A Collection of Exercises in Minimal Surfaces

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1 Introduction

These are a collection of exercises in minimal surfaces, most of which are taken from *A Course in Minimal Surfaces* by Colding and Minicozzi. I am still adding to this document, so please feel free to contact me if you find any errors.

2 Exercises

Problem 1. *Let* $\Sigma^{n-1} \subset S^n$ *be a minimal hypersurface. Prove that* $|\Sigma^{n-1}| \geq |S^{n-1}|$ *with equality if and only if* $\Sigma = S^{n-1}$.

Proof. **Claim**: If $\Sigma^{n-1} \subset S^n$ is a minimal hypersurface, then $C(\Sigma)$ is minimal in $\mathbb{R}^{n+1}$. **Proof**: Consider a parametrization $\Phi : \Omega \subset \mathbb{R}^{n-1} \to S^n$ of Σ, and note that cone can be parametrized as

$$C(t, x) = t\Phi(x)$$

where $t > 0$ and $x \in \Omega$. Note that $C_t = \Phi(x)$ and $C_{x_i} = t\Phi_{x_i}$ gives the tangent vectors of $C(t, x)$, which yields metric coefficients

$$g_{tt} = \langle \Phi(x), \Phi(x) \rangle = 1$$

as $\Phi(x) \in S^n$,

$$g_{ti} = \langle C_t, C_{x_i} \rangle = t\langle \Phi, \Phi_{x_i} \rangle = 0$$

since Φ and Φ_{x_i} are orthogonal, and lastly

$$g_{ij} = \langle C_{x_i}, C_{x_j} \rangle = t^2 \langle \Phi_{x_i}, \Phi_{x_j} \rangle$$

and so $ds^2 = dr^2 + r^2 g^{\Sigma}_{ij}$. For the second fundamental form, we have that the normal vector N to $C(\Sigma)$ is the same as the normal vector to Σ since $C(\Sigma)$ is rotationally symmetric, and therefore

$$h_{tt} = \langle C_{tt}, N \rangle = 0$$

as $C_{tt} = (\Phi(x))_t = 0$,

$$h_{ti} = \langle C_{tx_i}, N \rangle = \langle \Phi_{x_i}, N \rangle = 0$$

as Φ_{x_i} is tangent to Σ and hence $C(\Sigma)$, and lastly

$$h_{ij} = \langle C_{x_i x_j}, N \rangle = r h_{ij}^{\Sigma}$$

Now we compute the mean curvature H to be

$$H = \frac{1}{n}(g_{tt} + \sum_{i=1}^{n-1} h_{ii}) = \frac{1}{n}(0 + r \sum_{i=1}^{n-1} h_{ii}^{\Sigma}) = r \left(\frac{1}{n} \sum_{i=1}^{n-1} h_{ii}^{\Sigma} \right) = r H^{\Sigma} = 0$$

as Σ is minimal and has vanishing mean curvature. There, $C(\Sigma)$ is also minimal. $\square$
This is a consequence of the monotonicity formula for minimal surfaces and the upper semi-continuity. As $\Sigma \subset S^n$ is minimal, the cone over Σ, denoted $C(\Sigma)$ is also minimal. The volume element of $C(\Sigma) \cap B_0(r)$ is given by converting to polar coordinates:

$$d\mathrm{Vol}_{C(\Sigma)\cap B_0(r)} = r^{n-1} dr d\mathcal{H}^{n-1}(\theta)$$

and so for any $R > 0$, we have that

$$\mathrm{Vol}(C(\Sigma) \cap B_R(0)) = \int_0^R \int_{\Sigma} r^{n-1} d\mathcal{H}^{n-1}(\theta) dr = \mathrm{Vol}(\Sigma) \int_0^R r^{n-1} dr = \frac{R^n}{n} \mathrm{Vol}(\Sigma)$$

and so in computing the density, we have

$$\Theta(0, C(\Sigma)) := \lim_{R\to 0} \Theta(0, R, C(\Sigma)) = \lim_{R\to 0} \frac{\mathrm{Vol}(C(\Sigma) \cap B_R(0))}{\omega_n R^n} = \lim_{R\to 0} \frac{\frac{R^n}{n}\mathrm{Vol}(\Sigma)}{\omega_n R^n} = \frac{\mathrm{Vol}(\Sigma)}{n\omega_n}$$

