

1 Definition of one thing

Let $G \subseteq \mathrm{GL}_m(\mathbb{C})$ be a connected semisimple Lie group, and let $\mathfrak{g} \subseteq \mathfrak{gl}_m(\mathbb{C})$ be its Lie algebra. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the subset consisting of nilpotent elements. Let \mathcal{B} be the variety of Borel subalgebras of \mathfrak{g} . Let $\tilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$. Let $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ be the projection onto the second coordinate. For $n \in \mathcal{N}$, we call $\pi^{-1}(n)$ the *Springer fiber at n* .

2 Slodowy slice

A basis for $\mathfrak{sl}_2(\mathbb{C})$ is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra \mathfrak{g} , and a homomorphism $\phi : \mathfrak{sl}_2 \rightarrow \mathfrak{g}$ sending (e', h', f') to (e, h, f) , we say that (e, h, f) is an \mathfrak{sl}_2 -triple. Observe that e, f must be nilpotent, and h must be Cartan (???). If \mathfrak{g} is semisimple, then given any nilpotent $e \in \mathfrak{g}$, the Jacobson-Morozov theorem [1, 3.7.1] says that there exist $h, f \in \mathfrak{g}$ such that (e, h, f) is an \mathfrak{sl}_2 -triple.

Given (e, h, f) , we define the *Slodowy slice at e* as $\mathcal{S}_e := e + \ker \mathrm{ad}_f$. By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent $e \in \mathfrak{g}$, when \mathfrak{g} is semisimple.

2.1 Finding an \mathfrak{sl}_2 -triple in a simple case

Let

$$e = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \in \mathfrak{sl}_n,$$

where there are $n - 1$ ones. We will find the Slodowy slice at e .

As a first step, we will find an \mathfrak{sl}_2 -triple with e as its first element. Note that $[h', e'] = 2e'$, and $[e', f'] = h'$, and $[h', f'] = -2f'$. Thus e, h, f must obey the same relations. In particular, $he - eh = 2e$. The matrix eh is h shifted down one, and he is h shifted left one. Thus, $2e_{ij} = (he - eh)_{ij} =$

$h_{i,j+1} - h_{i-1,j}$. We can use this to show that $h_{ij} = 0$ when $i \neq j$. Then we can use it to show that $h_{ii} = h_{i-1,i-1} + 2$, so that $h_{ii} = h_{11} + 2(i-1)$.

Similarly, from $[e, f] = h$ we get that $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$. We can use this to show that $f_{ij} = 0$ when $j \neq i+1$. Then we can use it to show that $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$, that $f_{1,2} = -h_{1,1}$, and that $f_{n-1,n} = h_{n,n}$. From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^n h_{ii} \implies \sum_i h_{ii} = 0.$$

Oops, I didn't actually need to prove that, since $h \in \mathfrak{sl}_n$... it's nice to know that it's the unique solution in \mathfrak{gl}_n though? This shows that $h_{11} = n-1, h_{22} = n-3, \dots, h_{nn} = 1-n$. So we have determined h ; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for f in terms of h to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & \\ & 0 & (1-n) + (3-n) & & \\ & & \ddots & \ddots & \\ & & & 0 & (1-n) + \dots + (n-1) \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1(1-n) & & & \\ & 0 & 2(2-n) & & \\ & & 0 & (n-2)(-2) & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

2.2 Finding an \mathfrak{sl}_2 -triple in a slightly less simple case

Now, let

$$E = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{sl}_{m+n},$$

with e as in the previous subsection. We will find H, F so that (E, H, F) is an \mathfrak{sl}_2 -triple. Writing $F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & f \end{pmatrix}$, and similarly for H , we have

$$2E = [H, E] = \begin{pmatrix} 0 & H_{12}e \\ 0 & he \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ eH_{21} & eh \end{pmatrix},$$

$$H = [E, F] = \begin{pmatrix} 0 & 0 \\ eF_{21} & ef \end{pmatrix} - \begin{pmatrix} 0 & F_{12}e \\ 0 & fe \end{pmatrix}.$$

From these we observe that (e, h, f) must also be an \mathfrak{sl}_2 -triple, so h, f must be as in the previous subsection. We also see that $H_{11} = 0$. Recalling that left multiplication by e is a down-shift, and right multiplication is a left-shift, we see that H_{12} is all zeroes except for the leftmost column, and H_{21} is all zeroes except for the bottom row. Then our final constraint is that

$$-2F = [H, F] = \begin{pmatrix} H_{12}F_{21} & H_{12}F_{22} \\ H_{21}F_{11} + H_{22}F_{21} & H_{21}F_{12} + H_{22}F_{22} \end{pmatrix} - \begin{pmatrix} F_{12}H_{21} & F_{11}H_{12} + F_{12}H_{22} \\ F_{22}H_{21} & F_{21}H_{12} + F_{22}H_{22} \end{pmatrix}.$$

