Notes on Sobolev Spaces

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These are notes that I have compiled during my self-study for Sobolev Spaces. At the end, I have included several exercises, largely taken from *Partial Differential Equations* by Lawrence Evans. I am sure there may be errors in here, in which case feel free to email me.

Contents

1 Sobolev Spaces

1.1 Definitions

Let $U \subset \mathbb{R}^n$ be an open region.

Theorem 1.1 (Integration by Parts)**.** *Given $u, v \in C^1(\bar{U})$, we have that*

$$\int_U u_{x_i} v = -\int_U u v_{x_i} + \int_{\partial U} u v \eta^i$$

where $\eta = (\eta^1, ..., \eta^n)$ is the inward pointing unit normal vector.

From this, we notice that if $u \in C^1(U)$ and $\phi \in C_c^\infty(U)$,

$$\int_U u\phi_{x_i} = -\int_U u_{x_i} \phi dx$$

since $\int_{\partial U} u\phi\eta^i = 0$ as $\phi|_{\partial U} = 0$. From this, if α is a multi-index, we can apply this formula $|\alpha|$ times to obtain

$$\int_U uD^\alpha\phi dx = (-1)^{|\alpha|}\int_U D^\alpha u\phi dx$$

where definitionally

$$D^\alpha\phi = \frac{\partial^{\alpha_1}}{\alpha x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\alpha x_n^{\alpha_n}}\phi$$

From this we can formalize the notion of a weak derivative:

Definition 1.1 (Weak Derivative)**.** *Let $u, v \in L^1_{loc}(U)$, and α some multi-index. v is the α-th weak partial derivative of u, that is $D^\alpha u = v$ if*

$$\int_U uD^\alpha\phi dx = (-1)^{|\alpha|}\int_U v\phi dx$$

for all $\phi \in C_c^\infty(U)$. Further, we would say that "$D^\alpha u = v$ in the weak sense".

It is not hard to verify the uniqueness of weak derivatives. If $v, w \in L^1_{loc}$ are both α-th weak derivatives of u, then

$$\int_U uD^\alpha\phi dx = (-1)^{|\alpha|}\int_U v\phi dx = (-1)^{|\alpha|}\int_U w\phi dx$$

and hence

$$0 = \int_U (v - w)\phi dx$$

for all test functions ϕ, so by the fundamental lemma of the calculus of variations, we have that $v - w = 0$. Now, a simple example:

Example 1. *Let $n = 1$, $U = (0, 2)$, and*

$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1, \\ 1 & \text{if } 1 \leq x < 2. \end{cases}$$

Define

$$v(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1, \\ 0 & \text{if } 1 < x < 2. \end{cases}$$

We claim that $u' = v$ in the weak sense. Indeed, we have that for any $\phi \in C_c^\infty(U)$,

$$\int_0^2 u\phi' dx = \int_0^1 x\phi' dx + \int_1^2 \phi' dx = -\int_0^1 \phi dx + \phi(1) - \phi(1) = -\int_0^2 v\phi dx$$

as required.

Definition 1.2 (Sobolev Space)**.** *The Sobolev Space $W^{k,p}(U)$ consists of all locally functions $u \in L^p(U)$ such that for every multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| = a_1 + ... + a_n \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$. Moreover, we define the $W^{k,p}$ norm to be*

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p \, dx\right)^{1/p} & (1 \leq p < \infty) \\ \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_U |D^\alpha u| & (p = \infty). \end{cases}$$

Note: We often write

$$H^k(U) := W^{k,2}(U)$$

and one can verify that H^k is a Hilbert Space. From the definition of the norm, we can derive some nice identities, such as this one for H^1 functions:

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$$

Definition 1.3. *We denote*

$$W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{\|\cdot\|_{W^{k,p}}}$$

that is, the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

Thus, $u \in W_0^{k,p}(U)$ if and only if there are functions $\{u_m\} \in C_c^\infty(U)$ such that $u_m \to u$ in $W^{k,p}(U)$. We interpret $W_0^{k,p}(U)$ as the space of functions $u \in W^{k,p}(U)$ such that $D^\alpha u = 0$ on ∂U for al $|\alpha| \leq k-1$.

Example 2. *Let $\{r_k\}_{k=1}^\infty$ be a countable, dense subset of $U = B_1(0)$, the open unit ball. Define*

$$u(x) = \sum_{k=1}^\infty \frac{1}{2^k} |x - r_k|^{-\alpha}$$

Then we see that for any small ball centered around a fixed r_k, namely $B_\epsilon = \{y \in B_1(0) : |y - r_k| < \epsilon\}$, we have that the term of interest in the L^p norm is given by

$$\int_{B_\epsilon(r_k)} |x - r_k|^{-\alpha p} = \int_0^\epsilon r^{-\alpha p} r^{n-1} dr = \int_0^\epsilon r^{n-1-\alpha p} dr$$

converges to 0 if and only if $n - 1 - \alpha p > -1 \implies \alpha < \frac{n}{p}$. Similarly, for L^p norm of the gradient ∇u which behaves like $|x - r_k|^{-\alpha-1}$, we see that

$$\int_{B_\epsilon(r_k)} |x - r_k|^{-\alpha p} = \int_0^\epsilon r^{(-\alpha-1)p} r^{n-1} dr = \int_0^\epsilon r^{-p(\alpha+1)+n-1} dr$$

converges to 0 if and only if $-p(\alpha+1) + n - 1 > -1 \implies \alpha < \frac{n-p}{p}$. Therefore, $u \in W^{1,p}(U)$ for $\alpha < \frac{n-p}{p}$, but is unbounded for any open set in U.

Theorem 1.2 (Some properties of weak derivatives)**.** *Assume that $u, v \in W^{k,p}(U)$, $|\alpha| \leq k$, then*

(i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ for all multiindices α, β with $|\alpha| + |\beta| \leq k$.

(ii) For each $\lambda, \mu \in \mathbb{R}$*,* $\lambda u + \mu v \in W^{k,p}(U)$ *and* $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$*,* $|\alpha| \leq k$*.*

(iii) If V *is an open subset of* U*, then* $u \in W^{k,p}(V)$*.*

(iv) If $\zeta \in C_c^\infty(U)$*, then* $\zeta u \in W^{k,p}(U)$ *and*

Theorem 1.3 ($W^{k,p}$ is a Banach Space)**.** *For every* k, p*,* $W^{k,p}(U)$ *is a Banach Space.*

