

# Electromagnetism

When dealing with electrostatics, very small units are often required. It therefore bears revising some of the smaller SI prefixes.

- $\mu = 10^{-6}$  (micro)
- $n = 10^{-9}$  (nano)
- $p = 10^{-12}$  (pico)

## History

Electricity was first observed in ancient Greece, where static electricity was observed in amber. The name for electricity comes from the Greek word for amber, “elektron”. Magnetism was discovered around the same time, but it took many years for a connection between the two to be established. Only in 1820 did H.C. Oersted identify the connection. In the late 19th century, Maxwell’s equations quantified links between electricity and magnetism, and finally the two were unified by Einstein in 1905, through the theory of special relativity. This course only teaches up to slightly before 1905.

## Charge

Electromagnetism is one of the four fundamental forces of nature.

- Gravitational force
- Electromagnetism
- Strong force
- Weak force

The gravitational force is often modelled through Newton’s law.

$$F = \frac{Gm_1m_2}{r^2}\hat{r}$$

Electromagnetism mirrors this with Coulomb's Law, dating to 1798.

$$F = \frac{kq_1q_2}{r^2}\hat{r}$$

Where  $k$  is Coulomb's constant and  $q_1$  and  $q_2$  are the two charged particles interacting. Here,  $\hat{r}$  is the vectore direction defined by the two charges.

While gravity is only positive, electric charge comes in both positive and negative forms. This charge is measured in Coulombs.

$$1C = 1A \times 1s$$

A Coulomb is a large quantity of charge. Charge is often denoted with  $q$ . As an example, the charge of an electron is

$$1.6 \times 10^{-19}C$$

Charge is often generated by scraping electrons from surfaces, or by touching conductive surfaces together. Charge is quantised; it comes in integer quantities. It can be transferred, but it cannot be created or destroyed.

## Conductors

A conductor contains electrons no strongly bound to any particular nucleus within the conductor. These *free electrons* can move when under the influence of an electric field, creating an electric current. There are huge quantities of these free electrons available in an ordinary conductor. For example, copper contains around  $10^{22}$  free electrons per cubic centimetre. These are initially held by attractive forces with the nuclei, but can be moved along by external forces.

The counterpart to a conductor is an insulator, which have very few free electrons.

A process of charging by induction entails placing a charged body near to a neutral body. This will attract the opposite charges of the neutral body to one side. If the neutral body is then grounded, it will now be charged.

## Electric Force

Electric force is calculated, as touched on above, through Coulomb's law.

$$F = \frac{kq_1q_2}{r^2}\hat{r}$$

Here,  $F$  is the electric force in Newtons between the two charged bodies,  $q_1$  and  $q_2$  are the charges of the two bodies involved,  $r$  is the distance between the two,  $\hat{r}$  is a unit vector in the direction between  $q_1$  and  $q_2$  and  $k = 8.988 \times 10^9 \text{Nm}^2\text{C}^{-2}$ . If this value  $F$  is negative, the two are attracting each other, otherwise they are repelling each other.

$k$  is defined in terms of the *permittivity of free space*,  $\epsilon_0$  according to

$$k = \frac{1}{4\pi\epsilon_0}$$

Here,  $\epsilon_0 = 8.85 \times 10^{-12} \text{C}^2\text{N}^{-1}\text{m}^{-2}$ . This value is for the permittivity of free space, i.e. vacuum, and a different value is required to accurately model force between charges with material lying between them, known as the permittivity for that material. With this value, the behaviour of electric force for any material can be clearly defined.

The *superposition principle* allows us to add electric forces (or any forces) on a particle to determine the resultant force on that particle. This can be done in either a single direction or using vector addition in multiple directions. If two identical particles interact from opposite sides with a particle, their effects will cancel.

## Electric Fields

An application of superposition comes in electric fields, where we can use integration to apply the concept of superposition across a continuous field. For a linear charge distribution, along a single straight line in the  $x$  direction, this integral looks like

$$F_{q_0} = \int_{x_i}^{x_f} \frac{kq_0}{r_{0x}^2} \hat{r}_{0x} \lambda(x) dx$$

Here,  $x_i$  through  $x_f$  is the range of charges along the line,  $k$  is Coulomb's constant,  $q_0$  is the charge of the particle being considered,  $r_{0x}$  is the distance to the particle from each point  $x$ ,  $\vec{r}_{0x}$  is the vector between the particle and the point and  $\lambda(x)$  is the *linear charge density*, a function defining the charge of the line at any point  $x$ , measured in  $\text{Cm}^{-1}$ .

For a surface charge distribution, for example of a sphere, the integral looks like

$$F_{q_0} = \int_S \frac{kq_0}{r_{0dA}^2} \hat{r}_{0dA} \sigma(dA) dA$$

Here, we exchange linear position  $x$  for rate of change of area  $dA$ . We also exchange  $\lambda$  for  $\sigma$ , the surface charge density function for the body.

Finally, we can consider charge distribution throughout a volume

$$F_{q_0} = \int_V \frac{kq_0}{r_{0dV}^2} \hat{r}_{0dV} \rho(dV) dV$$

In this case, integration across area becomes integration across volume, and we exchange rate of change of area for rate of change of volume.  $\sigma$  becomes  $\rho$ , the volume charge density function.

We consider electric field lines through the construct of electric field lines. These are imaginary lines flowing away from positive charges toward negative charges, terminating at infinity or negative charges. The density of these lines is greatest around charges, which is where the electric field is strongest. Field lines are always perpendicular to the surface of a conductor. Using this concept, we can define the electric field  $\vec{E}$  for a point charge as

