Curvature, Stability, and Scalar Rigidity in Minimal Surface Theory

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Abstract

This talk will explore the interplay between energy estimates, curvature bounds, and rigidity results for minimal surfaces. The starting point will be Bernstein's theorem, which provides sufficient condition for a minimal graph of a function in R^3 to be a flat plane. Then, we will examine the relationship between energy bounds and curvature, including a key theorem of Schoen and Choi, which provide critical analytic control over the geometry of minimal surfaces. Building on this foundation, we will discuss Daniel Stern's recent work, which reframes stability in terms of harmonic maps and provides a novel perspective on scalar curvature and minimal surface analysis. Finally, we will examine the famous theorem of Schoen and Yau, which reveals topological constraints on stable minimal surfaces. We will compare two distinct proofs of this theorem: one using the stability inequality and the other relying purely on Stern's harmonic map framework. Together, these results highlight the deep connections between stability, curvature, and the analytic methods that underpin our understanding of minimal surfaces.

1 Notation and Preliminaries

Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, and we define

$$\mathrm{Graph}_u = \{(x, y, u(x, y)) : (x, y) \in \Omega\}$$

and the area functional

$$\mathrm{Area}(\mathrm{Graph}_u) = \int_\Omega |(1, 0, u_x) \times (0, 1, u_y)| = \int_\Omega \sqrt{1 + |\nabla u|^2}$$

and similarly the Dirichlet energy

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2$$

We would first like to define the notion of an area minimizer using the first variation of the area functional. Let η be a function such that $\eta|_{\partial\Omega} = 0$, and let $t \in \mathbb{R}$ be a parameter. We are looking for

$$\frac{d}{dt}\Big|_{t=0}\text{Area}(\text{Graph}_{u+t\eta}) = \frac{d}{dt}\Big|_{t=0}\int_\Omega \sqrt{1+|\nabla u + t\nabla\eta|^2}$$

$$= \int_\Omega \frac{\langle \nabla u, \nabla \eta\rangle}{\sqrt{1+|\nabla u|^2}} = -\int_\Omega \eta \,\text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

and since this must hold for every compactly supported variation η

$$\frac{d}{dt}\Big|_{t=0}\text{Area}(\text{Graph}_{u+t\eta}) \iff \text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

and this is the divergence form of the so-called *minimal surface equation.*

Now the case is different when we are looking at submanifolds. A few preliminaries on the geometry of submanifolds:

Definition 1 (Covariant Derivative and Second Fundamental Form)**.** *If X is a vector field on some submanifold $\Sigma \subset M$, let X^T and X^N denote the tangential and normal components of X, respectively. If ∇ is the covariant derivative on M, then the induced covariant derivative ∇_Σ is given by*

$$\nabla_\Sigma = (\nabla)^T$$

The second fundamental form A is a vector-valued bilinear form on Σ taking $X, Y \in T_x\Sigma$ to

$$A(X,Y) = (\nabla_X Y)^N$$

which essentially is the derivative of the unit normal ν.

From here, we define the **mean curvature vector** H to be trace of the second fundamental form:

$$\vec{H} = \sum_{i=1}^n A(E_i, E_i)$$

where $\{E_i\}$ is an orthonormal basis of $T_p\Sigma$. From this, we see that as $\vec{H} = H \cdot \mathbf{n}$, where $\mathbf{n}$ is the normal vector, we see that $H = \langle \vec{H}, \mathbf{n}\rangle$ is the **mean curvature**. Next, we would like to define the norm of the second fundamental form squared, given by

$$|A|^2 = \sum_{i,j=1}^n |A(E_i, E_j)|^2$$

lastly, we define the divergence over a manifold:

Definition 2 (Divergence over a manifold)**.** *Given a metric g, we have*

$$\text{div}_\Sigma X = \sum_{i=1}^{n-1} g(\nabla_{E_i} X, E_i)$$

where $\{E_i\}$ is an orthonormal basis as before. We also obtain a nice Leibniz rule:

$$\mathrm{div}_\Sigma(fX) = \langle \nabla_\Sigma f, X\rangle + f\,\mathrm{div}_\Sigma(X)$$

we can also define the Laplacian:

$$\Delta_\Sigma f = \mathrm{div}_\Sigma(\nabla_\Sigma f)$$

and f is harmonic if $\Delta_\Sigma f = 0$.