Proof. First, it is easy to verify that $\|\cdot\|_{W^{k,p}}$ is indeed a norm. From the definition, clearly $\|u\|_{W^{k,p}} = 0 \iff u = 0$, and $\|\lambda u\|_{W^{k,p}} = \lambda \|u\|_{W^{k,p}}$. For the triangle inequality, note that

$$\begin{aligned}
\|u + v\|_{W^{k,p}(U)} &= \left(\sum_{|\alpha| \leq k} \|D^\alpha u + D^\alpha v\|_{L^p(U)}^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha| \leq k} (\|D^\alpha u\|_{L^p} + \|D^\alpha v\|_{L^p(U)})^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p}^p \right)^{1/p} + \left(\sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^p}^p \right)^{1/p} \\
&= \|u\|_{W^{k,p}} + \|u\|_{W^{k,p}}
\end{aligned}$$

by Minkowski's inequality. Next, for showing completeness in the norm, we have that If $\{u_m\}$ is a Cauchy sequence in $W^{k,p}$, then $\exists u_\alpha \in L^p(U)$ such that $D^\alpha u_m \to u_\alpha \in L^p(U)$ and $u_m \to u \in L^p(U)$ for all $|\alpha| \leq k$. The last time we need to verify is that $D^\alpha u = u_\alpha$. Fix $\phi \in C_c^\infty(U)$, and see that

$$\int_U u D^\alpha \phi = \lim_{m \to \infty} \int_U u_m D^\alpha \phi = \lim_{m \to \infty} \int_U D^\alpha u_m \phi = (-1)^{|\alpha|} \int_U u_\alpha \phi$$

and so $D^\alpha u_m \to D^\alpha u \in L^p(U)$ as desired, and thus $u_m \to u \in W^{k,p}(U)$. □

1.2 Approximation by smooth functions

Here we will turn to mollifiers in order to approximate Sobolev functions be a sequence of smooth functions with nicer properties. For the entire section, for a given $\epsilon > 0$ we define the set $U_\epsilon = \{x \in U : \text{dist}(x, \partial U) < \epsilon\}$. Recall the standard mollifier:

Definition 1.4 (Standard Mollifier)**.** *Let* $\eta \in C_c^\infty(\mathbb{R}^n)$ *be defined as*

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right), & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

and then we define for each $\epsilon > 0$*,*

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right)$$

Note that from this definition, we have that $\eta_\epsilon \in C_c^\infty(B_\epsilon(0))$, and $\int_{\mathbb{R}^n} \eta_\epsilon = 1$. Given any $f \in L^1_{loc}(\mathbb{R}^n)$, we define the function

$$f^\epsilon := \eta_\epsilon * f = \int_{\mathbb{R}^n} \eta_\epsilon(x-y) f(y) dy$$

we would like to introduce the following theorem:

Theorem 1.4. *f^ϵ is smooth on $\mathbb{R}^n$.*

Proof. Let $e = (e_1, ..., e_n)$ be the standard basis, fix $x \in U_\epsilon$, and h small enough such that $x + he_i \in U_\epsilon$, then we have that

$$\frac{f^\epsilon(x+he_i) - f^\epsilon(x)}{h} = \frac{1}{\epsilon^n} \int_U \frac{1}{h} \left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] f(y)\,dy$$

$$= \frac{1}{\epsilon^n} \int_V \frac{1}{h} \left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] f(y)\,dy$$

for some open set $V \subset\subset U$. As

$$\frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] \to \frac{1}{\epsilon}\eta_{x_i}\left(\frac{x-y}{\epsilon}\right)$$

uniformly on V, the partial derivative $f^\epsilon_{x_i}(x)$ exists and equals

$$\int_U \eta_{\epsilon, x_i}(x-y) f(y)\,dy.$$

A similar argument shows that $D^\alpha f^\epsilon(x)$ exists, and

$$D^\alpha f^\epsilon(x) = \int_U D^\alpha \eta_\epsilon(x-y) f(y)\,dy \quad (x \in U_\epsilon).$$

□

Theorem 1.5 (Local Approximation by Smooth Functions)**.** *Let $u \in W^{k,p}(U)$ for some $p \in [1, \infty)$, and define $u^\epsilon = \eta_\epsilon * u$. Then we have that*

$$u^\epsilon \in C^\infty(U) \quad \forall \epsilon > 0$$

and

$$u^\epsilon \to u \in W^{k,p}(U_\epsilon) \quad \textit{as } \epsilon \to 0$$

Proof. The first claim follows by the previous theorem, and now we claim that if $|\alpha| \le k$, then $D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$ in U_ϵ. To see this, note that

$$D^\alpha u^\epsilon(x) = \int_U D^\alpha_x \eta_\epsilon(x-y) u(y) dy = (-1)^{|\alpha|} \int_U D^\alpha_y \eta_\epsilon(x-y) u(y) dy$$

However, since $\eta_\epsilon \in C_c^\infty(U)$, we have by integration by parts once more that

$$\int_U D^\alpha_y \eta_\epsilon(x-y) u(y) dy = (-1)^{|\alpha|} \int_U \eta_\epsilon(x-y) D^\alpha u(y) dy$$

so putting this all together, we have that

$$D^\alpha u^\epsilon(x) = (-1)^{|\alpha|+|\alpha|}\int_U \eta_\epsilon(x-y)D^\alpha u(y)dy = (\eta_\epsilon * D^\alpha u(y))(x)$$

which proves the claim. Now, we have that For $V \subset\subset U$, $D^\alpha u^\epsilon \to D^\alpha u$ in $L^p(V)$, so we are done. □

There are two more major theorems about approximation of Sobolev functions by smooth functions, so we will state both and cover the more important one.

Theorem 1.6 (Global Approximation by Smooth Functions)**.** *Assume U is a bounded set in $\mathbb{R}^n$, and that $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then $\exists u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that*

$$u_m \to u \in W^{k,p}(U)$$

Proof. This proof hinges on the result about local approximation by smooth functions, combined with a partition of unity. Here, we define $U_i := \{x \in U : \operatorname{dist}(x, \partial U) > \frac{1}{i}\}$, and note that $U = \cup_{i=1}^\infty U_i$. From this we set

$$V_i = U_{i+3} \setminus \bar{U}_{i+1}$$

and we now choose an open set V_0 such that $U = \cup_{i=0}^\infty V_i$. Now let $\{\zeta_i\}_{i=0}^\infty$ be a smooth partition of unity subordinate to $\{V_i\}_{i=0}^\infty$, so that the conditions

$$0 \leq \zeta_i \leq 1, \;\; \zeta_i \in C_c^\infty(V_i)$$

and

$$\sum_{i=0}^\infty \zeta_i = 1 \text{ on } U$$

are met. Now by the previous theorem, we have that for any $u \in W^{k,p}(U)$, that $\zeta_i u \in W^{k,p}(U)$ and $\operatorname{spt}(\zeta_i u) \subset V_i$. Define $W_i := U_{i+4} \setminus \bar{U}_i \supset V_i$, fix $\delta > 0$, and choose $\epsilon_i > 0$ small enough such that $u^i = \eta_{\epsilon_i} * (\zeta_i u)$ satisfies the conditions

$$\|u^i - \zeta_i u\|_{W^{k,p}(U)} \leq \frac{\delta}{2^{i+1}}$$

and

$$\operatorname{spt}(u^i) \subset W_i$$

are satisfied for every $i \in \mathbb{N}$. By definition, each $u^i \in C^\infty(U)$, and thus if we define the function $v := \sum_{i=0}^\infty u^i$, then v is also C^∞ since there are finitely many nonzero terms in the sum. Since $u = \sum_{i=0}^\infty \zeta_i u$, then for each $V \subset\subset U$ we have

$$\|v-u\|_{W^{k,p}(V)} \leq \sum_{i=0}^\infty \|u^i - \zeta_i u\|_{W^{k,p}(U)} \leq \delta \sum_{i=0}^\infty \frac{1}{2^{i+1}} = \delta$$

Finally, taking the supremum over all sets $V \subset\subset U$, we have that $\|v-u\|_{W^{k,p}(U)} \leq \delta$. □

It turns out that we can do even better, by approximating $W^{k,p}$ functions that are smooth up to the boundary:

Theorem 1.7. *Assume that U is bounded and that ∂U is C^1. If $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$, then there exist funcions $u_m \in C^\infty(\bar{U})$ such that $u_m \to u$ in $W^{k,p}(U)$.*

1.3 Sobolev Extension

Now we would like to extend functions from $W^{1,p}(U)$ to all of $W^{k,p}(\mathbb{R}^n)$. This is not as simple as setting the function equal to zero on $\mathbb{R}^n \setminus U$, as that may create some bad discontinuities that remove the possibility of the intended function having a weak first partial derivative.