Now for some regular point on the cone $q \in C(\Sigma)\setminus\{0\}$, we have the surface is smooth around this point and hence locally flat, so $\Theta(q, C(\Sigma)) = \lim_{R\to 0} \Theta(q, R, C(\Sigma)) = 1$. Now we can appeal the fact that the density function is upper semi-continuous, and hence for a sequence of points $\{p_k\} \to p$ on $C(\Sigma)$,

$$\Theta(p, C(\Sigma)) \geq \limsup_{j\to\infty} \Theta(p_k, C(\Sigma))$$

Letting q be regular point as before, we can choose a sequence of regular points explicitly $q_j = \frac{1}{j} q$ and we see that $q_j \to 0$. Thus,

$$\Theta(0, C(\Sigma)) \geq \limsup_{j\to\infty} \Theta(q_j, C(\Sigma)) = 1$$

$$\implies \frac{\mathrm{Vol}(\Sigma)}{n\omega_n} \geq 1 \implies \mathrm{Vol}(\Sigma) \geq n\omega_n = \mathrm{Vol}(S^{n-1})$$

which is what we desired to show.

For equality: Note that equality occurs if and only if

$$\frac{\mathrm{Vol}(\Sigma)}{\mathrm{Vol}(S^{n-1})} = 1 \implies \Theta(0, C(\Sigma)) = 1$$

and so the vertex locally resembles a flat n-plane. If this is the case, then the hypersurface Σ must be flat in S^n, and thus the great sphere S^{n-1} itself. $\square$

Problem 2 (Mantegazza Ex. 1.1.2)**.** *If a hypersurface $M \subset \mathbb{R}^n$ is locally the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$, that is $\varphi(x) = (x, f(x))$, then we have that*

$$
\begin{aligned}
g_{ij} &= \delta_{ij} + f_i f_j, \\
\nu &= -\frac{(\nabla f, -1)}{\sqrt{1+|\nabla f|^2}}, \\
h_{ij} &= \frac{Hess_{ij} f}{\sqrt{1+|\nabla f|^2}}, \\
H &= \frac{\Delta f}{\sqrt{1+|\nabla f|^2}} - \frac{Hess_f(\nabla f, \nabla f)}{(1+|\nabla f|^2)^{3/2}} = -div\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right),
\end{aligned}
$$

where $f_i = \partial_i f$ and $Hess_f$ is the Hessian of the function f.

Proof. First, the Riemannian metric for an immersed hypersurface is found by pulling back φ to the standard Euclidean Metric, that is,

$$g_i j = \langle \frac{\partial \phi}{\partial x_i} \mid \frac{\partial \phi}{\partial x_j} \rangle = (0, 0, ..., 1, 0, ..., f_i) \cdot (0, 0, ..., 1, 0, ..., f_j) = \delta_{ij} + f_i f_j$$

where δ_{ij} is the Kronecher delta. For the (inward) unit normal ν, we have that by definition

$$\nu = -\frac{(\nabla f, -1)}{\langle(\nabla f, -1)\rangle} = -\frac{(\nabla f, -1)}{\sqrt{(f_1^2 + f_2^2 + ... + f_n^2 + 1)}} = -\frac{(\nabla f, -1)}{\sqrt{1+|\nabla f|^2}}$$