Now $H_{12} = F_{12}e$, and $H_{21} = eF_{21}$, from the equation $H = [E, F]$. Substituting in the equation above then,

$$\begin{aligned} -2F &= \begin{pmatrix} F_{12}eF_{21} & F_{12}ef \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} F_{12}eF_{21} & F_{11}F_{12}e + F_{12}h \\ feF_{21} & F_{21}F_{12}e + fh \end{pmatrix} = \\ &= \begin{pmatrix} 0 & F_{12}(fe + h) \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e + F_{12}h \\ (ef - h)F_{21} & F_{21}F_{12}e + fh \end{pmatrix} = \\ &= \begin{pmatrix} 0 & F_{12}fe \\ eF_{21}F_{11} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e \\ efF_{21} & F_{21}F_{12}e + fh \end{pmatrix}. \end{aligned}$$

Now we see that $F_{11} = 0$, and consequently that $F_{12} = F_{21} = 0$ as well. This shows that $H_{12} = H_{21} = 0$. We conclude that H and F just have h and f in their bottom-right corners, respectively.

2.3 Finding the previous Slodowy slices

First we find $\ker \text{ad}_f$.

To find the Slodowy slice associated to the previous \mathfrak{sl}_2 -triple (E, H, F) , we just need to find $\ker \text{ad}_F$. We have

$$[F, X] = \begin{pmatrix} 0 & 0 \\ fX_{21} & fX_{22} \end{pmatrix} - \begin{pmatrix} 0 & X_{12}f \\ 0 & X_{22}f \end{pmatrix}.$$

3 Definition of springer fiber at slodowy slice

Let $\mathcal{N}_m \subseteq \mathfrak{gl}_m(\mathbb{C})$ be the subset consisting of nilpotent elements. For $X \in \mathcal{N}$ and $a, b \in \mathbb{C}^m$, let

$$A_{X,a,b} = \left(\begin{array}{c|ccc} X & & & b \\ \hline - & a & - & \\ \hline & & 0 & \\ & & 1 & 0 \\ & & & 1 & \ddots \\ & & & & \ddots & 0 \\ & & & & & 1 & 0 \end{array} \right) \in \mathfrak{gl}_{m+n}(\mathbb{C}).$$

Let \mathcal{M} be the variety of Borel subalgebras of $\mathfrak{gl}_{m+n}(\mathbb{C})$. Let

$$\tilde{\mathcal{N}} = \{(\mathfrak{m}, a, b, X) : A_{X,a,b} \in \mathfrak{m}\} \subseteq \mathcal{M} \times \mathbb{C}^m \times \mathbb{C}^m \times \mathcal{N}.$$

Let $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ be the projection onto the last coordinate. For $X \in \mathcal{N}$, we call $\pi^{-1}(X)$ the *other Springer fiber at X*.

4 Finding the Springer fibers of section 2

We have $\mathcal{M} = \{AHA^{-1} : A \in \text{GL}_{m+n}(\mathbb{C})\}$, where $H \subseteq \mathfrak{gl}_{m+n}(\mathbb{C})$ is the set of upper triangular matrices. We say a map $X : \mathfrak{gl}_{m+n} \rightarrow \mathfrak{gl}_{m+n}$ preserves a flag $V_0 \subseteq \cdots \subseteq V_{m+n}$ if $XV_i \subseteq V_i$ for each i . Let $E_0 \subseteq \cdots \subseteq E_{m+n}$ be the standard flag of \mathbb{C}^{m+n} . Since H is the set of X which preserve E ,

$$\mathcal{M} = \{\{X : \forall i. X(AE_i) \subseteq AE_i\} : A \in \text{GL}_n(\mathbb{C})\}.$$

So, for $X \in \mathcal{N}$,

$$\pi^{-1}(X) \cong \{(V, a, b) : \forall i. A_{X,a,b}V_i \subseteq V_i\}.$$

In this section we will find the irreducible components of $\pi^{-1}(X)$. Since $\pi^{-1}(X) \cong \pi^{-1}(AXA^{-1})$ for any invertible A , we will assume that X is in Jordan normal form.

Let λ be the shape of X , and let $(e_{ij})_{i \leq r, j \leq \lambda_i}$ be a Jordan basis for X , with $Xe_{ij} = e_{i,j-1}$. Let f_1, \dots, f_n be the standard basis for \mathbb{C}^n , with $A_{X,a,b}f_i = f_{i+1}$.

