Title

UROP+ Final Paper, Summer 2024
Owen Conoly
Mentor: Haoshuo Fu
Project Suggested by: TODO: also Haoshuo?

Abstract

1 Introduction

Let $\mathfrak{g} = \mathfrak{gl}_m \times \mathfrak{gl}_{m+n}$, with $n > 0$. Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N} \hookrightarrow \mathfrak{g}$ be the Springer resolution.

Let (e, h, f) be the principal $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n$. We have an embedding $\mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_m \times \mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_{m+n}$. Let (E, H, F) be the $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_{m+n}$ which is the image of (e, h, f) via this embedding. Let S be the Slodowy slice $E + \mathfrak{z}_{\mathfrak{gl}_m}(F)$. We have a map $\mathfrak{gl}_{m+n} \to \mathfrak{gl}_m$ given by coordinate projection. Restricting to S, we obtain $f_1 : S \to \mathfrak{gl}_m$. (It turns out that this map is surjective.) Then, we obtain $p_1 : S \to \mathfrak{g}$ by $x \mapsto (x, f_1(x))$. Taking the map $p_1 : S \to \mathfrak{g}$ and the Springer resolution $\widetilde{\mathcal{N}} \to \mathfrak{g}$, we obtain a fibred product $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$.

In this paper we study $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$. As a step towards studying $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$, we consider the map $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}} \to \mathfrak{gl}_m$ given by taking the map to $\mathfrak{g}$ and then projecting to $\mathfrak{gl}_m$. The main section of this paper will study the fibers of this map. Given $X \in \mathcal{N}_m \subseteq \mathfrak{gl}_m$, we call the fiber at X the *n-Slodowy-slice Springer fiber at X*.

Section 2 reviews some preliminary material. Section 3 finds the unique (up to similarity) principal $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n$. Section 4 embeds this $\mathfrak{sl}_2$-triple into $\mathfrak{gl}_{m+n}$ as described above. We compute the Slodowy slice and end up with a nice description of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$.

Section 5 discusses how to reduce the problem of finding the irreducible components of an n-Slodowy-slice Springer fiber at $X \in \mathcal{N}_m$ to an easier problem. Section 6 solves the easier problem. Section 7 finds the irreducible components of an n-Slodowy-slice Springer fiber. Section 8 applies the results of section 7 to find the irreducible components of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$. Section 9 discusses an interesting direction for future work. Finally, section 10 proves some linear algebra lemmas that were used in the paper.

1.1 Acknowledgment

TODO. UROP+ program and Haoshuo. MIT (provided funding)?

2 Preliminary Definitions and Facts

2.1 Conventions and Notations

We write $\mathrm{GL}_m, \mathfrak{sl}_m$ to denote $\mathrm{GL}_m(\mathbb{C}), \mathfrak{sl}_m(\mathbb{C})$, and so on. By J_m we refer to the nilpotent $m \times m$ Jordan block (which, by convention, has ones *below* the diagonal). Given a partition $\lambda = (\lambda_1, ..., \lambda_k)$, we write $J(\lambda)$ to denote the block matrix

$$\begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix}.$$

A partition is always indexed in nonincreasing order, even if it is defined differently. For example, if $\mu = (4, 6, 3)$, then $\mu_1 = 6, \mu_2 = 4, \mu_3 = 3$.

2.2 Springer fibers

Let $\mathfrak{g}$ be a Lie algebra. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the subset consisting of nilpotent elements. Let $\mathcal{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. Let $\widetilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$. Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the projection onto the second coordinate. We call this the *Springer resolution*. For $n \in \mathcal{N}$, we call $\pi^{-1}(n)$ the *Springer fiber at n*.

2.3 Springer fibers in $\mathfrak{gl}_m$

Now we let $\mathcal{N}_m$ be the set of nilpotent elements in $\mathfrak{gl}_m$, and $\mathcal{B}_m$ the variety of Borel subalgebras of $\mathfrak{gl}_m$. Let $H \subseteq \mathfrak{gl}_m$ be the subalgebra of upper triangular matrices. The variety of Borel subalgebras of $\mathfrak{gl}_m$ is $\mathcal{B} = \{gHg^{-1} : g \in \mathrm{GL}_m\}$. Thus, the Springer fiber at $X \in \mathcal{N}$ is

$$S_X = \{gHg^{-1} : X \in gHg^{-1}\}.$$

Definition 2.1. *A* flag (V_i) *of* $\mathbb{C}^m$ *is a sequence of subspaces*

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = \mathbb{C}^m,$$

where $\dim V_i = i$.

We say that $X \in \mathfrak{gl}_m$ *preserves* a flag (V_i) if $\forall i.\ XV_i \subseteq V_i$. Note that $X \in \mathcal{N}$ preserves a flag (V_i) if and only if $\forall i.\ XV_i \subseteq V_{i-1}$.

The simplest flag is the *standard flag* (E_i), where $E_i := \langle e_1, ..., e_i\rangle$. Note that the group H is exactly the subset of $\mathfrak{gl}_m$ which preserves E_i.

We can think of $\mathcal{B}$ as the set of flags of $\mathbb{C}^m$, via the correspondence

$$gHg^{-1} \leftrightarrow (gE_i).$$

Note that X preserves (gE_i) if and only if $X \in gHg^{-1}$. Thus, we may write the Springer fiber at $X \in \mathcal{N}$ in terms of flags, as

$$S_X = \{(gE_i) : X \in gHg^{-1}\} = \{(V_i) : \forall i.\ XV_i \subseteq V_{i-1}\}.$$

Theorem 2.2. *(Needs citation!) The irreducible components of the Springer fiber at $J(\mu)$ are in bijection with the standard Young tableaus of shape μ. Further, the irreducible components are equidimensional, of dimension* $\sum_{i<j} \min(\mu_i, \mu_j) = \sum_i (i-1)\mu_i$.

2.4 Slodowy Slices

A basis for $\mathfrak{sl}_2$ is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra $\mathfrak{g}$, and a homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ sending (e', h', f') to (e, h, f), we say that (e, h, f) is an $\mathfrak{sl}_2$-*triple*. If $\mathfrak{g}$ is semisimple, then given any nilpotent $e \in \mathfrak{g}$, the Jacobson-Morozov theorem [1, 3.7.1] says that there exist $h, f \in \mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple.

Given (e, h, f), we define the *Slodowy slice at* e as $\mathcal{S}_e := e + \ker \operatorname{ad}_f$. By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent $e \in \mathfrak{g}$, when $\mathfrak{g}$ is semisimple.

3 Principal $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n$

The unique nilpotent regular element of $\mathfrak{gl}_n$ (up to similarity) is J_n. In this section we use this to find that there is a unique principal $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n$. Write $e = J_n$.

Lemma 3.1. *There is exactly one way to choose $h, f \in \mathfrak{gl}_n$ so that (e, h, f) is an $\mathfrak{sl}_2$-triple.*

Proof. Note that $[h', e'] = 2e'$, and $[e', f'] = h'$, and $[h', f'] = -2f'$. Thus e, h, f must obey the same relations. In particular, $he - eh = 2e$. The matrix eh is h shifted down one, and he is h shifted left one. Thus, $2e_{ij} = (he - eh)_{ij} = h_{i,j+1} - h_{i-1,j}$. We can use this to show that $h_{ij} = 0$ when $i \neq j$. Then we can use it to show that $h_{ii} = h_{i-1,i-1} + 2$, so that $h_{ii} = h_{11} + 2(i-1)$.

Similarly, from $[e, f] = h$ we get that $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$. We can use this to show that $f_{ij} = 0$ when $j \neq i + 1$. Then we can use it to show that $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$, that $f_{1,2} = -h_{1,1}$, and that $f_{n-1,n} = h_{n,n}$. From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^{n} h_{ii} \implies$$

$$\sum_i h_{ii} = 0.$$

(That is, $h \in \mathfrak{sl}_n$.) This shows that $h_{11} = n-1, h_{22} = n-3, ..., h_{nn} = 1-n$. So we have determined h; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for f in terms of h to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & & \\ & 0 & (1-n)+(3-n) & & & \\ & & \ddots & \ddots & & \\ & & & 0 & (1-n)+\cdots+(n-1) \\ & & & & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1(1-n) & & & \\ & 0 & 2(2-n) & & \\ & & 0 & (n-2)(-2) & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

□

4 Finding $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$

4.1 Finding the Slodowy slice S

In the previous section we computed the principal $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{gl}_n$. Embedding this into $\mathfrak{gl}_{m+n}$ as described previously, we obtain

$$(E, H, F) = \left(\begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} \right).$$

Lemma 4.1. *$\mathfrak{z}_{\mathfrak{gl}_n}(f)$ is the set of upper-triangular $X \in \mathfrak{gl}_n$ such that for all i, j,*

$$X_{i+1,j+1} = \frac{j(n-j)}{i(n-i)} X_{ij}.$$

Proof. Let $X \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$. Looking at the definition of f from the previous section (and zero-padding the matrices), we see that $(fX)_{ij} = i(n-i)X_{i+1,j}$, and $(Xf)_{ij} = (j-1)(n-(j-1))X_{i,j-1}$. So,

$$\forall i, j \in \{1, ..., n\}.\ i(n-i)X_{i+1,j} = (j-1)(n-(j-1))X_{i,j-1}.$$

Taking $j = 1$, the condition above tells us that $\forall i \geq 2.\ X_{i,1} = 0$. Taking $j > 1$ and $i < n$, we obtain that

$$\forall i, j \in \{1, ..., n-1\}.\ X_{i+1,j+1} = \frac{j(n-j)}{i(n-i)} X_{ij}.$$

So, every $X \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$ is upper triangular and satisfies the condition above. Conversely, it is clear that for such X we have $fX = Xf$. $\square$

Lemma 4.2.

$$\mathfrak{z}_{\mathfrak{gl}_{m+n}}(F) = \left\{ \left(\begin{array}{c|c} X & \begin{array}{r} | \\ b \\ | \end{array} \\ \hline \text{— } a \text{ —} & \\ & Y \end{array} \right) : X \in \mathfrak{gl}_m; Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f); a, b \in \mathbb{C}^m \right\}.$$

Proof. Let $Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \in \mathfrak{z}_{m+n}(F)$. We have

$$\begin{pmatrix} 0 & Z_{12}f \\ 0 & Z_{22}f \end{pmatrix} = ZF = FZ = \begin{pmatrix} 0 & 0 \\ fZ_{21} & fZ_{22} \end{pmatrix}.$$

There is no restriction on Z_{11}. The condition $Z_{12}f = 0$ means that all but the last column of Z_{12} must be zero, and the condition $0 = fZ_{21}$ means that all but the first row of Z_{21} must be zero. And the condition $Z_{22}f = fZ_{22}$ means that $Z_{22} \in \mathfrak{z}_{\mathfrak{gl}_{m+n}}(f)$. $\square$

These lemmas provide an explicit characterization of the Slodowy slice $S = E + \mathfrak{z}_{\mathfrak{gl}_{m+n}}(F)$.

