

# 1 Definition of one thing

Let  $G \subseteq \mathrm{GL}_m(\mathbb{C})$  be a connected semisimple Lie group, and let  $\mathfrak{g} \subseteq \mathfrak{gl}_m(\mathbb{C})$  be its Lie algebra. Let  $\mathcal{N} \subseteq \mathfrak{g}$  be the subset consisting of nilpotent elements. Let  $\mathcal{B}$  be the variety of Borel subalgebras of  $\mathfrak{g}$ . Let  $\tilde{\mathcal{N}} = \{(\mathfrak{b}, n) : n \in \mathfrak{b}\} \subseteq \mathcal{B} \times \mathcal{N}$ . Let  $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  be the projection onto the second coordinate. For  $n \in \mathcal{N}$ , we call  $\pi^{-1}(n)$  the *Springer fiber at  $n$* .

## 2 Slodowy slice

A basis for  $\mathfrak{sl}_2(\mathbb{C})$  is

$$e' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Given a Lie algebra  $\mathfrak{g}$ , and a homomorphism  $\phi : \mathfrak{sl}_2 \rightarrow \mathfrak{g}$  sending  $(e', h', f')$  to  $(e, h, f)$ , we say that  $(e, h, f)$  is an  $\mathfrak{sl}_2$ -triple. Observe that  $e, f$  must be nilpotent, and  $h$  must be Cartan (???). If  $\mathfrak{g}$  is semisimple, then given any nilpotent  $e \in \mathfrak{g}$ , the Jacobson-Morozov theorem [1, 3.7.1] says that there exist  $h, f \in \mathfrak{g}$  such that  $(e, h, f)$  is an  $\mathfrak{sl}_2$ -triple.

Given  $(e, h, f)$ , we define the *Slodowy slice at  $e$*  as  $\mathcal{S}_e := e + \ker \mathrm{ad}_f$ . By the Jacobson-Morozov theorem, we can always find a Slodowy slice at any nilpotent  $e \in \mathfrak{g}$ , when  $\mathfrak{g}$  is semisimple.

### 2.1 Finding an $\mathfrak{sl}_2$ -triple in a simple case

Let

$$e = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \in \mathfrak{sl}_n,$$

where there are  $n - 1$  ones. We will find the Slodowy slice at  $e$ .

As a first step, we will find an  $\mathfrak{sl}_2$ -triple with  $e$  as its first element. Note that  $[h', e'] = 2e'$ , and  $[e', f'] = h'$ , and  $[h', f'] = -2f'$ . Thus  $e, h, f$  must obey the same relations. In particular,  $he - eh = 2e$ . The matrix  $eh$  is  $h$  shifted down one, and  $he$  is  $h$  shifted left one. Thus,  $2e_{ij} = (he - eh)_{ij} =$

$h_{i,j+1} - h_{i-1,j}$ . We can use this to show that  $h_{ij} = 0$  when  $i \neq j$ . Then we can use it to show that  $h_{ii} = h_{i-1,i-1} + 2$ , so that  $h_{ii} = h_{11} + 2(i-1)$ .

Similarly, from  $[e, f] = h$  we get that  $h_{ij} = (ef - fe)_{ij} = f_{i-1,j} - f_{i,j+1}$ . We can use this to show that  $f_{ij} = 0$  when  $j \neq i+1$ . Then we can use it to show that  $f_{i,i+1} = f_{i+1,i+2} + h_{i+1,i+1}$ , that  $f_{1,2} = -h_{1,1}$ , and that  $f_{n-1,n} = h_{n,n}$ . From the first equation we find that

$$-h_{1,1} = f_{1,2} = f_{n-1,n} + \sum_{i=1}^{n-1} h_{ii} = \sum_{i=1}^n h_{ii} \implies \sum_i h_{ii} = 0.$$

Oops, I didn't actually need to prove that, since  $h \in \mathfrak{sl}_n$ ... it's nice to know that it's the unique solution in  $\mathfrak{gl}_n$  though? This shows that  $h_{11} = n-1, h_{22} = n-3, \dots, h_{nn} = 1-n$ . So we have determined  $h$ ; it is

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & 3-n & \\ & & & & 1-n \end{pmatrix}.$$

