

# 1 Introduction

Let  $\mathfrak{g} = \mathfrak{gl}_m \times \mathfrak{gl}_{m+n}$ , with  $n > 0$ . Let  $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N} \hookrightarrow \mathfrak{g}$  be the Springer resolution.

Let  $(e, h, f)$  be the principal  $\mathfrak{sl}_2$ -triple in  $\mathfrak{gl}_n$ . We have an embedding  $\mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_m \times \mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_{m+n}$ . Let  $(E, H, F)$  be the  $\mathfrak{sl}_2$ -triple in  $\mathfrak{gl}_{m+n}$  which is the image of  $(e, h, f)$  via this embedding. Let  $S$  be the Slodowy slice  $F + \mathfrak{z}_{\mathfrak{gl}_m}(E)$ . We have a map  $\mathfrak{gl}_{m+n} \rightarrow \mathfrak{gl}_m$  given by coordinate projection. Restricting to  $S$ , we obtain  $f_1 : S \rightarrow \mathfrak{gl}_m$ . (It turns out that this map is surjective.) Then, we obtain  $p_1 : S \rightarrow \mathfrak{g}$  by  $x \mapsto (x, f_1(x))$ . Taking the map  $p_1 : S \rightarrow \mathfrak{g}$  and the Springer resolution  $\tilde{\mathcal{N}} \rightarrow \mathfrak{g}$ , we obtain a fibred product  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ .

In this paper we study  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ . As a step towards studying  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ , we consider the map  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \rightarrow \mathfrak{gl}_m$  given by taking the map to  $\mathfrak{g}$  and then projecting to  $\mathfrak{gl}_m$ . The main section of this paper will study the fibers of this map. Given  $X \in \mathcal{N}_m \subseteq \mathfrak{gl}_m$ , we call the fiber at  $X$  the *n-Slodowy-slice Springer fiber at X*.

In section 2, we review some preliminary material. In section 3, we find the unique (up to similarity) principal  $\mathfrak{sl}_2$ -triple in  $\mathfrak{gl}_n$ . In section 4, we embed this  $\mathfrak{sl}_2$ -triple into  $\mathfrak{gl}_{m+n}$  as described above. We compute the Slodowy slice, and we end up with a nice description of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ .

In section 5, we discuss how to reduce the problem of finding the irreducible components of an  $n$ -Slodowy-slice Springer fiber at  $X \in \mathcal{N}_m$  to an easier problem. In section 6 we solve the easier problem. In section 7 we find the irreducible components of an  $n$ -Slodowy-slice Springer fiber. In section 8 we apply the results of section 7 to find the irreducible components of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ . In section 9 we discuss an interesting direction for future work. Then in section 10 we prove some linear algebra facts that were used in the paper.

## 2 Preliminary Definitions and Facts

### 2.1 Conventions and Notations

We write  $\mathrm{GL}_m, \mathfrak{sl}_m$  to denote  $\mathrm{GL}_m(\mathbb{C}), \mathfrak{sl}_m(\mathbb{C})$ , and so on. By  $J_m$  we refer to the nilpotent  $m \times m$  Jordan block (which, by convention, has ones *below* the diagonal). Given a partition  $\lambda = (\lambda_1, \dots, \lambda_k)$ , we write  $J(\lambda)$  to denote the

block matrix

$$\begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix}.$$

## 2.2 Springer fibers

Let  $\mathfrak{g}$  be a Lie algebra. Let  $\mathcal{N} \subseteq \mathfrak{g}$  be the subset consisting of nilpotent elements. Let  $\mathcal{B}$  be the variety of Borel subalgebras of  $\mathfrak{g}$ . Let  $\tilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$ . Let  $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  be the projection onto the second coordinate. We call this the *Springer resolution*. For  $n \in \mathcal{N}$ , we call  $\pi^{-1}(n)$  the *Springer fiber at  $n$* .

## 2.3 Springer fibers in $\mathfrak{gl}_m$

Now we let  $\mathcal{N}_m$  be the set of nilpotent elements in  $\mathfrak{gl}_m$ , and  $\mathcal{B}_m$  the variety of Borel subalgebras of  $\mathfrak{gl}_m$ . Let  $H \subseteq \mathfrak{gl}_m$  be the subalgebra of upper triangular matrices. The variety of Borel subalgebras of  $\mathfrak{gl}_m$  is  $\mathcal{B} = \{gHg^{-1} : g \in \mathrm{GL}_m\}$ . Thus, the Springer fiber at  $X \in \mathcal{N}$  is

$$S_X = \{gHg^{-1} : X \in gHg^{-1}\}.$$

**Definition 2.1.** A flag  $(V_i)$  of  $\mathbb{C}^m$  is a sequence of subspaces

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = \mathbb{C}^m,$$

where  $\dim V_i = i$ .

We say that  $X \in \mathfrak{gl}_m$  preserves a flag  $(V_i)$  if  $\forall i. XV_i \subseteq V_i$ . Note that  $X \in \mathcal{N}$  preserves a flag  $(V_i)$  if and only if  $\forall i. XV_i \subseteq V_{i-1}$ .

The simplest flag is the *standard flag*  $(E_i)$ , where  $E_i := \langle e_1, \dots, e_i \rangle$ . Note that the group  $H$  is exactly the subset of  $\mathfrak{gl}_m$  which preserves  $E_i$ .

We can think of  $\mathcal{B}$  as the set of flags of  $\mathbb{C}^m$ , via the correspondence

$$gHg^{-1} \leftrightarrow (gE_i).$$

Note that  $X$  preserves  $(gE_i)$  if and only if  $X \in gHg^{-1}$ . Thus, we may write the Springer fiber at  $X \in \mathcal{N}$  in terms of flags, as

$$S_X = \{(gE_i) : X \in gHg^{-1}\} = \{(V_i) : \forall i. XV_i \subseteq V_{i-1}\}.$$

**Theorem 2.2.** *(Needs citation!) The irreducible components of the Springer fiber at  $J(\mu)$  are in bijection with the standard Young tableaux of shape  $\mu$ . Further, the irreducible components are equidimensional, of dimension  $\sum_{i \neq j} \min(\mu_i, \mu_j) = \sum_i (i-1)\mu_i$ .*

## 2.4 Slodowy Slices

A basis for  $\mathfrak{sl}_2$  is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra  $\mathfrak{g}$ , and a homomorphism  $\phi : \mathfrak{sl}_2 \rightarrow \mathfrak{g}$  sending  $(e', h', f')$  to  $(e, h, f)$ , we say that  $(e, h, f)$  is an  $\mathfrak{sl}_2$ -triple. Observe that  $e, f$  must be nilpotent, and  $h$  must be Cartan (TODO: what?) If  $\mathfrak{g}$  is semisimple, then given any nilpotent  $e \in \mathfrak{g}$ , the Jacobson-Morozov theorem [1, 3.7.1] says that there exist  $h, f \in \mathfrak{g}$  such that  $(e, h, f)$  is an  $\mathfrak{sl}_2$ -triple.

Given  $(e, h, f)$ , we define the *Slodowy slice at  $e$*  as  $\mathcal{S}_e := e + \ker \text{ad}_f$ . By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent  $e \in \mathfrak{g}$ , when  $\mathfrak{g}$  is semisimple.

## 3 Principal $\mathfrak{sl}_2$ -triple in $\mathfrak{gl}_n$

The unique nilpotent regular element of  $\mathfrak{gl}_n$  (up to similarity) is  $J_n$ . In this section we use this to find that there is a unique principal  $\mathfrak{sl}_2$ -triple in  $\mathfrak{gl}_n$ . Write  $e = J_n$ .

**Lemma 3.1.** *There is exactly one way to choose  $h, f \in \mathfrak{gl}_n$  so that  $(e, h, f)$  is an  $\mathfrak{sl}_2$ -triple.*

*Proof.* Note that  $[h', e'] = 2e'$ , and  $[e', f'] = h'$ , and  $[h', f'] = -2f'$ . Thus  $e, h, f$  must obey the same relations. In particular,  $he - eh = 2e$ . The matrix  $eh$  is  $h$  shifted down one, and  $he$  is  $h$  shifted left one. Thus,  $2e_{ij} = (he - eh)_{ij} = h_{i,j+1} - h_{i-1,j}$ . We can use this to show that  $h_{ij} = 0$  when  $i \neq j$ . Then we can use it to show that  $h_{ii} = h_{i-1,i-1} + 2$ , so that  $h_{ii} = h_{11} + 2(i-1)$ .

