Notation

In this subject, a variety of notation is used.

- $|$ read "such that" used largely in set definitions: $\{x \in \mathbb{R} \mid x \geq 2\}$
- $\forall$ read "for all".
- $\exists$ read "there exists".
- $\equiv$ read "is equivalent to' used to signify function equality, etc.
- $\ll$ read "much less than".
- $\log$ denotes the natural logarithm, $\ln$.
- Inverse trigonometric functions are written $\arcsin$ rather than $\sin^{-1}$.

Some common sets have specific symbols associated with them.

- Natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$. Exclusive of 0 in this course.
- Integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rational numbers, $\mathbb{Q} = \left\{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\right\}$
- Real numbers, $\mathbb{R}$
- Complex numbers, $\mathbb{C} = \left\{x + iy \mid x, y \in \mathbb{R}, i^2 = -1\right\}$
- xy plane, $\mathbb{R}^2 = \left\{(x, y) \mid x, y \in \mathbb{R}\right\}$
- Three dimensional space, $\mathbb{R}^3 = \left\{(x, y, z) \mid x, y, z \in \mathbb{R}\right\}$

Limits, Continuity, Sequences, Series

Limits

Limits form the fundamental concept behind the definition of a derivative; the instantaneous rate of change of a function at a point is defined in terms of a limit.

A limit is defined as follows. For a function $f(x)$, the limit of $f(x)$ as x approaches a is L, written as

$$\lim_{x \to a} f(x) = L$$

If $f(x)$ gets continually closer to L but $x \neq a$. If a limit exists, it must be a unique finite real number.

Examples

If $f(x)$ is defined as follows

$$f(x) = \begin{cases} 2x, x \neq 1 \\ 4, x = 1 \end{cases}$$

Evaluate the following

$$\lim_{x \to 1} f(x)$$

As $f(x)$ gets arbitrarily close to 2 whenever x is close to but not equal to 1, therefore the limit as $x \to 1$ is 2.

Evaluate the following limit.

$$\lim_{x \to 0} f(x) = \frac{1}{x^2}$$

As $f(x)$ is unbounded as $x \to 0$. Therefore, $f(x)$ cannot be made arbitrarily close to any one number and the limit does not exist.

Evaluate the following limit.

$$\lim_{x \to 0} \begin{cases} 1, x < 0 \\ 2, x \geq 0 \end{cases}$$

As x approaches 0 from the right, it grows arbritrarily close to 2, while as it approaches from the left it grows arbritrarily close to 1. As it doesn't grow arbritrarily close to a single value, no limit exists here.

Additional Limt Notation

In the previous examples, we noticed that a common way to examine limits is by considering the values the function approaches from each side. Because this is such a common construct, notation exists for the left and right limits independently. The right and left limits of the function f approaching 0, examined in the final example above are written as

$$\lim_{x\to 0^+} f(x) = 2$$

$$\lim_{x\to 0^-} f(x) = 1$$

In general, for a limit to exist the following statement must be true

$$\lim_{x\to a} = L \Leftrightarrow \lim_{x\to a^-} = L \text{ and } \lim_{x\to a^+} = L$$

i.e. for the limit to be L, both the left and right limits must be L.

Limit Laws

For two real valued functions f and g, and $c \in \mathbb{R}$ a constant. If the limits $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist, then the following limit laws apply.

$$\lim_{x\to a} \big[f(x) + g(x)\big] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \big[cf(x)\big] = c \lim_{x\to a} f(x)$$

$$\lim_{x\to a} \big[f(x)g(x)\big] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \left(\lim_{x\to a} g(x) \neq 0\right)$$

$$\lim_{x\to a} c = c$$

$$\lim_{x\to a} x = a$$

Example

Using the limit laws, evaluate the limit

$$\lim_{x\to 2} \frac{x^3+2x^2-1}{5-3x}$$

$$\lim_{x\to 2} \frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to 2} x^3+2x^2-1}{\lim_{x\to 2} 5-3x} = \frac{8+8-1}{5-6} = \frac{15}{-1} = -15$$

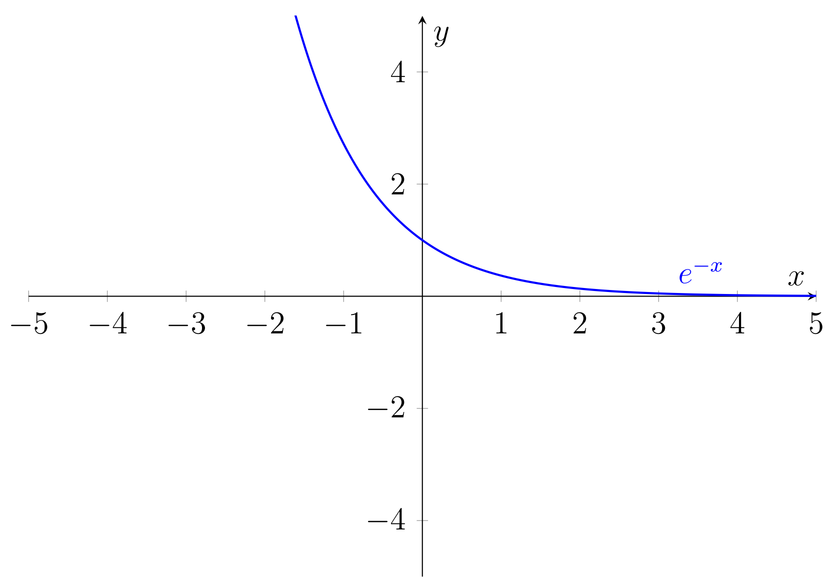
One could use more of the limit laws to break this down further; for instance breaking the added terms into individual limits, or breaking the x^n terms down using the product law.

Limits and Infinity

We can talk about a limit as $x \to \infty$, referring to what happens to the limit as x is made arbritrarily large. So the limit

$$\lim_{x\to\infty} f(x) = L$$

Is stating that as x is made arbritrarily larger, $f(x)$ becomes arbitrarily closer to L. L must be finite for $f(x)$ to take it's value.



For example, in the above plot we can see that as x becomes arbritrarily close to infinity, e^{-x} becomes arbritrarily close to 0. Therefore,

$$\lim_{x \to \infty} e^{-x} = 0$$

While this process of examining a graph and noting its convergence is practical for certain examples, it is a little laborious in the long term, so we use the constructs of standard limits to solve many problems.

Standard Limits

The following limits can be used without proof in this subject, with their truth taken as gospel.

$$\lim_{x \to \infty} \tfrac{1}{x^p} = 0 \quad (p > 0)$$

$$\lim_{x \to \infty} r^x = 0 \quad (0 \leq r < 1)$$

$$\lim_{x \to \infty} \tfrac{1}{x^p} = 0 \quad (p > 0)$$

$$\lim_{x \to \infty} r^x = 0 \quad (0 \leq r < 1)$$

$$\lim_{x \to \infty} a^{\frac{1}{x}} = 1 \quad (a > 0)$$

$$\lim_{x \to \infty} x^{\frac{1}{x}} = 1$$

$$\lim_{x \to \infty} \tfrac{\log(x)}{x^p} = 0 \quad (p > 0)$$

$$\lim_{x \to \infty} \left(1 + \tfrac{a}{x}\right)^x = e^a \quad (a \in \mathbb{R})$$

$$\lim_{x \to \infty} \tfrac{x^p}{a^x} = 0 \quad (p \in \mathbb{R}, a > 1)$$

For example, in the case of e^{-x}, we can use the second limit from above by taking $r = \frac{1}{e}$.

Terminology

If a limit exists, we can state that $f(x)$ *converges* as x approaches a. Inversely, we can state $f(x)$ *diverges* as x approaches a.

For example, as $\sin(x)$ oscillates between -1 and 1, it cannot approach a single number and therefore diverges as $x \to \infty$.

It is important to note that ∞ isn't a number; in general it denotes "any arbitrarily large number". A limit cannot be equal to infinity. Because infinity is not a number, certain cases become indeterminate.

$$\lim_{x\to\infty} \frac{3x^2 - 2x + 3}{x^2 + 4x + 4}$$

Here, we can see that both the numerator and the denominator approach infinity as $x \to \infty$. We cannot therefore divide their individual limits to find the overall limit and must instead alter the form to find the limit.

In this case, we can do this by dividing numerator and denominator by $\frac{1}{x^2}$.

$$\frac{\frac{1}{x^2}}{\frac{1}{x^2}} \cdot \frac{3x^2 - 2x + 3}{x^2 + 4x + 4} = \frac{3 - \frac{2}{x} + \frac{3}{x^2}}{1 - \frac{4}{x} + \frac{4}{x^2}}$$

$$\frac{\lim\limits_{x\to\infty} 3 - \frac{2}{x} + \frac{3}{x^2}}{\lim\limits_{x\to\infty} 1 - \frac{4}{x} + \frac{4}{x^2}} = \frac{3}{1} = 3$$

By modifying the fraction and then applying limit laws, we can solve this initially indeterminate limit.

The Sandwhich Theorem

The Sandwhich Theorem states that if $g(x) \leq f(x) \leq h(x)$ when $x \approx a$ but $x \neq a$ and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

Then $\lim\limits_{x\to a} f(x) = L$. In essence, it states that if a function lies between two other functions who each converge to L at a, then that function must also converge to a. This theorem can also be used to solve functions of the indeterminate form $\infty - \infty$. For example, $\lim\limits_{x\to\infty} f(x) = \sqrt{x^2+1} - x$ is of this form. We can simplify to some degree, but we need the Sandwhich Theorem to finish the problem.

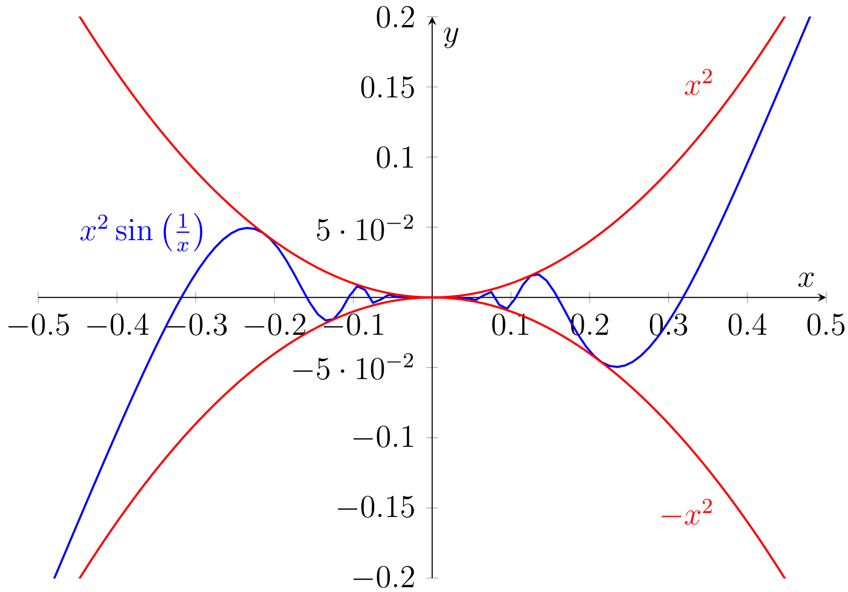
$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

Looking at this function, it looks like both $\sqrt{x^2+1}$ and x become arbitrarily large as $x\to\infty$. Thus, we would expect this function to converge to 0 as it approaches ∞. To prove this with the Sandwhich Theorem, we must find a lower bound and upper bound that each converge to 0.

