Motion

Galiliean Mechanics

Galilieo's Principle can describe an event using two variables (x, t) and ($x\prime$, $t\prime$). In this case, $x\prime = x + vt$ and $t\prime = t$. This treats the laws of mechanics as the same in any inertial frame; a frame moving at a constant speed. While these laws function at low velocities, they start to break down as v approaches c.

Fundamental Quantities

There are seven fundamental quantities, of which three are relevant; length, time and mass. These seven can be used to define all other physical quantities.

Dimensional Analysis

Dimensonal analysis is the process of taking a formula, breaking it down into input units, and solving this equation to ascertain whether the formula is accurate. e.g.

Fundamental quantities: length (L) and time (T).

$$v = v_0 + \frac{1}{2}at^2$$

$$\frac{L}{T} = \frac{L}{T} + \frac{L}{T^2}T^2$$

$$\frac{L}{T} = \frac{L}{T} + L \Rightarrow \text{incorrect}$$

Kinematics

- Position is defined as a vector indicating the distance from a reference point, the origin. It is directional.
- Displacement is the change in x or position. i.e. the Δx.
- Velocity is the ratio of displacement to time.
- Acceleration is the rate of change of velocity.

Velocity

On a graph of x vs t, average velocity is the slope of the line that connects two points.

Average speed is total distance divided by time. Thus, it is always positive.

Instantaneous velocity is obtained at a single moment in time, it is given by the slope of a curve at that point in time, i.e.

$$v_t = \frac{dx}{dt}$$

Acceleration

Acceleration is a vector; it has direction and magnitude. Acceleration can be expressed in ms^{-2} or in units of g: $9.8ms^{-2}$.

$$a_{\text{avg}} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

$$a = \frac{dv}{dt}$$

For many cases acceleration is constant, such as for objects falling due to the influence of gravity. In these instances, the equations of constant acceleration can be used. Where $s = x - x_0$. These include:

$$v = v_0 + at$$

$$s = v_0 + \frac{1}{2}at^2$$

$$s = vt - \frac{1}{2}at^2$$

Vectors

A vector is a mathematical object with magnitude and direction. Position, velocity and acceleration are examples of vector quantity. Scalar quantities, such as speed, don't have a direction.

Operations

The vector sum, or resultant vector is the net displacement (or velocity, acceleration, etc) of two or more vectors.

$$\vec{s} = \vec{a} + \vec{b}$$

Vector addition is commutative. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ is the commutative law. Vector addition is also associative. Any order of addition will yield the same result.
A negative sign reverse vector direction.

$$\vec{b} + (-\vec{b}) = 0$$

We use this to define vector subtraction.

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

These rules hold for all vectors, irrespective of what quantity they depict. Obviously, only vectors of the same type, with the same units can be added. This can be checked with dimensional analysis.

Components

Rather than adding them graphically, one can add vectors by breaking them down into their components. Components in two dimensions can be found with:

$$a_x = a\cos(\theta) \text{ and } a_y = a\sin(\theta)$$

$$a = \sqrt{a_x^2 + a_y^2}$$

In 3 dimensions we need more components so we use a, θ, ϕ or a_x, a_y, a_z. Unit vectors. A unit vector has the following properties:

- Has magnitude 1

- Has a particular direction
- Lacks dimension and unit
- is labeled with a hat: $\hat{i}$

$$\vec{a} = a_x\hat{i} + a_y\hat{j}\ (+a_z\hat{k})$$

a_x and a_y alone are scalar components. Vectors can be added with these components: $r_x = a_x + b_x$ etc. To subtract two vectors, we substract components: $r_x = a_x - b_x$ etc.

Rotations

Because vectors are independent of their coordinate system, we can rotate the system while maintaining the vector.

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2}$$

Scalar Multiplication

To multiply a vector by a scalar, we simply multiply each component by the scalar. The direction is unchanged unless the scalar is negative, in which case it is reversed.

$$3a = 3a_x\hat{i} + 3a_y\hat{j}$$

Dot Product

The dot product or scalar product of two vectors results in a scalar where a and b are magnitudes and ϕ is the angle between the directions of the tow vectors.

$$\vec{a} \cdot \vec{b} = ab\cos(\phi)$$

Dot product is the projection of one vector onto another. When $\theta = 90^\circ$ the dot product is 0. $\hat{i}, \hat{j}, \hat{k}$ are described as *orthonormal*:

$$\hat{i} \cdot \hat{i} = 1$$

$$\hat{i} \cdot \hat{j} = 0$$

The dot product is commutative. $a \cdot b = b \cdot a$.

Vector Product

The vector product is another ay of multiply two vectors, also known as the vector product.

$$c = ab\ \sin(\phi)$$

If a and b are parallel or antiparallel the vector product is 0. It is at a maximum when they are perpendicular. The vector product is not commutative. The direction of c is perpendicular to both a and b. So for example:

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{i} \times \hat{i} = 0$$

Position Vector

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Change is position vector is displacement:

$$\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$$

Instantaneous Values

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

In general, the instantaneous value of a changing quantity is given by the relevant derivative, just as with scalars.

Projectile Motion

$$\vec{v_0} = v_{0x}\hat{i} + v_{0y}\hat{j}$$

$$v_{0x} = v_0 \cos\theta \text{ and } v_{0y} = v_0 \sin\theta$$

A projectile has a duration of flight equal to twice the time taken from the highest point to the ground. Thus, when comparing projectiles, the one with the lower highest point will have a shorter period of flight. The range is given by:

$$R = \frac{v_0^2}{g}\sin 2\theta_0$$

Uniform Circular Motion

In uniform circular motion, velocity and acceleration each have constant magnitude but changing direction. This acceleration is called centripetal acceleration. The velocity is always at tangent to the circular path, while the acceleration is always inward toward the centre of the circle.

$$a = \frac{v^2}{R} \Rightarrow F = m\frac{v^2}{R}$$

$$T = \frac{2\pi R}{v}$$

The centripetal acceleration can come from many sources, such as gravity in a space station, friction for a car drive in a circle, or tension for a ball whirled on a string.

Consider a bicycle going around a vertical loop. At the top of the loop, the forces acting on the cyclists are the normal force (downward, opposing the centripetal force) and gravity, also downward.

$$-F_N - mg = m(-\frac{v^2}{R})$$

For the cyclist to not fall, normal force must be at minimum 0; centripetal force must cancel out with gravity. Thus, for a loop of radius $2.7\mathrm{m}$:

$$v = \sqrt{gR} = \sqrt{9.8 \times 2.7} = 5.1\mathrm{m/s}$$

Relative Motion

If reference frames are moving relative to each other, they may each observe differing velocities of an event, because the frames velocity will be added to the velocity of what they observe. Each observer will, however, observe the same acceleration of the event, assume both reference frames are inertial. If P is the event being observed, A is our primary reference frame, stationary with respect to P and B is our moving reference frame:

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$$

Force

A force is a "push or pull" on an object, which causes acceleration. Newton's laws of motion are applicable to objects which aren't moving at near c or at atomic scale.

Newton's first law states that: if no net force acts on a body, the body's velocity cannot change. These laws also only apply in inertial frames; over long enough distances, even the surface of the earth is non-inertial.

Mass is inversely proportional to acceleration due to force. Thus we arrive at Newton's second law:

$$\vec{F} = m\vec{a}$$

Acceleration along an axis is affected only by forces along the same axis. Thus, complex problems can be solved through decomposition.

Weight is mass under gravitation force. Given by:

$$W = mg$$

Normal Force

The normal force is the pushback of a surface against a force (such as weight) exerted on it. This force is always opposite to the force applied to the surface, and is always equal to the force applied to the surface, so that the forces on the object are in balance and thus the object remains stationary. This force is described by Newton's third law of motion.

$$\vec{F}_{BC} = -\vec{F}_{CB}$$

These forces are described as a third law force pair; a concept which arises any time two objects interact.

Friction

This occurs when an object is attempting to slide over another. It opposes the direction of motion of the moving object. If the friction is significant enough to stop the objects movement, then it will be equivalent to the force moving the object. Friction exists because surfaces are slightly rough, thus causing them to stick together a little.

Friction is essential for tasks such as picking things up for for propulsion, but overcoming it is also important for applications like efficiency in engines or motion through means such as roller skates. In general, anything intended to remain in motion must overcome friction.

Two types of frictional force exist; static frictional force acts as a kind of "normal" frictional force, preventing the motion of objects. It can increase from 0 to some maximum force. Thus, the maximum static frictional force must be overcome to begin motion for an object. Kinetic frictional force is the opposing force that acts on an object in motion, and is constant and usually weaker than static force. To maintain constant velocity, a constant force equal to the kinetic frictional force must be applied.

The magnitude of static friction is given by:

$$f_{s,\max} = \mu_s F_N$$

Where μ_s is the *coefficient of static friction*. If the applied force exceeds $f_{s,\max}$, sliding begins. The coefficient depends on the two surfaces involved. For example, rubber on dry concrete has a coefficient of static friction of $\mu_s = 1.0$ and of kinetic friction $\mu_k = 0.8$. Kinetic frictional force in general is given by:

$$f_k = \mu_k F_N$$

In general, $\mu_s > \mu_k$. The force required to overcome frictional is known as F_C, and is related linearly to mass. To determine μ_s we can use a setup with an object on a slope. If we slowly increase the angle, at the moment when the block just begins to move:

$$Mg\sin(\theta) = \mu_s N$$

$$Mg\cos(\theta) = N$$

$$\Rightarrow \tan(\theta) = \mu_s$$

Where θ is the angle between the surface and the flat. If we further increase the angle, the block will accelerate down the slope with acceleration:

$$a = (g\sin(\theta) - \mu_k g\cos(\theta))$$

For a force applied at an angle, we need to decompose into x and y components to address frictional forces.

Drag Force

A fluid is anything that can flow, i.e. a gas or liquid. Where there is a relative velocity between fluid and object, there is a resultant drag force. This force will oppose the relative motion, and will point in the direction of flow, relative to the body.

Drag force for a moving object with constant velocity is given by:

$$D = \frac{1}{2} C \rho A v^2$$

Where:

- v is relative velocity.
- ρ is the air density.
- C is an experimentally determined drag coefficient.
- A is the cross-sectional area of the body.

In practice, C will not be constant for all values of v. For a falling object:

$$D - F_g = ma$$

Once the drag force equals F_g, the object will reach terminal velocity at:

$$v_t = \sqrt{\frac{2F_g}{C \rho A}}$$

If A is increased, v_t decreases; the principle behind parachutes.

Tension

Tension occurs when a cord or rope is attached to an object and pulled to apply a force to the object. The cord will apply force to the object equal to it's tension force. If a system has multiple attachment points to a tensioned rope, each point pulling upward constitutes a force of T upward on the system; for example, with 2 attachment points, There is a mechanical advantage factor of 2.

Example

A car is moving with an acceleration of $1.2ms^{-2}$. In this car, a pendulum hangs. Find the angle at which this pendulum hangs relative to the normal.

$$a = 1.2ms^{-2}$$

$$x : ma = F_T \sin(\theta)$$
$$y : 0 = F_T \cos(\theta) \ - mg$$
$$\frac{F_T \sin(\theta)}{F_T \cos(\theta)} = \frac{ma}{mg}$$
$$\tan(\theta) = 0.122$$
$$\theta = 7^\circ$$

Perception of Forces

We percieve forces due to our awareness of our inertia and our sensing of the normal force of the ground against us. Thus, we are in freefall $(F_N = 0)$, we feel weightless because out acceleration is simply g and there is no normal force. When we are in a lift accelerating upward, we feel heavier because we are experiencing a greater normal force due to our greater acceleration. The *otic labyrinth* is an organ inside the ear, which uses hairs connected to an elastically suspended membrane. As the hairs move, cells detect their force and report this to the brain.