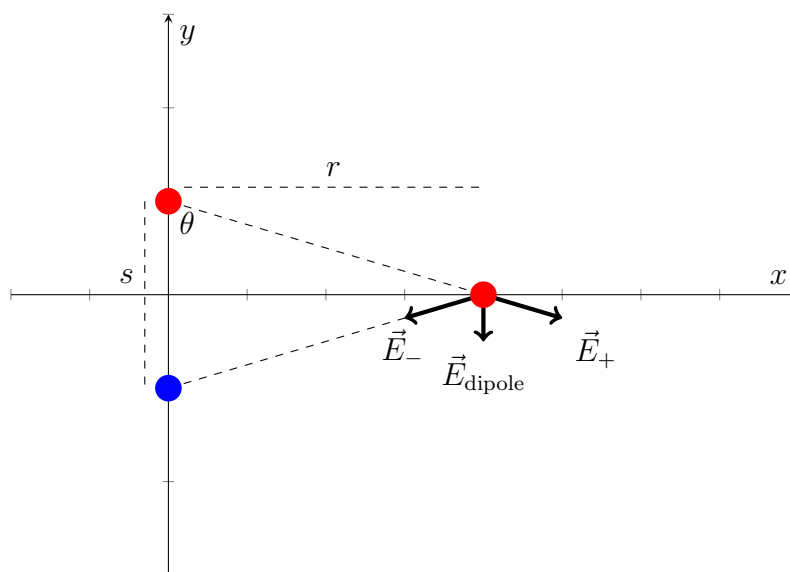
$$\vec{E} = \frac{\vec{F}}{q}$$

If we use  $\vec{F} = \frac{kq_1q_2}{r^2} \hat{r}$ , we can find the value for the electric field strength at a distance  $r$  to be

$$\vec{E} = \frac{kq}{r^2} \hat{r}$$

This yields a value in units of  $\text{NC}^{-1}$  or equivalently  $\text{Vm}^{-1}$ . Here  $q$  is the charge of the particle creating the field. For a system of charges, we can sum across the effects of individual charges on a charge in the resultant field.

For a dipole system, with a positive and negative charge separated by a distance  $s$ , as shown below, a special equation can be used to calculate the field strength at a distance  $r$ .



Here,  $\vec{E}_{\text{dipole}}$  can be calculated through

$$\frac{kqs}{r^3}$$

Returning again to a linear line of charge, we can consider a case where  $\lambda$  is a constant function; that is the charge per unit length of the body is constant throughout. In this case, for a point  $p$  at some distance  $r$  from the line, the force from two points an equal distance perpendicular to  $r$  will cancel in the perpendicular direction, leaving only the force in the direction of the radius. The resultant force vector on  $p$  will therefore be directly away from the linear charge line. The magnitude of this field can be calculated through

$$\vec{E} = \frac{2k\lambda}{r} \hat{r}$$

## Gauss' Law

Coulomb's law works best for point charges, and can be generalised through integration to a broader range of charges. Gauss' law is another law for force due to electric charge, and can be applied to all charge distributions. Gauss' law is one of Maxwell's equations; in a way it is a fundamental law of nature.

Gauss' law requires the relevant problem to have symmetry, perhaps cartesian or cylindrical. The symmetry may be translational, rotational or reflective. For a point charge, Gauss' law is the same as Coulomb's law:

$$\vec{E} = \frac{kQ}{r^2} \hat{r} = \frac{1}{4\pi\epsilon} \frac{Q}{r^2} \hat{r}$$

This can be equivalently expressed as

$$\vec{E} 4\pi r^2 = \frac{Q}{\epsilon}$$

Which can be intuitively understood as the surface area of a sphere multiplied by the field strength at any point on the surface of the sphere being equal to the charge of the particle divided by the permittivity of the material.

Flux is an important concept in electric fields. In fluid mechanics, flux describes the volume per unit time flowing through an area. If the area described is perpendicular to the flow, flux through it is at maximum. If perpendicular, the flux is 0. If we define the angle of the area to the perpendicular direction as  $\theta$ , then we can define flux through that area in an electric field as

$$\Phi = EA \cos(\theta)$$

Where  $E$  is the field strength. This can be equivalently expressed if we define a vector  $\vec{A}$  to be the direction of the area and  $\vec{v}$  to be the flow of material as

$$\Phi = \vec{v} \cdot \vec{A}$$

With electric field as the flow of material this becomes  $E \cdot \vec{A}$ . Using Gauss' law, this generalises from simple planar areas to more complex *Gaussian surfaces*, which are simply surfaces of any three dimensional body. The electric field through these surfaces can then be calculated by integration across the surface,

considering the electric field at each infinitesimal point by taking the dot product of the surface vector at that point and the electric field at that point, yielding

$$\Phi = \int \vec{E} \cdot d\vec{A}$$

The units of electric flux are  $\text{Nm}^2\text{C}^{-1}$ .  $d\vec{A}$  is a vector with direction given by the surface normal and magnitude of the area of the surface. In the case of a closed surface another simplification applies

$$\Phi = \oint_{\text{surface}} \vec{E} \cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

This states that the electric flux through a closed surface is equal to the enclosed charge divided by  $\epsilon_0$ . Thus we can simply sum the internal charges to find the charge through the surface.

For an infinite line of charge, with charge density  $\lambda$ , the electric field at a distance  $r$  from the line is given by

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0} \hat{r}$$

For an insulating sheet of charge, with charge density  $\sigma$ , the electric field strength at a distance  $r$  from the sheet has direction normal to the sheet and magnitude given by

$$E = \frac{\sigma}{2\epsilon_0}$$

For a charged conductor, the electric field projected will always be normal to the surface, and will always be zero internally. This is because the conductor will inherently balance internal forces until 0 net field exists. Near to the surface of such a conductor, the electric field, normal to the surface as stated, will have field strength given by

$$E = \frac{\sigma}{\epsilon_0}$$

This is applicable on in areas where the conductor is sheet-like; either on a flat area or on a small enough section of a curve to make the curve irrelevant. The

reason for the behaviour of the conductor in this way is that the like charges repel each other, forcing an even distribution across the outer surface of the conductor.

This applies equally to a hollow conductor. For a torus shaped conductor, the inner radius will have no charges upon it. For this reason, lightning poses minimal risk to a person inside a conducting vehicle like a car. Here the car acts as a *Faraday cage*.

## Electric Potential Energy

It becomes quite complex to accurately describe the electric field at a specific point in space. It can therefore be useful to uniquely define a potential energy for a point in space; an analogue to the gravitational potential energy at a given height.

For example, if a positive charge is placed near another positively charged object, the charge now has potential to move away from the object. Work will be required to move it closer, and it will gain kinetic energy as it is repelled. Electric force is *conservative*, just as gravity is. It is this property that allows potential to be uniquely defined with respect to space.

Work in an electric field is described just as mechanical work is, as force times distance, though in this case our force is electrical force,  $qE$ .

$$W = q\vec{E} \cdot \vec{d}$$

$$W = qEd$$

Work done against the field is negative. When work is done against the field, the electric potential  $U$  increases, i.e.  $\Delta U > 0$ . When potential energy is lost, kinetic energy is gain, work done is positive. The dot product in the above equation tells us that force at right angles to the field does not entail a loss in potential energy, and also allows us to use the equation in the general case through the use of vector arguments.