Corollary 4.3.

$$S = \left\{ \left(\begin{array}{c|c} X & \begin{array}{r} | \\ b \\ | \end{array} \\ \hline \text{— } a \text{ —} & \\ & e + Y \end{array} \right) : a, b \in \mathbb{C}^m, X \in \mathfrak{gl}_m, Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f) \right\}.$$

4.2 A description of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$

Recall that we have the map $S \to \mathfrak{g}$ given by $Z \mapsto (p(Z), Z)$, where $p : \mathfrak{gl}_{m+n} \to \mathfrak{gl}_m$ is the coordinate projection. We also have the Springer resolution $\widetilde{\mathcal{N}} \to \mathfrak{g}$. From these two maps we define the fibered product $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$.

To obtain an explicit description of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$, we will begin by finding the image of the projection $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}} \to S$. As the image of the Springer resolution is $\mathcal{N} = \mathcal{N}_m \times \mathcal{N}_{m+n}$, the image of the projection $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}} \to S$ is simply $S' = \{Z \in S : p(Z) \in \mathcal{N}_m, Z \in \mathcal{N}_{m+n}\}$.

Lemma 4.4.

$$S' = \left\{ A_{X,a,b} := \left(\begin{array}{c|ccccc} & & & & & | \\ X & & & & & b \\ & & & & & | \\ \hline — \quad a \quad — & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & & 1 & 0 \end{array} \right) \in \mathcal{N}_{m+n} : a, b \in \mathbb{C}^m, X \in \mathcal{N}_m \right\}.$$

Proof. As every element of $\mathfrak{z}_{\mathfrak{gl}_n}(f)$ is upper triangular, Corollary 10.10 tells us that if a matrix of the form given in Corollary 4.3 is nilpotent, and the upper-left block X is nilpotent as well, then $e + Y$ must be nilpotent. Then, again using that every element $Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$ is upper triangular, Lemma 10.8 tells us that if $e + Y$ is nilpotent, then $Y = 0$.

Hence every element of S' must simply have e in its bottom-right block. So, every element of S' is of the desired form. □

This is not a fully explicit characterization of S', since we don't say which choices of X and $a, b \in \mathbb{C}^m$ lead to $A_{X,a,b}$ being nilpotent. We could use Lemma 10.9 to find a necessary and sufficient condition on X, a, b; however, the description above of S' will be good enough for our purposes.

Corollary 4.5.

$$\begin{aligned} S \times_{\mathfrak{g}} \widetilde{\mathcal{N}} = \\ \{((X, A_{X,a,b}), ((X, A_{X,a,b}), \mathfrak{b})) : \mathfrak{b} \in \mathcal{B}, (X, A_{X,a,b}) \in \mathcal{N} \cap \mathfrak{b}\} \cong \\ \{(A_{X,a,b}, \mathfrak{b}) : \mathfrak{b} \in \mathcal{B}, (X, A_{X,a,b}) \in \mathfrak{b} \cap \mathcal{N}\}. \end{aligned}$$

We have a map $\pi : S \times_{\mathfrak{g}} \widetilde{\mathcal{N}} \to \mathcal{N}_m$ given by $(A_{X,a,b}, \mathfrak{b}) \mapsto X$. We define the *$n$-Slodowy-slice Springer fiber* at $X \in \mathcal{N}_m$ to be the fiber $\pi^{-1}(X)$. Because $\mathcal{B} = \mathcal{B}_m \times \mathcal{B}_{m+n}$,

$$\pi^{-1}(X) \cong \{(A_{X,a,b}, \mathfrak{b}_{m+n}) : A_{X,a,b} \in \mathfrak{b}_{m+n} \cap \mathcal{N}_{m+n}\} \times \{\mathfrak{b}_m : X \in \mathfrak{b}_m\}.$$

The right factor of the product is simply the usual Springer fiber at X.

Let

$$\mathcal{P}_X = \{(A_{X,a,b},(V_i)) : \forall i.\ A_{X,a,b}V_i \subseteq V_{i-1}\}.$$

Using our correspondence between Springer fibers in $\mathcal{B}_{m+n}$ and flags of $\mathbb{C}^{m+n}$, we see that $\mathcal{P}_X$ is isomorphic to the left factor of $\pi^{-1}(X)$. In the next few sections, we will find the irreducible components of $\mathcal{P}_X$. Then we will use this, along with the result about the usual Springer fiber at X, to find the irreducible components of $\pi^{-1}(X)$ and then of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$.

5 Strategy for finding components of $\mathcal{P}_{J(\lambda)}$

Let $X = J(\lambda) \in \mathfrak{gl}_m$, where $\lambda = (\lambda_1, ..., \lambda_k)$. In this section we write $\mathcal{P} := \mathcal{P}_{J(\lambda)}$ and $A_{a,b} := A_{J(\lambda),a,b}$. Let $(e_{ij})_{1\le i\le k, 1\le j\le \lambda_i}$ be the standard basis for $\mathbb{C}^m$. For convenience we write $e_{ij} := 0$ for $j < 1$; now we may express the fact that (e_{ij}) is a Jordan basis by writing $\forall i. Xe_{ij} = e_{i,j-1}$.

In this section we begin finding the irreducible components of

$$\mathcal{P} = \{(A_{a,b},(V_i)) : \forall i.\ A_{a,b}V_{i+1} \subseteq V_i\}.$$

For $1 \le w \le k$ and $0 \le r \le \lambda_w$ (note that we allow $r = 0$), define

$$\mathcal{P}_{w,r} := \{(A_{a,b}, V) \in \mathcal{P} : \exists P \in \mathrm{GL}_m .\ \exists b'.\ (P^{-1}, I_n)A_{a,b}(P, I_n) = A_{e_{wr},b'}\}.$$

Lemma 5.1. $\mathcal{P} = \bigcup_{1\le w\le k, 0\le r\le \lambda_w} \mathcal{P}_{wr}$.

Proof. For any $a \in \mathbb{C}^m$, define $f_a : \mathbb{C}^{m+n} \to \mathbb{C}$ by $e_{ij} \mapsto a_{ij}$ and $f_i \mapsto 0$. For this proof we take the coordinate-free view of $A_{a,b}$ as the linear transformation $\mathbb{C}^{m+n} \to \mathbb{C}^{m+n}$ that sends e_{ij} to $e_{i,j-1} + f_a(e_{ij})f_n$, sends f_{i+1} to f_i, and sends f_1 to $(b, 0)$.

Let $(A_{a,b},(V_i)) \in \mathcal{P}$. Since X is nilpotent, Theorem 10.7 says that there is a change of basis $P : \mathbb{C}^m \to \mathbb{C}^m$ such that Pe_{ij} forms a Jordan basis for X, and for all but one pair (i, j) we have $f(Pe_{ij}) = 0$.

Now let $Q = (P^{-1}, I_n)A_{a,b}(P, I_n)$. Since Pe_{ij} is a Jordan basis, we have $Q = A_{a',b'}$ for some $a', b' \in \mathbb{C}^m$. All that is left is to show that a' is of the form e_{wr} for some w and r. Indeed, this results from the fact that $f_{a'}(e_{ij}) = f_a(Pe_{ij}) = 0$ for all but one pair (i, j). ∎

Lemma 5.2. $\mathcal{P}_{w_1r_1} = \mathcal{P}_{w_2r_2}$ *exactly when either* $r_1 = r_2 = 0$, *or* $(\lambda_{w_1}, r_1) = (\lambda_{w_2}, r_2)$.

Proof. If $r_1 = r_2 = 0$, we have $e_{wr_1} = e_{wr_2} = 0$, so clearly $\mathcal{P}_{wr_1} = \mathcal{P}_{wr_2}$. And if $(\lambda_{w_1}, r_1) = (\lambda_{w_2}, r_2)$, then clearly any matrix of the form $A_{e_{w_1 r_1} b'}$ can be transformed to a matrix of the form $A_{e_{w_2 r_2} b'}$ by making the change of basis that swaps $e_{w_1 j}$ with $e_{w_2 j}$.

Conversely, suppose $\mathcal{P}_{w_1 r_1} = \mathcal{P}_{w_2 r_2}$. Let $k \geq 0$ be minimal so that $A_{e_{w_1 r_1},0} f_1 \neq 0$. Obviously $k = r_1$. Since $A_{e_{w_1 r_1},0}$ and $A_{e_{w_2 r_2},0}$ differ only by a change of basis of $\mathbb{C}^m$, the same reasoning shows that $k = r_2$. Hence $r_1 = r_2$. Now, if $r_1 = 0$, we are done.

