ALL THE λ_1'S ON CYCLIC ADMISSIBLE COVERS

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ABSTRACT. We compute the degrees of the first Chern class of the Hodge bundle λ_1 and of Hurwitz-Hodge classes λ_1^e on one dimensional moduli spaces of cyclic admissible covers of a rational curve. In higher dimension, we express the divisor class λ_1 as a linear combination of ψ classes and boundary strata; we detail a computational scheme, and show some infinite family of examples for the classes λ_1^e.

1. INTRODUCTION

A cyclic cover is the quotient map of a curve C by the effective action of a cyclic group $\mathbb{Z}/d\mathbb{Z}$; for example the projection to the x-axis from the curve $\{y^d = p(x)\} \subset \mathbb{A}^2$ gives an affine model determining by the Riemann existence theorem a cyclic cover of $\mathbb{P}^1$. Since the complex structure of a cyclic cover of $\mathbb{P}^1$ is essentially determined by the location and the monodromies of its branch points, families of cyclic covers are among the most classical ways to construct subvarieties of moduli spaces of curves.

The perspective of Hurwitz spaces, together with the Harris-Mumford admissible covers compactification in [HM82], proposes to study families of cyclic covers with fixed discrete invariants (genus, degree, and monodromy data around the branch points) as standalone moduli spaces, connected to moduli spaces of curves by natural source and branch morphisms. With the development of the language of orbifolds and stacks, [ACV01] interpret the normalization of the Harris-Mumford space as a (connected component of a) smooth stack of *twisted stable maps* from some orbifold modification of the base curves to a quotient stack $[pt./G]$. We use the terminology *moduli space of cyclic admissible covers* $Adm_d(m_1, \ldots, m_n)$ to denote this smooth stack rather than the Harris-Mumford version.

The branch and source morphisms from spaces of admissible covers create a correspondence that connects the geometry of moduli spaces of higher genus (cover) curves with the combinatorics of the configuration space of branch points, thus making explicit and accessible certain geometric information, see [BP00]. We illustrate this philosophy by studying the Hodge bundle $\mathbb{E}$, whose fiber over a moduli point consists of the global sections of the relative dualizing sheaf of the cover curve, in terms of combinatorial information from $\overline{M}_{0,n}$. Further, the cyclic action on the curve induces an action on the Hodge bundle. The language of twisted stable maps is very well attuned to study the Chern classes of subrepresentations $\mathbb{E}_e$ of the Hodge bundle, called *Hurwitz-Hodge classes* in [BGP08]. Hurwitz-Hodge integrals [JPT11], i.e. intersection numbers of these classes, are used in the computations of orbifold Gromov-Witten invariants [CC09, BG09a], and were a key tool in the development and study of the crepant resolution conjecture in the Hard Lefschetz case [BG09b].

While the orbifold version of the Grothendieck-Riemann-Roch theorem [Toe99] provides a powerful technique to compute individual Hurwitz-Hodge integrals [Zho07], it conceals the rich algebraic and combinatorial structure that Hurwitz-Hodge classes have when considered in families.

The first computation of this article highlights this structure in the case of the first Chern class of the bundles $\mathbb{E}_e$, denoted by λ_1^e. We use Atiyah-Bott localization to compute the degree of these classes on one dimensional moduli spaces of cyclic admissible covers. After the completion of this work it was pointed out to us that this computation already appeared, albeit with different language and proof technique, in a paper of Eskin, Kontsevich and Zorich [EKZ11]. While we can't claim originality for this result, we hope that our new proof might be a useful illustration of the power of Atiyah-Bott localization for these kind of problems.

Theorem 1.1 ([EKZ11], Theorem 1)**.** *Let d be a positive integer, m_1, m_2, m_3, m_4 a monodromy datum as in Definition 2.5, and e an integer between 0 and $d-1$. Assume, without loss of generality, that $\left\langle \frac{em_1}{d} \right\rangle \leq \left\langle \frac{em_2}{d} \right\rangle \leq \left\langle \frac{em_3}{d} \right\rangle \leq \left\langle \frac{em_4}{d} \right\rangle$, where the angle brackets mean fractional part. The degree of the orbifold class λ_1^e on the one-dimensional space of degree d cyclic admissible covers of a rational curve with monodromies m_i is given by the* *following formula:*

$$\int_{Adm_d(m_1,m_2,m_3,m_4)} \lambda_1^e = \begin{cases} 0 & \text{if } \sum_{i=1}^4 \left\langle \frac{em_i}{d} \right\rangle = 0,1,3 \\ \frac{1}{d}\left\langle \frac{em_1}{d} \right\rangle & \text{if } \sum_{i=1}^4 \left\langle \frac{em_i}{d} \right\rangle = 2 \text{ and } \left\langle \frac{em_1}{d} \right\rangle + \left\langle \frac{em_4}{d} \right\rangle \leq 1 \\ \frac{1}{d}\left(1 - \left\langle \frac{em_4}{d} \right\rangle\right) & \text{if } \sum_{i=1}^4 \left\langle \frac{em_i}{d} \right\rangle = 2 \text{ and } \left\langle \frac{em_1}{d} \right\rangle + \left\langle \frac{em_4}{d} \right\rangle > 1 \end{cases} \tag{1}$$

As an application of Theorem 1.1, in Theorem 3.2 we show that knowing the degrees of the classes λ_1^e for one dimensional spaces of cyclic admissible covers allows one, in principle, to compute the class λ_1^e as a linear combination of boundary divisors for moduli spaces of arbitrary dimension. It amounts to inverting the intersection matrix of a chosen basis of boundary divisors and a chosen basis of boundary curve classes.

In any given case this is a reasonable linear algebraic computation. Obtaining formulas for infinite families of moduli spaces is complicated by the fact that while strata give a natural set of generators for the groups of divisor and curve classes in admissible cover spaces, such sets are far from being bases. Choosing a basis often results in breaking symmetries and hides the natural structure that a general formula might have. A study group at the workshop *COMOC* [BCH$^+$24] has been investigating the problem of finding a good combinatorial description for an inverse of the intersection matrix for curves and divisors on moduli spaces of curves $\overline{M}_{0,n}$ with promising initial computations. Here we restrict our attention to a couple infinite families of moduli spaces, one where a simple vanishing holds and the other where the large amount of symmetry allows us to give an elegant formula for the classes λ_1^e. It is in fact thanks to the explorations of [BCH$^+$24] as well as the results in [AGSS12] that we could guess a formula valid for all d.

Theorem 1.2. *Let $Adm_d(1^n)$ denote the moduli space of cyclic admissible covers with n marked branch points all of monodromy one. By Δ_i we denote the boundary divisor consisting of the sum of all boundary strata where one of the vertices has exactly i marks, and κ_1 is Arbarello-Cornalba's kappa class. Then:*

(1) for $n = d$, $\lambda_1^1 = 0$;
(2) for $n = 2d$,

$$\lambda_1^1 = \frac{1}{2}\left(-\kappa_1 + \sum_{i=2}^{d} (i-1)\,\Delta_i\right).$$

The next result computes the degree of the class λ_1, the first Chern class of the full Hodge bundle $\mathbb{E}$, for all one dimensional moduli spaces of cyclic admissible covers.

Theorem 1.3. *Let d be a positive integer and m_1, m_2, m_3, m_4 a monodromy datum. The degree of the class λ_1 on the one-dimensional space of degree d cyclic admissible covers of a rational curve with monodromies m_i is given by the following formula, indexed by the power set of $[4] = \{1, 2, 3, 4\}$.*

$$\int_{Adm_d(m_1,m_2,m_3,m_4)} \lambda_1 = \frac{1}{24d^2}\left(\sum_{I\in\mathcal{P}([4])} (-1)^{|I|}\mathrm{gcd}^2\left(\sum_{i\in I} m_i, d\right)\right). \tag{2}$$

In the case of the full Hodge bundle, the class λ_1 can be described as a linear combination of boundary strata, ψ classes and the class κ_1; these are considered the standard generators for the tautological ring of the moduli spaces of curves. The general formula for λ_1 is a natural generalization of the one-dimensional case.

Theorem 1.4. *Let d be a positive integer and $m_1, \ldots, m_n$ be integers with $0 \leq m_i < d$. The class λ_1 on the space $Adm_d(m_1, \ldots, m_n)$ of cyclic admissible covers of a rational curve with monodromies m_i is equivalent to the following tautological expression:*

$$\lambda_1 = \frac{1}{24d}\left(\sum_{J\in\mathcal{P}([n])} \mathrm{gcd}^2\left(\sum_{j\in J} m_i, d\right)\Delta_J\right), \tag{3}$$

where

- *for $2 \leq |J| \leq n-2$, Δ_J denotes the boundary divisor generically parameterizing one-nodal curves with the branch points labelled by J sitting on one component, and the branch points in J^c on the other;*
- *for $J = \{j\}, [n] \smallsetminus \{j\}$, $\Delta_J := -\psi_j$;*
- *for $J = \phi, [n]$, $\Delta_J := \kappa_1$.*

Theorems 1.1 and 1.3 are proved using Atiyah-Bott localization, following a strategy introduced for ordinary Hodge integrals in [FP00b] and imported to the setting of admissible covers in [Cav06, Cav07]. The idea is to set-up a vanishing auxiliary integral on a moduli spaces of admissible covers of a parameterized $\mathbb{P}^1$: these are moduli spaces that admit a torus action, and hence the vanishing integral can be evaluated by restricting it to fixed loci for the torus action, giving rise to a relation among Hurwitz-Hodge integrals. By choosing the auxiliary integral carefully, one can arrange for the principal part of the relation to contain the degree of λ_1 (or λ_1^e), and for all other terms to be either zero dimensional moduli spaces, or functions of the degree of a ψ class, which is well-known.

