Mathematics HL: Analysis and Approaches

Complex Number-Based Physics Simulations as a Framework for Generative Artwork

20 Pages

0.1 Introduction

Generative art is the process of programming computers to create art: hand-crafting algorithms that can produce as many visual pieces as possible inputs (Wikipedia contributors, 2023b). These algorithms can be as simple as placing random squares on a screen to detailed maps of fictional cities built on commercial flight data. A generative artist's only limit is their programming skill. For me, generative art is the perfect medium. It allows me to be creative with my passions for math and science in a way that other mediums fall short, giving me a much-needed creative outlet among my math and science studies in school.

Given the mathematical nature of computers and generative algorithms, I wondered how effective complex numbers are as the basis of a generative art simulation. My favourite simulations for my own generative artworks are physics-based, as I can creatively leverage my passion for physics. Complex numbers are uniquely suited for this, as they can represent quasi-$\mathbb{R}^2$ space, which 2-dimensional vectors also represent. Complex numbers also have arguments and moduli, equivalent to the directions and magnitudes of 2D vectors, but are more integrated into the algebra behind the system. This led me to my research question: how effective are complex numbers as the framework of a physics-based generative art simulation?

The methods and behaviours made possible through the use of complex numbers in generative art are fascinatingly different from those of the real number system. The techniques discussed in this exploration include:

- The challenges and solutions to solving complex differentials
- Complex roots
- The use of modulus and arguments in visualization
- The way complex numbers create alternative formulae for similar calculations to other number systems
- The effects of complex multiplication
- The various methods of taking real, computer-readable values for complex numbers

The definition of an effective generative art simulation will be based on the unique creative possibilities offered by the complex framework, the variability in the possible outputs of the simulation, and my own opinion on the art produced from my perspective as an artist. These criteria ensure that the idea of a complex number framework is being measured on its usefulness as a tool for other generative artists to employ in different ways to create a wide variety of artworks, and for the potential for these artworks to be visually appealing.

This exploration will also frequently display the artwork produced via these techniques. This is done to illustrate the progression of the simulation as new techniques made possible by complex numbers are introduced. Continuous improvements are made in efforts to understand the possibilities of this framework for artists.

0.2 Process

0.2.1 Foundational Math

The simulation that I will construct in this exploration is an n-body gravitational simulation of orbiting particles, based on Isaac Newton's theory of gravity (Wikipedia contributors, 2023e), defined as:

$$\vec{F} = G\frac{m_1 m_2}{\vec{d}^2}$$

Where F is the force of gravity, G is the gravitational constant, d is the distance between the two bodies and m_1, m_2 are the masses of said bodies. This model is effective for the purposes of the simulation as it presents a mathematically simple, relative to many other models, and allows for the fine-tuning of variables in an easy-to-visualize manner.

To satisfy the research question, the above definition for the force acting on a particle will be reworked to:

$$\vec{F_t} = \sum_{n=1}^{k}(\frac{m_n}{(\vec{d_n})^2})$$

Where F_t is the total force on the particle, k is the number of gravitational bodies, m_n is the mass of a given body, d_n is the distance from the particle to the given gravitational body.

> ***This Exploration's Definition of a Particle:*** *In this exploration, a particle is a physics body with no gravitational pull of its own. A particle occupies no volume and is defined by a point in space. However, particles do have inertia.*

The above equations are not yet in a form suitable for complex numbers. The conversion is as follows:

Let α be the set, with size x, of the positions of all non-particle bodies with equal mass that imparts gravitational force to the particles.

Let m be an arbitrary attraction coefficient of a given particle where $m \in \mathbb{R}$

Let k be the position of a given particle in the complex plane where $k \in \mathbb{C}$

Let A be the acceleration imparted on the given particle where $A \in \mathbb{C}$

$$\vec{F_t} = \sum_{n=1}^{k}(\frac{m_n}{(\vec{d_n})^2})$$

Becomes

$$A(k) = \sum_{n=1}^{x} \left(\frac{m}{|\alpha_n - k|^2} \cdot \frac{\alpha_n - k}{|\alpha_n - k|} \right)$$

Where the first term is the magnitude of the acceleration, and the second is the normalized direction

$$A(k) = \sum_{n=1}^{x} \left(\frac{m \cdot (\alpha_n - k)}{|\alpha_n - k|^3} \right)$$

Note that the equation now results in the acceleration of the particle, as the mass of affecting bodies are fixed and equal. In the place of the mass of the bodies, each particle is given an arbitrary attraction coefficient that has a positive linear relation to the acceleration experienced due to gravity.

Since evaluating a second-order complex differential equation, such as the above, analytically is far outside of the scope of the curriculum, numerical methods must be employed. To solve this, a numerical solution based on Euler's method is employed.

The First-Order Euler's method is as follows (Wikipedia contributors, 2023a):
Given initial values for x_0 and y_0, and that $f(x, y(x)) = y'(x)$

$$y_{n+1} = y_n + hf(x_n, y_n)$$

And

$$x_{n+1} = x_n + h$$

The above expression may be thought of as n being time, y being the position of a body, and y' the velocity of the body, defined by a function f given the position and time passed. In the case of my simulation, I took a nested approach to Euler's method.
Let n be the time passed where $n \in \mathbb{N}$
Let p, p' and p'' be the position, velocity, and acceleration of a particle on the complex plane where $p, p', p'' \in \mathbb{C}$

$$p'' = A(p)$$

Where A is the acceleration as defined above

$$\therefore p'_{n+1} = p'_n + hA(p)$$

$$\therefore p_{n+1} = p_n + h\frac{p'_{n+1} + p'_n}{2}$$

Which may be simplified to

$$p_{n+1} = p_n + hp'_n + \frac{h^2 A(p)}{2}$$

This is remarkably similar to the kinematics equation, $\Delta x = vt + \frac{at^2}{2}$ (Wikipedia contributors, 2022), differing only in that it is numerical and therefore must account for the previous position of the particle, and that it is complex rather than real. This numerical solution to the field equation works because it simulates real Newtonian physics step by step, where a physics calculation is performed under the presumption that the change in acceleration across the time step is constant, this can then be repeated an arbitrary number of times to extend the solution.

A metaphor for this could be a frog jumping in a stream, the frog is only accelerated by the forces in the current when it lands between jumps. While the frog can only interpolate the minute changes in force over the current, a rope has the perfect solution in its shape for every minute change in the forces across its length. In this metaphor, the frog is the numerical solution to the field equation, the stream, while the rope is the solution derived from the field equation itself.

This numerical technique also reminds me of how MacLaurin series are used to approximate the outputs of trigonometric, exponential and logarithmic functions.

Numerical analysis is not without its downsides, however. It suffers from many inaccuracies and often requires millions of repetitions to form a solution (Wikipedia contributors, 2023a). These errors are due to the nature of numerical analysis, a method prone to the propagation of truncation errors and errors caused by the discretization of the time steps when in reality they are infinitely small, such as with the frog in the stream (Wikipedia contributors, 2024). However, in the case of generative artwork, these factors are not of concern. Modern computers can run billions of calculations in a second, and art as a whole rarely mandates mathematically perfect realism. Thus, with this foundation, a simulation may be constructed.

