The Monotonicity Formula for Energy Minimizing Maps

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Abstract

In this talk, we will define energy minimizing maps and introduce key examples. Then, we will derive the associated variational formulae. A central theme of the talk will be the monotonicity formula: we will explore its significance and highlight its appearance in related geometric contexts such as minimal surfaces, mean curvature flow, and Ricci flow. We will then prove the monotonicity formula for energy minimizing maps. Finally, we will define the density function, introduce tangent maps, and discuss the role of monotone quantities in the study of regularity and singularities.

1 Definitions and Examples

The framework for this chapter will be this: let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an open region, and let N be a smooth, compact Riemannian manifold embedded in $\mathbb{R}^p$ of dimension greater than or equal to 2. Now we look at maps $u : \Omega \to N \subset \mathbb{R}^p$.

Definition 1. *We define the Sobolev Space*

$$W^{1,2}(\Omega; N) := \{u(x) \in W^{1,2}(\Omega; \mathbb{R}^p) : u(x) \in N \ \ a.e\ x \in \Omega\}$$

Now we will define three types of harmonic maps: weakly harmonic (W), stationary harmonic (S), and energy minimizing (E). We will see that $(E) \to (S) \to (W)$, yet $(W) \not\to (S) \not\to (E)$.

Definition 2 (Weakly Harmonic)**.** *A map $u \in W^{1,2}(\Omega; \mathbb{R}^p)$ is called weakly harmonic if*

$$\int_\Omega \langle \nabla u, \nabla v \rangle = 0$$

for every test function $v \in C_c^\infty(\Omega; \mathbb{R}^p)$.

Definition 3 (Dirichlet Energy in a Ball)**.** *The energy $E_{B_r(y)}(u)$ for a function $u \in W^{1,2}(\Omega; N)$ in a ball $B_r(y) := \{x : |x - y| < r\}$ with $\bar{B}_r(y) \subset \Omega$ is defined by*

$$E_{B_r(y)}(u(x)) = \int_{B_r(y)} |Du(x)|^2 dx$$

We say that $u \in W^{1,2}(\Omega; N)$ is a *stationary harmonic map* if it is a critical point of the Euler-Lagrange functional for E. It is easy to check the implication stationary harmonic $\implies$ weakly harmonic. In this talk, we will mostly study specific kinds of harmonic maps called energy minimizers:

Definition 4 (Energy Minimizing Maps)**.** *We say that $u \in W^{1,2}(\Omega; N)$ is an energy minimizing map if for each $B_r(y) \subset \Omega$,*

$$E_{B_r(y)}(u) \leq E_{B_r(y)}(w)$$

for every $w \in W^{1,2}(B_r(y); N)$ with $w \equiv u$ in a neighborhood of $\partial B_r(y)$.

One trivial example is if $P \in N$ is some fixed point, and we set $u(x) = P \implies E(u) = 0$ so u is trivially energy minimizing. For a more interesting example, we define the map $u(x) : B_1 \subset \mathbb{R}^n \to S^{n-1}$ by

$$u(x) = \frac{x}{|x|}$$

it is easy to verify that u is harmonic, but it turns out that u is energy minimizing as well for $n \geq 3$. To find the energy of u, we must first find the square of the gradient Du:

$$\frac{\partial u^i}{\partial x_j} = \frac{\delta_{ij}|x| - x_i \frac{x_j}{|x|}}{|x|^2} = \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}$$

and thus

$$|Du|^2 = \sum_{i,j} \left(\frac{\partial u^i}{\partial x_j} \right)^2 = \sum_{i,j} \left(\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3} \right)^2 = \frac{1}{|x|^2} \left(n - \frac{|x|^2}{|x|^2} \right) = \frac{n-1}{|x|^2}$$

Now to find the total energy, we have that

$$E(u) = \int_{B_1} \frac{n-1}{|x|^2} dx = \int_0^1 r^{-2} r^{n-1} dr = \int_0^1 r^{n-3} dr = \frac{1}{n-2} < \infty$$

and it requires an additional argument to show that u is energy minimizing, but it is only true for $n \geq 3$ by the calculation above.