Since the full Hodge bundle is the direct sum of its subrepresentations, we have the relation $\lambda_1 = \sum_{e=0}^{d-1} \lambda_1^e$, hence in principle one may deduce Theorem 1.3 as a corollary of Theorem 1.1. While it is certainly straightforward to do so for any individual case, to obtain a general result we found it in the end more convenient to approach the non-orbifold computation independently. One may therefore use these results to deduce an elementary, albeit mysterious to us, arithmetic identity.

The formula for the degree of λ_1 from Theorem 1.3 naturally generalizes to higher dimensional moduli spaces to produce a *graph formula*: a representation of λ_1 in terms of combinatorially decorated strata classes, in a fashion similar to [JPPZ17]. Once one is able to guess a graph formula, one may prove it simply by computing intersection numbers with all boundary curves, using the fact that boundary curves generate $A_1(Adm_d(m_1, \ldots, m_n))$. The study of a graph formula description of λ_1 was initiated in [Tro07, BR11], which independently computed the hyperelliptic case. The second and third authors studied the $d = 3$ case in [OSC21]. Theorem 1.4 concludes this analysis for all spaces of cyclic covers of $\mathbb{P}^1$. While we are currently unable to produce a graph formula for the orbifold classes λ_1^e, our observations led us to the following question.

Question 1.5. *With notation as in Theorem 1.4, the following is a graph formula for the classes λ_1^e in the 4-pointed case:*

$$\lambda_1^e = \frac{1}{2}\left(\sum_{I \in \mathcal{P}([4])} \min\left\{0, 1 - \sum_{i \in I}\left\langle \frac{em_i}{d}\right\rangle\right\}\Delta_I\right).$$

Is there a natural generalization that yields a graph formula in the general case?

One of the goals of this manuscript is to communicate both to the readers versed in a more classical algebraic geometric language, as well as with the readers steeped in orbifold technology. We feel that there is potential for fruitful interactions between the two communities, and dedicate a good part of the background section in recalling (albeit briefly) the main connections between the two languages.

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2. Background

In this section we collect some background needed for the computations in the later sections. While there is no pretense to make this work self-contained, we provide basic information for translating between the languages of covers and orbifold maps, and suggest references for readers interested in more details.

2.1. **Orbifolds.** Orbifolds (or stacks in the algebraic category) are geometric objects generalizing the notion of orbit spaces to the case of non-free group actions. We refer to [ALR07],[Fan01] for a comprehensive introduction.

Global quotient orbifolds. Given a space X and a group G acting on it, one denotes by $[X/G]$ the orbifold quotient of X by G (see [FG03]). The most useful way to define this concept is through its functor of points; in simple terms, this means that one gives geometric structure to $[X/G]$ by describing the functions to it.

Definition 2.1. A function $f : B \to [X/G]$ consists of a pair:

- $\pi_f : E \to B$, a principal G-bundle over B;
- $F : E \to X$, a G-equivariant map.

Definition 2.1 produces a fiber diagram analogous to the universal property of orbit spaces in the case of a free action.

One may still think of geometric points of $[X/G]$ as orbits $[x]$ of points of X under the G action; each point $[x]$ comes with the additional information of an *isotropy group* G_x, isomorphic to the stabilizer of any point $x \in [x]$.

In the extreme case when $X = pt.$, we denote the quotient orbifold $[pt./G]$ by BG, and call it the classifying space for principal G-bundles (as the datum of the map F becomes in this case trival). This orbifold consists of a point with isotropy group G. A *line bundle* on BG consists of a one dimensional representation of G.

Definition 2.2. For d a non-negative integer, and $0 \leq e < d-1$ we denote by $\mathcal{O}_e$ the line bundle on $B(\mathbb{Z}/d\mathbb{Z})$ corresponding to the representation

$$[1] \cdot z = \exp\left(e\frac{2\pi i}{d}\right) z.$$

For $j \in \mathbb{Z}$, we denote by L_j the line bundle on $B\mathbb{C}^*$ corresponding to the representation

$$\alpha \cdot z = \alpha^j z.$$

Remark 2.3. The following observations will be useful later:

- we denote by $\mathcal{O}_e$ and L_j also line bundles that are pulled back from BG to other spaces. These consist of trivial line bundles with a G-action;
- in the case of $B\mathbb{C}^*$, we need to work with $\mathbb{Q}$-divisors; the index j in L_j will then be a rational number.[1]

Twisted curves. Global quotient orbifolds are the local models for the construction of more general orbifolds. A treatment of one-dimensional orbifolds and orbifold line bundles is in [Joh14], to which we refer the reader interested in more details. A *twisted curve* $\mathcal{C}$ is obtained from a Riemann surface C by replacing a finite number of disjoint open discs of C with (open sets of) global quotient orbifolds $[\mathbb{C}/(\mathbb{Z}/d\mathbb{Z})]$; the curve $\mathcal{C}$ and its so called coarse moduli space C have the same geometric points, but the twisted curve has a finite number of points, called *twisted points*, with non-trivial, cyclic isotropy groups.

A line bundle $\mathcal{L}$ on a twisted curve $\mathcal{C}$ contains the information of a representation of the isotropy group G_x over every twisted point, describing a lift of the group action to the fiber of the line bundle. The isotropy group G_x is identified with $\mathbb{Z}/d\mathbb{Z}$ by requiring that the generator acts by multiplication by $\mathrm{e}^{\frac{2\pi i}{d}}$ on the tangent space T_x; if $[1] \cdot w = \mathrm{e}^{k\frac{2\pi i}{d}} w$ for w in the fiber L_x, then the rational number k/d is called the *age* of $\mathcal{L}$ at x. The age of a line bundle at the twisted points contributes to the orbifold version of the Riemann-Roch theorem.

Theorem 2.4 (Orbifold Riemann-Roch, [AGV08], 7.2.1)**.** *Let $\mathcal{C}$ be a twisted curve, and $\mathcal{L}$ a line bundle on it; then:*

$$h^0(\mathcal{C}, \mathcal{L}) - h^1(\mathcal{C}, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g_{\mathcal{C}} - \sum_{x \in \mathcal{C}} \mathrm{age}_x(\mathcal{L}).$$

Maps of orbifolds. A map of orbifolds $f : \mathcal{X} \to \mathcal{Y}$ contains additional information with respect to the function on the coarse spaces $f : X \to Y$. For every point $x \in \mathcal{X}$, f determines a group homomorphism $\phi_{f,x} : G_x \to G_{f(x)}$. The map f is called *representable* if all the group homomorphisms $\phi_{f,x}$ are injective.

[1] One may interpret an $L_{p/q}$ as a virtual representation, i.e. a formal object that pulls-back to a representation via the power map $\cdot^q : B\mathbb{C}^* \to B\mathbb{C}^*$, but we will not need this degree of sophistication.

2.2. **Cyclic admissible covers and twisted stable maps.** Admissible covers were introduced in [HM82] to compactify the Hurwitz spaces. In [ACV01], the authors show that connected components of the smooth stack of twisted stable maps to BS_d realize the normalization of spaces of admissible covers. We now introduce these spaces in the case where the target is $B(\mathbb{Z}/d\mathbb{Z})$, and adopt the convention of calling *admissible covers* the smooth stack of twisted stable maps.

Let $\mathcal{C}$ be a twisted curve whose coarse space $C \cong \mathbb{P}^1$ is smooth. By Definition 2.1, a map $f : \mathcal{C} \to B\mathbb{Z}/d\mathbb{Z}$ consists of a principal $\mathbb{Z}/d\mathbb{Z}$-bundle $\mathcal{E} \to \mathcal{C}$. Passing to the corresponding coarse spaces one obtains a cyclic cover $\pi_f : E \to C$, that ramifies only over the (image of the) twisted points of $\mathcal{C}$. To be precise, if $\phi_{x,f}([1]) = m \in \mathbb{Z}/d\mathbb{Z}$, then $\pi_f^{-1}(x)$ consists of $\gcd(m,d)$ points of ramification order $d/\gcd(m,d)$; further, if one chooses a generic point $x_0 \in C$ as a base point and a labelling of its d inverse images that is compatible with the cyclic group action on E, then the image of a small loop around x in the monodromy representation associated to the cover π_f is precisely m. Hence we say that x is a point of monodromy m for π_f.

Definition 2.5. Let d be a positive integer. A set of integers $m_1, \dots, m_n$ with $0 \leq m_i < d$ are called **a monodromy datum** for degree d cyclic admissible covers of rational curves if $\sum_{i=1}^n m_i = 0 \mod d$ and $\gcd(m_1, \dots, m_n, d) = 1$.

Remark 2.6. The first condition in Definition 2.5 guarantees that admissible covers exist; the second one requires them to be connected.

Definition 2.7. Given a monodromy datum $m_1, \dots, m_n$, the moduli space of admissible covers $Adm_d\left(m_1, \dots, m_n\right)$ parameterizes representable maps $\mathcal{C} \to B(\mathbb{Z}/d\mathbb{Z})$ such that:

- the coarse moduli space C is a rational, nodal curve;
- $\mathcal{C}$ has exactly n twisted points labeled $x_1, \dots, x_n$ in the smooth locus of C, of monodromies $m_1, \dots, m_n$;
- the nodes of $\mathcal{C}$ may be twisted; in that case, the two shadows of a node (i.e. the preimages of the node via the normalization map) have opposite monodromies.