Now, we can use our expression for  $f$  in terms of  $h$  to obtain

$$f = \begin{pmatrix} 0 & 1-n & & & \\ & 0 & (1-n) + (3-n) & & \\ & & \ddots & \ddots & \\ & & & 0 & (1-n) + \dots + (n-1) \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1(1-n) & & & \\ & 0 & 2(2-n) & & \\ & & 0 & (n-2)(-2) & \\ & & & 0 & (n-1)(-1) \\ & & & & 0 \end{pmatrix}.$$

## 2.2 Finding an $\mathfrak{sl}_2$ -triple in a slightly less simple case

Now, let

$$E = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{sl}_{m+n},$$

with  $e$  as in the previous subsection. We will find  $H, F$  so that  $(E, H, F)$  is an  $\mathfrak{sl}_2$ -triple. Writing  $F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & f \end{pmatrix}$ , and similarly for  $H$ , we have

$$2E = [H, E] = \begin{pmatrix} 0 & H_{12}e \\ 0 & he \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ eH_{21} & eh \end{pmatrix},$$

$$H = [E, F] = \begin{pmatrix} 0 & 0 \\ eF_{21} & ef \end{pmatrix} - \begin{pmatrix} 0 & F_{12}e \\ 0 & fe \end{pmatrix}.$$

From these we observe that  $(e, h, f)$  must also be an  $\mathfrak{sl}_2$ -triple, so  $h, f$  must be as in the previous subsection. We also see that  $H_{11} = 0$ . Recalling that left multiplication by  $e$  is a down-shift, and right multiplication is a left-shift, we see that  $H_{12}$  is all zeroes except for the leftmost column, and  $H_{21}$  is all zeroes except for the bottom row. Then our final constraint is that

$$-2F = [H, F] = \begin{pmatrix} H_{12}F_{21} & H_{12}F_{22} \\ H_{21}F_{11} + H_{22}F_{21} & H_{21}F_{12} + H_{22}F_{22} \end{pmatrix} - \begin{pmatrix} F_{12}H_{21} & F_{11}H_{12} + F_{12}H_{22} \\ F_{22}H_{21} & F_{21}H_{12} + F_{22}H_{22} \end{pmatrix}.$$

Now  $H_{12} = F_{12}e$ , and  $H_{21} = eF_{21}$ , from the equation  $H = [E, F]$ . Substituting in the equation above then,

$$\begin{aligned} -2F &= \begin{pmatrix} F_{12}eF_{21} & F_{12}ef \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} F_{12}eF_{21} & F_{11}F_{12}e + F_{12}h \\ feF_{21} & F_{21}F_{12}e + fh \end{pmatrix} = \\ &= \begin{pmatrix} 0 & F_{12}(fe + h) \\ eF_{21}F_{11} + hF_{21} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e + F_{12}h \\ (ef - h)F_{21} & F_{21}F_{12}e + fh \end{pmatrix} = \\ &= \begin{pmatrix} 0 & F_{12}fe \\ eF_{21}F_{11} & eF_{21}F_{12} + hf \end{pmatrix} - \begin{pmatrix} 0 & F_{11}F_{12}e \\ efF_{21} & F_{21}F_{12}e + fh \end{pmatrix}. \end{aligned}$$

Now we see that  $F_{11} = 0$ , and consequently that  $F_{12} = F_{21} = 0$  as well. This shows that  $H_{12} = H_{21} = 0$ . We conclude that  $H$  and  $F$  just have  $h$  and  $f$  in their bottom-right corners, respectively.

### 3 Definition of springer fiber at slodowy slice

Let  $\mathcal{N}_m \subseteq \mathfrak{gl}_m(\mathbb{C})$  be the subset consisting of nilpotent elements. For  $X \in \mathcal{N}$  and  $a, b \in \mathbb{C}^m$ , let

$$A_{X,a,b} = \left( \begin{array}{c|ccc} X & & & & b \\ \hline - & a & - & & \\ \hline & & 0 & & \\ & & 1 & 0 & \\ & & & 1 & \ddots \\ & & & & \ddots & 0 \\ & & & & & 1 & 0 \end{array} \right) \in \mathfrak{gl}_{m+n}(\mathbb{C}).$$

Let  $\mathcal{M}$  be the variety of Borel subalgebras of  $\mathfrak{gl}_{m+n}(\mathbb{C})$ . Let

$$\tilde{\mathcal{N}} = \{(\mathfrak{m}, a, b, X) : A_{X,a,b} \in \mathfrak{m}\} \subseteq \mathcal{M} \times \mathbb{C}^m \times \mathbb{C}^m \times \mathcal{N}.$$

Let  $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  be the projection onto the last coordinate. For  $X \in \mathcal{N}$ , we call  $\pi^{-1}(X)$  the *other Springer fiber at X*.