2 The Variational Formulae

There are two different classes of variational formulae for harmonic maps. The first uses nearest point projection, and will not be covered extensively. The second does not, and the derivation of the Euler-Lagrange will be shown in detail here. This the setup: Suppose $u \in W^{1,2}(\Omega; N)$ is energy minimizing, $\bar{B}_r(y) \subset \Omega$, and suppose that for some $\epsilon > 0$, there is a 1-parameter family $\{u_s\}_{s \in (-\epsilon, \epsilon)}$ of maps $B_r(y)$ into N such that $u_0 \equiv u$, $Du_s \in L^2(\Omega)$, and $u_s \equiv u$ in a neighborhood of $\partial B_r(y)$ for each s. By definition, u is energy minimizing and hence

$$\frac{dE_{B_r(y)}}{ds}|_{s=0} = 0$$

and the left hand side is called the *first variation* of $E_{B_r(y)}$, and the family $\{u_s\}$ is called the a *variation* of u.

2.1 Class 1: The extrinsic variation

Let $\zeta \in C_c^\infty(B_r(y))$. We define the variation $\{u_s\}$ by

$$u_s = \Pi \circ (u + s\zeta)$$

and substituting into the first variation formula yields

$$\int_\Omega \sum_{i=1}^n (D_i u \cdot D_i \zeta - \zeta \cdot A_u(D_i u, D_i u)) = 0 \tag{1}$$

where A_u is the second fundamental form induced by u.

2.2 Class 2: The intrinsic variation

This variation works by shifting around values in the domain of u in order to find a minimizer. More precisely, letting $\zeta \in C_c^\infty$ as before, we now define

$$u_s(x) = u(x + s\zeta(x))$$

We claim that the Euler-Lagrange equations for this intrinsic variation is given by

$$\int_{B_r(y)} \sum_{i,j=1}^n (|Du|^2 \delta_{ij} - 2D_i u \cdot D_j u) D_i \zeta^i = 0 \tag{2}$$

and we will show the full calculation. We have that

$$\begin{aligned}
D_i u_s(x) =& \sum_{j=1}^n \frac{\partial}{\partial \zeta_j} u(x + s\zeta(x)) \cdot \frac{\partial}{\partial x_i}(x_j + s\zeta_i(x)) \\
=& \sum_{j=1}^n \frac{\partial}{\partial \zeta_j} u(x + s\zeta(x)) \cdot (\delta_{ij} + s\frac{\partial}{\partial x_i}\zeta_i(x)) \\
=& \sum_{j=1}^n D_j(u(x + s\zeta(x))) \cdot (\delta_{ij} + sD_i\zeta^j(x)) \\
=& (D_i u(x + s\zeta)) + \sum_{j=1}^n sD_i\zeta^j D_j u(x + s\zeta(x))
\end{aligned}$$

Now call $A := (D_i u(x+s\zeta))$ and $B := \sum_{j=1}^n sD_i\zeta^j D_j u(x+s\zeta(x))$ and using that $|D_j u_s(x)|^2 = |A + sB|^2 = A \cdot A + 2sA \cdot B + o(s^2)$, and note that

$$\begin{aligned}
|D_i u_s(x)|^2 =& A \cdot A + 2s(A \cdot B) + o(s^2) \\
=& |D_i u(x + s\zeta)|^2 + 2s(D_i u(x + s\zeta) \cdot \sum_{j=1}^n D_i\zeta^j D_j u(x + s\zeta(x))) + o(s^2)
\end{aligned}$$

and as $|Du_s(x)|^2 = \sum_{j=1}^n |D_i u_s(x)|^2$, we have after making the substitution $w = x + s\zeta$ (which gives a C^∞ diffeomorphism of B_ρ onto itself for $|s|$ small enough) that

$$\begin{aligned}
|Du_s(x)|^2 =& \sum_{j=1}^n [|D_j u(w)|^2 + 2s(D_j u(w) \cdot \sum_{k=1}^n D_j\zeta^k D_k u(w))] + o(s^2) \\
=& |Du(w)|^2 + 2s \sum_{j,k=1}^n (D_j u \cdot D_k u)(w) D_j \zeta^k(w) + o(s^2)
\end{aligned}$$

