1 Introduction

The class of models for aggregate relational data that we consider all involve modeling responses from a negative binomial distribution, with the mean μ_{ik} equal to the number of expected response for individual i about knowing a number of people in a group of interest $\mathcal{G}_k$.

$$\begin{aligned} y_{ik} &\sim \text{NegBin}(\omega_k \mu_{ik}, \omega_k) \\ E(y_{ik}) &= \mu_{ik} \\ Var(y_{ik}) &= \mu_{ik} + \frac{\mu_{ik}}{\omega_k} \end{aligned} \tag{1}$$

1.1 Random Mixing

The most basic model treats this expectation as simply d_i, the degree of individual i, multiplied by the proportion of the population that is in $\mathcal{G}_k$. Letting $\delta_{jk} = \mathbb{I}\{j \in \mathcal{G}_k\}$, we can derive this expression as follows:

$$\begin{aligned} \mu_{ik} &= \sum_{j=1}^{d_i} \mathbb{E}[\delta_{jk} | i \to j] \\ &= \sum_{j=1}^{d_i} \mathbb{P}(j \in \mathcal{G}_k | i \to j) \\ &= d_i \mathbb{P}(j \in \mathcal{G}_k | i \to j) \\ &= d_i \mathbb{P}(j \in \mathcal{G}_k) \\ &= d_i \left(\frac{N_k}{N} \right) \end{aligned} \tag{2}$$

1.2 Non-Random Age and Gender Mixing

If we believe that egos of certain ages and genders mix differently with alters of other ages and genders, then, we can model non-random age and gender mixing. Suppose the each individual i belongs to some age category a_i (e.g. 0-17, 18-24, etc.).

$$\begin{aligned}
\mu_{ik} &= \sum_{j=1}^{d_i} \mathbb{P}(j \in \mathcal{G}_k | a_i, g_i, i \to j) \\
&= d_i \mathbb{P}(j \in \mathcal{G}_k | a_i, g_i, i \to j) \\
&= d_i \sum_{a_j, g_j} \mathbb{P}(j \in \mathcal{G}_k, a_j, g_j | a_i, g_i, i \to j) \\
&= d_i \sum_{a_j, g_j} \mathbb{P}(j \in \mathcal{G}_k | a_j, g_j, a_i, g_i, i \to j) p(a_j, g_j | a_i, g_i, i \to j) \\
&= d_i \sum_{a_j, g_j} \mathbb{P}(j \in \mathcal{G}_k | a_j, g_j) \rho_{(a_i, g_i)(a_j, g_j)} \\
&= d_i \sum_{a_j, g_j} \rho_{(a_i, g_i)(a_j, g_j)} \mathbb{P}(j \in \mathcal{G}_k | a_j, g_j) \\
&= d_i \sum_{a_j, g_j} \rho_{(a_i, g_i)(a_j, g_j)} \left(\frac{N_{k, a_j, g_j}}{N_{a_j, g_j}} \right)
\end{aligned} \tag{3}$$

Here $\rho_{(a_i, g_i)(a_j, g_j)}$ is then a latent variable that can be inferred as the mixing rate between egos of age category a_i, gender g_i and alters of age category a_j, gender g_j.

1.3 Issues

In our experiments, the non-random age-gender mixing model suffered from bias/variance and identifiability issues because the mixing rate parameters lacked constraints (other than summing to 1). In an effort to resolve this issue, we now propose a model with more structure and fewer parameters.

2 Kernel Model

In this new approach, we first assume that age is continuous ($a_i \in (-\infty, \infty)$) rather than just binning age into categories. This allows us to model the mixing rate of an ego with age a_i with an alter of age a_j as a Gaussian kernel defined smoothly over all possible a_j.

Additionally, we also model the alter degree d_j for the first time. Interestingly, this modeling yields different results depending on whether we set up the model from the perspective of the alter or from the perspective of the ego (as we've done in all previous models above).