### 4 Finding the Springer fibers of section 2

We have  $\mathcal{M} = \{AHA^{-1} : A \in \mathrm{GL}_{m+n}(\mathbb{C})\}$ , where  $H \subseteq \mathfrak{gl}_{m+n}(\mathbb{C})$  is the set of upper triangular matrices. We say a map  $X : \mathfrak{gl}_{m+n} \rightarrow \mathfrak{gl}_{m+n}$  preserves a flag  $V_0 \subseteq \cdots \subseteq V_{m+n}$  if  $XV_i \subseteq V_i$  for each  $i$ . Let  $E_0 \subseteq \cdots \subseteq E_{m+n}$  be the standard flag of  $\mathbb{C}^{m+n}$ . Since  $H$  is the set of  $X$  which preserve  $E$ ,

$$\mathcal{M} = \{\{X : \forall i. X(AE_i) \subseteq AE_i\} : A \in \mathrm{GL}_n(\mathbb{C})\}.$$

So, for  $X \in \mathcal{N}$ ,

$$\pi^{-1}(X) \cong \{(V, a, b) : \forall i. A_{X,a,b}V_i \subseteq V_i\}.$$

In this section we will find the irreducible components of  $\pi^{-1}(X)$ . Since  $\pi^{-1}(X) \cong \pi^{-1}(AXA^{-1})$  for any invertible  $A$ , we will assume that  $X$  is in Jordan normal form.

Let  $\lambda$  be the shape of  $X$ , and let  $(e_{ij})_{i \leq r, j \leq \lambda_i}$  be a Jordan basis for  $X$ , with  $Xe_{ij} = e_{i,j-1}$ . Let  $f_1, \dots, f_n$  be the standard basis for  $\mathbb{C}^n$ , with  $A_{X,a,b}f_i = f_{i+1}$ .

## 4.1 A necessary condition for $A_{X,a,b}$ to be nilpotent

Suppose  $A := A_{X,a,b}$  is nilpotent. Define the *height* of a vector  $v \in \mathbb{C}^{m+n}$  as the smallest  $k \geq 0$  such that  $A_{X,a,b}^k v = 0$ . Clearly, for  $i \leq j$ , we have that the height of  $f_i$  is geq the height of  $f_j$ . Thus, for each  $k \geq 0$ , we must have  $A_{X,a,b}^k f_n \in \langle e_{ij} \rangle_{ij}$ . We have  $Af_n = \sum_{ij} b_{ij} e_{ij}$ . Then  $A^2 f_n = \sum_{ij} b_{ij} (e_{i,j-1} + a_{ij} f_1)$ , so we must have  $\sum_{ij} b_{ij} a_{ij} = 0$ . Similarly, since  $A^3 f_n \in \langle e_{ij} \rangle$ , we see that  $\sum_{ij} b_{ij} a_{i,j-1} = 0$ . Continuing in this way, we obtain that for all  $k \geq 0$ ,

$$\sum_{ij} b_{ij} a_{i,j-k} = 0. \quad (1)$$

In fact this is in fact a sufficient condition for  $A_{X,a,b}$  to be nilpotent, as can be seen from the characteristic polynomial of  $A_{X,a,b}$ . However, we will not do this calculation here, instead just showing that it is sufficient by giving a Jordan basis.

