APPM 2360 Project 2

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1 Introduction

In this project, we were tasked with examining two problems from civil engineering to apply techniques from linear algebra and differential equations. The first problem involves the distribution of forces in a simple bridge, and the second is concerned with the deflection of a beam under uniform loading.

2 Task Set A

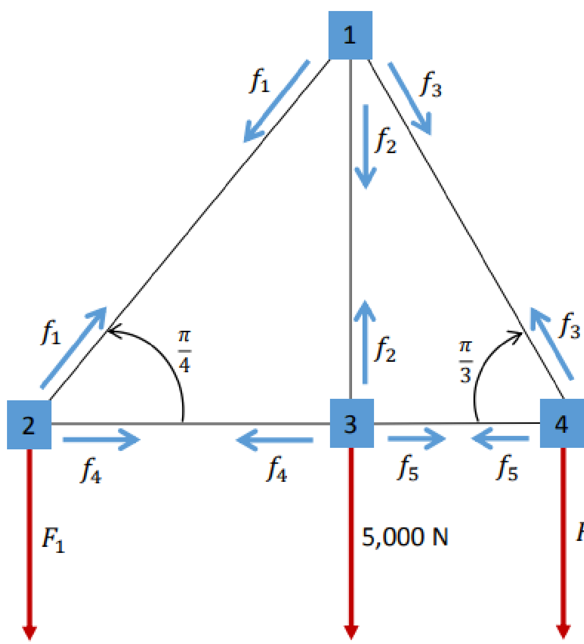

Figure 1: Distribution of the forces in a bridge

The first problem focused on examining the distribution of forces in a bridge. Using the diagram in Figure 1, we found the following equations for the horizontal and vertical force components for the 4 different joints of the bridge, displayed in the table below:

From the table, you can form equation (1) as a linear system of equations in the form of $\boldsymbol{A}\vec{\boldsymbol{f}} = \vec{\boldsymbol{b}}$, where $\vec{\boldsymbol{f}} = [f_1 f_2 f_3 f_4 f_5 F_1 F_2]^T$.

Joint	Horizontal	Vertical
$\boxed{1}$	$\frac{\sqrt{2}}{2}f_1 = \frac{1}{2}f_3$	$\frac{\sqrt{2}}{2}f_1 + f_2 + \frac{\sqrt{3}}{2}f_3 = 0$
$\boxed{2}$	$\frac{\sqrt{2}}{2}f_1 + f_4 = 0$	$\frac{\sqrt{2}}{2}f_1 = F_1$
$\boxed{3}$	$f_4 = f_5$	$f_2 = 5000N$
$\boxed{4}$	$\frac{1}{2}f_3 + f_5 = 0$	$\frac{\sqrt{3}}{2}f_3 = F_2$

Table 1: Equations of vertical and horizontal force components of the bridge

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5000 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The order $\boldsymbol{A}$ is (8x7). The orders of $\vec{\boldsymbol{f}}$ and $\vec{\boldsymbol{b}}$ are (7x1) and (8x1), respectively. The system is overdetermined, meaning there are more equations than unknown variables. It is not possible to calculate the determinant of $\boldsymbol{A}$ because you can only determine the determinant of a matrix that square (nxn). The augment matrix formed from equation (1) can be put in reduced row echelon form in Matlab to form equation (2).

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2588.19 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 5000 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -3660.25 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1830.13 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1830.13 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1830.13 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3169.87 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{2}$$

The rank of the RREF matrix is 7 because it has 7 pivot columns. The linear system is consistent because there aren't any rows that indicate that one of the variables multiplied by zero equals a non-zero answer.

In Table 2, you can see the tensions and external forces in the bridge.

Tensions		External Forces	
f_2	$5000N$	F_1	$-1830.13N$
f_4	$1830.13N$	F_2	$-3169.87N$
f_5	$1830.13N$		

Table 2: tensions and external forces in the bridge

From the Table 2 and the RREF matrix, you can see that the piece of the bridge under the most tension, being stretched the most is f_2. The piece that is being compressed the most is f_3.