Now, we have that

$$
\begin{aligned}
h_{ij} &= \left\langle \nu, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle \\
&= \left\langle -\frac{(\nabla f, -1)}{\sqrt{1+|\nabla f|^2}}, \frac{\partial}{\partial x_j}[(0, \ldots, 0, 1, 0, \ldots, 0, f_i)] \right\rangle \\
&= -\frac{1}{\sqrt{1+|\nabla f|^2}}[(f_1, \ldots, f_n, -1) \cdot (0, \ldots, 0, \frac{\partial^2 f}{\partial x_i \partial x_j})] \\
&= \frac{\mathrm{Hess}_{ij} f}{\sqrt{1+|\nabla f|^2}}
\end{aligned}
$$

Finally, using the fact that H is the trace of the second fundamental form, that is $H = g^{ij} h_{ij}$, we first compute that

$$g_{ij} = \delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2}$$

we compute that

$$
\begin{aligned}
H &= \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}h_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(\delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2})(\frac{\text{Hess}_{ij} f}{\sqrt{1+|\nabla f|^2}}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\delta_{ij}\text{Hess}_{ij} f}{\sqrt{1+|\nabla f|^2|}} - \frac{\text{Hess}_{ij} f f_i f_j}{(1+|\nabla f|^2)^{3/2}} \\
&= \frac{\Delta}{\sqrt{1+|\nabla f|^2}} + \sum_{i,j=1}^{n}\frac{\text{Hess}_{ij} f f_i f_j}{(1+|\nabla f|^2)^{3/2}} \\
&= \frac{\Delta f}{\sqrt{1+|\nabla f|^2}} - \frac{\text{Hess}_f(\nabla f, \nabla f)}{(1+|\nabla f|^2)^{3/2}}
\end{aligned}
$$

one can verify that this expression for H is indeed equal to $-\text{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)$ as well. $\square$

Problem 3 (Giaquinta and Martinazzi Ex. 11.14)**.** *Show that if*

$$F(x,y) = (x, y, u(x,y), v(x,y)), \quad (x,y) \in \Omega \subset \mathbb{R}^2,$$

with $u, v : \Omega \to \mathbb{R}$ locally Lipschitz continuous, then

$$dF^* dF = \begin{pmatrix} 1+u_x^2+v_x^2 & u_x u_y + v_x v_y \\ u_x u_y + v_x v_y & 1+u_y^2+v_y^2 \end{pmatrix},$$

(11.12)

and use (11.12) to prove that

$$\det(dF^* dF) = 1 + |dU|^2 + (\det(dU))^2,$$

where $U := (u,v) : \Omega \to \mathbb{R}^2$, so that for $\Sigma := F(\Omega)$ we have

$$\mathcal{H}^2(\Sigma) = \int_\Omega \sqrt{1+|dU|^2 + (\det(dU))^2}\, dx.$$

Proof. Using the definition $g_{ij} = \frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_j}$, we have that

$$g_{11} = \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial x} = (1, 0, u_x, v_x) \cdot (1, 0, u_x, v_x) = 1 + u_x^2 + v_x^2$$

$$g_{12} = g_{21} = \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial y} = (1, 0, u_x, v_x) \cdot (0, 1, u_y, v_y) = u_x u_y + v_x v_y$$

$$g_{22} = \frac{\partial F}{\partial y} \cdot \frac{\partial F}{\partial y} = (0, 1, u_y, v_y) \cdot (0, 1, u_y, v_y) = 1 + u_y^2 + v_y^2$$

Thus we have that $dF^*dF = g = \begin{pmatrix} 1+u_x^2+v_x^2 & u_xu_y+v_xv_y \\ u_xu_y+v_xv_y & 1+u_y^2+v_y^2 \end{pmatrix}$ and so

$$\det(dF^*dF) = (1+u_x^2+v_x^2)(1+u_y^2+v_y^2) - (u_xu_y+v_xv_y)^2$$

$$= 1+v_x^2+v_y^2+u_x^2+u_y^2+u_x^2u_y^2+v_x^2v_y^2+u_x^2v_y^2+u_y^2v_x^2-(u_xu_y)^2-2u_xu_yv_xv_y-(v_xv_y)^2$$