2.1 Ego Perpsective

Modifying the derivation from 1.3, the mean can be derived as follows:

$$
\begin{aligned}
\mu_{ik} &= d_i \mathbb{P}(j \in \mathcal{G}_k | a_i, g_i, i \to j) \qquad (4)\\
&= d_i \sum_{g_j} \int_{a_j} \int_{d_j} \mathbb{P}(j \in \mathcal{G}_k, a_j, g_j, d_j | a_i, g_i, i \to j) dd_j da_j \\
&= d_i \sum_{g_j} \int_{a_j} \int_{d_j} \mathbb{P}(j \in \mathcal{G}_k | a_j, g_j, d_j, a_i, g_i, i \to j) p(a_j, g_j, d_j | a_i, g_i, i \to j) dd_j da_j \\
&= d_i \sum_{g_j} \int_{a_j} \int_{d_j} \mathbb{P}(j \in \mathcal{G}_k | a_j, g_j) p(d_j | a_j, g_j, a_i, g_i, i \to j) p(a_j | g_j, a_i, g_i, i \to j) p(g_j | a_i, g_i, i \to j) dd_j da_j \\
&= d_i \sum_{g_j} \int_{a_j} \mathbb{P}(j \in \mathcal{G}_k | a_j, g_j) \left(\int_{d_j} p(d_j | a_j, g_j, a_i, g_i, i \to j) dd_j \right) p(a_j | g_j, a_i, g_i, i \to j) p(g_j | g_i, i \to j) da_j \\
&= d_i \sum_{g_j} \int_{a_j} \mathbb{P}(j \in \mathcal{G}_k | a_j, g_j) p(a_j | g_j, a_i, g_i, i \to j) p(g_j | g_i, i \to j) da_j \\
&= d_i \sum_{g_j} \int_{a_j} \left(\frac{\int_a \mathbb{P}(j \in \mathcal{G}_k | a, g_j) da}{\int_a \mathbb{P}(j \in \mathcal{G}_k | a, g_j) da} \right) \mathbb{P}(j \in \mathcal{G}_k | a_j, g_j) p(a_j | a_i, g_i, g_j, i \to j) p(g_j | g_i, i \to j) da_j \\
&= d_i \sum_{g_j} p(g_j | g_i, i \to j) \left(\int_a \mathbb{P}(j \in \mathcal{G}_k | a, g_j) da \right) \int_{a_j} \left(\frac{\mathbb{P}(j \in \mathcal{G}_k | a_j, g_j)}{\int_a \mathbb{P}(j \in \mathcal{G}_k | a, g_j) da} \right) p(a_j | a_i, g_i, g_j, i \to j) da_j \\
&\approx d_i \sum_{g_j} \rho_{g_i g_j} \left(\sum_a \mathbb{P}(j \in \mathcal{G}_k | a, g_j) \right) \int_{a_j} \mathcal{N}(a_j | \mu_{g_j,k}, \sigma^2_{g_j,k}) \mathcal{N}(a_j | a_i, \lambda_{g_i g_j}) da_j \\
&= d_i \sum_{g_j} \rho_{g_i g_j} \left(\sum_{a_j} \frac{N_{k,a_j,g_j}}{N_{a_j,g_j}} \right) \frac{e^{-\frac{(a_i - \mu_{g_j,k})^2}{2(\lambda_{g_i g_j} + \sigma^2_{g_j,k})}}}{\sqrt{2\pi(\lambda_{g_i g_j} + \sigma^2_{g_j,k})}}
\end{aligned}
$$

Here $\rho_{g_i g_j}$ is the same latent variable as in 1.2 that can be inferred as the mixing rate between egos of gender g_i and alters of gender g_j. Additionally, $\lambda_{g_i g_j}$ is a latent variable that can be inferred as the kernel bandwidth of the age mixing kernel. Essentially, small values of $\lambda_{g_i g_j}$ indicate that egos of gender g_i only know alters of gender g_j that are close to the ego in age (whereas larger values indicate the egos know alters of a wide range of ages, not necessarily just those close in age).

Additionally, $\mu_{g_j,k}$ and $\sigma^2_{g_j,k}$ are just estimated from the population data about group G_k. These values are analogous (though not exactly equal) to the mean and standard deviation of the ages of alters in group G_k with gender g_j.