4.1 A necessary condition for $A_{X,a,b}$ to be nilpotent

Suppose $A := A_{X,a,b}$ is nilpotent. Define the *height* of a vector $v \in \mathbb{C}^{m+n}$ as the smallest $k \geq 0$ such that $A_{X,a,b}^k v = 0$. Clearly, for $i \leq j$, we have that the height of f_i is geq the height of f_j . Thus, for each $k \geq 0$, we must have $A_{X,a,b}^k f_n \in \langle e_{ij} \rangle_{ij}$. We have $Af_n = \sum_{ij} b_{ij} e_{ij}$. Then $A^2 f_n = \sum_{ij} b_{ij} (e_{i,j-1} + a_{ij} f_1)$, so we must have $\sum_{ij} b_{ij} a_{ij} = 0$. Similarly, since $A^3 f_n \in \langle e_{ij} \rangle$, we see that $\sum_{ij} b_{ij} a_{i,j-1} = 0$. Continuing in this way, we obtain that for all $k \geq 0$,

$$\sum_{ij} b_{ij} a_{i,j-k} = 0. \quad (1)$$

In fact this is in fact a sufficient condition for $A_{X,a,b}$ to be nilpotent, as can be seen from the characteristic polynomial of $A_{X,a,b}$. However, we will not do this calculation here, instead just showing that it is sufficient by giving a Jordan basis.

4.2 When X is all zeroes

In this case we have $A(y, z) = (z_n b, (a \cdot y, z_1, \dots, z_{n-1}))$, and the condition becomes

$$\sum_i b_i a_i = 0.$$

In this case we will be able to write down explicitly the irreducible components of $F := \{(V, a, b) : \forall i. A_{X,a,b}V_i \subseteq V_i\} \cong \pi^{-1}(X)$. For any nonnegative $\delta_0, \delta_1, \dots, \delta_n, \delta_{n+1}$ summing to m , define the corresponding sequence $i_0 = \delta_0$, $i_n = \delta_{n+1} + i_n$, and for $j \in \{1, \dots, n\}$, $i_j = i_{j-1} + 1 + \delta_j$. Let E be the span of the e_i 's, and let $E' = \{(x, 0) \in E : x \cdot a = 0\}$, where the dot is the m -dimensional dot product. Then define F_δ as the set of $(V, a, b) \in F$ such that

- $b \in V_{i_0} \subseteq E'$
- for all $j \in \{1, \dots, n\}$, we have $f_j \in V_{i_j} \subseteq E' + \langle f_1, \dots, f_j \rangle$

I claim that the F_δ 's are the irreducible components of F . To begin, I show that their union is F .

Lemma 4.1. *Let $(V, a, b) \in \mathcal{F}$. Write $f_0 = b$, and $F = \langle f_0, f_1, \dots, f_n \rangle$. For each i , either $F \subseteq V_i$, or else there exists j such that $e_j \notin V_i$, but $\langle e_0, \dots, e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_1, \dots, e_j \rangle$.*

Proof. For $i = 0$, we may take $j = 0$. Now assume the statement holds for i , and we will prove it for $i + 1$. If $F \subseteq V_i$, then $F \subseteq V_{i+1}$, and we are done.

So, suppose there is j such that $e_j \notin V_i$, but $\langle e_0, \dots, e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_0, \dots, e_j \rangle$. We have two cases: either $V_{i+1} = V_i + \langle e_j \rangle$, or not.

- If so, then either $j = n$, in which case $F \subseteq V_{i+1}$, or else $j \neq n$, in which case $e_{j+1} \notin V_{i+1}$, but $\langle e_0, \dots, e_j \rangle \subseteq V_{i+1} \subseteq E' + \langle e_0, \dots, e_{j+1} \rangle$.
- If not, then $e_j \notin V_{i+1}$. Let v_{i+1} be so that $V_{i+1} = V_i + \langle v_{i+1} \rangle$. I just need to show that $v_{i+1} \in Y + \langle e_1, \dots, e_j \rangle$. It suffices to show that $v_{i+1}^\top e_k = 0$ for $k > j$. And to do this, it suffices to show that $(Av_{i+1})^\top e_{k-1} = 0$ for $k > j$.