0.2.2 Simulation

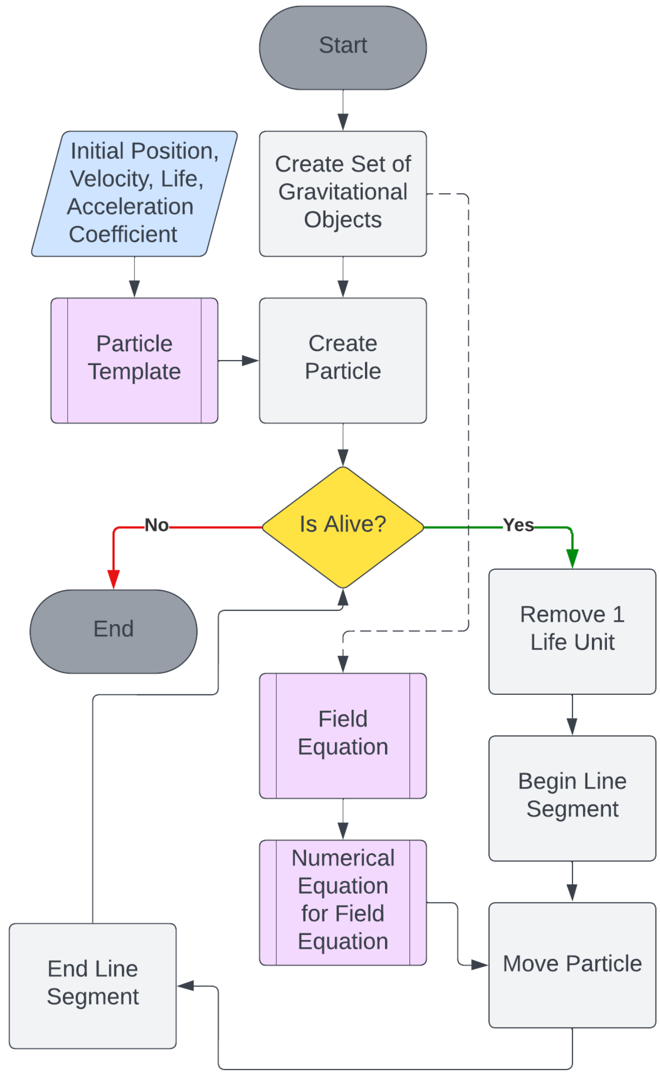

Figure 1: High-level program structure

The structure of the program is described in Figure 1. The pink rectangles are the functions that we can modify to fundamentally change the simulation. These include the field equation and the equations that numerically adjust a particle's position and velocity based on the field equation. Finally, the template of the particle and the inputs themselves may be adjusted.

Each particle has five basic properties:

- Position
- Velocity

- Acceleration coefficient
- Current life
- Maximum lifespan

These properties form the basic needs of this particle simulation. As shown in Figure 1. a particle-dependent constant, “Maximum Lifespan,” is set as the maximum lifespan of a particle, and an internal “Life” variable is set to this value upon particle initialization. At each timestep, this value decreases by one until it reaches zero and the particle is deleted.

Given this structure, the first simulation may be constructed, with the results shown in Figure 2. There is one gravitational body placed at 1+0i and a particle at -20+10i. The particle has an initial velocity of 0+0i, a maximum lifespan of 100,000 timesteps, and an acceleration coefficient of 100.0. Note that the gravitational body is indicated by a red circle for clarity.

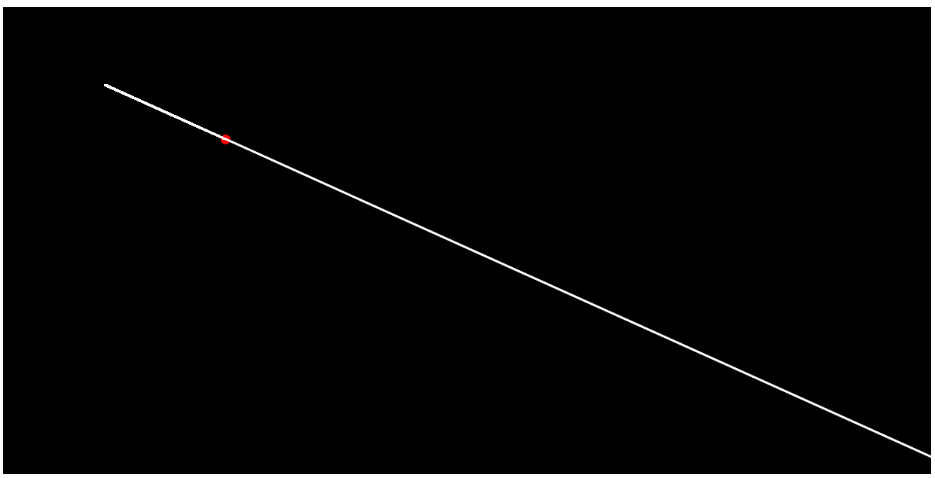

Figure 2: Note that empty space was cropped for clarity

The particle is travelling in a straight line, following the gravitational field outlined by the field function. While this piece may be considered abstract art, I do not believe it comes close to showing the advantages of the complex framework in generative art.

On the side of physics simulation, a disadvantage of numerical analysis is revealed. The particle’s acceleration reaches extraordinary values when near the gravitational body, and it increases its velocity so much that it flies off through the body and off the image. A more accurate result would be the particle being erased as it collides with the body. This may be done by creating a case in the simulation that the particle is destroyed, ie. its remaining life is set to zero, when the modulus of its acceleration is over an arbitrary number as the acceleration is very large when a particle is near the body.

The particle may also be given an initial, arbitrary, velocity to hopefully create a curved, dynamic line that is more interesting to look at.

Figure 3 has the same simulation parameters as those in Figure 2, but the initial velocity is set to 4+1i and the particle is deleted when the modulus of its acceleration is over 3000.