Theorem 1.8. *Assume that U is bounded and ∂U is C^1. Select a bounded open set V such that $U \subset\subset V$. They there exists a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(U)$:

(i) $Eu = u$ *a.e. in* U,

(ii) Eu *has support within* V,

and

(iii)
$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on $p, U,$ and V.

Proof. We would like to choose $x^0 \in \partial U$ such that ∂U is flat near x^0. Define $B := B_r(x_0)$, and further

$$\begin{cases} B^+ := B \cap \{x_n \geq 0\} \subset \bar{U} \\ B^- := B \cap \{x_n \leq 0\} \subset \mathbb{R}^n \setminus U \end{cases} \tag{1.1}$$

Given a function $u \in C^1(\bar{U})$, we define the *higher-order reflection* of u from B^+ to B^- to be

$$\begin{cases} u(x) & \text{if } x \in B^+ \\ -3u(x_1, ..., x_{n-1}, -x_n) + 4u(x_1, ..., x_{n-1}, -\frac{x_n}{2}) & \text{if } x \in B^- \end{cases}$$

From here, we define $u^- := \bar{u}|_{B^-}$ and $u^+ := \bar{u}|_{B^+}$. Note that

$$u^-_{x_n}(x) = 3u_{x_n}(x_1, ..., x_{n-1}, -x_n) - 2(x_1, ..., x_{n-1}, -\frac{x_n}{2})$$

and so

$$u^-_{x_n}|_{\{x_n=0\}} = u^+_{x_n}|_{\{x_n=0\}}$$

from this, we can also verify $u^-_{x_i}|_{\{x_n=0\}} = u^+_{x_i}|_{\{x_n=0\}}$ which implies that

$$D^\alpha u^-|_{\{x_n=0\}} = D^\alpha u^+|_{\{x_n=0\}}$$

for each $|\alpha| \leq 1$, and so $\bar{u} \in C^1(B)$. Now we are ready to verify that

$$\|\bar{u}\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B^+)}$$

for some constant C which does not depend on U. We must also consider the case where ∂U is not flat near the point x^0. We can find a C^1 mapping Φ, with inverse Ψ, such that

Φ "straightens out" ∂U near x^0. We will denote $y = \Phi(x)$, $x = \Phi(y)$, and $u'(y) := u(\Psi(y)$. Using the same process as before by choosing a small ball B centered at x^0, we can extend u' from B^+ to a function u' defined on all of B such that u' is C^1 and we have the estimate

$$\|\bar{u}'\|_{W^{1,p}(B)} \leq C\|u'\|_{W^{1,p}(B^+)}$$

Letting $W := \Psi(B)$, then converting back to the x-variables, we obtain an extension of $\bar{u}$ to u with

$$\|\bar{u}\|_{W^{1,p}(W)} \leq C\|u\|_{W^{1,p}(U)}$$

As ∂U is compact, there are finitely many points $x_i^0 \in \partial U$, open sets W_i and extensions $\bar{u}_i$ of u to W_i such that $\partial U \subset \cup_{i=1}^N W_i$. Take $W_0 \subset\subset U$ such that $U \subset \cup_{i=0}^N W_i$, and let $\{\zeta_i\}_{i=0}^N$ be the associated partition of unity. Write $\bar{u} := \sum_{i=0}^N \zeta_i \bar{u}_i$, where $\bar{u}_0 = u$. We then obtain the bound

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$$

for some constant $C = C(U, p, n)$. Moreover, we choose the support of $\bar{u}$ to lie within $V \supset\supset U$. Lastly, we define $Eu = \bar{u}$, and it can be verified that $u \mapsto Eu$ is linear. Up until this point, we have presupposed that $u \in C^\infty(\bar{U})$, but suppose instead that $u \in W^{1,p}(U)$ for some $1 \leq p < \infty$. Choose a sequence $u_m \in C^\infty(\bar{U})$ converging to u in $W^{1,p}(U)$, so by the previous estimate we have

$$\|Eu_m - Eu_l\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u_m - u_l\|_{W^{1,p}(U)}$$

Hence, $\{Eu_m\}_{m=1}^\infty$ is a Cauchy sequence, and thus converges to $\bar{u} = Eu$. Therefore, the extension operator E satisfies all of the conditions of the theorem. □

1.4 The Trace Operator

Theorem 1.9 (Trace Theorem). *Assume U is bounded and ∂U is C^1, then there exists a bounded linear operator*

$$T : W^{1,p}(U) \to L^p(\partial U)$$

such that

$$Tu = u|_{\partial U}$$

if $u \in W^{1,p}(U) \cap C(\bar{U})$, and

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$$

We call Tu the *trace* of U.

Proof. Assume that $u \in C^1(\bar{U})$. Pick a point $x^0 \in \partial U$ and assume that ∂U is flat near this point, that is, lies in the plane $\{x_n = 0\}$ Choose an open ball B as in the proof of the Sobolev Extension theorem, and define $\hat{B} := B_{r/2}(x^0) \subset B$, and denote Γ as the portion of ∂U within $\hat{B}$. Let $x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$. Then using the fact that $\zeta \equiv 1$ on $\hat{B}$, $\zeta \geq 0$ on $B \setminus \hat{B}$, and $\Gamma \subset \{x_n = 0\}$, we have that

$$\int_\Gamma |u|^p dx' \leq \int_{\{x_n=0\}} \zeta|u|^p dx' = -\int_{B^+} (\zeta|u|^p)_{x_n} dx \tag{1.2}$$

by divergence theorem. Expanding out and using the fact that ζ is compactly supported along with Young's inequality for the second term, we have that

$$-\int_{B^+}(\zeta|u|^p)_{x_n}dx = \int_{B^+}|u|^p\zeta_{x_n}+p|u|^{p-1}(\text{sgn}(u))u_{x_n}\zeta dx \leq C\int_{B^+}|u|^p+|Du|^p dx = C\|u\|_{W^{1,p}(U)}$$

Note that this estimate still holds when $x^0 \in \partial U$ but ∂U is not flat near x^0, after a change of variables, we find that

$$\int_\Gamma |u|^p dS \leq C\int_U |u|^p + |Du|^p dx$$

where Γ is an open subset of ∂U containing x^0. Next, we can use a partition to prove the desired estimate in the second part of theorem. As ∂U is compact, there exists finitely many points x_i^0 and finitely many open subsets $\Gamma_i \subset \partial U$ such that $\partial U = \cup_{i=1}^N \Gamma_i$, and

$$\|u\|_{L^p(\Gamma_i)} \leq C\|u\|_{W^{1,p}(U)}$$

Therefore, writing $Tu = u|_{\partial U}$, then by the statement above, we obtain that