Otherwise, we have $r_1 > 0$. Let $k' \geq 0$ be maximal so that $\exists v \in \mathbb{C}^m.\ A^{k'}_{e_{w_1 r_1},0} v = A_{e_{w_1 r_1},0} f_1$. Obviously $k' = \lambda_{w_1} - r_1$. Clearly k' is independent of $\mathbb{C}^m$-basis, so we see that $\lambda_{w_1} - r_1 = k' = \lambda_{w_2} - r_2$. □

Lemma 5.3. *When $\mathcal{P}_{w_1 r_1} \neq \mathcal{P}_{w_2 r_2}$, we have $\mathcal{P}_{w_1 r_1} \cap \mathcal{P}_{w_2 r_2} = \emptyset$.*

Proof. It suffices to remark that $\{(x, y) : \exists w, r.\ x, y \in \mathcal{P}_{wr}\}$ is an equivalence relation. Reflexivity is the statement of Lemma 5.1, and associativity and transitivity are obvious from the definition of $\mathcal{P}_{wr}$. □

Now, fix any w and r. We will find the irreducible components of $\mathcal{P}_{w,r}$. These will all happen to be equidimensional (with dimensions independent of w and r), so their closures in $\mathcal{P}$ will be the irreducible components of $\mathcal{P}$.

Let

$$G := \{P \in \mathrm{GL}_m : P^{-1} X P = X\},$$

and

$$G_{wr} = \{A \in G : e_{wr} A = e_{wr}\}.$$

Now, define

$$\mathcal{Q}_{wr} = \{(A_{e_{wr},b}, (V_i)) \in \mathcal{P}_{wr}\}.$$

Let G act on $\mathcal{P}_{wr}$ by

$$P \cdot (A_{e_{wr},b}, (V_i)) := ((P, I_n) A_{e_{wr},b} (P, I_n)^{-1}, (P, I_n)(V_i)) = (A_{e_{wr} P^{-1}, Pb}, (P, I_n)(V_i)).$$

Note that for any $x \in \mathcal{Q}_{wr}$, we have $G_{wr} = \{g \in G : g \cdot x \in \mathcal{Q}_{wr}\}$. So, by restriction of G to G_{wr} and $\mathcal{P}_{wr}$ to $\mathcal{Q}_{wr}$, we obtain an action of G_{wr} on $\mathcal{Q}_{wr}$.

Consider the map $\varphi : \mathcal{Q}_{wr} \times G \to \mathcal{P}_{wr}$ defined by

$$(x, P) \mapsto P \cdot x.$$

Then, letting G_{wr} act on G by $g \cdot h := h g^{-1}$, we obtain an action of G_{wr} on $\mathcal{Q}_{wr} \times G$.

Lemma 5.4. *The map φ is a principal G_{wr}-bundle.*

Proof. We need to show that G_{wr} acts freely and transitively on the fibers of φ. It is obvious that G_{wr} acts freely on $\mathcal{Q}_{wr} \times G$; it is enough to note that it acts freely on G.

Let $y \in \mathcal{P}_{wr}$. By definition of $\mathcal{P}_{wr}$, there is $P_y \in G$ with $P_y \cdot y \in \mathcal{Q}_{wr}$. We have

$$\begin{aligned} \varphi^{-1}(y) = \{(x, P) : P \cdot x = y\} = \\ \{(P^{-1}y, P) : P^{-1}y \in \mathcal{Q}_{wr}\} = \\ \{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \cdot (P_y \cdot y) \in \mathcal{Q}_{wr}\}. \end{aligned}$$

Using that $G_{wr} = \{g \in G : g \cdot (P_y \cdot y) = P_y \cdot y\}$, the expression above becomes

$$\{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \in G_{wr}\}.$$

Setting $Q := P^{-1}P_y^{-1}$, so that $P = P_y^{-1}Q^{-1}$, the above becomes

$$\begin{aligned} \{(Q \cdot (P_y \cdot y), P_y^{-1}Q^{-1}) : Q \in G_{wr}\} = \\ \{Q \cdot (P_y \cdot y, P_y^{-1}) : Q \in G_{wr}\}. \end{aligned}$$

Thus the fibers are exactly the G_{wr}-orbits; or in other words, G_{wr} acts transitively on the fibers, as desired. □

Our strategy is to find the irreducible components $X \subseteq \mathcal{Q}_{wr}$, and we will then argue that the irreducible components of $\mathcal{P}_{wr}$ are of the form $\varphi(X \times G)$. So, we will now find the irreducible components of $\mathcal{Q}_{wr}$.

Actually $\mathcal{Q}_{wr}$ might be unnecessarily difficult to think about; it is easiest in the case $r = 0$. So, we will change basis to make $r = 0$. Let $m' = m - r$, and $n' = n + r$. Let $\lambda' = (\lambda_1, ..., \lambda_{w-1}, \lambda_w - r, \lambda_{w+1}, ..., \lambda_k)$. Let $X' = J(\lambda)$. (TODO: explain that A' is like A, but with m', n' taking the place of m, n.) Let $\mathcal{Q}' = \{(A'_{X',0,b}, (V_i)) : \forall i.\ A_{X',0,b}V_{i+1} \subseteq V_i\}$.

Lemma 5.5. $\mathcal{Q}_{wr} \cong \mathcal{Q}'$

Proof. Let e'_{ij} be the standard Jordan basis for X'. Let $f'_1, ..., f'_{n'}$ be a basis for $\mathbb{C}^n$, with $A_{X',0,b}f'_{i+1} = f'_i$, and $A_{X',0,b}f_1 = (b, 0)$. Set $m' := m - r$, and $n' := n + r$. Define the linear map $Q_{wr} : \mathbb{C}^{m'+n'} \to \mathbb{C}^{m+n}$ by:

- For all i, $e'_{ij} \mapsto e_{ij}$.

- For $j = 1, ..., r$, $f'_{n+j} \mapsto e_{w,(\lambda_w - r)+j}$.
- For $j = 1, ..., n$, $f'_j \mapsto f_j + e_{w,(\lambda_w - r)-n+j}$.

Change of basis by Q_{wr} maps $\mathcal{Q}_{wr}$ to $\mathcal{Q}'$, and conjugation by Q_{wr}^{-1} maps $\mathcal{Q}'$ to $\mathcal{Q}_{wr}$. $\square$

The next section finds the irreducible components of $\mathcal{Q}'$. To avoid death by primes, we will refer to it as $\mathcal{Q}$, and refer to m', n' as m, n, and so on, throughout the next section.

6 The components of $\mathcal{Q}$

6.1 Setup

Let $X = J(\lambda) \in \mathfrak{gl}_m$. Let $(e_{ij})_{1 \leq i \leq k, 1 \leq j \leq \lambda_i}$ be the standard (Jordan) basis for $\mathbb{C}^m$. Let

$$\mathcal{Q} = \{(A_{0,b}, (V_i)) : \forall i.\ A_{0,b} V_{i+1} \subseteq V_i\}.$$

In this section we find the irreducible components of $\mathcal{Q}$.

We write b_{ij} to denote the projection of $b \in \mathbb{C}^m$ onto e_{ij}. For each row i, let $p_i(b) = \max\{j : b_{ij} \neq 0\}$ (the maximum of the empty set is zero). Then set $q_i(b) = \lambda_i - p_i(b)$. When it is clear enough from context where the b is coming from, we will just write p_i and q_i instead of $p_i(b)$ and $q_i(b)$.

Let $I = \{i_1 < \cdots < i_r\} \subseteq \{1, ..., k\}$, and let $(\rho_i)_{i \in I}$ be any map $I \to \mathbb{N}_{>0}$ such that (1) $\rho_i \leq \lambda_i$, (2) ρ_i is decreasing with i, (3) $\lambda_i - \rho_i$ is decreasing with i, and (4) $\rho_i < n$. For notational convenience (although we assign meaning to neither i_0 nor i_{r+1}), we define $q_{i_0} := n$, and $p_{i_{r+1}} := 0$. Then, we define $B_{I,(\rho_i)}$ as the set of $b \in \mathbb{C}^m$ satisfying the following conditions.

- For all $k \in \{1, ..., r\}$, $p_i = \rho_i$.
- For all $k \in \{0, ..., r\}$, $p_{i_{k+1}} = \max_{i : q_i < q_{i_k}} p_i$.

Note that for any $b \in B_{I,(\rho_i)}$ we have $p_{i_1} > \cdots > p_{i_r} > p_{i_{r+1}}$, and also $q_{i_0} > q_{i_1} > \cdots > q_{i_r}$.

Lemma 6.1. $\mathbb{C}^m = \bigcup_{I,(\rho_i)} B_{I,(\rho_i)}$, *where I ranges over all subsets of $\{1, ..., k\}$, and (ρ_i) ranges over all maps $I \to \mathbb{N}_{>0}$ satisfying the conditions (1),(2),(3),(4). Further, none of the $B_{I,(\rho_i)}$ is contained in the union of the others.*

Proof. Let $b \in \mathbb{C}^m$. If $\{i : q_{i_0} > q_i\}$ is the empty set, then stop. Otherwise, take any $i_1 \in \arg\max_{i:q_{i_0}>q_i} p_i$, and set $\rho_{i_1} := p_{i_1}$. If $\{i : q_{i_1} > q_i\} = \emptyset$, then stop. Otherwise, take any $i_2 \in \arg\max_{i:q_{i_1}>q_i} p_i$, and set $\rho_{i_2} := p_{i_2}$. Continuing on in this way, eventually we reach a k where $\{i : q_{i_k} > q_i\} = \emptyset$. Then we set $I = \{i_1, ..., i_k\}$. Note that $I, (\rho_i)$ satisfy conditions (1)–(4), and furthermore $b \in B_{I,(\rho_i)}$.