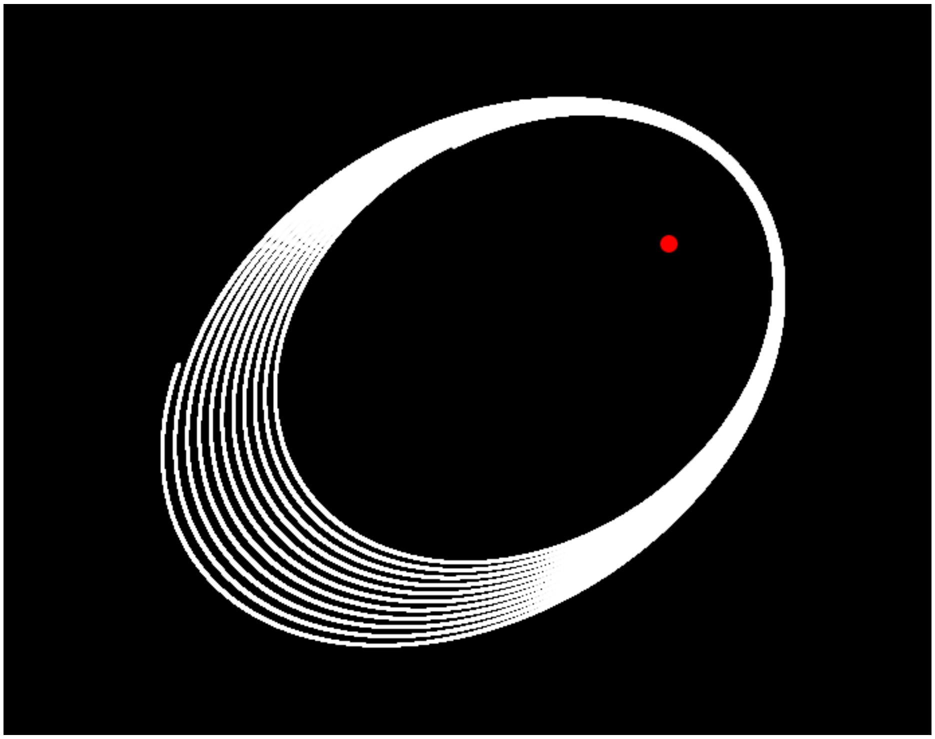

Figure 3: Note that empty space was cropped for clarity

The simulation is now producing artwork that strongly resembles physical phenomena, and the orbiting behaviour is much more interesting to look at compared to a straight line. However, the simulation does not yet visually show any advantage or uniqueness when compared to a vector-based simulation. For this, I will be constructing a new method of producing the positions of the gravitational bodies: complex roots.

When $x^n = z$ and where z is a constant, $z \in \mathbb{C}$ and $n \in \mathbb{Z}^+$ there will be n solutions for x. There are rare edge cases where there are fewer, such as when $z = 0$, but they are not pertinent. In many cases, some of the solutions are imaginary. This property of roots is ideal for my simulation and represents one of the key advantages of the complex-numbered foundation of my simulation: the unique properties of complex numbers.

De Moivre's Theorem will be used to find the roots of a given input, and those points will form the set of positions for the gravitational bodies in the simulation.

De Moivre's Theorem:

$$\sqrt[n]{z} = \sqrt[n]{|z|}(\cos(\frac{\theta + 2\pi k}{n}) + i\sin(\frac{\theta + 2\pi k}{n}))$$

Where n is the desired root to be taken, z is the number one would like to find the root of and k is the set of numbers from 0 to $n-1$ where $z \in \mathbb{C}$ and $n \in \mathbb{Z}^{+}$

$$k = 0, 1, 2, ..., (n-1)$$

$$\alpha = \sqrt[n]{z}$$

$$|\alpha| = n$$

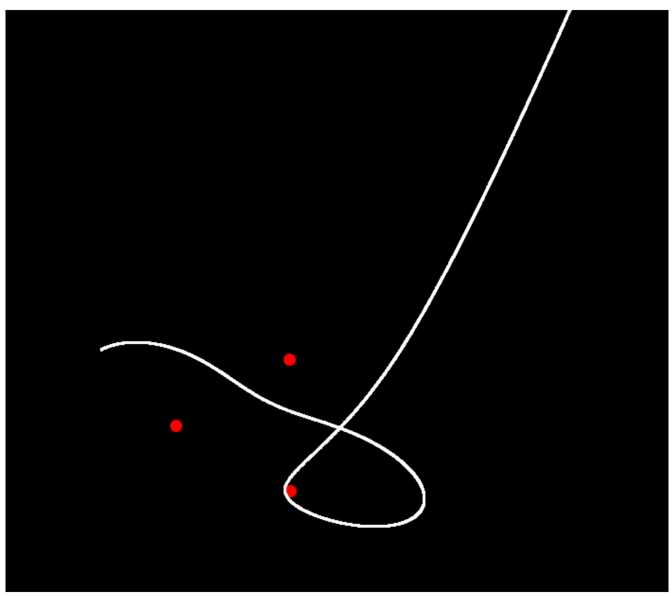

Figure 4: Note that empty space was cropped for clarity

Multiple roots allow for significant complexity and variation in the image produced. While the line is unlike an orbit, it is dynamic and complex, drawing the attention of the viewer. Because the simulation is run on a computer, many particles can be rendered at the same time, and show the uniqueness of the complex field equation. In the following image, everything is kept the same, except there are 100 particles placed between the points $-20 + 10i$ and $-20 + 11i$, the stroke weight is reduced, and the red circles are removed.