Note that the volume form is now given by $|\det(\partial x^i/\partial w^j)|dw = |\det(\partial \zeta^i/\partial x^j)|^{-1}dw = (1 - s\,\mathrm{div}\,\zeta + O(s^2))dw = (1 - s\sum_k D_k\zeta^k + O(s^2))dw$. Plugging everything in and disregarding any higher order terms, we have

$$\begin{aligned}
E(u_s) =& \int_{B_r(y)} |Du_s|^2 dx \\
=& \int_{B_r(y)} (|Du(w)|^2 + 2s \sum_{j,k=1}^n (D_j u \cdot D_k u)(w) D_j\zeta^k(w))(1 - s\sum_l D_l\zeta^l) dw \\
=& \int_{B_r(y)} \left(|Du|^2 + 2s \sum_{j,k=1}^n (D_j u \cdot D_k u)(w) D_j\zeta^k(w) - s|Du|^2 \sum_l D_l \zeta^l \right) dw + o(s^2) \\
=& E(u) + s\int_{B_r(y)} \left(2\sum_{j,k=1}^n (D_j u \cdot D_k u)(w) D_j\zeta^k(w) - |Du|^2 \sum_l D_l\zeta^l \right) dw
\end{aligned}$$

Finally, the Euler-Lagrange equation for E yields

$$\frac{d}{ds}\Big|_{s=0} E(u_s) = \int_{B_r(y)} \left(2\sum_{j,k=1}^n (D_j u \cdot D_k u)(w) D_j\zeta^k(w) - |Du|^2 \sum_l D_l\zeta^l(w) \right) dw = 0$$

From here, we rearrange indices to yield the equation

$$\begin{aligned}
& \int_{B_r(y)} \left(2\sum_{i,j=1}^n (D_i u \cdot D_j u)(w) D_i\zeta^j(w) - |Du|^2 \sum_i D_i\zeta^i(w) \right) dw = 0 \\
\implies & \int_{B_r(y)} \left(2\sum_{i,j=1}^n (D_i u \cdot D_j u)(w) D_i\zeta^j(w) - |Du|^2 \sum_{i,j=1}^n \delta_{ij} D_i\zeta^i(w) \right) dw = 0 \\
\implies & \int_{B_r(y)} \left(\sum_{i,j=1}^n 2(D_i u \cdot D_j u)(w) D_i\zeta^j(w) - |Du|^2 \delta_{ij} D_i\zeta^i(w) \right) dw = 0 \\
\implies & \boxed{\int_{B_r(y)} \sum_{i,j=1}^n (|Du|^2\delta_{ij} - 2D_i u \cdot D_j u) D_i\zeta^i = 0}
\end{aligned}$$

which is the desired intrinsic variational formula. Note that if u satisfies (1), then u is merely *weakly harmonic*. If u is C^2, we can integrate by parts in (1) to deduce that (1) $\implies$ (2). However, this not true in general. If u satisfies both (1) and (2), then u is *stationary harmonic*. It is also not true that (2) $\implies$ (1).

3 The Monotonicity Formula

Theorem 1 (Monotonicity Formula for Energy Minimizing Maps). *Let $u \in W^{1,2}(B_\rho(y);N)$ be an energy minimizing map. If $y \in \Omega$ and $\bar{B}_\rho(y) \subset \Omega$, then for all $0 < \sigma < \rho < \rho_0$ we have that*

$$\rho^{2-n}\int_{B_\rho(y)}|Du|^2 - \sigma^{2-n}\int_{B_\sigma(y)}|Du|^2 = 2\int_{B_\rho(y)\setminus B_\sigma(y)} R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2$$

where $\frac{\partial u}{\partial R}$ denotes the radial derivative in the direction $\frac{x-y}{|x-y|}$.

Proof. **Claim**: If $a = (a^1, ...a^n)$ are integrable functions on $B_{\rho_0}(y)$ and

$$\int_{B_{\rho_0}} \sum_{j=1}^{\infty} a^j D_j\zeta = 0$$

for all $\zeta \in C_c^\infty(B_{\rho_0}(y))$, then for almost every $\rho \in (0,\rho_0)$, we have that

$$\int_{B_\rho(y)} \sum_{j=1}^{n} a^j D_j\zeta = \int_{\partial B_\rho(y)} \eta \cdot a\zeta$$