Now we can calculate the first variation formula. I will not show all the explicit details, but we take $F : \Sigma \times (-\epsilon, \epsilon) \to M$ to be a variation of Σ with compact support and fixed boundary so that $F = \mathrm{Id}$ outside a compact set,

$$F(x, 0) = x$$

and for all $x \in \partial\Sigma$,

$$F(x, t) = x$$

we call F_t (partial derivative w.r.t. t) restricted to Σ the variational vector field. The idea is that given local coordinates x_i we define

$$g_{ij} = g(F_{x_i}, F_{x_j})$$
$$\nu(t) = \sqrt{\det(g_{ij}(t))}\sqrt{\det(g^{ij}(0))}$$

and define the volume to be the integral over the volume form

$$\mathrm{Vol}(F(\Sigma, t)) = \int \nu(t)\sqrt{\det(g_{ij}(0))}$$

and after some calculation, we find that the first variation formula yields

$$\frac{d}{dt}|_{t=0}\,\mathrm{Vol}(F(\Sigma, t)) = -\int_\Sigma \langle F_t, \vec{H}\rangle = \int_\Sigma \mathrm{div}_\Sigma F_t$$

and thus Σ is a critical point of the area functional if and only if H vanishes identically.

Definition 3 (Minimal Submanifold)**.** *An immersed submanifold $\Sigma^k \subset M^n$ is said to be minimal is the mean curvature H vanishes identically.*

2 Bernstein's Theorem

This section contains the proof of Bernstein's Theorem by Leon Simon. First, we start with a lemma relating curvature and energy:

Lemma 1. *If $u : \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, then for all nonnegative Lipschitz functions η with supp$(\eta) \subset \Omega \times \mathbb{R}$, we have that*

$$\int_{Graph_u} |A|^2\eta^2 \leq C \int_{Graph_u} |\nabla_{Graph_u}\eta|^2$$

While the proof will not be given here, it involves using the pullback of the Gauss map combined with Cauchy-Schwarz. In short, we are able to bound the total curvature $\int |A|^2$ by the energy of some cutoff function η.

2.1 The Logarithmic Cutoff Trick

Combined with the lemma above, we see that when the graph of u is entire, we can create a sequence of η's converging to 1 with Dirichlet energy converging to 0. Let N be some large fixed integer, and define η on $B_{e^{2N}} \subset \mathbb{R}^2$ as

$$\eta = \begin{cases} 1 & \text{if } r \leq e^N, \\ 2 - \frac{\log(r)}{N} & \text{if } e^N < r \leq e^{2N}, \\ 0 & \text{if } e^{2N} < r. \end{cases}$$

where $r = |x|$. Since $|\nabla \eta| = \frac{1}{Nr}$ is radial, we compute the energy as

$$\int_{\mathbb{R}^2} |\nabla \eta|^2 = 2\pi \int_{e^N}^{e^{2N}} \frac{1}{N^2 r^2} r dr = \frac{2\pi}{N^2} \int_{e^N}^{e^{2N}} \frac{dr}{r}$$
$$= \frac{2\pi}{N^2} [\log(e^{2N}) - \log(e^{2N})] = \frac{2\pi}{N}$$

so by taking a sequence of N's tending toward infinity, we see that our sequence of cutoffs converges to 1 with energy tending toward 0. The same argument works on a manifold Σ with

$$\mathrm{Vol}(B_r) \leq Cr^2$$

by breaking the integral up over annuli with bounded ratio between the inner an outer circles, that is since

$$\sup_{B_{e^\ell} \setminus B_{e^{\ell-1}}} |\nabla \eta|^2 \leq N^{-2} e^{2-2\ell},$$