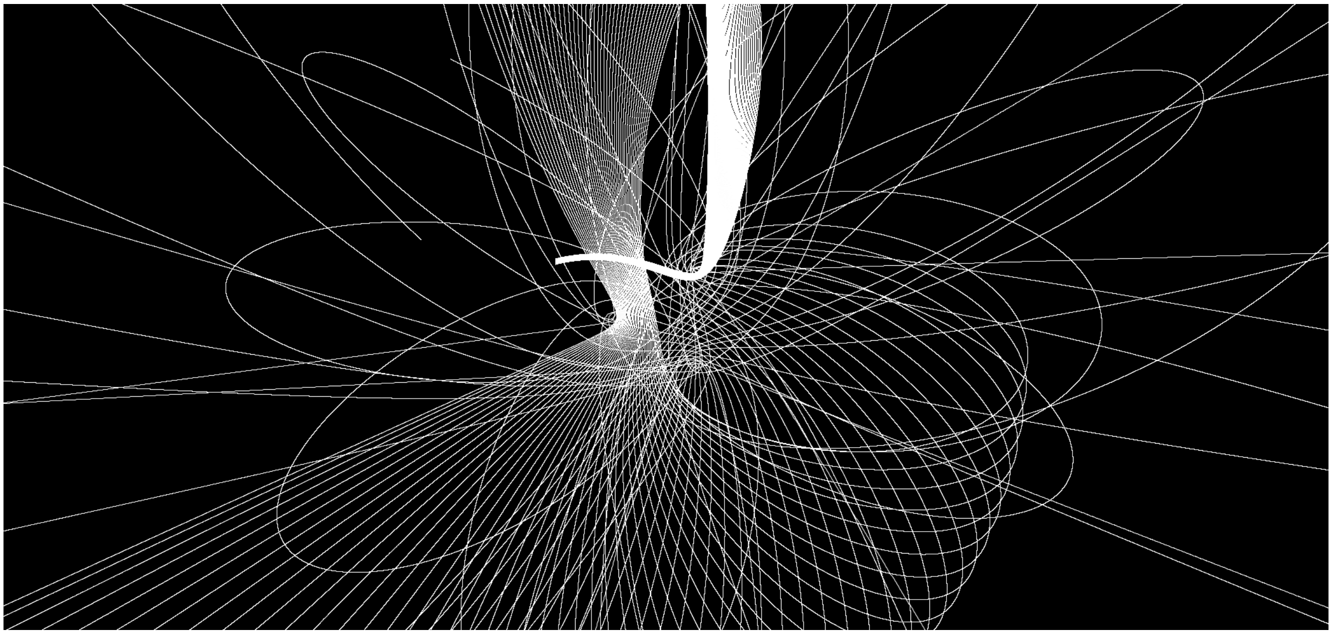

Figure 5: More particles

Figure 5 shows how visually interesting the complex framework can be because of its chaotic nature, each line follows a different path and the paths come together to make greater structures in the piece. Chaotic systems (Wikipedia contributors, 2023c) are defined by outputs that vary dramatically and highly unpredictably from a tiny variation in inputs, a common analogy is weather systems, where a tornado in Africa could be caused by the air currents of a butterfly in South America. Equally so, the same flap could prevent that tornado. Chaotic behaviour is seen in the simulation becuase the small range of values ($-20 + 10i$ to $-20 + 11i$) produces a wide variety of paths that could not have been predicted. In the next iteration, the number of particles is increased to 1000, and a gradient colour is applied such that the particles closer to $-20 + 11i$, in the beginning, are brighter than the others. This is done to create a sort of identification of the paths so viewers can see how chaotic the system can be.

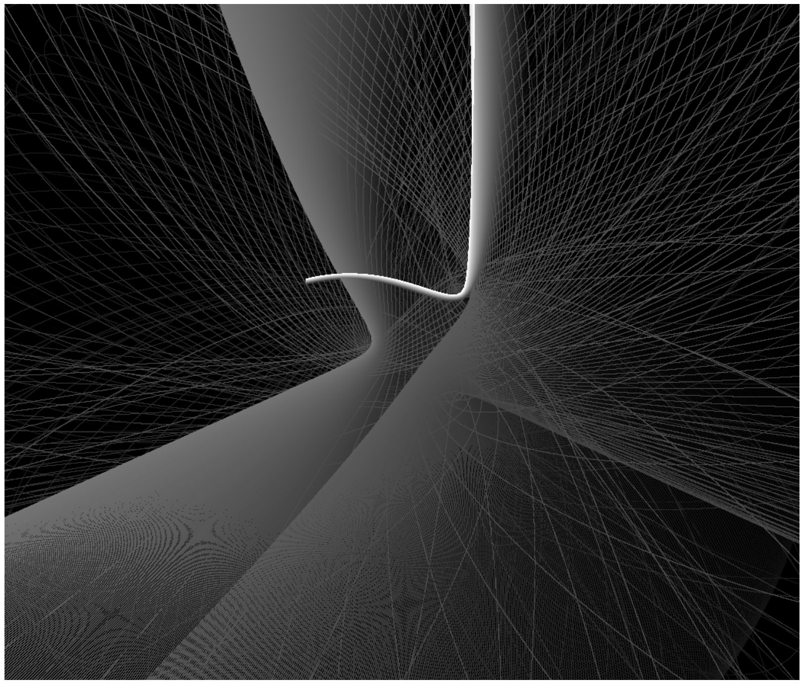

Figure 6: Note that the image is cropped for clarity

A very intriguing gradient-like effect is produced by the method, and it is visually appealing by being more gentle on the eyes. However, it highlights the fact that many particles are flying off the screen, never to be seen again. While it is possible to change the scale of the screen, much detail would be lost. Another solution is adding a drag-like force to the particles, that accelerates them in the opposite direction of their velocities, and keeps them closer to the centre of the screen.

Such an equation would look like this:

$$D = \frac{v}{|v|} \cdot \frac{|v|^2 \cdot C}{2}$$

Where D is the acceleration due to drag, $\frac{v}{|v|}$ is the normalized complex value indicating the direction of the acceleration and the second part of the product is the magnitude of the acceleration, with C being the drag coefficient of the particle. $D \in \mathbb{C}$, $C \in \mathbb{R}$

Simplified:

$$D = v(\frac{|v| \cdot C}{2})$$

This can be combined with the acceleration for the field equation to give:

$$A_{total} = A_{field} - D$$

Where $A_{total}, A_{field}, D \in \mathbb{C}$

This new acceleration formula may now be trivially integrated into the simulation. This is the result once the drag acceleration is added with a drag coefficient of 0.01. Note that the gradient effect is still in place.

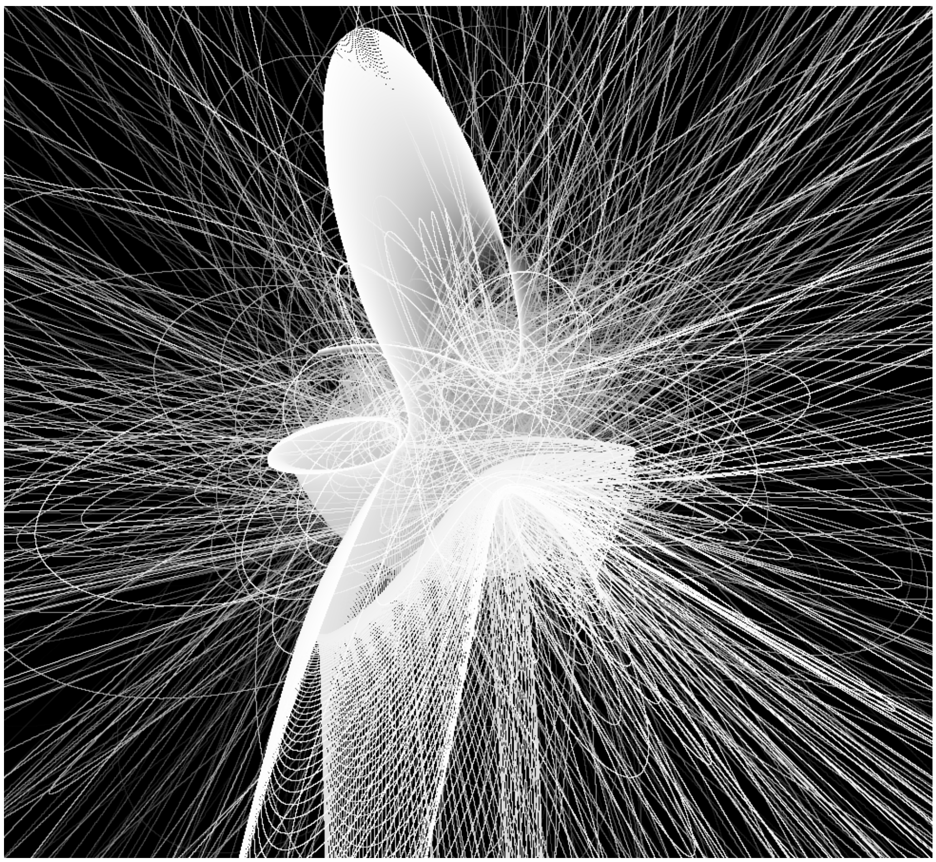

Figure 7: Note that the image is cropped for clarity

While the results have incredible complexity and detail, it is distracting and busy. It would be easier to see the points of high visual interest, when the paths are orbiting the centre, if the paths going straight away or straight towards it are dimmed. Another unique feature of the complex number system allows this to be easily accomplished. When the particles are travelling away or towards the centre, their velocities are more co-linear with the gravitational field lines. While the areas of interest are often travelling more perpendicular to the field. While in $\mathbb{R}^2$ space, the magnitude of a cross product of the field and velocity would be used, a simpler formula is found in complex algebra (Bogomolny, n.d.):

$$|z \times w| = \frac{\overline{z}w - z\overline{w}}{2}$$

This may be proven by first letting z and w be repressed in Euler's form:

$$z = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$w = r_2(\cos\theta_2 + i\sin\theta_2)$$