Now, we check that no $B_{I,(\rho_i)}$ is contained in the union of the others. Fix I and (ρ_i). Take any $b \in B_{I,(\rho_i)}$ with $p_i = \rho_i$ for $i \in I$ and $p_i = 0$ for $i \notin I$. It is clear that $b \notin B_{I',(\rho'_i)}$ whenever $I' \neq I$ or $(\rho'_i) \neq (\rho_i)$. □

Let $\mathcal{Q}_{I,(\rho_i)} := \{(A_{0,b}, (V_i)) \in \mathcal{Q} : b \in B_{I,(\rho_i)}\}$. We will show that $\mathcal{Q}_{I,(\rho_i)} \cong B_{I,(\rho_i)} \times$(some springer fiber). Then we will use the result about the irreducible components of a Springer fiber to find the irreducible components of $\mathcal{Q}_{I,(\rho_i)}$, and the closures of these will be the irreducible components of $\mathcal{Q}$.

6.2 Study of $B_{I,(\rho_i)}$

Fix any I and (ρ_i) satisfying the conditions (1)–(4) of Lemma 6.1. As before, we write $\{i_1 < \cdots < i_r\} := I$, and $q_{i_0} := n$, and $p_{i_{r+1}} := 0$.

First, we provide an alternative characterization of $B_{I,(\rho_i)}$.

Lemma 6.2. *$B_{I,(\rho_i)}$ is the set of $b \in \mathbb{C}^m$ satisfying the following conditions.*

- *For all $k \in \{1, ..., r\}$, $p_{i_k} = \rho_{i_k}$.*
- *For all $i \notin I$,*
  - *For all $k \in \{0, ..., r\}$ such that $\lambda_i > q_{i_k} + p_{i_{k+1}}$, we have $q_{i_k} \leq q_i$.*
  - *For all $k \in \{1, ..., r+1\}$ such that $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$, we have $p_i \leq p_{i_k}$.*

Proof. First we show that every element of $B_{I,(\rho_i)}$ satisfies those conditions. Let $b \in B_{I,(\rho_i)}$. It is clear that $\forall k.\ p_{i_k} = \rho_{i_k}$.

Take $i \notin I$ and $k \in \{0, ..., r\}$ such that $\lambda_i > q_{i_k} + p_{i_{k+1}}$. Suppose for contradiction that $q_i < q_{i_k}$. Then $p_i \leq \max_{j:q_j<q_{i_k}} p_j = p_{i_{k+1}}$. Then $\lambda_i = p_i + q_i < q_{i_k} + p_{i_{k+1}}$, a contradiction. So we must have $q_{i_k} \leq q_i$, as desired.

Now take $i \notin I$ and $k \in \{1, ..., r+1\}$ such that $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$. Suppose for contradiction that $p_i > p_{i_k}$. Then, putting this together with the first inequality, $\lambda_i - p_i < q_{i_{k-1}} + p_{i_k} - p_{i_k}$; that is, $q_i < q_{i_{k-1}}$. Consequently, $p_i \leq \max_{j:q_j<q_{i_{k-1}}} p_j = p_{i_k}$, as desired.

Now we have shown that every element of $B_{I,(\rho_i)}$ satisfies the conditions of the lemma, and we proceed to the converse. Let $b \in \mathbb{C}^m$ satisfy the conditions. Let $k \in \{0, ..., r\}$. We need to show that $\max_{j:q_j<q_{i_k}} p_j = p_{i_{k+1}}$. Given the conditions (1)–(4) on ρ_i, it suffices to show that for each $i \notin I$ with $q_i < q_{i_k}$, we have $p_i \leq p_{i_{k+1}}$. Indeed, given $i \notin I$ with $q_i < q_{i_k}$, we cannot have $\lambda_i > q_{i_k} + p_{i_{k+1}}$, as that would imply (by hypothesis) that $q_{i_k} \leq q_i$. Hence $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$, and consequently (by hypothesis) $p_i \leq p_{i_k}$. □

Corollary 6.3. *$B_{I,(\rho_i)}$ is the set of $b \in \mathbb{C}^m$ satisfying the following conditions.*

- *For all $k \in \{1, ..., r\}$, $p_{i_k} = \rho_{i_k}$.*
- *For $i \notin I$,*
  - *If $\lambda_i \geq q_{i_0} + p_{i_1}$, then $p_i \leq \lambda_i - q_{i_0}$.*
  - *If there is $k \in \{1, ..., r\}$ with $q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}$, then $p_i \leq \min(p_{i_k}, \lambda_i - q_{i_k})$.*
  - *If $q_{i_r} + p_{i_{r+1}} > \lambda_i$, then $p_i \leq p_{i_{r+1}}$.*

Proof. Both q_{i_k} and p_{i_k} are decreasing as k increases, so this follows directly from Lemma 6.2. (Note that $p_i \leq \lambda_i - q_{i_k}$ iff $q_{i_k} \leq q_i$.) □

Corollary 6.4.

$$B_{I,(\rho_i)} \cong \prod_{k=1}^{r} (\mathbb{C}^{\rho_{i_k}-1} \times (\mathbb{C} \setminus \{0\})) \times \prod_{i:\lambda_i \geq q_{i_0}+p_{i_1}} \mathbb{C}^{\lambda_i - q_{i_0}} \times$$
$$\prod_{k=1}^{r} \prod_{\{i \notin I: q_{i_{k-1}}+p_{i_k} > \lambda_i \geq q_{i_k}+p_{i_{k+1}}\}} \mathbb{C}^{\min(p_{i_k}, \lambda_i - q_{i_k})}.$$

Proof. Here we use the notation $x \times y = (x, y)$, and so on. The isomorphism sends $b \in B_{I,(\rho_i)}$ to

$$\prod_{k=1}^{r} (b_{i_k 1}, ..., b_{i_k \rho_{i_k}}) \times \prod_{i:\lambda_i \geq q_{i_0}+p_{i_1}} (b_{i1}, ..., b_{i,\lambda_i - q_{i_0}}) \times$$
$$\prod_{k=1}^{r} \prod_{\{i \notin I: q_{i_{k-1}}+p_{i_k} > \lambda_i \geq q_{i_k}+p_{i_{k+1}}\}} (b_{i1}, ..., b_{i,\min(p_{i_k}, \lambda_i - q_{i_k})}).$$

Corollary 6.3 says that this is an isomorphism. □

Corollary 6.5.

$$\dim B_{I,(\rho_i)} = \sum_{i:\lambda_i \geq q_{i_0}+\rho_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{r} \sum_{\{i: q_{i_{k-1}}+p_{i_k} > \lambda_i \geq q_{i_k}+p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}).$$

Proof. Immediate from Corollary 6.4. □

6.3 Study of $\mathcal{Q}_{I,(\rho_i)}$

Fix any I and (ρ_i) satisfying the conditions (1)–(4) of Lemma 6.1. In this subsection we find the irreducible components of $\mathcal{Q}_{I,(\rho_i)}$.

We claim that there is some $\mu(I,(\rho_i))$ such that every $(A_{0,b}, U) \in \mathcal{Q}_{I,(\rho_i)}$ is similar to $J(\mu)$. By finding a algebraic map taking $b \in B_{I,(\rho_i)}$ to a Jordan basis for $A_{0,b}$, we will put $\mathcal{Q}_{I,(\rho_i)}$ in isomorphism with the product $B_{I,(\rho_i)} \times (Springer fiber at J(\mu))$. Then we will use our result about the usual Springer fiber at $J(\mu)$ to find the irreducible components of $\mathcal{Q}_{I,(\rho_i)}$.

So, now we find a Jordan basis for $A_{0,b}$. As before, we write $\{i_1, ..., i_r\} := I$, and $q_{i_0} := n$, and $p_{i_{r+1}} := 0$. We also write, somewhat abusively, $b \in \mathbb{C}^{m+n}$ to refer to the vector $(b, 0) \in \mathbb{C}^{m+n}$.

Lemma 6.6. *The following vectors give a Jordan basis for $A_{0,b}$. (For convenience, we write $A := A_{0,b}$ in this lemma and proof.)*

- *For $i \notin I$, the chain of length $p_i + q_i$ beginning with e_{i,p_i+q_i}*
- *The chain of length $n + p_{i_1}$ beginning with $f_n - (A^{n+p_{i_1}} f_n >> n + p_{i_1})$*
- *For $k \in \{1, ..., r\}$, the chain of length $q_{i_k} + p_{i_{k+1}}$ beginning with $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} >> q_{i_k} + p_{i_{k+1}})$, where $v_{i_k} := A^{n-q_{i_k}} f_n - \sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i,l+q_{i_k}}$*

Proof. There are three things to check: (1) the chains are no longer than their claimed lengths, (2) the claimed lengths sum to $m + n$, and (3) the span of the chains is $\mathbb{C}^{m+n}$.

Proof of (1). It is obvious that a chain beginning with e_{i,p_i+q_i} has length $p_i + q_i$.

Now consider the chain beginning with $f_n - (A^{n+p_{i_1}} f_n >> n + p_{i_1})$. Note that $A^n f_n = b$, so $A^{n+p_{i_1}} f_n = A^{p_{i_1}} b = b << p_{i_1}$. By shifting b left p_{i_1} times, we zero out all the rows i where $q_i < n$. This ensures that the operation of shifting $b \lll p_{i_1}$ right $n+p_{i_1}$ times is invertible by shifting left $n+p_{i_1}$ times. That is,

$$\begin{aligned} A^{n+p_{i_1}} f_n &= \\ b \lll p_{i_1} &= \\ ((b \lll p_{i_1}) \ggg n+p_{i_1}) \lll n+p_{i_1} &= \\ A^{n+p_{i_1}}(A^{n+p_{i_1}} f_n \ggg n+p_{i_1}). \end{aligned}$$

This shows that the chain has length at most $n+p_{i_1}$, as desired.