we get

$$\int |\nabla \eta|^2 \leq \sum_{\ell=N+1}^{2N} \int_{B_{e^\ell} \setminus B_{e^{\ell-1}}} \left[N^{-2} e^{2-2\ell} \, \mathrm{Vol}(B_{e^\ell}) \right]$$
$$\leq Ce^2 N^{-2} \sum_{\ell=N+1}^{2N} \int_{B_{e^\ell} \setminus B_{e^{\ell-1}}} = \frac{Ce^2}{N}.$$

so clearly the argument still holds when $N \to \infty$. What follows is a corollary that will make the proof of Bernstein's theorem one line.

Corollary 1. *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, $\kappa > 1$, and Ω contains a ball of radius κR centered at the origin, then*

$$\int_{B_{\sqrt{\kappa} R} \cap Graph_u} |A|^2 \leq \frac{C}{\log \kappa}.$$

Proof. Let $\Sigma = \mathrm{Graph}_u$, and define η on all of $\mathbb{R}^3$ and restrict it to Σ in the following way: let r be the distance function from the origin in $\mathbb{R}^3$ and define η to be:

$$\eta = \begin{cases} 1 & \text{if } r^2 \leq \kappa R^2, \\ 2 - \frac{2\log(rR^{-1})}{\log \kappa} & \text{if } \kappa R^2 < r^2 \leq \kappa^2 R^2, \\ 0 & \text{if } r^2 > \kappa^2 R^2. \end{cases}$$

and note that $|\nabla_\Sigma r| \leq |\nabla r| = 1$ and thus $|\nabla_\Sigma \eta| \leq \frac{2}{r \log \kappa}$. Apply the lemma from before on curvature bounds, we see that

$$\int_{B_{\sqrt{\kappa}R} \cap \Sigma} |A|^2 \leq \int_\Sigma \eta^2 |A|^2 \leq C \int_\Sigma |\nabla_\Sigma \eta|^2 \leq \frac{4C}{(\log \kappa)^2} \int_{B_{\kappa R} \cap \Sigma} r^{-2}$$

$$\leq \frac{4C}{(\log \kappa)^2} \sum_{\ell = (\log \kappa)/2}^{\log \kappa} \int_{(B_{e^\ell R} \setminus B_{e^{\ell-1} R}) \cap \Sigma} r^{-2}$$

$$\leq \frac{4C}{(\log \kappa)^2} \sum_{\ell = (\log \kappa)/2}^{\log \kappa} 2\pi e^2 \leq \frac{4\pi C e^2}{\log \kappa}.$$

□

Now the famous theorem of Bernstein:

Theorem 1 (Bernstein (1917))**.** *If $u : \mathbb{R}^2 \to \mathbb{R}$ is an entire solution to the minimal surface equaiton, then $u(x, y) = ax + by + c$ for some constants $a, b, c \in \mathbb{R}$.*

Proof. The previous corollary, we have that for all $R > 1$,

$$\int_{B_{\sqrt{R}} \cap \mathrm{Graph}_u} |A|^2 \leq \frac{C}{\log R}.$$

and now we let $R \to \infty$ (u is entire), to obtain that $|A|^2 \equiv 0$, and therefore

$$u_{xx} = u_{xy} = u_{yy} = 0$$

and therefore $u = ax + by + c$ for appropriate constants $a, b, c \in \mathbb{R}$. □

3 Small Energy Curvature Estimates and a Theorem of Heinz

Here we would like to introduce the foremost result of Choi and Schoen relating small energy and small (total) curvature. Similar results in the domain of harmonic maps have been proven by Sacks and Uhlenbeck.