As before, a map $\mathcal{C} \to B(\mathbb{Z}/d\mathbb{Z})$ corresponds to a cyclic cover $E \to C$ of the rational nodal curve C. The nodes of E are precisely the inverse images of nodes of C, and the ramification orders of pairs of shadows of any node must match.

Spaces of admissible covers have universal morphisms denoted as in the following diagram:

$$\begin{tikzcd}
\mathcal{U}_E \arrow[d] \arrow[rr, bend left, "F"] \arrow[ddr, bend right, "\Pi"'] & \mathcal{U}_{\mathcal{E}} \arrow[l] \arrow[r] \arrow[d] & pt. \arrow[d] \\
\mathcal{U}_C & \mathcal{U}_{\mathcal{C}} \arrow[l] \arrow[r, "f"] \arrow[d, "\pi"'] & B(\mathbb{Z}/d\mathbb{Z}) \\
 & Adm_d\left(m_1, \dots, m_n\right) &
\end{tikzcd} \tag{4}$$

By the Riemann-Hurwitz formula, the genus of the cover curve E is

$$g = 1 + \frac{(n-2)d - \sum_{i=1}^n \gcd(m_i, d)}{2}. \tag{5}$$

There are two natural morphisms from the space of admissible covers. The *source* morphism $s : Adm_d\left(m_1, \dots, m_n\right) \to \overline{M}_g$ remebers the cover curve E. The *branch* morphism $br : Adm_d\left(m_1, \dots, m_n\right) \to \overline{M}_{0,n}$ records the base curve C together with the images of the n-twisted points of $\mathcal{C}$. The branch morphism is a bijection on geometric points, but every point of $Adm_d\left(m_1, \dots, m_n\right)$ has an order d cyclic isotropy group, and hence $\deg(br) = \frac{1}{d}$.

Boundary stratification. Boundary strata for the space $Adm_d\left(m_1, \dots, m_n\right)$ are in canonical bijection with boundary strata of $\overline{M}_{0,n}$, and they can therefore be indexed by the dual graphs of the base curves.

Any subset $I \in \mathcal{P}([n])$ with $2 \leq |I| \leq n-2$ identifies a boundary divisor Δ_I, whose set of points is isomorphic to the product $Adm_d\left(\{m_i\}_{i\in I}, \left[-\sum_{i\in I} m_i\right]_d\right) \times Adm_d\left(\{m_i\}_{i\in I^c}, \left[\sum_{i\in I} m_i\right]_d\right)$; as a stack, however Δ_I is isomorphic to a fiber product over $B\mathbb{Z}/d\mathbb{Z}$ of the two factors above. This causes, when integrating along Δ_I, a factor of d often referred to as the gluing factor (see [CC09, Section 1.6] for a discussion). To remember this, we abuse notation and write

$$\Delta_I \cong d \cdot Adm_d\left(\{m_i\}_{i\in I}, \left[-\sum_{i\in I} m_i\right]_d\right) \times Adm_d\left(\{m_i\}_{i\in I^c}, \left[\sum_{i\in I} m_i\right]_d\right). \tag{6}$$

We are especially interested in one-dimensional boundary strata, or boundary curves, in $Adm_d(m_1,\ldots,m_n)$. Their dual graphs are trees that have a unique vertex v of valence 4, and all other vertices trivalent. Removing v the set of indices is partitioned into four sets X, Y, Z, W. The rational equivalence class of a boundary curve depends only on such partition, and hence we denote a boundary curve class by $C_{(X,Y,Z,W)}$. We observe that, keeping in account gluing factors as well as automorphism factors coming from the zero dimensional moduli spaces in the product expression of a boundary curve, in the end one has:

$$C_{(X,Y,Z,W)} \cong Adm_d\left(\left[\sum_{x\in X} m_x\right]_d, \left[\sum_{y\in Y} m_y\right]_d, \left[\sum_{z\in Z} m_z\right]_d, \left[\sum_{w\in W} m_w\right]_d\right). \tag{7}$$

Parameterized admissible covers. Given a monodromy datum, we denote by $Adm_d\left(\mathbb{P}^1|m_1,\ldots,m_n\right)$ the space of *admissible covers of a parameterized* $\mathbb{P}^1$ ([Cav07]); using orbifold language these are twisted stable maps of degree $(1,0)$ to $\mathbb{P}^1\times B(\mathbb{Z}/d\mathbb{Z})$. In terms of the geometry of the covers $E\to C$, this means that one component $\mathbb{P}^1\subseteq C$ is chosen, and for two covers to be isomorphic, the isomorphism of the base curves must restrict to the identity of the special component $\mathbb{P}^1$. A universal diagram analogous to (4) holds for spaces of parameterized admissible covers, where one takes the cartesian product with $\mathbb{P}^1$ of every space in the rightmost column.

Given a parameterized admissible cover $f:\mathcal{C}\to\mathbb{P}^1\times B(\mathbb{Z}/d\mathbb{Z})$ and a bundle $\mathcal{L}=\mathcal{O}_{\mathbb{P}^1}(n)\otimes\mathcal{O}_e$, the cohomology groups of $f^*\mathcal{L}$ may be described in terms of the geometry of the cover $E\to C\to\mathbb{P}^1$. Unraveling the appropriate orbifold definitions (as in [BGP08]), one obtains

$$H^i(\mathcal{C}, f^*\mathcal{L}) = \left(H^i(E, F^*\mathcal{O}_{\mathbb{P}^1}(n))\right)_{d-e}, \tag{8}$$

where F is the composition of the two maps above, and the subscript $(d-e)$ denotes the subrepresentation of the cohomology group corresponding to the character $(d-e)$.

2.3. Tautological classes.

Intuitively, tautological classes on a family of moduli spaces are elements of the Chow (or cohomology) ring that are constructed using the intrinsic geometry of the objects parameterized (e.g. the structure sheaf or the dualizing sheaf), via a series of operations that involve push-forwards and pull-backs via tautological morphisms. We refer the reader to [Vak08] for a friendly introduction to the subject, and to [Mum83, FP00a, SvZ20, Lia21] for the reader interested in more details.

Chern classes of bundles. See [Ful98] for a less skeletal introduction to the subject. Given a vector bundle $E\to X$ of rank r, the *total Chern class* of E is a Chow class $c(E) := 1 + c_1(E) + \ldots + c_r(E)$, where $c_i(E)\in A^i(X)$ is called the i-th Chern class. Perhaps the single most important formal property in this theory is that the total Chern class is multiplicative with respect to extensions, in the sense that given a sequence $0\to F\to E\to Q\to 0$, one has $c(E) = c(F)c(Q)$; this readily implies that the first Chern class is additive, i.e. $c_1(E) = c_1(F) + c_1(Q)$.

Given a bundle E of rank r, its *Chern roots* $\alpha_1,\ldots,\alpha_r$ are graded symbols of degree one with the defining property that the i-th Chern class of E is the i-th elementary symmetric function in the Chern roots.

Chern roots allow to treat arbitrary bundles as if they split as the direct sum of line bundles, and are hence useful tools to compute Chern classes. For example, if E is as above and L a line bundle, then the Chern roots of the bundle $E\otimes L$ are $\alpha_i + c_1(L)$ and this fact allows to readily compute the Chern classes of the tensor product in terms of the Chern classes of the factors. As an application, we show a computation needed in Section 4.

Lemma 2.8.

$$c_1(E^{\oplus n}) = nc_1(E), \quad c_2(E^{\oplus n}) = nc_2(E) + \binom{n}{2}c_1^2(E).$$

Proof. These statements follow immediately from the multiplicativity of total Chern classes. ☐

Hodge bundles. The main object of study of this work is the first Chern class of the Hodge bundle and its orbifold variants on spaces of admissible covers, which we now introduce.

Definition 2.9. Consider a space of admissible covers $Adm_d\,(m_1,\ldots,m_n)$ and the universal morphisms from diagram (4). We define:

$$\mathbb{E} := (R^1\Pi_*F^*(\mathcal{O}))^\vee.$$

One observes that $\mathbb{E} = s^*(\mathbb{E})$, i.e. the Hodge bundle on the space of admissible covers is the pull-back of the homonymous bundle on the moduli space of curves via the source morphism [2]. Its fibers over a general point $\mathcal{C} \to B\mathbb{Z}/d\mathbb{Z}$ corresponding to a smooth cover $E \to C$ may be identified with the vector space of holomorphic one-forms on E. It follows that the rank of $\mathbb{E}$ is equal to the genus of E.

Definition 2.10. Consider a space of admissible covers $Adm_d\,(m_1,\ldots,m_n)$ and the universal morphisms from diagram (4). We define:

$$\mathbb{E}_e := (R^1\pi_*f^*(\mathcal{O}_e))^\vee.$$

It is immediate that $\mathbb{E}_0$ has rank 0, as generically points of $\mathbb{E}_0$ correspond to invariant holomorphic one-forms on E, which are pulled-back from $\mathbb{P}^1$ via the cover map. For $e \neq 0$, the rank of the bundle $\mathbb{E}_e$ is computed using the orbifold Riemann-Roch theorem:

$$\mathrm{rk}(\mathbb{E}_e) = \mathrm{rk}((\mathbb{E}^\vee)_{d-e}) = \mathrm{rk}(R^1\pi_*f^*\mathcal{O}_e) = h^1(C, f^*(\mathcal{O}_e)) = -1 + \sum_{i=1}^{n} \mathrm{age}_{x_i}(f^*(\mathcal{O}_e)) = -1 + \sum_{i=1}^{n} \left\langle \frac{em_i}{d} \right\rangle \tag{9}$$

The notation $\mathbb{E}_e$ follows from the interpretation of its fibers in terms of the geometry of the covers $E \to C$. Applying (8), one sees that the fiber of $\mathbb{E}_e$ over a general moduli point corresponding to a cover $E \to C$ equals the e-subrepresentation of the space of holomorphic one forms on E.