When a person is in a high acceleration vehicle, the perceive the angle to be tilting upward even when it isn't. This is because the resultant force vector of the normal force due to gravity and the normal force from the acceleration points upward relative to the flat plane.

Energy

Energy is required for any sort of motion. It is a scalar quantity. The transfer of energy is known as work.

Kinetic Energy

The faster an object moves, the higher its kinetic energy (E_k). For a stationary object, $E_k = 0$. For an object moving at non-relativistic speeds:

$$E_k = \frac{1}{2}mv^2$$

Kinetic energy is measured in joules (J).

$$1\,\text{joule} = 1\text{J} = 1\text{kgm}^2\text{s}^{-2}$$

If an objects kinetic energy changes, then work has been done on that object by a force. If energy is transferred to an object, positive work has been done, if energy is transferred from an object, negative work. For example, in a descending lift, gravity is doing positive work, by increasing the kinetic energy of the elevator, while the tension in the cable is doing negative work. Work can be determined from velocity through:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = Fd$$

Where d denotes distance. To calculate work done with vector quantities:

$$W = Fd\cos(\phi)$$

$$W = \vec{F} \cdot \vec{d}$$

Work is also measured in joules. For two or more forces, the net work is the sum of work done. When working with vectors, we can either calculate work for each vector independently and sum these, or take a vector sum and calculate work done once.

$$\Delta E_k = E_{k,f} - E_{k,i} = W$$

i.e. the change in kinetic energy is equal to the net work done.

Hooke's Law

When compressed or stretched, springs will attempt to return to the origin, exerting force given by:

$$\vec{F_s} = -k\vec{d}$$

Where k is the springs experimentally determined *spring constant*. k is a measure of the stiffness of the spring. F_s is a variable force, and exhibits a linear relationship between F and d.

We can find the work done by integrating:

$$W_s = \int_{x_i}^{x_f} -F_x \, dx$$

Plugging in kx for F_x:

$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

If the spring ends up closer to the origin $(x = 0)$, W_s will be positive. If it is further away, $W_s < 0$. For an initial posisition of $x = 0$:

$$W_s = \frac{1}{2}kx^2$$

Power

Power is the time rate at which a force does work. It is measured in joules per second or watts (W). If a force does W work in time Δt the average power due to the force is:

$$P_{\text{avg}} = \frac{W}{\Delta t}$$

and the instantaneous power at a given moment is:

$$P = \frac{dW}{dt}$$

$$P = \vec{F} \cdot \vec{v}$$

Using our understanding of conservation of energy, we can understand average and instantaneous power in terms of energy.

$$P_{\text{avg}} = \frac{\Delta E}{\Delta t}$$

$$P = \frac{dE}{dt}$$

Potential Energy

Potential energy, denoted by U or E_u is energy associated with the configuration of a system of objects that exert forces on one another. As an example, we can consider the earth and a bungee jumper as a system. In this system, gravitational potential energy accounts for the increase in kinetic energy during the fall, while elastic potential energy, held in the bonds between the particles in the rubber, accounts for the deceleration at the bottom of the fall. Elastic potential energy can be calculated with:

$$U(x) = \frac{1}{2}kx^2$$

When thinking about energy, it is important to understand that a system consists of two or more objects. A force acts between a particle and the rest of the system. When this force changes the configuration, the force does work denoted W_1. If the configuration change is reversed, the force reverses the energy transfer, doing work W_2.

For *conservative* forces $W_1 = -W_2$ is always true. Gravitational force, spring force and magnetic force are conservative. For these forces we can speak of potential energy. *Nonconservative* forces include kinetic friction force and drag force. For kinetic friction, this is because some energy is released as heat, which cannot be recovered into kinetic energy by friction.

For conservative forces, when an ideal object is moved around a closed path back to the beginning, no work is done. This implies that moving between two points costs the same amount of work done irrespective of the paths taken. This means that for calculations involving conservative forces, complex paths can be simplified. For an object being raised of lowered:

$$\Delta U = -W$$

This also applies to an elastic block-spring system: when a spring is compressed, elastic potential energy is stored, which is released as the spring returns to origin. In general, work can be calculated as:

$$W = \int_{x_i}^{x_f} F(x)dx$$

$$\Rightarrow \Delta U = -\int_{x_i}^{x_f} F(x)dx$$

In one dimension (x) force and E_u are related by work through:

$$F(x) = -\frac{dU(x)}{dx}$$

This allows us to find force from a graph of potential energy through finding the slope.

Mechanical Energy

The mechanical energy of a system is the sum of its potential and kinetic energy.

$$E_{\text{mec}} = E_k + E_u$$

Work done by conservative forces increases E_k and decreases E_u, so:

$$\Delta E_k = -\Delta E_u$$

If we take two instants of time 1 and 2:

$$E_{k,2} + E_{u,2} = E_{k,1} + E_{u,1}$$

i.e. the system has constant mechanical energy. This is known as the conservation of mechanical energy:

$$\Delta E_{\text{mec}} = \Delta E_k + \Delta E_u = 0$$

Work on a *system* can be considered to be equal to ΔE_{mec}. Although energy must be conserved within the system, the total energy of the system can be increased through external forces, for example by raising it, engendering an increase in gravitational potential energy.

However, most systems aren't ideal and thus suffer from friction, causing a release of thermal energy. Thus, work done is given in reality by:

$$\Delta E_{\text{th}} = f_k d$$

$$W = \Delta E_{\text{mec}} + \Delta E_{\text{th}}$$

Where E_{th} is thermal energy and f_k is the frictional coefficient of the surface involved.

Equilibrium

Points where $E_k = 0$ will be turning points in the motion of an object. At these points, objects have maximum potential energy and reverse in direction. A particle is said to be in *neutral equilibrium* if it is stationary and has only potential energy, with net force zero. An example of this is a marble on a flat surface.

Unstable equilibrium occurs when a particle is stationary with only potential and net force 0. However, if it is displaced slightly it will continue to move in that direction under a new force. An example of this is a marble balanced on a curved surface.

Stable equilibrium occurs when a particle is under the same conditions, but if displaced, will return to its original position. For example, consider a marble at the base of the ball. This is often observed in oscillating systems like pendulums.

Conservation of Energy

Energy transferred between systems can always be accounted for. Thus, the total energy E of a system, is constant. This includes mechanical, thermal and all other internal energy.

$$W = \Delta E = \Delta E_{\text{mec}} + \Delta E_{\text{th}} + \Delta E_{\text{int}}$$

In an isolated system, there is no external energy transfer. The total energy of this system cannot change.

$$\Delta E_{\text{mec}} + \Delta E_{\text{th}} + \Delta E_{\text{int}} = 0$$

Or, for two instants of time we can observe that:

$$E_{mec,2} = E_{mec,1} - \Delta E_{\text{th}} - \Delta E_{\text{int}}$$

The total energy of an isolated system is a function of its state; the intervening events are irrelevant.

Centre of Mass

For complex rotating objects, it can be extremely complex to consider their motion. However, for a specific point, the centre of mass, the motion is as of a particle, and thus is simple to analyse. For example, a cricket bat tossed spinning will have a very complex pattern of movement, however if we simply consider its centre of mass, the particle will transcribe a parabola, just as a ball would, with the rest of the bat rotating about this centre point.

For a system of particles, the centre of mass is the point that moves as though all of the system's mass were concentrated there. For two particles d units apart, where the origin is taken to be at the position of particle 1:

$$x_{\text{com}} = \frac{m_2}{m_1 + m_2} d$$

For two particles, with an arbitrary choice of origin:

$$x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

This generalises easily to any number of particles. If M is taken to be the sum of the masses of all of the particles:

$$\begin{aligned} x_{\text{com}} &= \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots + m_n x_n}{M} \\ &= \frac{1}{M} \sum_{i=1}^{n} m_i x_i \end{aligned}$$

For three dimensions we can use:

$$y_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i \ \text{ and } z_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i$$

This can be written in terms of vectors, where $\vec{r}_i$ is the position vector of the i'th particle:

$$\vec{r}_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i$$

For solid bodies, we assume an infinite sum of infinitely small particles. We can solve this through integration.

$$x_{\text{com}} = \frac{1}{M} \int x dm$$

Which, of course, generalises to each dimension. If we assume that our object is of uniform density (ρ)), then:

$$\rho = \frac{dm}{dV} = \frac{M}{V}$$

We can substitute this into our centre of mass formula to find that:

$$x_{\text{com}} = \frac{1}{V} \int x dV$$

If the object has symmetry, such as in a sphere or cube, the centre of mass lies at this point. So it lies at the centre of a sphere or cube, or for a sphere, it lies along the centre line of the circle and halfway down, etc. Note that the centre of mass need not be on the object; for a torus the centre of mass is in the centre of the whole. Because of this property, it is often possible to simply deduce the centre of mass of an object.

For systems with multiple objects, we can take centre mass of each object and calculate the centre mass of the resultant system. This approach can allow us to determine the result of removing a section of an object; the effect will be the inverse of adding a second copy of the object. For example, consider removing a smaller circle for a larger circle. One could find the centre mass of the smaller circle and the larger circle, then invert this result across the axes to find the resultant centre of mass of the larger circle.

For a system as a whole, the motion of the centre of mass will continue unaffected by internal forces. This assumes a closed system.

$$\vec{F}_{net} = M\vec{a}_{\text{com}}$$

Linear Momentum

Linear momentum is given by:

$$\vec{p} = m\vec{v}$$

Momentum points in the same direction as velocity, and can only be changed by a net external force for a system. The rate of change of momentum a particle or system with respect to time is equal to the net force on that object and is in the direction of that force.

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}$$

This is essentially Newton's second law ($f = ma$). Without external force, linear momentum cannot change. This is known as the law of conservation of linear momentum.

Collisions and Impulse

During a collision, momentum of a particle can change, as a net external force is acting on the particle. This is described as the application of an impulse (J) to the system or particle.

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt$$

The applied impulse is equal to the change in momentum during the collision, a face which is known as the linear momentum-impulse theorem.

$$\Delta\vec{p} = \vec{J}$$

Because momentum is conserved, we know that for a closed system:

$$(m_1v_1)_i + (m_2v_2)_i = (m_1v_1)_f + (m_2v_2)f$$

Elastic and Inelastic Collisions

When a tennis ball is placed atop a basketball and dropped, the tennis ball will rebound with a velocity in excess of what it's velocity would suggest, while the basketball will bounce very little.

Across subsequent bounces, the momentum of the ball will diminish, because it is not ideally elastic and thus loses energy as heat or sound with each bounce. The coefficient of restitution e is given by:

$$e = \frac{\text{relative velocity after collision}}{\text{relative velocity before collision}}$$

When a ball is bouncing in parabolic arcs, the square root of the ratio of the height of one bounce as compared to the next is a direct measure of e.

- $e = 1$ is perfectly elastic
- $e = 0$ is perfectly inelastic

A basketball would have a reasonably high coefficient of restitution, while a material like mud or gelatin would have a very low coefficient of restitution.

Collisions come in several varieties.