3 Simulation

3.1 Data

We simulate responses to questions about 12 names using estimated age means/variances (for each name), degree regression coefficients, simulated respondent degrees, name overdispersions, kernel lengthscales, and gender mixing rates in an attempt to determine whether our model can recover the parameters.

$$\beta = [6.256, -0.005, -3] \qquad \eta = 0.5 \qquad d_j \sim \log\mathcal{N}(\beta_1 + \beta_2 g_j - e^{\beta_3}(a_j - \bar{a})^2, \eta^2)$$

Var	Linda	Jen.	Karen	Kim.	Emily	Steph.	Mark	Jacob	Kevin	Kyle	Adam	Bruce
μ_k	63.3	37.4	56.1	39.8	28.0	35.6	49.2	22.5	38.8	25.6	31.0	62.4
σ_k	10.4	10.6	13.9	13.0	22.9	14.3	14.9	18.0	16.2	10.8	16.2	16.7
ω_k	4.23	8.42	7.65	3.83	9.79	10.91	5.01	2.80	2.22	2.60	12.69	4.95

$$\lambda = \begin{pmatrix} \lambda_{FF} & \lambda_{FM} \\ \lambda_{MF} & \lambda_{MM} \end{pmatrix} = \begin{pmatrix} 225 & 144 \\ 100 & 256 \end{pmatrix} \qquad \rho = \begin{pmatrix} \rho_{FF} & \rho_{FM} \\ \rho_{MF} & \rho_{MM} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.4 \\ 0.45 & 0.55 \end{pmatrix}$$

3.2 Priors

The following priors were used:

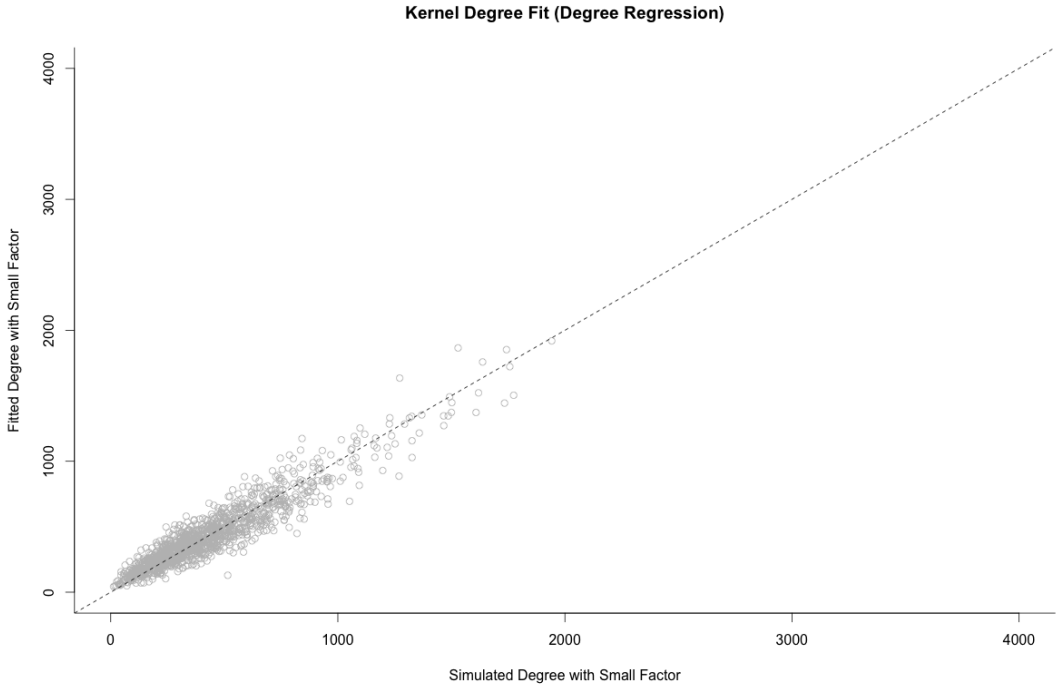
$$\beta \sim \mathcal{N}(0, 2I) \qquad \eta \sim \log\mathcal{N}(-0.7, 0.1) \qquad d_j \sim \log\mathcal{N}(\beta_1 + \beta_2 g_j - e^{\beta_3}(a_j - \bar{a})^2, \eta^2)$$

$$\rho_{g_i \cdot} \sim Alpha(5, 5) \qquad \lambda_{g_i g_j} \sim \log\mathcal{N}(\log 100, 0.5) \qquad \frac{1}{\frac{1}{\omega_k} + 1} \sim Beta(4.5, 0.5)$$

