The Monotonicity Formula for Energy Minimizing Maps and Monotone Quantities

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For this entire paper, let $\Omega \subset \mathbb{R}^n$ be an open region, and let N be a smooth, compact target manifold embedded in $\mathbb{R}^p$.

1 Main Theorems

Theorem 1.1 (Monotonicity Formula for Energy Minimizing Maps)**.** *Let $u \in W^{1,2}(B_\rho(y); N)$ be an energy minimizing map. If $y \in \Omega$ and $\bar{B}_\rho(y) \subset \Omega$, then for all $0 < \sigma < \rho < \rho_0$ we have that*

$$\rho^{2-n} \int_{B_\rho(y)} |Du|^2 - \sigma^{2-n} \int_{B_\sigma(y)} |Du|^2 = 2 \int_{B_\rho(y) \setminus B_\sigma(y)} R^{2-n} \left| \frac{\partial u}{\partial R} \right|^2$$

where $\frac{\partial u}{\partial R}$ denotes the radial derivative in the direction $\frac{x-y}{|x-y|}$.

Proof. **Claim**: If $a = (a^1, ...a^n)$ are integrable functions on $B_{\rho_0}(y)$ and

$$\int_{B_{\rho_0}} \sum_{j=1}^{\infty} a^j D_j \zeta = 0$$

for all $\zeta \in C_c^\infty(B_{\rho_0}(y))$, then for almost every $\rho \in (0, \rho_0)$, we have that

$$\int_{B_\rho(y)} \sum_{j=1}^{n} a^j D_j \zeta = \int_{\partial B_\rho(y)} \eta \cdot a\zeta$$

for any $\zeta \in C^\infty(\bar{B}_{\rho_0}(y))$, where $\eta \equiv \frac{x-y}{\rho}$ is outward unit normal of $\partial B_\rho(y)$
Proof: We define a cutoff function $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi \equiv 1$ on $B_\rho(y)$, $0 < \phi < 1$ on $B_{\rho_0}(y) \setminus B_\rho(y)$, and $\phi \equiv 0$ outside $B_\rho(y)$. Keep ζ as before and using that $a \in L^1(B_\rho(y))$, we can convolve a with ϕ to obtain

$$\int_{B_\rho(y)} \sum_{j=1}^{n} (a^j(x) * \phi(x)) D_j \zeta(x) = \int_{B_\rho(y)} \sum_{j=1}^{\infty} D_j (a^j(x) * \phi(x)) \zeta$$

and now we can apply divergence theorem to obtain

$$\int_{B_\rho(y)} \sum_{j=1}^{n} D_j(a^j(x) * \phi(x))\zeta = \int_{B_\rho(y)} \sum_{j=1}^{n} \eta^j(x) \cdot a^j(x)\zeta(x) = \int_{B_\rho(y)} \eta(x) \cdot a(x)\eta(x)$$

where $\eta = (\eta^1, ..., \eta^n)$ is the outward pointing unit normal as before. $\square$
Now since we assume u is an energy minimizer, u satisfies the variational equation given by

$$\int_{B_\rho(y)} \sum_{i,j=1}^{n} \left(|Du|^2\delta_{ij} - 2D_iu(x)D_ju(x)\right) D_i\zeta^j(x)$$

and using the above claim, we have that this expression is equivalent to

$$\int_{\partial B_\rho(y)} \sum_{i,j=1}^{n} \left(|Du(x)|^2\delta_{ij} - 2D_iu(x)D_ju(x)\right) \rho^{-1}(x_i - y_i)\zeta^j(x)$$

Notice that if $\zeta(x) = |x^j - y^j|$, where x^j picks out the j-th coordinate of x, then $D_i\zeta^j(x) = \delta_{ij}$, and δ_{ij} picks out n terms after summing i, j from 1 to n, and hence the first expression can be simplified to

$$\int_{B_\rho(y)} \sum_{i,j=1}^{n} \left(|Du|^2\delta_{ij} - 2D_iu(x)D_ju(x)\right) D_i\zeta^j(x) = \int_{B_\rho(y)} n|Du|^2 - 2|Du|^2 = (n-2)\int_{B_\rho(y)} |Du|^2$$