4. Suppose the load of 5000 Newtons at joint 3 is replaced by the "free" (unknown) force, $\mathbf{F}_3$

Joint	Horizontal	Vertical
$\boxed{1}$	$\frac{\sqrt{2}}{2}f_1 = \frac{1}{2}f_3$	-$\frac{\sqrt{2}}{2}f_1 - f_2 - \frac{\sqrt{3}}{2}f_3 = 0$
$\boxed{2}$	$\frac{\sqrt{2}}{2}f_1 + f_4 = 0$	$\frac{\sqrt{2}}{2}f_1 = F_1$
$\boxed{3}$	$f_4 = f_5$	$f_2 = F_3$
$\boxed{4}$	-$\frac{1}{2}f_3 - f_5 = 0$	$\frac{\sqrt{3}}{2}f_3 = F_2$

Table 3: Equations of vertical and horizontal force components of the bridge when the 5000N force is replaced by F_3

$$\begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

The order $\boldsymbol{A}$ is (8x8). The orders of $\vec{f}$ and $\vec{b}$ are (8x1) and (8x1), respectively. The system is neither overdetermined or underdetermined, because there is the same amount of equations as variables. It is possible to calculate the determinant of $\boldsymbol{A}$ because it is a square (nxn) matrix. The determinant of $\boldsymbol{A}$ is 0.

$$\begin{vmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & -1 & 0 \end{vmatrix} = \mathbf{0}$$

No the $\boldsymbol{A}$ matrix cannot be inverted because the determinant is 0. This also means that the linear system will not have a unique solution. This means there will either be no solution or infinitely many solutions to the equation $\boldsymbol{A}\vec{f} = \vec{b}$. Solving the new augmented matrix yields the following matrix.

$$
\left[\begin{array}{cccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & .518 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & .732 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -.366 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -.366 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & .366 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & .634 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{4}
$$

This has 7 pivot columns and 1 free variable. Let's call the free variable F_3 and then you write the solution in the form

$$
\vec{f} = \begin{bmatrix} -.518 \\ 1 \\ -.732 \\ .366 \\ .366 \\ -.366 \\ -.634 \end{bmatrix} F_3 \tag{5}
$$

5. If F1 and F2 turn out to be negative, what does this mean physically?
This means that the joints 2 and 4 are being pushed upward by external forces F_1 and F_2

3 Task Set B

$$
y'' = \frac{S}{EI}y + \frac{Qx}{2EI}(x-L),\ 0 \le L,\ y(0)=0,\ y(L)=0 \tag{6}
$$

3.1 Classify the ODE for the deflection of a uniform beam, given in (1). What is the order of the ODE? Is it linear? If so, is it homogeneous or nonhomogeneous? Are the coefficients constant or variable?

The differential equation can be rewritten in the form:

$$
y'' - \frac{S}{EI}y = \frac{Qx}{2EI}(x-L),\ 0 \le L,\ y(0)=0,\ y(L)=0 \tag{7}
$$

This is a second order, linear, homogeneous differential equation with two constant coefficients. The coefficients involved in this ODE are 1 and $\frac{S}{EI}$

Here, E is the modulus of elasticity, S is the stress at the endpoints, and I is the central moment of inertia. Under a uniform load, these variables would not change and therefore this would be a constant coefficient.

3.2

$$
y'' = p(x)y' + q(x)y + r(x),\ a \le x \le b,\ y(a) = a,\ y'(a) = 0, \tag{8}
$$

$$
y'' = p(x)y' + q(x)y,\ a \le x \le b,\ y(a) = 0,\ y'(a) = 1. \tag{9}
$$

$$
y(x) = y_1(x) + Cy_2(x) \tag{10}
$$

Where $y_1(x)$ is the solution to (8) and $y_2(x)$ is the solution to (9).

3.2.1 Explain how Eq. (10) is a manifestation of the non-homogeneous principle.

The non-homogeneous principle states that a non-homogeneous differential equation can be written as the sum of the solution to the homogeneous version of the differential equation and the particular solution to the given differential equation. In equation (10), $y_2(x)$ is the solution to the homogeneous equation and $y_1(x)$ is the particular solution to the differential equation including the non-homogeneous element $r(x)$, thus this is an example of an application of the non-homogeneous principle.