Similarly, from  $[e, f] = h$  we get that  $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$ . We can use this to show that  $f_{ij} = 0$  when  $j \neq i+1$ . Then we can use it to

show that  $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$ , that  $f_{1,2} = -h_{1,1}$ , and that  $f_{n-1,n} = h_{n,n}$ . From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^n h_{ii} \implies$$

$$\sum_i h_{ii} = 0.$$

(That is,  $h \in \mathfrak{sl}_n$ .) This shows that  $h_{11} = n-1, h_{22} = n-3, \dots, h_{nn} = 1-n$ . So we have determined  $h$ ; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for  $f$  in terms of  $h$  to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & \\ & 0 & (1-n) + (3-n) & & \\ & & \ddots & \ddots & \\ & & & 0 & (1-n) + \dots + (n-1) \\ & & & & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1(1-n) & & & \\ & 0 & 2(2-n) & & \\ & & 0 & (n-2)(-2) & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

□

## 4 Finding $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$

### 4.1 Finding the Slodowy slice $S$

In the previous section we computed the principal  $\mathfrak{sl}_2$ -triple  $(e, h, f)$  in  $\mathfrak{gl}_n$ . Embedding this into  $\mathfrak{gl}_{m+n}$  as described previously, we obtain

$$(E, H, F) = \left( \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} \right).$$

**Lemma 4.1.**  $\mathfrak{z}_{\mathfrak{gl}_n}(f)$  is the set of upper-triangular  $X \in \mathfrak{gl}_n$  such that for all  $i, j$ ,

$$X_{i+1,j+1} = \frac{j(n-j)}{i(n-i)} X_{ij}.$$

*Proof.* Let  $X \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$ . Looking at the definition of  $f$  from the previous section (and zero-padding the matrices), we see that  $(fX)_{ij} = i(n-i)X_{i+1,j}$ , and  $(Xf)_{ij} = (j-1)(n-(j-1))X_{i,j-1}$ . So,

$$\forall i, j \in \{1, \dots, n\}. i(n-i)X_{i+1,j} = (j-1)(n-(j-1))X_{i,j-1}.$$

Taking  $j = 1$ , the condition above tells us that  $\forall i \geq 2. X_{i,1} = 0$ . Taking  $j > 1$  and  $i < n$ , we obtain that

$$\forall i, j \in \{1, \dots, n-1\}. X_{i+1,j+1} = \frac{j(n-j)}{i(n-i)} X_{ij}.$$

So, every  $X \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$  is upper triangular and satisfies the condition above. Conversely, it is clear that for such  $X$  we have  $fX = Xf$ .  $\square$

**Lemma 4.2.**

$$\mathfrak{z}_{\mathfrak{gl}_{m+n}}(F) = \left\{ \left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - & a & - \end{array} & Y \end{array} \right) : X \in \mathfrak{gl}_m; Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f); a, b \in \mathbb{C}^m \right\}.$$

*Proof.* Let  $Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \in \mathfrak{z}_{m+n}(F)$ . We have

$$\begin{pmatrix} 0 & Z_{12}f \\ 0 & Z_{22}f \end{pmatrix} = ZF = FZ = \begin{pmatrix} 0 & 0 \\ fZ_{21} & fZ_{22} \end{pmatrix}.$$

We see that there is no restriction on  $Z_{11}$ . The condition  $Z_{12}f = 0$  means that all but the last column of  $Z_{12}$  must be zero, and the condition  $0 = fZ_{21}$  means that all but the first row of  $Z_{21}$  must be zero. And the condition  $Z_{22}f = fZ_{22}$  means that  $Z_{22} \in \mathfrak{z}_{\mathfrak{gl}_{m+n}}(f)$ .  $\square$

These lemmas provide an explicit characterization of the Slodowy slice  $S = E + \mathfrak{z}_{\mathfrak{gl}_{m+n}}(F)$ .

**Corollary 4.3.**

$$S = \left\{ \left( \begin{array}{c|c} X & b \\ \hline - & a & - \\ \hline e + Y & \end{array} \right) : a, b \in \mathbb{C}^m, X \in \mathfrak{gl}_m, Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f) \right\}.$$

## 4.2 A description of $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$

Recall that we have the map  $S \rightarrow \mathfrak{g}$  given by  $Z \mapsto (p(Z), Z)$ , where  $p : \mathfrak{gl}_{m+n} \rightarrow \mathfrak{gl}_m$  is the coordinate projection. We also have the Springer resolution  $\tilde{\mathcal{N}} \rightarrow \mathfrak{g}$ . And from these two maps we define the fibered product  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ .

To obtain an explicit description of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ , we will begin by finding the image of the projection  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \rightarrow S$ . As the image of the Springer resolution is  $\mathcal{N} = \mathcal{N}_m \times \mathcal{N}_{m+n}$ , the image of the projection  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \rightarrow S$  is simply  $S' = \{Z \in S : p(Z) \in \mathcal{N}_m, Z \in \mathcal{N}_{m+n}\}$ .

**Lemma 4.4.**

$$S' = \left\{ A_{X,a,b} := \left( \begin{array}{c|ccc} X & & & & b \\ \hline - & a & - & & \\ & & & 0 & \\ & & & 1 & 0 \\ & & & & \ddots & \ddots \\ & & & & & 1 & 0 \end{array} \right) \in \mathcal{N}_{m+n} : a, b \in \mathbb{C}^m, X \in \mathcal{N}_m \right\}.$$

*Proof.* As every element of  $\mathfrak{z}_{\mathfrak{gl}_n}(f)$  is upper triangular, Corollary 10.9 tells us that if a matrix of the form given in Corollary 4.3 is nilpotent, and the upper-left block  $X$  is nilpotent as well, then  $e + Y$  must be nilpotent. Then, again using that every element  $Y \in \mathfrak{z}_{\mathfrak{gl}_n}(f)$  is upper triangular, Lemma 10.7 tells us that if  $e + Y$  is nilpotent, then  $Y = 0$ .

Hence every element of  $S'$  must simply have  $e$  in its bottom-right block. We see that every element of  $S'$  is of the desired form.  $\square$

This is not a fully explicit characterization of  $S'$ , since we don't say which choices of  $X$  and  $a, b \in \mathbb{C}^m$  lead to  $A_{X,a,b}$  being nilpotent. We could use Lemma 10.8 to find a necessary and sufficient condition on  $X, a, b$ ; however, the description above of  $S'$  will be good enough for our purposes.

**Corollary 4.5.**

$$\begin{aligned} S \times_{\mathfrak{g}} \tilde{\mathcal{N}} = \\ \{((X, A_{X,a,b}), ((X, A_{X,a,b}), \mathfrak{b})) : \mathfrak{b} \in \mathcal{B}, (X, A_{X,a,b}) \in \mathcal{N} \cap \mathfrak{b}\} \cong \\ \{(A_{X,a,b}, \mathfrak{b}) : \mathfrak{b} \in \mathcal{B}, (X, A_{X,a,b}) \in \mathfrak{b} \cap \mathcal{N}\}. \end{aligned}$$

We have a map  $\pi : S \times_{\mathfrak{g}} \tilde{\mathcal{N}} \rightarrow \mathcal{N}_m$  given by  $(A_{X,a,b}, \mathfrak{b}) \mapsto X$ . We define the *n-Slodowy-slice Springer fiber* at  $X \in \mathcal{N}_m$  to be the fiber  $\pi^{-1}(X)$ . Because  $\mathcal{B} = \mathcal{B}_m \times \mathcal{B}_{m+n}$ ,

$$\pi^{-1}(X) \cong \{(A_{X,a,b}, \mathfrak{b}_{m+n}) : A_{X,a,b} \in \mathfrak{b}_{m+n} \cap \mathcal{N}_{m+n}\} \times \{\mathfrak{b}_m : X \in \mathfrak{b}_m\}.$$

The right factor of the product is simply the usual Springer fiber at  $X$ .

Let

$$\mathcal{P}_X = \{(A_{X,a,b}, (V_i)) : \forall i. A_{X,a,b} V_i \subseteq V_{i-1}\}.$$

Using our correspondence between Springer fibers in  $\mathcal{B}_{m+n}$  and flags of  $\mathbb{C}^{m+n}$ , we see that  $\mathcal{P}_X$  is isomorphic to the left factor of  $\pi^{-1}(X)$ . In the next few sections, we will find the irreducible components of  $\mathcal{P}_X$ . Then we will use this, along with the result about the usual Springer fiber at  $X$ , to find the irreducible components of  $\pi^{-1}(X)$  and then of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ .

## 5 Strategy for finding components of $\mathcal{P}_{J(\lambda)}$

Let  $X = J(\lambda) \in \mathfrak{gl}_m$ , where  $\lambda = (\lambda_1, \dots, \lambda_k)$ . In this section we write  $\mathcal{P} := \mathcal{P}_{J(\lambda)}$  and  $A_{a,b} := A_{J(\lambda),a,b}$ . Let  $(e_{ij})_{1 \leq i \leq k, 1 \leq j \leq \lambda_i}$  be the standard basis for  $\mathbb{C}^m$ . For convenience we write  $e_{ij} := 0$  for  $j < 1$ ; now we may express the fact that  $(e_{ij})$  is a Jordan basis by writing  $\forall i. X e_{ij} = e_{i,j-1}$ .

In this section we begin finding the irreducible components of

$$\mathcal{P} = \{(A_{a,b}, (V_i)) : \forall i. A_{a,b} V_{i+1} \subseteq V_i\}.$$

For  $1 \leq w \leq k$  and  $0 \leq r \leq \lambda_w$  (note that we allow  $r = 0$ ), define

$$\mathcal{P}_{w,r} := \{(A_{a,b}, V) \in \mathcal{P} : \exists P \in \mathrm{GL}_m. \exists b'. (P^{-1}, I_n) A_{a,b} (P, I_n) = A_{e_{wr}, b'}\}.$$

**Lemma 5.1.**  $\mathcal{P} = \bigcup_{1 \leq w \leq k, 0 \leq r \leq \lambda_w} \mathcal{P}_{wr}$ .

*Proof.* Let  $(A_{a,b}, (V_i)) \in \mathcal{P}$ . Define  $f : \mathbb{C}^m \rightarrow \mathbb{C}$  by  $f(e_{ij}) = a_{ij}$ . We take the coordinate-free view of  $A_{a,b}$  as the linear transformation  $T$  that sends  $e_{ij}$  to  $e_{i,j-1} + f(e_{ij})$ , sends  $f_{i+1}$  to  $f_i$ , and sends  $f_1$  to  $(b, 0)$ . Since  $X$  is nilpotent, Theorem 10.6 says that there is a change of basis  $P$  such that  $P e_{ij}$  forms a Jordan basis for  $X$ , and for all but one pair  $(i, j)$  we have  $f(P e_{ij}) = 0$ .