A lower bound for this function is easy; it never drops below 0, so $g(x)\equiv 0$ will do nicely. To find an upper bound, we can simply make the denominator smaller, so perhaps $h(x)=\frac{1}{x}$ is a good fit. As we know that both of these functions converge to 0 as $x\to\infty$ through the limit laws and standard limits, we can confidently state per the Sandwhich Theorem that $f\to 0$ as $x\to\infty$.

Example

A function which lends itself to use of the Sandwhich Theorem is $x^2\sin(\frac{1}{x})$



As shown on the above plot, the function never strays beyond the bounds of two parabolas. We can therefore evaluate its limit through the limits of the two bounding functions.

$$\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0 \Rightarrow \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

Continuity

Continuity is a property of a function which essentially describes the "smoothness" of the function. For a function f to be continuous at a point x, the limit

$$\lim_{x\to a} f(x) = f(a)$$

Must be true; i.e. the value of f at x must be the value that f approaches as it becomes arbritrarily close to x. As a simple example, let us check if f is continuous at $x = 1$.

$$f(x) = \begin{cases} 2x, x \neq 1 \\ 4, x = 1 \end{cases}$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} 2x = 2$$

$$f(1) = 4 \neq 2$$

$$\therefore f \text{ is not continuous at } 1$$

Continuity Theorems

If f and g are real valued functions and c is a constant, then assuming f and g are continuous at $x = a$, the following functions are additionally continuous at $x = a$.

- $f + g$
- cf
- fg
- $\frac{f}{g}$ if $g(a) \neq 0$

If f is continuous at $x = a$ and g is continuous at $x = f(a)$, then $g \circ f$ is continuous at $x = a$. Thus if two continuous functions are composed, the resultant function will in addition be continuous.

All of the following function types are continuous across their domain.

- Polynomials
- Trigonometric functions
- Exponential functions
- Logarithmic functions
- nth root functions
- Hyperbolic functions

Example

For which values of x is f continuous?

$$f(x) = \frac{\sin(x^2 + 1)}{\log(x)}$$

We know that $\sin$ is continuous across $\mathbb{R}$, as is $x^2 + 1$. $\log(x)$ is defined for $\mathbb{R}^+$. Using function composition, we know that $\sin(x^2 + 1)$ is continuous on $\mathbb{R}$, and using division we can see that f will be continuous for all values in the domain of $\log(x)$ where $\log(x) \neq 0$, i.e. $\mathbb{R}^+ \setminus \{1\}$.

If f is continuous at b and $\lim\limits_{x \to a} g(x) = b$ then

$$\lim_{x \to a} f\left[g(x)\right] = f\left[\lim_{x \to a} g(x)\right] = f(b)$$

Example

$$\lim_{x \to \infty} \sin\left(e^{-x}\right) = \sin\left(\lim_{x \to \infty} e^{-x}\right) = \sin(0) = 0$$

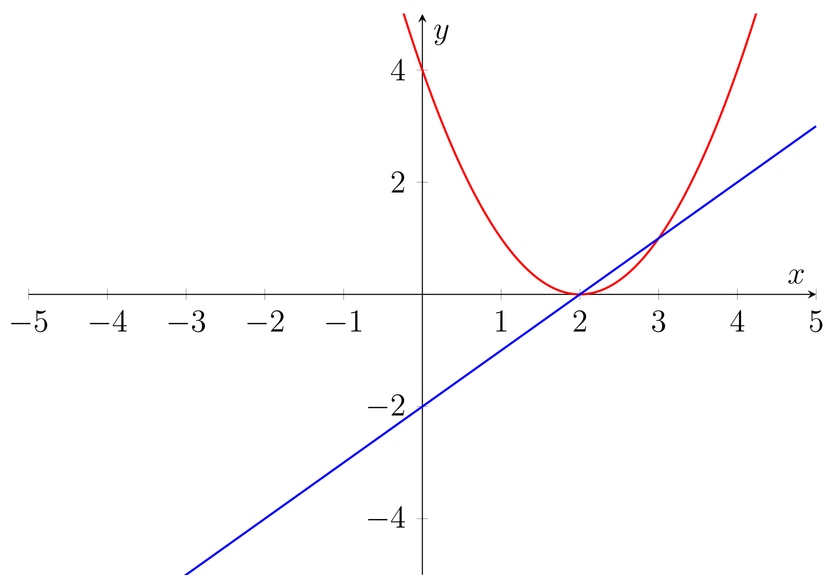
We can only do this because $\sin$ is continuous on $\mathbb{R}$.

Derivatives

The derivative is defined using a limit.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If this limit exists, the function is differentiable at a. Geometrically, this implies that a tangent line can be drawn at a on the graph with gradient yielded by the above limit.



If a function is differentiable at $x = a$, the the function is continuous at that point.

L'Hôpital's Rule

Given f and g are differentiable functions near some value $x = a$ and $g'(x) \neq 0$ near a but $\neq a$. If the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Has an indeterminate form of

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

Then L'Hôpital's Rule is applicable.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

For example, we can solve limits like the one below much more easily.

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \frac{\cos(0)}{1} = 1$$

L'Hôpital's Rule can be applied repeatedly to the same function as long as its conditions still hold. Sometimes the form of a function must be modified to have a quotient before the rule can be applied to it.

Rigour

Thus far, we have used the phrase *arbritrarily close* without a proper definition of what that means. A more formal definition of this exists, where we pick an arbritrary positive number ϵ. When we do this, there is another positive number δ for which $\left|f(x) - L\right| < \epsilon$ when $0 < |x - a| < \delta$.

Sequences

A sequence is a function which maps the natural numbers to $\mathbb{R}$ i.e. of the form $f : \mathbb{N} \to \mathbb{R}$. They can be thought of as ordered lists of real numbers, or as functions where $f(n)$ is the nth number in the sequence. A common notation for a sequence a is $\{a_n\}$, where a_n is a function defining the nth element.

A sequence has the limit L if a_n can be made arbritrarily close to L by making n sufficiently large. To indicate this, we write

$$\lim_{n \to \infty} a_n = L$$

If a limit L exists, then the sequence converges, otherwise it diverges. The only major difference between limits on sequences and limits on functions is that sequences deal with discrete rather than continuous values. This matters significantly in cases like trigonometric functions. If a function converges, then a sequence with that function will also converge.

The limit laws and Sandwhich Theorem apply to limits for sequences as well. For the Sandwhich Theorem, the sandwhiching sequences can consider only a range of n, greater than some value N.

Examples

For each of the following sequences, determine whether they converge or diverge.

- $\left\{\frac{1}{n}\right\}$ converges to 0.
- $\left\{-1^{n-1}\right\}$ diverges, oscillating between 1 and -1.
- $\{n\}$ diverges to ∞.

Standard Limits of Sequences

A set of standard limits of sequences exist, which can be used to solve other limits involving sequences.

$$\lim_{n\to\infty} \tfrac{1}{n^p} = 0 \quad (p > 0)$$

$$\lim_{n\to\infty} r^n = 0 \quad (|r| < 1)$$

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad (a > 0)$$

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} \tfrac{a^n}{n!} = 0 \quad (a \in \mathbb{R})$$

$$\lim_{n\to\infty} \tfrac{\log(n)}{n^p} = 0 \quad (p > 0)$$

$$\lim_{n\to\infty} \left(1 + \tfrac{a}{n}\right)^n = e^a \quad (a \in \mathbb{R})$$

$$\lim_{n\to\infty} \tfrac{n^p}{a^n} = 0 \quad (p \in \mathbb{R}, a > 1)$$

Examples

$$\lim_{n\to\infty} \left[\left(\frac{n-2}{n}\right)^n + \frac{4n^2}{3^n}\right] = \lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n + \lim_{n\to\infty} \frac{4n^2}{3^n}$$

$$\left(\frac{n-2}{n}\right)^n = \left(1-\frac{2}{n}\right)^n \Rightarrow \lim_{n\to\infty}\left(\frac{n-2}{n}\right)^n = e^{-2} \Rightarrow$$
$$\lim_{n\to\infty}\left(\frac{n-2}{n}\right)^n + \lim_{n\to\infty}\frac{4n^2}{3^n} = e^{-2}+0 = e^{-2}$$

Using standard limits, we can easily find the solution to this fairly complex-looking limit. The key here is recognising that the first term can be easily rearranged to the exponential standard limit, with the second term fairly naturally simplifying to another standard limit.

$$a_n = \frac{3^n+2}{4^n+2^n}, n \geq 1$$
$$\frac{3^n+2}{4^n+2^n} \cdot \frac{\frac{1}{4^n}}{\frac{1}{4^n}} = \frac{\left(\frac{3}{4}\right)^n + \frac{2}{4^n}}{1+\left(\frac{1}{2}\right)^n}$$
$$\lim_{n\to\infty} a_n = \frac{0+0}{1+0} = 0$$

By dividing through by the largest term, we find out that it outweighs the terms in the denominator, an that the whole limit collapses to 0. A tool for doing this is the *order hierachy*, which indicates the growth rate of various forms for large n

$$\log(n) \ll n^p \ll a^n \ll n!$$

A situation where the Sandwhich Theorem is very applicable is one where the function in question clearly falls into some well defined range. For example

$$\lim_{n\to\infty}\frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}}$$
$$0 \leq \sin^2\left(\frac{n\pi}{3}\right) \leq 1 \Rightarrow 1 \leq 1+sin^2\left(\frac{n\pi}{3}\right) \leq 2$$
$$\frac{1}{\sqrt{n}} \leq \frac{\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} \leq \frac{2}{\sqrt{n}}$$
$$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0 \Rightarrow$$
$$\lim_{n\to\infty}\frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} = 0$$

Here, we attack the most complex part of the problem and construct two equations which elegantly border it. We then use the limits of these equations to solve the overall limit.

Series

A series arises when one attempts to sum up the values in a sequence. For a sequence $\{a_n\}$, if we add each a_n in order, we create another sequence $\{s_n\}$.

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_2 + a_2 + a_2$$

This *sequence of partial sums* $\{s_n\}$ may or may not converge. In the case that it does, we describe the sum S as

$$S = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(a_1 + a_2 + \ldots + a_n)$$

For example, we can find the sum of a sequence like so:

$$a_n = \left(\frac{1}{2}\right)^n, n \geq 1$$

$$s_1 = a_2 = \frac{1}{2}$$

$$s_2 = a_1 + a_2 = \frac{3}{4}$$

$$s_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1 - 0 = 1$$

This is example is a *geometric series*. More generally, a series with terms a_n is denoted with the sum

$$s_n = \sum_{n=1}^{\infty} a_n$$

Where the value on the bottom $(n = 1)$ is the starting value of n and the value on the top is the value we limit towards. If the limit of s_n exists, the series converges. If it does not, the series diverges. For the sequence $\{n\} = 1, 2, 3, 4, \ldots$ we have the series

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + \ldots$$

Because the sequence and series both diverge to infinity, the series diverges. Decimal representations of numbers can be thought of as a series. If we take

$$\left\{\frac{1}{10^n}\right\} = 0.1, 0.01, 0.001, \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{10^n} = 0.1 + 0.01 + 0.001 + \ldots = 0.1111\ldots$$

A general case exists, for a number $x \in (0,1)$ with the decimal digits $d_1, d_2, \ldots$ then

$$x = 0.d_1 d_2 d_3 \ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

It is important to note that a sequence differs from a sequence of partial sums differs from a series. The sequence of partial sums represents a *finite approximation* of the sum of the sequence.