## 4.2 When $X$ is all zeroes

In this case we have  $A(y, z) = (z_n b, (a \cdot y, z_1, \dots, z_{n-1}))$ , and the condition becomes

$$\sum_i b_i a_i = 0.$$

In this case we will be able to write down explicitly the irreducible components of  $F := \{(V, a, b) : \forall i. A_{X,a,b} V_i \subseteq V_i\} \cong \pi^{-1}(X)$ . For any nonnegative  $\delta_0, \delta_1, \dots, \delta_n, \delta_{n+1}$  summing to  $m$ , define the corresponding sequence  $i_0 = \delta_0$ ,  $i_n = \delta_{n+1} + i_n$ , and for  $j \in \{1, \dots, n\}$ ,  $i_j = i_{j-1} + 1 + \delta_j$ . Let  $E$  be the span of the  $e_i$ 's, and let  $E' = \{(x, 0) \in E : x \cdot a = 0\}$ , where the dot is the  $m$ -dimensional dot product. Then define  $F_\delta$  as the set of  $(V, a, b) \in F$  such that

- $b \in V_{i_0} \subseteq E'$
- for all  $j \in \{1, \dots, n\}$ , we have  $f_j \in V_{i_j} \subseteq E' + \langle f_1, \dots, f_j \rangle$

I claim that the  $F_\delta$ 's are the irreducible components of  $F$ . To begin, I show that their union is  $F$ .

**Lemma 4.1.** *Let  $(V, a, b) \in \mathcal{F}$ . Write  $f_0 = b$ , and  $F = \langle f_0, f_1, \dots, f_n \rangle$ . For each  $i$ , either  $F \subseteq V_i$ , or else there exists  $j$  such that  $e_j \notin V_i$ , but  $\langle e_0, \dots, e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_1, \dots, e_j \rangle$ .*

*Proof.* For  $i = 0$ , we may take  $j = 0$ . Now assume the statement holds for  $i$ , and we will prove it for  $i + 1$ . If  $F \subseteq V_i$ , then  $F \subseteq V_{i+1}$ , and we are done.

So, suppose there is  $j$  such that  $e_j \notin V_i$ , but  $\langle e_0, \dots, e_{j-1} \rangle \subseteq V_i \subseteq E' + \langle e_0, \dots, e_j \rangle$ . We have two cases: either  $V_{i+1} = V_i + \langle e_j \rangle$ , or not.

- If so, then either  $j = n$ , in which case  $F \subseteq V_{i+1}$ , or else  $j \neq n$ , in which case  $e_{j+1} \notin V_{i+1}$ , but  $\langle e_0, \dots, e_j \rangle \subseteq V_{i+1} \subseteq E' + \langle e_0, \dots, e_{j+1} \rangle$ .
- If not, then  $e_j \notin V_{i+1}$ . Let  $v_{i+1}$  be so that  $V_{i+1} = V_i + \langle v_{i+1} \rangle$ . I just need to show that  $v_{i+1} \in Y + \langle e_1, \dots, e_j \rangle$ . It suffices to show that  $v_{i+1}^\top e_k = 0$  for  $k > j$ . And to do this, it suffices to show that  $(Av_{i+1})^\top e_{k-1} = 0$  for  $k > j$ .

Note that  $Av_{i+1} \in V_i \cap A\mathbb{C}^{2n} \subseteq \langle e_1, \dots, e_j \rangle$ . So, for  $k > j + 1$  it is clear that  $(Av_{i+1})^\top e_{k-1} = 0$ . Now, suppose for contradiction that  $(Av_{i+1})^\top e_j \neq 0$ . Then  $Av_{i+1}$  is linearly independent of  $e_1, \dots, e_{j+1}$ . Since  $Av_{i+1} \in \langle e_1, \dots, e_j \rangle$ , it follows that  $e_j \in \langle e_1, \dots, e_{j-1}, Av_{i+1} \rangle \subseteq V_i$ , a contradiction.

□

**Corollary 4.2.**  $\mathcal{F} = \bigcup_\delta \mathcal{F}_\delta$

*Proof.* Let  $(V, a, b) \in F$ . Let  $i_0 = \min\{i : b \in V_i\}$ , and let  $i' = \max\{i : V_i \subseteq E'\}$ . We want that  $b \in V_{i_0} \subseteq E'$ , so we want that  $i_0 \leq i'$ . For contradiction, suppose  $i' < i_0$ .

□

**Lemma 4.3.** Each  $\mathcal{F}_\delta$  is a closed subvariety of  $\mathcal{F}$ .

*Proof.*

□

**Lemma 4.4.** Each  $\mathcal{F}_\delta$  is irreducible of dimension  $m$ .

*Proof.* Let  $\mathcal{E}$  be the variety of partial flags of  $E$  of shape  $(\delta_0, \dots, \delta_{n+1})$ . Define  $g : \mathcal{F}_\delta \rightarrow \mathcal{E}$  by

$$(V, a, b) \mapsto 0 \subseteq V_{i_0} \cap E \subseteq V_{i_1} \cap E \subseteq \dots \subseteq V_{i_{n+1}} \cap E = E.$$

It is clear that  $g$  is surjective, as the definition of  $\mathcal{F}_\delta$  places no restriction on the intersections  $V_i \cap E$ . Now let's look more closely at the fibers  $g^{-1}(U)$ .