3.2.2 Given that $y_1(x)$ and $y_2(x)$ are solutions to (8) and (9), respectively, find a value for the constant C so that (10) is the solution to the BVP given in (3) (from the project 2 worksheet).

$y(x) = y_1(x) + Cy_2(x)$

$y(\alpha) = y_1(\alpha) + Cy_2(\alpha)$

$\alpha = \alpha + 0$

$\alpha = \alpha$

$y(\beta) = y_1(\beta) + Cy_2(\beta)$

$$C = \frac{y(\beta) - y_1(\beta)}{y_2(\beta)} \tag{11}$$

3.3

3.3.1 Set z(x) = y'(x), and rewrite (4) as a system of first order ODEs, together with the appropriate initial conditions

$$z' = p(x)z + q(x)y + r(x),\ a \leq x \leq b,\ y(a) = a,\ y'(a) = 0, \tag{12}$$

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} z \\ p(x)z + q(x)y + r(x) \end{bmatrix}$$

$$\begin{bmatrix} y(a) \\ z(a) \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

3.3.2 Similarly, rewrite (5) as a system of first order IVPs.

$$z' = p(x)z + q(x)y,\ a \leq x \leq b,\ y(a) = 1,\ z(a) = 0, \tag{13}$$

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} z \\ p(x)z + q(x)y \end{bmatrix}$$

$$\begin{bmatrix} y(a) \\ z(a) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3.4 Modify the "INPUTS" section of the given Matlab code, linearShooting.m, to approximate the deflection of a uniform beam every 12 inches with the following characteristics:

Parameter	Value
Modulus of Elasticity (E)	5×10^7 lb/in^2
Moment of Inertia (I)	60 in^4
Beam Length (L)	360 in
Uniform Load Intensity (Q)	50 lb/in
End Stress (S)	900 lb