Properties of Series

If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are converging series, and $c \in \mathbb{R} \backslash \{0\}$ is a constant then

- $\sum\limits_{n=1}^{\infty} (a_n + b_n) = \sum\limits_{n=1}^{\infty} a_n + \sum\limits_{n=1}^{\infty} b_n$also converges.
- $\sum\limits_{n=1}^{\infty} (ca_n) = c \sum\limits_{n=1}^{\infty} a_n$ and still converges. If $\sum\limits_{n=1}^{\infty} a_n$ diverges, then $\sum\limits_{n=1}^{\infty} (ca_n)$ also diverges.

Geometric Series

A geometric series has the form

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$$

Where $a \in \mathbb{R} \setminus \{0\}$ and $r \in \mathbb{R}$. If $|r| < 1$, the series will converge while if $|r| \geq 1$ the series will diverge. In the case that $|r| < 1$, we have that

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

In addition, there is a general form for a geometric series (though it is somewhat less elegant)

$$\sum_{k=0}^{n} ar^k = \frac{a\left(1-r^{n+1}\right)}{1-r}$$

Though it must be noted that this formula is only valid for $r \neq 1$.

Harmonic p Series

A harmonic p series has the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

These series converge if $p > 1$ and diverge if $p \leq 1$. For instance

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Converges while

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Diverges.

It is worth noting that thanks to the previously explored properties of series, we know that any multiple of a harmonic p series has the same divergence behaviour as it's base.

Divergence Test

A test exists to determine whether a series diverges. If the limit of the sequence the series is based on is not zero, then the series diverges. More precisely

$$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

If the limit is exactly 0, the series may converge or diverge and another test must be used to identify convergence or divergence.

Comparison Test

For two positive series (all elements non-negative) of the forms

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n$$

- if $a_n \leq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- if $a_n \geq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

In general, we compare a given unknown series (a above) to a harmonic p series or a geometric series, because we know the behaviour of these in some detail. As an example, we can try and identify the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{3+\frac{5}{n}}{2n^2+n+2}$$

$$\frac{3+\frac{5}{n}}{2n^2+n+2} \approx \frac{3}{2n^2}$$

By identifying the most significant terms in the denominator and the numerator, we have identified that we expect the series to converge.

$$\frac{3+\frac{5}{n}}{2n^2+n+2} \leq \frac{8}{2n^2} = \frac{4}{n^2}$$

Here, we wanted to simplify the denominator into something that we recognise and can deal with while making it larger. To do that, we would ideally like to reduce the term to have only a single term in both the numerator and denominator. To simplify and enlarge the denominator, we can simply remove the n denominator of the 5, as we know $n \geq 1$. Removing terms from the denominator inherently makes the fraction larger, so we can just remove the $n+2$ term, rearranging into a harmonic p series.

$$\sum_{n=1}^{\infty} \frac{4}{n^2} \text{ converges} \Rightarrow \sum_{n=1}^{\infty} \frac{3+\frac{5}{n}}{2n^2+n+2} \text{ converges}$$

Ratio Test

For a positive term series of the form

$$\sum_{n=1}^{\infty} a_n$$

With the limit

$$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$

Then, this series has the following properties.

- If $L < 1$, the series converges
- If $L > 1$, the series diverges
- If $L = 1$, the ratio test is inconclusive.

The ratio test is most useful in situations where a_n contains an exponential or factorial function of n. For instance, we can figure out whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{10^n}{n!}$$

$$\lim_{n \to \infty} \frac{10^{n+1}}{(n+1)!} \div \frac{10^n}{n!} = \lim_{n \to \infty} \frac{10^{n+1}}{(n+1)!} \times \frac{n!}{10^n} = \lim_{n \to \infty} \frac{10^1}{1} \times \frac{n!}{(n+1)!}$$

$$= \lim_{n \to \infty} \frac{10}{n+1} = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{10^n}{n!} \text{ converges}$$

Hyperbolic Functions

To deal with hyperbolic functions, it can help to revise a few function properties. Even functions have the property that

$$f(-x) = f(x)$$

For example, x^2 is an even function, as is $cos(x)$. An odd function has the property that

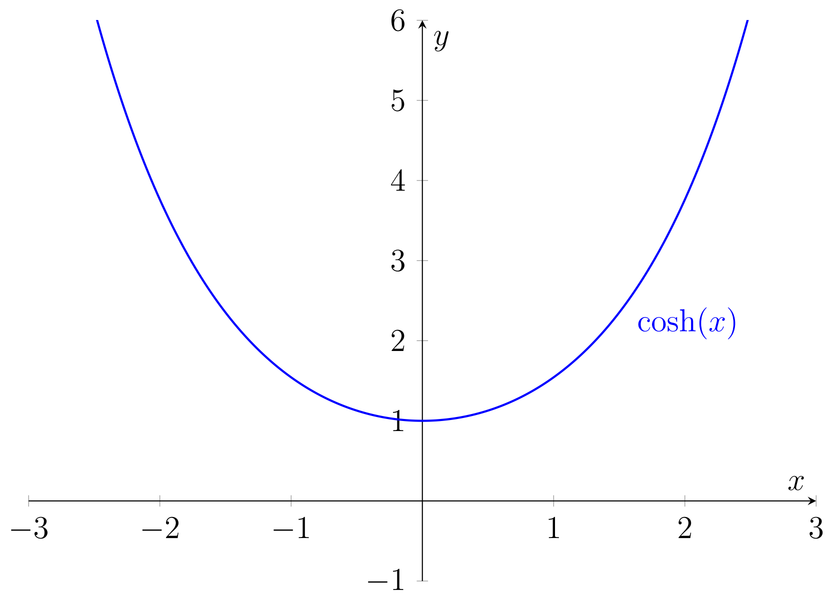
$$f(-x) = -f(x)$$

They have a kind of rotational symmetry, where a rotation by π doesn't alter the graph. $sin(x)$ and x^3 are odd functions.

The first hyperbolic function is the hyperbolic cosine function, $\cosh$, often pronounced as it's written ("cosh").

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

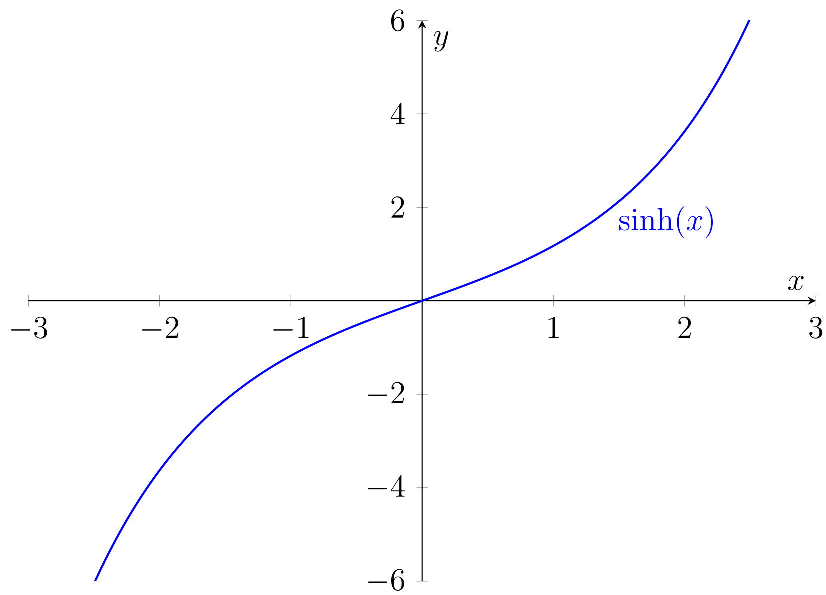
$\cosh$ is an even function. At 0, it is equal to one, and the two sides rise evenly on each side. It essentially looks like a parabola. The plot of $\cosh$ is below.



The second hyperbolic function is hyperbolic sine, $\sinh$, often read as "shine". It is defined as

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

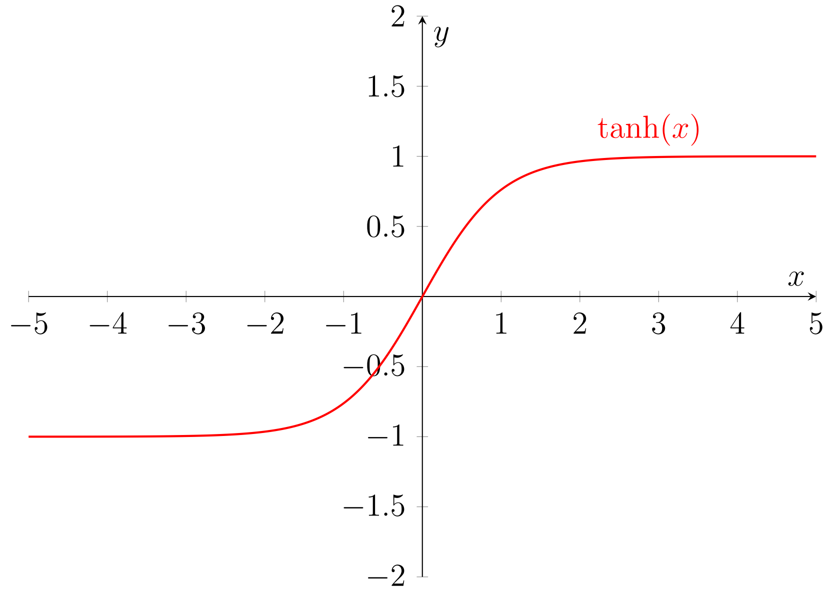
$\sinh$ has a x-intercept at 0. It is an odd function. The plot of $\sinh$ is below.



To complete the set, we have hyperbolic tangent, $\tanh$, often read as "than". It is defined as the quotient of $\sinh$ on $\cosh$, i.e.

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$\tanh$ has the range $(-1, 1)$, with an x-intercept at 0. It is an odd function, due to the presence of $\cosh$ in it's definition. The plot of $\tanh$ is below.

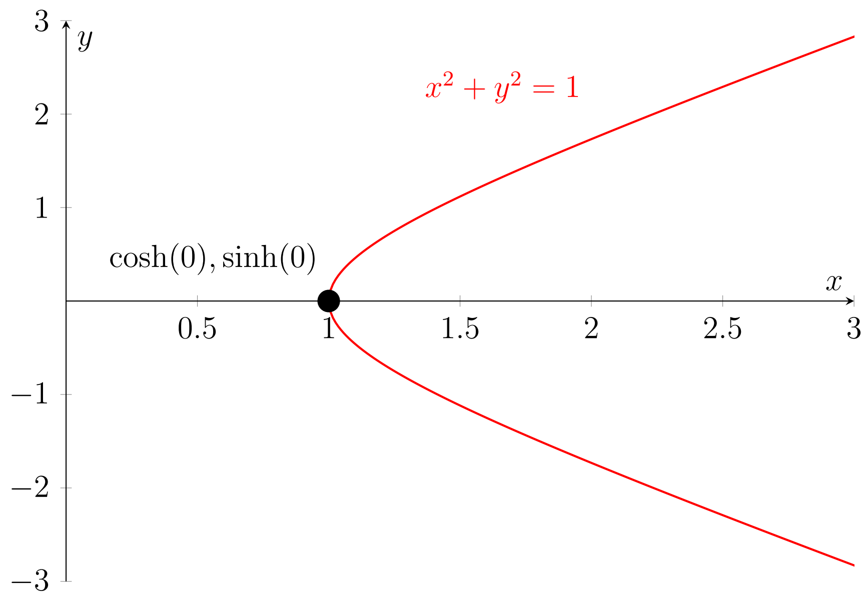


The trigonometric functions have the trigonometric identity

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Which mirrors the form of the equation of a circle, $x^2 + y^2 = r$. This tells us that a $(\sin(\theta), \cos(\theta))$ pair encodes a position on the unit circle.