$$= 1+v_x^2+v_y^2+u_x^2+u_y^2+u_x^2v_y^2+u_y^2v_x^2-2u_xu_yv_xv_y$$

Now letting $U := (u(x,y), v(x,y)) : \Omega \to \mathbb{R}^2$ as before, we have that

$$dU = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and so $|dU|^2 = u_x^2+u_y^2+v_x^2+v_y^2$, and $|\det(dU)| = u_xv_y - u_yv_x$, so by grouping terms together, we have that

$$\det(dF^*dF) = 1 + (u_x^2+u_y^2+v_x^2+v_y^2) + (u_x^2v_y^2 - 2u_xu_yv_xv_y + u_y^2v_x^2) = 1 + |dU|^2 + |\det(dU)|^2$$

and thus the area functional (and n-dim Hausdorff measure) is given by integrating the Riemannian volume form:

$$\mathcal{H}^n(\Sigma) = \text{Area}(\Sigma) = \int_\Sigma \sqrt{\det(dF^*dF)} = \int_\Sigma \sqrt{1+|dU|^2+|\det(dU)|^2}$$

as we desired to show. $\square$

Problem 4. *Prove the uniqueness of minimal graphs, that is, if two functions $u, v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ solve the minimal surface equation and $u|_{\partial\Omega} = v|_{\partial\Omega}$, then $u = v$ on all of Ω.*

Proof. Given that both u and v solve the minimal surface equation, we have that

$$\text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = 0$$

Thus, $\forall \eta \in C_c^2(\Omega)$,

$$\int_\Omega \eta \cdot \text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = 0 \implies -\int_\Omega \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) \cdot \nabla\eta = 0$$

by Divergence theorem. Choosing $\eta = u - v \in C_c^2(\Omega)$, and defining the function $\Phi(p) = \frac{p}{\sqrt{1+|p|^2}}$, we see that

$$(\Phi(p) - \Phi(q)) \cdot (p - q) \geq 0$$

since the function Φ is positive semidefinite (see Problem 5). Thus, settng $p = \nabla u$, $q = \nabla v$, and $\nabla\eta = p - q = \nabla u - \nabla v$, we have that

$$(\Phi(p) - \Phi(q)) \cdot (p - q) = \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) \cdot \nabla\eta \geq 0$$

This, this quantity is nonnegative and its integral over Ω is 0, which forces

$$\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}-\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)\cdot\nabla\eta=0\implies\nabla u=\nabla v\quad\text{on }\Omega$$

and hence $u=v$ on all of Ω as well. □

Problem 5 (C-M Ex. 1)**.** *Prove that the area functional on graphs is convex.*

Proof. A sufficient convexity condition is that the Hessian of the integrand of the area functional is positive definite. For $u:\mathbb{R}^2\to\mathbb{R}$ and $\text{Graph}_u=\{(x,y,u(x,y)):x,y\in\mathbb{R}\}$ the area functional is defined as

$$\text{Area}(\text{Graph}_u)=\int_{\text{Graph}_u}\sqrt{1+|\nabla u|^2}$$

Define $F(p)=\sqrt{1+|p|^2}$, and we have that

$$\nabla F(p)=\frac{p}{\sqrt{1+|p|^2}}$$

and so

$$\nabla^2F(p)=\nabla[p\cdot(1+|p|^2)^{-1/2}]=\frac{1}{\sqrt{1+|p|^2}}I-\frac{pp^T}{(1+|p|^2)^{3/2}}$$

For the positive semi-definite condition, we require that $v^T\nabla^2F(p)v\geq 0$ for all $p\in\mathbb{R}^n$. We have that

$$v^T\nabla^2F(p)v=\frac{1}{(1+|p|^2)^{3/2}}[(1+|p|^2)|v|^2-(p\cdot v)^2]$$

By Cauchy-Schwarz, we see that $(p\cdot v)^2\leq|p|^2|v|^2$, and so

$$(1+|p|^2)|v|^2-(p\cdot v)^2\geq(1+|p|^2)|v|^2-|p|^2|v|^2=|v|^2\geq 0$$

so the integrand F is convex, so is the area functional. □

Problem 6 (C-M Ex. 2)**.** *Prove that the catenoid and helicoid are minimal surfaces.*