- In *elastic collisions* total kinetic energy is conserved. A useful approximation for common situations, such as on a pool table. In real situations, some energy is always transferred; as sound or friction in our billiards example.
- *Inelastic collisions* transfer some energy.
- *Completely inelastic collisions* result in the two objects stuck together, such as in the case of a bullet lodging in another object.

For two moving particles undergoing an inelastic collision, the resultant transfer of energy will be described by

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{if} + m_2 v_{2f}$$

For a completely inelastic collision with m_2 at rest,

$$m_1 v_{1i} = (m_1 + m_2) v_f$$

In the case of a completely inelastic collision, the velocity of the centre of mass will remain unchanged.

$$\vec{v}_{\text{com}} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2}$$

For elastic collisions, the total kinetic energy is always conserved. In the case of a stationary m_2 for momentum:

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

and for kinetic energy:

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

For the individual objects:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \qquad\qquad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

This makes perfect sense when we consider with an example. For a pool ball, $m_1 \approx m_2$, all of the velocity of m_1 will be transferred to m_2, which will continue with that speed, exactly the situation observed in pool. As another example, consider a ball colliding with a building. Here, $m_1 \ll m_2$ and so v_{1f} will be roughly equal to $-v_{1i}$, i.e. the ball has bounced back with roughly its initial velocity, as we would expect. What this tells us is that when two masses collide, their final velocities are determined by the ratio of their masses.

For higher dimensions, these formulas can be used by breaking the problem down into components for each dimension.

Rockets

Rockets and their exhaust products form an isolated system. Thus momentum within the system must be conserved. However, because the rocket is ejecting mass out, its mass is not a constant. Thus it will be accelerating with time.

$$M_v = -dMU + (M + dM)(v + dv)$$

This can be simplified using relative speed, given by

$$U = v + dv - v_{\text{rel}}$$

This leads us to the first rocket equation:

$$Rv_{\text{rel}} = Ma$$

Where:

- R is the mass rate of fuel consumption (kgs^{-1})
- v_{rel} is the relative velocity of the ejected fuel
- M is the (time dependent) mass of the rocket
- a is the acceleration of the rocket

Thus, Rv_{rel} is the *thrust*, T. Which is the force exerted by the rocket engine. This eventually gives us the second rocket equation:

$$v_f - v_i = v_{\text{rel}} \ln \frac{M_i}{M_f}$$

Which in words states that the change in velocity of the rocket will be equal to the relative velocity multiplied by the logarithm of the ration between the initial and final mass of the rocket. This logarithmic relationship is why it is so expensive to launch spacecraft; the fuel must make up a significant proportion of the mass to achieve sufficient velocity. This is why rockets often drop stages to increase this ratio.

Rotational Motion

Rotational motion occurs with rigid bodies, which rotate as units, locked together without changing in shape or other properties. In this subject rotation around a fixed axis, usually an axis of symmetry. Thus the rotation of a star, where layers of gas rotate independently is excluded. In addition a rolling bowling ball is excluded as both rotation and translation are ocurring.

Rotation of bodies around an axis of rotation is measured by angular position from the x-axis. Rotation is measured in radians, which are dimensionless.

$$1\,\text{revolution} = 360^\circ = 2\pi\,\text{rad}$$

Radians do not reset to 0 after passing 2π, they continue to increase. If we know $\theta(t)$, the rotation of the object with respect to time, we know all we need to about its kinematics. We can define angular displacement as the size of the angle between the initial position (the x-axis) and the current angle:

$$\Delta\theta = \theta_2 - \theta_1$$

It is worth noting that "clocks are negative"; a counterclockwise rotation in the context of angular rotation is positive.

Rotations are *non-commutative*, that is the order in which rotations are performed is significant. Rotation in the y followed by in the x is not equivalent to the inverse.

Average angular velocity is given by the average angular displacement during a time interval.

$$\omega_{\text{avg}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous angular velocity can be found by taking the limit of this:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The magnitude of angular velocity is the angular speed. Average angular acceleration is given by the change in angular velocity.

$$\alpha_{\text{avg}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

We can of course take the limit of this for instantaneous angular acceleration. For a rigid body, these statements will be true for all points; all points will have the same angular velocity. Angular velocity and acceleration can be written as vectors. The *right hand rule* is used to determine the direction of these vectors. If one wraps the fingers of their right hand around the axis of rotation, with the fingers pointing in the direction of rotation, the extended thumb will point in the direction of the resultant vector.

Equations

The system of equations of constant acceleration can be converted simply into rotation by simply considering angle where one would consider position; our first derivative of position is angular velocity, our second angular acceleration.

Position	Angle
$v = v_0 + at$	$\omega = \omega_0 + at$
$\Delta x = v_0 t + \frac{1}{2}at^2$	$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$\Delta x = vt - \frac{1}{2}at^2$	$\Delta\theta = \omega t - \frac{1}{2}\alpha t^2$
$\Delta x = \frac{1}{2}(v_0 + v)t$	$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t$
$v^2 = v_0^2 + 2a\Delta x$	$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$

To relate angular variables with linear variables, we need to use the angle and the radius. For a θ in radians and r in metres:

- For position, $s = \theta r$
- For speed, $v = \omega r$
- Tangential acceleration is given by $a_t = \alpha r$
- Radial acceleration, $a_r = \frac{v^2}{r} = \omega^2 r$
- For period, $T = \frac{2\pi}{\omega}$

Note that tangential acceleration is the acceleration at a tangent to the rotation while radial acceleration is the centripetal acceleration back in toward the axis of rotation.

Energy

For a rotation system, the rotation kinetic energy is given by summing the individual kinetic energies of all particles in the system.

$$E_k = \sum_{i=0}^{n} \frac{1}{2} m_i v_i^2$$

For a rigid body, all particles have the same rotational velocity. We can use this fact to express the kinetic energy as follows:

$$E_k = \frac{1}{2} \left(\sum_{i=0}^{n} m_i r_i^2 \right) \omega^2$$

The term in brackets here is known as the *rotational inertia* or *moment of inertia* of the body in the axis of rotation, denoted I. This is a property of the body being considered. The formula for kinetic energy can then be rewritten:

$$E_k = \frac{1}{2} I \omega^2$$

Relating this to linear quantities, I relates to mass while ω, as elsewhere relates to velocity. I in effect tells us how difficult it is to change the speed of rotation of a body. The moment of inertia can be considered in a general sense by taking the limit across a solid body:

$$I = \int r^2 dm$$

Rather than calculate the moment of inertia in this course, it is more common to use one of a few common shapes which have intuitive moments.

- A thin hoop around an axis with radius r has a moment of inertia given by $I = mr^2$ where m is the mass of the hoop. This is the maximum motion of inertia for a given mass.
- An annulus, which is a hoop with thickness, the moment of inertia is given by $I = \frac{1}{2}m(r_1^2 + r_2^2)$, where r_1 is the inner radius and r_2 is the outer radius. The limit of this case is $r_1 = r_2$, the thin hoop.
- A cylinder has moment given by $\frac{1}{2}mr^2$, the other limit of the annulus, where $r_1 = 0$.
- A sphere of uniform density has $I = \frac{2}{5}mr^2$.
- A hollow sphere has $I = \frac{2}{3}mr^2$, because the mass is more concentrated further away from the center.

These concepts are used often in machinery, where objects with a high moment of intertia are used to store energy. A classical example is in a flywheel.

If we want to consider different axes of rotation for the same object, as long as they are parallel and one of them is through the center of mass, we can use the *parallel-axis theorem* to relate the center of mass' moment of inertia to that of another parallel axis.

$$I = I_{\text{com}} + mh^2$$

Where I_{com} is the moment of inertia of the axis through the center of mass, m is the mass of the body and h is the distance to the other axis of rotation.

Force

The force necessary to rotate an object depends on the angle of the force and where on the object it is applied. When force is applied to an object, only force applied tangentially will cause a rotation. The magnitude of the rotation will be dependent on the radial distance at which the force is applied. The construct used to understand this *torque*.

$$\tau = rF\sin(\phi)$$

Here, r is the radial distance, F is the magnitude of the force applied and ϕ is the angle between the force and a line from the rotation axis. The perpendicular between the force and the axis if one extends out the force as a line is known as the *moment arm* of the force. Torque is expressed in Nm, the same as Joules, but as it is not an energy not *in* Joules. A torque is defined as positive if it would cause a counterclockwise rotation and negative if it would cause a clockwise rotation. For several torques, we can sum them to find the net or resultant torque.

With torque, we now have the tools to rewrite $F = ma$ with our rotational variables. Torque will take the role of force, with I replacing mass and α acceleration.

$$\tau = I\alpha$$

We can understand rotational work as the change in rotational kinetic energy.

$$\Delta E_k = E_{k,f} - E_{k,i} = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W$$

For rotation in a fixed axis, work is the area under a torque vs angle graph.

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta$$

When we have a constant torque, this is simply $W = \tau(\theta_f - \theta_i)$, the rotational equivalent of $W = Fd$. Work can be related to power through.

$$P = \frac{dW}{dt} = \tau\omega$$

Analogy Summary

Translation	Rotation
x	θ
$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$
m	$I = mr^2$
$F_{\text{net}} = ma$	$\tau_{\text{net}} = I\alpha$
$W = \int F dx$	$W = \int \tau d\theta$
$E_K = \frac{1}{2}mv^2$	$E_K = \frac{1}{2}I\omega^2$
$P = Fv$	$P = \tau\omega$
$W = \Delta E_K$	$W = \Delta E_K$

Rolling Objects

For a rolling object, both translational and rotational motion is occurring. For a wheel for instance, the centre of mass is always moving in a straight line, with the rest of the object rotating. For a wheel rolling in this way, one period is the time taken for a full rotation to occur. The displacement between the starting position of a rotation and the end position will be given by $s = 2\pi R$, the circumfrence of the wheel. For an arc of size θ the distance covered while the wheel turns this angle will be $s = \theta R$. We can take the derivative to find the velocity of the centre of the mass:

$$v_{\text{com}} = \omega R$$

The translational velocity of the wheel is directly related to its speed of rotation. This is only true for a wheel with no slippage. For a turning wheel, a point on one side of the wheel as a velocity exactly opposite to the velocity of a point on the other side. For a wheel which is purely translating, these points will have an equal velocity $= v_{\text{com}}$. To find the velocity of these points for a rolling wheel, we can add the vectors, finding that the point at the top has a velocity of twice the centre of mass, while the point at the bottom has no velocity.

For this system, the kinetic energy must be the sum of the translational and rotational kinetic energy.

$$E_K = \frac{1}{2} I_{\text{com}} \omega^2 + \frac{1}{2} m v_{\text{com}}^2$$

For a wheel to accelerate, its angular speed must change. This can be seen simply from the derivation of the velocity formula, and yields:

$$a_{\text{com}} = \alpha R$$

If a slip occurs, an object is not rolling smoothly. In this case, the object has a frictional force exerting a torque on the object. For all rolling objects, we can write the moment of inertia in the form

$$I = cmr^2$$

This can be seen in the earlier examples of a sphere ($c = \frac{2}{5}$) or the hoop ($c = 1$). We can accomodate any other rolling object by adjusting to a different value of c. For a point particle $c = 0$, because there is no accumulation of rotational motion with a very small radius. For an object rolling down a ramp of height h:

$$mgh = \frac{1}{2} I_{\text{com}} \omega^2 + \frac{1}{2} m v_{\text{com}}^2$$

Here, we can replace ω with $\frac{v_{\text{com}}}{r}$ to simplify.

$$\frac{1}{2} cmr^2 \left(\frac{v_{\text{com}}}{r} \right)^2 + \frac{1}{2} m v_{\text{com}}^2 = mgh$$

$$\frac{1}{2} m (1 + c) v_{\text{com}}^2 = mgh$$

Rearranging for v_{com} we find

$$v_{\text{com}} = \sqrt{\frac{2gh}{1 + c}}$$

This tells us that the final velocity of the object is independent of its mass and of its radius, and depends only on the geometry of the object, represented by c. This makes sense; the large the value of c, the more rotational energy the object gains.