Theorem 2 (Choi-Schoen)**.** *There exist $\epsilon, \rho > 0$ (depending on M) so that if $r_0 < \rho$, $\Sigma^2 \subset M$ is a compact minimal surface with $\partial \Sigma \subset \partial B_{r_0}(x)$, $0 < \delta \leq 1$, and*

$$\int_{B_{r_0} \cap \Sigma} |A|^2 < \delta \epsilon,$$

then for all $0 < \sigma \leq r_0$ and $y \in B_{r_0 - \sigma}(x)$,

$$\sigma^2 |A|^2(y) \leq \delta.$$

Proof. It will suffice to prove this for $M^n = \mathbb{R}^n$, as there are only a few additional modifications for the general case. Define $F = (r_0 - r)^2 |A|^2$ on $B_{r_0} \cap \Sigma$, and notice that $F|_{\partial B_{r_0}} = 0$. Suppose for the sake of a contradiction that $F(x_0) \geq 0$, and define $\sigma > 0$ by

$$\sigma^2 |A|^2(x_0) = \frac{\delta}{4}$$

As $F(x_0) \geq \delta$, we have that

$$2\sigma \leq r_0 - r(x_0)$$

Therefore, by triangle inequality, we see that on $B_\sigma(x_0)$

$$\frac{1}{2} \leq \frac{r_0 - r}{r_0 - r(x_0)} \leq 2$$

Since F achieves its maximum at x_0, we conclude that

$$\begin{aligned}(r_0 - r(x_0))^2 \sup_{B_\sigma(x_0)\cap\Sigma} |A|^2 &\leq 4 \sup_{B_\sigma(x_0)\cap\Sigma} F(x) = 4F(x_0) \\ &= 4(r_0 - r(x_0))^2 |A|^2(x_0). \end{aligned} \tag{2.34}$$

Dividing through by $(r_0 - r(x_0))^2$ and using the definition of σ gives

$$\sup_{B_\sigma(x_0)\cap\Sigma} |A|^2 \leq 4|A|^2(x_0) = \delta\sigma^{-2}. \tag{2.35}$$

After rescaling $B_\sigma(x_0) \cap \Sigma$ to unit size (and still calling it Σ!), we have

$$\sup_{B_1(x_0)\cap\Sigma} |A|^2 \leq 4|A|^2(x_0) = \delta \leq 1. \tag{2.36}$$

By Simons' inequality on $B_1(x_0) \cap \Sigma$,

$$\Delta |A|^2 \geq -2|A|^2. \tag{2.37}$$

The desired contradiction now follows from the mean value inequality. Namely, Corollary 1.16 implies that

$$\frac{\delta}{4} = |A|^2(x_0) \leq \frac{\int_{B_1(x_0)} |A|^2}{\pi} < \frac{e}{\pi}\epsilon = \frac{\delta e}{\pi},$$

which is a contradiction provided that ϵ is chosen sufficiently small.

□

As an immediate consequence we have the curvature estimate of Heinz for minimal graphs.

Theorem 3 (Heinz Curvature Estimate)**.** *Let D_{r_0} be a disk of radius r_0 centered at the origin in $\mathbb{R}^2$, $u : D_{r_0} \to \mathbb{R}$ a solution to the minimal surface equation, and $\Sigma = Graph_u$. If $0 < \sigma \leq r_0$, then we have that*

$$\sigma^2 \sup_{D_{r_0 - \sigma}} |A|^2 \leq C$$

Proof. It suffices to prove the case where $\sigma = r_0$, as the curvature $|A|^2$ will only decrease as we take σ smaller. Using the corollary previously, we have that for any $\kappa > 1$,

$$\int_{B_{r_0/\sqrt{\kappa}} \cap \mathrm{Graph}_u} |A|^2 \leq \frac{C}{\log \kappa}$$

and $\kappa > 1$ can be chosen such that

$$\frac{C}{\log \kappa} < \epsilon$$

where ϵ is given by the theorem of Choi-Schoen, and we're done. □

4 A Modern Rigidity Result of Stern for Scalar Curvature

While we will not have a long discussion about stability of minimal surfaces, we will need the stability inequality to prove a famous theorem of Schoen and Yau.