It follows that

$$\mathbb{E} = \bigoplus_{e=0}^{d-1} \mathbb{E}_e. \tag{10}$$

We need to work with subrepresentations of the dual of the Hodge bundle as well. The natural way to induce an action on a dual space implies:

$$(\mathbb{E}^\vee)_e \cong (\mathbb{E}_{d-e})^\vee. \tag{11}$$

We recall the G-Mumford relation, introduced in [BGP08]:

$$c(\mathbb{E}_e \oplus (\mathbb{E}^\vee)_e) = 1. \tag{12}$$

Definition 2.11. We define:

$$\lambda_1 := c_1(\mathbb{E}), \quad \lambda_1^e := c_1(\mathbb{E}_e).$$

Lemma 2.12. *For any choice of positive integers d, e, with $0 \le e < d$, and monodromy datum $m_1,\ldots,m_n$,*

$$\lambda_1^e = \lambda_1^{d-e}. \tag{13}$$

Proof. We temporarily denote by $\hat{\lambda}_1^e$ the first Chern class of the e-eigenbundle of the dual of the Hodge bundle on the space $Adm_d\,(m_1,\ldots,m_n)$. Equation (11) implies that $\hat{\lambda}_1^e = -\lambda_1^{d-e}$; the G-Mumford relation (12) implies $\lambda_1^e + \hat{\lambda}_1^e = 0$. Combining the two equations the statement follows. $\square$

2.3.1. *Psi classes.* We recommend [Koc01] for an introduction to ψ classes on moduli spaces of curves.

Definition 2.13. Consider the moduli space $Adm_d\,(m_1,\ldots,m_n)$. For $1 \le i \le n$ we denote by ψ_i the pullback $br^*(\psi_i)$, where we assume the notion of ψ classes on $\overline{M}_{0,n}$.

[2]only rational components of the curve E may get contracted in the source morphism and therefore $H^1(E,\mathcal{O}) \cong H^1(s(E),\mathcal{O})$.

To attach some meaning to this definition for readers who are completely unfamiliar with ψ classes, the class ψ_i is the first Chern class of a line bundle on $Adm_d\,(m_1,\ldots,m_n)$ whose fiber over a moduli point $E\to C$ is canonically identified with the cotangent line of C at the i-th branch point.

By the projection formula, we have

$$\int_{Adm_d(m_1,\ldots,m_n)}\prod\psi_i^{k_i}=\frac{1}{d}\int_{\overline{M}_{0,n}}\prod\psi_i^{k_i}=\frac{1}{d}\binom{n-3}{k_1,\ldots,k_n}. \tag{14}$$

2.4. **Atiyah-Bott localization.** We give a brief account of localization and develop some details geared to our application of it. We follow the language and notations in [HKK+03, chapters 4 and 27]. A complete reference for this technique for moduli spaces of maps from orbifold curves is [Liu13].

Consider the one-dimensional algebraic torus $\mathbb{C}^*$, and recall that the $\mathbb{C}^*$-equivariant Chow ring of a point is a polynomial ring in one variable: $A^*_{\mathbb{C}^*}(\{pt\},\mathbb{C})=\mathbb{C}[t]$, with $t=c_1(L_1)$.

Let $\mathbb{C}^*$ act on a smooth, proper stack X, denote by $i_k:F_k\hookrightarrow X$ the irreducible components of the fixed locus for this action and by N_{F_k} their normal bundles. The natural map:

$$\begin{array}{ccc} A^*_{\mathbb{C}^*}(X)\otimes\mathbb{C}(t) & \to & \sum_k A^*_{\mathbb{C}^*}(F_k)\otimes\mathbb{C}(t) \\ \\ \alpha & \mapsto & \dfrac{i_k^*\alpha}{c_{top}(N_{F_k})}. \end{array}$$

is an isomorphism. Pushing forward equivariantly to the class of a point, one has the Atiyah-Bott integration formula:

$$\int_{[X]}\alpha=\sum_k\int_{[F_k]}\frac{i_k^*\alpha}{c_{top}(N_{F_k})}. \tag{15}$$

Let $\mathbb{C}^*$ act on a two-dimensional vector space V via $t\cdot(z_0,z_1)=(tz_0,z_1)$. This action descends to $\mathbb{P}^1$ with fixed points $0=(1:0)$ and $\infty=(0:1)$. An equivariant lift of the $\mathbb{C}^*$ action to a line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$ over $\mathbb{P}^1$ is uniquely determined by the representations (i.e. line bundles over $B\mathbb{C}^*$) $L_{j(0)},L_{j(\infty)}$ of the fibers over the fixed points. One may check that the weights $(j(0),j(\infty))$ satisfy $j(0)-j(\infty)=d$.

3. The degree of Hurwitz-Hodge classes λ_1^e

In this section we compute the degree of the classes λ_1^e on moduli spaces of cyclic covers with exactly four branch points. We use the Atiyah-Bott localization formula (15) to obtain a relation that allows us to determine the desired degrees in terms of the known degrees of ψ classes and of zero-dimensional boundary strata.

We begin with a vanishing lemma aimed at simplifying the proof of Theorem 1.1.

Lemma 3.1. *Let d be a positive integer, $0\le e<d$ and m_1,m_2,m_3,m_4 be a monodromy datum. Assume that there exists an i between 1 and 4 such that $\left\langle\frac{em_i}{d}\right\rangle=0$. The orbifold class λ_1^e on the one-dimensional space of degree d cyclic admissible covers of a rational curve with monodromies m_i is 0.*

Proof. There is a natural morphism of moduli spaces

$$\varphi_e:Adm_d(m_1,\ldots,m_4)\to Adm_d(em_1,\ldots,em_4), \tag{16}$$

given by postcomposing an admissible cover, thought of as a map from a twisted curve to the stack $B\mathbb{Z}/d\mathbb{Z}$, with the endomorphism $\varphi_e:B\mathbb{Z}/d\mathbb{Z}\to B\mathbb{Z}/d\mathbb{Z}$ induced by multiplication by e. One may see that

$$\varphi_e^*(\lambda_1^1)=\lambda_1^e, \tag{17}$$

and therefore proving the Lemma is equivalent to showing that $\lambda_1^1=0$ when one of the monodromies is 0.

Assume without loss of generality that $m_4=0$. We have a forgetful morphism

$$\pi_4:Adm_d(m_1,m_2,m_3,0)\to Adm_d(m_1,m_2,m_3), \tag{18}$$

and $\lambda_1^1=\pi_4^*(\lambda_1^1)$. The vanishing follows immediately from λ_1^1 being a degree one class and $Adm_d(m_1,m_2,m_3)$ a zero-dimensional space.

□

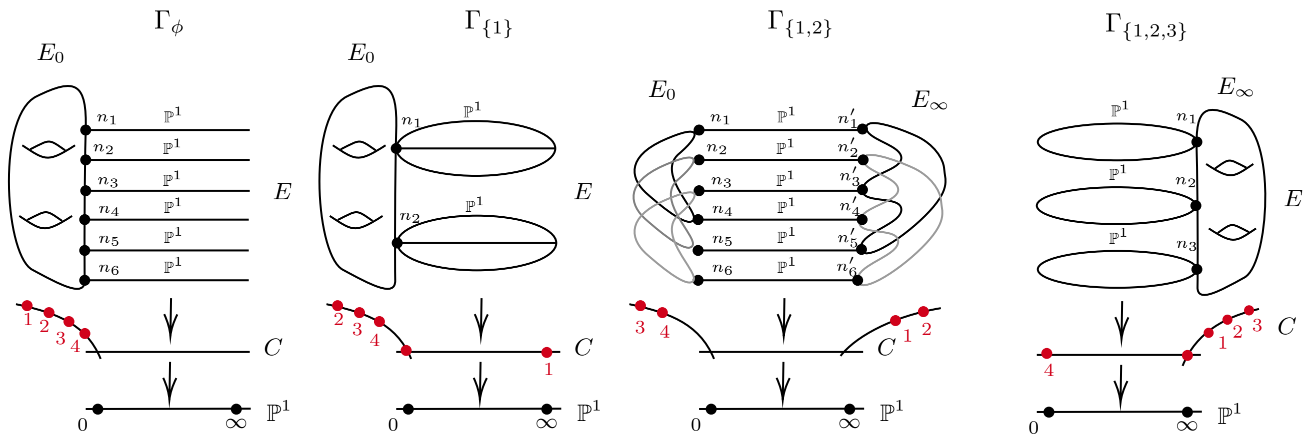

FIGURE 1. An example of the four types of fixed loci. We depicted the case $Adm_6\,(2,4,3,3)$ to illustrate the most complicated situation that may happen: in the fixed locus $\Gamma_{\{1,2\}}$ the curves contracting over 0 and ∞ may be disconnected even though the cover E is connected.