for any $\zeta \in C^\infty(\bar{B}_{\rho_0}(y))$, where $\eta \equiv \frac{x-y}{\rho}$ is outward unit normal of $\partial B_\rho(y)$
Proof: Extend a outside $B_{\rho_0}(y)$ by zero to obtain $\tilde{a} \in L^1(\mathbb{R}^n)$, and let $a_\epsilon = \phi_\epsilon * \tilde{a}$, where ϕ_ϵ is a standard mollifer. If $\mathrm{div}(a) = 0$, then $\mathrm{div}(a_\epsilon) = \mathrm{div}(\tilde{a} * \phi_\epsilon) = \mathrm{div}(\tilde{a}) * \phi_\epsilon = 0$ for all $x \in B_{\rho_0 - \epsilon}(y)$. Fix a radius $\rho \in (0,\rho_0)$, and for ϵ small enough such that $\epsilon < \rho_0 - \rho$, we have that $\mathrm{div}(a_\epsilon) = 0$ on the closure $\bar{B}_\rho(y)$. Take any test function $\zeta \in C^\infty(\bar{B}_\rho(y))$, and if we let $F := a_\epsilon\zeta \in C^1(\bar{B}_\rho(y))$, then note that by divergence theorem

$$\int_{B_\rho(y)} \mathrm{div}(F)dx = \int_{B_\rho(y)} \mathrm{div}(a_\epsilon\zeta) = \int_{\partial B_\rho(y)} (a_\epsilon\zeta)\cdot\eta dS$$

using the product rule for divergence and the fact that $\mathrm{div}(a_\epsilon) = 0$ in $B_\rho(y)$, note that

$$\int_{B_\rho(y)} \mathrm{div}(a_\epsilon\zeta) = \int_{B_\rho(y)} \zeta\mathrm{div}(a_\epsilon) + a_\epsilon\cdot\nabla\zeta = \int_{B_\rho(y)} a_\epsilon\cdot\nabla\zeta = \int_{\partial B_\rho(y)} (a_\epsilon\zeta)\cdot\eta$$

Now, we can let $\epsilon \to 0$ to yield the desired result. □
Now since we assume u is an energy minimizer, u satisfies the variational equation given by

$$\int_{B_\rho(y)} \sum_{i,j=1}^{n} \left(|Du|^2\delta_{ij} - 2D_iu(x)D_ju(x)\right) D_i\zeta^j(x)$$

and using the above claim, we have that this expression is equivalent to

$$\int_{\partial B_\rho(y)} \sum_{i,j=1}^{n} \left(|Du(x)|^2\delta_{ij} - 2D_iu(x)D_ju(x)\right) \rho^{-1}(x_i - y_i)\zeta^j(x)$$

Notice that if $\zeta(x) = |x^j - y^j|$, where x^j picks out the j-th coordinate of x, then $D_i\zeta^j(x) = \delta_{ij}$, and δ_{ij} picks out n terms after summing i, j from 1 to n, and hence the first expression can be simplified to

$$\int_{B_\rho(y)} \sum_{i,j=1}^{n} \left(|Du|^2\delta_{ij} - 2D_iu(x)D_ju(x)\right) D_i\zeta^j(x) = \int_{B_\rho(y)} n|Du|^2 - 2|Du|^2 = (n-2)\int_{B_\rho(y)} |Du|^2$$

Moreover, in the second expression, we have that this simplifies to

$$\int_{\partial B_\rho(y)} \sum_{i,j=1}^{n} \left(|Du(x)|^2\delta_{ij} - 2D_iu(x)D_ju(x)\right) \rho^{-1}(x_i - y_i)\zeta^j(x) = \int_{\partial B_\rho(y)} \rho \left(|Du(x)|^2 - 2\left|\frac{\partial u}{\partial R}\right|^2\right)$$

since $|\frac{\partial u}{\partial R}|^2 = \sum_{i,j=1}^{n} D_iu(x)D_ju(x)\frac{|x^i-y^i||x^j-y^j|}{\rho^2}$ Therefore, we have

$$(n-2)\int_{B_\rho(y)} |Du|^2 = \rho\int_{\partial B_\rho(y)} \left(|Du(x)|^2 - 2\left|\frac{\partial u}{\partial R}\right|^2\right)$$

and now, seeing that $\int_{\partial B_\rho(y)} f = \frac{\partial}{\partial\rho}\int_{B_\rho(y)} f$ by coarea formula, we have that after multiplying both sides by ρ^{1-n} and computing derivatives, we find that

$$\frac{d}{d\rho}(\rho^{2-n}\int_{B_\rho(y)} |Du|^2) = (2-n)\rho^{1-n}\int_{B_\rho(y)} |Du|^2 + \rho^{2-n}\int_{\partial B_\rho(y)} |Du|^2$$