Note that $Av_{i+1} \in V_i \cap A\mathbb{C}^{2n} \subseteq \langle e_1, \dots, e_j \rangle$. So, for $k > j + 1$ it is clear that $(Av_{i+1})^\top e_{k-1} = 0$. Now, suppose for contradiction that $(Av_{i+1})^\top e_j \neq 0$. Then Av_{i+1} is linearly independent of e_1, \dots, e_{j+1} . Since $Av_{i+1} \in \langle e_1, \dots, e_j \rangle$, it follows that $e_j \in \langle e_1, \dots, e_{j-1}, Av_{i+1} \rangle \subseteq V_i$, a contradiction.

□

Corollary 4.2. $\mathcal{F} = \bigcup_\delta \mathcal{F}_\delta$

Proof. Let $(V, a, b) \in F$. Let $i_0 = \min\{i : b \in V_i\}$, and let $i' = \max\{i : V_i \subseteq E'\}$. We want that $b \in V_{i_0} \subseteq E'$, so we want that $i_0 \leq i'$. For contradiction, suppose $i' < i_0$.

□

Lemma 4.3. *Each \mathcal{F}_δ is a closed subvariety of \mathcal{F} .*

Proof.

□

Lemma 4.4. *Each \mathcal{F}_δ is irreducible of dimension m .*

Proof. Let \mathcal{E} be the variety of partial flags of E of shape $(\delta_0, \dots, \delta_{n+1})$. Define $g : \mathcal{F}_\delta \rightarrow \mathcal{E}$ by

$$(V, a, b) \mapsto 0 \subseteq V_{i_0} \cap E \subseteq V_{i_1} \cap E \subseteq \dots \subseteq V_{i_{n+1}} \cap E = E.$$

It is clear that g is surjective, as the definition of \mathcal{F}_δ places no restriction on the intersections $V_i \cap E$. Now let's look more closely at the fibers $g^{-1}(U)$.

First let's find the flags V such that there exist a, b with $g(V, a, b) = U$. We see that, for instance, $V_{i_0} \cap E = U_1$. In fact $V_{i_0} \subseteq E$, so $V_{i_0} = U_1$. But we are free to choose the vector spaces between 0 and V_{i_0} however we wish, so we get some degrees of freedom like \mathcal{F}_{δ_0} , the complete flag variety on $\mathbb{C}^{\delta_0} \cong V_{i_0}/0$. Similarly, for every $j = 1, \dots, n+1$, we can choose the vector spaces between $V_{i_{j-1}}$ and V_{i_j} arbitrarily, so we get degrees of freedom like $\mathcal{F}_{\delta_{i_j}}$, the complete flag variety on $\mathbb{C}^{\delta_{i_j}} \cong V_{i_j}/V_{i_{j-1}}$. Finally, to meet the constraint of (V, a, b) being in \mathcal{F}_δ , we can choose any $b \in U_1$ and any a such that $\bar{a} \in U_n^\perp$. Thus we get an isomorphism

$$g^{-1}(U) \cong \mathcal{F}_{\delta_0} \times \dots \times \mathcal{F}_{\delta_{n+1}} \times \mathbb{C}^{\delta_0} \times \mathbb{C}^{\delta_{n+1}}.$$

□

Theorem 4.5. *The \mathcal{F}_δ 's are the irreducible components of \mathcal{F} .*

4.3 When X is a Jordan block

Suppose X is a single Jordan block of size m . In this case we just write the basis of \mathbb{C}^m as e_1, \dots, e_m , and the above condition on a and b simply becomes

$$\forall k \geq 0. \sum_{i=k+1}^m b_i a_{i-k} = 0. \quad (2)$$

This condition can be simplified even more.

Lemma 4.6. *The condition (1) holds iff there exist nonnegative m_1, m_2, m_3 , a_1, \dots, a_{m_3} , and b_1, \dots, b_{m_3} satisfying the following conditions.*

- $m = m_1 + m_2 + m_3$
- $a = (0, 0, \dots, 0, a_1, \dots, a_{m_3})$

- $b = (b_1, \dots, b_{m_1}, 0, 0, \dots, 0)$
- If $m_1 \neq 0$, then $a_{m_1} \neq 0$
- If $m_3 \neq 0$, then $b_1 \neq 0$

Proof. If a and b are of this form, then for every $k \geq 0$ we have $b_i a_{i-k} = 0$, so clearly the condition holds.

Now suppose (1) holds. If a or b is zero, this is trivial. Otherwise, let $m_1 = \max\{i : a_i \neq 0\}$, and let $m_3 = \min\{i : b_i \neq 0\}$. We just need to show that $m_1 < m_3$. For contradiction, suppose $m_1 - m_3$ is nonnegative. Then by (1), $0 = \sum_i b_i a_{i-(m_1-m_3)} = b_1 a_{m_1}$, contradicting that b_1 and a_{m_1} are nonzero. \square

4.4 In the case of general X

In the case where X was a single Jordan block, it was helpful to have the condition that all the nonzero a 's were to the right of all the nonzero b 's. In the general case, it is not clear that such a thing should be true. And indeed it is not. However, we will see that we can change basis of \mathbb{C}^m so that a similar thing is true.

