$(H^p(\mathbb{D}), \|\cdot\|_{H^p(\mathbb{D})})$ is a Banach Space

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Definition of H^p Spaces

Definition [H^p Space]

H^p, or **Hardy spaces**, are classes of complex-analytic functions. Here, we will work solely with the (open) unit disc:

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}.$$

Let $F : \mathbb{D} \to \mathbb{C}$. For $0 < r < 1$, define $F_r : [0, 2\pi] \to \mathbb{C}$ by

$$F_r(\theta) = F\left(re^{i\theta}\right).$$

Definition of H^p Spaces (Continued)

Definition [H^p Space (Continued)]

- For $1 \leq p < \infty$, define the p-norm:

$$\|F\|_{H^p(\mathbb{D})} = \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| F\left(re^{i\theta}\right) \right|^p \, d\theta \right)^{1/p}.$$

- For $p = \infty$, define:

$$\|F\|_{H^\infty(\mathbb{D})} = \sup_{z \in \mathbb{D}} |F(z)|.$$

The Hardy space $H^p(\mathbb{D})$ is then defined as:

$$H^p(\mathbb{D}) = \{F : \mathbb{D} \to \mathbb{C} \mid \|F\|_{H^p} < \infty\}.$$

Some Interesting Results about H^p Spaces

- $(H^p, \|\cdot\|_{H^p})$ is a **Banach space** for all $1 \leq p \leq \infty$.
- H^p spaces are nested; that is, for $p \leq q$:

$$H^\infty \subset H^q \subset H^p \subset H^1.$$

- **Fatou's Theorem:** Every function in H^p has radial limits almost everywhere on the unit circle.
- **Poisson Representation**: Let $f \in H^1(\mathbb{D})$ with boundary function $f(e^{i\theta}) = \phi(\theta) \in L^1([0, 2\pi])$. Then for $z = re^{i\theta}$:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} \, \phi(t) \, dt.$$

This formula expresses $f(z)$ explicitly in terms of its boundary values $\phi(t)$.

Proof that H^p is a Banach Space

Theorem

$(H^p(\mathbb{D}), \|\cdot\|_{H^p(\mathbb{D})})$ *is a* **Banach space** *for all* $1 \leq p \leq \infty$

Proof.

It is rather straightforward to show $(H^p, \|\cdot\|_{H^p})$ is a normed vector space as is L^p. Next, we must show that every Cauchy Sequence converges in H^p. For ease of notation, we will define from the outset

$$N_p(r, F) = \left(\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^p d\theta \right)^{1/p}$$

Let $\{F_n\}_{n \in \mathbb{N}}$ be a Cauchy Sequence in H^p. Then for any $\varepsilon > 0$, $\exists N_\varepsilon$ such that $\forall n, m \geq N_\varepsilon$, we have

$$\|F_n - F_m\|_{H^p} = \sup_{0 \leq r < 1} N_p(r, F_n - F_m) < \varepsilon$$

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Proof.

Claim: For any fixed $z \in \mathbb{D}$ and $n \in \mathbb{N}$, the sequence $F_n(z)$ is Cauchy in $\mathbb{C}$ and thus $F_n(z) \to F(z)$ in $\mathbb{C}$.

Proof of Claim: Since F_n is analytic in $\mathbb{D}$, then by Cauchy's Integral formula we can write

$$F_n(z) = \frac{1}{2\pi i} \int_{C_r} \frac{F_n(\omega)}{\omega - z} d\omega$$

where $C_r := \{|\omega| = r\}$ with $|z| < r < 1$.

$\square$

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Proof.

What follows is the series of calculations after parametrization $\omega = re^{i\theta} \implies d\omega = ire^{i\theta} d\theta$:

$$\begin{aligned}
|F_n(z)| &\leq \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{F_n\left(re^{i\theta}\right)}{re^{i\theta} - z} re^{i\theta}\, d\theta \right| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\left|F_n\left(re^{i\theta}\right)\right|}{|re^{i\theta} - z|} r\, d\theta \\
&\leq \frac{r}{r - |z|} \frac{1}{2\pi} \int_0^{2\pi} |F_n(re^{i\theta})| d\theta
\end{aligned}$$

$\square$

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Proof.

Observe now that by Holder's Inequality with $p \geq 1$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$ we have that

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |F_n(re^{i\theta})| d\theta &\leq \frac{1}{2\pi}\left(\int_0^{2\pi} |F_n(re^{i\theta})|^p d\theta\right)^{1/p} \left(\int_0^{2\pi} |1|^{p'} d\theta\right)^{1/p'} \\
&= \frac{1}{2\pi} N_p(r, F_n)(2\pi)^{1/p'} \\
&= (2\pi)^{\frac{1}{p'}-1} N_p(r, F_n) \\
&\leq N_p(r, F_n)
\end{aligned}
$$

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Proof.