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$$

for some constant C not depending on u. For this equality to hold, we assume that $u \in C^1(\bar{U})$. Assume now that $u \in W^{1,p}(U)$, then there exist $u_m \in C^\infty(\bar{U})$ converging to $u \in W^{1,p}(U)$. Using the estimate from above, we have

$$\|Tu_m - Tu_l\|_{L^p(\partial U)} \leq \|u_m - u_l\|_{W^{1,p}(U)}$$

as T is a bounded linear operator, and so $\{T_m\}$ is a Cauchy sequence in $L^p(\partial U)$. We now define

$$Tu = \lim_{m\to\infty} Tu_m$$

which does not depend on the choice of smooth functions approximating u. If $u \in W^{1,p}(U) \cap C(\bar{U})$, the u_m approximating u converge uniformly to u on $\bar{U}$. Thus, $Tu = u|_{\partial U}$. □

Theorem 1.10. *Assume U is bounded and ∂U is C^1. Suppose also that $u \in W^{1,p}(U)$, then*

$$u \in W_0^{1,p}(U) \iff Tu = 0 \text{ on } \partial U$$

Proof. We will show only the forward direction here. Let $u \in W_0^{1,p}(U)$, and let $u_m \in C_c^\infty(U)$ such that $u_m \to u$ in $W^{1,p}(U)$. As $Tu_m = 0$ on ∂U and $T : W^{1,p}(U) \to L^p(\partial U)$ is a bounded linear operator, then $Tu = 0$ on ∂U. □

1.5 Sobolev Inequalities

Here we will present some motivation for the Gagliardo-Nirenburg-Sobolev Inequality. What will be proven is a statement of this form: Let $u \in C_c^\infty(\mathbb{R}^n)$, and let $n \in \mathbb{N}$ be fixed. If $1 \leq p < n$, and $1 \leq q < \infty$, then if we have the estimate

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

for some $C > 0$, then $q = \frac{np}{n-p}$.

Let $u \in C_c^\infty(\mathbb{R}^n)$, and consider the rescaling $u_\lambda(x) := u(\lambda x)$. Note that

$$\|u_\lambda\|_{L^q}^q = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(t)|^q dt$$

and similarly

$$\|Du_\lambda\|_{L^p}^p = \lambda^p \int_{\mathbb{R}^n} |Du(\lambda x)|^p dx = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |Du(t)|^p dt$$

so we obtain

$$\frac{1}{\lambda^{n/q}} \|u\|_{L^q} \leq \frac{\lambda}{\lambda^{n/p}} \|Du\|_{L^p} \implies \|u\|_{L^q} \leq C \lambda^{1+\frac{n}{q}-\frac{n}{p}} \|Du\|_{L^p}$$

This implies that $1 + \frac{n}{q} - \frac{n}{p} = 0$ or else we can take $\lambda \to 0$ or $\lambda \to \infty$ and obtain a contradiction. Furthermore,

$$1 + \frac{n}{q} - \frac{n}{p} = 0 \implies \frac{1}{q} = \frac{1}{p} - \frac{1}{n} = \implies q = \frac{np}{n-p}$$

Definition 1.5 (Sobolev Conjugate)**.** *If $1 \leq p < n$, we define the Sobolev Conjugate to be*

$$p^* = \frac{np}{n-p}$$

and as a consequence,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

Theorem 1.11 (Gagliardo-Nirenburg-Sobolev Inequality)**.** *If $1 \leq p < n$, then there exists a constant C only depending on p and n such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p}(\mathbb{R}^n)$$

Proof. We first prove the case of $p = 1$. Since u has compact support, for each $i = 1, ...n$ and $x \in \mathbb{R}^n$ we have that

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) dy_i$$

and hence

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du_(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i$$

so we have that

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du(x_1, ..., y_i, ..., x_n)| dy_i \right)^{\frac{1}{n-1}}$$

and now by integrating with respect to x_1, we have

$$\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 \leq \int_{-\infty}^{\infty} \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du| \, dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$= \left(\int_{-\infty}^{\infty} |Du|\, dy_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i\right)^{\frac{1}{n-1}} dx_1$$

$$\leq \left(\int_{-\infty}^{\infty} |Du|\, dy_1\right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dy_i\right)^{\frac{1}{n-1}},$$

where we employed the general Holder Inequality in the last inequality. Then, doing the same thing and integrating with respect to x_2 now, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 dx_2$$

$$\leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dy_2\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{\substack{i=1 \\ i\neq 2}}^{n} I_i^{\frac{1}{n-1}} dx_2,$$

for

$$I_1 := \int_{-\infty}^{\infty} |Du|\, dy_1, \quad I_i := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dy_i \quad (i = 3, \dots, n).$$

Applying once more the extended Hölder inequality, we find

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 dx_2$$

$$\leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dy_2\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dy_1 dx_2\right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dx_2 dy_i\right)^{\frac{1}{n-1}}.$$

We continue by integrating with respect to $x_3, \dots, x_n$, eventually to find

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |Du|\, dx_1 \dots dy_i \dots dx_n\right)^{\frac{1}{n-1}}$$

$$= \left(\int_{\mathbb{R}^n} |Du|\, dx\right)^{\frac{n}{n-1}}$$

This is the estimate for $p = 1$.
For the other cases, namely when $1 < p < n$, we apply the estimate to the function $v := |u|^{\gamma}$, where $\gamma > 1$ is to be chosen. Note that

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |Du|^{\gamma} dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du| dx$$

$$\leq \gamma \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^{p} dx\right)^{\frac{1}{p}}.$$

We choose γ so that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1}$. That is, we set

$$\gamma := \frac{p(n-1)}{n-p} > 1,$$

In this case, $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^*$, and thus $\frac{n-1}{n} = \frac{1}{p^*}$. Therefore, dividing by $\gamma \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}$ on both sides, the exponent of the term on the left hand side becomes $\frac{n-1}{n} - \frac{p-1}{p} = \frac{n-p}{np} = \frac{1}{p^*}$, so finally we have that

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^n} |Du|^p\right)^{\frac{1}{p}}$$

□

Theorem 1.12 (An estimate on $W^{1,p}$)**.** *Let $U \subset \mathbb{R}^n$ be bounded and open, and suppose ∂U is C^1. Then for $1 \leq p < n$, and $u \in W^{1,p}(U)$, then $u \in L^{p^*}(U)$ with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{W^{1,p}}$$

Proof. As ∂U is C^1, by Sobolev extension, $\exists Eu =: \bar{u} \in W^{1,p}(\mathbb{R}^n)$, and

- $\bar{u} = u$, $\bar{u}$ has compact support, and
- $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$

and as $\bar{u}$ has compact support, $\exists u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$u_m \to u \in W^{1,p}(\mathbb{R}^n)$$

and thus by Gagliardo-Nirenburg-Sobolev,

$$\|u_m - u_l\|_{L^{p^*}} \leq C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$$

for all $l, m \geq 1$. Therefore, $u_m \to \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$, and by G-N-S we also have