## Electric Potential

Electric potential energy,  $U$  differs from electric potential  $V$ .



- While  $U$  is a property of a system of charges and interacting electric fields,  $V$  is independent of interacting charges.
- $U$  at a point is not a property of in space, nor is the difference in potential between two points.  $V$  is a property of two points in space and the difference in potential between them.
- $U$  is dependent on the charge,  $q$ , which  $V$  is made independent by dividing through  $U$  by  $q$ .

$$V(P) = \frac{U(P)}{q}$$

Whenever a charge moves in a field, the change in  $U$  is proportional to the charge, while the change in potential per unit charge is independent.

$$\Delta V = V_f - V_i = \frac{\Delta U}{q}$$

i.e. the change in electric potential is equal to initial potential minus final potential is equal to change in electric potential energy divided by charge. The unit of electric potential is the Volt V,  $1V = 1JC^{-1}$ . Electric fields can also be equivalently measured in  $Vm^{-1}$  rather than  $NC^{-1}$ .

Potential itself has no meaning; only differences in potential have meaning. Therefore, we can define zero according to convenience. Usually this is done as either a charge at infinite distance or the charge of the earth.

In general  $E$  points towards regions of low  $V$  and away from high  $V$ . If one moves perpendicular to a field line through an electric field, one is traversing an equipotential line, moving about without changing potential. For a point charge, the electric potential at a distance  $r$  is given by

$$\frac{1}{4\pi\epsilon_0} \frac{Q}{r} = \frac{kQ}{r}$$

It is directly proportional to distance. For an infinite line of charge, the change in potential when moving from position  $r_1$  to position  $r_2$  is given by

$$V_2 - V_1 = \Delta V = -\frac{\lambda}{2\pi\epsilon_0} \log\left(\frac{r_2}{r_1}\right)$$

For a hollow conductor, such as a hollow sphere, the potential inside the conductor must be the same as the potential at the surface.

To find the electric potential at a point  $P$  affected by multiple charges, one can simply sum the potential of the individual charges to find the resultant potential at  $P$ .

$$V(P) = V_1(P) + V_2(P) + \dots + V_n(P) = \frac{U}{q}$$

This is equivalent to the potential divided by the charge.

### Pointy Conductors

Consider a conductor which at one end comes to a point. This can be approximated by considering a large sphere in proximity to a smaller sphere, connected by a conductor between the two. In this case, the potential of the system must be uniform, and therefore a greater proportion of the system's charge must be distributed around the radius of the larger sphere. In fact, the ratio of charges is equivalent to the ratio of the radii, i.e.

$$\frac{q_1}{q_2} = \frac{r_1}{r_2}$$

If we then consider the electric field around each sphere, we find that for a larger sphere 1 and a smaller sphere 2

$$\frac{E_1}{E_2} = \frac{q_2 r_2^2}{q_2 r_1^2} = \frac{r_2}{r_1}$$

$$\therefore r_2 < r_1 \Rightarrow E_2 > E_1$$

To summarise, a smaller radius implies a larger electric field. It is for this reason that lightning tends to strike sharp points, such as the Eiffel Tower. For the same reason it is dangerous to be atop a mountain during a thunderstorm.

### Electric Dipoles

Due to the strength of the electric force, it is rare to encounter free charges in nature. More common are dipoles, conductors with a positive and negative pole.

A dipole with two charges of equal magnitude in a uniform electric field will experience equal force to each pole. If the dipole is not parallel to the direction of the field, it will experience a torque. The magnitude of this torque will be

$$\tau = Fd \sin(\theta) = qdE \sin(\theta)$$

Where  $d$  is the distance between the two poles of the dipole, each of which have charge magnitude  $q$ . A dipole is often defined as a vector  $p$  where

$$p = qd$$

and  $p$  points from the negative to positive terminal of the dipole. Using this vector representation, we find that the torque is given by

$$\vec{\tau} = \vec{p} \times \vec{E}$$

To rotate against this field, work must be done on the dipole. When the dipole is parallel with the field,  $\theta = 0 \Rightarrow \tau = 0$ . For the dipole to rotate to some other angle, work must be done, and the magnitude of this work is given by

$$W = \tau\theta$$

The potential energy of the system at any angle is given by the dot product

$$U = -\vec{p} \cdot \vec{E}$$

## Capacitors

A capacitor stores the energy of an electric field. It does this by separating positive and negative electric charges within itself. Generally, capacitors use two sheets of electric foil, often coiled into a cylinder. The total charge within the capacitor is also zero, though often with a large positive and a large negative charge. Between these two plates, a potential difference given by

$$V = \frac{qd}{A\epsilon_0}$$

Where  $A$  is the area of the two plates and  $d$  is the distance between them. In the case that there is air or another material rather than vacuum between the two plates, a different  $\epsilon$  must of course be used. This expression for the potential difference within the capacitor leads to a definition of the *capacitance* of the capacitor.

$$C = \frac{\epsilon_0 A}{d}$$

$$\Rightarrow q = CV$$

Here  $C$  is measured in units of  $\text{CV}^{-1}$  or Farads, F. The definition above is for a parallel plate capacitor; other shapes may have subtly different expressions. To create capacitors with high capacitance, it is common to use a material with a different  $\epsilon$  rather than change the separation or area.  $d$  can be extremely small; in applications like DRAM it is often as little as 50nm.

## Energy Storage in Capacitors

Charging a capacitor requires energy, as electrons are forced onto the plates. The internal energy of a capacitor is given by

$$U = \frac{1}{2} \frac{q^2}{C} = \frac{1}{2} CV^2$$

If potential is fixed, say by a battery, the energy stored can be increased through higher capacitance. If the charge is fixed, increasing the capacitance decreases the potential energy. Increasing voltage is a very effective way to increase the energy stored of a capacitor. It is useful to be able to talk about energy density of a capacitor, the formula for which is

$$\frac{1}{2} \epsilon_0 E^2$$

Assuming a vacuum, otherwise an appropriate  $\epsilon$  must be used. Interestingly, this formula can be used for *all* electric fields, with units of  $\text{Jm}^{-3}$ . When changing capacitance by using a non-vacuum filling material, we use the formula

$$C = \frac{\kappa \epsilon_0 A}{d} = \frac{\epsilon A}{d}$$

Where  $\kappa$  is the *dielectric constant*, a coefficient to  $\epsilon_0$  which relates the permittivity of the material to that of free space. This higher  $\epsilon$  allows for a significantly higher storage of energy. However, if the field becomes too strong, the dielectric will fail. We therefore have a concept of dielectric strength, which informs of the maximum field which can be created across the material without destruction.