Because  $P e_{ij}$  is a Jordan basis,  $A_{a',b'}$  of  $T$  in this new basis has the same upper-left corner  $X$ ; and since  $f(P e_{ij}) = 0$  for all but one pair  $(i, j)$ , we have  $a'_{ij} = 0$  for all but one pair  $(i, j)$ .  $\square$

**Lemma 5.2.**  $\mathcal{P}_{w_1 r_1} = \mathcal{P}_{w_2 r_2}$  exactly when either  $r_1 = r_2 = 0$ , or  $(\lambda_{w_1}, r_1) = (\lambda_{w_2}, r_2)$ . When  $\mathcal{P}_{w_1 r_1} \neq \mathcal{P}_{w_2 r_2}$ , we have  $\mathcal{P}_{w_1 r_1} \cap \mathcal{P}_{w_2 r_2} = \emptyset$ .

*Proof.* Now, if  $r_1 = r_2 = 0$ , we have  $e_{w_1 r_1} = e_{w_2 r_2} = 0$ , so clearly  $\mathcal{P}_{w_1 r_1} = \mathcal{P}_{w_2 r_2}$ . And if  $(\lambda_{w_1}, r_1) = (\lambda_{w_2}, r_2)$ , then clearly any matrix of the form  $A_{e_{w_1 r_1}, b'}$  can be transformed to a matrix of the form  $A_{e_{w_2 r_2}, b'}$  by making the change of basis that swaps  $e_{w_1 j}$  with  $e_{w_2 j}$ . Conversely, suppose  $\mathcal{P}_{w_1 r_1} = \mathcal{P}_{w_2 r_2}$ . Then there is  $P$  such that  $(P^{-1}, I_n) A_{e_{w_1 r_1}, b} (P, I_n) = A_{e_{w_2 r_2}, b'}$ .



Then, either there is  $k$  such that (for any  $b'_1, b'_2$ )  $A_{e_{w_1 r_1} b'_1}^k$

The second observation, that the nonequal sets are disjoint, just follows from the fact that (this restricted form of) similarity is an equivalence relation.  $\square$

Now, fix any  $w$  and  $r$ . We will find the irreducible components of  $\mathcal{P}_{w,r}$ . These will all happen to be equidimensional (with dimensions independent of  $w$  and  $r$ ), so their closures in  $\mathcal{P}$  will be the irreducible components of  $\mathcal{P}$ .

Let

$$G := \{P \in \mathrm{GL}_m : P^{-1}XP = X\},$$

and

$$G_{wr} = \{A \in G : e_{wr}A = e_{wr}\}.$$

Now, define

$$\mathcal{Q}_{wr} = \{(A_{e_{wr},b}, (V_i)) \in \mathcal{P}_{wr}\}.$$

Let  $G$  act on  $\mathcal{P}_{wr}$  by

$$P \cdot (A_{e_{wr},b}, (V_i)) := ((P, I_n)A_{e_{wr},b}(P, I_n)^{-1}, (P, I_n)(V_i)) = (A_{e_{wr}P^{-1}, Pb}, (P, I_n)(V_i)).$$

Note that for any  $x \in \mathcal{Q}_{wr}$ , we have  $G_{wr} = \{g \in G : g \cdot x \in \mathcal{Q}_{wr}\}$ . So, by restriction of  $G$  to  $G_{wr}$  and  $\mathcal{P}_{wr}$  to  $\mathcal{Q}_{wr}$ , we obtain an action of  $G_{wr}$  on  $\mathcal{Q}_{wr}$ .

Consider the map  $\varphi : \mathcal{Q}_{wr} \times G \rightarrow \mathcal{P}_{wr}$  defined by

$$(x, P) \mapsto P \cdot x.$$

Then, letting  $G_{wr}$  act on  $G$  by  $g \cdot h := hg^{-1}$ , we obtain an action of  $G_{wr}$  on  $\mathcal{Q}_{wr} \times G$ .

**Lemma 5.3.** *The map  $\varphi$  is a principal  $G_{wr}$ -bundle.*

*Proof.* We need to show that  $G_{wr}$  acts freely and transitively on the fibers of  $\varphi$ . It is obvious that  $G_{wr}$  acts freely on  $\mathcal{Q}_{wr} \times G$ ; it is enough to note that it acts freely on  $G$ .

Let  $y \in \mathcal{P}_{wr}$ . By definition of  $\mathcal{P}_{wr}$ , there is  $P_y \in G$  with  $P_y \cdot y \in \mathcal{Q}_{wr}$ . We have

$$\begin{aligned} \varphi^{-1}(y) &= \{(x, P) : P \cdot x = y\} = \\ &= \{(P^{-1}y, P) : P^{-1}y \in \mathcal{Q}_{wr}\} = \\ &= \{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \cdot (P_y \cdot y) \in \mathcal{Q}_{wr}\}. \end{aligned}$$

Using that  $G_{wr} = \{g \in G : g \cdot (P_y \cdot y) = P_y \cdot y\}$ , the expression above becomes

$$\{((P^{-1}P_y^{-1}) \cdot (P_y \cdot y), P) : (P^{-1}P_y^{-1}) \in G_{wr}\}.$$

Setting  $Q := P^{-1}P_y^{-1}$ , so that  $P = P_y^{-1}Q^{-1}$ , the above becomes

$$\begin{aligned} \{(Q \cdot (P_y \cdot y), P_y^{-1}Q^{-1}) : Q \in G_{wr}\} = \\ \{Q \cdot (P_y \cdot y, P_y^{-1}) : Q \in G_{wr}\}. \end{aligned}$$

We see that the fibers are exactly the  $G_{wr}$ -orbits; or in other words,  $G_{wr}$  acts transitively on the fibers, as desired.  $\square$

Our strategy is to find the irreducible components  $X \subseteq \mathcal{Q}_{wr}$ , and we will then argue that the irreducible components of  $\mathcal{P}_{wr}$  are of the form  $\varphi(X \times G)$ . So, we will now find the irreducible components of  $\mathcal{Q}_{wr}$ .

Actually  $\mathcal{Q}_{wr}$  might be unnecessarily difficult to think about; it is easiest in the case  $r = 0$ . So, we will change basis to make  $r = 0$ . Let  $m' = m - r$ , and  $n' = n + r$ . Let  $\lambda' = (\lambda_1, \dots, \lambda_{w-1}, \lambda_w - r, \lambda_{w+1}, \dots, \lambda_k)$ . Let  $X' = J(\lambda)$ . (TODO: explain that  $A'$  is like  $A$ , but with  $m', n'$  taking the place of  $m, n$ .) Let  $\mathcal{Q}' = \{(A'_{X',0,b}, (V_i)) : \forall i. A_{X',0,b}V_{i+1} \subseteq V_i\}$ .

**Lemma 5.4.**  $\mathcal{Q}_{wr} \cong \mathcal{Q}'$

*Proof.* Let  $e'_{ij}$  be the standard Jordan basis for  $X'$ . Let  $f'_1, \dots, f'_{n'}$  be a basis for  $\mathbb{C}^{n'}$ , with  $A_{X',0,b}f'_{i+1} = f'_i$ , and  $A_{X',0,b}f'_1 = (b, 0)$ . Set  $m' := m - r$ , and  $n' := n + r$ . Define the linear map  $Q_{wr} : \mathbb{C}^{m'+n'} \rightarrow \mathbb{C}^{m+n}$  by:

- For all  $i$ ,  $e'_{ij} \mapsto e_{ij}$ .
- For  $j = 1, \dots, r$ ,  $f'_{n+j} \mapsto e_{w,(\lambda_w-r)+j}$ .
- For  $j = 1, \dots, n$ ,  $f'_j \mapsto f_j + e_{w,(\lambda_w-r)-n+j}$ .

Observe that conjugation by  $Q_{wr}$  maps  $\mathcal{Q}_{wr}$  to  $\mathcal{Q}'$ , and conjugation by  $Q_{wr}^{-1}$  maps  $\mathcal{Q}'$  to  $\mathcal{Q}_{wr}$ .  $\square$

In the next section we will find the irreducible components of  $\mathcal{Q}'$ . To avoid death by primes, we will refer to it as  $\mathcal{Q}$ , and refer to  $m', n'$  as  $m, n$ , and so on, throughout the next section.

## 6 The components of $\mathcal{Q}$

### 6.1 Setup

Let  $X = J(\lambda) \in \mathfrak{gl}_m$ . Let  $(e_{ij})_{1 \leq i \leq k, 1 \leq j \leq \lambda_i}$  be the standard (Jordan) basis for  $\mathbb{C}^m$ . Let

$$\mathcal{Q} = \{(A_{0,b}, (V_i)) : \forall i. A_{0,b}V_{i+1} \subseteq V_i\}.$$

In this section we find the irreducible components of  $\mathcal{Q}$ .

