta$  places no restriction on the intersections  $V_i \cap E$ . Now let's look more closely at the fibers  $g^{-1}(U)$ .

First let's find the flags  $V$  such that there exist  $a, b$  with  $g(V, a, b) = U$ . We see that, for instance,  $V_{i_0} \cap E = U_1$ . In fact  $V_{i_0} \subseteq E$ , so  $V_{i_0} = U_1$ . But we are free to choose the vector spaces between 0 and  $V_{i_0}$  however we wish, so we get some degrees of freedom like  $\mathcal{F}_{\delta_0}$ , the complete flag variety on  $\mathbb{C}^{\delta_0} \cong V_{i_0}/0$ . Similarly, for every  $j = 1, \dots, n+1$ , we can choose the vector spaces between  $V_{i_{j-1}}$  and  $V_{i_j}$  arbitrarily, so we get degrees of freedom like  $\mathcal{F}_{\delta_{i_j}}$ , the complete flag variety on  $\mathbb{C}^{\delta_{i_j}} \cong V_{i_j}/V_{i_{j-1}}$ . Finally, to meet the constraint of  $(V, a, b)$  being in  $\mathcal{F}_\delta$ , we can choose any  $b \in U_1$  and any  $a$  such that  $\bar{a} \in U_n^\perp$ . Thus we get an isomorphism

$$g^{-1}(U) \cong \mathcal{F}_{\delta_0} \times \dots \times \mathcal{F}_{\delta_{n+1}} \times \mathbb{C}^{\delta_0} \times \mathbb{C}^{\delta_{n+1}}.$$

□

**Theorem 9.7.** *The  $\mathcal{F}_\delta$ 's are the irreducible components of  $\mathcal{F}$ .*

## 9.5 In the case that $X$ is a Jordan block

This case seems harder to work with explicitly than the case that  $X$  is zero, so our strategy is to reuse

## 10 A different variety

Define  $R = \{(X, \mathfrak{b}_1, \mathfrak{b}_2) : X \in \mathfrak{b}_1 \wedge u(X) \in \mathfrak{b}_2\} \subseteq \mathcal{S}'_{m,n} \times \mathcal{B}_{m+n} \times \mathcal{B}_m$ . We can obtain a subvariety of  $R$  by requiring that  $u(X)$  is in some fixed similarity class. (TODO: why is this a subvariety? Is it? Is this even the right way of explaining the significance of the  $m^2$ ?) We expect that each of these subvarieties is an irreducible component of dimension  $m^2$ . We will verify these things using our previous computations of springer fibers.

## 10.1 TODO

- Why are SOn flags what they are.

## References

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