Proof of Theorem 1.1. The formulas in Theorem 1.1 are compatible with Lemma 3.1: the only case to check is when $\sum_{i=1}^4 \left\langle \frac{em_i}{d} \right\rangle = 2$; in this case $\left\langle \frac{em_1}{d} \right\rangle = 0$ implies that we are in the second line of (1), and the theorem predicts $\lambda_1^e = 0$. Therefore Lemma 3.1 proves the theorem when $\left\langle \frac{em_1}{d} \right\rangle = 0$. We can assume for the remainder of the proof that $\left\langle \frac{em_i}{d} \right\rangle \neq 0$ for all i.

By the orbifold Riemann-Roch computation (9), the rank of $\mathbb{E}_e$ is zero when $\sum_{i=1}^4 \left\langle \frac{em_i}{d} \right\rangle = 1$, which implies that $\lambda_1^e = 0$ in this case. Since $\lambda_1^e = \lambda_1^{d-e}$ by Lemma 2.12 and $\sum_{i=1}^4 \left\langle \frac{em_i}{d} \right\rangle = 1$ if an only if $\sum_{i=1}^4 \left\langle \frac{(d-e)m_i}{d} \right\rangle = 3$, we obtain that the class vanishes when the sum of the ages is 3. This completes the proof of the first line of (1).

The second and third lines are also equivalent: e, m_1, m_2, m_3, m_4 satisfy the two numerical conditions of the second line if and only if $d-e, m_1, m_2, m_3, m_4$ satisfy the conditions from the third line; the ordering of the fractional parts of $(d-e)m_i/d$ is reversed hence the smallest term is

$$\left\langle \frac{(d-e)m_4}{d} \right\rangle = 1 - \left\langle \frac{em_4}{d} \right\rangle .$$

Thus establishing that the third line in (1) holds completes the proof of the theorem.

The auxiliary integral. Given a monodromy datum (m_1, m_2, m_3, m_4) and an integer $1 \leq e \leq d-1$ satisfying the numerical conditions in the fourth line of (1), we consider the space of parameterized admissible covers, which we denote by $Adm_d\left(\mathbb{P}^1 | m_1, m_2, m_3, m_4\right)$. Letting f, π denote the tautological morphisms as in (4), we have

$$\int_{Adm_d(\mathbb{P}^1|m_1,m_2,m_3,m_4)} ev_4^*(c_1(\mathcal{O}_{\mathbb{P}^1}(1))) \cdot c_2(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_e)) = 0; \tag{19}$$

the integral (19) vanishes for dimension reasons: we are integrating a class of degree 3 on a space of dimension 4.

The natural $\mathbb{C}^*$ action on $\mathbb{P}^1$ induces a torus action on $Adm_d\left(\mathbb{P}^1 | m_1, m_2, m_3, m_4\right)$ by post-composition. We may therefore consider equivariant lifts of the integrands and evaluate the integral using the localization formula (15). Integration in equivariant cohomology yields a polynomial in the equivariant parameter, hence the fact that the degree of the integrand is stricly less than the dimension of the space implies that the vanishing of (19) continues to hold, regardless of the choice of equivariant lifts of the classes.

We choose the linearization $(j(0), j(\infty)) = (1, 0)$ for the bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ so that the first Chern class is represented by the class of the fixed point $0 \in \mathbb{P}^1$. For the bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ we choose the linearization $(0, 1)$.

Fixed loci. An admissible cover is fixed under the $\mathbb{C}^*$-action when the four evaluation morphisms have image contained in the fixed locus of $\mathbb{P}^1$, i.e. the two points $0, \infty$. Fixed loci may be indexed by elements of the power set $\mathcal{P}([4])$, assigning to a subset I the locus Γ_I of maps where the marked points in I are mapped to ∞, and those in I^c to 0.

Observe that if $4 \in I$, then $\Gamma_I \cap ev_4^*(c_1(\mathcal{O}_{\mathbb{P}^1}(1))) = \Gamma_I \cap ev_4^*(0) = \phi$, and therefore such fixed loci do not contribute to the localization computation. For all remaining fixed loci, we have $ev_4^*(0)_{|\Gamma_I} = c_1(\mathcal{O}_{\mathbb{P}^1}(1)_{|0}) = t$.

Normal bundles. The Euler class of the normal bundle $e(N_{\Gamma_I})$ has two types of contributions, that may be described in terms of the geometry of the base curves parameterized by the fixed locus, see Figure 1: if there is a marked point or a component contracting at p, one of the fixed points of $\mathbb{P}^1$, then a normal direction to the fixed locus may be identified with $T_p\mathbb{P}^1$. If a component $\tilde{C}$ of C contracts at p, then another normal direction is identified with the deformation space of the node; denoting by $\tilde{p} \in \tilde{C}$ the shadow of the node in the normalization of the curve, the deformation space of the node is described as $T_p\mathbb{P}^1 \boxtimes T_{\tilde{p}}\tilde{C}$.

Restriction of the integrand to fixed loci. The computation of the restriction of the class $c_2(R^1\pi_*f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_e))$ to a fixed locus Γ_I is slightly more sophisticated: the fibers of this bundle over a nodal cover $E \to B$ may be analyzed using the normalization sequence. Rather than attempting a general discussion here, we carry out such analysis explicitly for each fixed locus. Refer to Figure 1 where each type of fixed locus is depicted.

$\mathbf{\Gamma}_\phi \cong Adm_d\,(m_1, m_2, m_3, m_4, 0)$. A general point in this fixed locus corresponds to a cover $F : E \to C$, where $C = \mathbb{P}^1 \cup \tilde{C}$ is a nodal curve with exactly two components, one of which is mapped with degree 1 to $\mathbb{P}^1$ (and therefore denoted $\mathbb{P}^1$), while the other contracts over 0 . The cover E consists of a connected cyclic cover of $E_0 \to \tilde{C}$ and d copies of $\mathbb{P}^1$ mapping with degree 1 to $\mathbb{P}^1$. Tensoring the normalization sequence

$$0 \to \mathcal{O}_E \to \mathcal{O}_{E_0} \oplus \bigoplus_{i=1}^{d} \mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{i=1}^{d} \mathbb{C}_{n_i} \to 0 \tag{20}$$

by the invertible sheaf $f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_e)$, taking the long exact sequence in cohomology and using (8), we obtain:

$$0 \to L_0 \to H^1(E, F^*(\mathcal{O}_{\mathbb{P}^1}(-1))_{d-e} \to H^1(E_0, \mathcal{O})_{d-e} \to 0. \tag{21}$$

Globalizing the fiberwise computation in (21), we obtain:

$$R^1\pi_*f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_e))_{|\Gamma_\phi} = ((\mathbb{E}^\vee)_{d-e} \oplus L_0)\,. \tag{22}$$

The contribution of Γ_ϕ to the localization computation of (19) is then:

$$\mathrm{Cont}(\Gamma_\phi) = \int_{\Gamma_\phi} \frac{t\ c_2\left((\mathbb{E}^\vee)_{d-e} \oplus L_0\right)}{t(t-\psi_0)} = 0 \tag{23}$$

where the vanishing holds because the bundle $(\mathbb{E}^\vee)_{d-e}$ has rank 1 by (9).

$\mathbf{\Gamma_{\{j\}}} \cong Adm_d\,(m_1, m_2, m_3, m_4)$. For a general point in this fixed locus, the base $C = \mathbb{P}^1 \cup \tilde{C}$ is a nodal curve with exactly two components, as in the previous case. The cover E consists of a connected cyclic cover of $E_0 \to \tilde{C}$ and $q_j := \gcd(m_j, d)$ copies of $\mathbb{P}^1$ mapping with degree $r_j := d/\gcd(m_j, d)$ to $\mathbb{P}^1$. The normalization sequence

$$0 \to \mathcal{O}_E \to \mathcal{O}_{E_0} \oplus \bigoplus_{i=1}^{q_j} \mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{i=1}^{q_j} \mathbb{C}_{n_i} \to 0 \tag{24}$$

tensored by $f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_e)$ gives rise to the long exact sequence in cohomology[3]:

(25)

$$0 \to L_0 \otimes \left(\bigoplus_{i=1}^{q_j} \mathcal{O}_{i\frac{d}{q_j}}\right)_{d-e} \to H^1(E, F^*(\mathcal{O}_{\mathbb{P}^1}(-1)))_{d-e} \to H^1(E_0, \mathcal{O})_{d-e} \oplus \left(\bigoplus_{i=1}^{q_j} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-r_j)) \otimes \mathcal{O}_{i\frac{d}{q_j}}\right)_{d-e} \to 0.$$

The first non-zero term in (25) is the decomposition of the permutation representation induced by the action of $\mathbb{Z}/d\mathbb{Z}$ on the nodes of C. The generator [1] cyclically permutes the nodes hence the element $[q_j]$ fixes all points. In choosing the $(d-e)$-th eigenspaces of the various representations in (25), we must consider two distinct cases.
If d divides q_je, (25) reduces to:

$$0 \to L_0 \to H^1(E, F^*(\mathcal{O}_{\mathbb{P}^1}(-1)))_{d-e} \to H^1(E_0, \mathcal{O})_{d-e} \to 0; \tag{26}$$

[3]Note that in (24) and in the following long exact sequence, $\mathbb{P}^1$ refers to the rational components in the cover curve, and not the source.

