Research Statement

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1 Introduction

My mathematical interests lie in geometric analysis and differential geometry. Specifically, my research is concerned with the analysis of singularities of energy minimizing maps and geometric flows. In geometric analysis, the question of the uniqueness of tangent maps is notoriously opaque, and is the subject of much modern research. My thesis consists of two parts, the first of which focuses on the analysis of harmonic maps following the pioneering work of Leon Simon[1]. Here, I explicate the theory of singularities and tangent maps of energy minimizers on Riemannian manifolds. Furthermore, I will explore the question of the uniqueness of tangent maps, discussing results of both Simon and Brian White[2]. There, I will discuss the application of the Lojasciewicz inequalties as a crucial tool to ensure uniqueness with maps with isolated singularities. In the discussion where the domain is a Riemannian Manifold, in particular S^{n-1}, I will outline the variational techniques and estimates arising from the Lojasiewicz inequality as well. In the second half, we will discuss the application of Lojasiewicz inequalities in the Mean Curvature Flow setting following the work of Felix Schulze[4]. By discussing both theoretical insights and practical analytic methods, my work seeks to contribute to these critical areas in differential geometry, potentially influencing further research and discussion.

2 Energy Minimizing Maps

Energy minimizing maps play a crucial role in mathematical analysis and model phenomenon in physics where systems natural evolve to states of minimal energy. The general setting throughout the whole paper is this: let $\Omega \subset \mathbb{R}^n$ be an open region, and let N be a compact C^∞ Riemannian manifold isometrically embedded into $\mathbb{R}^p$ for some p. We give the following definition for such an energy minimizing map:

Definition 2.1. *(Energy Minimizing Map) Given a function $u \in W^{1,2}(\Omega;\mathbb{R})$, we define the energy of u on a ball $B_\rho(y) = \{x : |x-y| < \rho\}$ to be*

$$\mathcal{E}_{B_\rho(y)}(u) = \int_{B_\rho(y)} |Du|^2$$

We say that a map u is energy minimizing if for each ball $B_\rho(y)$, we have that

$$\mathcal{E}_{B_\rho(y)}(u) \leq \mathcal{E}_{B_\rho(y)}(w)$$

for all $w \in W^{1,2}(\Omega, N)$ with $w = u$ in a neighborhood of $\partial B_\rho(y)$.

From this, we provide an simple example of an energy minimizing map:

Example 2.1. *(Radial Projection) For $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^{n-3}$, we define the map $u : \mathbb{R}^n \to S^2$ to be*

$$u(x, y) = \frac{x}{|x|}$$

It is an easy exercise in computation to show that u is energy minimizing, but this example demonstrates the presence of singularities of energy minimizing maps. As we see, a singularity forms at the point $x = 0$. This, we say that the *singular set* of u, denoted by $\text{sing}(u)$ is $\{0\}$. More formally:

Definition 2.2. *(Regular and Singular Set) We denote the regular set of an energy minimizer $u \in W^{1,2}(\Omega; N)$ to be*

$$\textit{reg}(u) := \{x \in \Omega : u \textit{ is } C^\infty \textit{ in a neighborhood of } x\}$$

and

$$\textit{sing}(u) = \Omega \setminus \textit{reg}(u)$$

The singular set of energy minimizing maps holds great importance for geometric analysts and has been studied extensively over the past 50 years. One question that had gone unanswered for years until the work of Aaron Naber and Daniele Volterra bears on the nature of $\textit{sing}(u)$ in codimension three. The question was this: is $\mathcal{H}^{n-3}(\text{sing}(u) < \infty$ and is $\text{sing}(u)$ $(n-3)$-rectifiable? The answer turns out to be yes in both cases, thanks to the pioneering work of Naber and Volterra in *Rectifiable-Reifenberg and the Regularity of Stationary and Energy Minimizing Maps.* Their work features the techique of quantifiable rectifiability in Geometric Measure theory. One of the main tools in understanding the singular set are tangent maps, and particularly, their uniqueness, for which we give the following definition:

Definition 2.3. *(Tangent Maps) Given $u : \Omega \to \mathbb{R}^p$, and $B_{\rho_0}(y) \subset \Omega$, for $\rho > 0$, we consider the rescaling*

$$u_{y,\rho} = u(y + \rho x)$$

by choosing a sequence ρ_j such that $\rho_j \downarrow 0$ and then by passing to a subsequence $\rho_{j'}$, we have that $u_{y,\rho_{j'}} \to \phi$ locally in $\mathbb{R}^n$ with respect to the $W^{1,2}$ norm, and we call ϕ the ***tangent map*** *of u at y.*