A defibrillator is an example of a very high capacitance application, where a material boasting a very high  $\kappa$  is utilised.

## Capacitance in Circuits

Capacitors in parallel add their capacitance. Effectively, the surface areas of the individual capacitors add.

$$C_T = \sum_{i \rightarrow n} C_i$$

For capacitors in series, the capacitance adds similarly to resistors in parallel.

$$\frac{1}{C_T} = \sum_{i \rightarrow n} \frac{1}{C_i}$$

This is because the charge between the negative terminal of one must be adjacent to the positive terminal of another, and the charge between the two must be neutral.

## Electric Current

Around an atom, energy exists in quantised levels. Electrons exist in shells around a nucleus, and each shell has a different binding energy. The number of electrons that can fit within each shell is different, beginning with two and increasing moving toward outer shells.

Insulators are more tightly bound than are conductors. In conductors, electrons can move around more freely and conduct to other particles. This implies that there is a relatively large gap between energy bands of the particle.

Semiconductors have a smaller gap between bands than conductors, but larger than insulators. By changing their conductivity by introducing traces of conductors or altering the temperature, one can control their conducting behaviour.

Electrons naturally move around rapidly and largely randomly, but by applying a force to them with an electric field, this random motion can be skewed toward one end of the circuit resulting in a flow of electricity. Electric current is the rate of transport of charge along a conductor.

$$I = \frac{dq}{dt}$$

This value  $i$  can be calculated by considering the movement of individual electrons within a conductor, according to the following formula.

$$\frac{dq}{dt} = enAv_d = I$$

Here,  $e$  is the charge of an electron,  $n$  is the density of electrons in the conducting material,  $A$  is the cross sectional area of the flow direction in the conductor and  $v_d$  is the drift speed of an electron. Current is measured in Amps, A equal to  $Cs^{-1}$ .

A material with a lower  $v_d$  generally has more things for a flowing electron to “bump in to”, causing it to slow down, and might be said to have a higher resistance. This resistance is defined more formally through

$$R = \frac{V}{I}$$

Generally,  $R$  is dependent on temperature, voltage and current. A superconductor has 0 resistance, implying a free flow of electrons. A device with a linear relationship between  $I$  and  $V$  is an Ohmic resistor.

Power is defined as the product of current and voltage, i.e.

$$P = VI$$

When electricity passes through a material, electrons are bumping into the material, releasing heat into it, causing the material to heat up and wasting power. For a resistor with resistance  $R$ , the power dissipated is

$$P_{\text{lost}} = I^2 R$$

It is for this reason that power lines are run at very high voltages like 500kV. Because power loss is related to the square of current, running at a high voltage allows the same delivery of power with a much lower power loss.

## Circuits

A battery in a circuit is a source of electromotive force, electric potential. It drives current around a circuit. The electric potential in the circuit is dissipated

across the components of the circuit according to their resistances. In the case of the resistor, the potential is converted to heat.

$$\epsilon - IR = 0$$

$$I = \frac{\epsilon}{R}$$

Here,  $\epsilon$  is electromotive, equivalent to  $V$ . To understand complex circuits, we use Kirchhoff's Rules for Circuit Analysis. These rules essentially dictate conservation of charge and conservation of energy within circuits. This states that any current within a circuit must be going somewhere; it must traverse from one end of potential back to the same.

Mathematically, at any node, any combination of path, the sum of current must be 0. This applies along a straight line; if the current all passes through one point, the outflow equals the inflow so the sum is 0.

$$\sum I = 0$$

Around any loop, the following statement must be true.

$$\pm \sum \epsilon \pm \sum IR \pm \sum \frac{q}{C} = 0$$

So if we add up all of the voltage input (say battery), voltage loss (say resistors) and capacitors, the sum must be 0, i.e. the circuit is conservative; any gain in charge is matched by an equivalent loss. This allows us to, for example figure out how to calculate cumulative resistors in series as

$$R_T = \sum_{i \rightarrow n} R_i$$

It can also be used to solve for resistors in parallel, yielding the equation

$$\frac{1}{R_T} = \sum_{i \rightarrow n} \frac{1}{R_i}$$

These equations are determined by tracing the paths and considering the splitting of current that must occur for the circuit to obey the two conservations.

## Magnetic Fields

In electric fields, electric monopoles exert forces on each other. Magnetic fields behave differently, as all magnetic fields are dipoles. No “magnet charge” exists, instead moving charges create magnetic fields. The simplest example of a magnet is a bar magnet. A bar magnet is a magnetic dipole, with a north and south pole.

Field lines are drawn from the north pole to the south pole, and the density of these field lines is known as magnetic flux. This density is highest at either of these poles.

The idea of all magnetic fields being dipoles can be expressed through the previously explored idea of a closed surface integral. This is Gauss’ Law of magnetism, and is one of Maxwell’s equations.

$$\oint \vec{B} \cdot d\vec{A} = 0$$

This states that the magnetic flux through any given three dimensional surface must be 0, which can be understood as the fact that any field lines which emerge from the surface must also return through the sphere to the opposite pole.

An electric charge moving through a magnetic field will tend to move perpendicular to that field, in an arc. This is because it has a certain force acting on it which causes it to move in a parabola as it accelerates. The magnitude and direction of this force is given by

$$\vec{F} = q\vec{v} \times \vec{B}$$

Where  $\vec{F}$  is the force vector on the particle with charge  $q$  moving with velocity vector  $\vec{v}$  through electric field with strength and direction defined by  $\vec{B}$ . The units of  $\vec{B}$  are  $T$ , Tesla. Interestingly a faster moving charge will have a greater force exerted on it. The magnitude of this force can be calculated through

$$F = qvB \sin(\phi)$$

Where  $\phi$  is the angle between  $v$  and  $B$ . This force will always be perpendicular to the plane defined by the velocity and magnetic field directions. This can be simulated with the right hand rule; if one curls their fingers from  $v$  to  $B$  and extends the thumb, it will be in the direction of the force.