If $d \not| \, q_j e$, then we have:

$$0 \to H^1(E, F^*(\mathcal{O}_{\mathbb{P}^1}(-1)))_{d-e} \to H^1(E_0, \mathcal{O})_{d-e} \oplus L_{\left\langle \frac{em_j}{d} \right\rangle} \to 0. \tag{27}$$

In both cases one may deduce:

$$R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_e))_{|\Gamma_{\{j\}}} = ((\mathbb{E}^\vee)_{d-e}) \oplus L_{\left\langle \frac{em_j}{d} \right\rangle}. \tag{28}$$

The contribution of $\Gamma_{\{j\}}$ to the localization computation of (19) is then:

$$\mathrm{Cont}(\Gamma_{\{j\}}) = \int_{\Gamma_{\{j\}}} \frac{t\ c_2(((\mathbb{E}^\vee)_{d-e}) \oplus L_{\left\langle \frac{em_j}{d} \right\rangle})}{-t^2(t-\psi_0)} = \frac{\left\langle \frac{m_j e}{d} \right\rangle}{t} \int_{Adm_d(m_1,m_2,m_3,m_4)} \lambda_1^e. \tag{29}$$

$\mathbf{\Gamma_{\{1,2,3\}}} \cong Adm_d\,(m_1, m_2, m_3, m_4)$. The analysis for this fixed locus is very similar to the previous case, the main difference being that the contracting component E_∞ now lies over $\infty \in \mathbb{P}^1$. Denoting $q_4 := \gcd(m_4, d)$ and $r_4 := d/\gcd(m_4, d)$, the long exact sequence in cohomology is then:

$$0 \to L_1 \otimes \left(\bigoplus_{i=1}^{q_4} \mathcal{O}_{i\frac{d}{q_4}} \right)_{d-e} \to H^1(E, F^*(\mathcal{O}_{\mathbb{P}^1}(-1)))_{d-e} \to H^1(E_\infty, \mathcal{O})_{d-e} \otimes L_1 \oplus \left(\bigoplus_{i=1}^{q_4} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-r_4)) \otimes \mathcal{O}_{i\frac{d}{q_4}} \right)_{d-e} \to 0. \tag{30}$$

As in the previous case one must analyse separately the cases when d does or doesn't divide $q_4 e$, but in both cases one may write:

$$R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_e))_{|\Gamma_{\{1,2,3\}}} = ((\mathbb{E}^\vee)_{d-e}) \otimes L_1 \oplus L_{1-\left\langle \frac{em_4}{d} \right\rangle}. \tag{31}$$

The contribution of $\Gamma_{\{1,2,3\}}$ to the localization computation of (19) is then:

$$\mathrm{Cont}(\Gamma_{\{1,2,3\}}) = \int_{\Gamma_{\{1,2,3\}}} \frac{t\ c_2(((\mathbb{E}^\vee)_{d-e}) \otimes L_1 \oplus L_{1-\left\langle \frac{m_4 e}{d} \right\rangle})}{t^2(t+\psi_\infty)} = \frac{\left\langle \frac{m_4 e}{d} \right\rangle - 1}{t} \int_{Adm_d(m_1,m_2,m_3,m_4)} (\lambda_1^e + \psi_\infty). \tag{32}$$

$\mathbf{\Gamma_{\{i,j\}}} \cong d \cdot Adm_d\,(m_k, m_4, 2d - m_k - m_4) \times Adm_d\,(m_i, m_j, d - m_i - m_j)$. These are zero dimensional fixed loci parameterizing covers with contracting components over both 0 and ∞, so the only non-zero contributions in the localization computation arise from integrating classes which are multiples of some power of the equivariant parameter. However, because we are in the numerical situation $\left\langle \frac{em_1}{d} \right\rangle + \left\langle \frac{em_4}{d} \right\rangle > 1$, we can conclude from orbifold Riemann-Roch (Theorem 2.4) that for every choice of $k \in \{1, 2, 3\}$, $p_1^*(\mathbb{E}^\vee)_{d-e}$ has rank one. Analyzing the long exact sequence in cohomology arising from tensoring the normalization sequence with $f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_e)$, one sees that $p_1^*(\mathbb{E}^\vee)_{d-e}$ is a summand for the rank two bundle $R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_e))_{|\Gamma_{\{i,j\}}}$. It follows that $c_2(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_e))_{|\Gamma_{\{i,j\}}})$ is a multiple of λ_1^e and therefore it has no term which is a pure multiple of t^2, forcing $\mathrm{Cont}(\Gamma_{\{i,j\}}) = 0$.

Evaluation of auxiliary integral. Adding all contributions (and ignoring the global factor of $1/t$), we obtain:

$$\begin{aligned} 0 &= \sum_{I \in [3]} \mathrm{Cont}(\Gamma_I) \\ &= \int_{Adm_d(m_1,m_2,m_3,m_4)} \left(\sum_{j=1}^{3} \left\langle \frac{em_j}{d} \right\rangle \lambda_1^e \right) + \left(\left\langle \frac{em_4}{d} \right\rangle - 1 \right) (\lambda_1^e + \psi_\infty) \\ &= \left(\int_{Adm_d(m_1,m_2,m_3,m_4)} \lambda_1^e \right) + \frac{\left\langle \frac{em_4}{d} \right\rangle - 1}{d}, \end{aligned} \tag{33}$$

where we have used that $\sum_{j=1}^4 \left\langle \frac{em_j}{d} \right\rangle = 2$ and that the degree of the class ψ_∞ is $1/d$. The result follows immediately solving for the degree of λ_1^e. $\square$

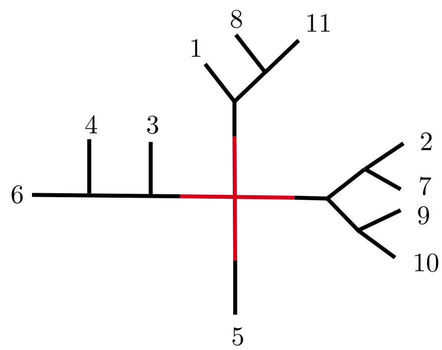

FIGURE 2. A boundary curve representing the curve class $C(\{1,8,11\},\{2,7,9,10\},\{3,4,6\},\{5\})$. Note that when a subset is a singleton, by trivalent tree we intend a single end and no vertices.

3.1. **The classes λ_1^e for higher dimensional spaces.** We now investigate the class λ_1^e as a divisor on an arbitrary moduli space of cyclic covers of a rational curve. We describe a general procedure to express this class as a linear combination of strata, and compute explicitly some infinite families of examples where the result gives an especially nice formula.

Theorem 3.2. *Let $(m_1,\dots,m_n)$ be a monodromy datum and consider the moduli space $Adm_d(m_1,\dots,m_n)$. The class λ_1^e can be computed as a linear combination of boundary strata using the results of Theorem 1.1.*

Proof. Let $\{X,Y,Z,W\}$ be a four part partition of the set $[n]$ with none of the four subsets being empty. By $C_{(X,Y,Z,W)} \in A_1(Adm_d(m_1,\dots,m_n))$ we denote a class of boundary curves on the moduli space $Adm_d(m_1,\dots,m_n)$ given as the pull-back via the coarse moduli map of the curve class in $\overline{M}_{0,n}$ we now describe. Start with a four valent star graph and attach to the four ends four trivalent trees, each one with ends labeled by all the marks in one of the four subsets X,Y,Z and W, see Figure 2. It is well-known that picking different trivalent trees yields linearly equivalent cycles: any two trivalent graphs may be connected by a sequence of Whitehead moves (see [GdPK23, Figure 1]), which in turn give a linear equivalence pushed-forward via a boundary gluing morphism.

Denote by $x = (\sum_{m_i\in X} m_i) \mod d$, and define similarly y,z,w. Any boundary curve in the class $C_{(X,Y,Z,W)}$ is isomorphic to $Adm_d(x,y,z,w)$. By the natural splitting properties of the Hodge bundle on strata we have

$$\int_{Adm_d(m_1,\dots,m_n)} C_{(X,Y,Z,W)} \cdot \lambda_1^e = \int_{Adm_d(x,y,z,w)} \lambda_1^e, \tag{34}$$

and therefore the intersections of λ_1^e with all boundary curves in $Adm_d(m_1,\dots,m_n)$ is given by Theorem 1.1.

The Chow group of curves $A_1(Adm_d(m_1,\dots,m_n))$ is generated by the curve classes $C_{(X,Y,Z,W)}$, and the group of codimension one classes $A^1(Adm_d(m_1,\dots,m_n))$ is generated by boundary divisors. Working with rational coefficients, this follows from the fact that the coarse moduli space of the space of cyclic admissible covers is $\overline{M}_{0,n}$, and the presentation for the Chow ring of $\overline{M}_{0,n}$ in [Kee92]. In order to express λ_1^e as a linear combination of boundary divisors, it suffices to choose subsets of the boundary curves and of the boundary divisors that provide bases for the two respective Chow groups, invert the intersection matrix among these chosen classes, and multiply by the vector of intersection numbers of the chosen boundary curves with λ_1^e. $\square$

Proof of Theorem 1.2. Since the class λ_1^1 is invariant under the S_n-action permuting the branch points of $Adm_d(1^n)$, we seek a formula in terms of the symmetric boundary divisors Δ_i, i.e. the sum of all divisors parameterizing nodal curves where at least one of the components has exactly i marked points. It is well known, see for example [AGSS12, Section 2.2], that the symmetrized boundary divisors are a basis for $A^1(\overline{M}_{0,n}^{S_n})$. We denote by $C_{(1,1,j,n-j-2)}$ any boundary curve $C_{(X,Y,Z,W)}$, where $|X|=|Y|=1, |Z|=j$ and $|W|=n-j-2$ (since we will be doing intersections with symmetic boundary divisors, only the sizes of the four subsets matters). In [AGSS12, Section 4] it is shown that the intersection matrix of the symmetric divisors with these curves is invertible. Therefore, to prove the theorem it suffices to show that the two sides of the equations in (1) and (2) from Theorem 1.2 have the same intersection with all boundary curves of type $C_{(1,1,j,n-j-2)}$.