Now, let $k \in \{1,...,r\}$. We have $v_{i_k} = A^{n-q_{i_k}} f_n - \sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i_k, l+q_{i_k}}$. First we note that $q_{i_k} < n$ by the definition of $\mathcal{Q}_{I,(\rho_i)}$, so the definition of v_{i_k} makes sense. We consider the chain beginning with $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \ggg q_{i_k} + p_{i_{k+1}})$. Note that $A^{q_{i_k}} v_{i_k}$ is just b with row i_k zeroed out. For brevity, we write $b_{i_k} := A^{q_{i_k}} v_{i_k}$. Note that $b_{i_k} \lll p_{i_{k+1}}$ has rows l zeroed out, for all l with $q_l < q_{i_k}$. This ensures that shifting $b_{i_k} \lll p_{i_{k+1}}$ right $q_{i_k} + p_{i_{k+1}}$ times can be inverted by shifting left $q_{i_k} + p_{i_{k+1}}$ times. That is,

$$\begin{aligned} A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} &= \\ b_{i_k} \lll p_{i_{k+1}} &= \\ ((b_{i_k} \lll p_{i_{k+1}}) \ggg q_{i_k} + p_{i_{k+1}}) \lll q_{i_k} + p_{i_{k+1}} &= \\ A^{q_{i_k}+p_{i_{k+1}}}(A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \ggg q_{i_k} + p_{i_{k+1}}). \end{aligned}$$

This shows that the chain has length at most $q_{i_k} + p_{i_{k+1}}$, as desired. □

Proof of (2). The sum of the lengths is

$$\sum_{i \notin I} (p_i + q_i) + \sum_{k=0}^{r} (q_{i_k} + p_{i_{k+1}}) = \sum_i (q_i + p_i) = m + n.$$

□

Proof of (3). Let W be the span of the chains listed. We need to show that $W = \mathbb{C}^{m+n}$. Because every $i \in I$ satisfies $q_i < n$, clearly $\langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$.

We claim that $f_n \in W$ as well. To see this, consider the chain beginning with $f_n - (A^{n+p_{i_1}} f_n \ggg n+p_{i_1})$. As explained in the proof of (1), we have $A^{n+p_{i_1}} f_n \in \langle e_{ij} \rangle_{i,j:q_i \geq n}$. Consequently, $(A^{n+p_{i_1}} f_n \ggg n+p_{i_1}) \in \langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$. Because $f_n - (A^{n+p_{i_1}} f_n \ggg n+p_{i_1}) \in W$, this implies that $f_n \in W$.

Because $AW \subseteq W$ (obvious, since W is a span of chains), the fact that $f_n \in W$ implies that $f_i \in W$ for each i, and also $b \lll l \in W$ for each $l \geq 0$.

Now we are left with showing that $\langle e_{ij}\rangle_{i,j:q_i<n} \subseteq W$. It suffices to show that $e_{i,\lambda_i} \in W$ for each i with $q_i < n$. This is obvious for $i \notin I$. So, we just need to show that $e_{i_k\lambda_{i_k}} \in W$ for each $k \in \{1, ..., r\}$. We do this inductively; fix k, and suppose we have already shown that $e_{i_{k'},\lambda_{i_{k'}}} \in W$ for $k' < k$. We will show that $e_{i_k,\lambda_{i_k}} \in W$.

To see this, consider the chain beginning with $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} >> q_{i_k} + p_{i_{k+1}})$. (Recall $v_{i_k} = A^{n-q_{i_k}} f_n - \sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i_k,l+q_{i_k}}$.) Since $b_{i_k,p_{i_k}} \neq 0$ (by definition of p_{i_k}), it suffices to show that $\sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i_k,l+q_{i_k}} \in W$. As explained in the proof of (1), we have $A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \in \langle e_{lj}\rangle_{q_l \geq q_{i_k}}$. And since $A^{q_{i_k}+p_{i_{k+1}}} v_{i_k}$ has row i_k zeroed out, in fact $A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \in \langle e_{lj}\rangle_{l:l\neq i_k \wedge q_l \geq q_{i_k}}$. Hence, $A^{q_i+P_i} v_i >> q_i + P_i \in \langle e_{lj}\rangle_{l:l\neq i\wedge q_l\geq q_i}$. By our inductive hypothesis, $\langle e_{lj}\rangle_{l:l\neq i_k\wedge q_l\geq q_{i_k}} \subseteq W$, and consequently $A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} >> q_{i_k} + p_{i_{k+1}} \in W$. Since we know $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} >> q_{i_k} + p_{i_{k+1}}) \in W$, this implies that $v_{i_k} \in W$. Because $A^{n-q_{i_k}} f_n \in W$, this then implies that $\sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i_k,l+q_{i_k}} \in W$, as desired. □

We proved (1), (2), (3), so we are done. □

Let $\mu(I, (\rho_i))$ be the shape of the Jordan basis given in the lemma. Let X_μ be the Springer fiber at $J(\mu)$, and let $(X_{\mu,\alpha})_{\alpha\in\mathrm{SYT}(\mu)}$ be the irreducible components.

Given a zero-indexed list $L = [L_0, ..., L_{l-1}]$, we define $\gamma(L) = \sum_i iL_i$. We are interested in this thing because for any α, the dimension of $X_{\mu,\alpha}$ is $\gamma([\mu_1, ..., \mu_{k+1}])$.

Lemma 6.7.

$$\gamma([\mu_1, ..., \mu_{k+1}]) = \gamma([0, \lambda_1, ..., \lambda_k]) -$$
$$\left[\sum_{i:\lambda_i\geq q_{i_0}+\rho_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{r} \sum_{\{i:q_{i_{k-1}}+p_{i_k}>\lambda_i\geq q_{i_k}+p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k})\right].$$

Proof. Let $L = [q_{i_0}, \lambda_1, ..., \lambda_k]$. Note that

$$\mu = [..., q_{i_0} + p_{i_1}, ..., q_{i_1} + p_{i_2}, ..., ..., q_{i_r} + p_{i_{r+1}}, ...].$$

Let L' be the result of taking μ and, for each x, replacing one occurence of $q_{i_x} + p_{i_{x+1}}$ by $q_{i_x} + p_{i_x}$; that is,

$$L' = [..., q_{i_0}, ..., q_{i_1} + p_{i_1}, ..., q_{i_2} + p_{i_2}, ..., q_{i_r} + p_{i_r}].$$

We can transform μ into L' by just 'moving' each p_{i_k} to the right by $1+\#\{i \notin I : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}$ slots. So,

$$\gamma(L') - \gamma(\mu) = \sum_{k=1}^{r} p_{i_k}(1 + \#\{i \notin I : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}) =$$

$$\sum_{k=1}^{r} p_{i_k} \cdot \#\{i : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}.$$

Now we consider how to transform L' into L. First we shift q_{i_0} to the left by $\#\{i \notin I : \lambda_i \geq q_{i_0} + p_{i_1}\}$ slots. Then we leave q_{i_0} in place and sort the rest of the list. This entails shifting each $q_{i_x} + p_{i_x}$ to the left by $\#\{i \notin I : q_{i_x} + p_{i_x} > \lambda_i \geq q_{i_x} + p_{i_{x+1}}\}$ slots. Shifting $q_{i_x} + p_{i_x}$ to the left one slot, by swapping it with λ_i, changes the value of γ by $\lambda_i - (q_{i_x} + p_{i_x})$. To go from L' to L, we can just make these swaps repeatedly. So,

$$\gamma(L) - \gamma(L') =$$

$$\sum_{i \notin I : \lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{s} \sum_{\{i \notin I : q_{i_k} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} (\lambda_i - (q_{i_k} + p_{i_k})).$$

Now, we put these two results together to get $\gamma(L) - \gamma(\mu)$.

$$
\begin{aligned}
&\gamma(L) - \gamma(\mu) = \\
&[\gamma(L) - \gamma(L')] + [\gamma(L') - \gamma(\mu)] = \\
&\left[\sum_{i:\lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{s} \sum_{\{i: q_{i_k} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} (\lambda_i - (q_{i_k} + p_{i_k})) \right] + \\
&\left[\sum_{k=1}^{s} \sum_{\{i: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} p_{i_k} \right] = \\
&\sum_{i:\lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{s} \sum_{\{i: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_k}\}} p_{i_k} + \\
&\sum_{k=1}^{s} \sum_{i: q_{i_k} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}} (\lambda_i - q_{i_k}) = \\
&\sum_{i:\lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^{s} \sum_{\{i: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}).
\end{aligned}
$$

Since $\gamma(L) = \gamma([0, \lambda_1, ..., \lambda_k])$, the equation above implies the desired result. □

Lemma 6.8. $\mathcal{Q}_{I,(\rho_i)} \cong B_{I,(\rho_i)} \times X_\mu$.

Proof. For $b \in B_{I,(\rho_i)}$, let P_b be the change-of-basis matrix, with columns given by the Jordan basis of the previous lemma, so that $J(\mu) = P_b^{-1} A_{0,b} P_b$. From looking at the Jordan basis, it is clear that the map $P : B_{I,(\rho_i)} \to \mathrm{GL}_{m+n}$, given by $b \mapsto P_b$, is algebraic.