The Lorentz Force Law tells us how to combine the effects of magnet and electric fields on a moving charge. Intuitively enough, it essentially just says "add them you Drongo"

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Because a charge in a field will tend to move in an arc, it is clear that if the area the charge is in is large enough, it will eventually trace out a circle. If we want to find the radius of that circle, perhaps for designing a Cyclotron or similar, we can use the equation

$$r = \frac{mv}{qB}$$

This assumes a charge of mass  $m$  with velocity  $v$  moving in a field at right angles to its velocity plane. The period of this rotation is given by

$$T = \frac{1}{f} = \frac{2\pi m}{qB}$$

Interestingly enough, independent of  $v$ .

## Lorentz Force

When working with both an electric field and a magnetic field, it is often useful to have the two perpendicular, such as in the case of a cathode ray tube.

Another application of these perpendicular fields is an ion velocity filter, where one uses the fact that magnetic force is proportional to velocity to filter out charged particles of other velocities. This is done by balancing the electrical and magnetic fields such that ions of the desired velocity will have balanced forces from the two, while other velocities will have larger or smaller forces, causing them to crash into the side of the chamber. To balance in this way, one simply needs to solve the equality

$$qE = qvB \Rightarrow v = \frac{E}{B}$$

This same process can be used to construct a simple mass spectrometer, a device which measures the charge to mass ratio of ions. By accelerating charges in an electric field, they will end up with

$$v \propto \sqrt{\frac{q}{m}}$$

And can then be passed through an ion filter to measure velocity.

## The Hall Effect

If a magnetic field is running through a wire with a current, the deflection caused by the magnetic field will result in a build up of negative charges on one side, creating a potential difference between the two sides of the wire. This is known as the Hall Effect. The magnitude of this effect continues to increase until the force exerted by the created electric field is equivalent to the external magnetic field within the wire, i.e.

$$F_E = F_B$$

is the condition for the process to end. This can be used to measure the drift velocity within the material because the force due to the magnetic field is proportional to the velocity of the electrons. The final voltage across the wire is known as the Hall Voltage for the material.

## Origin of a Magnetic Field

Magnetic fields can be created in two ways. The first of these is by magnetic materials, and the second is by currents. A current produces a magnetic field according to the Biot-Savart Law, which takes the form

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{s} \times \hat{r}}{r^2}$$

Where  $d\vec{B}$  is the section of magnetic field at a distance  $r$  in a direction  $\hat{r}$  from the current carrier,  $\mu_0$  is the vacuum permeability and  $d\vec{s}$  is the rate of change of the current carrying surface at the relevant point.

$\mu_0$ , the vacuum permeability is rather like the vacuum permittivity we use for electric fields. Like permittivity, we can replace  $\mu_0$  with a determined  $\mu$  for a non-vacuum material.

This law gives us the right hand rule for a wire. If one places their thumb along the direction of current in a wire, and wraps their fingers around, the fingers will indicate the direction of magnetic field.

Much as we try to use Gauss' Law rather than Coulomb's law in electrostatics, the complexity of the Biot-Savart law means it is often better to use Ampere's law when dealing with magnetic fields.

## Ampere's Law

Ampere's law states that

$$\oint B \cdot ds = \mu_0 I$$

i.e. for a closed loop around a current carrying path  $s$ , the sum of the magnetic field  $B$  dotted with the rate of change of the path  $ds$  will be equal to the vacuum permeability multiplied by the enclosed current. For the magnetic field around a wire, this yields

$$B = \frac{\mu_0 I}{2\pi r}$$

For an internal section of radius  $r$  within a larger wire of radius  $R$ , Ampere's law tells us that the magnetic field enclosed will be

$$|B| = \frac{\mu_0 I r}{2\pi R^2}$$

When working with Ampere's law it is important to consider net current; if one wire carrying 3A in one direction is enclosed alongside another carrying 1A in the opposite direction, the total current is 2A in the first direction.

For a long solenoid, with  $n$  turns per metre, we can find the internal magnetic field using Ampere's law.

$$B = \mu_0 n I$$

This is independent of the diameter of the solenoid. For a real solenoid, this only applies near the centre.

For an arc of current  $I$  at a radial distance  $r$  traversing an angle  $\theta$  (in radians), the magnetic field strength at the centre is given by

$$B = \frac{\mu_0 I \theta}{4\pi r}$$

Two adjacent current carrying wires will exert a force on each other due to their induced magnetic fields. The magnitude of this force by one of these wires  $b$  on the other  $a$  is given by

$$F_{ba} = \frac{\mu_0 I_a I_b L_b}{2\pi d}$$

Where  $d$  is the distance between the two. If the two currents are in the same direction they will attract each other, while if they are opposite they will repel. This is the result of the more general equations

$$\vec{F} = q\vec{v} \times \vec{B}$$

$$\vec{F} = i\vec{L} \times \vec{B}$$

For the magnetic force on a moving charge and the magnetic force on a current respectively.

### Faraday's Law

Faraday's Law of induction is one of Maxwell's equations. The law states that a change in magnetic flux causes a potential difference. In practice, this means that moving a magnet towards or away from a loop of wire causes a current and voltage to appear in the loop. The direction of this current is such that the magnetic field induced by it opposes the change in flux.

Magnetic flux through an area is defined as

$$\Phi_B = \int_{\text{surface}} \vec{B} \cdot d\vec{A}$$

The magnetic flux is the product of the normal area with magnetic field. In the case that  $B$  and  $A$  are parallel, this is simply

$$\Phi_B = BA$$

Faraday's law can be intuited by considering a metal bar being moved at some velocity  $v$  through a constant magnetic field of strength  $B$ . The effect of this magnetic field will be to induce a Hall effect within the bar, creating a potential difference between its two ends. If one then adds a circuit connecting the ends of the bar, it acts like a battery, dissipating its voltage across the wire. Thus a current is created. This current induces a magnetic field, which opposes the direction of motion. This system can be described by

$$IR = LBv$$

Where  $I$  is the current created by the Hall effect,  $R$  is the resistance of the external circuit and  $L$  is the length of the metal bar. Thus we see the power induced is dependent on the velocity. The induced EMF is given by

$$V = -IR = -LBv$$

More generally this can be expressed as

$$V = -\frac{dBA}{dt}$$

Where  $A = Ldx$ . Thus the potential difference induced by this system will be  $V$ . It turns out we have a definition for  $BA$ .

$$\epsilon = -\frac{d\Phi_B}{dt}$$

$\epsilon$  is conventionally used rather than  $V$  for this induced EMF.