CASE $n = d$: for all i we have $C_{(1,1,j,d-j-2)} \cong Adm_d(1, 1, j, d-j-2)$; since the monodromies at the four points add to d, the intersection of λ_1^1 with all boundary curves is equal to 0 (first line of Theorem 1.1). This proves that $\lambda_1^1 = 0$.

CASE $n = 2d$: if $j < d-1$, then $2d - j - 2 > d$ and $C_{(1,1,j,2d-j-2)} \cong Adm_d(1, 1, j, d-2-j)$. The same argument as before applies and the intersection with λ_1^1 vanishes.

When $j = d-1$, we have $C_{(1,1,d-1,d-1)} \cong Adm_d(1, 1, d-1, d-1)$ and by the second line of Theorem 1.1, $C_{(1,1,d-1,d-1)} \cdot \lambda_1^1 = 1/d$.

We use the following intersection numbers, which are obtained by multiplying by $1/d$ (global isotropy factor in admissible cover spaces) the corresponding well-known intersection numbers in $\overline{M}_{0,n}^{S_n}$:

$$d\kappa_1 \cdot C_{(1,1,j,2d-j-2)} = 1, \qquad d\Delta_i \cdot C_{(1,1,j,2d-j-2)} = \begin{cases} 3 & \text{if } i = 2, j = 1 \\ 2 & \text{if } i > 2, j = i-1 \\ 1 & \text{if } i = 2, j \geq 2 \\ -1 & \text{if } i = j+2 \text{ or } (i = j, i \neq 2, d-1) \\ -2 & \text{if } i = d-1, j = d-1 \\ 0 & \text{else} \end{cases} \tag{35}$$

With the intersection numbers in (35) it is immediate to verify:

$$\frac{1}{2}\left(-\kappa_1 + \sum_{i=2}^{d} (i-1)\,\Delta_i\right) \cdot C_{(1,1,j,2d-j-2)} = \begin{cases} \frac{1}{2d}(-1+3-2) = 0, & \text{if } j = 1 \\ \frac{1}{2d}(-1+1-(j-1)+2j-(j+1)) = 0, & \text{if } 1 < j < d-1 \\ \frac{1}{2d}(-1+1-2(d-2)+2(d-1)) = \frac{1}{d} & \text{if } j = d-1 \end{cases}, \tag{36}$$

which concludes the proof. □

Remark 3.3. Theorem 1.2 does not give a formula for λ_1^1 exclusively in terms of boundary divisors. This can be achieved by using the relation $\kappa_1 = \sum_{i=1}^{n} \psi_i - \sum_{i=2}^{\lfloor n/2 \rfloor} \Delta_i$, plus well known boundary expressions for ψ classes. We chose to leave the class κ_1 in our formula, as it makes the expression especially simple.

4. The first Chern class of the full Hodge bundle $\mathbb{E}$

4.1. **The one dimensional case.** In this section we prove Theorem 1.3. Before we begin the computation, we rephrase the statement of Theorem 1.3 in a less symmetric but more compact way.

Corollary 4.1. *The degree of λ_1 on the space $Adm_d(m_1, m_2, m_3, m_4)$ may be expressed as:*

$$\int_{Adm_d(m_1,m_2,m_3,m_4)} \lambda_1 = \frac{1}{12d^2}\left(d^2 - \sum_{i=1}^{4} \gcd^2(m_i, d) + \sum_{i=1}^{3} \gcd^2(m_i + m_4, d)\right). \tag{37}$$

Proof. Since $m_1 + m_2 + m_3 + m_4 = 0 \in \mathbb{Z}/d\mathbb{Z}$ we have that $\gcd\left(\sum_{i \in I} m_i, d\right) = \gcd\left(\sum_{i \in I^c} m_i, d\right)$. Formula (37) is obtained from (2) by taking only one representative for each pair I, I^c and doubling its contribution. □

We also specialize the theorem to the case of prime degree as the result is especially elegant in this case.

Corollary 4.2. *If $d = p$ is a prime number, then the degree of λ_1 on the space $Adm_p(m_1, m_2, m_3, m_4)$ is a rational function of p. Precisely:*

$$\int_{Adm_p(m_1,m_2,m_3,m_4)} \lambda_1 = \begin{cases} 0 & \text{if } 0 \in \{m_1, m_2, m_3, m_4\}, \\ \frac{p^2-1}{12p^2} & \text{if } m_i + m_j \neq 0 \in \mathbb{Z}/p\mathbb{Z} \text{ for all } i, j, \\ \frac{p^2-1}{6p^2} & \text{if } \{m_1, m_2, m_3, m_4\} = \{i, p-i, j, p-j\}, \text{ all distinct}, \\ \frac{p^2-1}{4p^2} & \text{if } \{m_1, m_2, m_3, m_4\} = \{i, p-i\}. \end{cases} \tag{38}$$

Proof of Theorem 1.3. Consider the auxiliary vanishing integral:

$$\int_{Adm_d(\mathbb{P}^1|m_1,m_2,m_3,m_4)} ev_4^*(c_1(\mathcal{O}_{\mathbb{P}^1}(1)))\cdot c_2(R^1\Pi_*F^*(\mathcal{O}_{\mathbb{P}^1}(-1)))=0; \tag{39}$$

We adopt the same localization set-up as in Section 3, with the only difference that the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ is pull-pushed via the morphisms from the universal cover curve $\mathcal{U}_E$. The fixed loci, their normal bundles, and the restriction of the class $ev_4^*(c_1(\mathcal{O}_{\mathbb{P}^1}(1)))$ are the same as in Section 3. Here we analyze the restrictions of the class $c_2(R^1\Pi_*F^*(\mathcal{O}_{\mathbb{P}^1}(-1)))$. These computations have been coded and the code is available upon request.

$\mathbf{\Gamma}_\phi \cong Adm_d\,(m_1,m_2,m_3,m_4,0)$. Tensoring the normalization sequence (20) by the invertible sheaf $F^*(\mathcal{O}_{\mathbb{P}^1}(-1))$ and taking the long exact sequence in cohomology, we obtain:

$$0\to L_0\to L_0^{\oplus d}\to H^1(E,F^*(\mathcal{O}_{\mathbb{P}^1}(-1)))\to H^1(E_0,\mathcal{O})\to 0. \tag{40}$$

Globalizing the fiberwise computation in , we obtain:

$$c_2(R^1\Pi_*F^*(\mathcal{O}_{\mathbb{P}^1}(-1))_{|\Gamma_\phi})=c_2(((\mathbb{E}^\vee)\oplus\mathcal{O}^{\oplus d-1}))=\lambda_2. \tag{41}$$

The contribution of Γ_ϕ to the localization computation of (39) is then:

$$\mathrm{Cont}(\Gamma_\phi)=\int_{\Gamma_\phi}\frac{t\ \lambda_2}{t(t-\psi_0)}=0 \tag{42}$$

where the vanishing holds by the projection formula because the class λ_2 is obtained by pull-back of the homonymous class via the forgetful morphism $\pi_0 : Adm_d\,(m_1,m_2,m_3,m_4,0)\to Adm_d\,(m_1,m_2,m_3,m_4)$ and $\lambda_2=0$ on $Adm_d(m_1,m_2,m_3,m_4)$ for dimension reasons.

$\mathbf{\Gamma_{\{j\}}}\cong Adm_d\,(m_1,m_2,m_3,m_4)$. Denoting $q_j=\gcd(m_j,d)$ the number of nodes over $0\in\mathbb{P}^1$ and $r_j=d/q_j$, the relevant long exact sequence in cohomology is:

$$0\to L_0\to\bigoplus_{i=1}^{q_j}L_0\to H^1(E,F^*(\mathcal{O}_{\mathbb{P}^1}(-1)))\to H^1(E_0,\mathcal{O})\oplus\bigoplus_{i=1}^{q_j}H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-r_j))\to 0. \tag{43}$$

Globally one observes that the bundles with fiber $H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-r_j))$ are trivial, but not equivariantly trivial. Computing the torus weights (see [Cav06, Section 2.2, Term 2], for example) one obtains

$$H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-r_j))=\bigoplus_{i=1}^{r_j-1}L_{\frac{i}{r_j}}$$

Globally,

$$R^1\Pi_*F^*(\mathcal{O}_{\mathbb{P}^1}(-1))_{|\Gamma_{\{j\}}}=(L_0^{\oplus q-1}\oplus\mathbb{E}^\vee)\oplus\left(\bigoplus_{i=1}^{r_j-1}L_{\frac{i}{r_j}}\right)^{\oplus q_j}. \tag{44}$$

The second Chern class of the bundle in (55) is computed using the formal properties of Chern classes from Section 2.3. Since $c_t^{eq}(L_0)=1$ and we will be integrating on a one dimensional space, the only contributing part of c_2 is given by

$$c_1(\mathbb{E}^\vee)c_1\left(\left(\bigoplus_{i=1}^{r_j-1}L_{\frac{i}{r_j}}\right)^{\oplus q_j}\right)+c_2\left(\left(\bigoplus_{i=1}^{r_j-1}L_{\frac{i}{r_j}}\right)^{\oplus q_j}\right). \tag{45}$$