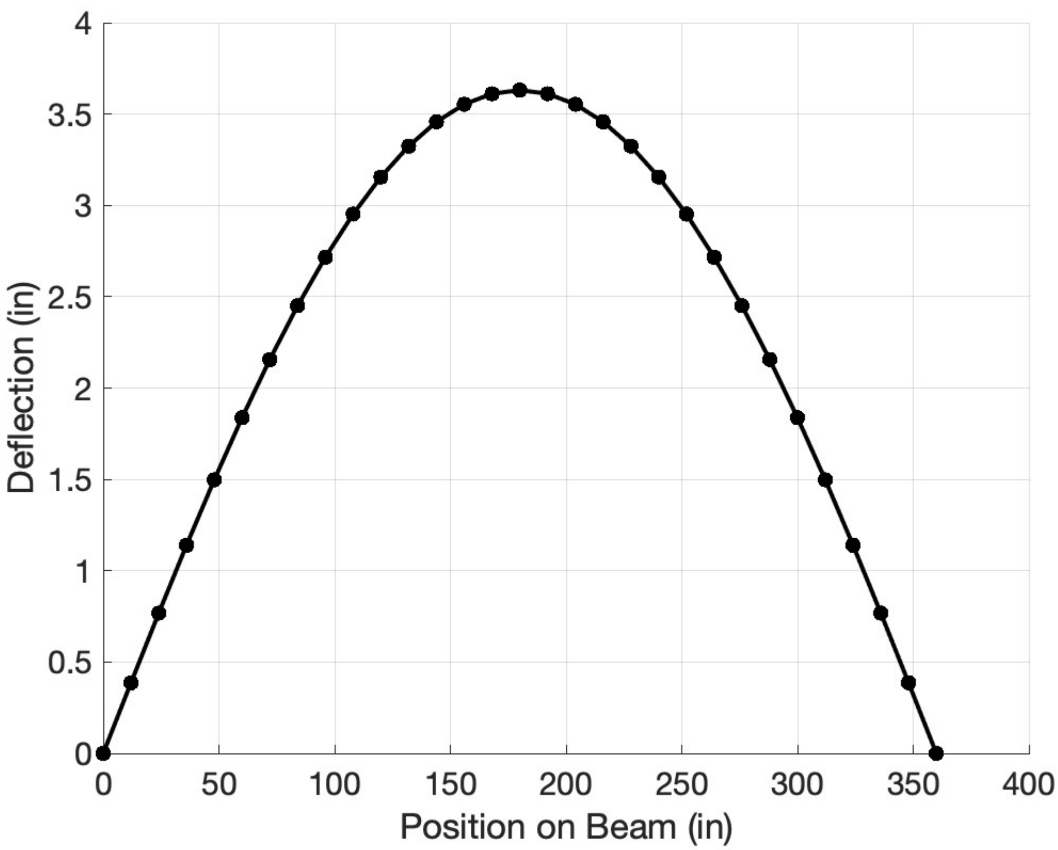

Figure 2: The deflection in inches of a uniform beam of length 360 vs the position on the beam (inches).

3.4.1 What is the maximum deflection of the beam and where does it occur? What is the average deflection of the beam?

The maximum deflection of the beam is 3.63065 inches and it occurs at the center of the beam (180 inches). The average deflection of the beam is 2.2466 inches.

3.4.2 Leaving all other parameters unchanged, find the maximum deflection of the beam and where it occurs if the length of the beam is doubled to 720. What is the average deflection of the beam?

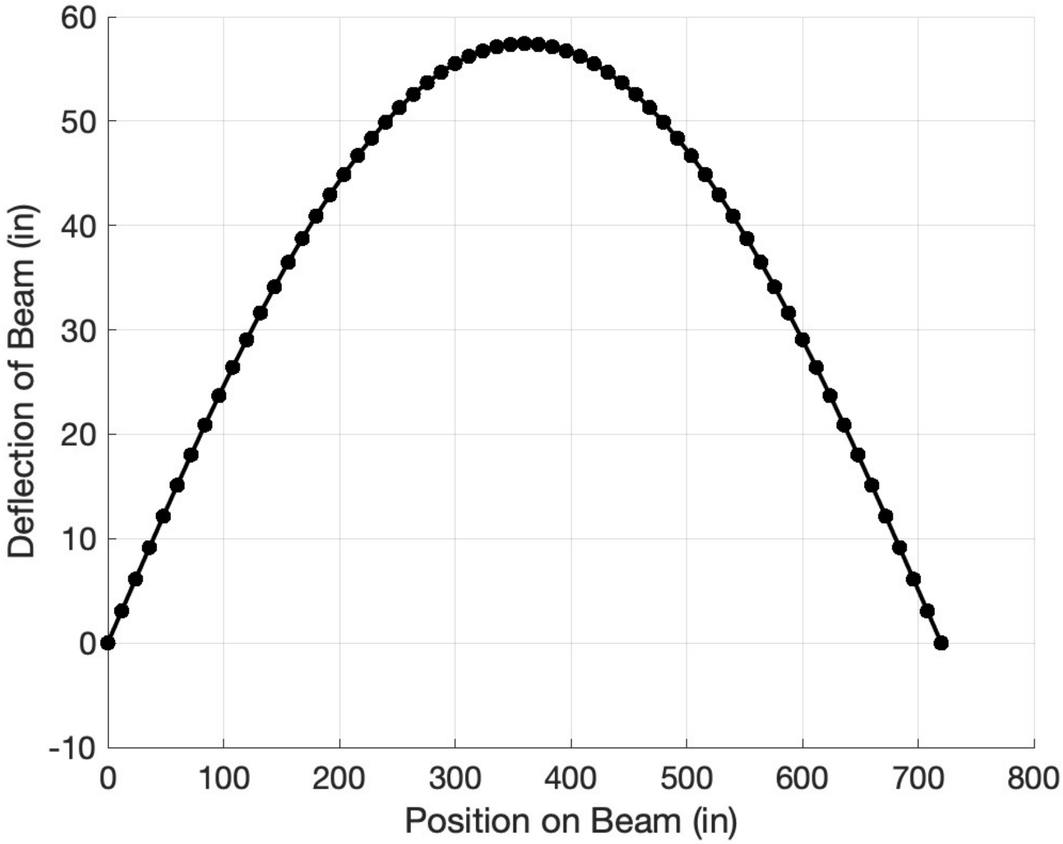

Figure 3: The deflection in inches of a uniform beam of length 720 vs the position on the beam (inches).

