0.1   Momentum Equation

Conservation of mass is stated in the usual way. For low Reynolds number conservation of momentum becomes a force balance statement for respective phases:

$$\nabla \cdot [\phi \sigma_{\mathbf{f}}] + \phi \rho_f \mathbf{g} + \mathbf{F_f} = \mathbf{0}$$
$$\nabla \cdot [(1 - \phi)\sigma_{\mathbf{s}}] + (1 - \phi)\rho_s \mathbf{g} + \mathbf{F_s} = \mathbf{0} \tag{1}$$

It is assumed that the interphase force between phases respects Newton's third law of motion, i.e. $\mathbf{F_f} = -\mathbf{F_s}$. We assume the following form for $\mathbf{F_s}$:

$$\mathbf{F_s} = d(\mathbf{v} - \mathbf{V}) - P\nabla\phi + \mathbf{F}_\sigma, \tag{2}$$

where $\mathbf{F}_\sigma$ is the force that arises from interfacial surface tension. For the time being we make no additional assumptions about the form of this term. The second unknown in the above is the interfacial pressure, P. This is conventionally taken to be equal to the mechanical pressure in the fluid, but instead we will take the general form $P = P_f + P_b$. The fluid momentum equation becomes (after taking $\sigma_{\mathbf{f}} = -P_f \mathbb{I}$)

$$-\nabla \cdot [\phi P_f \mathbb{I}] + \phi \rho_f \mathbf{g} - d(\mathbf{v} - \mathbf{V}) + P\nabla\phi - \mathbf{F}_\sigma$$
$$= -\nabla \cdot [\phi P_f \mathbb{I}] + \phi \rho_f \mathbf{g} - d(\mathbf{v} - \mathbf{V}) + (P_f + P_b)\nabla\phi - \mathbf{F}_\sigma = 0, \tag{3}$$

which, after some rearrangement becomes

$$\phi(\mathbf{v} - \mathbf{V}) = -\frac{\phi^2}{d}\left[\nabla P_f - \rho_f \mathbf{g} - \frac{P_b}{\phi}\nabla\phi + \frac{\mathbf{F}_\sigma}{\phi}\right], \tag{4}$$

which is similar the standard Darcay equation but with some extra terms. Similarly, the solid momentum equation becomes (after taking $\sigma_{\mathbf{s}} = \tau - P_s \mathbb{I}$)

$$\nabla \cdot [(1 - \phi)\tau_{\mathbf{s}}] - \nabla \cdot [(1 - \phi)P_s \mathbb{I}] + (1 - \phi)\rho_s \mathbf{g} + d(\mathbf{v} - \mathbf{V}) - (P_f + P_b)\nabla\phi + \mathbf{F}_\sigma = 0. \tag{5}$$

To obtain the desired solid momentum equationwe add the equations in (1), by which the interphase force terms drop out entirely, and we are left with

$$\nabla \cdot [(1 - \phi)\tau_{\mathbf{s}}] + \nabla[\phi(P_s - P_f)] - \nabla P_s + \bar{\rho}\mathbf{g} = 0, \tag{6}$$

where $\bar{\rho}$ is the phase-avergaed density. We now introduce another closure:

$$P_f - P_s = \zeta\nabla \cdot \mathbf{V} + P_a, \tag{7}$$

so that

$$\nabla \cdot [(1 - \phi)\tau_{\mathbf{s}}] - \nabla[\phi(\zeta\nabla \cdot \mathbf{V} + P_a)] - \nabla P_s + \bar{\rho}\mathbf{g} = 0, \tag{8}$$

which is no different in content to the standard momentum equation (all of our other modifications have dropped out), except that there is an extra $-\nabla(\phi P_a)$ terms floating about.

0.2   Surface Energy

We assume that $\mathbf{F}_\sigma = \nabla E_c$, where E_c is the interfacial crack energy per unit volume. Following Riley and Kohstedt (1991) we write

$$E_c = \frac{1}{V}(A_{ss}\gamma_{ss} + A_{sl}\gamma_{sl}) \tag{9}$$

where γ_{ss} and γ_{sl} are the respective solid–solid and solid–liquid interfacial free energies. V is the volume of a representative cube of length d, so that $V = d^3$. A_{ss} is the average solid–solid surface

area within the volume – the average of surface area of dry cracks times the total number of dry cracks. A_{sl} is the average surface area of fluid filled cracks within the volume times the number of fluid filled cracks. We assume that wet and dry cracks have the same dimensions, and that the average surface area of a crack is c^2. We let $N_t = N_d + N_w$ be the average number of cracks within a volume, such that N_d and N_w are the respective average numbers of dry and wet cracks. Then

$$A_{ss} = (N_t - N_w) \times c^2 \tag{10}$$

$$= N_t c^2 \left(1 - \frac{N_w}{N_t}\right) \tag{11}$$

$$= N_t c^2 (1 - f_w), \tag{12}$$

where f_w is the ratio of wet cracks to the total number of cracks. Similarly

$$A_{sl} = N_t c^2 f_w. \tag{13}$$

So that

$$E_c = \frac{c^2 N_t}{d^3} \left((1 - f_w)\gamma_{ss} + f_w \gamma_{sl}\right). \tag{14}$$

We let $\alpha = \frac{c^2 N_t}{d^3}$, the specific surface area of cracks within the volume.

Now we want to relate the defined quantities to the porosity. if δ is a characteristic crack length, then it makes sense to say that

$$\phi = \frac{N_w c^2 \delta}{d^3} = f_w \alpha \delta. \tag{15}$$

Assumption: all fluid exists in cracks, and nowhere else – is this right? Writing E_c with porosity terms gives

$$E_c = \frac{1}{\delta} \left((\alpha\delta - \phi)\gamma_{ss} - \phi\gamma_{sl}\right). \tag{16}$$

Then

$$\mathbf{F}_\sigma = \nabla E_c = -\frac{\gamma_{ss} + \gamma_{sl}}{\delta} \nabla \phi \tag{17}$$