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n)}$$

$$\implies \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

Finally, this combined with the fact that $\bar{u} = u$ on U yields that

$$\|u\|_{W^{1,p}(U)} \leq C\|u\|_{W^{1,p}(U)}$$

□

We will also include one last estimate for $W_0^{1,p}$:

Theorem 1.13 (Poincare Inequality)**.** *Assume $U \subset \mathbb{R}^n$ is bounded and open, and let $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. We have*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

In particular, $\forall p \in [1, \infty]$, we have that

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

Proof. As $u \in W_0^{1,p}(U)$, there exist functions $u_m \in C_c^\infty$ converging to u in $W^{1,p}$. We extend u_m to be zero on $\mathbb{R}^n \setminus \bar{U}$, and thus

$$\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}$$

and since U is bounded, by L^p interpolation, we have that

$$\|u\|_{L^p(U)} \leq C\|u\|_{L^{p^*}(U)}$$

if $1 \leq q \leq p^*$ □

1.6 Rellich-Kondrachov Compactness Theorem

Definition 1.6. *Let X, Y be Banach spaces, $X \subset Y$, then we say that X is compactly embedded in Y if*

- *If $u \in X$, then $\|u\|_Y \leq C\|u\|_X$ for some constant C*
- *If $\{u_k\}_{k=1}^\infty$ is a sequence in X with $\sup_k \|u_k\|_X \leq \infty$, then some subsequence $\{u_{k_j}\}_{j=1}^\infty \subseteq \{u_k\}_{k=1}^\infty$ converges in Y with some limit u:*

$$\lim_{j\to\infty} \|u_{k_j} - u\|_Y = 0$$

From this definition, we are ready to state the Rellich-Kondrachov Compactness Theorem

Theorem 1.14 (Rellich-Kondrachov Compactness Theorem)**.** *Assume $U \subset \mathbb{R}^n$ is bounded, open, and ∂U is C^1. Suppose $1 \leq p < n$, then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $1 \leq q < p^*$.

Proof. By G-N-S, we already have that $\|u\|_{L^q} \leq C\|u\|_{W^{1,p}}$ for each $1 \leq q < p^*$, and so $W^{1,p}(U) \subset L^q(U)$. This covers the first condition for compact embedding, so now we must show that each bounded sequence in $W^{1,p}$ is precompact in L^q. Let $\{u_m\}$ be a sequence in $W^{1,p}(U)$, and by the extension theorem, we assume that all u_m have support contained in some open set V with $V \supset U$, and that

$$\sup_m \|u_m\|_{W^{1,p}(V)} < \infty$$

Now, given $\epsilon > 0$ we define the mollified functions $u_m^\epsilon := \eta_\epsilon * u_m$, where η_ϵ is the standard mollifier. These functions $\{u_m^\epsilon\}$ also satisfy $\mathrm{spt}(u_m^\epsilon) \subset V$ for every $m \in \mathbb{N}$ as well. We claim that

$$u_m^\epsilon \to u_m \in L^q(V)$$

as $\epsilon \to 0$ uniformly in m.

$$\begin{aligned}
u_m^\epsilon(x) - u_m(x) =& \int_{\mathbb{R}^n} \eta_\epsilon(x-z)u_m(z)dz - u_m(x) \\
=& \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \eta\left(\frac{x-z}{\epsilon}\right)u_m(z)dz - u_m(z) \\
=& \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \left(\frac{x-z}{\epsilon}\right)(u_m(z) - u_m(x))dz \\
=& \int_{B(0,1)} \eta(y)(u_m(x-\epsilon y) - u_m(x))dy \\
=& \int_{B(0,1)} \eta(y)\int_0^1 \frac{d}{dt}(u_m(x-\epsilon ty))dtdy \\
=& -\epsilon\int_{B(0,1)} \eta(y)\int_0^1 Du_m(x-\epsilon ty)\cdot ydtdy
\end{aligned}$$

Taking absolute values on both sides and integrating with respect to x yields the estimate

$$\int_V |u_m^\epsilon(x) - u_m(x)|dx \leq \epsilon\int_{B(0,1)} \eta(y)\int_0^1\int_V |Du_m(x-\epsilon ty)|dxdtdy \leq \epsilon\int_V |Du_m(z)|dz$$

and this holds if $u_m \in W^{1,p}(V)$

$$\implies \|u_m^\epsilon - u_m\|_{L^1(V)} \leq \epsilon\|Du_m\|_{L^1(V)} \leq \epsilon C\|Du_m\|_{L^p(V)}$$

Letting $\epsilon \to 0$, we have that

$$u_m^\epsilon \to u_m \in L^1$$

uniformly in m. Now since $1 \leq q < p^*$, we have by L^p interpolation that

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta}$$

where $\frac{1}{q} = \theta + \frac{(1-\theta)}{p^*}$, and $\theta \in (0,1)$. Since we assumed that $\{u_m\}$ is uniformly bounded in $W^{1,p}(V)$, by G-N-S we have that $\|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta} < \infty$, so

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq C\|u_m^\epsilon - u_m\|_{L^1(V)}^\theta$$

and so $u_m^\epsilon \to u_m$ uniformly in $L^q(V)$. Next, for the sake of use Arzela-Ascoli, we claim that for each $\epsilon > 0$ fixed, the sequence $\{u_m^\epsilon\}_{m=1}^\infty$ is uniformly bounded and equicontinuous. For the uniform boundedness, we indeed see that

$$|u_m^\epsilon(x)| \leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u_m(y)|dy \leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq \frac{C}{\epsilon^n} < \infty$$

and similarly

$$|Du_m^\epsilon(x)| \leq \int_{B(x,\epsilon)} |D\eta_\epsilon(x-y)||u_m(y)|dy \leq \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq \frac{C}{\epsilon^{n+1}} < \infty$$

To show the equicontinuity condition, we fix $\delta > 0$, and we need to show that there exists a subsequence $\{u_{m_j}\} \subset \{u_m\}$ such that

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta$$

Now since $u_\epsilon \to u \in L^q$, we choose $\epsilon > 0$ small enough that

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \frac{\delta}{2}$$

Now for every $m \in \mathbb{N}$, $\mathrm{spt}(u_m) \subset V \subset \mathbb{R}^n$, where V is fixed and bounded, and so $\mathrm{spt}(u_m^\epsilon) \subset V$ as well, so by Arzela-Ascoli, we can obtain a subsequence $\{u_{m_j}^\epsilon\} \subset \{u_m^\epsilon\}$ which converges uniformly on V. Thus,

$$\limsup_{j,k\to\infty} \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} = 0$$

and therefore

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta$$

which proves our initial claim. Lastly, we use this claim, and choose $\delta = 1, \frac{1}{2}, \frac{1}{3}, ...$ and a standard diagonal argument to extract a subsequence $\{u_{m_l}\} \subset \{u_m\}$ satisfying