3.3 Results

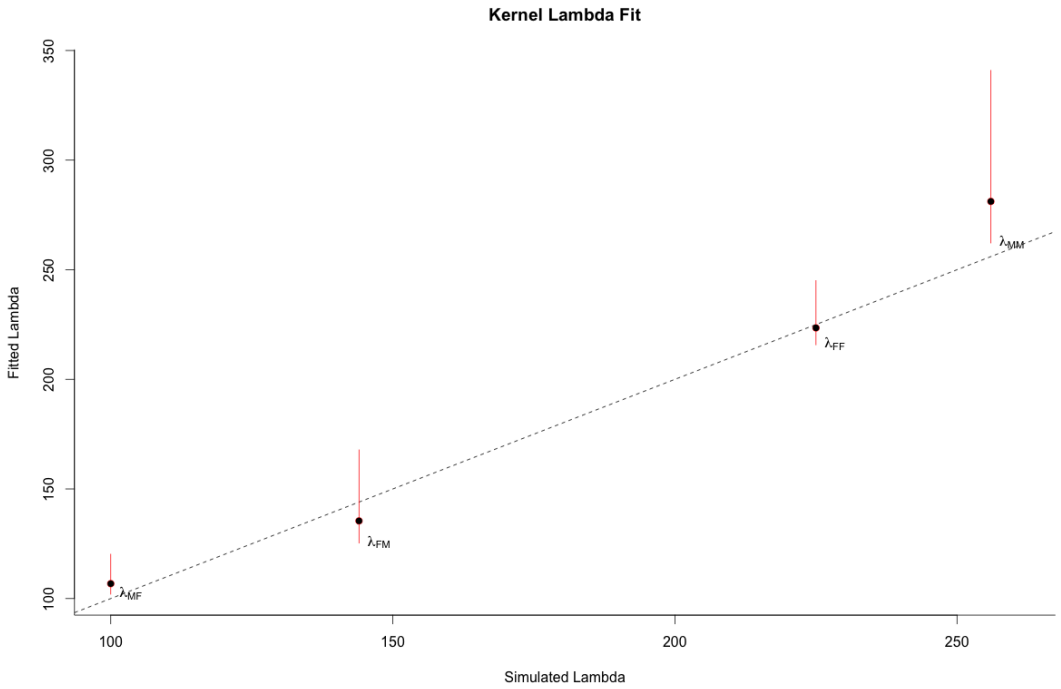
Respondent Degrees (d_i)

The degrees are recovered well.