The existence of the such tangent maps is guaranteed by the monotonicity formula for energy minimizers and a nice compactness theorem for a sequence of energy minimizers $\{u_j\}$ in $W^{1,2}$ ensuring convergence. Inuitively, the tangent map blows up the local behavior of the map at point, as if looking at the function through a microscope. What is far less obvious is the uniqueness of such maps, and not much progress has been made with regard to this question in recent time. However, some results about the *non-uniqueness* of such maps is known, and notably an example for which uniqueness fails is featured in the 1992 paper *Nonuniqueness Tangent Maps at Isolated Singularities of Harmonic Maps* by Brian White. Part of my research interest includes this analysis of uniqueness of tangent maps, potentially discovering new instances of nonuniqueness. As it pertains to tangent flows, while I cover the work of Felix Schulze as it pertains to the uniqueness of compact tangent flows in the mean curvature flow setting, it is worth mentioning that Antonio Ache proves a similar result in the Ricci Flow setting in his paper *On the uniqueness of asymptotic limits of the Ricci flow.*

3 Uniqueness of Tangent Maps of Energy Minimizers

Leon Simon is the main pioneer of using variational techniques to study the uniqueness of tangent maps. This approach involves defining an appropriate functional over a C^1 Riemannian Manifold and obtaining certain estimates. Notably, when examining maps $u \in C^\infty(S^{n-1}; N)$, where N is smooth, compact Riemannian manifold of dimension $m \geq 2$. Specifically, we examine vector bundles over Riemannian manifolds (S^{n-1} in this case), which will provide a framework for the application of one inequality in particular, the Lojasiewicz Inequality, for which we will be able to prove certain estimates. Developed in 1965[3], the Lojasiewicz Inequality (often named the Lojasiewicz-Simon Inequality now), is the following

Theorem 3.1 (Lojasiewicz). *Let Ω and f be as above. For every critical point $x \in \Omega$ of f, $\exists$ a neighborhood V of x, and an exponent $\theta \in [1/2, 1]$ and a constant $C > 0$ such that*

$$|f(x) - f(y)|^\theta \leq C|\nabla f(y)| \quad \forall y \in V$$

Thus, we can always provide upper bound, given by the magnitude of the gradient of f for the distance between any two points in a suitable neighborhood of x, given appropriate constants θ and C. In Simon's approach, we define a functional

$$\mathcal{F}(u) = \int_{S^{n-1}} (F(\omega, u, \nabla^{\mathbf{V}} u) - F(\omega, 0, 0))$$

where $F(\omega, z, \eta) \in C^\infty(S^{n-1} \times \mathbb{R}^p \times \mathbb{R}^{np}; \mathbb{R})$ and u is a C^2 function with norm smaller than δ, for $\delta > 0$ given. From here, given a suitable function $\tilde{u}$, we can relate this functional to the energy functional $\mathcal{E}_{S^{n-1}}$ by the following relation:

$$\mathcal{E}_{S^{n-1}}(u) - \mathcal{E}_{S^{n-1}}(\tilde{\varphi_0}) = \mathcal{F}(\tilde{u})$$

where $\varphi_0 \in C^\infty(S^{n-1}; N)$ is a fixed harmonic function which is C^3 close to u, that is $|u - \varphi_0| < \delta$. As we will see, with the additional assumption that the target manifold N is real analytic, the functional $\mathcal{F}$ satisfies the necessary conditions for the application of the Lojasiewicz inequality. As a result, this provides a certain inequality relating $\mathcal{E}_{S^{n-1}}$ to its first variation, or Euler-Lagrange operator:

$$|\mathcal{E}_{S^{n-1}}(u) - \mathcal{E}_{S^{n-1}}(\varphi_0)|^{1-\frac{\alpha}{2}} \leq C\|\mathcal{M}_{\mathcal{E}_{S^{n-1}}}\|_{L^2}$$

This L^2 estimate will enable us to prove the following essential result, we still remains one of the only knows about uniqueness of tangent maps:

Theorem 3.2 (Simon). *If φ is a tangent map of u at singular point $y \in sing(u)$ and $sing(\varphi) = 0$, then φ is the unique tangent maps of y at p, and*

$$u(y + r\omega) = \varphi(\omega) + \epsilon(r, \omega)$$

where $\omega \in S^{n-1}$, and

$$\lim_{r \downarrow 0} |\log r|^\alpha \sup_{\omega \in S^{n-1}} |\epsilon(r, \omega)| = 0$$

The proof hinges heavily on the Lojasiewicz inequality, variational techniques applied to the energy functional, and a few other key results pertaining to regularity of EMM's. What is also know is that even for isolated singularities, if the assumption that N is real-analytic is removed, ex. N is merely smooth, the result is false. Part of my research interest includes exploring new conditions for which the same variational techniques can be applied, and perhaps extending Simon's approach outside of the energy minimizing map setting.