Now, we remark that the Springer fiber at $A_{0,b}$ is simply $\{P_b(V_i) : (V_i) \in X_\mu\}$. This gives the isomorphism $B_{I,(\rho_i)} \times X_\mu \to \mathcal{Q}_{I,(\rho_i)}$ defined by

$$(b, (V_i)) \mapsto (P_b J(\mu) P_b^{-1}, P_b(V_i)),$$

with inverse

$$(A_{0,b}, (V_i)) \mapsto (b, P_b^{-1}(V_i)).$$

□

Corollary 6.9. *For $\alpha \in \mathrm{SYT}(\mu)$, let $\mathcal{Q}_{I,(\rho_i),\alpha}$ be the subvariety of $\mathcal{Q}_{I,(\rho_i)}$ which corresponds to $B_{I,(\rho_i)} \times X_{\mu,\alpha}$ via the isomorphism of Lemma 6.8. The subvarieties $(\mathcal{Q}_{I,(\rho_i),\alpha})_{\alpha\in\mathrm{SYT}(\mu)}$ are the irreducible components of $\mathcal{Q}_{I,(\rho_i)}$. Each has dimension $\gamma([0,\lambda_1,...,\lambda_k])$.*

Proof. We know that $B_{I,(\rho_i)}$ is irreducible by Corollary 6.4. Then, the fact that the $B_{I,(\rho_i)} \times X_{\mu,\alpha}$ are the irreducible components of $B_{I,(\rho_i)} \times X_\mu$ just follows from the fact that the $X_{\mu,\alpha}$ are the irreducible components of X_μ.

To get the dimension, we add the dimension of $B_{I,(\rho_i)}$ to the dimension of $X_{\mu,\alpha}$. We get the dimension of $B_{I,(\rho_i)}$ from Corollary 6.5, and we get the dimension of $X_{\mu,\alpha}$ from Lemma 6.7. Adding them together, things cancel out and we get $\gamma([0,\lambda_1,...,\lambda_k])$. □

6.4 Conclusion

Theorem 6.10. *The irreducible components of $\mathcal{Q}$ are the closures of the subvarieties $\mathcal{Q}_{I,(\rho_i),\alpha}$, as we let $I,(\rho_i)$ range over all possibilities satisfying the conditions (1)–(4) of Lemma 6.1, and we let $\alpha \in \mathrm{SYT}(\mu(I,(\rho_i)))$.*

Proof. By Corollary 6.9, the $\mathcal{Q}_{I,(\rho_i),\alpha}$ are irreducible and equidimensional. Then Lemma 6.1 says that their union is $\mathcal{Q}$, and in addition that none is contained in the union of the others. □

7 The components of $\mathcal{P}_{J(\lambda)}$

As in section 5, we write $X = J(\lambda) \in \mathfrak{gl}_m$, where $\lambda = (\lambda_1,...,\lambda_k)$. We write $\mathcal{P} := \mathcal{P}_{J(\lambda)}$, and $A_{a,b} := A_{J(\lambda),a,b}$. And as before, $(e_{ij})_{ij}$ and $(f_j)_j$ form the standard basis for $\mathbb{C}^{m+n}$.

7.1 The components of $\mathcal{Q}_{wr}$

Recall from section 5 the varieties

$$\mathcal{Q}_{wr} = \{(A_{e_{wr},b},(V_i)) \in \mathcal{P}\}.$$

Fix any w,r. Lemma 5.5 tells us that

$$\mathcal{Q}_{wr} \cong \mathcal{Q}' := \{(A'_{X',0,b},(V_i)) : \forall i.\ A'_{X',0,b}V_{i+1} \subseteq V_i\},$$

where $X' = J(\lambda')$, and $\lambda' = (\lambda_1, ..., \lambda_w - r, ..., \lambda_k)$, and $m' = m - r$, and $n' = n + r$.

Let $(\mathcal{Q}'_{I,(\rho_i),\alpha})_{I,(\rho_i),\alpha}$ be the irreducible components of $\mathcal{Q}'$ give by Theorem 6.10. Write $\mathcal{Q}_{w,r,I,(\rho_i),\alpha}$ to denote the irreducible component of $\mathcal{Q}_{wr}$ corresponding to $\mathcal{Q}'_{I,(\rho_i),\alpha}$ via the isomorphism $\mathcal{Q}_{wr} \cong \mathcal{Q}'$ of Lemma 5.5.

Theorem 7.1. *The irreducible components of $\mathcal{Q}_{wr}$ are the subvarieties $\mathcal{Q}_{w,r,I,(\rho_i),\alpha}$. Each has dimension $\sum_{i \le j} \min(\lambda'_i, \lambda'_j)$, where $\lambda' = (\lambda_1, ..., \lambda_{w-1}, \lambda_w - r, \lambda_{w+1}, ..., \lambda_k)$.*

Proof. From the foregoing discussion, it is clear that they are indeed the irreducible components. To calculate the dimension, we refer to Corollary 6.9, which says the dimension is $\gamma([0, \lambda'_1, ..., \lambda'_k])$. □

7.2 The varieties G_{wr} and G

Fix w, r. Recall from section 5 the groups $G = \{P \in \mathrm{GL}_m : P^{-1}XP = X\}$ and $G_{wr} = \{A \in G : e_{wr}A = e_{wr}\}$.

Lemma 7.2. *G is irreducible and has dimension $\sum_{ij} \min(\lambda_i, \lambda_j)$.*

Proof. The closure of G in $\mathfrak{gl}_m$ is $\mathfrak{z}_{\mathfrak{gl}_m}(X)$. Lemma 10.3 says that $\mathfrak{z}_{\mathfrak{gl}_m}(X)$ is isomorphic to $\mathbb{C}^{\sum_{ij} \min(\lambda_i, \lambda_j)}$. □

Lemma 7.3. *G_{wr} is irreducible and has dimension $\sum_{ij} \min(\lambda_i, \lambda'_j)$, where $\lambda' = (\lambda_1, ..., \lambda_{w-1}, \lambda_w - r, \lambda_{w+1}, ..., \lambda_k)$.*

Proof. The closure of G_{wr} in $\mathfrak{gl}_m$ is $V = \{Y \in \mathfrak{z}_{\mathfrak{gl}_m}(X) : e_{wr}Y = e_{wr}\}$. In the case that $r = 0$, the constraint that $e_{wr}Y = e_{wr}$ is no constraint at all, so we have $G_{wr} = G$, and the result follows from Lemma 7.2.

In the case that $r > 0$, we observe that $V = \{Y + I : Y \in \mathfrak{z}_{\mathfrak{gl}_m}(X), e_{wr}Y = 0\}$. The constraint $e_{wr}Y = 0$ is just saying that a certain row of Y must be all zeroes. So, the set of Y such that $Y + I \in V$ is the set described by Corollary 10.4. Hence $V \cong \mathbb{C}^{\sum_{ij} \min(\lambda'_i, \lambda'_j)}$. □

7.3 The components of $\mathcal{P}_{wr}$

Recall from section 5 the subvarieties $\mathcal{P}_{wr} \subseteq \mathcal{P}$. From Lemma 5.4 we have the principal G_{wr}-bundle $\varphi_{wr} : \mathcal{Q}_{wr} \times G \to \mathcal{P}_{wr}$.

Theorem 7.4. *Every irreducible component of $\mathcal{P}_{wr}$ is the closure of some subvariety of the form $\varphi_{wr}(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G)$, and the closure of each subvariety of this form is an irreducible component. Each has dimension $\sum_{i \leq j} \min(\lambda_i, \lambda_j)$.*

Remark 7.5. *Theorem 7.4 does not say "the irreducible components of $\mathcal{P}_{wr}$ are the closures of the subvarieties $\varphi_{wr}(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G)$", as that would seem to suggest some claim about distinctness. Further analysis is required to determine which ones are distinct.*

Proof. Together, Theorem 6.10 and Lemma 7.2 tell us that the $\mathcal{Q}_{w,r,I,(\rho_i),\alpha}$ are irreducible, and their union is $\mathcal{Q}_{wr}$. Hence their images are irreducible, and the surjectivity of φ_{wr} implies that their union of their images is $\mathcal{P}_{wr}$.

Now, to verify that each image is an irreducible component, we need only verify that they are equidimensional. We use the result of Lemma 5.4, namely that φ_{wr} is a principal G_{wr}-bundle.

Let $V_{w,r,I,(\rho_i),\alpha}$ be the closure of $\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G$ in $\mathcal{Q}_{wr} \times G$ under action by G_{wr}. Clearly $\varphi_{wr}(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G) = \varphi_{wr}(V_{w,r,I,(\rho_i),\alpha})$. So, we have only to compute the dimension of $\varphi(V_{w,r,I,(\rho_i),\alpha})$.