Proof. Given the parametrization of the catenoid $X(u,v)=(\cosh v\cos u,\cosh v\sin u,v)$, the most straightforward way to demonstrate this is to compute the mean curvature, for which we will first need the first and second fundamental forms, respectively. The metric coefficients (first fundamental form) are given by

$$D=\langle X_u,X_u\rangle=(-\cosh v\sin u,\cosh v\cos u,0)\cdot(-\cosh v\sin u,\cosh v\cos u,0)=\cosh^2 v$$

$$E=\langle X_u,X_v\rangle=(-\cosh v\sin u,\cosh v\cos u,0)\cdot(\sinh v,\cos u,\sinh v\sin u,1)=0$$

$$F=\langle X_v,X_v\rangle=(\sinh v,\cos u,\sinh v\sin u,1)\cdot(\sinh v,\cos u,\sinh v\sin u,1)=1+\sinh^2 v$$

and thus

$$\mathrm{I}=\begin{pmatrix}\cosh^2 v & 0\\ 0 & 1+\sinh^2 v\end{pmatrix}$$

Now for the second fundamental form, we see that the unit normal vector is given by

$$\nu = \frac{X_u \times X_v}{||X_u \times X_v||} = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v)$$

and so the second fundamental form has coefficients

$$L = \langle \nu, X_{uu} \rangle = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v) \cdot (\cosh v \cos u, -\cosh v, \sin u, 0) = -1$$

$$M = \langle \nu, X_{uv} \rangle = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v) \cdot (\sinh v \sin u, -\sin hv \cos u, 0) = 0$$

$$N = \langle \nu, X_{vv} \rangle = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v) \cdot (\cosh v \cos u, \cosh v \sin u, 0) = 1$$

so

$$\mathrm{II} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now to compute the mean curvature, we can use the formula

$$2H = \mathrm{Trace}((\mathrm{II})(\mathrm{I}^{-1}))$$

and as

$$\mathrm{I}^{-1} = \frac{1}{\cosh^2 v}\mathrm{Id}$$

we have that

$$2H = \mathrm{Trace}\big((\mathrm{II})(\mathrm{I})^{-1}\big) = \mathrm{Trace}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\cosh^2(v)} & 0 \\ 0 & \frac{1}{\cosh^2(v)} \end{pmatrix}\right) = -\frac{1}{\cosh^2(v)} + \frac{1}{\cosh^2(v)} = 0.$$

and so $H = 0$ as required, so the catenoid is minimal For the helicoid, we have the parametrization $\Phi(u, v) = (v \cos u, v \sin u, u)$, which gives us metric coefficients

$$D = \langle X_u, X_u \rangle = (-v \sin u, v \cos u, 1) \cdot (-v \sin u, v \cos u, 1) = v^2 + 1$$

$$E = \langle X_u, X_v \rangle = (-v \sin u, v \cos u, 1) \cdot (\cos u, \sin u, 0) = 0$$

$$F = \langle X_v, X_v \rangle = (\cos u, \sin u, 0) \cdot (\cos u, \sin u, 0) = 1$$

and so

$$\mathrm{I} = \begin{pmatrix} v^2 + 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Moreover, the unit normal ν is given by

$$\nu = \frac{X_u \times X_v}{||X_u \times X_v||} = \frac{1}{\sqrt{1 + v^2}}(-\sin u, \cos u, -v)$$

and the second fundamental form coefficients are

$$L = \langle \nu, \Phi_{uu} \rangle = \frac{1}{\sqrt{1 + v^2}}(-\sin u, \cos u, -v) \cdot (-v \cos u, -v \sin u, 0) = 0$$