The magnitude of the cross-product is equivalent to:

$$z \times w = |z||w|\sin(arg(z) - arg(w))$$

Or:

$$z \times w = r_1 r_2 \sin(\theta_1 - \theta_2)$$

Expand and simplify:

$$|z \times w| = \frac{\overline{z}w - z\overline{w}}{2}$$

$$z \times w = \frac{r_1(\cos\theta_1 - i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2) - r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 - i\sin\theta_2)}{2}$$

$$z \times w = \tfrac{r_1 r_2}{2}((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(-\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)) - (\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 - \cos\theta_2\sin\theta_1)))$$

$$z \times w = \frac{r_1 r_2}{2} 2i(-\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)$$

$$z \times w = r_1 r_2 i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)$$

$$z \times w = r_1 r_2 i\sin(\theta_1 - \theta_2)$$

Note that $r_1 r_2 i\sin(\theta_1 - \theta_2) \neq r_1(\cos\theta_1 + i\sin\theta_1)$, so really $i|z \times w| = \frac{\overline{z}w - z\overline{w}}{2}$. However, in some cases, it is worth noting that the imaginary value is more useful. Nonetheless, a simple function will still convert it into a positive real number:

$$|z \times w| = |i(\frac{\overline{z}w - z\overline{w}}{2})|$$

The equation may now be added into the program, with a and v being the acceleration and velocity of a particle where $a, v \in \mathbb{C}$. However, the following equation allows for a smoother, quasi-normalized look to the lines:

$$opacity = (1 - (|v \times a| + 1)^{-1}) \cdot 100\%$$

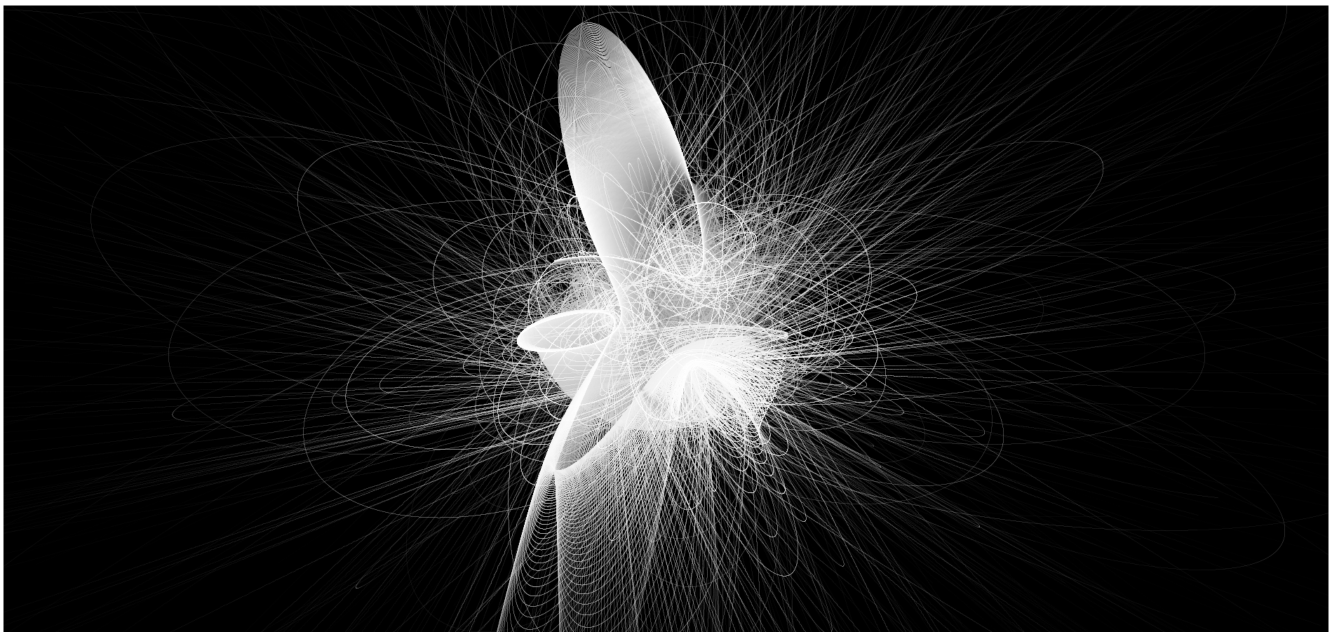

Figure 8:

While this is a significant improvement, the cross-product linearly correlates to the product of the moduli of the inputs, resulting in dimmer lines away from the center, where the gravitational field and by extension velocities are largest. This does not improve the image as much as I hoped, as the outer paths are dim to the point of practical invisibility, and the center structure is still too busy. This may be corrected by dividing the original cross-product by twice the product of the moduli of the acceleration and velocity variables. This is shown as:

$$opacity = (1 - (\frac{|v \times a|}{2|v||a|} + 1)^{-1}) \cdot 100\%$$

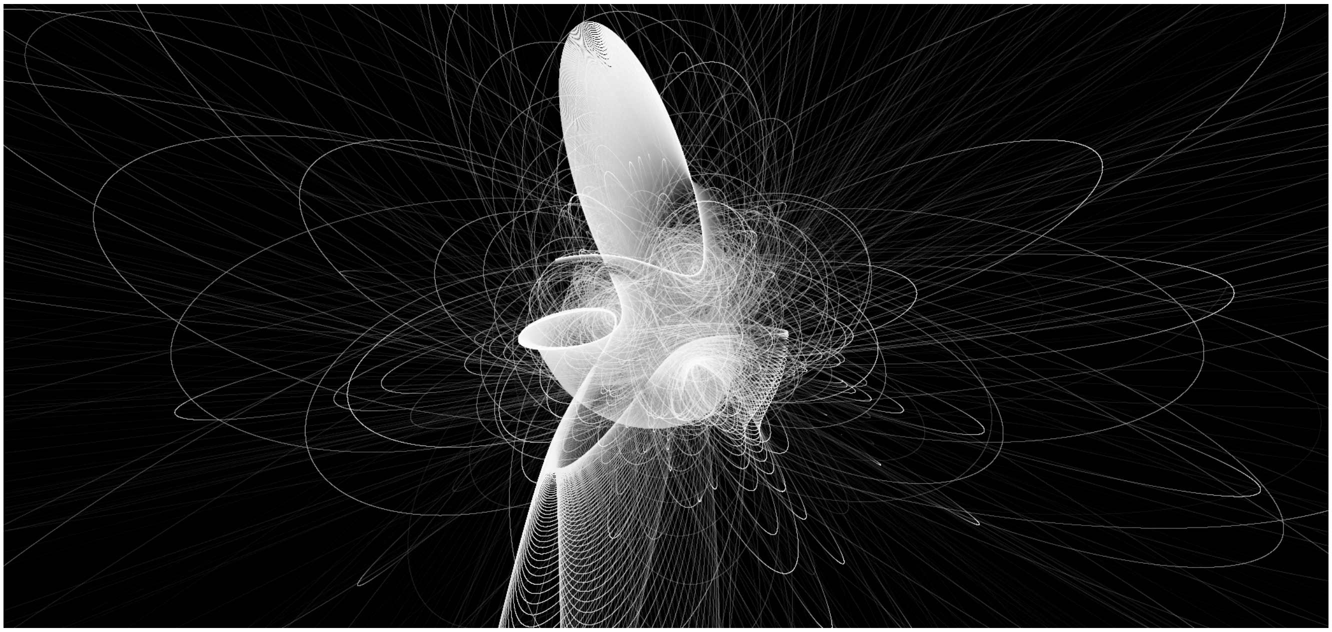

Figure 9:

This is a significant improvement, every part of the image is visible, yet the chaotic character of the inner orbits is satisfyingly salient. I am surprised that I was able to achieve this effect with so few modifications, and it shows the power of complex numbers used in this way.

Now, the differences in opacity can be extended to changes in hue. This is accomplished via a linear mapping function:

Let $f(i, a, b, x, y)$ be the linearly-mapped output of the function
Let i be the input value that is to be mapped
Let a, b be the lower and upper bounds of the domain of the input value, respectively
Let x, y be the lower and upper bounds of the intended range of the output, respectively. Where $i, a, b, x, y \in \mathbb{R}$

Illustrated visually:

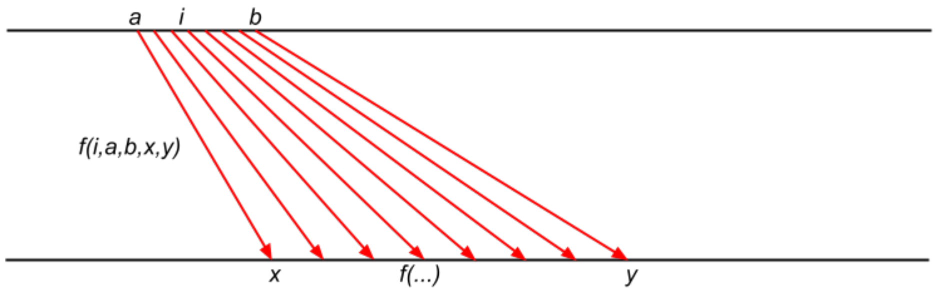

Figure 10: A graphical model of the mapping function

First, i is normalized relative to the values of a and b:

$$\hat{i} = \frac{i - a}{b - a}$$

Note that $\hat{i}$ has a range of $[0, 1]$, thus it is normalized.

The mapped value is now given by:

$$f(i, a, b, x, y) = \hat{i}(y - x) + x$$

Substituting $\frac{i-a}{b-a}$ for $\hat{i}$:

$$f(i, a, b, x, y) = \frac{i - a}{b - a}(y - x) + x$$

In this case, $f(i, a, b, x, y)$ may become $hue = f(opacity, 0, 1, x, y)$, where x and y are two hue values in the Hue-Saturation-Brightness or HSB colour space. A similar function is used to evenly space the particles along their initial range of positions, inducing their initial chaotic behaviour.

Another unique feature of complex numbers, integral to their usefulness as tools of real-world physics and in Euler's formula, is the fact that many complex operations result in a pseudo-rotation of the complex value about the origin. This is another way that chaos theory emerges in the simulation. Each point has its initial velocity made equivalent to the product of its initial position and a complex number.

With these techniques combined, the hue of the lines is set to map between blue-green and purple, and the initial velocities of the particles are set to be $0 + \frac{1}{10}i$ times their initial position. For visual purposes, the initial range of positions of the particles is from $-30 + 0i$ to $-30 + 20i$ and the number of particles is reduced to 200:

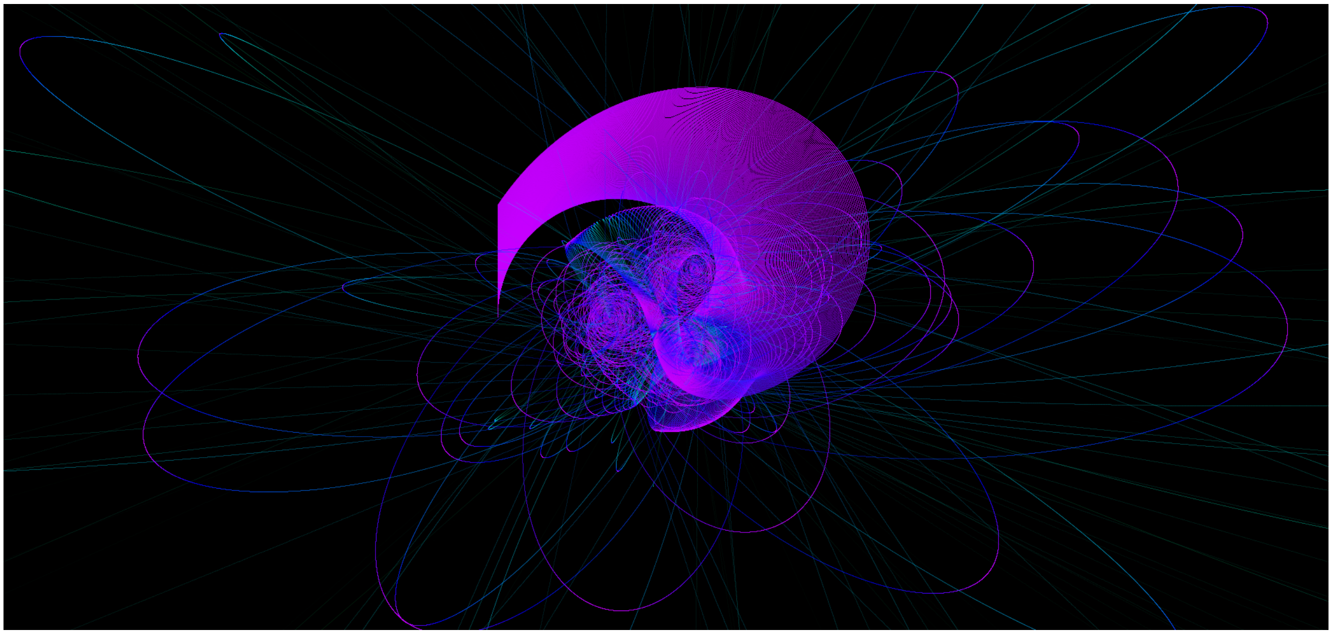

Figure 11:

With colour, even more creative possibilities emerge for the simulation. The dynamic hues also highlight the abstract interference patterns of the pixelated lines. These patterns are very similar to Moiré patterns, a phenomena "produced when a partially opaque ruled pattern with transparent gaps is overlaid on another similar pattern" (Wikipedia contributors, 2023d). These patterns create complexity from the addition and interference of two simple patterns and lend additional visual interest to the images produced by this simulation.