Let $c_{i',j'}^i \in \mathbb{C}$ be some arbitrary coefficients. Given our Jordan basis e_{ij} of \mathbb{C}^m , we can define a new Jordan basis e'_{ij} of \mathbb{C}^m by

$$e'_{i,\lambda_i} = e_{i,\lambda_i} + \sum_{\{(i',j') : i' \geq i\} \setminus \{(i,j)\}} c_{i',j'}^i e_{i',j'}.$$

Then we define

$$e'_{i,j} = X^{\lambda_i-j} e'_{i,\lambda_i} = e_{ij} + \sum_{\{(i',j') : i' \geq i\} \setminus \{(i,j)\}} c_{i',j'}^i e_{i',j'-(\lambda_i-j)}.$$

Let us calculate the new value a' of a in this new basis. That is, we want the values a'_{ij} that satisfy $Ae'_{ij} = e'_{i,j-1} + a'_{ij}f_1$. Looking at the definition of e'_{ij} above, we see that

$$a'_{ij} = a_{ij} + \sum_{\{(i',j') : i' \geq i\} \setminus \{(i,j)\}} c_{i',j'}^i a_{i',j'-(\lambda_i-j)}.$$

Now, for each i , choose the coefficients $c_{i',j'}^i$ to maximize the value $\max\{j : (a'_{i,1}, \dots, a'_{i,j}) = 0\}$. In other words, we choose $c_{i',j'}^i$ to zero out the largest possible prefix of $(a_{i,1}, \dots, a_{i,\lambda_i})$.

Lemma 4.7. *If the condition of Equation (1) on a' and b' holds, then there exist nonnegative $p_i, q_i, r_i, \alpha_{i,j}, \beta_{i,j}$ such that for each i , the following conditions are satisfied.*

- $\lambda_i = p_i + q_i + r_i$
- $(a'_{i,1}, \dots, a'_{i,\lambda_i}) = (0, 0, \dots, 0, \alpha_{i,1}, \dots, \alpha_{i,r_i})$
- $(b'_{i,1}, \dots, b'_{i,\lambda_i}) = (\beta_{i,1}, \dots, \beta_{i,p_i}, 0, 0, \dots, 0)$
- If $r_i \neq 0$, then $\alpha_{i,1} \neq 0$
- If $p_i \neq 0$, then $\beta_{i,p_i} \neq 0$

Proof. We begin with the first row of λ , namely $i = 1$. Let $k = \min\{j : a'_{1,j} \neq 0\}$ and $l = \max\{j : b'_{1,j} \neq 0\}$. I just need to show that $l < k$. Suppose not. Then $(l - k)$ is nonnegative, and by Equation (1) we have

$$0 = \sum_{ij} b'_{ij} a'_{j-(l-k)} = b'_{1,l} a'_{1,k} + \sum_{i' \geq 1, (i', j') \neq (1, \lambda_1)} b'_{i'j} a'_{j-(l-k)}.$$

This suggests defining

$$e''_{1,\lambda_1} = b'_{1,l} e'_{1,\lambda_1} + \sum_{i' \geq 1, (i', j') \neq (1, \lambda_1)} b'_{i', j' - (\lambda_1 - l)} e'_{i'j'}.$$

Then for $j \leq k$,

$$\begin{aligned} a''_{1,j} &= b'_{1,l} a'_{1,j} + \sum_{i' \geq 1, (i', j') \neq (1, \lambda_1)} b'_{i', j' - (\lambda_1 - l)} a'_{i', j' - (\lambda_1 - j)} = \\ &= \sum_{i', j'} b'_{i', j' - (\lambda_1 - l)} a'_{i', j' - (\lambda_1 - j)} = \sum_{i', j'} b'_{i', j'} a'_{i', (j' + (\lambda_1 - l)) - (\lambda_1 - j)} = \sum_{i', j'} b'_{i', j'} a'_{i', j' - (l - j)}. \end{aligned}$$

This last thing is zero by Equation (1), using that $l - j \geq 0$. But this contradicts the definition of the $c^1_{i'j'}$, because we made it so that $(a''_{1,1}, \dots, a''_{1,k}) = 0$. \square

4.5 TODO

- Why are SOn flags what they are.

References

- [1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.