The maximum deflection of the beam is 57.4122 inches and it occurs at the center of the beam (360 inches). The average deflection of the beam is 36.1358 inches.

3.4.3 Do your solutions make physical sense? Explain briefly.

Yes, my solutions do make physical sense. We would expect that a beam would have the maximum deflection at its center. The graphs appear how we would expect to see a beam bend under stress when its ends are held in place. We also would expect a longer beam to have a much greater deflection. The load on each beam is 50lbs/inch and thus the load on the beam of length 360 is $50 * 360 = 18000$lbs and the load on the beam of length 720 is $50 * 720 = 36000$lbs.

4 Conclusion

This project examined two problems in civil engineering, to which we applied techniques of linear algebra and differential equations. We used linear algebra to explore the forces within a bridge. We found that the piece under the most tension was directly over the load on the bridge, which makes sense. We also found that the two external forces F_1 and F_2 were negative, meaning they were acting to hold the bridge up. We then explored what would happen if we replaced the value of 5000 N with an unknown variable F_3. Working with this matrix, we found that there was not a unique solution, but we were able to solve a for $\vec{f}$ to find an answer with infinitely many solutions. Specifically, this equation gives a relation to any force applied to the bridge at and the other forces experienced within the bridge. Given any force for F_3, you can easily find the value of other forces using the equation (5). For the bridge deflection problem, we successfully converted our second-order differential equations into a system of two first-order differential equations, which we could analyze using Matlab's ODE functions. We found that the most deflection in the bridge would be at the center and that longer beams would deflect more, which makes sense intuitively.

5 Appendix: (Code)

5.1 Code for Figures 2 and 3 in task set B problem 4

```
function linearShooting

%Solves the BVP y'' = p(x)y' + q(x)y + r(x), for a<x<b, with the boundary
%conditions y(a)=alpha and y(b)=beta.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%INPUTS.  Change these to adjust for the problem you are solving.

a = 0;  b = 720;             %the endpoints of the interval, a<x<b.
h = 12;                     %space between points on x axis.
alpha = 0;  beta = 0;       %boundary values.  y(a)=alpha, y(b)=beta.
p = @(x) 0;     %continuous function
q = @(x)  3 * 10^-7;      %positive continuous function
r = @(x)  ((50*x)/(2*5*10^7*60))*(x-720);       %continuous function

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Main part of the code.  Solves numerically the two IVP systems with
%ode45, and then combines the results to form the solution y to the BVP.

t = a:h:b;

[ ~, y1 ] = ode45( @odefun1, t, [alpha,0] );
[ ~, y2 ] = ode45( @odefun2, t,     [0,1] );

y1 = y1(:,1);  y2 = y2(:,1);

y = y1 + (beta-y1(end)) / y2(end) * y2;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Plots the numerical solution y

figure(1), clf, hold('on')
plot( t, y, 'k', 'lineWidth', 2 )
plot( t, y, 'k.', 'markerSize', 20 )
set( gca, 'fontSize', 15 )
xlabel('x'), ylabel('y(x)')
grid('on')
drawnow, hold('off')

L = mean(y);
disp(L)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%The two ODE functions that are passed into ode45

    function u = odefun1(t,y)
        u = zeros(2,1);
        u(1) = y(2);
        u(2) = p(t)*y(2) + q(t)*y(1) + r(t);
    end

    function u = odefun2(t,y)
        u = zeros(2,1);
        u(1) = y(2);
```

```
        u(2) = p(t)*y(2) + q(t)*y(1);
    end

end
```