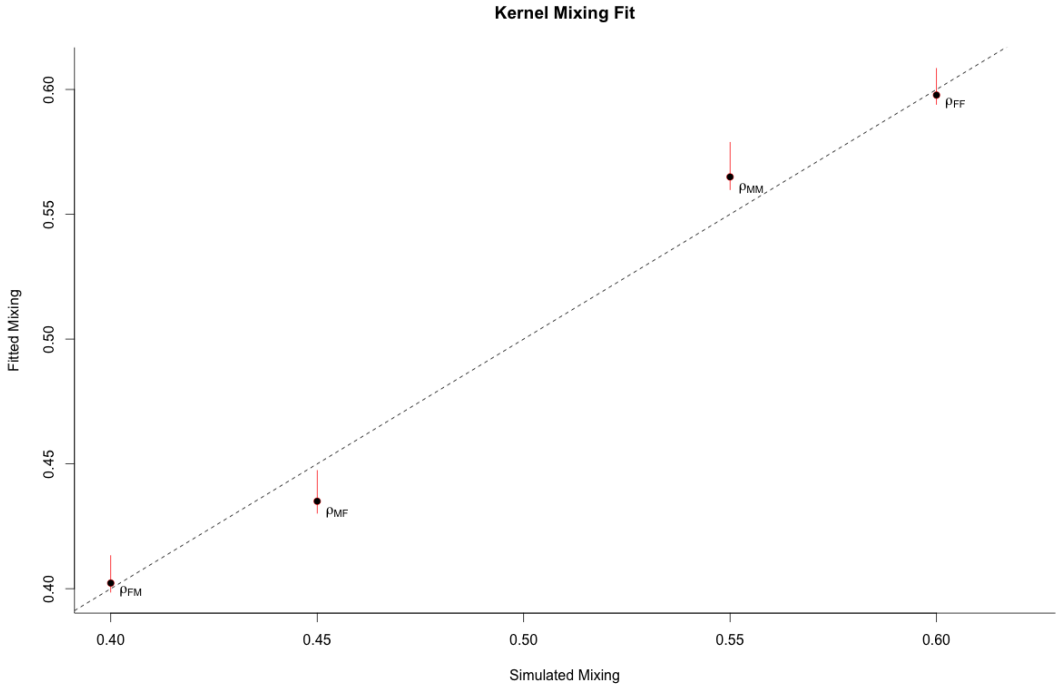
Kernel Lengthscale ($\lambda_{g_i g_j}$)

The continuous model does a decent job of recovering the kernel lengthscales.



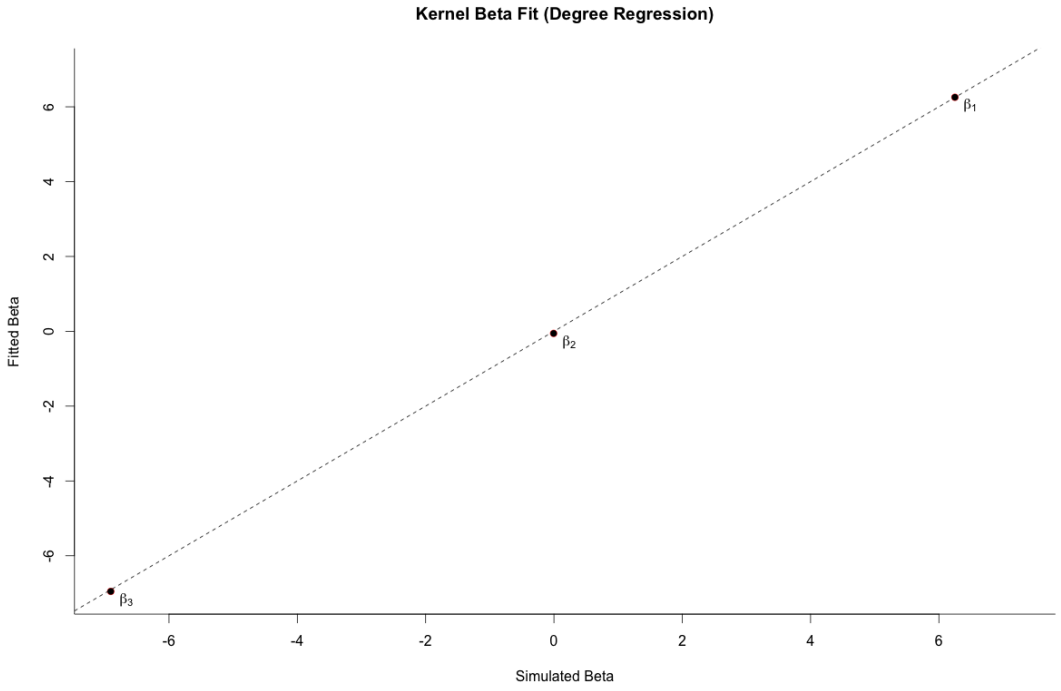
Gender Mixing Rates ($\rho_{g_i g_k}$)

the gender mixing rates are recovered decently well.