Using equations of constant acceleration, we can find the acceleration of the object, and after some rearranging, this yields the following expression, where θ is the angle of incline down which the object rolls.

$$a_{\text{com}} = \frac{g\sin(\theta)}{1+c}$$

This tells us that a point particle ($c = 0$), will have the greatest acceleration of any shape. This makes sense as for a point particle, no energy is gained as rotational kinetic energy, and so all gain in kinetic energy is as downward velocity. A sphere ($c = \frac{2}{5}$) will have a greater acceleration than a hoop ($c = 1$). The relationship between these accelerations can be analysed through

$$a_{\text{com}} = \frac{a_{\text{particle}}}{1+c} \text{ for } c > 0$$

Torque

Previously, we defined torque for a rotating body and a fixed axis. We can now redefined this for any particle moving along a path. As this path need not be circle, torque will be defined here as a vector. The direction of torque is defined with the right hand rule, if a is the first vector and b is the second, the right hand rule tells us the direction of torque by taking a as the index finger and b as the middle finger, and taking the direction of the extended thumb.

If we take $\vec{r}$ to be a vector from the origin to the particles current position and $\vec{F}$ to be the force on the particle, the torque is then given by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

And the magnitude of the torque is given by

$$\tau = rF\sin(\phi)$$

Thus, the torque will be a maximum when the force and $\vec{r}$ are at $\phi = \frac{\pi}{2}$ from each other, with fits with our understanding of reality. If $\vec{F}$ is in the same direction as the position vector, no torque will be exerted. This can be expressed in two equivalent ways.

$$\tau = rF_{\perp} = r_{\perp}F$$

The direction of torque will be perpendicular to the plane defined by $\vec{r}$ and $\vec{F}$. The cross product is calculated as follows.

$$\vec{a} = 2\hat{\imath} + 3\hat{\jmath}$$

$$\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$$

$$\vec{a} \times \vec{b} = 4(\hat{\imath} \times \hat{\imath}) + 6(\hat{\imath} \times \hat{\jmath}) + 4(\hat{\imath} \times \hat{k}) + 6(\hat{\jmath} \times \hat{\imath}) + 9(\hat{\jmath} \times \hat{\jmath}) + 6(\hat{\jmath} \times \hat{k})$$

Because the cross product of two vectors which are proportional to each other is 0, $\hat{\imath} \times \hat{\imath} = 0$, so most of these terms disappear. In addition, if we reverse the order of vectors in a cross product, we reverse the direction of the resultant vector. So $\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath} = \hat{k}$. Thus, each of these pairs produces another unit vector: $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$ and $\hat{k} \times \hat{\imath} = \hat{\jmath}$. This fact is known as the *cyclic permutation*; if we write these letters in a triangle and read them clockwise, we can remember these equalities. Applying this knowledge to $\vec{a} \times \vec{b}$:

$$\vec{a} \times \vec{b} = 0 + 6(\hat{\imath} \times \hat{\jmath}) + 4(\hat{\imath} \times \hat{k}) + 6(\hat{\jmath} \times \hat{\imath}) + 0 + 6(\hat{\jmath} \times \hat{k})$$

$$= 6(\hat{\imath} \times \hat{\jmath}) + 4(\hat{\imath} \times \hat{k}) - 6(\hat{\imath} \times \hat{\jmath}) + 6(\hat{\jmath} \times \hat{k})$$

$$= -4(\hat{k} \times \hat{\imath}) + 6(\hat{\jmath} \times \hat{k})$$

$$= 6\hat{\imath} - 4\hat{\jmath}$$

This, however is a little bit long winded, and the value can instead be calculated using a *determinant*. To find the determinant of $\vec{r} \times \vec{F}$ we can use the following matrix, where (x, y, z) is $\vec{r}$, the position vector and (F_x, F_y, F_z) is $\vec{F}$, the force vector.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

What this asks us to do, in essence, is to take the top left cell, eliminate its row and column, and multiply the top left (y) and bottom right (F_z) cells of the resultant matrix, then subtract the product of the top right (z) and bottom left (F_y) from this, then subtract from this term the result of doing the same with the middle top cell and add the result of the top right cell. As an equation, this becomes:

$$\tau = (yF_z - zF_y)\hat{\imath} - (xF_z - zF_x)\hat{\jmath} + (xF_y - yF_x)\hat{k}$$

$$= (yF_z - zF_y)\hat{\imath} + (zF_x - xF_z)\hat{\jmath} + (xF_y - yF_x)\hat{k}$$

This structure can of course be used more generally to solve any cross product. The first vector goes on the second line and the second on the third line. With the previous example:

$$\vec{a} = 2\hat{\imath} + 3\hat{\jmath}$$
$$\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$$
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 0 \\ 2 & 3 & 2 \end{vmatrix}$$
$$\vec{a} \times \vec{b} = (6-0)\hat{\imath} + (0-4)\hat{\jmath} + (6-6)\hat{k}$$
$$= 6\hat{\imath} - 4\hat{\jmath}$$

A property of determinants is that any pair of rows or columns can be swapped with the result of multiplying the result by -1. This makes clear that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

Angular Momentum

This understanding can be extended to a concept of angular momentum. Angular momentum is denoted $\vec{l}$ and can be calculated from position and momentum or equivalently position, mass and velocity.

$$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

It bears noting that the particle doesn't need to be rotating around the origin to have angular momentum around it. Angular momentum is measured in the same units as momentum, in kgms^{-1}. The same properties of torque hold here, with the magnitude of l being given by

$$l = rmv\sin(\phi) = rp_\perp = rmv_\perp = r_\perp p = r_\perp mv$$

It is worth reiterating the angular momentum has meaning only with respect to the specified origin; if the origin is moved, angular momentum will change. In addition, as a vector product, angular momentum is always perpendicular to the plane formed by the position and momentum vectors. If position and momentum are in the same direction, there is no angular velocity.

Using the tool of angular momentum, we can rewrite Newton's second law as

$$\vec{\tau}_{\text{net}} = \frac{d\vec{l}}{dt}$$

As angular momentum is a vector, we can find the angular momentum of a system by summing the angular momentum of particles in the systems. Thus the rate of change of the total angular momentum in the system is the total net torque on all particles in the system. We denote angular momentum for a system as L as opposed to the l for a single particle.

When considering these equations, it is essential that both torque and angular momentum are measured relative to the same origin. In addition, when considering a system, if the centre of mass of that system is accelerating, the origin taken must be the centre of mass, less unbalanced acceleration of the particles disrupt the solution.

The angular momentum of a rigid body can be found through summation. For a body rotating about the z axis:

$$L_z = \sum_{i=1}^{n} l_{iz} = \sum_{i=1}^{n} m_i v_i r_{\perp i} = \sum_{i=1}^{n} m_i(\omega r_{\perp i}) r_{\perp i}$$

$$= \omega \left(\sum_{i=1}^{n} m_i r_{\perp i}^2 \right) = \omega I$$

i.e. the total linear momentum of the body is equal to the sum of the linear momentums of its particles, is equal to the moment of inertia of the body multiplied by its rate of rotation.

As with translational momentum, angular momentum is conserved. This implies that if the moment of inertia of an object decreases, the angular velocity must increase, and that if the angular velocity decreases this must coincide with an increase in moment of inertia. It also suggests that if an object with angular momentum is introduced into a system, it will impart some of its momentum to the system. For example, a person on a rotation capable chair is handed a spinning wheel. If the person tilts the wheel such that it is spinning in the same axis as the rotation of the chair, the chair must rotate slightly in the opposite direction to maintain the 0 angular velocity it had before the wheel was introduced. This can be expressed like so

$$I_i\omega_i = I_f\omega_f$$

Because angular momentum is a vector, we can seperate it into components and consider that the net angular momentum *on an axis* remains constant. So even if the angular momentum in the x direction changes, the angular momentum in the y and z directions must remain constant. The conservation of linear momentum allows us to determine an equation for the increase of ω in terms of the change in linear momentum.

$$\omega_f = \omega_i\frac{l_i}{l_f}$$

Because angular velocity increases, the rotational kinetic energy of a system increases when its moment of inertia decreases. This doesn't violate conservation of energy as work needed to be done to decrease the moment of inertia.

Precession

If a spinning object, such as a bicycle wheel, is suspended relative to an axis and has an external force such as gravity acting on it, a torque will be generated relative to the axis causing it to rotate about that axis. This phenomena is known as precession. If the angular momentum is initially outward, it will continue to point outward throughout the rotation. Its magnitude will remain constant, but its direction will change constantly due to the torque. In this situation, the bicycle wheel is a gyroscope. If the wheel were not fixed, it would topple, but by giving it angular momentum, it is stabilised. This fact is widely utilised in navigation technologies. The same phenomena is observed in the motion of a spinning top.

For a spinning top at an angle ϕ to the ground, with mass m, spinning with angular velocity ω and with moment of inertia I, under gravitational acceleration g, the angular speed of precession, or *precession rate* Ω is given by

$$\Omega = \frac{mgr}{I\omega}$$

Note that this is independent of ϕ, the angle with the ground, and, due to the m term in I, the mass of the top. This implies that any top of similar geometry will have the same behavior in terms of precession. This formula will not hold for a gyroscope spinning too slowly.

Equilibrium

It is common that we want objects to remain static despite forces acting on them. For example a ceiling fan, a sliding ice puck or a rolling bicycle wheel have a constant linear momentum and a constant angular momentum. These objects are said to be in *equilibrium*, their linear and angular momentums are each constant. Objects with $\vec{P} = \vec{L} = 0$, such as a book on a table are said to be in *static equilibrium*. If an object would return to this equilibrium after a disturbance, it is in the previously visited stable equilibrium. If a disturbance disrupts the equilibrium, the object is in unstable equilibrium.

We can define requirements of equilibrium using Newton's second law, for both the rotational and linear cases.

$$\vec{F}_{\text{net}} = 0$$

$$\vec{\tau}_{\text{net}} = 0$$

Often we will only consider equilibrium in a single plane, for example in the xy plane.

The gravitational force on a body is the sum of the forces on the particles making up the body. In general, the force due to gravity on an object can be assumed to apply to the *center of gravity* of the body, which is for most purposes well approximated by the center of mass of the body. However this is an approximation; in reality the attraction of each particle is slightly different than for its neighbours.