Moreover, in the second expression, we have that this simplifies to

$$\int_{\partial B_\rho(y)} \sum_{i,j=1}^{n} \left(|Du(x)|^2\delta_{ij} - 2D_iu(x)D_ju(x)\right) \rho^{-1}(x_i - y_i)\zeta^j(x) = \int_{\partial B_\rho(y)} \rho \left(|Du(x)|^2 - 2\left|\frac{\partial u}{\partial R}\right|^2\right)$$

since $|\frac{\partial u}{\partial R}|^2 = \sum_{i,j=1}^{n} D_iu(x)D_ju(x)\frac{|x^i - y^i||x^j - y^j|}{\rho^2}$ Therefore, we have

$$(n-2)\int_{B_\rho(y)} |Du|^2 = \rho \int_{\partial B_\rho(y)} \left(|Du(x)|^2 - 2\left|\frac{\partial u}{\partial R}\right|^2\right)$$

and now, seeing that $\int_{\partial B_\rho(y)} f = \frac{\partial}{\partial \rho}\int_{B_\rho(y)} f$ by coarea formula, we have that after multiplying both sides by ρ^{1-n} and computing derivatives, we find that

$$\frac{d}{d\rho}(\rho^{2-n}\int_{B_\rho(y)} |Du|^2) = (2-n)\rho^{1-n}\int_{B_\rho(y)} |Du|^2 + \rho^{2-n}\int_{\partial B_\rho(y)} |Du|^2$$

combined with the fact that

$$(2-n)\rho^{1-n}\int_{B_\rho(y)} |Du|^2 = -\rho^{2-n}\int_{\partial B_\rho(y)} \left(|Du|^2 - 2\left|\frac{\partial u}{\partial R}\right|^2\right)$$

from the first equation, substituting this quantity in cancels out the $\rho^{2-n}\int_{B_\rho(y)} 2\left|\frac{\partial u}{\partial R}\right|^2$ terms, and we left with

$$\frac{d}{d\rho}\left(\rho^{2-n}\int_{B_\rho(y)}|Du|^2\right)=\rho^{2-n}\frac{d}{d\rho}\int_{B_\rho(y)}2\left|\frac{\partial u}{\partial R}\right|^2=2\frac{d}{d\rho}\left(\int_{B_\rho(y)\setminus B_\tau(y)}R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2\right)$$

for a fixed choice of $\tau\in(0,\rho)$. Then, by integrating on both sides from σ to ρ and using the fundamental theorem of calculus, we have that

$$\int_\sigma^\rho\frac{d}{d\rho}\left(\rho^{2-n}\int_{B_\rho(y)}|Du|^2\right)=\int_\sigma^\rho 2\frac{d}{d\rho}\left(\int_{B_\rho(y)\setminus B_\tau(y)}R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2\right)$$
$$\implies \sigma^{2-n}\int_{B_\sigma(y)}|Du|^2-\tau^{2-n}\int_{B_\tau(y)}|Du|^2=2\int_{B_\sigma(y)\setminus B_\tau(y)}R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2$$

□

Definition 1.1 (Density)**.** *Given any map $u\in W^{1,2}(\Omega;N)$, we define the density function $\Theta:\Omega\to\mathbb{R}$ to be*

$$\Theta_u(y)=\lim_{\rho\downarrow 0}\rho^{2-n}\int_{B_\rho(y)}|Du|^2$$

Note that from the monotonicity formula, we have that since $R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2$ is clearly strictly nonnegative,

$$\sigma^{2-n}\int_{B_\sigma(y)}|Du|^2\geq\tau^{2-n}\int_{B_\tau(y)}|Du|^2$$

for all $0<\tau<\sigma$, and thus

$$\sigma^{2-n}\int_{B_\sigma(y)}|Du|^2$$

is an increasing function of σ for $\sigma\in(0,\sigma_0)$. Therefore, the density function exists, and we obtain the immediate fact:

Corollary 1.1. *Θ_u is an upper semi-continuous function on Ω, that is, if $y_j\to y\in\Omega$, then*

$$\Theta_u(y)\geq\limsup_{j\to\infty}\Theta_u(y_i)$$

Proof. Let $\epsilon>0$ and $\rho>0$ with $\rho+\epsilon<\operatorname{dist}(y,\partial\Omega)$. By the monoticity formula, we have that

$$\Theta_u(y_j)\leq\rho^{2-n}\int_{B_\rho(y_j)}|Du|^2$$

for j sufficiently large to ensure $\rho<\operatorname{dist}(y_j,\partial\Omega)$. Now since $B_\rho(y_j)\subset B_{\rho+\epsilon}(y)$ for all j large enough, we have that

$$\Theta_u(y_j)\leq\rho^{2-n}\int_{B_{\rho+\epsilon}(y)}|Du|^2$$

for sufficiently large j, so we obtain that $\limsup_{j\to\infty}\Theta_u(y_j)\leq\rho^{2-n}\int_{B_{\rho+\epsilon}(y)}|Du|^2$. By letting $\epsilon\downarrow 0$, we conclude that

$$\limsup_{j\to\infty}\Theta_u(y_j)\leq\rho^{2-n}\int_{B_\rho(y)}|Du|^2\implies\limsup_{j\to\infty}\Theta_u(y_j)\leq\Theta_u(y)$$