0.2.3 More Results

While the same base values have been used for demonstrative purposes throughout this exploration, the real power of the complex numbers powering this simulation lies in the many degrees of freedom one has with their choice of what values to pick, and how the systemic chaos of the system allows for infinitely many distinct outputs. Shown are several different images produced by the simulation as a result of changing the techniques and numbers shown previously in this exploration.

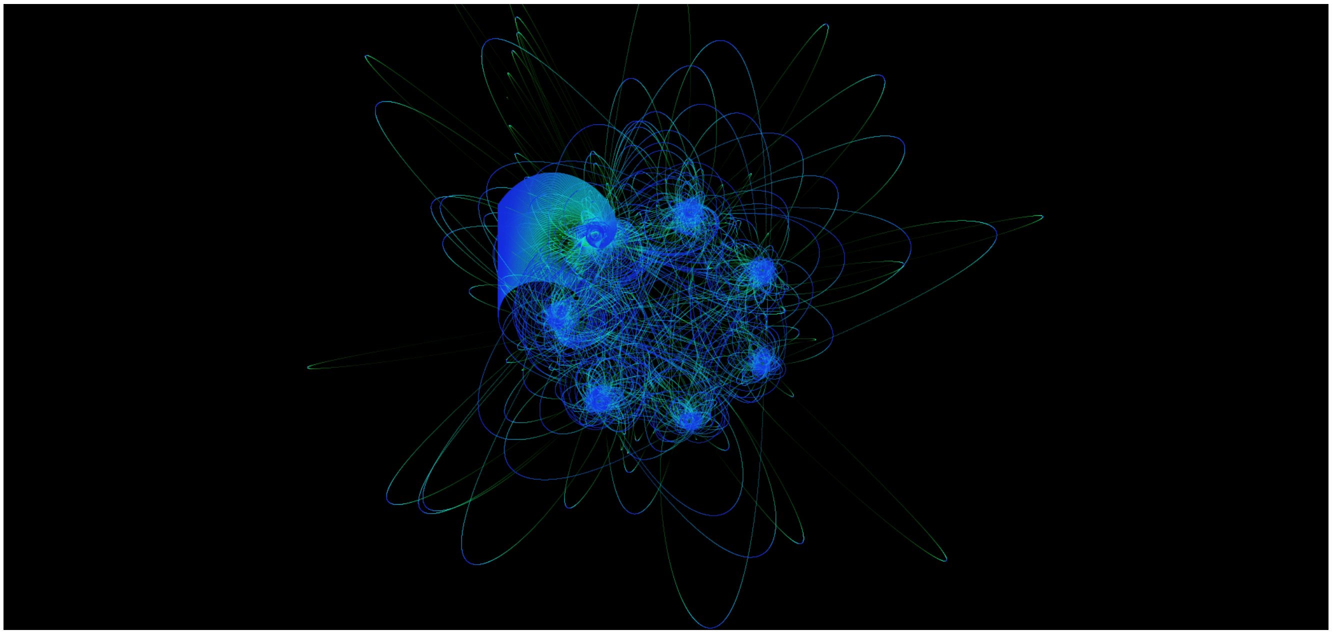

Figure 12:

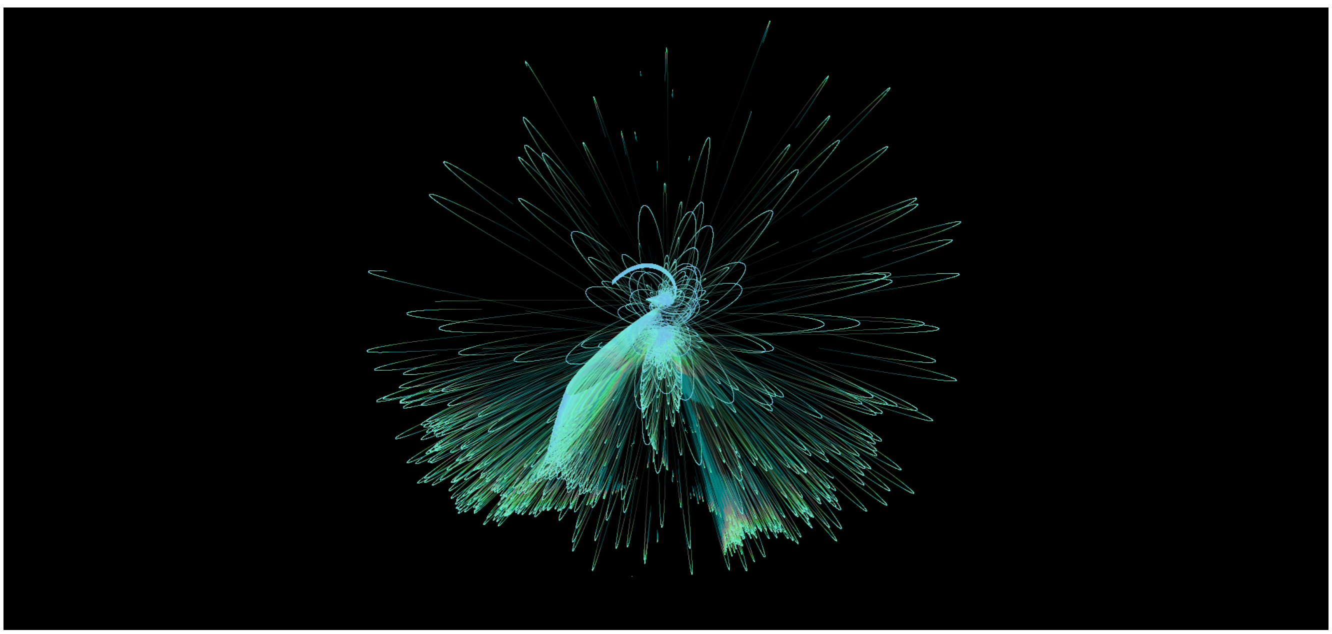

Figure 13:

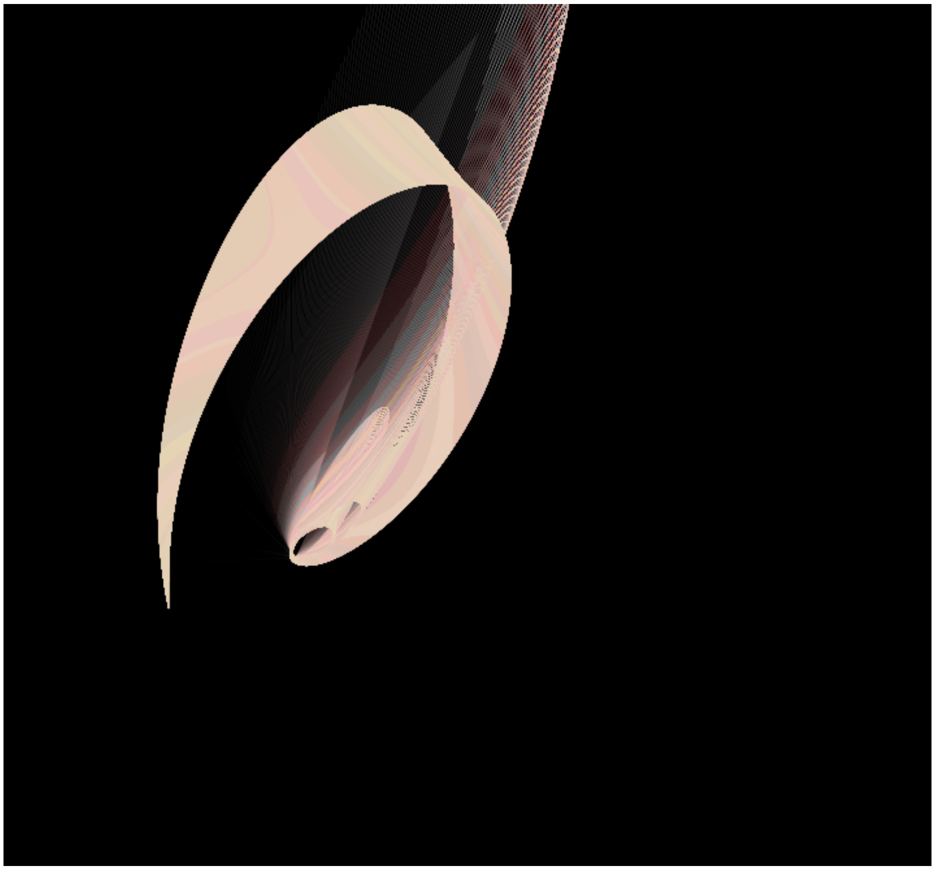

Figure 14: Note that the image is cropped for clarity

0.3 Conclusion

Overall, this exploration has shown that complex numbers can form a powerful and unique foundation for generative artworks, especially in the space of physics simulations often dominated $\mathbb{R}^2$ vector systems. The unique features of complex numbers, their roots, multiplication, etc., are seen as mathematical solutions to creative problems, rather than limitations. The simulation constructed in this exploration alone shows the diverse and beautiful, abstract worlds of particles and physics made only possible by a simple change in the way we represent numbers.

Mathematically speaking, this system is significantly purer when compared to an equivalent vector-based system, as everything can be represented as equations devoid of tall brackets and separated variables. It is essentially a point moving on a number line, something many young children would understand, with an added dimension that never suspends the purity of the equations and calculus beyond view.

Complex numbers fully fulfill my criteria for an excellent generative art tool. I explored the wide variety of creative opportunities made available by a complex framework, including physics simulations, complex roots, drag simulations and an opacity algorithm that creates fascinating colours and shades in the artwork. Even from my single simulation, I was able to create a plethora of distinct pieces by simply altering the starting variables of the simulation. It also fulfills my final criterion, that the art produced has the potential for beauty from my perspective as an artist. For these reasons, the answer to the research question is that complex numbers are an extremely effective framework for creating art in a physics-based generative art simulation.

0.3.1 Improvements

While the overall exploration was a success, there are numerous improvements and further discoveries to be made within this approach.

Comparatively simple avenues of further exploration include finding new techniques for representing different properties of the particles, such as the plotted lines becoming thinner as the particles age. Interesting images could be produced by overlaying two species of particles with different properties as they create vastly different patterns around the same gravitational bodies.

But the improvements can also become much more involved to give justice to the possibilities this approach offers. While for the scope of this exploration, a numerical solution to the foundational equations of motion worked out fine creatively, an analytical solution would have been faster to compute and have created even more opportunities for the quirks and benefits of complex numbers to emerge. This means that the paths of the particles could have easily been computed in a manner where each particle is moving simultaneously, rather than this exploration's approach of simulating each particle in its entirety before simulating the next, to save memory. This approach would allow a real-time model of the particles, bringing an element of time to the pieces and enhancing the viewer's experience. A viewer could interact with the artwork by changing the position of the gravitational bodies, watching the particles' motion evolve in real time.

Even in still images, a simulation may be formed that draws a polygon out of select particles and creates an amorphous, unpredictable structure similar to those produced in Anders Hoff's "Sand Spline" project:

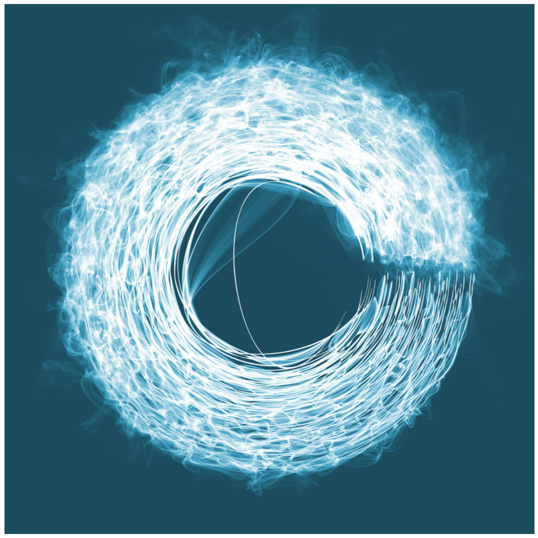

Figure 15: Inconvergent, 2015

It would also be worthwhile to use complex numbers in other simulations less physical in nature. Perhaps a piece created by passing the points of the shape existing in the complex plane through a complex function, and then plotting lines between transformed and original points.

Or, in a similar vein, one could recursively pass the points through the function and plot the modulus of the points after an arbitrary number of recursions, similar to this rendition of the Mandelbrot set by Wolfgang Beyer:

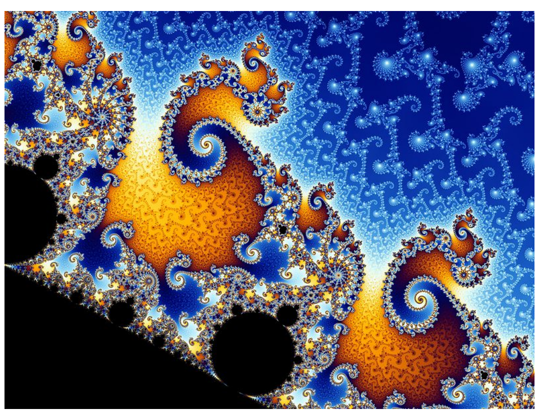

Figure 16: Beyer, 2005

In the end, this exploration was a resounding success in proving the effectiveness of a novel approach to generative art: complex numbers. The initial artworks produced in this exploration give mere hints of what is possible with this approach, while the suggested improvements pave the way for future, more exciting ways to fully employ the effectiveness of this approach.

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