We write  $b_{ij}$  to denote the projection of  $b \in \mathbb{C}^m$  onto  $e_{ij}$ . For each row  $i$ , let  $p_i(b) = \max\{j : b_{ij} \neq 0\}$  (the maximum of the empty set is zero). Then set  $q_i(b) = \lambda_i - p_i(b)$ . When it is clear enough from context where the  $b$  is coming from, we will just write  $p_i$  and  $q_i$  instead of  $p_i(b)$  and  $q_i(b)$ .

Let  $I = \{i_1 < \dots < i_r\} \subseteq \{1, \dots, k\}$ , and let  $(\rho_i)_{i \in I}$  be any map  $I \rightarrow \mathbb{N}_{>0}$  such that (1)  $\rho_i \leq \lambda_i$ , (2)  $\rho_i$  is decreasing with  $i$ , (3)  $\lambda_i - \rho_i$  is decreasing with  $i$ , and (4)  $\rho_i < n$ . For notational convenience (although we assign meaning to neither  $i_0$  nor  $i_{r+1}$ ), we define  $q_{i_0} := n$ , and  $p_{i_{r+1}} := 0$ . Then, we define  $B_{I,(\rho_i)}$  as the set of  $b \in \mathbb{C}^m$  satisfying the following conditions.

- For all  $k \in \{1, \dots, r\}$ ,  $p_i = \rho_i$ .
- For all  $k \in \{0, \dots, r\}$ ,  $p_{i_{k+1}} = \max_{i: q_i < q_{i_k}} p_i$ .

Note that for any  $b \in B_{I,(\rho_i)}$  we have  $p_{i_1} > \dots > p_{i_r} > p_{i_{r+1}}$ , and also  $q_{i_0} > q_{i_1} > \dots > q_{i_r}$ .

**Lemma 6.1.**  $\mathbb{C}^m = \bigcup_{I,(\rho_i)} B_{I,(\rho_i)}$ , where  $I$  ranges over all subsets of  $\{1, \dots, k\}$ , and  $(\rho_i)$  ranges over all maps  $I \rightarrow \mathbb{N}_{>0}$  satisfying the conditions (1),(2),(3),(4). Further, none of the  $B_{I,(\rho_i)}$  is contained in the union of the others.

*Proof.* Let  $b \in \mathbb{C}^m$ . If  $\{i : q_{i_0} > q_i\}$  is the empty set, then stop. Otherwise, take any  $i_1 \in \arg \max_{i: q_{i_0} > q_i} p_i$ , and set  $\rho_{i_1} := p_{i_1}$ . If  $\{i : q_{i_1} > q_i\} = \emptyset$ , then stop. Otherwise, take any  $i_2 \in \arg \max_{i: q_{i_1} > q_i} p_i$ , and set  $\rho_{i_2} := p_{i_2}$ . Continuing on in this way, eventually we reach a  $k$  where  $\{i : q_{i_k} > q_i\} = \emptyset$ . Then we set  $I = \{i_1, \dots, i_k\}$ . Note that  $I, (\rho_i)$  satisfy conditions (1)–(4), and furthermore  $b \in B_{I,(\rho_i)}$ .

Now, we check that no  $B_{I,(\rho_i)}$  is contained in the union of the others. Fix  $I$  and  $(\rho_i)$ . Take any  $b \in B_{I,(\rho_i)}$  with  $p_i = \rho_i$  for  $i \in I$  and  $p_i = 0$  for  $i \notin I$ . It is clear that  $b \notin B_{I',(\rho'_i)}$  whenever  $I' \neq I$  or  $(\rho'_i) \neq (\rho_i)$ .  $\square$

Let  $\mathcal{Q}_{I,(\rho_i)} := \{(A_{0,b}, (V_i)) \in \mathcal{Q} : b \in B_{I,(\rho_i)}\}$ . We will show that  $\mathcal{Q}_{I,(\rho_i)} \cong B_{I,(\rho_i)} \times (\text{some Springer fiber})$ . Then we will use the result about the irreducible components of a Springer fiber to find the irreducible components of  $\mathcal{Q}_{I,(\rho_i)}$ , and the closures of these will be the irreducible components of  $\mathcal{Q}$ .

## 6.2 Study of $B_{I,(\rho_i)}$

Fix any  $I$  and  $(\rho_i)$  satisfying the conditions (1)–(4) of Lemma 6.1. As before, we write  $\{i_1 < \dots < i_r\} := I$ , and  $q_{i_0} := n$ , and  $p_{i_{r+1}} := 0$ .

First, we provide an alternative characterization of  $B_{I,(\rho_i)}$ .

**Lemma 6.2.**  *$B_{I,(\rho_i)}$  is the set of  $b \in \mathbb{C}^m$  satisfying the following conditions.*

- For all  $k \in \{1, \dots, r\}$ ,  $p_{i_k} = \rho_{i_k}$ .
- For all  $i \notin I$ ,
  - For all  $k \in \{0, \dots, r\}$  such that  $\lambda_i > q_{i_k} + p_{i_{k+1}}$ , we have  $q_{i_k} \leq q_i$ .
  - For all  $k \in \{1, \dots, r+1\}$  such that  $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$ , we have  $p_i \leq p_{i_k}$ .

*Proof.* First we show that every element of  $B_{I,(\rho_i)}$  satisfies those conditions. Let  $b \in B_{I,(\rho_i)}$ . It is clear that  $\forall k. p_{i_k} = \rho_{i_k}$ .

Take  $i \notin I$  and  $k \in \{0, \dots, r\}$  such that  $\lambda_i > q_{i_k} + p_{i_{k+1}}$ . Suppose for contradiction that  $q_i < q_{i_k}$ . Then  $p_i \leq \max_{j: q_j < q_{i_k}} p_j = p_{i_{k+1}}$ . Then  $\lambda_i = p_i + q_i < q_{i_k} + p_{i_{k+1}}$ , a contradiction. So we must have  $q_{i_k} \leq q_i$ , as desired.

Now take  $i \notin I$  and  $k \in \{1, \dots, r+1\}$  such that  $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$ . Suppose for contradiction that  $p_i > p_{i_k}$ . Then, putting this together with the first inequality,  $\lambda_i - p_i < q_{i_{k-1}} + p_{i_k} - p_{i_k}$ ; that is,  $q_i < q_{i_{k-1}}$ . Consequently,  $p_i \leq \max_{j: q_j < q_{i_{k-1}}} p_j = p_{i_k}$ , as desired.

Now we have shown that every element of  $B_{I,(\rho_i)}$  satisfies the conditions of the lemma, and we proceed to the converse. Let  $b \in \mathbb{C}^m$  satisfy the conditions. Let  $k \in \{0, \dots, r\}$ . We need to show that  $\max_{j: q_j < q_{i_k}} p_j = p_{i_{k+1}}$ . Given the conditions (1)–(4) on  $\rho_i$ , it suffices to show that for each  $i \notin I$  with  $q_i < q_{i_k}$ , we have  $p_i \leq p_{i_{k+1}}$ . Indeed, given  $i \notin I$  with  $q_i < q_{i_k}$ , we cannot have  $\lambda_i > q_{i_k} + p_{i_{k+1}}$ , as that would imply (by hypothesis) that  $q_{i_k} \leq q_i$ . Hence  $\lambda_i \leq q_{i_{k-1}} + p_{i_k}$ , and consequently (by hypothesis)  $p_i \leq p_{i_k}$ .  $\square$

**Corollary 6.3.**  *$B_{I,(\rho_i)}$  is the set of  $b \in \mathbb{C}^m$  satisfying the following conditions.*

- For all  $k \in \{1, \dots, r\}$ ,  $p_{i_k} = \rho_{i_k}$ .

- For  $i \notin I$ ,
  - If  $\lambda_i \geq q_{i_0} + p_{i_1}$ , then  $p_i \leq \lambda_i - q_{i_0}$ .
  - If there is  $k \in \{1, \dots, r\}$  with  $q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}$ , then  $p_i \leq \min(p_{i_k}, \lambda_i - q_{i_k})$ .
  - If  $q_{i_r} + p_{i_{r+1}} > \lambda_i$ , then  $p_i \leq p_{i_{r+1}}$ .

*Proof.* Both  $q_{i_k}$  and  $p_{i_k}$  are decreasing as  $k$  increases, so this follows directly from Lemma 6.2. (Note that  $p_i \leq \lambda_i - q_{i_k}$  iff  $q_{i_k} \leq q_i$ .)  $\square$

**Corollary 6.4.**

$$B_{I,(\rho_i)} \cong \prod_{k=1}^r (\mathbb{C}^{p_{i_k}-1} \times (\mathbb{C} \setminus \{0\})) \times \prod_{i:\lambda_i \geq q_{i_0} + p_{i_1}} \mathbb{C}^{\lambda_i - q_{i_0}} \times \prod_{k=1}^r \prod_{\{i \notin I: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} \mathbb{C}^{\min(p_{i_k}, \lambda_i - q_{i_k})}.$$

*Proof.* Here I use the notation  $x \times y = (x, y)$ , and so on. The isomorphism sends  $b \in B_{I,(\rho_i)}$  to

$$\prod_{k=1}^r (b_{i_k 1}, \dots, b_{i_k \rho_{i_k}}) \times \prod_{i:\lambda_i \geq q_{i_0} + p_{i_1}} (b_{i 1}, \dots, b_{i, \lambda_i - q_{i_0}}) \times \prod_{k=1}^r \prod_{\{i \notin I: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} (b_{i 1}, \dots, b_{i, \min(p_{i_k}, \lambda_i - q_{i_k})}).$$

Corollary 6.3 says that this is an isomorphism.  $\square$

**Corollary 6.5.**

$$\dim B_{I,(\rho_i)} = \sum_{i:\lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^r \sum_{\{i: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}).$$

*Proof.* Immediate from Corollary 6.4.  $\square$

### 6.3 Study of $\mathcal{Q}_{I,(\rho_i)}$

Fix any  $I$  and  $(\rho_i)$  satisfying the conditions (1)–(4) of Lemma 6.1. In this subsection we find the irreducible components of  $\mathcal{Q}_{I,(\rho_i)}$ .