First let's find the flags  $V$  such that there exist  $a, b$  with  $g(V, a, b) = U$ . We see that, for instance,  $V_{i_0} \cap E = U_1$ . In fact  $V_{i_0} \subseteq E$ , so  $V_{i_0} = U_1$ . But we are free to choose the vector spaces between 0 and  $V_{i_0}$  however we wish, so we get some degrees of freedom like  $\mathcal{F}_{\delta_0}$ , the complete flag variety on  $\mathbb{C}^{\delta_0} \cong V_{i_0}/0$ . Similarly, for every  $j = 1, \dots, n+1$ , we can choose the vector spaces between  $V_{i_{j-1}}$  and  $V_{i_j}$  arbitrarily, so we get degrees of freedom like  $\mathcal{F}_{\delta_{i_j}}$ , the complete flag variety on  $\mathbb{C}^{\delta_{i_j}} \cong V_{i_j}/V_{i_{j-1}}$ . Finally, to meet the constraint of  $(V, a, b)$  being in  $\mathcal{F}_\delta$ , we can choose any  $b \in U_1$  and any  $a$  such that  $\bar{a} \in U_n^\perp$ . Thus we get an isomorphism

$$g^{-1}(U) \cong \mathcal{F}_{\delta_0} \times \dots \times \mathcal{F}_{\delta_{n+1}} \times \mathbb{C}^{\delta_0} \times \mathbb{C}^{\delta_{n+1}}.$$

□

**Theorem 4.5.** *The  $\mathcal{F}_\delta$ 's are the irreducible components of  $\mathcal{F}$ .*

### 4.3 When $X$ is a Jordan block

Suppose  $X$  is a single Jordan block of size  $m$ . In this case we just write the basis of  $\mathbb{C}^m$  as  $e_1, \dots, e_m$ , and the above condition on  $a$  and  $b$  simply becomes

$$\forall k \geq 0. \sum_{i=k+1}^m b_i a_{i-k} = 0. \quad (2)$$

This condition can be simplified even more.

**Lemma 4.6.** *The condition (1) holds iff there exist nonnegative  $m_1, m_2, m_3$ ,  $a_1, \dots, a_{m_3}$ , and  $b_1, \dots, b_{m_3}$  satisfying the following conditions.*

- $m = m_1 + m_2 + m_3$
- $a = (0, 0, \dots, 0, a_1, \dots, a_{m_3})$
- $b = (b_1, \dots, b_{m_1}, 0, 0, \dots, 0)$
- If  $m_1 \neq 0$ , then  $a_{m_1} \neq 0$
- If  $m_3 \neq 0$ , then  $b_1 \neq 0$

*Proof.* If  $a$  and  $b$  are of this form, then for every  $k \geq 0$  we have  $b_i a_{i-k} = 0$ , so clearly the condition holds.

Now suppose (1) holds. If  $a$  or  $b$  is zero, this is trivial. Otherwise, let  $m_1 = \max\{i : a_i \neq 0\}$ , and let  $m_3 = \min\{i : b_i \neq 0\}$ . We just need to show that  $m_1 < m_3$ . For contradiction, suppose  $m_1 - m_3$  is nonnegative. Then by (1),  $0 = \sum_i b_i a_{i-(m_1-m_3)} = b_1 a_{m_1}$ , contradicting that  $b_1$  and  $a_{m_1}$  are nonzero.  $\square$

#### 4.4 In the case of general $X$

In the case where  $X$  was a single Jordan block, it was helpful to have the condition that all the nonzero  $a$ 's were to the right of all the nonzero  $b$ 's. In the general case, it is not clear that such a thing should be true. And indeed it is not. However, we will see that we can change basis of  $\mathbb{C}^m$  so that a similar thing is true.