$$M = \langle \nu, \Phi_{uv} \rangle = \frac{1}{\sqrt{1+v^2}}(-\sin u, \cos u, -v) \cdot (-\sin u, \cos u, 0) = \frac{1}{\sqrt{v^2+1}}$$

$$N = \langle \nu, \Phi_{vv} \rangle = \frac{1}{\sqrt{1+v^2}}(-\sin u, \cos u, -v) \cdot (0,0,0) = 0$$

and so we have that

$$\mathrm{II} = \frac{1}{\sqrt{v^2+1}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and thus

$$2H = \mathrm{Trace}\big((\mathrm{II})(\mathrm{I})^{-1}\big) = \mathrm{Trace}\Big(\frac{1}{\sqrt{v^2+1}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{v^2+1} \end{pmatrix}\Big) = 0.$$

and so $H = 0$ for the helicoid as well, and hence it is a minimal surface. □

Problem 7 (C-M Ex. 5)**.** *Compute the total curvature, i.e., $\int K$ of the catenoid.*

Proof. Having already computed the first and second fundamental forms of the catenoid as in Ex. 2, we have

$$\mathrm{I} = \begin{pmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{pmatrix}$$

and

$$\mathrm{II} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the Gaussian curvature is given by

$$K = \frac{\det(\mathrm{II})}{\det(\mathrm{I})} = \frac{LN - M^2}{EG - F^2} = -\frac{1}{\cosh^4 v}$$

Now $u \in [0, 2\pi]$, and $v \in (-\infty, \infty)$, and we have that the volume form is given by $d\mathrm{Vol} = \sqrt{\det(\mathrm{I})} dudv = \cosh^2 v dudv$ and integrating we obtain

$$TC = \int_{-\infty}^{\infty} \int_0^{2\pi} K d\mathrm{Vol} = \int_{-\infty}^{\infty} \int_0^{2\pi} -\frac{1}{\cosh^2 v} dudv = -2\pi \int_{-\infty}^{\infty} \frac{1}{\cosh^2 v} dv = -4\pi$$

and thus

$$\boxed{TC(\mathrm{Catenoid}) = -4\pi}$$

□

Problem 8 (C-M Ex. 18)**.** *Compute the morse index of the catenoid.*

Proof. Here, note that the catenoid, denoted $\mathcal{C}$ is conformal to the sphere minus two points via the Gauss map $N : \mathcal{C} \to S^2$, which is diffeomorphism in this case. We see that the quadratic form associated with the morse index

$$\mathcal{Q}(\varphi) = \int_\Sigma (|\nabla \varphi|^2 - |A|^2 \varphi^2)$$

is a conformal invariant since the Dirichlet energy and $|A|^2$ scale in exactly the right way to ensure invariance for minimal surfaces. It is indeed a fact that for minimal surfaces, the eigenvalue problem $Lu = \lambda u$ retains the same set of λ under a conformal transformation. Thus, the spectrum of L is independent of the conformal model of the minimal surface. Thus, we compute the stability operator L for S^2 (viewed as minimal inside S^3) to be

$$L = \Delta_{S^2} + |A|^2 + \mathrm{Ric}_{S^3}(N, N) = \Delta_{S^2} + 2$$

Since S^2 is totally geodesic in S^3 and hence $|A|^2 \equiv 0$. For the Ricci term, as S^3 has constant sectional curvature 1, each tangent vector has $n-1$ orthogonal directions with sectional curvature equal to one, so $\mathrm{Ric}_{S^n} = (n-1)g \implies \mathrm{Ric}_{S^3}(X, X) = 2\langle X, X\rangle$, and so for the unit vector N, $\mathrm{Ric}_3(N, N) = 2$ identically. It is well known that the second eigenvalue of Δ_{S^2} is 2, and thus

$$\boxed{\mathrm{Index}(\mathcal{C}) = 1}$$

□

Problem 9 (C-M Ex. 15)**.** *Suppose that $\Sigma^2 \subset \mathbb{R}^3$ is a minimal surface so that for all R,*

$$Area(B_R \cap \Sigma) \leq 2\pi R^2$$

Prove that either Σ is embedded or the union of two planes.