4 The Mean Curvature Flow Setting

Recall the definition of a family of hypersurfaces moving by mean curvature flow (MCF):

Definition 4.1. *(Mean Curvature Flow) Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an n-dimensional manifold. The mean curvature flow of φ_0 is a family of smooth immersions $\varphi_t : M \to \mathbb{R}^{n+1}$ for $t \in [0, T)$ such that setting $\varphi(p, t) = \varphi_t(p)$ the map $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ is a smooth solution of the following system of PDE's*

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = H(p,t)\nu(p,t), \\ \varphi(p,0) = \varphi_0(p), \end{cases} \tag{1}$$

where $H(p, t)$ and $\nu(p, t)$ are respectively the mean curvature and the unit normal of the hypersurface φ_t at the point $p \in M$.

Intuitively, MCF leads to the flattening a surface and smooths out irregularities over time since the hypersuface is evolving in the direction steepest descent of the volume element at each point. For example, if one evolves the n-Sphere S^n by mean curvature flow, the surface shrinks to a point and then becomes *extinct*, and such a point is a singularity of the flow.Further, a round cylinder $S^{n-1} \times \mathbb{R} \subset \mathbb{R}^n$ contracts over the course of the flow to a line, and a torus contracts to a circle. In each of these examples, the initial hypersurface exhibits radically different singular sets. In MCF, analysis of singularities is a crucial element of the field, just as with energy minimizers, or harmonic maps more broadly. In the same way as one uses tangent maps to analysis the behavior near singularities of energy minimizers, one can also use tangent maps to study the behavior of singularities of MCF (or any geometric flow). The definition of a tangent flow is analogous to that of tangent maps and can be stated in the following manner:

Definition 4.2 (Tangent Flow)**.** *For $\lambda > 0$, define the rescaling $\mathcal{D}_\lambda$ as*

$$\begin{aligned} \mathcal{D}_\lambda : \mathbb{R}^{n+k} \times \mathbb{R} &\to \mathbb{R}^{n+k} \times \mathbb{R} \\ (x,t) &\mapsto (\lambda t, \lambda^2 t) \end{aligned}$$

The rescaled flow will be given by

$$\mathcal{M}_t^\lambda = \mathcal{D}_\lambda(\mathcal{M}_{t_0+\lambda^{-2}t})$$

*and finally, we define the **tangent flow** as*

$$\lim_{\lambda\to\infty} \mathcal{M}_t^\lambda$$

In his 2014 paper *Uniqueness of Compact Tangent Flows in Mean Curvature Flow*, Felix Schulze ingeniously found an application of Lojasiewicz inequality in the MCF setting, and the approach is quite similar to Simons'. Here, the appropriate functional is given by the Gaussian density ratio

Definition 4.3 (Backward Heat Kernel and Gaussian Density). *Let $\mathcal{M}$ be a smooth mean curvature flow of n-dimensional hypersurfaces embedded in $\mathbb{R}^{n+k}$. We define*

$$\rho_{x_0,t_0}(x,t) = \frac{1}{(4\pi(t_0-t))^{n/2}} \exp(-\frac{|x-x_0|^2}{4(t_0-t)}),\, t < t_0$$

is the ***backward heat kernel centered*** *at (x_0, t_0). From this, we define*

$$\Theta_{x_0,t_0}(\mathcal{M}, t) := \int_{M_t} \rho_{x_0,t_0}(x,t) d\mathcal{H}^n$$

to be the ***Gaussian Density ratio****.*

Due to the key result *Huisken's Monotonicity Formula*, which states that for $t < t_0$

$$\frac{d}{dt}\int_{M_t} \rho_{x_0,t_0}(x,t) d\mathcal{H}^n = -\int_{M_t} |\vec{H} + \frac{\vec{x}}{2(t_0-t)}|^2 \rho_{x_0,t_0}(x,t) d\mathcal{H}^n$$

we see that the Gaussian density ratio $\Theta_{x_0,t_0}(\mathcal{M}, t)$ is a decreasing function of time. Therefore, for $t \in (t_1, t_2]$, we have that

$$\Theta_{x_0,t_0}(\mathcal{M}) = \lim_{t\to t_0} \Theta_{x_0,t_0}(t)$$

exists for all $x_0 \in \mathbb{R}^{n+k}$ and is the **Gaussian density** of (x_0, t_0). Based on the derived monotonicity formula, we define the appropriate energy functional for surfaces that can be written over as normal graphs over a closed, smooth, and embedded n-surface Σ, and for v the smooth section of the normal bundle $T^{\perp}\Sigma$ such that $M = \text{graph}_{\Sigma}(v)$, we have

$$\mathcal{E}(v) = \int_M \rho(x) d\mathcal{H}^n = \int_{\Sigma} \rho(y + v(y)) J(y, v, \nabla^{\sigma} v) d\mathcal{H}^n$$