The hyperbolic functions mirror this behaviour for a hyperbola $(x^2 - y^2 = 1)$), having the property that $\cosh^2(t) - \sinh^2(t) = 1$. i.e. for a pair $(\cosh(t), \sinh(t))$, a point on the hyperbola $x^2 - y^2 = 1$ is uniquely encoded. Only points on the right hand side of the hyperbola are encoded in this way.



Hyperbolic Identities

We can use the relationship between $\cosh$ and $\sinh$ to solve equations. For example, in the following case we can find $\sinh$ and $\tanh$ from the value of $\cosh$ and the fact that $x < 0$.

$$\cosh(x) = \frac{13}{12}$$

$$\cosh^2(x) - \sinh^2(x) = 1 \Rightarrow \sinh^2(x) = \left(\frac{13}{12}\right)^2 - 1 \Rightarrow$$

$$\sinh(x) = \sqrt{\frac{169}{144} - 1} = \sqrt{\frac{25}{144}} = \pm\frac{5}{12}$$

$$x < 0 \Rightarrow \sinh(x) = -\frac{5}{12}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{-\frac{5}{12}}{\frac{13}{12}} = -\frac{5}{13}$$

The key identity that allowed us to solve this was the fundamental hyperbolic identity, that of

$$\boxed{\cosh^2(x) - \sinh^2(x) = 1}$$

A variety of other more specialised identities also exist, beginning with the addition formulae.

$$\boxed{\begin{gathered}
\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \\
\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \\
\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y) \\
\sinh(x+y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)
\end{gathered}}$$

We also have the double angle formulae, just as with trigonometric functions.

$$\boxed{\begin{gathered}
\sinh(2x) = 2\sinh(x)\cosh(x) \\
\cosh(2x) = \cosh^2(x) + \sinh^2(x) \\
\cosh(2x) = 2\cosh^2(x) - 1 \\
\cosh(2x) = 2\sinh^2(x) + 1
\end{gathered}}$$

Reciprocal Hyperbolic Functions

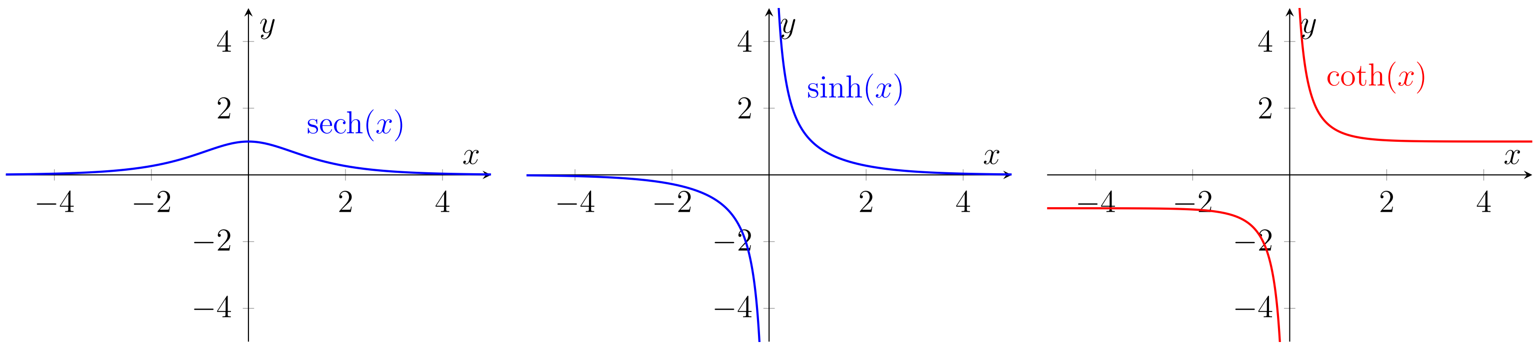
Three reciprocal hyperbolic functions exist. These are

$$\mathrm{sech}(x) = \frac{1}{\cosh(x)}$$

$$\mathrm{cosech}(x) = \frac{1}{\sinh}(x)$$

$$\coth(x) = \frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)}$$

The plots of these functions are below.



These functions have the associated identities

$$\begin{gathered}
\cosh^2(x) - \sinh^2(x) = 1 \\
\coth^2(x) - 1 = \operatorname{cosech}^2(x) \\
1 - \tanh^2(x) = \operatorname{sech}^2(x)
\end{gathered}$$

Derivatives of Hyperbolic Functions

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\cosh(x)\right] = \sinh(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\sinh(x)\right] = \cosh(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\tanh(x)\right] = \operatorname{sech}^2(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{sech}(x)\right] = -\operatorname{sech}(x)\tanh(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{cosech}(x)\right] = -\operatorname{cosech}(x)\coth(x)\ (x \neq 0) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\coth(x)\right] = -\operatorname{cosech}^2(x)\ (x \neq 0
\end{gathered}$$

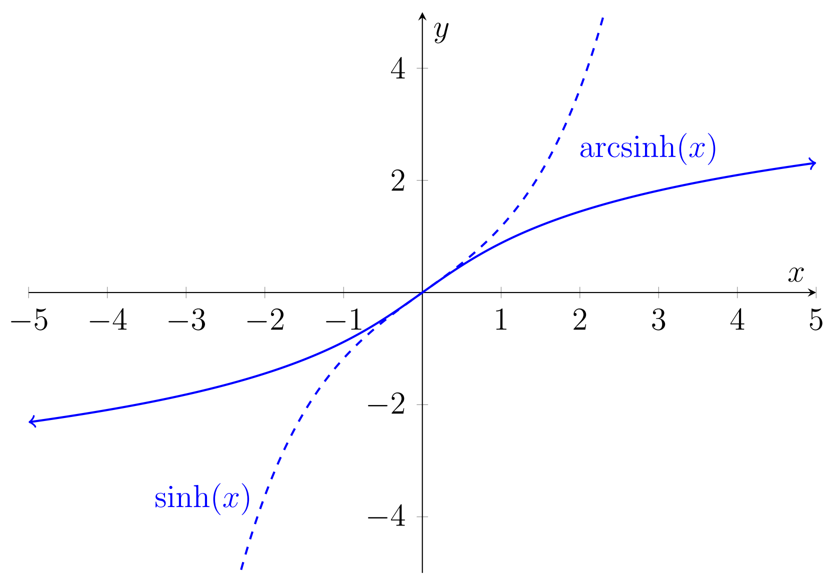
These derivatives bear some resemblance to trigonometric derivatives, however it is worth noting that no negative signs appear when taking derivatives of $\sinh$ and $\cosh$ as they do with trigonometric functions.

Inverse Hyperbolic Functions

Finally, inverses of some of the hyperbolic functions exist. For $\sinh(x)$, we have the inverse function $\operatorname{arcsinh}$. This function can also be written in terms of a logarithm, which makes sense as it is the inverse of an exponential function.

$$\operatorname{arcsinh} = \log(x + \sqrt{x^2 + 1})$$

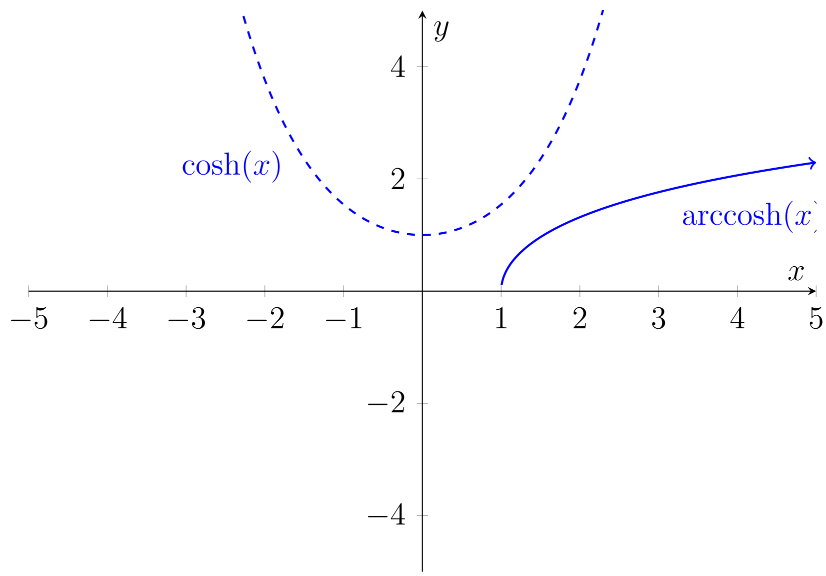
Because $\sinh$ is a nice one-to-one function, it has a simple inversion function, pictured below.



For $\cosh$, the domain of $\cosh$ must be restricted to yield a one-to-one function. The domain is therefore restricted to $[0, \infty)$. Because the range of $\cosh$ is $[1, \infty)$, this is the domain of $\operatorname{arccosh}$.

$$\operatorname{arccosh} = \log(x + \sqrt{x^2 - 1}),\ (x \geq 1)$$

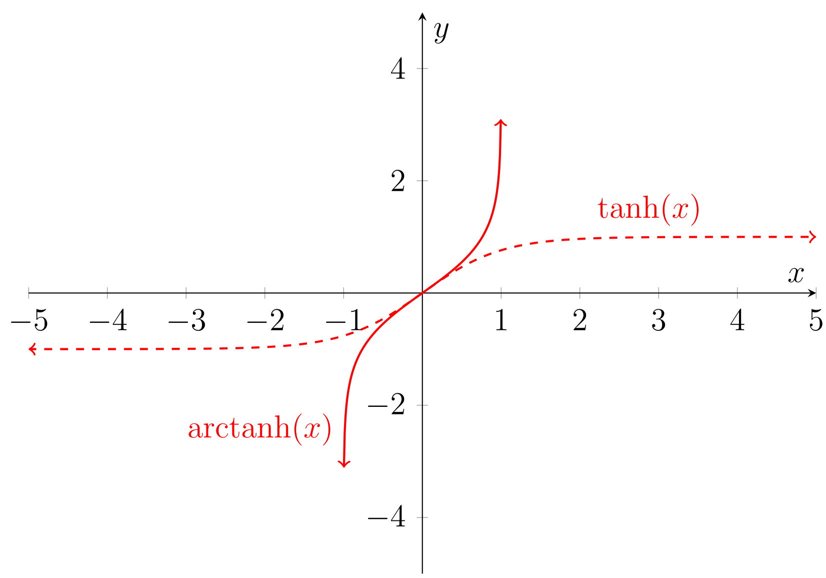
The plot of $\operatorname{arccosh}$ is below.



This means that $\mathrm{arccosh}(\cosh(x)) = |x|$ rather than x. The final inverse is $\mathrm{arctanh}$. As $\tanh$ is one-to-one, this function is a perfect inverse.