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)}$$

and thus $W^{1,p}$ can be compactly embedded in L^q. □

1.7 Poincare Inequalities and More

First as a matter of notation, we define the **average value of** u over a region U to be

$$(u)_U := \frac{1}{|U|}\int_U u dy$$

Theorem 1.15. *Let $U \subset \mathbb{R}^n$ be bounded, connected, and open, and suppose ∂U is C^1. Assume $1 \leq p \leq \infty$, then $\exists C := C(n,p,U)$ such that*

$$\|u - (u)_U\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

for every $u \in W^{1,p}(U)$

Proof. Assume not, so $\exists \{u_k\} \in W^{1,p}(U)$ satisfying

$$\|u_k - (u_k)_U\|_{L^p}(U) > k\|Du_k\|_{L^p(U)}$$

Now we renormalize by defining a new sequence of functions:

$$v_k = \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p}}$$

then we have that $(v_k) = \frac{(u_k)_U - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p}} = 0$, and $\|v_k\|_{L^p(U)} = 1$. Thus, $\|Dv_k\|_{L^p(U)} < \frac{1}{k} \implies \{v_k\}_{k=1}^{\infty}$ are bounded in $W^{1,p}(U)$. By the Rellich-Kondrachov compactness theorem, there exists a subsequence $\{v_{k_j}\} \subset \{v_k\}$ such that $\{v_{k_j}\} \to v \in L^p(U)$. Therefore, from before, we can verify that $(v)_U = 0$ and $\|v\|_{L^p(U)} = 1$. From the sequential bound on Dv_k from before, we have that $\forall \phi \in C_c^{\infty}(U)$,

$$\int_U v\phi_{x_i} dx = \lim_{k_j \to \infty} \int_U v_{k_j} \phi_{x_i} dx = -\lim_{k_j \to \infty} \int_U (v_{k_j})_{x_i} \phi dx = 0$$

and thus $v \in W^{1,p}(U)$, with $Dv = 0$ almost everywhere, and thus $v \equiv$ const. However, as $(v)_U = 0$, it must be the case that $v \equiv 0$, and therefore $\|v\|_{L^p(U)} = 0$, and this is a contradiction. $\square$

Theorem 1.16 (Poincare's Inequality for a Ball)**.** *Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n and p, such that*

$$\|u - (u)_{x,r}\|_{L^p(B(x,r))} \leq Cr\|Du\|_{L^p(B(x,r))} \tag{1.3}$$

for each ball $B(x,r) \subset \mathbb{R}^n$ and each function $u \in W^{1,p}(B^0(x,r))$.

Proof. The case where $U = B^0(0,1)$ is covered by the classical Poincare Inequality, and so if $u \in W^{1,p}(B^0(x,r))$, we can write $v(y) = u(x+ry)$, where $y \in B(0,1)$, so that $v \in B^0(0,1)$, and we have that

$$\|v - (v)_{B(0,1)}\|_{L^p(B(0,1))} \leq C\|Dv\|_{L^p(B(0,1))}.$$
$$\implies \|u(x+ry) - (u(x+ry))_{B(x,r))}\|_{L^p(B(x,r))} \leq C\|Du(x+ry)\|_{L^p(B(x,r))}$$
$$\implies \|u - (u)_{B(x,r))}\|_{L^p(B(x,r))} \leq Cr\|Du\|_{L^p(B(x,r))}$$

$\square$

Theorem 1.17 (Hardy's Inequality)**.** *Assume $n \geq 3$, and $r > 0$. Suppose $u \in H^1(B(0,r))$, then $\frac{u}{|x|} \in L^2(B(0,r))$, with*

$$\int_{B(0,r)} \frac{u^2}{|x|^2} dx \leq C \int_{B(0,r)} |Du|^2 + \frac{u^2}{r^2} dx$$

Proof. Assume that $u \in C^{\infty}(B(0,r))$, and note that $D(\frac{1}{|x|}) = -\frac{x}{|x|^3}$, and thus

$$\int_{B(0,r)} \frac{u^2}{|x|^2} dx = -\int_{B(0,r)} u^2 D\left(\frac{1}{|x|}\right) \frac{x}{|x|} dx$$

and combining integration parts and divergence theorem by taking the gradient of $u^2 \frac{x}{|x|}$ and integrating $D\left(\frac{1}{|x|}\right)$, we obtain

$$\int_{B(0,r)} 2uDu\frac{x}{|x|} + \left(\frac{n-1}{|x|^2}\right) u^2 dx + \int_{\partial B(0,r)} u^2 \frac{x}{|x|^2} \nu dx$$

$$\implies (2-n)\int_{B(0,r)} \frac{u^2}{|x|^2}dx = 2\int_{B(0,r)} uDu\frac{x}{|x|}dx + \frac{1}{r}\int_{\partial B(0,r)} u^2 dS$$

From this, we see further that

$$\int_{B(0,r)} \frac{u^2}{|x|^2}dx \leq C\int_{B(0,r)} |Du|^2 dx + \frac{C}{r}\int_{\partial B(0,r)} u^2 dx$$

and further, by divergence theorem, we have that

$$\begin{aligned} r\int_{\partial B(0,r)} u^2 dx = \int_{B(0,r)} \operatorname{div}(xu^2)dx &= \int_{B(0,r)} nu^2 + 2uDu\cdot x dx \\ &\leq C\int_{B(0,r)} u^2 + r^2|Du|^2 dx \end{aligned}$$

and now we divide through by r^2 to obtain

$$\frac{1}{r}\int_{\partial B(0,r)} u^2 dS \leq C\int_{B(0,r)} |Du|^2 + \frac{u^2}{r^2}dx$$

Substituting this expression in for the derived inequality for $\int_{B(0,r)} \frac{u^2}{|x|^2}dx$, we obtain the desired result. $\square$

Lastly, we would like to provide the definitions for other types of Sobolev spaces we have yet to discuss:

Theorem 1.18 (Characterization of H^k by Fourier Transform). *Given $k \in \mathbb{R}$, we say that a function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1+|y|^k)\hat{u} \in L^2(\mathbb{R}^n)$$

In addition, there exists a positive constant C such that

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \leq \|(1+|y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)}$$

Definition 1.7 (Fractional Sobolev Spaces). *Given any $s \in \mathbb{R}_+$, if $u \in L^2(\mathbb{R}^n)$, then $u \in H^s(\mathbb{R}^n)$ if $(1+|y|^s)\hat{u} \in L^2(\mathbb{R}^n)$. Morever, we define the H^s norm by*

$$\|u\|_{H^s(\mathbb{R}^n)} := \|(1+|y|^s)\hat{u}\|_{L^2(\mathbb{R}^n)}$$

And last but not least, we define the space H^{-1}.

Definition 1.8 (The Dual Space to H_0^1). *We denote by $H^{-1}(U)$ the dual space to H_0^1. If $f \in H^{-1}(U)$, we define the norm*

$$\|f\|_{H^{-1}(U)} := \sup\{\langle f, u\rangle : u \in H_0^1(U), \|u\|_{H_0^1(U)} \leq 1\}$$

1.8 Exercises

Exercise 1.1. *Let $U \subset \mathbb{R}^n$ be a connected region. Suppose $u \in W^{1,p}$ satisfies*

$$Du = 0$$

almost everywhere. Prove that u is constant.