Let  $c_{i',j'}^i \in \mathbb{C}$  be some arbitrary coefficients. Given our Jordan basis  $e_{ij}$  of  $\mathbb{C}^m$ , we can define a new Jordan basis  $e'_{ij}$  of  $\mathbb{C}^m$  by

$$e'_{i,\lambda_i} = e_{i,\lambda_i} + \sum_{\{(i',j') : i' \geq i\} \setminus \{(i,j)\}} c_{i',j'}^i e_{i',j'}.$$

Then we define

$$e'_{i,j} = X^{\lambda_i-j} e'_{i,\lambda_i} = e_{ij} + \sum_{\{(i',j') : i' \geq i\} \setminus \{(i,j)\}} c_{i',j'}^i e_{i',j'-(\lambda_i-j)}.$$

Let us calculate the new value  $a'$  of  $a$  in this new basis. That is, we want the values  $a'_{ij}$  that satisfy  $Ae'_{ij} = e'_{i,j-1} + a'_{ij}f_1$ . Looking at the definition of  $e'_{ij}$  above, we see that

$$a'_{ij} = a_{ij} + \sum_{\{(i',j') : i' \geq i\} \setminus \{(i,j)\}} c_{i',j'}^i a_{i',j'-(\lambda_i-j)}.$$

Now, for each  $i$ , choose the coefficients  $c_{i',j'}^i$  to maximize the value  $\max\{j : (a'_{i,1}, \dots, a'_{i,j}) = 0\}$ . In other words, we choose  $c_{i',j'}^i$  to zero out the largest possible prefix of  $(a_{i,1}, \dots, a_{i,\lambda_i})$ .

**Lemma 4.7.** *If the condition of Equation (1) on  $a'$  and  $b'$  holds, then there exist nonnegative  $p_i, q_i, r_i, \alpha_{i,j}, \beta_{i,j}$  such that for each  $i$ , the following conditions are satisfied.*



- $\lambda_i = p_i + q_i + r_i$
- $(a'_{i,1}, \dots, a'_{i,\lambda_i}) = (0, 0, \dots, 0, \alpha_{i,1}, \dots, \alpha_{i,r_i})$
- $(b'_{i,1}, \dots, b'_{i,\lambda_i}) = (\beta_{i,1}, \dots, \beta_{i,p_i}, 0, 0, \dots, 0)$
- If  $r_i \neq 0$ , then  $\alpha_{i,1} \neq 0$
- If  $p_i \neq 0$ , then  $\beta_{i,p_i} \neq 0$

*Proof.* We begin with the first row of  $\lambda$ , namely  $i = 1$ . Let  $k = \min\{j : a'_{1,j} \neq 0\}$  and  $l = \max\{j : b'_{1,j} \neq 0\}$ . I just need to show that  $l < k$ . Suppose not. Then  $(l - k)$  is nonnegative, and by Equation (1) we have

$$0 = \sum_{ij} b'_{ij} a'_{j-(l-k)} = b'_{1,l} a'_{1,k} + \sum_{i' \geq 1, (i', j') \neq (1, \lambda_1)} b'_{i'j} a'_{j-(l-k)}.$$

This suggests defining

$$e''_{1,\lambda_1} = b'_{1,l} e'_{1,\lambda_1} + \sum_{i' \geq 1, (i', j') \neq (1, \lambda_1)} b'_{i', j' - (\lambda_1 - l)} e'_{i'j'}.$$

Then for  $j \leq k$ ,

$$a''_{1,j} = b'_{1,l} a'_{1,j} + \sum_{i' \geq 1, (i', j') \neq (1, \lambda_1)} b'_{i', j' - (\lambda_1 - l)} a'_{i', j' - (\lambda_1 - j)} =$$

$$\sum_{i', j'} b'_{i', j' - (\lambda_1 - l)} a'_{i', j' - (\lambda_1 - j)} = \sum_{i', j'} b'_{i', j'} a'_{i', (j' + (\lambda_1 - l)) - (\lambda_1 - j)} = \sum_{i', j'} b'_{i', j'} a'_{i', j' - (l - j)}.$$

This last thing is zero by Equation (1), using that  $l - j \geq 0$ . But this contradicts the definition of the  $c_{i'j'}^1$ , because we made it so that  $(a''_{1,1}, \dots, a''_{1,k}) = 0$ .  $\square$

## 4.5 TODO

- Why are SOn flags what they are.

## References

- [1] N. Chriss and victor ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.