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right),\ (-1 < x < 1)$$

The range of $\tanh$ because the domain of $\mathrm{arctanh}$, so $\mathrm{arctanh}$ accepts values in the range $(-1, 1)$ and spits out values in $\mathbb{R}$.



The inverse reciprocal hyperbolic functions are also defined, though they are seldom used. The formulas for the inverse functions are obtained in the usual way of obtaining inverse functions, by taking $y = \operatorname{arcsinh}(x)$ and then solving for x in terms of y.

Derivatives of Inverse Hyperbolic Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{arcsinh}(x)\right] = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{arccosh}(x)\right] = \frac{1}{\sqrt{x^2-1}},\ (x > 1)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{arctanh}(x)\right] = \frac{1}{1-x^2},\ (-1 < x < 1)$$

Complex Numbers

Complex numbers extend the natural, integer, rational and real numbers. By introducing the number i with the property that $i^2 = -1$, we gain access to another set of numbers of the form

$$\mathbb{C} = \left\{x + yi | x, y \in \mathbb{R}\right\}$$

This allows us to do some powerful things, like to solve any polynomial function with complex coefficients. Their basic operations follow. The most basic is of course addition, where one just adds the relevant terms.

$$(2 + 3i) + (4 + i) = 6 + 4i$$

Multiplication is performed piecewise in F.O.I.L. style.

$$(1 + i)(2 + i) = 2 + i + 2i + i^2 = 1 + 3i$$

The complex conjugate of a complex number is found by multiplying the imaginary part (the multiple of i) by -1.

$$z = x + iy \Rightarrow \bar{z} = x - iy$$

The modulus of a complex number can be thought of as the vector norm of the number and is calculated as

$$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$$

The argument of a complex number is the angle it forms with the positive real axis (usually the x-axis). This argument θ has the property that

$$\tan(\theta) = \frac{y}{x}$$

Using this argument we can write a complex number in trigonometric polar form, so where r is the norm of the complex number it can be written as

$$z = r(\cos(\theta) + i\sin(\theta))$$

The argument θ is non-unique, and so the concept of the principal argument as the single possible θ which lies in the interval $(-\pi, \pi]$ is called the *principal argument* and is unique.

The Complex Exponential

The complex exponential is an alternate way of representing a complex number. According to Euler's formula, a complex number has the following property.

$$re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$$

According to this formula, we can see Euler's identity.

$$e^{i\pi} = -1$$

Multiplication of the complex exponential results in addition of angles.

$$r_1e^{i\theta}r_2e^{i\phi} = r_1r_2e^{i(\theta+\phi)}$$

Likewise division results in subtraction.

$$\frac{z}{w} = \frac{r_1}{r_2}e^{i(\theta-\phi)}$$

To raise a complex exponential to a positive power, we can use De Moivre's theorem, which states.

$$z^n = \left(re^{i\theta}\right)^n = r^ne^{in\theta}$$

Exponential Form of $\sin$ and $\cos$

The real and imaginary parts of complex numbers have certain interesting interactions with the conjugate operation.

$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\mathrm{Re}(z) \Rightarrow \mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$$

$$z - \bar{z} = (x + iy) - (x - iy) = 2iy = 2i\mathrm{Im}(z) \Rightarrow \mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

Applying this to the exponential form, we can find some interesting representations of $\sin$ and $\cos$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \Rightarrow \cos(-\theta) + i\sin(\theta) \Rightarrow e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

This tells us that

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta) \Rightarrow \cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

Applying the same working to $\sin$ we find

$$\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Referring back to our imaginary and real part representations earlier, we see that this tells us what we already knew in a different way; $\cos$ is the real part of a complex number and $\sin$ is the imaginary part.

These formulas bear an interesting resemblance to the hyperbolic functions we explored earlier. Plugging $i\theta$ into $\cosh$ and $\sinh$ one finds

$$\cosh(i\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos(\theta)$$

$$\sinh(i\theta) = \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) = \sin(\theta)$$

Differentiation via the Complex Exponential

For a complex number $z = x + iy, x, y \in \mathbb{R}$, we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

To take a derivative of a complex number, we take the derivative of real parts and imaginary parts seperately. For example.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(x)+(x^2+1)i\right]=-\sin(x)+2xi$$

For a complex exponential function of the form e^{zx} where z the derivative is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{zx}\right]=ze^{zx}$$

This can be proven by splitting the function as follows

$$e^{(a+ib)x}=e^{ax}e^{ibx}=e^{ax}(\cos(bx)+i\sin(bx))$$

And using the product rule to solve the derivative. This can be used to solve some fairly complex derivatives quite efficiently.

$$\frac{\mathrm{d}^{56}}{\mathrm{d}t^{56}}\left[e^{-t}\cos(t)\right]$$

$$e^{-t}\cos(t)=e^{-t}\mathrm{Re}\left(e^{it}\right)=\mathrm{Re}\left(e^{-t}e^{it}\right)=\mathrm{Re}\left(e^{(-1+i)t}\right)$$

$$\frac{\mathrm{d}^{56}}{\mathrm{d}t^{56}}\left[\mathrm{Re}\left(e^{(-1+i)t}\right)\right]=\mathrm{Re}\left((-1+i)^{56}e^{(-1+i)t}\right)$$

$$(-1+i)^{56}=\left(\sqrt{2}e^{\frac{3\pi}{4}i}\right)^{56}=\sqrt{2}^{56}e^{56\frac{3\pi}{4}i}=2^{28}e^{0}=2^{28}$$

$$\frac{\mathrm{d}^{56}}{\mathrm{d}t^{56}}\left[e^{-t}\cos(t)\right]=\mathrm{Re}\left(2^{28}e^{-(1+i)t}\right)=\mathrm{Re}\left(2^{28}e^{-t}(\cos(t)+i\sin(t))\right)$$
$$=2^{28}e^{-t}\cos(t)$$

Here, we used the fact that taking a complex derivative simply entails doing the real and imaginary parts seperately, in combination with the fact that the form of the original term was the real part of a complex number to perform the derivative much more rapidly than would ordinarily be possible.

Knowledge of this derivative makes simple the inverse; it's integral.

$$\int e^{kx}\,\mathrm{d}x=\frac{1}{k}e^{kx}+D$$

Where k is a complex number and x is the real input to the function. D may be a complex number. Like in the previous example, the real and imaginary parts of complex numbers can be considered distinctly.

Integral Calculus

Derivative Substitution

For a function $F'(x)$ we may know $f(x) = F'(x)$. We then know that

$$\int f(x)\,\mathrm{d}x = F(t) + C$$

Applying the chain rule, we also know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[F(g(x))\big] = F'(g(x))g'(x)$$

In reverse, this means that

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C$$

$$\int f(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = F(u) + c$$

This fact is known as *derivative substitution*, and is a useful technique for calculating integrals. The general process for using derivative substitution is

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

An example

$$\int (6x^2 + 10)\sinh(x^3 + 5x - 2)\,\mathrm{d}x$$

$$u = x^3 + 5x - 2, \quad u' = 3x^2 + 5$$

$$\int 2(3x^2 + 5)\sinh(x^3 + 5x - 2)\,\mathrm{d}x = \int 2u'\sinh(x)\,\mathrm{d}x = 2\int \sinh(u)\,\mathrm{d}u =$$

$$2\cosh(u) + C = 2\cosh(x^3 + 5x - 2) + C$$

So with the key observation that $6x^2 + 10$ is twice the derivative of the $\sinh$ term, we can rewrite the integal in terms of u and integrate the simpler resultant function using a standard integral. Another example

$$\int \frac{\operatorname{sech}^2(3x)}{10 + 2\tanh(3x)}\,\mathrm{d}x$$

$$u = 10 + 2\tanh(3x), \quad u' = 12\,\mathrm{sech}^2(3x)$$

$$\int \frac{\mathrm{sech}^2(3x)}{10 + 2\tanh(3x)}\,\mathrm{d}x = \frac{1}{12}\int \frac{u'}{u}\,\mathrm{d}x = \frac{1}{12}\int \frac{1}{u}\,\mathrm{d}u$$

$$= \frac{1}{12}\log(|u|) + C = \frac{1}{12}\log\left(\left|12\,\mathrm{sech}^2(3x)\right|\right) + C$$

Trigonometric and Hyperbolic Substitution

For expressions of the form

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \sqrt{x^2 - a^2}$$

We can use a trick where we substitute x for a function $g(\theta)$.

$$\int f(x)\,\mathrm{d}x = \int f(g(\theta))g'(\theta)\,\mathrm{d}\theta$$

This entails replace x and then using derivative substitution to simplify the resultant expression. The appropriate substitution in a few cases is contained in the below table.

$\sqrt{a^2 - x^2}, \quad \frac{1}{\sqrt{a^2-x^2}}, \quad (a^2 - x^2)^{\frac{3}{2}}, \quad \text{etc}$	$x = a\sin(\theta) \text{ or } x = a\cos(\theta)$
$\sqrt{a^2 + x^2}, \quad \frac{1}{\sqrt{a^2+x^2}}, \quad (a^2 + x^2)^{-\frac{3}{2}}, \quad \text{etc}$	$x = a\sinh(\theta)$
$\sqrt{x^2 - a^2}, \quad \frac{1}{\sqrt{x^2-a^2}}, \quad (x^2 - a^2)^{\frac{5}{2}}, \quad \text{etc}$	$x = a\cosh(\theta)$
$\frac{1}{a^2+x^2}, \quad \frac{1}{(a^2+x^2)^2}, \quad \text{etc}$	$x = a\tan(\theta)$

This is useful because these functions have useful identities when squared. An example

$$\int \frac{1}{\sqrt{x^2 - 25}}\,\mathrm{d}x$$

$$x = 5\cosh(\theta) \Rightarrow \theta = \mathrm{arccosh}\left(\frac{x}{5}\right) \Rightarrow x \geq 5$$

Because we are rewriting x as a hyperbolic function, we must be careful that the values of x are valid for the function. Here, for $g(\theta)$ to output x, $x \geq 5$ must be true. In addition, for the overall function to be valid we need $\sqrt{(}x^2 - 25) \neq 0$, thus $x > 5 \Rightarrow \theta > 0$.

$$x = 5\cosh(\theta) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = 5\sinh(\theta)$$

$$\frac{1}{\sqrt{x^2 - 25}} = \frac{1}{\sqrt{25\cosh^2(\theta) - 25}} = \frac{1}{\sqrt{25(\cosh^2(\theta) - 1)}} = \frac{1}{5\sqrt{\sinh^2(\theta)}}$$

$$= \frac{1}{5\left|\sinh(\theta)\right|}, \quad \theta > 0 \Rightarrow \frac{1}{5\sinh(\theta)}$$

Here, we were able to use the identity $\cosh^2(\theta) - \sinh^2(\theta) = 1$ to simplify the denominator. We then used the established fact of $\theta > 0$ to remove the absolute value from the denominator.

$$\int \frac{1}{\sqrt{x^2 - 25}}\,\mathrm{d}x = \frac{1}{5\sinh(\theta)} 5\sinh(\theta)\mathrm{d}\theta = \int 1\mathrm{d}\theta = \theta + C = \operatorname{arccosh}\left(\frac{x}{5}\right) + C$$

Here we see the result of the derivative substitution, as we multiply by $5\sinh(\theta)$. We use the value of θ we calculated earlier to put the final answer back in terms of x.