I claim that there is some  $\mu(I, (\rho_i))$  such that every  $(A_{0,b}, U) \in \mathcal{Q}_{I,(\rho_i)}$  is similar to  $J(\mu)$ . By finding an algebraic map taking  $b \in B_{I,(\rho_i)}$  to a Jordan basis for  $A_{0,b}$ , we will put  $\mathcal{Q}_{I,(\rho_i)}$  in isomorphism with the product  $B_{I,(\rho_i)} \times (\text{Springer fiber at } J(\mu))$ . Then we will use our result about the usual Springer fiber at  $J(\mu)$  to find the irreducible components of  $\mathcal{Q}_{I,(\rho_i)}$ .

So, now we find a Jordan basis for  $A_{0,b}$ . As before, we write  $\{i_1, \dots, i_r\} := I$ , and  $q_{i_0} := n$ , and  $p_{i_{r+1}} := 0$ . We also write, somewhat abusively,  $b \in \mathbb{C}^{m+n}$  to refer to the vector  $(b, 0) \in \mathbb{C}^{m+n}$ .

**Lemma 6.6.** *The following vectors give a Jordan basis for  $A_{0,b}$ . (For convenience, we write  $A := A_{0,b}$  in this lemma and proof.)*

- For  $i \notin I$ , the chain of length  $p_i + q_i$  beginning with  $e_{i,p_i+q_i}$
- The chain of length  $n + p_{i_1}$  beginning with  $f_n - (A^{n+p_{i_1}} f_n \gg n + p_{i_1})$
- For  $k \in \{1, \dots, r\}$ , the chain of length  $q_{i_k} + p_{i_{k+1}}$  beginning with  $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \gg q_{i_k} + p_{i_{k+1}})$ , where  $v_{i_k} := A^{n-q_{i_k}} f_n - \sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i,l+q_{i_k}}$

*Proof.* There are three things to check: (1) the chains are no longer than their claimed lengths, (2) the claimed lengths sum to  $m + n$ , and (3) the span of the chains is  $\mathbb{C}^{m+n}$ .

*Proof of (1).* It is obvious that a chain beginning with  $e_{i,p_i+q_i}$  has length  $p_i + q_i$ .

Now consider the chain beginning with  $f_n - (A^{n+p_{i_1}} f_n \gg n + p_{i_1})$ . Note that  $A^n f_n = b$ , so  $A^{n+p_{i_1}} f_n = A^{p_{i_1}} b = b \ll p_{i_1}$ . By shifting  $b$  left  $p_{i_1}$  times, we zero out all the rows  $i$  where  $q_i < n$ . This ensures that the operation of shifting  $b \ll p_{i_1}$  right  $n + p_{i_1}$  times is invertible by shifting left  $n + p_{i_1}$  times. That is,

$$\begin{aligned} A^{n+p_{i_1}} f_n &= \\ b \ll p_{i_1} &= \\ ((b \ll p_{i_1}) \gg n + p_{i_1}) \ll n + p_{i_1} &= \\ A^{n+p_{i_1}} (A^{n+p_{i_1}} f_n \gg n + p_{i_1}). \end{aligned}$$

This shows that the chain has length at most  $n + p_{i_1}$ , as desired.

Now, let  $k \in \{1, \dots, r\}$ . We have  $v_{i_k} = A^{n-q_{i_k}} f_n - \sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i_k, l+q_{i_k}}$ . First we note that  $q_{i_k} < n$  by the definition of  $\mathcal{Q}_{I,(\rho_i)}$ , so the definition of  $v_{i_k}$  makes sense. We consider the chain beginning with  $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \gg q_{i_k} + p_{i_{k+1}})$ . Note that  $A^{q_{i_k}} v_{i_k}$  is just  $b$  with row  $i_k$  zeroed out. For brevity, we write  $b_{i_k} := A^{q_{i_k}} v_{i_k}$ . Note that  $b_{i_k} \ll p_{i_{k+1}}$  has rows  $l$  zeroed out, for all  $l$  with  $q_l < q_{i_k}$ . This ensures that shifting  $b_{i_k} \ll p_{i_{k+1}}$  right  $q_{i_k} + p_{i_{k+1}}$  times can be inverted by shifting left  $q_{i_k} + p_{i_{k+1}}$  times. That is,

$$\begin{aligned} A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} &= \\ b_{i_k} \ll p_{i_{k+1}} &= \\ ((b_{i_k} \ll p_{i_{k+1}}) \gg q_{i_k} + p_{i_{k+1}}) \ll q_{i_k} + p_{i_{k+1}} &= \\ A^{q_{i_k}+p_{i_{k+1}}} (A^{q_{i_k}} v_{i_k} \gg q_{i_k} + p_{i_{k+1}}). \end{aligned}$$

This shows that the chain has length at most  $q_{i_k} + p_{i_{k+1}}$ , as desired.  $\square$

*Proof of (2).* The sum of the lengths is

$$\sum_{i \notin I} (p_i + q_i) + \sum_{k=0}^r (q_{i_k} + p_{i_{k+1}}) = \sum_i (q_i + p_i) = m + n.$$

$\square$

*Proof of (3).* Let  $W$  be the span of the chains listed. We need to show that  $W = \mathbb{C}^{m+n}$ . Because every  $i \in I$  satisfies  $q_i < n$ , clearly  $\langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$ .

I claim that  $f_n \in W$  as well. To see this, we consider the chain beginning with  $f_n - (A^{n+p_{i_1}} f_n \gg n + p_{i_1})$ . As explained in the proof of (1), we have  $A^{n+p_{i_1}} f_n \in \langle e_{ij} \rangle_{i,j:q_i \geq n}$ . Consequently,  $(A^{n+p_{i_1}} f_n \gg n + p_{i_1}) \in \langle e_{ij} \rangle_{i,j:q_i \geq n} \subseteq W$ . Because  $f_n - (A^{n+p_{i_1}} f_n \gg n + p_{i_1}) \in W$ , this implies that  $f_n \in W$ .

Because  $AW \subseteq W$  (obvious, since  $W$  is a span of chains), the fact that  $f_n \in W$  implies that  $f_i \in W$  for each  $i$ , and also  $b \ll l \in W$  for each  $l \geq 0$ .

Now we are left with showing that  $\langle e_{ij} \rangle_{i,j:q_i < n} \subseteq W$ . It suffices to show that  $e_{i,\lambda_i} \in W$  for each  $i$  with  $q_i < n$ . This is obvious for  $i \notin I$ . So, we just need to show that  $e_{i_k, \lambda_{i_k}} \in W$  for each  $k \in \{1, \dots, r\}$ . We do this inductively; fix  $k$ , and suppose we have already shown that  $e_{i_{k'}, \lambda_{i_{k'}}} \in W$  for  $k' < k$ . We will show that  $e_{i_k, \lambda_{i_k}} \in W$ .

To see this, consider the chain beginning with  $v_{i_k} - (A^{q_{i_k}+p_{i_{k+1}}} v_{i_k} \gg q_{i_k} + p_{i_{k+1}})$ . (Recall  $v_{i_k} = A^{n-q_{i_k}} f_n - \sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i_k, l+q_{i_k}}$ .) Since  $b_{i_k, p_{i_k}} \neq 0$

(by definition of  $p_{i_k}$ ), it suffices to show that  $\sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i_k, l+q_{i_k}} \in W$ . As explained in the proof of (1), we have  $A^{q_{i_k}+p_{i_k+1}} v_{i_k} \in \langle e_{lj} \rangle_{q_l \geq q_{i_k}}$ . And since  $A^{q_{i_k}+p_{i_k+1}} v_{i_k}$  has row  $i_k$  zeroed out, in fact  $A^{q_{i_k}+p_{i_k+1}} v_{i_k} \in \langle e_{lj} \rangle_{l: l \neq i_k \wedge q_l \geq q_{i_k}}$ . Hence,  $A^{q_i+P_i} v_i \gg q_i + P_i \in \langle e_{lj} \rangle_{l: l \neq i \wedge q_l \geq q_i}$ . By our inductive hypothesis,  $\langle e_{lj} \rangle_{l: l \neq i_k \wedge q_l \geq q_{i_k}} \subseteq W$ , and consequently  $A^{q_{i_k}+p_{i_k+1}} v_{i_k} \gg q_{i_k} + p_{i_k+1} \in W$ . Since we know  $v_{i_k} - (A^{q_{i_k}+p_{i_k+1}} v_{i_k} \gg q_{i_k} + p_{i_k+1}) \in W$ , this implies that  $v_{i_k} \in W$ . Because  $A^{n-q_{i_k}} f_n \in W$ , this then implies that  $\sum_{l=1}^{p_{i_k}} b_{i_k l} e_{i_k, l+q_{i_k}} \in W$ , as desired.  $\square$

We proved (1), (2), (3), so we are done.  $\square$

Let  $\mu(I, (\rho_i))$  be the shape of the Jordan basis given in the lemma. Let  $X_\mu$  be the Springer fiber at  $J(\mu)$ , and let  $(X_{\mu, \alpha})_{\alpha \in \text{SYT}(\mu)}$  be the irreducible components.