We can use equilibrium to gain insight into some situations. Consider as an example a beam of length L and mass $m = 1.8\text{kg}$ placed such that each end is supported by a fulcrum. The beam has a block of mass $M = 2.7\text{kg}$ placed at $\frac{L}{4}$ units along it. We can determine the force on each of the fulcrums by considering that the momentum of the system must be 0 for the beam to be stationary. In that case, the net torque on the beam must be 0. If we take the leftmost fulcrum as our rotation axis we can find the net torque due to gravity through

$$\tau = \frac{1}{4}MgL + \frac{1}{2}mgL$$

And we know that this torque must be equal to the torque from the normal force exerted by the right fulcrum. Thus

$$F_r L = \frac{1}{4}MgL + \frac{1}{2}mgL$$

$$F_r = \frac{1}{4}Mg + \frac{1}{2}mg$$

$$F_r = \frac{1}{4} \times 2.7 \times 9.8 + \frac{1}{2} \times 1.8 \times 9.8$$

$$F_r = 15.4\mathrm{N}$$

This accounts for an angular momentum of 0, so we can use this figure to find the force on the left fulcrum by observing that the linear momentum must also be 0. For this to be the case, the total normal force must cancel with the total downward force from gravity.

$$F_l = (M + m)g - F_r$$

$$F_l = (2.7 + 1.8) \times 9.8 - 15.4$$

$$F_l = 28.7\mathrm{N}$$

Elasticity

Some problems suffer from a wide range of unknowns which make them unnapproachable with the techniques studied thus far, and these problems are described as *indeterminate*. Often this is due to a fault assumption; that the bodies involved are rigid and do not deform. In reality, bodies do not behave this way and it is through the concept of *elasticity* that we are to solve some more complex problems.

A *stress*, which is a deforming force applied per unit area, produces a *strain* or a unit deformation. Three main types of stress exist.

- Tensile stress occurs when an object is stretched or squished in a single dimension.
- Shearing stress occurs when an object has opposite forces applied to each end, causing it to warp.
- Hydraulic stress occurs when an object is compressed from all sides.

For a given object, an *elastic range* for which the relatioship between stress and strain is roughly linear. Beyond this range we reach the *yield strength* and enter the range of *permanent deformation* which is the case until the object eventually ruptures at its *ultimate strength*. This linear relationship is defined by the *modulus of elasticity* of the object related through

$$\text{stress} = \text{modulus} \times \text{strain}$$

For simple tension or compression, stress is given by force divided by area. The strain is a dimension quantity given by the change in length of the object divided by its total length. The objects modulus of elasticity is also known as its *Young's modulus* denoted E.

$$\frac{F}{A} = E\frac{\Delta L}{L}$$

Many materials have very different tensile and compressive strengths, even though they will use the same modulus for both calculations. Concrete for example has an extremely high compressive strength and very low tensile strength. The region of elastic deformation is characterised by Hooke's law.

Strain guages are objects used to measure strain as their resistance changes due to strains placed on them. They are commonly used in large buildings to monitor the strains on them.

When a force is exerted on an axis perpendicular to the long axis of an object we use the objects *shear modulus* G to calculate the results. When an object has one end stretched by a distance Δx as compared to its inital state

$$\frac{F}{A} = F\frac{\Delta x}{L}$$

Where L is the objects long axis length. For hydraulic compression, the *bulk modulus* B is used. Where ΔV is the change in volume of the objects volume V. There pressure p is found through

$$p = B\frac{\Delta V}{V}$$

Special Relativity

Galiliean Mechanics

Classical mechanics are useful for many everyday phenomena, but the reality is somewhat different in reference frames moving at a significant proportion of the speed of light, c. The speed of light is determined through the calculation

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \mathrm{ms}^{-1}$$

Here, ϵ_0 is the *permittivity of free space* and μ_0 is the *permeability of free space*. Light in space is in reality a related electric and magnetic field, perpendicular to each other and self propagating across space. Maxwell's equations are used to explain this phenomena, telling us that electromagnetic waves travel at c in a vacuum and that they *can* travel in a vacuum; they don't require a medium.

Einstein postulated that the speed of light is the same in all inertial reference frame, a fact fundamental to modern physics. The theory is known as the *Special Theory of Relativity* because it deals exclusively with inertial reference frames. He extended this to accelerating reference frames, however this is not studied in this subject. Although the surface of the earth is not technically an inertial reference frame, the effects of its acceleration are very minor and can usually be ignored.

A reference frame is inertial if it moves with a constant velocity with respect to an inertial reference frame. For instance, a car is an inertial reference frame. Observing it from the outside, we might say it is moving at $10\mathrm{ms}^{-1}$. We can also take the perspective from the inside; in this case the surroundings of the car appear to be moving at $-10\mathrm{ms}^{-1}$. If an event occurs in the car, such as a ball being tossed, for an external observer, the velocity of the car is added to the velocity of the ball to adapt it to the external reference frame. This is a *Galiliean transformation*.

In general, for an event occuring at time t and position x' in a moving reference frame at speed v relative to an external observer, the event appears to occur at

$$x = x' + vt$$

If an object in a moving reference frame gains an additional velocity u', the velocity in the external reference frame will be given by

$$u = u' + v$$

These can be performed as vector additions if necessary. Acceleration is constant between frames. The laws of mechanics are the same in all reference frames.

Relativity

Einstein's principle of relativity states that the speed of light is constant in all inertial reference frames. Indeed it states that all laws of physics are the same in all inertial reference frames. If all laws are the same, and Maxwell's equations yield the same speed for c in all frames, it makes sense that c must be a constant. If this is the case, we must adjust our understanding of space and time to match.

Consider a cyclist moving at speed u. His velocity is given by

$$u = \frac{\Delta x}{\Delta t}$$

If we instead measured $\Delta x'$ from a moving car, it would appear that the cyclist had moved less distance over the same interval Δt, and thus that they had a lower velocity. This makes sense; the relative velocity of the cyclist compared to the car is lower than the relative velocity of the cyclist as compared with a stationary reference frame; say the road.

If, however, we replace the cyclist with a beam of light we find:

$$u = \frac{\Delta x'}{\Delta t} = \frac{\Delta x}{\Delta t} = c$$

A problematic result; despite a smaller distance covered ($\Delta x > \Delta x'$), we know that the velocity of the light is unchanged. This implies that the time passed in the car must be lower than in the stationary reference frame; we have run up against time dilation.

To implement this, we need to add a new quality to our events; where before something happened at a given coordinate (x, y, z) at a given time, we must now consider that the event occured at a given *spacetime coordinate* (x, y, z, t), all of which are prone to transformation between reference frames.

An important fact to note is that t is not the time an event was observed, but the time at which it occurred. If an event occurs at $t = 1\mu\text{s}$ and light takes an additional two microseconds to reach one observer and 1 further microsecond to reach another, each observer should nonetheless state the same value of t. Indeed, the delay could be used to calculate x in some circustances.

Simultaneity Examples

Two events can be said to be simultaneous only if they take place at the same time in the same inertial frame. They might be at different positions, but must be measured to have the same time. For example, with the knowledge that light travels at 300m per μs, consider an event 1200m away observed at $3\mu\text{s}$ and a second event 600m away observed at $5\mu\text{s}$ are they simultaneous?

$$t_1 = 5\mu\text{s} - \frac{1200\text{m}}{300\text{m}\mu\text{s}^{-1}} = 1\mu\text{s}$$

$$t_2 = 3\mu\text{s} - \frac{600\text{m}}{300\text{m}\mu\text{s}^{-1}} = 1\mu\text{s}$$

The two events are simultaneous, because they occurred at the same time t. Events that are simultaneous in a given frame aren't necessarily simultaneous in a different reference frame.

As an example to show this, consider a light emitted from a central point 17.5km from each of two detectors. In the stationary reference frame of the detectors, the light takes $58.33\mu\text{s}$ to reach each. and so the two events of light reaching the two detectors are simultaneous. For an observer passing this scene at a speed of 600kms^{-1}, the light appears to travel only 17465m, taking $58.22\mu\text{s}$ in one direction but 17535m, taking $58.45\mu\text{s}$ in the other. Thus, the two events are not simultaneous in the moving observer's reference frame. Simultaneity of events is only possible in a single reference frame.

As another example; consider a moving train with two lights on it. A device on the train indicates from which light source it received light first. An external observer perceives the two lights to have flashed simultaneously. This means that on board the train, light from the light source in the direction of motion must have reached the detector first, as the train is moving in the direction of that light, effecting a higher velocity. Thus, the events are simultaneous in the external frame but at different times on board the train.

Time Dilation and Length Contraction

Consider a set up with a light emitting box which releases a pulse of light toward a mirror, which reflects the light back to the box which measures the time taken, Δt. This time in the (moving) reference frame of the box is

$$\Delta t' = \frac{2h}{c}$$

For a stationary observer, the light appears to have traversed a large distance because it also moved in the direction of the box, and thus a longer Δt is observed. This Δt is given by

$$\Delta t = \frac{\Delta t'}{\sqrt{\frac{1-v^2}{c^2}}}$$

To simplify this we define

$$\beta = \frac{v}{c}$$

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}}$$

Analysing this we can see that as v gets closer to 0, so too does β and so $\Delta t \approx \Delta t'$, while when $v \to c$, $\beta \to 1$, and so $\Delta t \to \infty$.

At this stage it is useful to add a notion of *proper time*, defined as the time between two events that occur at the same position and denoted $\Delta \tau$. In the above case, time in the frame of the emitting box is proper time. Proper time is considered to be in the relevant stationary reference frame.

If an object is moving very rapidly, time moves more slowly in its reference frame as compared to a stationary reference frame. This means, that for the object to measure the same velocity as an observer in an external frame, the distances covered must be shorter. For an object in the S' reference frame, with beta value β, the contracted length of a distance of prpoer length L is given by

$$L' = L\sqrt{1-\beta^2}$$

Proper length is the length between two objects in the frame at which those objects are at rest.

If we consider a distance d in a cartesian coordinate system, it can be said that

$$d^2 = \Delta x^2 + \Delta y^2$$

For some x and y. An interesting geometric property of this is that it is *invariant*; if we rotate our coordinate system, x and y might move but d will always stay the same. A similar invariant exists for relativistic physics:

$$s^2 = c^2(\Delta t)^2 - (\Delta x)^2 \quad (-(\Delta y)^2 - (\Delta z)^2)$$

This expression has the same value in all inertial reference frames. s is known as the *spacetime interval*. We can use the knowledge of this invariant to effect transformations of the spacetime coordinates of events between reference frames.

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

The *Lorentz factor* γ allows us to perform these transformations. For a stationary reference frame S and a second reference frame S', moving at speed v relative to S in the x direction, we transform the coordinates as follows.

S′	S
$x' = \gamma(x - vt)$	$x = \gamma(x' + vt')$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$t' = \gamma(t - \frac{vx}{c^2})$	$t = \gamma(t' + \frac{vx'}{c^2})$

It is very important to make sure the events one is comparing are equivalent and not e.g. the measurement of an event.

If we have stationary frame S, with an object inside it moving at speed u and a second frame S', moving relative to S with speed v then we can find u and u', the velocity as measured by an observer in S' as

$$u' = \frac{u-v}{1-\frac{uv}{c^2}} \quad \Big| \quad u = \frac{u'+v}{1+\frac{u'v}{c^2}}$$

Relativistic Momentum

Because relativistic velocity doesn't have simple transformations, we need to reconsider concepts such as momentum for this new framework. If we take velocity for a particle of mass m to be given by

$$p = m\frac{\Delta x}{\Delta \tau}$$

i.e. mass multiplied by velocity (using proper time). If u is the velocity of the particle as measured in a given frame s then p in that frame is given by

$$p = \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma mu$$

With this definition, momentum is conserved in all intertial reference frames. It can be seen in this formula that as the velocity of the particle increases closer to c it's momentum approaches infinity. This is an intuitive way to understand why particles can't reach the speed of light; it would require an infinite quantity of energy.