combined with the fact that

$$(2-n)\rho^{1-n}\int_{B_\rho(y)} |Du|^2 = -\rho^{2-n}\int_{\partial B_\rho(y)} \left(|Du|^2 - 2\left|\frac{\partial u}{\partial R}\right|^2\right)$$

from the first equation, substituting this quantity in cancels out the $\rho^{2-n}\int_{\partial B_\rho(y)} |Du|^2$ terms, and we left with

$$\frac{d}{d\rho}\left(\rho^{2-n}\int_{B_\rho(y)} |Du|^2\right) = \rho^{2-n}\frac{d}{d\rho}\int_{B_\rho(y)} 2\left|\frac{\partial u}{\partial R}\right|^2 = 2\frac{d}{d\rho}\left(\int_{B_\rho(y)\setminus B_\tau(y)} R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2\right)$$

for a fixed choice of $\tau \in (0, \rho)$. Then, by integrating on both sides from σ to ρ and using the fundamental theorem of calculus, we have that

$$\int_\sigma^\rho \frac{d}{d\rho}\left(\rho^{2-n}\int_{B_\rho(y)} |Du|^2\right) = \int_\sigma^\rho 2\frac{d}{d\rho}\left(\int_{B_\rho(y)\setminus B_\tau(y)} R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2\right)$$

$$\implies \sigma^{2-n}\int_{B_\sigma(y)} |Du|^2 - \tau^{2-n}\int_{B_\tau(y)} |Du|^2 = 2\int_{B_\sigma(y)\setminus B_\tau(y)} R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2$$

□

Definition 5 (Density). *Given any map $u \in W^{1,2}(\Omega; N)$, we define the density function $\Theta : \Omega \to \mathbb{R}$ to be*

$$\Theta_u(y) = \lim_{\rho \downarrow 0} \rho^{2-n} \int_{B_\rho(y)} |Du|^2$$

Note that from the monotonicity formula, we have that since $R^{2-n} \left| \frac{\partial u}{\partial R} \right|^2$ is clearly strictly nonnegative,

$$\sigma^{2-n} \int_{B_\sigma(y)} |Du|^2 \geq \tau^{2-n} \int_{B_\tau(y)} |Du|^2$$

for all $0 < \tau < \sigma$, and thus

$$\sigma^{2-n} \int_{B_\sigma(y)} |Du|^2$$

is an increasing function of σ for $\sigma \in (0, \sigma_0)$. Therefore, the density function exists, and we obtain the immediate fact:

Theorem 2. Θ_u *is an upper semi-continuous function on Ω, that is, if $y_j \to y \in \Omega$, then*

$$\Theta_u(y) \geq \limsup_{j \to \infty} \Theta_u(y_i)$$

Proof. Let $\epsilon > 0$ and $\rho > 0$ with $\rho + \epsilon < \operatorname{dist}(y, \partial\Omega)$. By the monoticity formula, we have that

$$\Theta_u(y_j) \leq \rho^{2-n} \int_{B_\rho(y_j)} |Du|^2$$

for j sufficiently large to ensure $\rho < \operatorname{dist}(y_j, \partial\Omega)$. Now since $B_\rho(y_j) \subset B_{\rho+\epsilon}(y)$ for all j large enough, we have that

$$\Theta_u(y_j) \leq \rho^{2-n} \int_{B_{\rho+\epsilon}(y)} |Du|^2$$

for sufficiently large j, so we obtain that $\limsup_{j \to \infty} \Theta_u(y_j) \leq \rho^{2-n} \int_{B_{\rho+\epsilon}(y)} |Du|^2$. By letting $\epsilon \downarrow 0$, we conclude that

$$\limsup_{j \to \infty} \Theta_u(y_j) \leq \rho^{2-n} \int_{B_\rho(y)} |Du|^2 \implies \limsup_{j \to \infty} \Theta_u(y_j) \leq \Theta_u(y)$$

after taking the limit $\rho \downarrow 0$. □

4 Tangent Maps, Monotone Quantities

In order for the density function to be meaningful, we would hope to see invariance upon rescaling u. By rescaling, we mean blowing up the function u around certain points to examine local behavior. This gives rise to the follow rigorous construction: Given $u : \Omega \to \mathbb{R}^p$ and $B_{\rho_0}(y)$ such that $\bar{B}_{\rho_0}(y) \subset \Omega$, and for any $\rho > 0$, consider the scaling function $u_{y,\rho}$ given by