Given a zero-indexed list  $L = [L_0, \dots, L_{l-1}]$ , we define  $\gamma(L) = \sum_i i L_i$ . We are interested in this thing because for any  $\alpha$ , the dimension of  $X_{\mu, \alpha}$  is  $\gamma([\mu_1, \dots, \mu_{k+1}])$ .

**Lemma 6.7.**

$$\gamma([\mu_1, \dots, \mu_{k+1}]) = \gamma([0, \lambda_1, \dots, \lambda_k]) - \left[ \sum_{i: \lambda_i \geq q_{i_0} + \rho_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^r \sum_{\{i: q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}) \right].$$

*Proof.* Let  $L = [q_{i_0}, \lambda_1, \dots, \lambda_k]$ . Note that

$$\mu = [\dots, q_{i_0} + p_{i_1}, \dots, q_{i_1} + p_{i_2}, \dots, \dots, q_{i_r} + p_{i_{r+1}}, \dots].$$

Let  $L'$  be the result of taking  $\mu$  and, for each  $x$ , replacing one occurrence of  $q_{i_x} + p_{i_{x+1}}$  by  $q_{i_x} + p_{i_x}$ ; that is,

$$L' = [\dots, q_{i_0}, \dots, q_{i_1} + p_{i_1}, \dots, q_{i_2} + p_{i_2}, \dots, q_{i_r} + p_{i_r}].$$

We can transform  $\mu$  into  $L'$  by just ‘moving’ each  $p_{i_k}$  to the right by  $1 + \#\{i \notin I : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}$  slots. So,

$$\gamma(L') - \gamma(\mu) = \sum_{k=1}^r p_{i_k} (1 + \#\{i \notin I : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}) =$$



$$\sum_{k=1}^r p_{i_k} \cdot \#\{i : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}.$$

Now we consider how to transform  $L'$  into  $L$ . First we shift  $q_{i_0}$  to the left by  $\#\{i \notin I : \lambda_i \geq q_{i_0} + p_{i_1}\}$  slots. Then we leave  $q_{i_0}$  in place and sort the rest of the list. This entails shifting each  $q_{i_x} + p_{i_x}$  to the left by  $\#\{i \notin I : q_{i_x} + p_{i_x} > \lambda_i \geq q_{i_x} + p_{i_{x+1}}\}$  slots. Shifting  $q_{i_x} + p_{i_x}$  to the left one slot, by swapping it with  $\lambda_i$ , changes the value of  $\gamma$  by  $\lambda_i - (q_{i_x} + p_{i_x})$ . To go from  $L'$  to  $L$ , we can just make these swaps repeatedly. So,

$$\gamma(L) - \gamma(L') =$$

$$\sum_{i \notin I : \lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^s \sum_{\{i \notin I : q_{i_k} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} (\lambda_i - (q_{i_k} + p_{i_k})).$$

Now, we put these two results together to get  $\gamma(L) - \gamma(\mu)$ .

$$\begin{aligned} & \gamma(L) - \gamma(\mu) = \\ & [\gamma(L) - \gamma(L')] + [\gamma(L') - \gamma(\mu)] = \\ & \left[ \sum_{i : \lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^s \sum_{\{i : q_{i_k} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} (\lambda_i - (q_{i_k} + p_{i_k})) \right] + \\ & \left[ \sum_{k=1}^s \sum_{\{i : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} p_{i_k} \right] = \\ & \sum_{i : \lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^s \sum_{\{i : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_k}\}} p_{i_k} + \\ & \sum_{k=1}^s \sum_{i : q_{i_k} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}} (\lambda_i - q_{i_k}) = \\ & \sum_{i : \lambda_i \geq q_{i_0} + p_{i_1}} (\lambda_i - q_{i_0}) + \sum_{k=1}^s \sum_{\{i : q_{i_{k-1}} + p_{i_k} > \lambda_i \geq q_{i_k} + p_{i_{k+1}}\}} \min(p_{i_k}, \lambda_i - q_{i_k}). \end{aligned}$$

Observing that  $\gamma(L) = \gamma([0, \lambda_1, \dots, \lambda_k])$ , the equation above gives the desired result.  $\square$

**Lemma 6.8.**  $\mathcal{Q}_{I,(\rho_i)} \cong B_{I,(\rho_i)} \times X_\mu$ .

*Proof.* For  $b \in B_{I,(\rho_i)}$ , let  $P_b$  be the change-of-basis matrix, with columns given by the Jordan basis of the previous lemma, so that  $J(\mu) = P_b^{-1}A_{0,b}P_b$ . From looking at the Jordan basis, it is clear that the map  $P : B_{I,(\rho_i)} \rightarrow \text{GL}_{m+n}$ , given by  $b \mapsto P_b$ , is algebraic.

Now, we remark that the Springer fiber at  $A_{0,b}$  is simply  $\{P_b(V_i) : (V_i) \in X_\mu\}$ . This gives the isomorphism  $B_{I,(\rho_i)} \times X_\mu \rightarrow \mathcal{Q}_{I,(\rho_i)}$  defined by

$$(b, (V_i)) \mapsto (P_b J(\mu) P_b^{-1}, P_b(V_i)),$$

with inverse

$$(A_{0,b}, (V_i)) \mapsto (b, P_b^{-1}(V_i)).$$

□

**Corollary 6.9.** *For  $\alpha \in \text{SYT}(\mu)$ , let  $\mathcal{Q}_{I,(\rho_i),\alpha}$  be the subvariety of  $\mathcal{Q}_{I,(\rho_i)}$  which corresponds to  $B_{I,(\rho_i)} \times X_{\mu,\alpha}$  via the isomorphism of Lemma 6.8. The subvarieties  $(\mathcal{Q}_{I,(\rho_i),\alpha})_{\alpha \in \text{SYT}(\mu)}$  are the irreducible components of  $\mathcal{Q}_{I,(\rho_i)}$ . Each has dimension  $\gamma([0, \lambda_1, \dots, \lambda_k])$ .*

*Proof.* We know that  $B_{I,(\rho_i)}$  is irreducible by Corollary 6.4. Then, the fact that the  $B_{I,(\rho_i)} \times X_{\mu,\alpha}$  are the irreducible components of  $B_{I,(\rho_i)} \times X_\mu$  just follows from the fact that the  $X_{\mu,\alpha}$  are the irreducible components of  $X_\mu$ .

To get the dimension, we add the dimension of  $B_{I,(\rho_i)}$  to the dimension of  $X_{\mu,\alpha}$ . We get the dimension of  $B_{I,(\rho_i)}$  from Corollary 6.5, and we get the dimension of  $X_{\mu,\alpha}$  from Lemma 6.7. Adding them together, things cancel out and we get  $\gamma([0, \lambda_1, \dots, \lambda_k])$ . □

## 6.4 Conclusion

**Theorem 6.10.** *The irreducible components of  $\mathcal{Q}$  are the closures of the subvarieties  $\mathcal{Q}_{I,(\rho_i),\alpha}$ , as we let  $I, (\rho_i)$  range over all possibilities satisfying the conditions (1)–(4) of Lemma 6.1, and we let  $\alpha \in \text{SYT}(\mu(I, (\rho_i)))$ .*

*Proof.* We see from Corollary 6.9 that the  $\mathcal{Q}_{I,(\rho_i),\alpha}$  are irreducible and equidimensional. Then Lemma 6.1 tells us that their union is  $\mathcal{Q}$ , and in addition that none is contained in the union of the others. □

## 7 The components of $\mathcal{P}_{J(\lambda)}$

As in section 5, we write  $X = J(\lambda) \in \mathfrak{gl}_m$ , where  $\lambda = (\lambda_1, \dots, \lambda_k)$ . We write  $\mathcal{P} := \mathcal{P}_{J(\lambda)}$ , and  $A_{a,b} := A_{J(\lambda),a,b}$ . And as before,  $(e_{ij})_{ij}$  and  $(f_j)_j$  form the standard basis for  $\mathbb{C}^{m+n}$ .

### 7.1 The components of $\mathcal{Q}_{wr}$

Recall from section 5 the varieties

$$\mathcal{Q}_{wr} = \{(A_{e_{wr},b}, (V_i)) \in \mathcal{P}\}.$$

Fix any  $w, r$ . Lemma 5.4 tells us that

$$\mathcal{Q}_{wr} \cong \mathcal{Q}' := \{(A'_{X',0,b}, (V_i)) : \forall i. A'_{X',0,b} V_{i+1} \subseteq V_i\},$$

where  $X' = J(\lambda')$ , and  $\lambda' = (\lambda_1, \dots, \lambda_w - r, \dots, \lambda_k)$ , and  $m' = m - r$ , and  $n' = n + r$ .

Let  $(\mathcal{Q}'_{I,(\rho_i),\alpha})_{I,(\rho_i),\alpha}$  be the irreducible components of  $\mathcal{Q}'$  give by Theorem 6.10. Write  $\mathcal{Q}_{w,r,I,(\rho_i),\alpha}$  to denote the irreducible component of  $\mathcal{Q}_{wr}$  corresponding to  $\mathcal{Q}'_{I,(\rho_i),\alpha}$  via the isomorphism  $\mathcal{Q}_{wr} \cong \mathcal{Q}'$  of Lemma 5.4.