Proof. By the monotonicity formula, the density function Θ is nondecreasing in r, and so

$$\Theta(x_0, \Sigma) = \lim_{r\to 0} \frac{\mathrm{Area}(B_r(x_0) \cap \Sigma)}{\mathrm{Area}(B_r(x_0) \subset \mathbb{R}^2)} \geq 1 \implies \frac{\mathrm{Area}(B_r(x_0) \cap \Sigma)}{\pi r^2} \geq 1$$

for all $r > 0$. By the assumption, we also have that

$$\Theta(x_0, r, \Sigma) = \frac{\mathrm{Area}(B_r \cap \Sigma)}{\pi r^2} \leq 2$$

and consequently,

$$1 \leq \Theta(x_0, r, \Sigma) \leq 2$$

Given that Σ is a codimension one surface in $\mathbb{R}^n$, the density function must be an integer since the tangent cone has integer multiplicity at almost every point. Thus, we have that $\Theta(x_0, r, \Sigma) \in \{1, 2\}$. If $\Theta(x_0, r, \Sigma) = 1$, then $\forall x_0 \in \Sigma$, Σ is a single sheet around each point, and hence must be embedded. If instead, $\Theta(x_0, r, \Sigma) = 2$, then using completeness and taking $r \to \infty$, we have that Σ is almost everywhere two-sheeted, and hence must be the union of two overlapping planes. □

Problem 10 (C-M Ex. 19)**.** *Prove that the morse index of the helicoid is infinite. Can you divide the helicoid into two stable pieces?*

Proof. Here, we can appeal to a theorem of Fischer-Colbrie which said that finite morse index implies finite total curvature and use the contrapositive. Let $\mathcal{H}$ be the helicoid parametrized by the map $\phi(s,t) = (t\cos(s), t\sin(s), s)$, where $s, t \in \mathbb{R}$. We have already computed the first and second fundamental form to be

$$\mathrm{I} = \begin{pmatrix} t^2+1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathrm{II} = \begin{pmatrix} 0 & \frac{1}{\sqrt{t^2+1}} \\ \frac{1}{\sqrt{t^2+1}} & 0 \end{pmatrix}$$

and we see that the Gauss curvature is given by

$$K = \frac{\det(\mathrm{II})}{\det(\mathrm{I})} = -\frac{1}{(1+t^2)^2}$$

and hence the total curvature is given by

$$TC(\mathcal{H}) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(1+t^2)^2} dtds = -\frac{\pi}{2}\int_{-\infty}^{\infty} ds = \infty$$

and thus the helicoid has non-finite total curvature, and hence

$$\mathrm{Index}(\mathcal{H}) = \infty$$

As a consequence, the helicoid cannot be divided into two stable pieces, as the total curvature for one of the pieces will still require integration of ds with noncompact bounds of integration, namely with one of the endpoints being $+\infty$ or ∞ (or both). $\square$

Problem 11 (C-M Ex. 21)**.** *What is the morse index of Enneper's surface?*

Proof. Here we can use the Weierstrass representation of Enneper's surface to show that is conformally equivalent to the sphere (minus one point). This is given by

$$g(z) = z, \quad \phi(z)dz = dz$$

The Gauss map N for the Weierstrass representation is given by $N(z) = \left(\frac{2\Re(g)}{1+|g(z)|^2}, \frac{2\Im(g)}{1+|g(z)|^2}, \frac{|g(z)|^2-1}{1+|g(z)|^2}\right)$, and setting $z = u + iv$, we have that for Enneper's surface the Gauss map is

$$N_{Enn}(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

Now recall that the standard stereographic projection map $\pi : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ is given explicitly by

$$\pi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

which is a differomorphism with inverse

$$\pi^{-1}(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

and we see that N_{Enn} is precisely π^{-1}, and thus is a diffeomorphism onto $S^2 \setminus \{(0,0,1)\}$. Hence, Enneper's surface is conformally equivalent to the sphere minus one point, and thus has the same morse index, namely 1 (see Ex. 18). Thus,

$$\boxed{\text{Index(Enneper)} = 1}$$

□

Problem 12 (C-M Ex. 32)**.** *Prove that the least possible area of a closed minimal surface in S^3 is 4π.*