This is easy, because φ_{wr} (and, consequently, the restriction of φ_{wr} to $V_{w,r,I,(\rho_i),\alpha}$) is a principal G_{wr}-bundle, and principal bundles play nicely with dimensions. That is, we can conclude that

$$\dim \varphi_{wr}(V_{w,r,I,(\rho_i),\alpha}) = \dim V_{w,r,I,(\rho_i),\alpha} - \dim G_{wr}.$$

Since $\dim(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G) = \dim(\mathcal{Q}_{wr} \times G)$ by Theorem 7.1, we know that $\dim V_{w,r,I,(\rho_i),\alpha} = \dim(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G) = \dim \mathcal{Q}_{w,r,I,(\rho_i),\alpha} + \dim G$. By the equation above then, we get

$$\dim \varphi_{wr}(V_{w,r,I,(\rho_i),\alpha}) = \dim \mathcal{Q}_{w,r,I,(\rho_i),\alpha} + \dim G - \dim G_{wr}.$$

We already calculated those dimensions in Theorem 7.1, Lemma 7.2, and Lemma 7.3 respectively. Referring to those, we get

$$\dim \varphi_{wr}(V_{w,r,I,(\rho_i),\alpha}) = \sum_{i \leq j} \min(\lambda'_i, \lambda'_j) + \sum_{ij} \min(\lambda_i, \lambda_j) - \sum_{ij} \min(\lambda_i, \lambda'_j). \quad (1)$$

Now,

$$\sum_{i \leq j} \min(\lambda'_i, \lambda'_j) = \sum_{i \leq j} \min(\lambda_i, \lambda_j) - \sum_{j} \min(\lambda_w, \lambda_j) + \sum_{j} \min(\lambda_w - r, \lambda_j),$$

and

$$\sum_{ij} \min(\lambda_i, \lambda'_j) = \sum_{ij} \min(\lambda_i, \lambda_j) - \sum_i \min(\lambda_i, \lambda_w) + \sum_i \min(\lambda_i, \lambda_w - r).$$

Substituting these into the RHS of Equation (1), things cancel out, and we get

$$\sum_{i \le j} \min(\lambda_i, \lambda_j) + \sum_{ij} \min(\lambda_i, \lambda_j) - \sum_{ij} \min(\lambda_i, \lambda_j) = \sum_{i \le j} \min(\lambda_i, \lambda_j).$$

□

7.4 The components of $\mathcal{P}$

Corollary 7.6. *Every irreducible component of $\mathcal{P}$ is the closure of some subvariety of the form $\varphi_{wr}(\mathcal{Q}_{w,r,I,(\rho_i),\alpha} \times G)$, and the closure of each subvariety of this form is an irreducible component. Each has dimension $\sum_{i \le j} \min(\lambda_i, \lambda_j)$.*

Proof. Follows directly from Theorem 7.4. □

8 The components of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$

Recall from Corollary 4.5 that

$$S \times_{\mathfrak{g}} \widetilde{\mathcal{N}} \cong \{(A_{X,a,b}, (U_i), (V_i)) : \forall i.\ XU_{i+1} \subseteq U_i; \forall i.\ A_{X,a,b} V_{i+1} \subseteq V_i\}.$$

We defined $\pi : S \times_{\mathfrak{g}} \widetilde{\mathcal{N}} \to \mathcal{N}_m$ by $(A_{X,a,b}, (U_i), (V_i)) \mapsto X$, and called $\pi^{-1}(X)$ the n-Slodowy-slice Springer fiber at X.

Theorem 8.1. $\pi^{-1}(J(\lambda)) \cong \mathcal{P}_{J(\lambda)} \times X_\lambda$. *Each irreducible component of $\pi^{-1}(J(\lambda))$ has dimension $\sum_{ij} \min(\lambda_i, \lambda_j)$.*

Proof. The isomorphism sends $(A_{X,a,b}, (U_i), (V_i))$ to $((A_{X,a,b}, (U_i)), (V_i))$. The irreducible components of $\mathcal{P}_{J(\lambda)}$ are given by Corollary 7.6. Those of X_λ are given by Theorem 2.2. Adding the dimensions gives

$$\sum_{i \le j} \min(\lambda_i, \lambda_j) + \sum_{i < j} \min(\lambda_i, \lambda_j) = \sum_{ij} \min(\lambda_i, \lambda_j).$$

□

Let λ be any partition of m. Let $\mathrm{GL}_m \times \{I_n\}$ act on $\pi^{-1}(J(\lambda))$ by conjugation; that is,

$$(P, I_n) \cdot (A_{X,a,b}, (U_i), (V_i)) :=$$
$$((P, I_n)A_{X,a,b}(P, I_n)^{-1}, (P, I_n)(U_i), (P, I_n)(V_i)).$$

Let

$$K = \{(g, I_n) \in \mathrm{GL}_m \times \{I_n\} : gXg^{-1} = X\}.$$

Define $\phi_\lambda : \pi^{-1}(J(\lambda)) \times (\mathrm{GL}_m \times \{I_n\}) \to S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ by $\phi_\lambda(x, g) = g \cdot x$.

Lemma 8.2. *For each partition λ of m, the map ϕ_λ is a principal K-bundle.*

Proof. Analogous to Lemma 5.4. □

Let $(C_{\lambda,\beta})_\beta$ be the irreducible components of $\pi^{-1}(J(\lambda))$. These are described by Theorem 8.1.

Theorem 8.3. *Every irreducible component of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ is the closure of some subvariety of the form $\phi_\lambda(C_{\lambda,\beta} \times (\mathrm{GL}_m \times \{I_n\}))$, and the closure of each subvariety of this form is an irreducible component. Each has dimension m^2.*

Proof. This is analogous to Theorem 7.4; the only difference is the dimension calculation. This time, we have

$$\begin{aligned} \dim \phi_\lambda(C_{\lambda,\beta} \times (\mathrm{GL}_m \times \{I_n\})) = \\ \dim C_{\lambda,\beta} + \dim \mathrm{GL}_m - \dim K = \\ \sum_{ij} \min(\lambda_i, \lambda_j) + m^2 - \sum_{ij} \min(\lambda_i, \lambda_j) = \\ m^2. \end{aligned}$$

The dimension of $C_{\lambda,\beta}$ comes from Theorem 7.4; the dimension of GL_m is obvious; and the dimension of K comes from Lemma 10.3. □

9 Future work

The paper (TODO: cite) describes a bijection between the Steinberg variety (which is blah) and pairs of standard Young tableaux (each having shape blah). The variety $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ looks something like the Steinberg variety. Maybe there is a similar correspondence, between elements of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ and pairs of standard Young tableaux: one having the shape of the Jordan form of X, one having the shape of the Jordan form of $A_{X,a,b}$.

Separately, this paper would benefit from an introductory section where we motivate the study of $S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$.

10 Linear Algebra Facts

In this section we prove linear algebra facts that were used earlier. They are confined to this section to avoid interrupting the rest of the paper.

10.1 The centralizer of a nilpotent matrix

Definition 10.1. *A matrix Y is* Toeplitz *if it is constant along bands parallel to the main diagonal. That is, $\forall i, j, k.\ Y_{ij} = Y_{i+k,j+k}$.*

Definition 10.2. *An $m \times n$ matrix Y is* lower-left Toeplitz *if it is Toeplitz and, in addition, we have $y_{n-i,j-1} = 0$ whenever $i + j \geq \min(m, n)$.*

That is, Y is lower-left Toeplitz if it is Toeplitz, and the only nonzero entries are those with Manhattan distance less than $\min(m, n)$ from the entry in the bottom-left corner. In yet other words, all but the leftmost (equivalently, bottommost) $\min(m, n)$ diagonal bands are zero.

Lemma 10.3. *Let $\lambda = (\lambda_1, ..., \lambda_k)$ be a partition of m. The centralizer of $J(\lambda)$ in $\mathfrak{gl}_m$ is the subalgebra consisting of matrices*

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix},$$

where each M_{ij} is a $\lambda_i \times \lambda_j$ matrix, such that each M_{ij} is lower-left Toeplitz.

Proof. Let

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix}.$$

We need to show that $J(\lambda)M = MJ(\lambda)$ if and only if each M_{ij} is lower-left Toeplitz.

We have

$$J(\lambda)M = \begin{pmatrix} J_{\lambda_1}M_{11} & \cdots & J_{\lambda_1}M_{1k} \\ \vdots & & \vdots \\ J_{\lambda_k}M_{k1} & \cdots & J_{\lambda_k}M_{kk} \end{pmatrix}, \text{ and } MJ(\lambda) = \begin{pmatrix} M_{11}J_{\lambda_1} & \cdots & M_{1k}J_{\lambda_k} \\ \vdots & & \vdots \\ M_{k1}J_{\lambda_1} & \cdots & M_{kk}J_{\lambda_k} \end{pmatrix}.$$

So, we have $J(\lambda)M = MJ(\lambda)$ if and only if $\forall i, j.\ J_{\lambda_i}M_{ij} = M_{ij}J_{\lambda_j}$. Multiplying on the left by J_{λ_i} just shifts each row down by one, and multiplying on the right by J_{λ_j} shifts each column left by one. The matrices for which left-shifting gives the same result as down-shifting are exactly the lower-left Toeplitz matrices. $\square$

Corollary 10.4. *Let $\lambda = (\lambda_1, ..., \lambda_k)$ be a partition of m. Let $w \in \{1, ..., k\}$ and $r \in \{1, ..., \lambda_w\}$. Let $i = [\sum_{j<w} \lambda_j] + r$. The set of $M \in \mathfrak{z}_{\mathfrak{gl}_m}(J(\lambda))$ such that the ith row of M is equal to zero is the set of matrices*

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix},$$

where each M_{ij} is a $\lambda_i \times \lambda_j$ matrix, such that:

- *For $i \neq w$, M_{ij} is lower-left Toeplitz*
- *Each M_{wj} is of the form*

  $$M_{wj} = \begin{pmatrix} 0 \\ M'_{wj} \end{pmatrix},$$

  where M'_{wj} is a $(\lambda_w - r) \times \lambda_j$ lower-left Toeplitz matrix.

10.2 A 'normalization' fact about Jordan bases

Lemma 10.5. *For any finite-dimensional V, nilpotent $A : V \to V$, and linear $f : V \to \mathbb{C}$, there is a Jordan basis e_{ij} for A such that there is at most one i such that there exists j such that $f(e_{ij}) \neq 0$.*

Proof. For any Jordan basis $(e_{ij})_{ij}$ of A, define

$$S((e_{ij})_{1 \leq i \leq k, 1 \leq j \leq \lambda_i}) := \sum_i \begin{cases} -1, & \forall j.\ f(e_{ij}) = 0 \\ \lambda_i - \min\{j : f(e_{ij}) \neq 0\}, & \text{otherwise} \end{cases}.$$

We proceed by induction on the measure S. That is, let $(e_{ij})_{ij}$ be a Jordan basis for A. Our inductive hypothesis is that if there exists a Jordan basis $(e'_{ij})_{ij}$ with $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$, then we get the desired conclusion.