Proof. Define the region

$$U_\epsilon = \{x \in U : \operatorname{dist}(x, \partial U) > \epsilon\}$$

and the mollified functions $u^\epsilon = \eta_\epsilon * u$, where η_ϵ is the standard mollifier. Note that by the definition of the weak derivative:

$$\begin{aligned}
Du^\epsilon(x) =& D\int_U \eta_\epsilon(x-y)u(y)dy \\
=& \int_U D_x\eta(x-y)u(y)dy \\
=& -\int_U D_y\eta(x-y)u(y)dy \\
=& \int_U \eta(x-y)Du(y)dy = 0 \\
&\implies Du^\epsilon(x) = 0
\end{aligned}$$

Thus, $u^\epsilon \equiv$ const a.e. in U_ϵ. Further, by the properties of mollifiers,

$$\|u^\epsilon - u\|_{L^p(U_\epsilon)} \to 0$$

as $\epsilon \to \infty$, we have that $u \equiv$ const as well. □

Exercise 1.2 (Chain Rule for $W^{1,p}$)**.** *Suppose $U \subset \mathbb{R}^n$ is bounded, and let $f \in C^1(\mathbb{R})$, $f' \in L^\infty(\mathbb{R})$, and $u \in W^{1,p}(U)$ for some $p \in [1, \infty]$. The $f \circ u \in W^{1,p}(U)$ and $D(f \circ u) = f'(u)Du$.*

Proof. Here, it suffices to prove the case when $p = 1$ since U is bounded and hence $u \in W^{1,p}(U) \implies u \in W^{1,1}(U)$, and the weak derivative of $f \circ u$ is $f'(u)Du$ which belongs to L^p, and hence $f \circ u \in W^{1,p}(U)$. Since $f' \in L^\infty(\mathbb{R})$, we have that $|f(t)| \leq C(1 + |t|)$ for some constant C, where $t \in \mathbb{R}$. From here, we see that

$$\int_U |f(u(x))|dx \leq C\int_U (1 + |u(x)|)dx = C(|U| + \|u\|_{L^1(U)}) < \infty$$

and thus $f \circ u \in L^1(U)$. Next, using the approximation theorem for Sobolev functions, we now choose a sequence $u_k \in C^\infty(U)$ such that

$$u_k \to u \in W^{1,1}_{loc}(U)$$

For all $\phi \in C_c^\infty(U)$, we have that

$$\int_U (f \circ u_k)D\phi dx = -\int_U D(f \circ u_k)\phi dx = -\int f'(u_k)Du_k\phi dx$$

Since $f \in C^1(\mathbb{R})$ and $f' \in L^\infty(\mathbb{R})$, f satisfies the mean value property given by

$$|f(u_k(x)) - f(u(k))| \le \sup_{t\in\mathbb{R}} |f'(t)||u_k(x) - u(x)|$$

and thus $f \circ u_k \to f \circ u$ in $L^1_{loc}(U)$, so it follows that

$$\lim_{k\to\infty} \int_U (f \circ u_k) D\phi dx = \int_U (f \circ u) D\phi dx$$

Moreover, up to extracting a subsequence $u_k \to u$ almost everywhere in U, we have that $f'(u_k) \to f'(u)$ almost everywhere also. Thus, by dominated convergence theorem, we have

$$\begin{aligned}\int_U \Big(f'(u_k)\, Du_k\, \varphi \;-\; f'(u)\, Du\, \phi\Big)\, dx \;\le\; &\int_U \big(f'(u_k) - f'(u)\big)\, Du\, |\phi|\, dx \\ &+ \sup_{t\in\mathbb{R}} |f'(t)| \int_U |Du_k - Du|\, |\phi|\, dx \\ &+ \int_U |f'(u_k) - f'(u)|\, |Du|\, |\phi|\, dx \;\to\; 0 \quad \text{as } k \to \infty.\end{aligned}$$

Thus, as the weak derivative of $f \circ u$ is $f'(u)Du$, and we concluded that $\int_U f'(u_k)Du_k\phi dx \to \int_U f'(u)Du\phi dx$, we have that

$$f \circ u \in W^{1,p}(U)$$

as desired. $\square$

Exercise 1.3 (A Variant of Hardy's Inequality). *Show that for each $n \ge 3$ there exists a constant C such that*

$$\int_{\mathbb{R}^n} \frac{u(x)^2}{|x|^2}\, dx \;\le\; C \int_{\mathbb{R}^n} |Du(x)|^2\, dx$$

holds for all $u \in H^1(\mathbb{R}^n)$.

Proof. First we note that $|Du + \lambda \frac{x}{|x|} u|^2 \ge 0$ for all $\lambda > 0$ and $u \in H^1(\mathbb{R}^n)$. By expanding out and integrating the inequality on both sides, we have that

$$\int_{\mathbb{R}^n} |Du|^2 dx + 2\lambda \int_{\mathbb{R}^n} \frac{x}{|x|^2} u \cdot Du dx + \lambda^2 \int \frac{u^2}{|x|^2} dx \ge 0$$

Next, we examine the middle term. Notice that $2\frac{x}{|x|^2} u \cdot Du = D(u^2) \cdot \frac{x}{|x|^2}$. Using integration my parts this quantity and using that $\operatorname{div}\left(\frac{x}{|x|^2}\right) = \frac{n-2}{|x|^2}$, we have that

$$\int_{\mathbb{R}^n} D(u^2) \cdot \frac{x}{|x|^2} = \lim_{R\to\infty} \int_{|x|=R} u^2 \frac{x}{|x|^2} \cdot \nu dS(x) - (n-2) \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx$$

Notice that for the boundary term, since $u \in H^1(\mathbb{R}^n)$, we have that

$$\int_{|x|=R} |u(x)|^2 dS(x) \le \int_{|x|\ge R} |u(x)|^2 dx \le \int_{\mathbb{R}^n} |u(x)|^2 dx \le |M|$$

Thus, as $\frac{x}{|x|^2}\cdot\nu = \frac{1}{|x|} = \frac{1}{R}$

$$\lim_{R\to\infty}\int_{|x|=R} u^2\frac{x}{|x|^2}\cdot\nu dS(x) = \lim_{R\to\infty}\frac{1}{R}\int_{|x|=R}|u(x)|^2dS(x) \leq \lim_{R\to\infty}\frac{1}{R}\int_{|x|\geq R}|u(x)|^2dx \leq \lim_{R\to\infty}\frac{|M|}{R} = 0$$

and so all in all, we are left with

$$\int_{\mathbb{R}^n}|Du|^2dx - \lambda(n-2)\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}dx + \lambda^2\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}dx \geq 0$$

$$\implies (\lambda(n-2)-\lambda^2)\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}dx \leq \int_{\mathbb{R}^n}|Du|^2dx$$

$$\implies \int_{\mathbb{R}^n}\frac{u(x)^2}{|x|^2}\,dx \quad\leq\quad C\int_{\mathbb{R}^n}|Du(x)|^2\,dx$$

□

Exercise 1.4. *Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$, with the bound*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