### Lenz's Law

A solid object in a changing magnetic field will have eddy currents within it.

The negative sign in the expression of Faraday's law is Lenz's Law. It tells us that the field induced by the current opposes the change in flux. This is important; if the opposite were true moving a magnet through a loop would cause it to be sucked through.

This is an important principles for an application like an AC generator, jumping rings where a conducting solenoid is used to induce an opposing current in a metal ring, which appears to jump away from the solenoid or a transformer, where the magnetic field induced by the current around one side of the transformer induces a current on the other side with voltage determined by

$$V_S = V_P \frac{N_S}{N_P}$$

Where  $V_S$  is the output or secondary voltage,  $V_P$  is the input or primary voltage,  $N_S$  is the number of coils on the output side and  $N_P$  is the number of coils on the input side. Transformers are essential for changing voltages for power delivery.

Lenz's Law is also the reason that a magnet falling through a metal pipe is slowed as it falls.

An adaptation of Ampere's law is needed to explain the phenomena of a magnetic field appearing between charging capacitor plates, in addition to other phenomena. This is necessary because the simple version of the law fails to describe magnetic fields induced by changing "electric flux" or *displacement current*. This adaptation yields

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

This is the fourth of Maxwell's equations, known as the Ampere-Maxwell Law.

Maxwell's equations are quite beautifully tied together by the equation for the speed of light. Because a photon is a parallel electric and magnetic field propagating through space, it is intuitive that it is in some way dependent on Maxwell's equations, and indeed the speed of light is given by

$$\frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

## Magnetic Materials

A variety of magnetic materials exist. All of these materials are magnetic dipoles, such as a current in a loop or a bar magnet. We define the magnetic dipole moment  $\vec{\mu}$  as

$$\vec{\mu} = I \vec{A}$$

This is similar to the electric dipole moment  $\vec{p} = q \vec{d}$  discussed earlier. In a magnetic field, it experiences the torque

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

$$U = -\vec{\mu} \cdot \vec{B}$$

Protons, neutrons and electrons all have a magnetic dipole moment. The magnetic properties of matter largely arise from these moments. The magnetic dipole moment of an electron is much larger than that of a proton, which is a little under three times that of a neutron. Thus, the overall moment of an

atom is largely determined by its electrons. We can conceptualise this as electrons moving around a nucleus, i.e. a current, creating a magnetic field. Therefore all matter is magnetic; though not necessarily very strongly. In many atoms however, the moments sum to zero. Even if a single atom doesn't, it may cancel with adjacent atoms.

Materials in which these particles do not cancel each others fields are more significantly magnetic. In a *paramagnetic material* the magnetic moment of an individual atom is non-zero. The individual atoms will try to align, but will be unable to due to thermal agitation. This will lead to a skew in the overall alignment, causing a magnetic field.

In *diamagnetism* the magnetic moment of each individual atom is zero unless there is an external field. If there is, a magnetic moment is induced in the opposite direction to the external field. Thus, the material will be repelled from the pole of a stron magnetic. This property actually exists for all atoms, but is generally quite weak.

*Ferromagnetism* is the most common form, seen in bar magnets and similar objects. In these materials, adjacent atoms “collaborate” via exchange coupling, causing the atoms to align and creating a strong magnetic field. There are few of these materials, with the most well known one being iron. Ferromagnetic materials are used in settings like MRI machines. This doesn't happen inherently; different regions within a lump of iron may be differently aligned. However, when a strong field is applied to the material, it can become aligned, creating a strong magnet. If the material is heated, thermal agitation can ruin this alignment. Ferromagnets can be wrapped in wire to create strong electromagnets.

$$B_{\text{total}} = \mu_0 n I + B_{\text{Fe}}$$

Is the equation for the magnetic field created by wrapping an iron magnet of strenght  $B_{\text{Fe}}$  in  $n$  turns of wire with current  $I$  through them. This can be alternately expressed as

$$B_{\text{total}} = \mu n I$$

Where  $\mu$  is a property of the iron and  $\mu \gg \mu_0$ .

## Fluids

A fluid is a substance that can flow; generally liquids and gases. They will conform to the boundaries of any container in which they are placed. Formally, they are defined as materials which cannot support shearing stresses.

### Elasticity

The capacity of a material to stretch is described by its *Young's modulus*,  $Y$

$$\frac{F}{A} = Y \frac{\Delta L}{L}$$

Where  $F$  is the force exerted across an area  $A$ ,  $\Delta L$  is the change in length of the object of initial length  $L$ . The atoms in the material behave similarly to ideal springs when below the deformation limit of the material.

Fluids are not elastic; when a shear is applied to a fluid, it flows aside.

Density of a fluid describes the mass of a unit per unit volume.

$$\rho = \frac{\Delta m}{\Delta V} \left( = \frac{dm}{dV} \right)$$

Where  $m$  is the mass contained in the volume  $V$ . For a body of uniform material

$$\rho = \frac{m}{V}$$

$\rho$  is measured in  $\text{kgm}^{-3}$ . The density of gasses varies significantly whereas liquids do not, because gases are dramatically more compressible.

Pressure exerted by a fluid on an object immersed in that fluid is given by

$$p = \frac{F}{A}$$

Where  $F$  is the force exerted and  $A$  is the surface area of the object. Pressure is measured in  $\text{Nm}^{-2}$ , also known as Pa, Pascals. Pressure occurs due to the net force of molecules of gas colliding with the walls of a body. For a liquid, it occurs due to gravity pulling the liquid against the walls.

$$p = p_0 + \rho dg$$



The above equation describes the pressure within a liquid with surface pressure  $p_0$ , density  $\rho$  at a depth  $d$  under gravity  $g$ . This formula holds for any incompressible fluid. It is worth noting that is not dependent on area in any way; due to the pressure from the walls, a higher or lower area does not effect the pressure.

$$\Delta p = \rho gh$$

Because of this, pressure is directly related to depth  $h$  by the above equation.

The density of air above the earth is higher near the surface, due to gravity. Most of the variation in air pressure is due to weather events.