Another example

$$\int \sqrt{9 - 4x^2}\,\mathrm{d}x, \;\; |x| \leq \frac{3}{2}$$

$$\int \sqrt{9 - 4x^2}\,\mathrm{d}x = 2\int \sqrt{\frac{9}{4} - x^2}$$

$$x = \frac{3}{2}\sin^2(\theta) \Rightarrow \theta = \arcsin\left(\frac{2x}{3}\right)$$

$$\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2} \Rightarrow \frac{-3}{2} \leq x \leq \frac{3}{2}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{3}{2}\cos(\theta)$$

$$\sqrt{\frac{9}{4}-x^2} = \sqrt{\frac{9}{4}-\frac{9}{4}\sin^2(\theta)} = \sqrt{\frac{9}{4}\cos^2(\theta)} = \frac{3}{2}\left|\cos(\theta)\right| = \frac{3}{2}\cos(\theta)$$

Because we know the range of θ is restricted, we can know that $\cos$ will always be positive here, which allows us to remove the absolute value.

$$2\int\sqrt{\frac{9}{4}-x^2} = 2\int\frac{3}{2}\cos(\theta)\cdot\frac{3}{2}\cos(\theta)\,\mathrm{d}\theta = \frac{9}{2}\int\cos^2(\theta)\,\mathrm{d}\theta$$

$$\frac{9}{2}\int\cos^2(\theta)\,\mathrm{d}\theta = \frac{9}{2}\frac{1}{2}\int\cos(2\theta)+1\,\mathrm{d}\theta = \frac{9}{4}\frac{1}{2}\sin(2\theta)+\theta+C$$

$$= \frac{9}{4}\big(\sin(\theta)\cos(\theta)+\theta\big)+C = \frac{9}{4}\left(\sin(\theta)\sqrt{1-\sin^2(\theta)}+\theta\right)+C$$

$$= \frac{9}{4}\left(\frac{2x}{3}\sqrt{1-\frac{4x^2}{9}}+\arcsin\left(\frac{2x}{3}\right)\right)+C = x\sqrt{\frac{9}{4}-x^2}+\frac{9}{4}\arcsin\left(\frac{2x}{3}\right)+C$$

A lot going on here, some of which is explained by the application of a couple of double angle formulae. The first instance is where we use the double angle formula $\cos(2x) = 2\cos^2(x)-1$ to remove the $\cos^2$ term and enable integration. Later, the formula $\sin(2x) = 2\sin(x)\cos(x)$ is used to remove the $\sin(2\theta)$ term. The $\cos$ is then replaced according to $\sin^2+\cos^2 = 1$ to enable substitution with the value for θ determined earlier, the $\arcsin$ term of which annihilates with the $\sin$ in the $\sin(\theta)$ expressions to yield $\frac{2x}{3}$.

Integration of Powers of Hyperbolic Functions

$$\int\sinh^m(x)\cosh^n(x)\,\mathrm{d}x$$

Consider the above integral with two integer powers of $\sinh$ and $\cosh$. There are two processes for this; one for the case of one power being odd and another for the case where both are even. First, an example of an even case.

$$\int\cosh^4(\theta)\,\mathrm{d}\theta$$

$$\cosh^4(\theta) = \left(\cosh^2(\theta)\right)^2 = \left(\frac{\cosh(2\theta)+1}{2}\right)^2 =$$

$$\frac{1}{4}\left(\cosh^2(2\theta) + 2\cosh(2\theta) + 1\right) = \frac{1}{4}\left(\frac{\cosh(4\theta)+1}{2} + 2\cosh(2\theta) + 1\right) \Rightarrow$$

$$\int \cosh^4(\theta)\,\mathrm{d}\theta = \frac{1}{4}\int \frac{1}{2}\cosh(4\theta) + 2\cosh(2\theta) + \frac{3}{2}\,\mathrm{d}\theta$$

$$= \frac{1}{4}\left(\frac{1}{8}\sinh(4\theta) + \sinh(2\theta) + \frac{3}{2}\theta\right) + C$$

Here, we rearranged an integral we didn't know ($\cosh^4$) to one that we did know ($\cosh$) through repeated application of the double angle formula

$$\cosh^2(\theta) = \frac{\cosh(2\theta)+1}{2}$$

For the odd power case, the approach is a little different.

$$\int \sinh^5(x)\cosh^6(x)\,\mathrm{d}x$$

$$\int \sinh\sinh^4(x)\cosh^6(x)\,\mathrm{d}x = \int \sinh\left(\sinh^2(x)\right)^2\cosh^6(x)\,\mathrm{d}x =$$

$$\int \sinh(x)\left(\cosh^2(x) - 1\right)^2\cosh^6(x)\,\mathrm{d}x$$

$$u = \cosh(x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \sinh(x)$$

$$\int \sinh(x)\left(\cosh^2(x) - 1\right)^2\cosh^6(x)\,\mathrm{d}x = \int (u^2-1)^2 u^6 \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x =$$

$$\int (u^4 - 2u^2 + 1)u^6\,\mathrm{d}x = \int u^{10} - 2u^8 + u^6\,\mathrm{d}x = u^{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C \Rightarrow$$

$$\int \sinh^5(x)\cosh^6(x)\,\mathrm{d}x = \frac{\cosh^1 1(x)}{11} - \frac{2\cosh^9(x)}{9} + \frac{\cosh^7(x)}{7} + C$$

Here, we took aside one $\sinh$ as our sacrifice to the integral gods and used it to change variable of integration from x to u. We did this by using the hyperbolic identity $\cosh^2(x) - \sinh^2(x) = 1$ to convert all of the other $\sinh$ terms into $\cosh$ terms and then substituting u for $\cosh(x)$.

Integration by Parts

The product rule used during differentiation is

$$\frac{\mathrm{d}}{\mathrm{d}x}[uv] = \frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

To perform integration by parts, this experssion is integrated to yield the rule

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

An example

$$\int x^2 \log(x)\,\mathrm{d}x, \quad (x > 0)$$

$$u = \log(x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^2 \Rightarrow v = \frac{x^3}{3}$$

$$\int x^2 \log(x)\,\mathrm{d}x = \frac{x^3}{3}\log(x) - \int \frac{x^3}{3}\frac{1}{x}\,\mathrm{d}x$$

$$= \frac{x^3\log(x)}{3} - \frac{x^3}{9} + C$$

This can also be used cleverly in some settings to rearrange integrals one doesn't know how to solve.

$$\int \log(x)\,\mathrm{d}x$$

$$u = \log(x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x$$

$$\int \log(x)\,\mathrm{d}x = x\log(x) - \int \frac{x}{x}\,\mathrm{d}x = x\log(x) - \int 1 = x\log(x) - x + C$$

Partial Fractions

For two polynomials $f(x)$, of degree n and $g(x)$, of degree d,

$$\frac{f(x)}{g(x)}$$

can be written as the sum of partial fractions. In the case that $n < d$, the process for this is to factorise g, write down the partial fraction expansion and finally to find unknown coefficients. To write down the partial fraction expansion for a denominator g, we use the expansions from the below table.

Denominator Factor	Partial Fraction Expansion
$(x-a)$	$\frac{A}{x-a}$
$(x-a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_r}{(x-a)^r}$
x^2+bx+c	$\frac{Ax+B}{x^2+bx+c}$
$(x^2+bx+c)^r$	$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \ldots + \frac{A_rx+B_r}{(x^2+bx+c)^r}$

As an example of using this process, let us attempt to integrate.

$$\int \frac{4}{x^2(x+2)}\,\mathrm{d}x, \quad (x \neq 0, -2)$$

$$\frac{4}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} =$$

$$\frac{Ax(x+2) + B(x+2) + Cx^2}{x^2(x+2)} = \frac{(A+C)x^2 + (2A+B)x + 2B}{x^2(x+2)} \Rightarrow$$

$$A + C = 0, \quad 2A + B = 0, \quad 4 = 2B \Rightarrow$$

$$A = -1, \quad B = 2, \quad C = 1 \Rightarrow$$

$$\int \frac{4}{x^2(x+2)}\,\mathrm{d}x = \int \frac{-1}{x} + \frac{2}{x^2} + \frac{1}{x+2}\,\mathrm{d}x = \log(|x|) - \frac{2}{x} + \log(|x+2|) + C$$

In the case that $n \geq d$, we must first use polynomial long division to reduce the degree of the denominator.

Differential Equations

Ordinary Differential Equations

An ordinary differential equation or ODE is an equation of the form

$$f\left(x, y\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \dots \frac{\mathrm{d}^ny}{\mathrm{d}x^n}\right) = 0$$

Here, there is only one independent variable x, and one dependent variable y. The equation is then expressed in terms of functions and derivatives of these two functions. The *order* of a differential equation is the highest derivative that appears in the equation (second, third, etc).

A solution of this equation is a function $y(x)$ which satisfies the equation for all x in some interval. For example

$$y(x) = x^2 + \frac{2}{x}$$

Is a solution for all $x \in \mathbb{R} \setminus \{0\}$ to

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = 3x$$

Which can be shown via substitution, replacing all y with $y(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 + \frac{2}{x}\right] + \frac{x^2 + \frac{2}{x}}{x} = 3x$$

$$2x - \frac{2}{x^2} + x + \frac{2}{x^2} = 3x$$
$$2x + x = 3x$$

First Order

A first order differential equation is a differential equation containing only at most first derivatives such as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3$$

This example can be easily solved through integration.

$$y = \int x^3 \,\mathrm{d}x = \frac{x^4}{4} + c$$

These solutions usually contain an arbitrary constant $c \in \mathbb{R}$, and the solutions represented by various values of c are known as the family of solutions to this differential equation. It is interesting to note that these solutions can never intersect, as the differ by some value at all x.

To find a *particular solution*, we need some starting conditions for our differential equation, and we can then solve to find the value of c. If for instance we knew that our previous example had $y(0) = 2$ we could solve to find

$$y(0) = \frac{0^4}{4} + c = 2 \Rightarrow c = 2$$

Seperable First Order

A seperable first order differential equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = M(x)N(y), \quad (M(x) \neq 0, N(y) \neq 0)$$

To solve, we seperate our variables like so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = M(x)N(y) \Rightarrow \frac{1}{N(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = M(x), \quad N(Y \neq 0) \Rightarrow$$

$$\int \frac{1}{N(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int M(x)\,\mathrm{d}x \Rightarrow \int \frac{1}{N(y)}\,\mathrm{d}y = \int M(x)\,\mathrm{d}x$$

This is essentially an application of derivative substitution to clean up the equation and seperate the two variables.