Proof. This follows immediately from Problem 1. For any minimal $\Sigma^2 \subset S^3$,

$$|\Sigma| \geq |S^2| = 4\pi.$$

□

Problem 13 (C-M Ex. 33, 34, 35)**.** *For the next three exercises, define a functional F on hypersurfaces Σ in $\mathbb{R}^{n+1}$ by*

$$F(\Sigma) = \int_\Sigma \exp\left(-\frac{|x|^2}{4}\right). \tag{1}$$

1. *Compute the first variation of F and the Euler-Lagrange equation for critical points of F. (These critical points are called self-shrinkers and come up in mean curvature flow; see [Hu], [CM25], and [CM6] for more details.)*

2. *Compute the second variation of F for a compactly supported normal variation.*

3. *Show that there are no closed hypersurfaces that are stable critical points for F.*

Proof. 1. Recall that if ϕ is a real-valued function on Σ and $\mathbf{n}$ is the unit normal, then the first variation formula yields

$$\delta\left(\int_\Sigma \phi\right) = \int_\Sigma (\nabla\phi \cdot \mathbf{n} - \phi H)\phi_{\text{speed}}$$

where ϕ_{speed} denotes the normal speed function, which in this case is synonymous with ϕ. Setting $\phi(x) = e^{-\frac{|x|^2}{4}}$, we have that

$$\nabla\phi = -\frac{x}{2}e^{-\frac{|x|^2}{4}}$$

and so we evaluate that

$$\delta(F) = \int_\Sigma (-\frac{1}{2}\langle x, \mathbf{n}\rangle e^{-\frac{|x|^2}{4}} - e^{-\frac{|x|^2}{4}} H)\phi = \int_\Sigma \phi^2(-\frac{1}{2}\langle x, \mathbf{n}\rangle - H)$$

and so we see that Euler-Lagrange equation yields

$$\delta(F) = 0 \iff H = -\frac{\langle x, \mathbf{n}\rangle}{2}$$

and this is the so called **self-shrinker** equation.

2. He, we can appeal to a classic result of Colding-Minicozzi, namely that if $\Sigma \subset \mathbb{R}^n$ is a critical point of the functional $\int_\Sigma e^{-f}$, then if $\phi\mathbf{n}$ is a normal variation, then we have

$$\delta^2 F(\phi, \phi) = \int_\Sigma e^{-f}[|\nabla^\Sigma \phi|^2 - |A|^2\phi^2 - (\nabla^2 f(\mathbf{n}, \mathbf{n}))\phi^2]$$

and by choosing $f = \frac{|x|^2}{4}$, we have that $\nabla^2 f = \nabla[\frac{|x|}{2}] = \frac{1}{2}$ we obtain that

$$\delta^2 F = \int_\Sigma e^{-\frac{|x|^2}{4}}[|\nabla^\Sigma \phi|^2 - |A|^2\phi^2 - \frac{1}{2}\phi^2]$$

A minimal hypersurface is called stable if $\delta^2 F(\Sigma) \geq 0$. Suppose this is the case. Then we can use the constant function $\phi \equiv 1$ and the second variation formula above to obtain

$$\delta^2 F(\Sigma) = \int_\Sigma e^{-\frac{|x|^2}{4}}[-|A|^2 - \frac{1}{2}]$$

and as $e^{-\frac{|x|^2}{4}} > 0$, and $|A|^2 + \frac{1}{2} \geq \frac{1}{2}$, and thus the integrand is strictly negative, which yields a negative integral when integrating over a closed hypersurface, and thus

$$\delta^2 F(\Sigma) < 0$$

for Σ closed, which is a contradiction. $\square$