If we use our equation for the spacetime invariant from earlier, the equation for relativistic momentum and apply some witchcraft, we find that the total energy of a particle is given by

$$E = \gamma mc^2 = E_0 + K = \text{rest energy} + \text{kinetic energy}$$

Where E_0, the rest energy is given my mc^2. Thus, the kinetic energy, K, is given by $(\gamma - 1)E_0$. This kinetic energy is roughly the same as $\frac{1}{2}mv^2$ for low velocitys, but approaches infinity as velocity approaches c. We can find another equation, known as the relativistic energy equation:

$$E^2 = E_0^2 + p^2c^2$$

As an example, let us find the energy of a Muon moving at $0.96c$, with a mass 207 times that of an electron.

$$E_0 = mc^2 = 207 \times 9.1 \times 10^{-31} \times (3 \times 10^8)^2 = 1.7 \times 10^{-11}\text{J}$$

$$\gamma_p = \frac{1}{\sqrt{1-\frac{(0.96c)^2}{c^2}}} = 3.57$$

$$E = \gamma_p mc^2 = 3.57 \times 1.7 \times 10^{-11} = 6 \times 10^{-11}\mathrm{J}$$

The energy levels here are almost absurdly miniscule; exponents like 10^{-11} are somewhat ridiculous. Thus we instead often use a different unit for energy levels of particles This value is the energy of an electron after being accelerated through a potential difference of $1\mathrm{V}$.

$$1\mathrm{eV} = 1.6 \times 10^{-19}\mathrm{J}$$

The electron volt (eV) is this unit. We also use mega-electron volts, which are simply greater by a factor of 10^6.

Using the relativistic energy equation, we can find a momentum for photons, despite their having 0 mass:

$$E^2 = E_0^2 + p^2c^2 = 0 + p^2c^2$$

$$\Rightarrow p = \frac{E}{c}$$

Thus the momentum of a photon is given by its energy divided by c. What happens when two particles collide and fuse together? Let us consider two equivalent particles, each with mass m and kinetic energy K.

$$E_i = 2E_0 + 2K = 2mc^2 + 2K$$

When they collide from opposite directions, their momentum cancels and we are left with a single particle of mass M

$$E_f = E_0 + 0 = Mc^2$$

We can figure out what this mass is using conservation of energy

$$M = 2m + \frac{2K}{c^2}$$

The final mass is greater than the initial mass, as some of the energy has been converted into mass during the collision. Energy in an isolated system is conserved, although mass is not.

$$E = \sum E_i$$

$$E_i = \gamma_i m_i c^2$$

Newton's Theory of Gravitation

Planet's move in elliptical orbits with the sun at one focus. The major axis of the ellipse describes the line across the widest point of the ellipse through the sun, while the minor axis is the perpendicular maximum width of the ellipse. An interesting property of orbital motion is that if we draw a line between the sun and the planet at some time and then a second line at some later time, the area traced out by the two lines will be a constant for a given Δt, irrespective of where the planet is in its orbit. Another property is that the square of a planet's orbital period is proportional to the cube of the semi-major axis (half of the major axis) length.

Most of the planets in our solar system have very close to circular orbits; mercury has the least circular orbit. While in orbit, a body is in free fall. It has some constant tangential speed, and accelerates towards the center, yielding the circular motion we now. The strength of gravity in this situation is inversely proportional to the squre of the orbital radius. The force of gravity between two masses is thus given by

$$F = \frac{Gm_1m_2}{r^2}$$

Where G is the gravitational constant, $6.67 \cdot 10^{-11}\mathrm{Nm}^2\mathrm{kg}^{-2}$. This value is extremely small, and explains why gravity is so small compared to some other forces. The only reason for the effectiveness of earths or the suns gravity is the huge masses involved. While one might think it would be necessary to calculate a force vector for each individual particle body in a gravitational interaction, for spheres this is equivalent to considering all the mass to be concentrated at a single point in the centre.

Gravitational potential energy can be reconsidered through the following formula for large scales where the gravity at the start and end heights is significantly different, making $U = mgh$ inaccurate.

$$U = -\frac{GMm}{r}$$

Where M is the central body (e.g. the earth) and m is the orbital body. This defines the work required to move away from a body exerting a gravitational force. This can be used, for example to determine the escape velocity for earth.

$$U_i + K_i = U_f + K_f = 0$$

$$K_i = -U_i = \frac{GMm}{R_E}$$

$$\frac{1}{2}mv^2 = \frac{GMm}{R_E} \Rightarrow v = \sqrt{\frac{2GM}{R_E}} = 1.12 \cdot 10^4 \text{ms}^{-1}$$

For an object orbiting in a circle, the kinetic energy will be given by

$$K = -\frac{1}{2}U_g$$

If the object is given a thrust to raise this kinetic energy, it's orbit will become elliptical. As an example, we can consider what speed would be required for an object to orbit the earth at the surface.

$$F = \frac{GMm}{R^2} = ma = \frac{mv^2}{R}$$

$$v = \sqrt{\frac{GM}{R}} = 7900\text{ms}^{-1}$$

Kepler's Third Law states that, for an object orbiting at radius r, this radius is related to the orbital period by

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

An orbit where the period matches the period of rotation of the earth is described as geosynchronous. Using Kepler's law, we can find that the radius needs to be $36,900\text{km}$.

Kepler's Second Law, that the area swept out across a time t by an orbiting body is a constant can be shown through conservation of angular momentum. Angular momentum of an orbiting body is given by

$$\vec{l} = \vec{r} \times \vec{p} = rmv\sin(\beta)$$

Where β is the angle between the radius and the velocity. Here the value

$$\frac{1}{2}\frac{L}{m}$$

Is a constant.

Oscillations

Simple Harmonic Motion occurs when a system is cycling through a cycle of states, such as a block on a spring, oscillating out and back as the spring compresses and depresses. A system in such a state has the properties T or period, which is the time taken for it to complete a full cycle, f, frequency, which is the inverse of period, measured in Hz or cycles per second and amplitude, the maximum displacement from its equilibrium position.

We can model oscillating systems using the tool of trigonometric functions. For a system with amplitude A and frequency f, we can use angular velocity ω to model the displacement at a time t.

$$x(t) = A\cos(\omega t)$$

Here, ω is simply $2\pi f$, the angular frequency. We can take a derivative of this to find the velocity at a given time.

$$v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t)$$

Through these equations, we can essentially consider oscillating systems to be examples of circular motion. If we have different starting conditions for particles we want to compare, we can use a phase constant ϕ:

$$x(t) = A\cos(\omega t + \phi)$$

If we want to compare two different sytems, we can take $\phi_2 - \phi_1$ to be the *phase difference*.

We can study the energy in an oscillating system by considering that it must be the sum of kinetic and potential energies.

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

At the extremes of the system, when $x = \pm A$, we know the value of K and U.

$$x = \pm A \Rightarrow v = 0 \Rightarrow E = U = \frac{1}{2}kA^2$$

$$x = 0 \Rightarrow U = 0 \Rightarrow E = K = \frac{1}{2}mv_{\max}^2$$

$$\Rightarrow \frac{1}{2}mv_{\max}^2 = \frac{1}{2}kA^2$$

This lets us derive an equation for maximum velocity in an oscillating system

$$v_{\max} = A\sqrt{\frac{k}{m}} = \omega A$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}} = 2\pi f = \frac{2\pi}{T}$$

$$\Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

This tells us that the frequency is independent of the amplitude, and is proportional to the ratio between k, the spring constant and m.

We can use our function for position to find equations for energy.

$$x(t) = A\cos(\omega t) \quad v(t) = -\omega A\sin(\omega t)$$

$$U(t) = \frac{1}{2}kA^2\cos^2(\omega t) \quad K(t) = \frac{1}{2}kA^2\sin^2(\omega t)$$

$$E(t) = U(t) + K(t) = \frac{1}{2}kA^2(\cos^2(\omega t) + \sin^2(\omega t))$$

$$E = \frac{1}{2}kA^2$$

Thus, we find what we would expect; the energy in the system is constant, due to the trigonometric identity cancelling the terms of the potential and kinetic energies.

Taking a derivative of the velocity function we can find $a(t)$

$$a(t) = -\omega^2 A\cos(\omega t) = -\omega^2 x(t)$$

And we see that the acceleration is related to the displacement. As the position grows further from the origin, its acceleration increases by the square of the systems angular velocity.

Example

Consider a 5kg block hanging at an equilibrium position from a spring with constant 2000Nm^{-1}. The block is pulled down 5cm and then given an initial velocity of 1ms^{-1} upward. Find the frequency, amplitude and total mechanical energy.

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2000}{5}} = \frac{20}{2\pi} = \frac{10}{\pi}\text{rads}^{-1}$$

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2} \times 2000 \times 0.05^2 + \frac{1}{2} \times 5 \times 1^2 = 5\text{J}$$

$$E = \frac{1}{2}kA^2 = 5 = \frac{1}{2} \times 2000 \times A^2 \Rightarrow \sqrt{\frac{10}{2000}} = A = 0.071\text{m}$$

For a pendulum hanging on a string of length L, oscillating slightly around the rest posisition, the displacement equation has a value ω given by

$$\omega = \sqrt{\frac{g}{L}} = 2\pi f$$

The period of oscillation is independent of both the amplitude and mass. This expression is only valid for small values of θ, less than around $10°$, because it relies on $\theta \approx \sin(\theta)$ to calculate τ as a linear force through

$$\tau = -Mgl\theta$$

As a general rule, any system with a linear restoring force, such as the horizontal component of gravity for the pendulum, or the restoring force of the spring in the previous example, will undergo simple harmonic motion around its origin position.

Damped Oscillation

While in an ideal system, oscillations would continue indefinitely, in reality they can only go on for so long before halting. This is due to damping in real world systems, due to forces like drag and friction. A damping force is usually of the form

$$\vec{D} = -b\vec{v}$$

Where $\vec{D}$ is the damping force and b is the *damping constant*. This changes the net force equation to

$$F_{\text{net}} = F_{\text{restorative}} + D = -kx - bv = ma$$

This yields a somewhat more complex equation of motion

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\Rightarrow x(t) = Ae^{-\frac{bt}{2m}}\cos(\omega t + \phi)$$

With ω given by

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

In this equation, a negative exponential function is added as a damping factor to the oscillating term. This makes the oscillations smaller and smaller with time, gradually bringing the energy lower. We can derive a new equation for the energy of a sytem with a damping factor.

$$E(t) = \frac{1}{2}k\left(Ae^{-\frac{bt}{2m}}\right)^2 = \frac{1}{2}kA^2e^{-\frac{bt}{m}}$$

The lost energy is usually as thermal energy. Knowing this allows us to write

$$E(t) = U + K \to 0 \Rightarrow E_{\text{total}} = E(t) + E_{\text{thermal}}$$

$$\Rightarrow E_{\text{thermal}} = E_{\text{total}} - E(t)$$

We take advantage of damping in structures like damping tanks in the Eureka Tower, where large tanks of water are used to dissipate oscillation due to wind by allow the water within to slosh about.

A counterpart to damped oscillation is driven oscillation. If a driving force, such as a person pushing a swing, is applied at a frequency matching the natural frequency of an oscillator, resonance occurs dramatically increasing amplitude of oscillation.

Waves and Particles

When a particle moves between two points, both matter and energy are transferred. On the other hand, when waves move only energy is transferred. Many types of waves exist. These include

- Mechanical waves, such as sound waves, which propagate physically through a medium such as air or water
- Electromagnetic which do not require a medium and self propagate through vacuum or a medium. Visible light is a small section of the electromagnetic spectrum.
- Matter waves which describe atomic particles such as electrons, governed by quantum mechanics.