Gauge pressure is the pressure measured by a tyre gauge, i.e. measured with respect to atmospheric pressure. Thus

$$p_{\text{gauge}} = p - 1\text{Atm}$$

## Hydraulics

A hydraulic mechanism like a car jack works because, given a system where one can pressurise water from one point to increase pressure below another point, we can modify the ratio of the areas of the two ends of the system to create a mechanical advantage.

$$p_1 = \frac{F_1}{A_1} = p_2 = \frac{F_2}{A_2} + \rho gh$$

$$\Rightarrow F_2 = \frac{A_2}{A_1} F_1 - \rho gh A_2$$

The above equation relate the force exerted by the person operating a hydraulic jack,  $F_1$  on an area  $A_1$  to the force exerted on the other end  $F_2$  across area  $A_2$ . Thus, for a small  $h$ , the mechanical advantage is given by

$$F_2 \approx \frac{A_2}{A_1} F_1$$

The force exerted by the jack on the vehicle it lifts is increased by factor  $\frac{A_2}{A_1}$  as compared to the force  $F_1$  input by the user.

## Barometer

A barometer is a device for measuring pressure. It works by using an inverted tube filled with water, and upending this tube into a basin which is partially filled with water. Due to gravity the water in the tube will press down into the basin, causing a rise the the water level, creating a vacuum at the top of the tube. The height of the water above the basin level, i.e. from the basin level to the beginning of the vacuum, will be dependent on atmospheric pressure.

$$p_{\text{atm}} = \rho gh$$

## Pascal's Principle

Pascal's Principle states that a change in pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the fluid, and to the walls of the containing vessel.

## Buoyancy and Archimede's Principle

Archimede's Principle states that a body wholly or partially immersed in a fluid will be buoyed upward by a force equal to the weight of the fluid displaced by the body. Thus for a body of mass  $m$

$$F_b = mg = \rho Vg$$

$$F_b = \rho_{\text{fluid}} V_{\text{submerged}} g$$

Essentially a body can float as long as it has a lower density (or effective density) than the liquid it is immersed in. For example, given a floating cube of ice, it has a buoyancy force given by

$$F_b = \rho_{\text{water}} L^3 g$$

When fully submerged, assuming a sidelength  $L$ , we can then find the carrying capacity of this cube through

$$\rho_{\text{ice}} L^3 g + C = F_b$$

Where  $C$  is the carrying capacity.

## Fluid Dynamics

When considering dynamic fluids rather than static fluids, one needs to consider conservation of energies. An ideal fluid is one in which the velocity if the fluid is constant at all points in time. An incompressible liquid is one with a constant density which cannot be changed. A nonviscous flow is a flowing fluid which doesn't resist the flow. Honey is an example of a viscous liquid.

The equations examined are targeted at liquids which have all three of these properties. Thus they work well for water, reasonable well for air under some conditions and poorly for many other fluids.

A streamline is a path traced out by a fluid element. The velocity of a fluid at a point is at a tangent to the streamline and streamlines may not cross in ideal cases.

There are two crucial equations for fluid dynamics.

$$A_1 v_1 = A_2 v_2$$

This is the *continuity equation*; this tells us that the area at a point multiplied by the liquids velocity is a constant, i.e. the volume flow rate is a constant. It is for this reason that the stream from a tap narrows as it falls, due to acceleration from gravity.

This is extended by Bernoulli's Equation, which gives us a conservation of energy for liquids.

$$p + \frac{1}{2}\rho v^2 + \rho g y = \text{constant}$$

Where  $p$  is the pressure of the liquid,  $\rho$  is its density,  $g$  is gravity and  $y$  is the height of the liquid. This tells us that the kinetic energy per unit volume plus the work per unit volume plus the potential energy per unit volume is a constant. If  $y$  is small we can state that

$$p + \frac{1}{2}\rho v^2 = \text{constant}$$

An example application of this is to consider the speed at which water will emerge from a hole  $h$  units down from the surface of a tank of water. If we take depth  $h$  to be at  $y = 0$ , and apply our knowledge that the water is stationary at the surface we can find

$$p_0 + \frac{1}{2}\rho 0^2 + \rho g h = p_0 + \frac{1}{2}\rho v^2 + 0$$

$$v = \sqrt{2gh}$$

Curiously this is the same as for water free-falling a vertical distance  $h$ .

## Thermal Physics

### Thermal Properties of Matter

There are three phases of matter; solid, liquid and gas. The major difference between these states is in the bonds between the atoms in the matter. They are never totally stationary, but they are relatively fixed in solids, somewhat more free in liquids and utterly independent in gases.

A state variable tells us about the state of a system (such as pressure, volume, etc). One of these is that we talk about the number density of atoms

$$\eta = \frac{N}{V}$$

$\eta$  is the number of atoms (or molecules or particles) per unit volume.

The atomic mass number  $A$  is given by the number of protons added to the number of neutrons. The atomic mass unit  $u$  is given by

$$1u = \frac{m(^{12}C)}{12} = 1.66 \times 10^{-27} \text{kg}$$

and is sometimes known as the Dalton. This is one twelfth of the mass of a carbon-12 atom.

Avogadro's number  $N_A$  is

$$N_A = 6.02 \times 10^{23}$$

And this refers to the number of "elementary units" per mole. These units could be atoms, molecules, etc. Thus one mole of carbon twelve is  $N_A$  carbon atoms. The molar mass of a substance is the mass in grams of one mole of a substance.

Temperature is measured in Kelvin (K), which is measured in units of the same size as Celsius, starting from  $-273.15^\circ\text{C}$ , absolute 0.

## Phases of Matter

We can think about the bonds between atoms as a kind of spring system. When atoms are pushed close together, a strong repulsive force repels them, while when they are further apart a weaker attractive force draws them together. If one increases the energy of the system, perhaps in the form of an increase in temperature, it becomes easier for atoms to break these bonds and change phase.

Within a gas, the temperature tells us something about the behaviour of the particles. At 0K, the particles are stationary.

When heating a liquid, the temperature will be unchanged as the phase change occurs. Temperature and pressure have a relationship for matter; higher pressures imply more solid matter, while higher temperature implies less solid matter. We can therefore find a point, the “triple point” on a plot of temperature against pressure where the matter could change to any phase.

Another consequence of this relationship is sublimation; when at very low pressure, matter will simply turn into a gas.

The critical point occurs at high pressure and high temperature, and at this point a material can be both liquid and gaseous. For most materials, the shape of the pressure versus temperature plot is similar, with the left hand side at low temperature solid, the bottom part a wedge of gas and liquid a wedge driven between the two at relatively high temperature and pressure.