**Corollary 7.1.** *The irreducible components of  $\mathcal{Q}_{wr}$  are exactly the subvarieties  $\mathcal{Q}_{w,r,I,(\rho_i),\alpha}$ . Each has dimension  $\sum_{ij} \min(\lambda'_i, \lambda'_j)$ , where  $\lambda' = (\lambda_1, \dots, \lambda_{w-1}, \lambda_w - r, \lambda_{w+1}, \dots, \lambda_k)$ .*

*Proof.* From the foregoing discussion, it is clear that they are indeed the irreducible components. To calculate the dimension, we refer to Corollary 6.9, which says the dimension is  $\gamma([0, \lambda'_1, \dots, \lambda'_k])$ .  $\square$

In section 5 we defined the subvarieties  $\mathcal{P}_{wr} \subseteq \mathcal{P}$  and the principal  $G_{wr}$ -bundle  $\varphi_{wr} : \mathcal{Q}_{wr} \times G \rightarrow \mathcal{P}_{wr}$ .

## 8 The components of $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$

Define  $R = \{(X, \mathfrak{b}_1, \mathfrak{b}_2) : X \in \mathfrak{b}_1 \wedge u(X) \in \mathfrak{b}_2\} \subseteq \mathcal{S}'_{m,n} \times \mathcal{B}_{m+n} \times \mathcal{B}_m$ . We can obtain a subvariety of  $R$  by requiring that  $u(X)$  is in some fixed similarity class. We expect that each of these subvarieties is an irreducible component of dimension  $m^2$ . We will verify these things using our previous computations of springer fibers.

## 9 Future work

The paper (TODO: cite) describes a correspondence between Steinberg variety and pairs of young tableaux. We observe that  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$  looks something like the Steinberg variety. Maybe there is a similar correspondence, between elements of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$  and pairs of young tableaux: one of shape of  $X$ , one of shape of  $A_{X,a,b}$ .

Separately, this paper could use a nice introductory section where we motivate the study of  $S \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ .

## 10 Linear Algebra Facts

In this section we prove linear algebra facts that were used earlier. They are confined to this section to avoid interrupting the flow of the paper.

### 10.1 The centralizer of a nilpotent matrix

**Definition 10.1.** *A matrix  $Y$  is Toeplitz if it is constant along bands parallel to the main diagonal. That is,  $\forall i, j, k. Y_{ij} = Y_{i+k, j+k}$ .*

**Definition 10.2.** *An  $m \times n$  matrix  $Y$  is lower-left Toeplitz if it is Toeplitz and, in addition, we have  $y_{n-i, j-1} = 0$  whenever  $i + j \geq \min(m, n)$ .*

That is,  $Y$  is lower-left Toeplitz if it is Toeplitz, and the only nonzero entries are those with Manhattan distance less than  $\min(m, n)$  from the entry in the bottom-left corner. In yet other words, all but the leftmost (equivalently, bottommost)  $\min(m, n)$  diagonal bands are zero.

**Lemma 10.3.** *Let  $\lambda = (\lambda_1, \dots, \lambda_k)$  be a partition of  $m$ . The centralizer of  $J(\lambda)$  in  $\mathfrak{gl}_m$  is the subalgebra consisting of matrices*

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix},$$

where each  $M_{ij}$  is a  $\lambda_i \times \lambda_j$  matrix, such that each  $M_{ij}$  is lower-left Toeplitz.

*Proof.* Let

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix}.$$

We need to show that  $J(\lambda)M = MJ(\lambda)$  if and only if each  $M_{ij}$  is lower-left Toeplitz.

We have

$$J(\lambda)M = \begin{pmatrix} J_{\lambda_1}M_{11} & \cdots & J_{\lambda_1}M_{1k} \\ \vdots & & \vdots \\ J_{\lambda_k}M_{k1} & \cdots & J_{\lambda_k}M_{kk} \end{pmatrix}, \text{ and } MJ(\lambda) = \begin{pmatrix} M_{11}J_{\lambda_1} & \cdots & M_{1k}J_{\lambda_k} \\ \vdots & & \vdots \\ M_{k1}J_{\lambda_1} & \cdots & M_{kk}J_{\lambda_k} \end{pmatrix}.$$

So, we have  $J(\lambda)M = MJ(\lambda)$  if and only if  $\forall i, j, J_{\lambda_i}M_{ij} = M_{ij}J_{\lambda_j}$ . Multiplying on the left by  $J_{\lambda_i}$  just shifts each row down by one, and multiplying on the right by  $J_{\lambda_j}$  shifts each column left by one. The matrices for which left-shifting gives the same result as down-shifting are exactly the lower-left Toeplitz matrices.  $\square$

For nilpotent  $X$  in Jordan form and  $a$  in ‘normalized’ form (i.e., with at most one nonzero element, which is a one), we will find the centralizer of  $A_{X,a,b}$  in

$$\{(A, I) : A \in \mathfrak{gl}_m\} \subseteq \mathfrak{gl}_{m+n}.$$

Note that an element of the form  $(A, I)$  commutes with  $A_{X,a,b}$  if and only if  $A$  commutes with  $X$ , and  $aA = a$ , and  $Ab = b$ . So, we just have to find

$$\{A \in \mathfrak{gl}_m : AX = XA, aA = a, Ab = b\}.$$

It is just the set of  $A = [A_{ij}]_{ij}$ , where each  $A_{ij}$  is lower-left-Toeplitz.

Let  $v_{ij}$  be the leftmost column of  $A_{ij}$ , so that

$$A_{ij} = \begin{pmatrix} v_{ij} \searrow 0 & \cdots & v_{ij} \searrow [\lambda_j - 1] \end{pmatrix}.$$

Since  $A_{ij}$  is lower-left-Toeplitz,  $v_{ij}$  is of the form

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where  $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$  can be chosen freely. Now we determine which matrices of this form satisfy  $aA = a$ . For simplicity, we will instead find the matrices

$A$  such that  $aA = 0$  (so that  $a(A + I) = a$ ). (Note that  $I \in C_1$ , and  $C_1$  is closed under addition, so this is really the same thing.)

In the case that  $a = 0$ , clearly every  $A$  works. Otherwise, let  $i_0, j_0$  be such that  $a_{i_0, j_0} = 1$ . Now, clearly the constraint that  $aA = 0$  is just saying that the  $(i_0, j_0)$ th row of  $A$  must be zero. That is, for each  $j$  the  $j_0$ th row of  $A_{i_0 j}$  must be zero. This just requires that for each  $j$ , we must have

$$v_{ij} = \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where  $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$  can be chosen freely. So, we have now found the set

$$C_2 := \{A \in \mathfrak{gl}_m : AX = XA, aA = a\}.$$

It is the matrices of the form  $I + A$ , where  $A_{ij}$  is a block matrix of size  $\lambda_i \times \lambda_j$  with

$$A_{ij} = T(v_{ij}) = T \begin{pmatrix} 0 \\ v'_{ij} \end{pmatrix},$$

where  $v'_{ij} \in \mathbb{C}^{\min(\lambda_i, \lambda_j)}$  in the case  $i \neq i_0$  and  $v'_{ij} \in \mathbb{C}^{\min(\lambda_i - j_0, \lambda_j)}$  if  $i = i_0$ .

## 10.2 A ‘normalization’ fact about Jordan bases

**Lemma 10.4.** *For any finite-dimensional  $V$ , nilpotent  $A : V \rightarrow V$ , and linear  $f : V \rightarrow \mathbb{C}$ , there is a Jordan basis  $e_{ij}$  for  $A$  such that there is at most one  $i$  such that there exists  $j$  such that  $f(e_{ij}) \neq 0$ .*

*Proof.* For any Jordan basis  $(e_{ij})_{ij}$  of  $A$ , define

$$S((e_{ij})_{1 \leq i \leq k, 1 \leq j \leq \lambda_i}) := \sum_i \begin{cases} -1, & \forall j. f(e_{ij}) = 0 \\ \lambda_i - \min\{j : f(e_{ij}) \neq 0\}, & \text{otherwise} \end{cases}.$$

We proceed by induction on the measure  $S$ . That is, let  $(e_{ij})_{ij}$  be a Jordan basis for  $A$ . Our inductive hypothesis is that if there exists a Jordan basis  $(e'_{ij})_{ij}$  with  $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$ , then we get the desired conclusion.

Now, we have two cases. In the first case,  $(e_{ij})_{ij}$  already satisfies the desired property. In this case we are done. In the other case, there exist  $i_1, j_1, i_2, j_2$  with  $i_1 \neq i_2$ , and  $f(e_{i_1 j_1}) \neq 0$ , and  $f(e_{i_2 j_2}) \neq 0$ . We let  $j_1, j_2$  be minimal with this property, so that  $\forall j < j_1. f(e_{i_1 j}) = 0$ , and  $\forall j < j_2. f(e_{i_2 j}) = 0$ . Wlog, we assume that  $\lambda_{i_1} - j_1 \leq \lambda_{i_2} - j_2$ .

By our inductive hypothesis, all we need to do is find a Jordan basis  $(e'_{ij})_{ij}$  with  $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$ . This is what we do. Define  $e'_{ij}$  as follows.