One computes

$$c_1\left(\bigoplus_{i=1}^{r_j-1}L_{\frac{i}{r_j}}\right)=\frac{r_j-1}{2}t\qquad c_2\left(\bigoplus_{i=1}^{r_j-1}L_{\frac{i}{r_j}}\right)=\frac{(r_j-1)(r_j-2)(3r_j-1)}{24r_j}t^2. \tag{46}$$

Using Lemma 2.8 and the relation $q_jr_j=d$, one then evaluates (45) to:

$$C_j:=-\frac{d-q_j}{2}\lambda_1t+\frac{(d-q_j)(3d^2-3dq_j-4d+2q_j)}{24d}t^2 \tag{47}$$

The contribution of $\Gamma_{\{i\}}$ to the localization computation of (39) is then:

$$\text{(48)}\quad \mathrm{Cont}(\Gamma_{\{j\}}) = \int_{\Gamma_{\{j\}}} \frac{t\ C_j}{-t^2(t-\psi_0)} = \frac{1}{t}\left[\left(\frac{d-q_j}{2}\int_{Adm_d(m_1,m_2,m_3,m_4)}\lambda_1\right) - \frac{(d-q_j)(3d^2-3dq_j-4d+2q_j)}{24d^2}\right].$$

$\mathbf{\Gamma_{\{1,2,3\}}} \cong Adm_d\,(m_1,m_2,m_3,m_4)$. This fixed locus parameterizes curves with one component contracting over $\infty \in \mathbb{P}^1$. We denote $q_4 = \gcd(m_4, d)$ the number of nodes over ∞ and $r_4 = d/q_4$. The long exact sequence in cohomology is:

$$\text{(49)}\qquad 0 \to L_1 \to \bigoplus_{i=1}^{q_4} L_1 \to H^1(E, F^*(\mathcal{O}_{\mathbb{P}^1}(-1))) \to H^1(E_\infty, \mathcal{O}) \otimes L_1 \oplus \bigoplus_{i=1}^{q_4} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-r_4)) \to 0.$$

Globally one obtains:

$$\text{(50)}\qquad R^1\Pi_* F^*(\mathcal{O}_{\mathbb{P}^1}(-1))_{|\Gamma_{\{1,2,3\}}} = (L_1^{q_4-1} \oplus (\mathbb{E}^\vee \otimes L_1)) \oplus \left(\bigoplus_{i=1}^{r_4-1} L_{\frac{i}{r_4}}\right)^{\oplus q_4}.$$

The bundle $L_1^{q_4-1} \oplus (\mathbb{E}^\vee \otimes L_1)$ has $q_4 - 1$ Chern roots equal to t, plus the Chern roots $-\alpha_i + t$, where α_i is a Chern root of the Hodge bundle $\mathbb{E}$. It follows that:

$$\text{(51)}\quad c_1(L_1^{q_4-1} \oplus \mathbb{E}^\vee \otimes L_1) = -\lambda_1 + (g+q_4-1)t \qquad c_2(L_1^{q_4-1} \oplus \mathbb{E}^\vee \otimes L_1) = \binom{g+q_4-1}{2}t^2 - (g+q_4-2)t\lambda_1 + \lambda_2.$$

Combining (46) and (51), and neglecting the λ_2 term which will not survive integration, one has that the relevant part of the second Chern class of (50) is

(52)

$$C_{1,2,3} := \frac{4-d-q_j-2g}{2}\lambda_1 t + \frac{3d^3+12gd^2+6d^2q_4-16d^2+12g^2d+12gdq_4+3{q_4}^2d-36gd-18dq_4+24d-2{q_4}^2}{24d}t^2$$

The contribution of $\Gamma_{\{1,2,3\}}$ to the localization computation of (39) is then:

$$\mathrm{Cont}(\Gamma_{\{1,2,3\}}) = \int_{\Gamma_{\{1,2,3\}}} \frac{t\ C_{1,2,3}}{t^2(t+\psi_\infty)} = \frac{1}{t}\left[\frac{4-d-q_4-2g}{2}\int_{Adm_d(m_1,m_2,m_3,m_4)}\lambda_1^e\right.$$

$$\text{(53)}\qquad \left. - \frac{3d^3+12gd^2+6d^2q_4-16d^2+12g^2d+12gdq_4+3{q_4}^2d-36gd-18dq_4+24d-2{q_4}^2}{24d^2}\right].$$

$\mathbf{\Gamma_{\{i,j\}}} \cong d \cdot Adm_d\,(m_k, m_4, 2d-m_k-m_4) \times Adm_d\,(m_i, m_j, d-m_i-m_j)$. These are zero dimensional fixed loci parameterizing covers with contracting components over both 0 and ∞. There are several discrete invariants associated to this fixed locus; we recall those that are used in the computation: there are $\gcd(m_i, m_j, d)$ components of the cover contracting to $\infty \in \mathbb{P}^1$; we denote by g_{ij} the genus of the (possibly disconnected) cover contracting to $\infty \in \mathbb{P}^1$; we denote $q_{ij} = \gcd(m_i + m_j, d)$ the number of rational components of the cover mapping onto $\mathbb{P}^1$, and $r_{ij} = d/q_{ij}$. Since the only non-zero contributions in the localization computation arise from integrating classes which are multiples of some power of the equivariant parameter, we may simplify the normalization sequence by setting all geometric Chern roots of the Hodge bundles appearing to 0, to obtain:

$$\text{(54)}\qquad 0 \to L_1^{\gcd(m_i,m_j,d)} \to L_1^{q_{ij}} \to H^1(E, F^*(\mathcal{O}_{\mathbb{P}^1}(-1))) \to L_1^{g_{ij}+\gcd(m_i,m_j,d)-1} \oplus \left(\bigoplus_{i=1}^{r-1} L_{\frac{i}{r_{ij}}}\right)^{q_{ij}} \to 0.$$

It follows that:

$$\text{(55)}\qquad R^1\Pi_* F^*(\mathcal{O}_{\mathbb{P}^1}(-1))_{|\Gamma_{\{i,j\}}} = L_1^{q_{ij}+g_{ij}-1} \oplus \left(\bigoplus_{i=1}^{r_{ij}-1} L_{\frac{i}{r_{ij}}}\right)^{\oplus q_{ij}}.$$

One can observe that (55) formally agrees with (50) upon substituting g with g_{ij} and q_4 with q_{ij}. The second Chern class of $R^1\Pi_* F^*(\mathcal{O}_{\mathbb{P}^1}(-1))_{|\Gamma_{\{i,j\}}}$ therefore may be read off from the t^2 coefficient of $C_{1,2,3}$.

The contribution of $\Gamma_{i,j}$ to the localization computation of (39) is then:

$$\mathrm{Cont}(\Gamma_{i,j}) = \frac{3d^3 + 12g_{ij}d^2 + 6d^2q_{ij} - 16d^2 + 12g_{ij}^2 d + 12g_{ij}dq_{ij} + 3{q_{ij}}^2 d - 36g_{ij}d - 18dq_{ij} + 24d - 2{q_{ij}}^2}{24d^2} \tag{56}$$

The theorem readily follows by solving the equation

$$0 \quad = \quad \sum_{I\in[3]} \mathrm{Cont}(\Gamma_I) \tag{57}$$

for λ_1 and using the Riemann-Hurwitz formula to substitute $g = d + 1 - (q_1 + q_2 + q_3 + q_4)/2$ and $g_{ij} = 1 - (q_i + q_j + q_{ij} - d)/2$. □

4.2. **Graph formula for λ_1.** In this section we prove Theorem 1.4, expressing the class λ_1 on a general space of cyclic admissible covers as a linear combination of boundary strata and of the classes ψ_i and κ_1.

Since the group $A_1(Adm_d\,(m_1, \ldots, m_n))$ is generated by classes of boundary curves $C_{(X,Y,Z,W)}$, to establish formula (3) it suffices to show the truth of the numerical equations following from intersecting with each boundary curve.

Choose a bijection $b : [4] \to \{X, Y, Z, W\}$. From Theorem 1.3, it follows:

$$C_{(X,Y,Z,W)} \cdot \lambda_1 \quad = \quad \frac{1}{24d^2}\left(\sum_{I\in\mathcal{P}([4])} (-1)^{|I|}\mathrm{gcd}^2\left(\sum_{j\in\left(\bigcup_{i\in I} b(i)\right)} m_j, d\right)\right), \tag{58}$$

where we don't worry about reducing anything modulo d as that operation is irrelevant when then taking a gcd with d itself.

From standard boundary intersection theory in $\overline{M}_{0,n}$, together with the fact that $Adm_d\,(m_1, \ldots, m_n)$ is a $B(\mathbb{Z}/d\mathbb{Z})$ gerbe over $\overline{M}_{0,n}$, one has:

$$C_{(X,Y,Z,W)} \cdot \Delta_J = \begin{cases} \frac{(-1)^{|I|}}{d} & \exists\, I \text{ in } \mathcal{P}([4]) \text{ s.t. } J = \bigcup_{i\in I} b(i), \\ 0 & \text{else.} \end{cases} \tag{59}$$

Observe that the formula holds also when some of the sets X, Y, Z, W are singletons because of the definition $\Delta_{\{j\}} = \Delta_{[n]\smallsetminus\{j\}} = -\psi_j$. It is now immediate to verify that intersecting $C_{(X,Y,Z,W)}$ with the left hand side of (3) produces the left hand side of (58), concluding the proof of Theorem 1.4.

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