## Ideal Gas Equation

An ideal gas obeys the *Ideal Gas Equation*

$$pV = Nk_B T$$

Where  $p$  is pressure,  $V$  is volume,  $N$  is the number of particles,  $T$  is temperature and  $k_B$  is *Boltzmann's constant*. Most “ideal” gases in the real world approximate this equation. This relationship between  $p$  and  $T$  when all other variables are held constant allows for the development of some very accurate thermometers.

Boltzmann's constant is given by

$$k_B = 1.38 \times 10^{-23} \text{J K}^{-1}$$

Using the *universal gas constant*  $R = 8.31 \text{ J K}^{-1} \text{ mol}^{-1}$  we can rewrite this as

$$pV = nRT$$

Where  $n$  is the number of moles of the gas.

### Example

What is the volume occupied by 1mol of helium atoms at a temperature of  $20^\circ\text{C}$  at a pressure of 1atm?

$$pV = nRT \Rightarrow V = \frac{nRT}{p} = \frac{1 \times 8.31 \times (273 + 20)}{1.013 \times 10^5} = 0.024 \text{ m}^3$$

Using this equation, we can find a variety of equilibria for a gas at different intersections of variables. For instance we can find that

$$\frac{p}{T} = \frac{nR}{V}$$

Implying that if the right hand side is a constant, a change in pressure must effect a change in temperature. We know about these through *quasi-static processes*, where we consider the effect of a tiny change in one variable. Three main types are talked about

- The example given above, with constant volume is an *isochoric process*
- A constant temperature implies an *isothermal process*
- A constant pressure implies an *isobaric process*

## First Law of Thermodynamics

The work-kinetic energy theorem states

$$\Delta K = W_c + W_{\text{diss}} + W_{\text{ext}}$$

i.e. the change in kinetic energy is the sum of work done by conservative forces (i.e.  $-\Delta U$ ) plus the work done by dissipative forces, such as friction plus the work done by external forces.

We define mechanical energy as the *macroscopic* energy of the system.

$$E_{\text{mech}} = K + U$$

We define thermal energy as the *microscopic* energy of the system.

$$E_{\text{th}} = K_{\mu} + U_{\mu}$$

The total energy of the system is then the sum of the mechanical and thermal energy of the system. There are multiple forms of microscopic energy; thermal energy, chemical energy, which is the energy stored in bonds between molecules and nuclear energy, the energy stored in an atomic nucleus.

The internal or microscopic energy is then given by

$$E_{\text{int}} = E_{\text{th}} + E_{\text{chem}} + E_{\text{nuc}} + \dots$$

While the overall system energy is given by

$$E_{\text{sys}} = E_{\text{mech}} + E_{\text{int}}$$

We often simplify and assume  $E_{\text{int}} = E_{\text{th}}$ . We can consider an increase in energy of a system through

$$\Delta E_{\text{sys}} = \Delta E_{\text{mech}} + \Delta E_{\text{th}} = W + Q$$

Where  $W$  is external work on the system and  $Q$  is external heat transfer to the system. An example of this can be seen in a diesel engine, where air is compressed on an upward stroke to a high pressure, at which stage fuel is injected, immediately igniting due to the high temperature, expanding downward and pushing the piston, doing work on the engine.

When the gas expands in this way, the work done is given by the area under the  $pV$  curve, from  $V_{\text{initial}}$  to  $V_{\text{final}}$ .

$$W = - \int_{V_i}^{V_f} p dV$$

For an expanding gas, one where  $V_f > V_i$ , negative work is done on the gas. This is an isothermal process. This integral comes out to be

$$-nRT \log \left( \frac{V_f}{V_i} \right)$$

In an isochoric process,  $dV = 0 \Rightarrow W = 0$ . For an isobaric (constant pressure) process, the work is simply given by  $-p\Delta V$ . Finally, we come to the First Law of Thermodynamics, that

$$\Delta E_{\text{th}} = W + Q$$

In an isothermal process,

$$\Delta T = 0 \Rightarrow \Delta E_{\text{th}} = 0 \Rightarrow Q = -W$$

In an isochoric process

$$\Delta V = 0 \Rightarrow W = 0 \Rightarrow \Delta E_{\text{th}} = Q = nc_V \Delta T$$

An adiabatic process is one in which  $Q = 0 \Rightarrow \Delta E_{\text{th}} = W$  thus there is no transfer of heat, and energy is transferred exclusively as work. This is a desirable property in engines and similar mechanisms.

## Joining Macroscopic and Microscopic

For particles in a gas, we find that the root mean squared speed will be given by

$$v_{\text{rms}} = \sqrt{\bar{v^2}} = \sqrt{\frac{3k_B T}{m}}$$

This gives us the equation for the kinetic energy of a molecule in a gas is given by

$$\bar{K}_{\text{molecule}} = \frac{1}{2} m \bar{v^2} = \frac{3}{2} k_B T$$

i.e. the average kinetic energy is dependent only on the temperature. This implies that thermal energy is evenly distributed among particles of a gas. We can use this to find thermal energy.

$$E_{\text{th}} = N \bar{K}_{\text{molecule}} = \frac{3}{2} N k_B T = \frac{3}{2} n R T$$

Thus, the thermal energy of a system is dependent only on the temperature and quantity of particles in the system. We can also find that

$$\Delta E_{\text{th}} = \frac{3}{2} n R \Delta T$$



$$\Delta E_{\text{th}} = nc_V \delta T \Rightarrow c_V = \frac{3}{2}R$$

Where  $c_V$  is the heat capacity of the gas.

We can consider the *mean free path* of a molecule in a gas as the distance the particle can travel without colliding with another particle or a side of the container. This yields an equation of

$$\lambda = \frac{1}{4\sqrt{2}\pi r^2 \eta}$$

Where  $\lambda$  is this mean free path length, and  $\eta$  is the previously discussed number density.

### Example

We can calculate the mean free path for nitrogen-2 at 300K at pressure 1atm, assuming a radius of  $r = 1 \times 10^{-10}\text{m}$  as follows. First, we find the number density  $\eta$  of the particles through

$$pV = Kk_B T \Rightarrow \eta = \frac{N}{V} = \frac{p}{k_B T} = \frac{1.013 \times 10^5}{1.38 \times 10^{-23} \times 300} = 2.44 \times 10^{25}\text{m}^{-3}$$

We can then use this to find  $\lambda$  through the previous equation.

$$\lambda = \frac{1}{4\sqrt{2}\pi(1 \times 10^{-10})^2 \times 2.44 \times 10^{25}} = 2.3 \times 10^{-7}\text{m}$$