Linear Equations

Systems of Linear Equations

Systems of equations and row operations.

For example, consider a network of flows; nodes with a given inflow and outflow. To compute flows in this network we can use a system linear equations.

Data fitting using a polynomial. Sometimes, we want to find a function of a certain form which fits to a set of data points. To find the relevant coefficients, we can use a system of linear equations.

In general, we will take the variables in a linear equation to be $x_1, x_2, \ldots, x_n$ and the coefficients to be $a_1, \ldots, a_n$.

A finite collection of linear equations of a given set of variables is called a *system of linear equations* or a *linear system*.

$$x_1 + 5x_2 + 6x_3 = 100$$

$$x_2 - x_3 = -1$$

$$-x_1 + x_3 = 11$$

Here, despite missing x_1 and x_2 respectively, the second and third equations are still part of the same system as they implicitly have a term with a 0 coefficient.

The organisation of the above system, with all variables on the right and constants at left is the standard form of presenting a system.

A *homogenous* linear system is one where all of the constants at right are 0. These systems are easier to solve, and by solving a homogenous version of a non-homogenous system, we can find a solution to the non-homogenous variant.

A solution to a linear system is a set of values for variables that cause all equations in the system to be true.

Solving by Elimination

$$(1): \ 2x - y = 3$$

$$(2): \ x + y = 0$$

$$(2) \Rightarrow y = -x$$
$$(2)\&(1) \Rightarrow 2x - (-x) = 3 \Rightarrow 3x = 3$$
$$x = 1 \Rightarrow y = -1$$

A key to the applicability of this method is that we can divide by the coefficients, which will not always be a valid assumption. This method can be implemented algorithmically and will always either yield a solution or tell you there is none.

Matrices

Really, the variables in a linear system aren't really important; it is simply the coefficients which define their relations. A matrix, a rectangular array of numbers, can be used to store these values. A $p \times q$ matrix has p rows and q columns.

A *augmented matrix* for a linear system is the matrix formed from the coefficients in the equations and the constant terms, separated by a vertical line. For example

$$\begin{array}{r} 2x - y = 3 \Rightarrow 2x + -1y = 3 \\ x + y = 0 \Rightarrow 1x + 1y = 0 \end{array} = \left[\begin{array}{cc|c} 2 & -1 & 3 \\ 1 & 1 & 0 \end{array}\right]$$

With coefficients at left and constants at right. The number of rows should be equal to the number of equations. Each column corresponds to a given variable in the equations.

We can perform some *elementary row operations* to such a matrix without changing its solutions. These are

- Interchanging two rows
- Multiplying a row by a non-zero constant
- Adding a multiple of a row to another row