after taking the limit $\rho\downarrow 0$. □

2 Tangent Maps, Monotone Quantities

In order for the density function to be meaningful, we would hope to see invariance upon rescaling u. By rescaling, we mean blowing up the function u around certain points to examine local behavior. This gives rise to the follow rigorous construction: Given $u : \Omega \to \mathbb{R}^p$ and $B_{\rho_0}(y)$ such that $\bar{B}_{\rho_0}(y) \subset \Omega$, and for any $\rho > 0$, consider the scaling function $u_{y,\rho}$ given by

$$u_{y,\rho}(x) = u(y + \rho x)$$

Note that on $B_{\rho_0}(0)$, $u_{y,\rho}$ is well defined. For $\sigma > 0$ and $\rho < \frac{\rho_0}{\sigma}$, after making a change of variables with $\tilde{x} = y + \rho x$ in the energy integral for $u_{y,\rho}$ and noting that $Du_{y,\rho}(x) = \rho D(u(y + \rho x))$, we have the domain changes from $B_\sigma(0) \to B_{\sigma\rho}(y)$, and has a Jacobian factor given by $dx = \rho^{-n} d\tilde{x}$

$$\sigma^{2-n} \int_{B_\sigma(0)} |Du_{y,\rho}(x)|^2 = (\sigma\rho)^{2-n} \int_{B_{\sigma\rho}(y)} |Du(\tilde{x})|^2 \le \rho_0^{2-n} \int_{B_{\rho_0}(y)} |Du|^2 \tag{2.1}$$

by the Monotonicity Formula (1.3). Therefore, if $\rho_j \downarrow 0$, then $\limsup_{j\to\infty} \int_{B_\sigma(0)} |Du_{y,\rho_j}|^2 < \infty$ for all $\sigma > 0$, and so by the Compactness Theorem for energy minimizers, there is a subsequence $\rho_{j'}$ such that $u_{y,\rho_{j'}} \to \varphi$ locally in $\mathbb{R}^n$ w.r.t. the $W^{1,2}$-norm.

Definition 2.1 (Tangent Map)**.** *Any φ obtained this way is called a tangent map of u at y. Moreover, $\varphi : \mathbb{R}^n \to N$ is an energy minimizing map with $\Omega = \mathbb{R}^n$.*

As a consequence:

Corollary 2.1. *Given an energy minimizer $u \in W^{1,2}(\Omega; N)$, the density function is invariant under rescaling of u, and furthermore, the tangent map of u at any point $y \in \Omega$ is constant along rays.*

Proof. We saw above that if $u_{y,\rho}(x) = u(y + \rho x)$, then given $B_\sigma(0)$, we have that

$$\Theta_{u_{y,\rho}}(0) = \lim_{\sigma\downarrow 0} \sigma^{2-n} \int_{B_\sigma(0)} |Du_{y,\rho}(x)|^2 dx = \lim_{\sigma\downarrow 0} (\sigma\rho)^{2-n} \int_{B_{\sigma\rho}(y)} |Du(\tilde{x})|^2$$

and setting $s = \sigma\rho$, we have

$$\lim_{\sigma\downarrow 0} \int_{B_{\sigma\rho}(y)} |Du(\tilde{x})|^2 d\tilde{x} = \lim_{s\downarrow 0} s^{2-n} \int_{B_s(y)} |Du|^2 = \Theta_u(y)$$

and thus $\Theta_{u_{y,\rho}}(0) = \Theta_u(y)$, so Θ_u is invariant under rescaling of u. Moreover, choosing a sequence $\rho_j \to 0$ such that u_{y,ρ_j} converges to a tangent map φ and taking $j \to \infty$ in the previous equation also yields

$$\sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2 = \Theta_u(y)$$

since $\mathcal{E}(u_{y,\rho_j}) \to \mathcal{E}(\varphi)$. In particular, $\sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2$ is a constant function of σ and since $\Theta_\varphi(0) = \lim_{\sigma\to 0} \sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2$, we have that

$$\Theta_u(y) = \Theta_\varphi(0) \equiv \sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2$$

Further, by the monotonicity formula,

$$0 = \sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2 - \tau^{2-n} \int_{B_\tau(0)} |D\varphi|^2 = \int_{B_\sigma(0)\setminus B_\tau(0)} R^{2-n} \left| \frac{\partial \varphi}{\partial R} \right|$$

$$\implies \frac{\partial \varphi}{\partial R} = 0$$

and as $\varphi \in W^{1,2}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^p)$, then by integration along rays, we have that

$$\varphi(\lambda x) \equiv \varphi(x) \quad \forall \lambda > 0, x \in \mathbb{R}^n$$

Namely, the tangent map φ is constant along rays. $\square$