Now, we have two cases. In the first case, $(e_{ij})_{ij}$ already satisfies the desired property. In this case we are done. In the other case, there exist i_1, j_1, i_2, j_2 with $i_1 \neq i_2$, and $f(e_{i_1 j_1}) \neq 0$, and $f(e_{i_2 j_2}) \neq 0$. We let j_1, j_2 be minimal with this property, so that $\forall j < j_1.\ f(e_{i_1 j}) = 0$, and $\forall j < j_2.\ f(e_{i_2 j}) = 0$. Wlog, we assume that $\lambda_{i_1} - j_1 \leq \lambda_{i_2} - j_2$.

By our inductive hypothesis, all we need to do is find a Jordan basis $(e'_{ij})_{ij}$ with $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$. This is what we do. Define e'_{ij} as follows.

- $e'_{i_1,\lambda_1} := e_{i_1,\lambda_1} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} e_{i_2,j_2+(\lambda_{i_1}-j_1)}$
- For $j < \lambda_1$, $e'_{i_1,j} := A^{\lambda_{i_1}-j} e'_{i_1,\lambda_{i_1}}$
- For $i \neq i_1$, $e'_{ij} := e_{ij}$.

Clearly this is a Jordan basis for A. Further, we claim that $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$. It suffices to show that $\forall j \leq j_1.\ f(e_{i_1,j}) = 0$. We have

$$f(e'_{i_1,j}) = f\left(A^{\lambda_{i_1}-j}\left(e_{i_1,\lambda_1} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} e_{i_2,j_2+(\lambda_{i_1}-j_1)}\right)\right) =$$

$$f\left(e_{i_1,j} - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} e_{i_2,j_2+(j-j_1)}\right) = f(e_{i_1,j}) - \frac{f(e_{i_1,j_1})}{f(e_{i_2,j_2})} f(e_{i_2,j_2+(j-j_1)}).$$

Clearly (by design), this expression is zero when $j = j_1$. And for $j < j_1$, we have $f(e_{i_1,j}) = f(e_{i_2,j_2+(j-j_1)}) = 0$, so it is zero then as well. Hence the measure S of this new basis is smaller, as desired. $\square$

Lemma 10.6. *For any n and linear $f : \mathbb{C}^n \to \mathbb{C}$, there is a Jordan basis e_j for J_n such that there is at most one j with $f(e_j) \neq 0$.*

Proof. Let e_j be a Jordan basis for J_n. If $\{j : f(e_j) \neq 0\}$ is the empty set, we are done. Otherwise, let $j_0 = \min\{j : f(e_j) \neq 0\}$. For any Jordan basis f_j with $j_0 = \min\{j : f(e_j) \neq 0\}$, define

$$S((f_j)_j) := \begin{cases} -1, & \{j > j_0 : f(e_j) \neq 0\} = \emptyset \\ n - \min\{j > j_0 : f(e_j) \neq 0\}, & \text{otherwise} \end{cases}.$$

We proceed by induction on S. That is, let $(e_j)_j$ be a Jordan basis for J_n with $j_0 = \min\{j : f(e_j) \neq 0\}$. Our inductive hypothesis is that if there exists a Jordan basis $(e'_j)_j$ with $j_0 = \min\{j : f(e'_j) \neq 0\}$ and $S((e'_j)_j) < S((e_j)_j)$, then the conclusion holds.

We have two cases: either $(e_j)_j$ satisfies the desired property, or not. If not, then let $j_1 = \min\{j > j_0 : f(e_j) \neq 0\}$, and define a new Jordan basis e'_j as follows.

- $e'_n := e_n - \frac{f(e_{j_1})}{f(e_{j_0})} e_{n-(j_1-j_0)}$
- For $j < n$, $e'_j := J_n^{n-j} e'_n$

It is straightforward to check that $j_0 = \min\{j : f(e'_j) \neq 0\}$, and that $S((e'_j)_j) \leq S((e_j)_j) - 1$. By our inductive hypothesis, we are done. □

Theorem 10.7. *For any finite-dimensional V, nilpotent $A : V \to V$, and linear $f : V \to \mathbb{C}$, there is a Jordan basis e_{ij} for A such that there is at most one pair (i, j) with $f(e_{ij}) \neq 0$.*

Proof. Lemma 10.5 provides a Jordan basis e_{ij} such that for all $i \neq i_0$ and all j, we have $f(e_{ij}) = 0$. Restricting A to $\langle e_{i_0 j} \rangle_{j \text{ arbitrary}}$ gives a Jordan block, and then applying Lemma 10.6 gives the desired result. □

10.3 Nilpotency Lemmas

Lemma 10.8. *Let $X \in \mathfrak{gl}_n$ be upper triangular. Then $J_n + X$ is nilpotent if and only if $X = 0$.*

Proof. Clearly if $X = 0$, then $J_n + X$ is nilpotent. Inversely, suppose $X \neq 0$. Let $e_1, ..., e_n$ be the standard basis, with $J_n e_i = e_{i+1}$. Let $i_1 = \max\{i : X e_i \neq 0\}$. As X is upper triangular, we have $X e_{i_1} = v + a e_{i_2}$, with $a \in \mathbb{C} \setminus \{0\}$, $i_2 \leq i_1$, and $v \in \langle e_1, ..., e_{i_1 - 1} \rangle$.

Now, $(J_n + X)^{i_1} e_1 = e_{i_1+1} + v + a e_{i_2}$. Then, $(J_n + X)^{i_1 + (n - i_1)} e_1 = 0 + (J_n + X)^{n - i_1}(v + a e_{i_2})$. Clearly $(J_n + X)^{n - i_1}(v + a e_{i_2}) = v' + a e_{i_2 + n - i_1}$, with $v' \in \langle e_1, ..., e_{i_2 + n - i_1 - 1} \rangle$. Now, since $i_2 \leq i_1$, we have $i_2 + n - i_1 \leq n$, and therefore $(J_n + X)^n e_1 \neq 0$. It follows that $J_n + X$ is not nilpotent. □

Lemma 10.9. *Let $X \in \mathfrak{gl}_m$, and let*

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ d_1 & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ & d_2 & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ & & \ddots & \cdots & \vdots & \vdots \\ & & & d_{n-2} & y_{n-1,n-1} & y_{n-1,n} \\ & & & & d_{n-1} & y_{nn} \end{pmatrix} \in \mathfrak{gl}_n .$$

For any $a, b \in \mathbb{C}^m$,

$$\det \left(\begin{array}{c|c} X & \begin{matrix} & | \\ & b \\ & | \end{matrix} \\ \hline \text{— } a \text{ —} & \\ & Y \end{array}\right) =$$

$$\det X \det Y + \left(\prod_i d_i\right) \det \left(\begin{array}{c|c} X & \begin{matrix} | \\ b \\ | \end{matrix} \\ \hline \text{— } a \text{ —} & 0 \end{array}\right)$$

Proof. By induction on n. In the case $n = 1$, expanding along the last row (taking the usual interpretation of the empty product) gives the desired result.

Now suppose $n > 1$. Expanding along the last row, we get

$$d_{n-1} \det \left(\begin{array}{c|c} X & \begin{matrix} & | \\ & b \\ & | \end{matrix} \\ \hline \text{— } a \text{ —} & \\ & Y_{n,n-1} \end{array}\right) - y_{nn} \det \left(\begin{array}{c|c} X & \\ \hline \text{— } a \text{ —} & \\ & Y_{n,n} \end{array}\right).$$

Using our inductive hypothesis for the first determiniant, and using that $\det \left(\begin{array}{c|c} A_{11} & 0 \\ A_{21} & A_{22} \end{array}\right) = \det A_{11} \det A_{22}$ for the second, the expression becomes

$$d_{n-1} \left(\det X \det Y_{n,n-1} + \left(\prod_{i \leq n-2} d_i\right) \det \left(\begin{array}{c|c} X & \begin{matrix} | \\ b \\ | \end{matrix} \\ \hline \text{— } a \text{ —} & 0 \end{array}\right) \right) - y_{nn} \det X \det Y_{nn} =$$

$$(d_{n-1}Y_{n,n-1} - y_{nn}\det Y_{nn})\det X + \left(\prod_i d_i\right)\det\left(\begin{array}{c|c} X & \begin{matrix}|\\ b\\ |\end{matrix} \\ \hline \text{—}\ \ a\ \ \text{—} & 0\end{array}\right) =$$

$$\det Y \det X + \left(\prod_i d_i\right)\det\left(\begin{array}{c|c} X & \begin{matrix}|\\ b\\ |\end{matrix} \\ \hline \text{—}\ \ a\ \ \text{—} & 0\end{array}\right).$$

□

Corollary 10.10. *If X is nilpotent, and*

$$\left(\begin{array}{c|c} X & \begin{matrix}|\\ b\\ |\end{matrix} \\ \hline \begin{matrix}\text{—}\ \ a\ \ \text{—}\\ \\ \\ \end{matrix} & Y\end{array}\right)$$

is nilpotent as well, then Y is nilpotent.

Proof. By the previous lemma, the characteristic polynomial of the big matrix is

$$g_X(\lambda)g_Y(\lambda) + f(\lambda),$$

where $g_X(\lambda) = \lambda^m$ is the characteristic polynomial of X, and $f(\lambda)$ is some polynomial of degree at most $m - 1$. □

References

[1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.