$$u_{y,\rho}(x) = u(y + \rho x)$$

Note that on $B_{\rho_0}(0)$, $u_{y,\rho}$ is well defined. For $\sigma > 0$ and $\rho < \frac{\rho_0}{\sigma}$, after making a change of variables with $\tilde{x} = y + \rho x$ in the energy integral for $u_{y,\rho}$ and noting that $Du_{y,\rho}(x) = \rho D(u(y + \rho x))$, we have the domain changes from $B_\sigma(0) \to B_{\sigma\rho}(y)$, and has a Jacobian factor given by $dx = \rho^{-n} d\tilde{x}$

$$\sigma^{2-n} \int_{B_\sigma(0)} |Du_{y,\rho}(x)|^2 = (\sigma\rho)^{2-n} \int_{B_{\sigma\rho}(y)} |Du(\tilde{x})|^2 \leq \rho_0^{2-n} \int_{B_{\rho_0}(y)} |Du|^2 \tag{3}$$

by the Monotonicity Formula (1.3). Therefore, if $\rho_j \downarrow 0$, then $\limsup_{j\to\infty} \int_{B_\sigma(0)} |Du_{y,\rho_j}|^2 < \infty$ for all $\sigma > 0$, and so by the Compactness Theorem for energy minimizers, there is a subsequence $\rho_{j'}$ such that $u_{y,\rho_{j'}} \to \varphi$ locally in $\mathbb{R}^n$ w.r.t. the $W^{1,2}$-norm.

Definition 6 (Tangent Map)**.** *Any φ obtained this way is called a tangent map of u at y. Moreover, $\varphi : \mathbb{R}^n \to N$ is an energy minimizing map with $\Omega = \mathbb{R}^n$.*

As a consequence:

Theorem 3. *Given an energy minimizer $u \in W^{1,2}(\Omega; N)$, the density function is invariant under rescaling of u, and furthermore, the tangent map of u at any point $y \in \Omega$ is constant along rays.*

Proof. We saw above that if $u_{y,\rho}(x) = u(y + \rho x)$, then given $B_\sigma(0)$, we have that

$$\Theta_{u_{y,\rho}}(0) = \lim_{\sigma\downarrow 0} \sigma^{2-n} \int_{B_\sigma(0)} |Du_{y,\rho}(x)|^2 dx = \lim_{\sigma\downarrow 0} (\sigma\rho)^{2-n} \int_{B_{\sigma\rho}(y)} |Du(\tilde{x})|^2$$

and setting $s = \sigma\rho$, we have

$$\lim_{\sigma\downarrow 0} \int_{B_{\sigma\rho}(y)} |Du(\tilde{x})|^2 d\tilde{x} = \lim_{s\downarrow 0} s^{2-n} \int_{B_s(y)} |Du|^2 = \Theta_u(y)$$

and thus $\Theta_{u_{y,\rho}}(0) = \Theta_u(y)$, so Θ_u is invariant under rescaling of u. Moreover, choosing a sequence $\rho_j \to 0$ such that u_{y,ρ_j} converges to a tangent map φ and taking $j \to \infty$ in the previous equation also yields

$$\sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2 = \Theta_u(y)$$

since $E(u_{y,\rho_j}) \to E(\varphi)$. In particular, $\sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2$ is a constant function of σ and since $\Theta_\varphi(0) = \lim_{\sigma\to 0} \sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2$, we have that

$$\Theta_u(y) = \Theta_\varphi(0) \equiv \sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2$$

Further, by the monotonicity formula,

$$0 = \sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2 - \tau^{2-n} \int_{B_\tau(0)} |D\varphi|^2 = \int_{B_\sigma(0)\setminus B_\tau(0)} R^{2-n} \left| \frac{\partial \varphi}{\partial R} \right|$$

$$\implies \frac{\partial \varphi}{\partial R} = 0$$

and as $\varphi \in W^{1,2}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^p)$, then by integration along rays, we have that

$$\varphi(\lambda x) \equiv \varphi(x) \quad \forall \lambda > 0, x \in \mathbb{R}^n$$

Namely, the tangent map φ is constant along rays. $\square$