Waves occur not only in one dimension, such as along a string, but also in two dimensions, such as the ripples on a lake, where each ripple corresponds to a peak and thus is a wavelength apart, or in three dimensions such as when sound waves are created at a point in the air and scurry away in all directions.

Waves can also be broken into transverse waves, sinusoidal waves and longitudinal waves. A transverse wave propagates a single peak through space, such as a wave on the ocean, while a sinusoidal wave resembles a sine or cosine pattern. Longitudinal waves have displacement parallel to the direction of motion, such as in the case of a tugged spring.

For a transverse wave propagating along a string, a peak will move along the string with a speed given by

$$v = \sqrt{\frac{\tau}{\mu}}$$

Where τ is the tension in the string, the restoring force term allowing oscillation and μ is the mass per unit length of the string, the inertial term enforcing oscillation. For a travelling sinusoidal wave, we have a general form for the displacement at a given time t and position x of

$$D(x,t) = A\sin(kx \pm \omega t + \phi_0)$$

Where k is the *angular wave number*. For this expression to yield the correct displacement at each point seperated by one wavelength, we know that k must be given by

$$k = \frac{2\pi}{\lambda}$$

Because we know that the displacement of a point on the wave should be constant as the wave propagates, we know that

$$kx - \omega t + \phi_0$$

Is a constant. This allows us to find a relationship between position x and time t for a point travelling at wave speed v.

$$\frac{d}{dt}(kx - \omega t + \phi_0)$$

$$k\frac{dx}{dt} - \omega = 0 \Rightarrow v = \frac{\omega}{k}$$

The vertical velocity of a point on the wave is given by

$$\frac{dy}{dt} = -\omega A \cos(kx - \omega t + \phi_0)$$

$$\Rightarrow v_{y,\mathrm{max}} = \omega A$$

The phase difference between two points on a sine wave is given by

$$\delta\phi = 2\pi\frac{\Delta x}{\lambda}$$

Thus, waves are out of phase when $\Delta x = \frac{\lambda}{2}$ and in phase when $\Delta x = \lambda$.

A quick summary of useful formulae:

$$\Big\| \; k = \tfrac{2\pi}{\lambda} \; \Big| \; \omega = \tfrac{2\pi}{T} \; \Big| \; f = \tfrac{1}{T} \; \Big| \; v = \tfrac{\omega}{k} \; \Big| \; v = f\lambda \; \Big\|$$

Superposition

For multiple travelling waves in the same medium, the total displacement in the medium is given by the sum of the displacements of the individual waves.

$$D(x,t) = D_1(x,t) + D_2(x,t) + ...$$

This will hold as long as D is below the elastic limit of the medium. If D exceeds this limit, deformation or breaking can occur. A consequence of this is that any wave can be generated with a sufficient number of sinusoidal waves, known as Fourier's Theorem.

An important consequence of superposition is standing waves, where two waves with equal amplitude, wavelength and frequency interact while travelling in opposite directions. In this situation, the two have the equations

$$D_1 = A\sin(kx - \omega t) \quad D_2 = A\sin(kx + \omega t)$$

$$D = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$

The resultant wave D will be a standing wave, which appears to be a stationary oscillating wave. The equation D can be rearranged to find

$$D = 2A\sin(kx)\cos(\omega t)$$

Which tells us that the wave doesn't travel, instead the amplitude is dependent on x position and the volume at a time changes. In addition, it tells us there are points which are always 0, where $\sin(kx) = 0$, at $kx = m\pi$. If we remember our relationship between k and λ, we can find that

$$kx = m\pi \Rightarrow x = \left\{ m\frac{\lambda}{2} | m \in \mathbb{Z} \right\}$$

So the *nodes* of the wave occur at integer multiples of a half wavelength. Nodes are points where the standing wave is always 0. Their inverse are *antinodes*, where the amplitude of the standing wave is a maximum, occuring when $\sin(kx) = \pm 1$. These points are given by

$$x = \left(m + \frac{1}{2} \right) \frac{\lambda}{2}$$

For two waves with a phase difference of ϕ, the resultant amplitude of the superposition of the two is given by

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi)}$$

One way which standing waves occur, such as in stringed instruments, is through reflection of waves. When a transverse wave in a string reaches the end of the string, the wave is reflected back with inverse displacement. This allows a standing wave to form along a string, the key principle behind stringed instruments.

Reflection of waves occurs not only at boundaries of strings or media, but also at discontinuities, places where the properties of the medium change. Because the velocity of the wave is affected by the mass of a string, if a string drops in mass the nature of the wave changes. For a wave moving from a heavier to a lighter string, a portion of the wave will be reflected backwards (with lower amplitude in the same vertical direction) at lower speed while another portion will continue onto the lighter string (with greater amplitude than the reflected portion due to the lower mass) and higher speed.

For a smaller string moving into a larger string, the larger string acts partially like a hard boundary, reflecting part of the wave with reduced inverse displacement and equal speed, and partially like a transition, with a slow, smaller wave with displacement in the same direction continuing along the thicker string.

This principle can be used to diffuse vibrations by including many changes in thickness of material to reduce transmission of sound or other mechanical waves in a structure.

A string of a certain length will have standing waves within it of wavelengths dependent on its length. For a string with both ends fixed, of length L, the *first harmonic* will have a wave length given by $2L$, the second wavelength of L and the third of $\frac{2}{3}L$. In general, the wavelength of the nth harmonic is given by

$$\lambda = \frac{2L}{n}$$

For example consider a violin string of length $32\mathrm{cm}$ and mass of $4.5 \cdot 10^{-4}\mathrm{kg}$. What is the tension in the string if its fundamental frequency (first harmonic) is $f_1 = 196\mathrm{Hz}$?

$$v = \sqrt{\frac{\tau}{\mu}}$$

$$\mu = \frac{m}{l} = \frac{4.5 \cdot 10^{-4}}{0.32} = 0.0014$$

$$v = f\lambda \Rightarrow v = 196 \times (2 \times 0.32) = 125.44$$

$$v^2\mu = \tau \Rightarrow 125.44^2 \times 0.0014 = \tau = 22.13\mathrm{N}$$

Sound Waves

Sound waves are longitudinal mechanical waves, which can travel through all media irrespective of state, operating by compressing the material ahead of them while the material behind them relaxes. Where transverse waves apply a shearing stress to a material, longitudinal waves apply a compressive stress. It is for this reason that fluids such as water or are cannot transmit a transverse wave while solids can. Because fluids can be compressed, they can transmit longitudinal waves.

Knowing that sound waves move through a medium through compression, we can find an equation for the speed of sound in a fluid. Once again, we find that it is of the form

$$\sqrt{\frac{\text{restoring force}}{\text{inertial term}}}$$

The requirement for oscillation which allows waves to transmit. In this case, the restoring force is the materials *Bulk modulus*, visited earlier when stresses were examined. Here, it is coupled with density to understand the behaviour of compressed materials and see how they can transmit waves.

$$v = \sqrt{\frac{B}{\rho}}$$

B is the Bulk modulus of the relevant fluid, while ρ is the density. As an example, we can consider air at $0°\mathrm{C}$. Here, air has a density of $1.29\mathrm{kgm}^{-3}$ and sound travels through it at $331\mathrm{ms}^{-1}$. What is the Bulk modulus of air at this temperature?

$$331 = \sqrt{\frac{B}{1.29}} \Rightarrow 331^2 \times 1.29 = B = 141333.7$$

For a solid, the speed of sound is given by a slightly different equation. Here, the restoring force is given by Young's modulus (E), rather than the Bulk modulus, and density is still used as the denominator. The resultant equation for velocity is

$$v = \sqrt{\frac{E}{\rho}}$$

Sound waves are the most common case where we observe wave interference. Constructive interference occurs when waves are offset by a whole number of wavelengths, which can occur due to a phase difference of 0 and a starting position offset by a whole number of wavelengths or in a situation where the phase difference compensates the position difference, e.g. $\phi = \pi$ and $\Delta x = \frac{\lambda}{2}$.

Destructive interference occurs when the waves are offset by $\frac{\lambda}{2}$, which can occur for the same reasons. Both of these interferences can occur in higher dimensions. To understand these situations, we use the concept of path difference. If r is the distance from a source to a point, we can say constructive interference occurs at that point if

$$\Delta r = m\lambda$$

Where Δr is the difference in distances from two sources and m is an integer. As long as the path difference is a whole number of wavelengths, constructive interference will occur. For the inverse case, when a crest of one wave meets another we have

$$\Delta r = \left(m + \frac{1}{2}\right)\lambda$$

Standing waves can also occur in situations less contrived than in an instrument string, such as in a simple room with a speaker at one end, due to the interference that occurs. The pressure at certain points will be at a maximum, offset by π from the points where pressure is at equilibrium levels, where the wave amplitude is at a maximum.

For a pipe which is closed at one end and open at the other, resonance can still occur. In this case, the first harmonic will occur at

$$L = \frac{\lambda}{4}$$

An interesting property of this set up is that only odd harmonics can occur, with the wavelengths of these harmonics given by the equation

$$\lambda_n = \frac{4L}{n}$$

For the nth harmonic, where n is always odd. The reason for this behaviour is because the air at the closed end of the tube must be fixed, which implies that the air at the open end must be at maximum displacement. Thus, we have a cycle from 0 to maximum across the length of the pipe, i.e. $\frac{\lambda}{4}$ across the length of the pipe.

This makes it simple to understand that for a closed pipe, the air must be stationary at each end, which gives us the same result as for a fixed string, with the first harmonic at $\lambda = 2L$ and the general form of

$$\lambda_n = \frac{2L}{n}$$

A third case can be considered, where both ends are open. This behaves (shockingly enough) as the inverse of the case of both ends closed, with antinodes at each end in contrast to the closed tubes nodes. The wavelengths are determined in the same fashion as for the doubly closed tube.

Doppler Effect

An interesting frequency related phenomena is the Doppler effect, which is the change in frequency which occurs when a sound source and an observer are moving relative to each other, such as the change in frequency of the siren of a passing emergency vehicle.

This effect occurs because the velocity of the source is added to the waves in front of it, resulting in a higher frequency ahead of the source and a lower frequency behind it, where the velocity is subtracted from that of the waves. The magnitude of the change in frequency can be calculated using the relative velocities of the waves and source. Here, v is the velocity of the wave, v_s is the velocity of source f_o is the frequency observed and f_s is the frequency emitted at the source. In this situation, the source is approaching the observer with velocity v. $f_o > f_s$.

$$f_o = \left(\frac{v}{v - v_s}\right) f_s$$

For a source moving away from the observer, the sign of the source velocity is flipped. Here, $f_o < f_s$.

$$f_o = \left(\frac{v}{v + v_x}\right) f_s$$

The same effect occurs when the observer is moving toward the source. For an observer approaching the source, the observed frequency is given by

$$f_o = \left(\frac{v + v_o}{v}\right) f_s$$

Here, the sign of v_o is flipped to find the observed frequency for an observer retreating from the source. As an example, let us find the speed with which one would need to move toward an observer to make a $4800\mathrm{Hz}$ sound shift up to $5200\mathrm{Hz}$. The speed of sound can be taken to be $343\mathrm{ms}^{-1}$.

$$f_o = \left(\frac{v}{v - v_s}\right) f_s$$

$$\frac{f_o}{f_s} = \left(\frac{v}{v - v_s}\right) \Rightarrow \frac{f_s}{f_o} = \left(\frac{v - v_s}{v}\right) \Rightarrow \frac{f_s}{f_o} = 1 - \frac{v_s}{v} \Rightarrow v_s = \left(1 - \frac{f_s}{f_o}\right) v$$

$$v_s = \left(1 - \frac{4800}{5200}\right) 343 = 26.4$$

In the case that a source of sound is moving faster than the sound waves which it emits, these sound waves will tend to build up on top of each other, increasing the size of the resultant waves through constructive interference. This is the cause of a sonic boom.