Degree Regression Parameters (β_j)

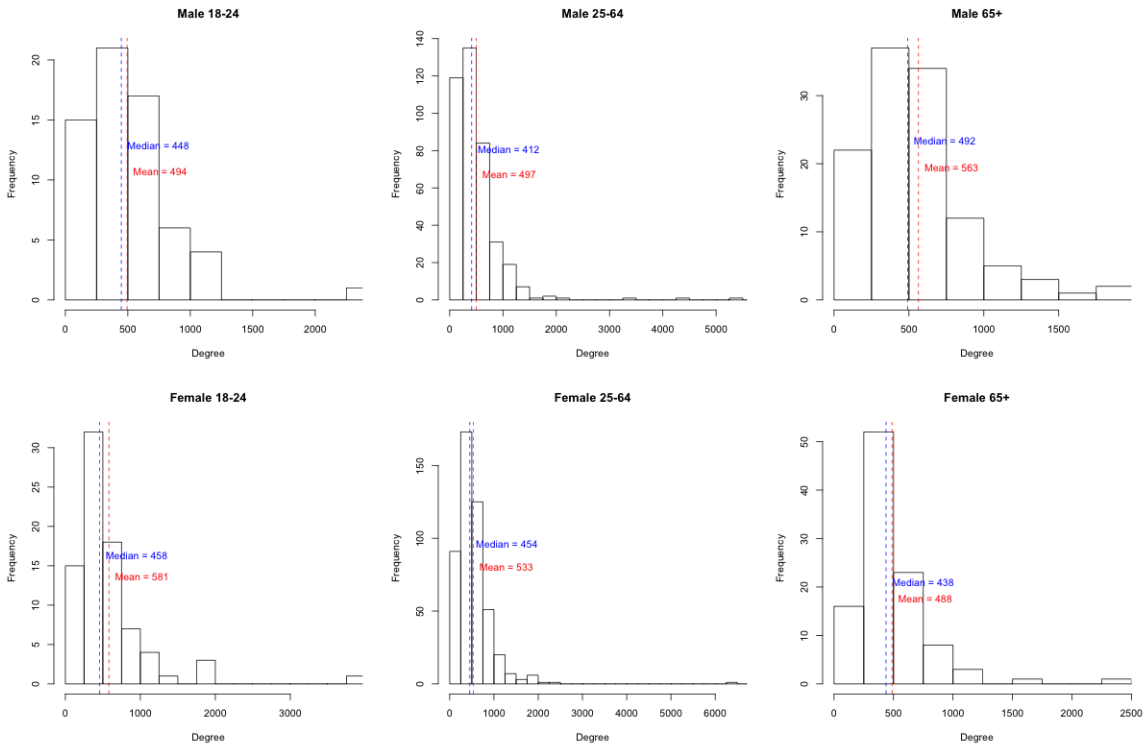
Lastly, the beta regression parameters are recovered very well.



4 Omni Data Results

Respondent Degrees

The continuous model estimates imply decreasing and then increasing network size by age for men, but monotonically decreasing network size by age for women.



Gender Mixing Rates

The gender mixing rates imply male-dominated networks. In general, about 56% of a female's network is female, while 59% of a male's network is male. This also implies that females mix more with males than the other way around.

$$\rho_{BAYES} = \begin{pmatrix} \rho_{FF} & \rho_{FM} \\ \rho_{MF} & \rho_{MM} \end{pmatrix} = \begin{pmatrix} 0.44 & 0.56 \\ 0.41 & 0.59 \end{pmatrix}$$

Kernel Lengthscales

The length scales estimated from the actual data are large, implying very flat kernels. The important distinction here, then, is perhaps the relative size of the lengthscales.

$$\lambda_{BAYES} = \begin{pmatrix} \lambda_{FF} & \lambda_{FM} \\ \lambda_{MF} & \lambda_{MM} \end{pmatrix} = \begin{pmatrix} 969 & 1253 \\ 1104 & 1153 \end{pmatrix}$$

Indeed, it seems that the female to female kernel is much tighter than the male to male kernel, implying that women tend to know a narrow age range of other women but men tend to know a wide age range of other men. Alternatively, the female to male kernel is much wider than the male to female kernel, implying that women know a wider age range of men while men known a relatively narrow age range of women.