Examples

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{1+x}, \quad (x \neq -1)$$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x} \Rightarrow \int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{1+x}\,\mathrm{d}x = \int \frac{1}{y}\,\mathrm{d}y \Rightarrow$$

$$\log(|1+x|) + c = \log(|y|) + c \Rightarrow e^{\log(|1+x|)+c} = e^{\log(|y|)} \Rightarrow$$
$$|y| = e^c|1+x| \Rightarrow y = \pm A(1+x)$$

In this example we made the assumption that y was non-zero when we divided by y initially. This works out in the end because we know$A = \pm e^c \neq 0$ so our solution covers us, however we should also check whether $y(x) \equiv 0$ is a solution, which in this case it is. We can incorporate this into our previous solution by allowing A to be 0.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2y\sqrt{1-x^2}}, \quad (-1 < x < 1, y \neq 0), \quad y(0) = 3$$
$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int 2y \,\mathrm{d}y = \int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x \Rightarrow$$
$$x = \cos(\theta) \Rightarrow \theta = \arccos(x)$$
$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin(\theta) \Rightarrow \int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x = \int \frac{1}{\sqrt{1-\cos^2(\theta)}} \times -\sin(\theta) \,\mathrm{d}\theta$$
$$\frac{1}{\sqrt{1-\cos^2(\theta)}} = \frac{1}{\sqrt{\sin^2(\theta)}} = \frac{1}{\sin(\theta)} \Rightarrow$$
$$\int 2y \,\mathrm{d}y = \int \frac{1}{\sin(\theta)} \times -\sin(\theta) \,\mathrm{d}\theta = \int -\frac{\sin(\theta)}{\sin(\theta)} \,\mathrm{d}\theta = \int -1 \,\mathrm{d}\theta \Rightarrow$$
$$y^2 = -\theta + c = -\arccos(x) + c \Rightarrow |y| = \sqrt{-\arccos(x) + c} \Rightarrow$$
$$y = \pm\sqrt{-\arccos(x) + c}$$
$$y(0) = 3 \Rightarrow 3 = \pm\sqrt{-\arccos(0) + c} = \pm\sqrt{-\frac{\pi}{2} + c} \Rightarrow$$
$$9 = -\frac{\pi}{2} + c \Rightarrow c = 9 + \frac{\pi}{2}$$

This is actually the wrong way to do it. The correct way to do it is to realise that the right hand side is a standard integral of $\arcsin$ and use that to find $c = 9$ for the solution

$$y = \pm\sqrt{\arcsin(x) + 9}$$

The two solutions are however equivalent so either works. $y(0) = 3$ tells us that the positive side of each of these two solutions are the correct one.

Integrating Factor and Linear First Order

By working through the following differential equation, we can make an observation which will be useful in solving a certain form of differential equations.

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^x$$

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \frac{\mathrm{d}}{\mathrm{d}x}\left[xy\right] \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[xy\right] = e^x \Rightarrow$$

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left[xy\right]\,\mathrm{d}x = \int e^x\,\mathrm{d}x \Rightarrow xy = e^x + c \Rightarrow$$

$$y = \frac{e^x + c}{x}, \quad x \neq 0$$

By obsering that $x\frac{\mathrm{d}y}{\mathrm{d}x} + y$ is the result of the application of the product rule in finding a derivative of xy, we were able dramatically simplify the solving of this equation. This process can be made more general as well.

For an ordinary differential equation to be linear it must be of the form

$$f_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + f_0(x)y + q(x) = 0$$

for example, the equation solved at the start of this section was linear, which becomes more clear when one rearranges it.

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^x \Rightarrow x\frac{\mathrm{d}y}{\mathrm{d}x} + y - e^x = 0$$

This is important, because it is the first step in solving differential equations using the product rule. We start by arranging our differential equation into the standrad form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

To solve it, we then multiply by our integrating factor $I(x)$, a function which we don't yet know.

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)P(x)y = I(x)Q(x)$$

We want I to cause our equation to satisfy the condition

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[y(x)I(x)\big] = Q(x)I(x)$$

To find an I with this property we need to solve such that

$$\frac{\mathrm{d}I}{\mathrm{d}x} = P(x)I \Rightarrow I = \pm e^{\int P(x)\,\mathrm{d}x}$$

We have observed that this condition yields a seperable differential equation solved it for the general form and used this to generate our I. Because we need only a single integrating factor I, we can discard the negative solutions to yield

$$I(x) = e^{\int P(x)\,\mathrm{d}x}$$

For the same reason, we can ignore any absolute values or constants of integration in the integral as well. Applying this to our first example:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = \frac{e^x}{x} \Rightarrow P(x) = \frac{1}{x}$$

$$I(x) = e^{\int \frac{1}{x}\,\mathrm{d}x} = e^{\log(x)} = x$$

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)\frac{y}{x} + I(x)\frac{e^x}{x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + y + e^x \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[xy\right] = e^x$$

Through our general formula, we found our way to the same solution we arrived at through a leap of intuition in the initial example.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = \sin(x), \quad (x \neq 0)$$

$$I(x) = e^{\int P(x)\,\mathrm{d}x} = e^{\int \frac{1}{x}\,\mathrm{d}x} = e^{\log(x)} = x$$

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x\sin(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[xy\right] \Rightarrow$$

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left[xy\right]\,\mathrm{d}x = \int x\sin(x)\,\mathrm{d}x \Rightarrow xy = -x\cos(x) + \int \cos(x)\,\mathrm{d}x \Rightarrow$$

$$xy = -x\cos(x) + \sin(x) + c \Rightarrow y = -\cos(x) + \frac{\sin(x)}{x} + \frac{c}{x}$$

Here we used an integrating factor to solve the left side, and then integrated the right side through derivative substitution.

Other First Order

It is sometimes possible to apply a substitution to simplify a differential equation down to a seperable or linear differential equation. A differentiable equation of a *homogenous type* has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$$

In this case, we can substitute $y = \frac{y}{x}$ to reduce to a seperable differential equation. Another case is differential equations with the form of Bernoulli's equation, that is

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

In this case, a substitution of $u = y^{1-n}$ reduces the equation to linear.

Homogenous Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right), \quad (-\frac{\pi}{2} < \frac{y}{x} < \frac{\pi}{2})$$

$$u = \frac{y}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + u$$

$$x\frac{\mathrm{d}u}{\mathrm{d}x} + u = u + \cos^2(u) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\cos^2(u)}{x} \Rightarrow$$

$$\sec^2(u)\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \Rightarrow \int \sec^2 \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x \Rightarrow \int \sec^2\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x \Rightarrow$$

$$\tan(u) = \log(|x|) + C \Rightarrow \tan\left(\frac{y}{x}\right) = \log(|x|) + C \Rightarrow$$

$$y = x\arctan(\log(|x| + C))$$

Bernoulli Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^{3x}y^4, \quad (y \neq 0)$$

$$u = y^{-3}$$

We want to express y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of u, x and $\frac{\mathrm{d}u}{\mathrm{d}x}$.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[y^{-3}\right] \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -3y^{-4}\frac{\mathrm{d}y}{\mathrm{d}x}$$

Because we are diffentiating an equation of y, we need to use implicit differentiation, hence the $\frac{\mathrm{d}y}{\mathrm{d}x}$ term. Now that we have this equation, we can multiply our other equation by the coefficient to replace.

$$-3y^{-4}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) = -3y^{-4}\left(e^{3x}y^4\right) \Rightarrow$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} - 3y^{-3} = -3e^{3x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 3u$$

At this stage we simple have a linear differential equation, and the solution is fairly easy to find.

Population Models

Malthus (Doomsday) Model

The Malthus or Doomsday model for population growth states that rate of growth at a time t of a population p is proportional to the population.

$$\frac{\mathrm{d}p}{\mathrm{d}t} \propto p \Rightarrow \frac{\mathrm{d}p}{\mathrm{d}t} = kp$$

Here, k is a proportionality constant which corresponds to net births per unit population per unit time. If we take $p(0) = p_0$ we find that

$$p(t) = p_0e^{kt}$$

Thus, the model is clearly an exponential relationship. Obviously this model is somewhat unrealistic; for a $k > 0$ it implies unending exponential growth.

Equilibrium Solutions

An equilibrium solution to a differential equation is a constant solution. For example, in the Malthusian example $p_0 = 0$ is an equilibrium solution of $p(t) \equiv 0$. For these to exist, we must have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

For a differential equation with variables x and t. For example

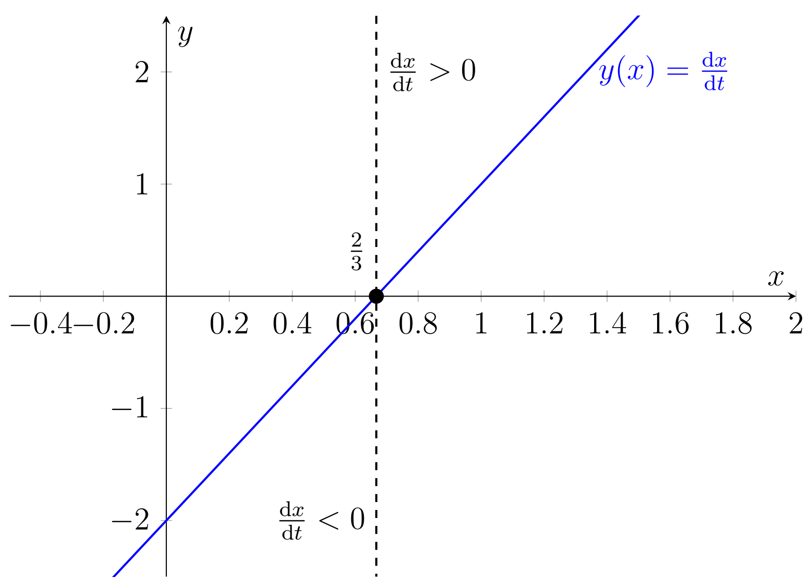
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \Rightarrow x = \frac{2}{3}$$

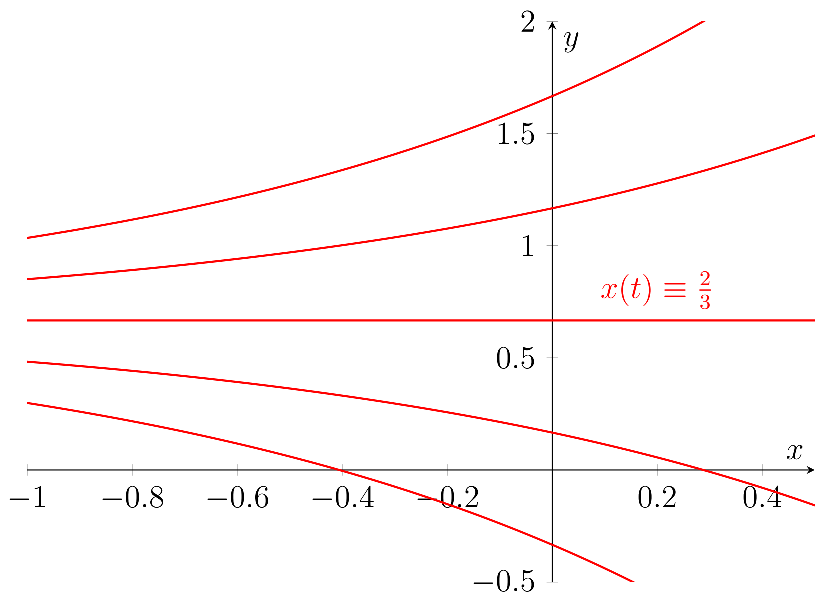
The equilibrium solution is $x(t) \equiv \frac{2}{3}$. A *phase plot* is a plot of the derivative of a function against the value of the function. On these plots, the equilibria are the horizontal axis intercepts. These plots are generally only useful for differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

This is known as an *autonomous* differential equation, because the right hand side is independent of t.



Examining this plot, we can see that the derivative of x will be positive for all values of $x > \frac{2}{3}$ while it will be negative for values less than $\frac{2}{3}$. We can also see that as x increases, the gradient of the function increases, as well as the inverse. Using this information, in combination with the constant solution learned from the intercet, and the fact that no solution may cross any other solution, we can plot the family of solutions to this differential equation.