Wave Model of Light

Wave properties are used to explain refraction, diffraction and interference of light. However, particle properties are needed to explain effects like the photoelectric and compton effects. Thus, light has a dual nature, displaying both sets of behaviours.

Dispersion occurs when white light moves between mediums. For a given material, each wavelength of light will have a different *index of refraction* n, which is related to the velocity of the light as it passes through the material through

$$v = \frac{c}{n}$$

The visible spectrum of light ranges from wavelengths of $400\mathrm{nm}$ to around $700\mathrm{nm}$, with violet at the low end and red at the high end. The refractive index of red light in glass is 1.520 while for violet light the index is 1.538; this implies that in general higher wavelengths suffer less diffraction. Because of this, white light, which is in reality a combination of numerous colours of light, moving between mediums will be decomposed into its component colours.

Interference

Interference between different waves generally only occurs when the wavelengths of the waves involved are within around two orders of magnitude of each other.

When waves pass through a gap of comparable size to their wavelength, diffraction occurs. If multiple adjacent waves undergo similar diffraction, an interference pattern emerges, with bands of contstructive and destructive interference appearing. If a screen is placed in the path of the light, a repeating pattern of light and dark waves will appear. The dark points are nodes, while the bright bands are antinodes. The further apart the sources are, the closer the bands will be.

The positions of these bands are determined by phase difference, which for sources that are in phase is determined by path difference of the two waves. For the central point, the two waves will be in phase as the length of the path will be the same. For the first dark band, the path difference is $\frac{\lambda}{2}$, then the first off center light band will be at λ. This can also be expressed in terms of the distance apart of the two slits (d) and the angle to the band on the screen θ. In this notation, constructive interference occurs when

$$d\sin(\theta) = m\lambda, m \in \mathbb{N}$$

And destructive interference occurs at points offset by half a wavelength.

$$d\sin(\theta) = \left(m + \frac{1}{2}\right)\lambda, m \in \mathbb{N}$$

Examining these expressions, we can see how as d increases, the θ values for a given band will decrease. When L, the distance from the slits to the screen is increased, the bands grow further apart. If the wavelength is lowered, the bands will move closer together.

Examples

- Find the angle of the first antinode for two sound waves of frequency $2000\mathrm{Hz}$ passing through slits seperated by $0.3\mathrm{m}$ with velocity $354\mathrm{ms}^{-1}$.

$$m = 1 \Rightarrow d\sin(\theta) = 1\lambda = \frac{v}{f}$$

$$\Rightarrow \theta = \arcsin\left(\frac{\frac{v}{f}}{d}\right) = 36^\circ$$

- What slit seperation would be required for $0.03\mathrm{m}$ microwaves to have the same angle to the first antinode?

$$\lambda = 0.03, \theta = 36^\circ$$

$$m = 1 \Rightarrow d\sin(\theta) = \lambda \Rightarrow d = \frac{\lambda}{\sin(\theta)} = \frac{0.03}{\sin(36^\circ)} = 0.05\mathrm{m}$$

- If the slit seperation is reduced to $1.00\mu\mathrm{m}$, what frequency of light should be used to maintain the angle?

$$m = 1, d = 1 \cdot 10^{-6}$$

$$d\sin(\theta) = \lambda = \frac{v}{f} \Rightarrow f = \frac{3 \cdot 10^8}{1 \cdot 10^6 \sin(36^\circ)} = 5 \cdot 10^{14}\mathrm{Hz}$$

If multiple slits are in the line, we have a diffraction grating, and the interference patterns will add, resulting in a very clear pattern on a screen. In natural examples such as feathers, *reflection gratings* often cause beautiful shining patterns through similar mechanisms.

Single Slit

A way to understand diffraction through a single slit is by considering a wavefront to be made up of an arbitrary number of infinetismal wavelets, each of which emits a spherical wave. The tangent line to these wavelets then forms the next wave front. In this way, light spreads around corners. The magnitude of diffraction is greatest when

$$d \approx \lambda$$

Diffraction occurs in single slit experiments because the wavelets act independently, and the path difference between their wavelets creates interference. The nodes of such a diffraction pattern occur depending on the spacing of the wavelets. For a slit of width a,

$$a\sin(\theta) = p\lambda, p \in \mathbb{N}^+$$

The central maximum will be of a width $2y$ where $\tan(\theta) = \frac{y}{L}$. If λ decreases, the pattern tightens as θ decreases. If white light is diffracted in this way, dispersion occurs at the edge of the interference pattern.

By applying similar concepts to circular light sources, it has been found that for two objects to be individually resolved in an image, they must have

$$\Delta\theta \geq 2.44\frac{\lambda}{D}$$

Where D is the diameter of the emitting circles, and $\Delta\theta$ is the difference in angle between the centres of the two circles.

Example

A laser light of wavelength 632.8nm is passed through a 0.3mm single slit. What is the width of the central maximum on a screen a distance one metre from the slit?

$$\lambda = 632.8 \cdot 10^{-9}, d = 0.3 \cdot 10^{-3}, L = 1$$

$$a\sin(\theta) = p\lambda \Rightarrow \theta = \arcsin\left(\frac{p\lambda}{a}\right) = \arcsin(\frac{1 \times 632.8 \cdot 10^{-9}}{0.3 \cdot 10^{-3}}) = 0.12^\circ$$

$$\tan(\theta) = \frac{y}{L} \Rightarrow 2y = 2L\tan(\theta) = 2L\tan(0.12^\circ) = 4.2\text{mm}$$

An application of interference of light is in an interferometer, which by measuring the brightness of resultant light from a split beam interfering itself, to identify the the additional distance taken by the long path of the light.

Ray Model of Light

A ray is a straight line perpendicular to a wave front. We can use rays to better understand phenomena such as specular reflection for a smooth mirror, or diffuse reflection for a rough surface. Diffuse reflection occurs when light bounces off a rough object, with the effective result of light being emitted from every point of the object as a point source.

When light bounces off a smooth surface, specular reflection occurs. In this case, the angle of incidence is equal to the angle of relfection. The incident ray is measured with respect to the normal.

$$\theta_i = \theta_r$$

Plane mirror produce virtual images behind their surfaces. These images occur because the reflected light, by nature of the relationship between angle of incidence and angle of reflection, appears to come from a single point. The object distance s, between the object and the mirror, is equal to the image distance, s', within the mirror the image appears.

Refraction occurs when a ray of light moves between two media of different index of refraction. Light moving between media will change speed, and therefore will need to change direction. For light moving from a low index of refraction to a higher one, the light will bend toward the normal ($\theta_r < \theta_i$). As the angle of incidence increases, i.e. the angle with the surface becomes shallower, more light will be reflected off the surface and less will be refracted. For light moving from a denser material (higher refractive index), the angle of refraction will be higher than the angle of incidence. If the angle of incidence is $90°$, no refraction occurs and only a change in wavelength occurs. This change is given by

$$n = \frac{c}{v} = \frac{\lambda_{\text{vacuum}}}{\lambda_{\text{medium}}}$$

For example, for yellow light of wavelength 600nm, moving from air into water $n = 1.33$, the new wavelength is given by

$$\frac{600}{1.33} = 451\text{nm}$$

Because we perceive colour through frequency, a change in wavelength doesn't matter. The ratio of angles for a ray moving from medium 1 to medium 2 is given by Snell's law

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1} \Rightarrow n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

For example, consider a diving who perceives the sun through water with a refractive index of 1.33 to be at $45°$ to the normal.

$$\begin{aligned} \theta_{\text{air}} &= \arcsin\left(\frac{n_{\text{water}} \sin(\theta_{\text{water}})}{n_{\text{air}}}\right) \\ &= \arcsin\left(\frac{1.33 \sin(45°)}{1}\right) = 70.5° \end{aligned}$$

When the angle of refraction reaches $90°$, no light escapes the medium. This is known as total internal refraction, and occurs when the angle of incidence is greater than or equal to the critical angle. This critical angle is given by

$$\sin(\theta_c) = \frac{n_2}{n_1}$$

Where n_2 is the material the light is trying to escape to, and n_1 is the medium it is trapped in. For this to occur $n_1 > n_2$ must be true. Objects beyond the critical angle outside the medium will be invisible to observers within the medium.

For an object at an apparent depth s' within some refractive material (e.g. water), the real depth is given by

$$s' = \frac{n_2}{n_1}s \Rightarrow s = s'\frac{n_1}{n_2}$$

Where $n_1 > n_2$ and n_1 is the material the object is in, with the observer looking in from n_2.

The reason for dispersion, the division of white light into its component colours, is also refraction; because light is of differening wavelengths, different components of the light refract by different amounts, thus spreading out as they change medium.

Lenses

Ray tracing is a technique where we follow the path of a ray to understand the behaviour of light through a lens or other surface. Using this technique and considering a single surved surface we can noticed that if a point source projects light onto all points of the surface of a curved lens equally, at each of these points the light will curve toward the normal of the surface at that point, resulting in all of the light being focussed at some point inside the glass. While this trait of focussing the light is desirable, we obviously can't observe from inside glass.

We get around this by using a symmetrical convex lens, which takes in light on one side at focusses it at a point on the other side at a focal length F. The inverse process of spreading out light can be undertaken with a symettric concave lens. A symettric concave lens appears to focus the image at a negative focal point F on the side of the light.

The *power* of a lens is described as the inverse of the focal length.

$$P = \frac{1}{F}$$

To find the resultant focal point of a system of lenses we can add the powers of the individual lenses. For example a convex lens with $F = 1\mathrm{m}$ is placed in contact with a concave lens with focal length $F = -0.5\mathrm{m}$. The focal length of the combined lens is given by

$$P = \frac{1}{f} \Rightarrow P_1 = 1, P_2 = -2$$
$$P = P_1 + P_2 = 1 - 2 = -1$$
$$F = \frac{1}{P} = -1\mathrm{m}$$

Light refracted through a convex lens will create a real, inverted image at a certain distance from the lens. This can be understood by seeing that light which passes through the center of a convex lens will effectively have no refraction affecting it, and will pass through the same point as two rays which will be each others inverse; these rays are the ray which travels parallel between the object and the lens and the ray which exits parallel from the lens. This applys for objects outside the focal length of the convex lens. In the case of an object *inside* the focal length, the resultant image will be vertical and will exist behind the real object. For an object outside the focal length of a concave lens, the virtual image which appears will be a smaller object closer to the lens.

The lens equation can be used to determine the focal length, and therefore power of a lens based on the distance of an object and its image. The formula, known as the *Thin Lens Formula* uses Snell's law to state

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

Where s is the distance from the object to the center of the lens, s' is the distance to the image and f is the focal length. If $f > 0$ the lens is converging, like a convex lens. $s' > 0$ when the image is real, while it is less than 0 is the image is vertical. The equation for the magnification of an object is

$$m = -\frac{s'}{s}$$

If half of a lens is blocked, the effect will be to make the resultant image fainter.

Chromatic aberration occurs as a result of dispersion in glass, as each wavelength has a different focal length. The solution is to use a glass which has very similar focal lengths for ever colour of light, or a combination of glasses which cancel out each others flaws.