- $e'_{i_1, \lambda_1} := e_{i_1, \lambda_1} - \frac{f(e_{i_1, j_1})}{f(e_{i_2, j_2})} e_{i_2, j_2 + (\lambda_{i_1} - j_1)}$
- For  $j < \lambda_1$ ,  $e'_{i_1, j} := A^{\lambda_{i_1} - j} e'_{i_1, \lambda_{i_1}}$
- For  $i \neq i_1$ ,  $e'_{ij} := e_{ij}$ .

Clearly this is a Jordan basis for  $A$ . Further, I claim that  $S((e'_{ij})_{ij}) < S((e_{ij})_{ij})$ . It suffices to show that  $\forall j \leq j_1$ ,  $f(e_{i_1, j}) = 0$ . We have

$$\begin{aligned} f(e'_{i_1, j}) &= f\left(A^{\lambda_{i_1} - j} \left(e_{i_1, \lambda_1} - \frac{f(e_{i_1, j_1})}{f(e_{i_2, j_2})} e_{i_2, j_2 + (\lambda_{i_1} - j_1)}\right)\right) = \\ &= f\left(e_{i_1, j} - \frac{f(e_{i_1, j_1})}{f(e_{i_2, j_2})} e_{i_2, j_2 + (j - j_1)}\right) = f(e_{i_1, j}) - \frac{f(e_{i_1, j_1})}{f(e_{i_2, j_2})} f(e_{i_2, j_2 + (j - j_1)}). \end{aligned}$$

Clearly (by design), this expression is zero when  $j = j_1$ . And for  $j < j_1$ , we have  $f(e_{i_1, j}) = f(e_{i_2, j_2 + (j - j_1)}) = 0$ , so it is zero then as well. Hence the measure  $S$  of this new basis is smaller, as desired.  $\square$

**Lemma 10.5.** *For any  $n$  and linear  $f : \mathbb{C}^n \rightarrow \mathbb{C}$ , there is a Jordan basis  $e_j$  for  $J_n$  such that there is at most one  $j$  with  $f(e_j) \neq 0$ .*

*Proof.* Let  $e_j$  be a Jordan basis for  $J_n$ . If  $\{j : f(e_j) \neq 0\}$  is the empty set, we are done. Otherwise, let  $j_0 = \min\{j : f(e_j) \neq 0\}$ . For any Jordan basis  $f_j$  with  $j_0 = \min\{j : f(e_j) \neq 0\}$ , define

$$S((f_j)_j) := \begin{cases} -1, & \{j > j_0 : f(e_j) \neq 0\} = \emptyset \\ n - \min\{j > j_0 : f(e_j) \neq 0\}, & \text{otherwise} \end{cases}.$$

We proceed by induction on  $S$ . That is, let  $(e_j)_j$  be a Jordan basis for  $J_n$  with  $j_0 = \min\{j : f(e_j) \neq 0\}$ . Our inductive hypothesis is that if there exists a Jordan basis  $(e'_j)_j$  with  $j_0 = \min\{j : f(e'_j) \neq 0\}$  and  $S((e'_j)_j) < S((e_j)_j)$ , then the conclusion holds.

We have two cases: either  $(e_j)_j$  satisfies the desired property, or not. If not, then let  $j_1 = \min\{j > j_0 : f(e_j) \neq 0\}$ , and define a new Jordan basis  $e'_j$  as follows.

- $e'_n := e_n - \frac{f(e_{j_1})}{f(e_{j_0})} e_{n - (j_1 - j_0)}$
- For  $j < n$ ,  $e'_j := J_n^{n-j} e'_n$

It is straightforward to check that  $j_0 = \min\{j : f(e'_j) \neq 0\}$ , and that  $S((e'_j)_j) \leq S((e_j)_j) - 1$ . By our inductive hypothesis, we are done.  $\square$

**Theorem 10.6.** *For any finite-dimensional  $V$ , nilpotent  $A : V \rightarrow V$ , and linear  $f : V \rightarrow \mathbb{C}$ , there is a Jordan basis  $e_{ij}$  for  $A$  such that there is at most one pair  $(i, j)$  with  $f(e_{ij}) \neq 0$ .*

*Proof.* Lemma 10.4 provides a Jordan basis  $e_{ij}$  such that for all  $i \neq i_0$  and all  $j$ , we have  $f(e_{ij}) = 0$ . Restricting  $A$  to  $\langle e_{i_0 j} \rangle_{j \text{ arbitrary}}$  gives a Jordan block, and then applying Lemma 10.5 gives the desired result.  $\square$

### 10.3 Nilpotency Lemmas

**Lemma 10.7.** *Let  $X \in \mathfrak{gl}_n$  be upper triangular. Then  $J_n + X$  is nilpotent if and only if  $X = 0$ .*

*Proof.* Clearly if  $X = 0$ , then  $J_n + X$  is nilpotent. Inversely, suppose  $X \neq 0$ . Let  $e_1, \dots, e_n$  be the standard basis, with  $J_n e_i = e_{i+1}$ . Let  $i_1 = \max\{i : X e_i \neq 0\}$ . As  $X$  is upper triangular, we have  $X e_{i_1} = v + a e_{i_2}$ , with  $a \in \mathbb{C} \setminus \{0\}$ ,  $i_2 \leq i_1$ , and  $v \in \langle e_1, \dots, e_{i_1-1} \rangle$ .

Now,  $(J_n + X)^{i_1} e_1 = e_{i_1+1} + v + a e_{i_2}$ . Then,  $(J_n + X)^{i_1+(n-i_1)} e_1 = 0 + (J_n + X)^{n-i_1}(v + a e_{i_2})$ . Clearly  $(J_n + X)^{n-i_1}(v + a e_{i_2}) = v' + a e_{i_2+n-i_1}$ , with  $v' \in \langle e_1, \dots, e_{i_2+n-i_1-1} \rangle$ . Now, since  $i_2 \leq i_1$ , we have  $i_2 + n - i_1 \leq n$ , and therefore  $(J_n + X)^n e_1 \neq 0$ . It follows that  $J_n + X$  is not nilpotent.  $\square$

**Lemma 10.8.** *Let  $X \in \mathfrak{gl}_m$ , and let*

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ d_1 & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\ & d_2 & y_{33} & \cdots & y_{3,n-1} & y_{3n} \\ & & \ddots & \cdots & \vdots & \vdots \\ & & & d_{n-2} & y_{n-1,n-1} & y_{n-1,n} \\ & & & & d_{n-1} & y_{nn} \end{pmatrix} \in \mathfrak{gl}_n.$$

For any  $a, b \in \mathbb{C}^m$ ,

$$\det \left( \begin{array}{ccc|c} & X & & b \\ \hline - & a & - & \\ & & Y & \end{array} \right) =$$



$$\det X \det Y + \left( \prod_i d_i \right) \det \left( \begin{array}{ccc|c} & & & | \\ & & & | \\ X & & & b \\ & & & | \\ \hline - & a & - & 0 \end{array} \right)$$

*Proof.* By induction on  $n$ . In the case  $n = 1$ , expanding along the last row (taking the usual interpretation of the empty product) gives the desired result.

Now suppose  $n > 1$ . Expanding along the last row, we get

$$d_{n-1} \det \left( \begin{array}{ccc|c} & & & | \\ & & & | \\ X & & & b \\ & & & | \\ \hline - & a & - & Y_{n,n-1} \end{array} \right) - y_{nn} \det \left( \begin{array}{ccc|c} & & & | \\ & & & | \\ X & & & \\ & & & | \\ \hline - & a & - & Y_{n,n} \end{array} \right).$$

Using our inductive hypothesis for the first determiniant, and using that  $\det \left( \begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array} \right) = \det A_{11} \det A_{22}$  for the second, the expression becomes

$$\begin{aligned} & d_{n-1} \left( \det X \det Y_{n,n-1} + \left( \prod_{i \leq n-2} d_i \right) \det \left( \begin{array}{ccc|c} & & & | \\ & & & | \\ X & & & b \\ & & & | \\ \hline - & a & - & 0 \end{array} \right) \right) - y_{nn} \det X \det Y_{nn} = \\ & (d_{n-1} Y_{n,n-1} - y_{nn} \det Y_{nn}) \det X + \left( \prod_i d_i \right) \det \left( \begin{array}{ccc|c} & & & | \\ & & & | \\ X & & & b \\ & & & | \\ \hline - & a & - & 0 \end{array} \right) = \\ & \det Y \det X + \left( \prod_i d_i \right) \det \left( \begin{array}{ccc|c} & & & | \\ & & & | \\ X & & & b \\ & & & | \\ \hline - & a & - & 0 \end{array} \right). \end{aligned}$$

□

**Corollary 10.9.** *If  $X$  is nilpotent, and*

$$\left( \begin{array}{c|c} X & \begin{array}{c} | \\ b \\ | \end{array} \\ \hline \begin{array}{c} - \quad a \quad - \end{array} & Y \end{array} \right)$$

*is nilpotent as well, then  $Y$  is nilpotent.*

*Proof.* By the previous lemma, the characteristic polynomial of the big matrix is

$$g_X(\lambda)g_Y(\lambda) + f(\lambda),$$

where  $g_X(\lambda) = \lambda^m$  is the characteristic polynomial of  $X$ , and  $f(\lambda)$  is some polynomial of degree at most  $m - 1$ .  $\square$

## References

- [1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.