By definition, $\|F_n(re^{i\theta})\|_{H^p} = \sup_{0 \leq r < 1} N_p(r, F_n)$, and hence

$$|F_n(z)| \leq \frac{r}{r - |z|} N_p(r, F_n) \leq \frac{r}{r - |z|} \|F_n\|_{H^p(\mathbb{D})}$$

Now since z is fixed, we can set $r = \frac{1+|z|}{2}$ and so

$$|F_n(z)| \leq \frac{1 + |z|}{1 - |z|} \leq \frac{2}{1 - |z|} \|F_n\|_{H^p(\mathbb{D})}$$

Now we can set $\alpha_z = \frac{2}{1-|z|}$

$\square$

Proof that H^p is a Banach Space

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Proof.

Thus, for $n, m \geq N_\varepsilon$, we have that

$$|F_n(z) - F_m(z)| \leq \alpha_z \|F_n - F_m\|_{H^p} < \alpha_z \varepsilon$$

Also so $\{F_n(z)\}$ is Cauchy in $\mathbb{C}$, completing the proof of the claim. $\mathbb{C}$ is complete, and thus

$$\exists F(z) \text{ s.t. } F_n(z) \to F(z) \in \mathbb{C}$$

ensuring pointwise convergence of F_n to F in $\mathbb{C}$.

$\square$

Proof that H^p is a Banach Space

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Proof.

To demonstrate that $\|F(z) - F_n(z)\|_{H^p(\mathbb{D})} \to 0$ as $n \to \infty$, we will appeal to the Dominated Convergence Theorem.

1. $F_n(re^{i\theta}) \to F(re^{i\theta})$ for each $\theta \in [0, 2\pi]$ and $r \in (0, 1)$
2. Uniform domination: we need an integrable function $D(\theta)$ such that $|F_n(re^{i\theta}) - F(re^{i\theta})| \leq D(\theta)$ for all $n \in \mathbb{N}$

Since $\|F_n\|_{H^p} \leq M$, we have that

$$|F_n(re^{i\theta})|, |F(re^{i\theta})| \leq \frac{M}{(1-r)^{1/p}}$$

□

Proof that H^p is a Banach Space

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Proof.

$\implies |F_n(re^{i\theta}) - F(re^{i\theta})| \leq \frac{2M}{(1-r)^{1/p}} = D(\theta)$. Indeed

$$\int_0^{2\pi} D(\theta) d\theta = \frac{4M\pi}{(1-r)^{1/p}} \leq \infty$$

since $r < 1$. Thus, by DCT

$$\lim_{n\to\infty} \int_0^{2\pi} |F_n(re^{i\theta}) - F(re^{i\theta})| d\theta = \int_0^{2\pi} \lim_{n\to\infty} |F_n(re^{i\theta}) - F(re^{i\theta})| d\theta = 0$$

which establishes that $N_p(r, F_n - F) \to 0$ as $n \to \infty$ for each r.

$\square$

Proof that H^p is a Banach Space

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Proof.

For any $r_0 < 1$, the convergence $N_p(r, F_n - F) \to 0$ as $n \to \infty$ is uniform for $r \in [0, r_0]$, so we conclude that

$$\|F_n - F\|_{H^p} = \sup_{0 \leq r < 1} N_p(r, F_n - F) \to 0$$

as $n \to \infty$. Lastly, we must establish the analyticity of the limit function F. Now recal the **Weierstrass Convergence Theorem** from complex analysis, which states that if a sequence of functions $\{f_n\}$ is analytic on an open set U and converges uniformly on every compact subset of U to a function f, then f is also analytic on U. $\square$

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Proof.

Consider a family of compact sets $K_r = \overline{B(0,r)}$, for $r < 1$. For all z such that $|z| < r$, since $1 - |z| \geq 1 - r > 0$

$$\implies |F_n(z) - F(z)| \leq \frac{2}{1-r} \|F_n - F\|_{H^p(\mathbb{D})} \to 0$$

and so $F_n \to F$ uniformly on every set K_r, and thus F is analytic on $\mathbb{D}$.

$\square$

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Proof.

Altogether, we have that

1. $\{F_n\} \to F \in \mathbb{C}$ pointwise.
2. $\|F_n - F\|_{H^p} \to 0$ as $n \to \infty$.
3. F is indeed analytic in $\mathbb{D}$.

and so $F \in H^p(\mathbb{D})$, so $H^p(\mathbb{D})$ is complete.
The case $p = \infty$ follows this proof closely and is left as an exercise to the viewer (Hint: Use that $H^\infty(\mathbb{D}) \subset BC(\mathbb{D})$ and Montel's Theorem).

□