Following this definition, we use the first variation to solve for the gradient of $\mathcal{E}$, and then we can apply the Lojasiewicz inequality, along with a few other estimates and decay rates to conclude the following result:

Theorem 4.1 (Schulze 2011). *If Σ is a unit multiplicity tangent flow that is compact, smooth, and embedded, arising from a compact and embedded mean curvature flow, then it is the unique tangent flow at that point, that is, it does not depend on the choice of rescaling.*

For more on uniqueness results of tangent flows in the MCF setting, one can reference the pioneering work of Minicozzi and Colding. In particular, a more thorough treatment of the applications Lojasiewicz Inequalities to MCF[5]. In particular, one remarkable result that they have proven is the following:

Theorem 4.2 (Minicozzi, Colding 2013). *If M_t is a MCF in $\mathbb{R}^{n+1}$, then at each cylindrical singular point, the tangent flow is unique.*

5 Research Directions

5.1 Examples of Nonuniqueness

As it pertains to the question of the uniqueness of tangent maps of energy minimizers, the conditions for uniqueness are seemingly quite rigid (e.g. a real-analytic target manifold), yet the question of nonuniqueness of tangent maps seems to be less known. While one can assume "nice" regularity conditions like smoothness and prove that uniqueness can fail, as Simon does, such examples of nonunique tangent maps are sparse, the most notable example coming from White. As such, I would like to examine instances of nonuniqueness more closely, and extrapolate this analysis to the mean curvature flow as well to construct specific examples of where nonuniqueness of tangent flows is ensured. In both settings, I would be interested in analyzing more complex target manifolds for energy minimizers, as well as higher codimension hypersurfaces in the MCF setting.

5.2 Lojasiewicz Inequalities in Other Settings

While the Lojasiewicz Inequalities have already found application to many areas within geometric, including Ricci Flow, gradient flows, and even in deep learning with the non-linear systems and neural networks. I would be eager to explore more potential applications of these inequalities and other variational methods to Ricci Flow, Yamabe flow, or Yang-Mills flow. This would also entail analysis of the the singular sets arising in these settings, with the intention of deriving certain uniqueness results. The major obstacle to invocation of these inequalities is some kind of compactness condition, which the results of Schulze, as well as Colding and Minicozzi hinge on. Perhaps finding a way to "expand" the preconditions for the inequalities could be of interest, as well as gaining a better understanding of the sharpest Lojasiewicz exponent.

5.3 Sobolev Maps and RCB Spaces

Since energy minimizers are Sobolev maps by definition, understanding regularity conditions of these maps in greater detail is essential. There are many directions one may go with this, including trace theory, the strong density problem, and examining boundary conditions for these maps. I am particularly interested in a paper of Honda and Sire[6] which studies Sobolev maps between Ricci-curvature bounded spaces and provides striking insight into the nature of harmonic maps, using techniques such as heat smoothing, Cheeger's differentiability theorem, and eigenmaps. While less germane to my Master's Thesis topic, this is another direction in the study of harmonic maps I would be glad to take.

References

[1] Leon Simon. *Theorems on Regularity and Singularity of Energy Minimizing Maps*. Birkhäuser Verlag, Basel, Boston, Berlin, 1996.

[2] Brian White. Nonunique tangent maps at isolated singularities of harmonic maps. *arXiv preprint math/9201270*, 1992. URL `https://arxiv.org/abs/math/9201270`.

[3] Stanisław Łojasiewicz, *Ensembles semi-analytiques*, preprint IHES, 1965.

[4] Felix Schulze. Uniqueness of compact tangent flows in Mean Curvature Flow. *arXiv preprint 1107.4643*, 2011. URL `https://arxiv.org/abs/1107.4643`.

[5] Tobias Holck Colding and William P. Minicozzi II. Uniqueness of blowups and Lojasiewicz inequalities. *arXiv preprint 1312.4046*, 2014. URL `https://arxiv.org/abs/1312.4046`.

[6] Shouhei Honda and Yannick Sire. Sobolev mappings between RCD spaces and applications to harmonic maps: a heat kernel approach. *arXiv preprint 2105.08578*, 2023. URL `https://arxiv.org/abs/2105.08578`.