We can see that for the initial condition of $x(0) = \frac{2}{3}$, the value will be constant, while for initial conditions greater than this value the equation diverges to infinity and for values less, it diverges to negative infinity.

It can be useful to examine if an equilibrium is stable. An equilibrium is *stable* if nearby solutions grow closer with increasing t, while it is unstable if solutions diverge, as they do in the above example. On a phase plot, a stable equilibrium will have a negative slope passing through it. On a phase plot, an unstable equilibrium has a positive slope through it, as in the example.

A semistable equilibrium occurs when one side of solutions approach the equilibrium. This occurs at a touching point in a phase plot.

Extending the Doomsday Model

To extend the Malthusian Doomsday model explored earlier, we can add a notion of "harvesting", the process of some quantity of the population being lost every unit time.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp - h, \quad h > 0$$

This has little effect on the exponential curve; because the curve is exponential, we can subtract h without changing its behaviour in the long term.

Another alteration we could make would be to add a "competition" term, creating a logistic model. This added term accounts for competition for resources by adding a negative term dependent on the population size.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp - \frac{k}{a}p^2 = kp\left(1 - \frac{p}{a}\right)$$

Here, a is the *carrying capacity*, the capacity for population in the environment. In the case that $p > a$, this yields a negative derivative i.e. a decreasing population, steadily approaching a. For example, a logistic model with $k = 1$ and $a = 4$ looks like

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left(1 - \frac{p}{4}\right)$$
$$\Rightarrow p(t) \equiv 0, p(t) \equiv 4$$

We can immediately see two equilibrium solutions thanks to the factorised form. We know that other solutions may never cross these, therefore all other solutions must be strictly less than 0, between 0 and 4 or strictly greater than 4.

Drawing a phase plot, one finds that the derivative is positive for all values between 0 and 4, and negative for outside values. It is at a maximum at 2; this can all be derived from the parabolic shape of the derivative. This showcases the classic logistic behaviour of an exponential curve tapering off as the population nears capacity. For values above a, the population decays rapidly to a.

In this case, 0 is an unstable equilibrium while 4 is a stable equilibrium. The point of inflection of the plot is when the gradient is at a maximum, i.e. at $p = 2$.

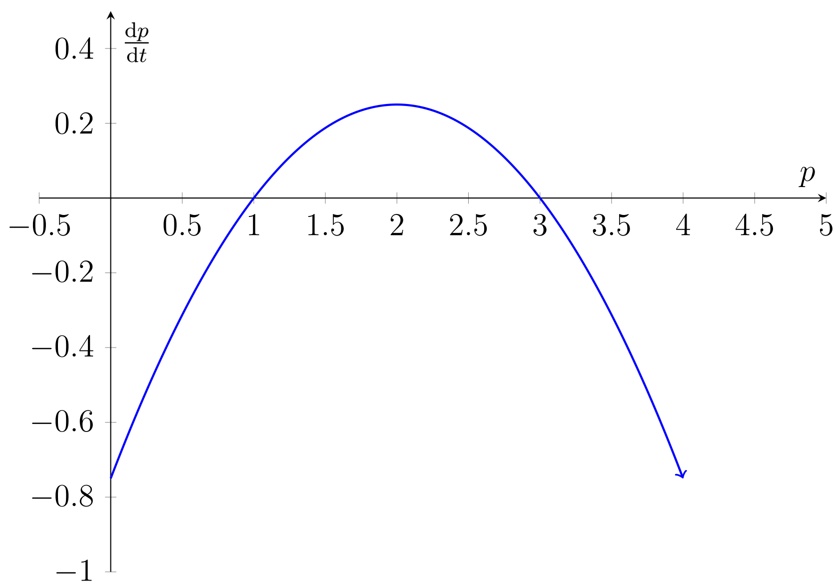
We can combine the logistic model with harvesting to produce a combined model.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp\left(1 - \frac{p}{a}\right) - h$$

As an example we can take

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left(1 - \frac{p}{4}\right) - \frac{3}{4}$$

Solving the quadratic we find equilibria of $p(t) \equiv 3, p(t) \equiv= 1$.



Examining the phase plot we once again find that population will increase while between these equilibria, with the upper equilibrium being a stable equilibrium and the lower unstable. The main difference is the narrowed bounds; because of the harvesting term a starting population of 1 or less will die out and a population will never grow beyond 3.

While this phase plot analysis is useful, it is of course much more useful to actually solve the differential equation. We can take $N(t) = 1$ to make this a seperable integral.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{1}{4}(p-3)(p-1)$$

$$\int \frac{1}{-\frac{1}{4}(p-3)(p-1)} \frac{\mathrm{d}p}{\mathrm{d}t}\,\mathrm{d}t = \int 1\,\mathrm{d}t \Rightarrow \int \frac{4}{(p-3)(p-1)}\,\mathrm{d}p = \int 1\,\mathrm{d}t$$

$$\frac{4}{(p-3)(p-1)} = \frac{A}{p-3} + \frac{B}{p-1} =$$

$$\frac{A(p-1)}{(p-3)(p-1)} + \frac{B(p-3)}{(p-3)(p-1)} = \frac{A(p-1)+B(p-3)}{(p-3)(p-1)} =$$

$$\frac{Ap - A + Bp - 3B}{(p-3)(p-1)} \Rightarrow -A - 3B = 4, \quad A + B = 0$$

$$A = -B \Rightarrow -A - 3B = 4 = B - 3B \Rightarrow B = 2 \Rightarrow A = -2 \Rightarrow$$

$$\int \frac{4}{(p-3)(p-1)}\,\mathrm{d}p = \int \frac{2}{(p-1)} - \frac{2}{(p-3)}\,\mathrm{d}p = \int 1\,\mathrm{d}t \Rightarrow$$

$$2\log(|p-1|) - 2\log(|p-3|) = t + C$$

$$\log(|p-1|) - \log(|p-3|) = \frac{t}{2} + C$$

$$\log\left(\frac{|p-1|}{|p-3|}\right) = \frac{t}{2} + C \Rightarrow \frac{p-1}{p-3} = \pm e^{\frac{t}{2}+C} = \pm A e^{\frac{t}{2}}$$

Mixing Problems

Differential equations can be used to consider the combination of liquids, for example considering a pollutant entering a pond. If x represents the total quantity of pollutant in the pool at time t, then we can take $C = \frac{x}{V}$ where C is the concentration of pollutant in the pool of volume V at time t. We have $\frac{\mathrm{d}x}{\mathrm{d}t}$ is equal rate of inflow minus rate of outflow.

$$\text{inflow} = C_{\text{in}} \times F_{\text{in}}$$

Where F_{in} is the flow rate of liquid into the pool with pollutant at concentration C_{in} in that liquid.

$$\text{outflow} = C \times F_{\text{out}} = \frac{x}{V} \times F_{\text{out}}$$

Here, because we have defined C in terms of x and V we can replace it. For values $V = 1000\text{m}^3$, $x_0 = 100\text{g}$, $F_{\text{in}} = F_{\text{out}} = 10\text{m}^3\text{min}^{-1}$, $C_{\text{in}} = 2\text{g m}^{-3}$ we find

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\text{g m}^{-3} \times 10\text{m}^3\text{min}^{-1} - \frac{x}{1000\text{m}^3} \times 10\text{m}^3\text{min}^{-1}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 20\mathrm{g\ min}^{-1} - \frac{x}{100}\mathrm{min}$$

So if we say t is in terms of minutes and x in terms of grams we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 20 - \frac{x}{100}$$

Solving this, one finds

$$x(t) = 2000 + ce^{\frac{-t}{100}}$$

$$x(0) = 100 \Rightarrow c = -1900$$

To find the concentration at a given time, we can use $c(t) = \frac{x(t)}{V}$ to find

$$c(t) = 2 - \frac{19}{10}e^{\frac{-t}{100}}$$

As t increases, the right hand term approaches 0, so we can say that in the long term the concentration will be 2. We could describe this right hand term as a *transient term* and the left hand (constant) term as a *steady state term*.

What does the equation look like if the inflow rate is instead $5\mathrm{m}^3\mathrm{min}^{-1}$? Inflow will now be

$$C_{\mathrm{in}} \times F_{\mathrm{in}} = 2\mathrm{g\ m}^{-3} \times 5\mathrm{m}^3\mathrm{min}^{-1} = 10\mathrm{g\ min}^{-1}$$

We now have a changing rather than constant volume. The volume at time t can be expressed as

$$V(t) = 1000 + 5t - 10t = 1000 - 5t$$

Thus the outflow is given by

$$\frac{x}{V} \times F_{\mathrm{out}} = \frac{x}{1000 - 5t} \times 10 = \frac{x}{100 - \frac{t}{5}}$$

And the overall equation is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 10 - \frac{x}{100 - \frac{t}{5}}$$

If $t > 200$, $V < 0$ which doesn't make much sense, so we can restrict the domain. This is a linear equation, so it can be solved through the integrating factor method.

Second Order Ordinary Differential Equations

A second order differential equation is of the form

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0$$

Generally a second order differential equation will have two arbitrary constants. Therefore, a second piece of information is necessary to solve an initial value problem. For example, $y(x_0) = y_0$ and $y'(x_0) = y_1$. By adding this second variable, we could instead have two values for y; if we have $y(a) = y_0$ and $y(b) = y_1$, we have a boundary value problem; two points through which the solution passes.

Linear Second Order

The general form of a linear second order ordinary differential equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = R(x)$$

If $R(x) = 0$, the equation is homogenous. If $R(x) \neq 0$ the equation is inhomogenous. The general solution to a homogenous equation of this form is

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

Here, y_1 and y_2 are *linearly independent solutions* of the equation and c_2 and c_2 are arbitrary constants. The term "linearly independent" implies that

$$c_1y_1(x) + c_2y_2(x) = 0 \Rightarrow c_2 = c_2 = 0$$

Another way of stating this is that neither function is a non-zero constant multiple of the other function. For instance, while e^{2x} and xe^{2x} are linearly independent, x^2 and $2x^2$ are not.

The reason this works is that both solutions can be understood to exist in the same space of solutions to the equation, and as linear solutions, they can be added while still being a solution to the equation.

Homogenous Second Order Linear

Equations of this form look like

$$ay'' + by' + cy = 0$$

Where a, b, c are constants. To find $y(x)$, we need to find two linearly independent solutions. Once we have these, we can use our theorem for the general solution.

The best way to start out this process is by considering exponential solutions. If we first try $y(x) = e^{\lambda x}$, we find

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

Substituting into the equation, we find

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$

$$\Rightarrow e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$$

Because $e^{\lambda x} \neq 0$, we need the term in brackets (known as the *characteristic equation* of the differential equation) to equal 0 and thus be a solution. To find λ we can simply use the quadratic formula, which conveniently yields a positive and negative value; i.e. two linearly independent solutions.

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the case that the discriminant, $b^2 - 4ac > 0$, this formula yields two distinct real values, each of which is a solution. Thus, the overall general solution is

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

Example

$$y'' + 7y' + 12y = 0$$

$$\Rightarrow a = 1, b = 7, c = 12$$

$$\Rightarrow \lambda = \frac{-7 \pm \sqrt{(-7)^2 - 4 \times 1 \times 12}}{2} = \frac{-7 \pm \sqrt{49 - 48}}{2} = \frac{-7 \pm 1}{2}$$

$$\Rightarrow \lambda = -4, \quad \lambda = -3$$

$$\Rightarrow y(x) = Ae^{-4x} + Be^{-3x}$$