Proof. Here, we use the inverse Fourier transform given by

$$u(x) = \int_{\mathbb{R}^n}\hat{u}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

and the immediate estimate

$$|u(x)| \leq \int_{\mathbb{R}^n}|\hat{u}(\xi)e^{2\pi i x\cdot\xi}|d\xi = \int_{\mathbb{R}^n}|\hat{u}(\xi)|d\xi$$

Now, we rewrite $\hat{u}(\xi) = (1+|\xi|^2)^{s/2}(1+|\xi|^2)^{-s/2}|\hat{u}(\xi)|$, so that we can obtain Cauchy-Schwarz to obtain

$$\begin{aligned}\int_{\mathbb{R}^n}|\hat{u}(\xi)|d\xi &= \int_{\mathbb{R}^n}(1+|\xi|^2)^{s/2}(1+|\xi|^2)^{-s/2}|\hat{u}(\xi)|d\xi \\ &\leq \left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-s}d\xi\right)^{1/2}\left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{s}|\hat{u}(\xi)|^2d\xi\right)^{1/2} \\ &= \left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-s}d\xi\right)^{1/2}\|u\|_{H^s(\mathbb{R}^n)}\end{aligned}$$

Now, to show that integral in front is finite, we make the substitution $r = |\xi| \implies d\xi = r^{n-1}drd\text{Vol}$ to yield

$$\int_{\mathbb{R}^n}(1+|\xi|^2)^{-s}d\xi = \int_{S^{n-1}}d\text{Vol}\int_0^\infty(1+r^2)^{-s}r^{n-1}dr = |S^{n-1}|\int_0^\infty(1+r^2)^{-s}r^{n-1}dr$$

To ensure the integral convergences, note that near $r = 0$, $(1+r^2)^{-s}r^{n-1} \approx r^{n-1}$, and $\int_0^1 r^{n-1}dr$ clearly convergences. As $r \to \infty$, we have that $(1+r^2)^{-s}r^{n-1} \approx r^{-2s+n-1}$, and

$$\int_1^\infty r^{-2s+n-1}dr = \frac{r^{-2s+n}}{-2s+n}\Big|_1^\infty$$

and we see that this quantity converges if and only if $-2s+n < 0 \implies s > \frac{n}{2}$, which is precisely what we assume. Thus $\int_0^\infty (1+r^2)^{-s}r^{n-1}dr < +\infty$ and its value depends only on the choice of n and s, and so we have that

$$|u(x)| \leq \int_{\mathbb{R}^n} |\hat{u}(\xi)|d\xi \leq C\|u\|_{H^s(\mathbb{R}^n)}$$

where $C := C(n,s)$, which is what we desired to show. Thus, $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ for $s > \frac{n}{2}$. □

Exercise 1.5. *Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $uv \in H^s(\mathbb{R}^n)$ and*

$$\|uv\|_{H^s(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)},$$

the constant C depending only on s and n.

Proof. Let $w = uv$, and note that $\hat{w}(\xi) = c_1(\hat{u} * \hat{v})(\xi) = c_1 \int_{\mathbb{R}^n} \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta$. Thus, in evaluating the integrand of the H^s norm, we have

$$(1+|\xi|^2)^{s/2}|\hat{w}(\xi)| \leq c_1 \int_{\mathbb{R}^n} (1+|\xi|^2)^{s/2}|\hat{u}(\xi-\eta)||\hat{v}(\eta)|d\eta$$

Moreover, we have the following splitting property, that for $\xi, \eta \in \mathbb{R}^n$,

$$(1+|\xi|^2)^{s/2} \leq c_2((1+|\xi-\eta|^2)^{s/2} + (1+|\eta|^2)^{s/2})$$

putting this together, we find that

$$(1+|\xi|^2)^{s/2}|\hat{w}(\xi)| \leq c_1c_2\left[\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^{s/2}|\hat{u}(\xi-\eta)||\hat{v}(\eta)|d\eta + \int_{\mathbb{R}^n}(1+|\xi|^2)^{s/2}|\hat{v}(\xi)||\hat{u}(\xi-\eta)|d\eta\right]$$

identifying $f(\zeta) := (1+|\zeta|^2)^{s/2}|\hat{u}(\zeta)|$, $g(\zeta) := \hat{v}(\zeta)$, $h := |\hat{u}(\zeta)|$, and $k(\zeta) = (1+|\zeta|^2)^{s/2}|\hat{v}(\zeta)|$, we can rewrite the above inequality as

$$(1+|\xi|^2)^{s/2}|\hat{w}(\xi)| \leq c_1c_2((f*g)(\zeta) + (h*k)(\zeta))$$

this yields that

$$\begin{aligned}\|(1+|\xi|^2)^{s/2}|\hat{w}(\xi)|\|_{L^2(\mathbb{R}^n)} =&\|w\|_{H^s(\mathbb{R}^n)} \\ \leq& c_1c_2\|(f*g)(\zeta) + (h*k)(\zeta)\|_{L^2(\mathbb{R}^n)} \\ \leq& c_1c_2(\|(f*g)(\zeta)\|_{L^2(\mathbb{R}^n)} + \|(h*k)(\zeta)\|_{L^2(\mathbb{R}^n)}) \\ \leq& c_1c_2(\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)} + \|h\|_{L^2(\mathbb{R}^n)}\|k\|_{L^1(\mathbb{R}^2)})\end{aligned}$$

Now desire to show that $g, k \in L^1(\mathbb{R}^n)$. Note that by Cauchy-Schwarz,

$$\begin{aligned}
\|g\|_{L^1(\mathbb{R}^n)} =& \int_{\mathbb{R}^n} (1+|\zeta|^2)^{-s/2}(1+|\zeta|^2)^{s/2}\hat{v}(\zeta)d\zeta \\
\leq& \int_{\mathbb{R}^n} (1+|\zeta|^2)^{-s/2}(1+|\zeta|^2)^{s/2}|\hat{v}(\zeta)|d\zeta \\
\leq& \left(\int_{\mathbb{R}^n} (1+|\zeta|^2)^{-s}d\zeta\right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\zeta|^2)^{s}|\hat{v}(\zeta)|^2 d\zeta\right)^{1/2} \\
=& C'\|v\|_{H^s(\mathbb{R}^n)} < +\infty
\end{aligned}$$

since the integral in front converges precisely because we assume $s > \frac{n}{2}$ (see previous exercise). In the same way, we also have the estimate $\|h\|_{L^1(\mathbb{R}^n)} \leq C'\|u\|_{H^s(\mathbb{R}^n)}$. Note as well that $\|f\|_{L^2(\mathbb{R}^n)} = \|u\|_{H^s(\mathbb{R}^n)}$ and $\|k\|_{L^2(\mathbb{R}^n)} = \|v\|_{H^s(\mathbb{R}^n)}$. Combining all of these estimates, we have that

$$\begin{aligned}
\|w\|_{H^s(\mathbb{R}^n)} \leq c_1c_2(\|u\|_{H^s(\mathbb{R}^n)}(C'\|v\|_{H^s(\mathbb{R}^n)}) + \|v\|_{H^s(\mathbb{R}^n)}(C'\|u\|_{H^s(\mathbb{R}^n)})) \\
\implies \|uv\|_{H^s(\mathbb{R}^n)} \leq C(\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)})
\end{aligned}$$

with C only depending on n and s. $\square$