Definition 4 (The Second Variation of Volume and Stability). *Given the volume functional defined before and a compactly supported variation F, we have that the second variation is given by*

$$\frac{d^2}{dt^2}|_{t=0} \operatorname{Vol}(F(\Sigma, t)) = -\int_\Sigma \langle F_t, LF_t \rangle$$

Where L is the stability operator. For the sake of simplicity, the stability operator simplifies considerably when Σ has a trivial normal bundle, and for a normal vector field $X = \eta N$, L is given by

$$L\eta = \Delta_\Sigma \eta + |A|^2 \eta + \mathrm{Ric}_M(N, N)$$

We say that a surface is stable of its second variation is positive, that is

$$\frac{d^2}{dt^2}|_{t=0} \operatorname{Vol}(F(\Sigma, t)) \geq 0$$

Here, we are most interested in stability and how it relates to minimal surfaces and area minimizers. There are many examples of unstable minimal surfaces, such as the catenoid, but stability tells us that a surface also minimizes area *locally*. Now we state a crucial statement about curvature estimates of stable minimal surfaces.

Theorem 4 (Stability inequality). *Let $\Sigma^{n-1} \subset M^n$ be a stable minimal hypersurface with trivial normal bundle. Then for all $\eta \in Lip(\Sigma) \cap C_c(\Sigma)$, we have*

$$\int_\Sigma (\inf_M \mathrm{Ric}_M + |A|^2) \leq \int_\Sigma |\nabla_\Sigma \eta|^2$$

Proof. Since we assume that Σ is stable, we have automatically that $-\int_\Sigma \langle \eta, L\eta \rangle \geq 0$, and so

$$0 \leq -\int_\Sigma \langle \eta, L\eta \rangle = -\int_\Sigma (\eta \nabla_\Sigma \eta + |A|^2 \eta^2 + \mathrm{Ric}_M(N, N))$$

$$\implies \int_\Sigma (|A|^2\eta^2 + \mathrm{Ric}_M) \le -\int_\Sigma \eta\nabla_\Sigma\eta$$

and now we integrate by parts on the right hand side and use the fact that η is compactly supported to show

$$\int_\Sigma (\inf_M \mathrm{Ric}_M + |A|^2) \le \int_\Sigma |\nabla_\Sigma\eta|^2$$

as desired. □

Now we are ready to prove the following theorem of Schoen and Yau:

Theorem 5 (Schoen-Yau)**.** *There does not exist a Riemannian metric on the 3-torus admitting strictly positive curvature*

Proof. Assume for the sake of contradiction that $\exists g$ on $(\mathbb{T}^3, g)$ such that $\mathrm{Scal}_g(\mathbb{T}^3) > 0$. $\mathbb{T}^3$ has non-trivial two-dimensional homology classes, so choose some $\alpha \in H_2(\mathbb{T}^3;\mathbb{Z})$. It is a fact that $\exists$ an embedded surface $\Sigma \subset \mathbb{T}^3$ of least area in its homology class, so all surfaces representing α, choose the one that is area-minimizing and call it Σ. As Σ is area-minimizing, it is stable, and so by stability inequality, $\forall \eta \in \mathrm{Lip}(\Sigma) \cap C_c(\Sigma)$ and $M := \mathbb{T}^3$, we have

$$\int_\Sigma (\inf_M \mathrm{Ric_M} + |A|^2)\eta^2 \le \int_\Sigma |\nabla_\Sigma\eta|^2$$

Now choose $\eta \equiv 1$ to be such a cutoff function, and observe

$$\int_\Sigma (\inf_M \mathrm{Ric_M} + |A|^2) \le 0$$

and by the Gauss equations, we have the identity

$$\mathrm{Ric}_M(N,N) = \frac{1}{2}\mathrm{Scal}_M - K_\Sigma - \frac{1}{2}|A|^2$$

were K_Σ is Gaussian curvature. Hence,

$$\int_\Sigma \frac{1}{2}\mathrm{Scal}_M - K_\Sigma + \frac{1}{2}|A|^2 \le 0 \implies \frac{1}{2}\int_\Sigma (\mathrm{Scal}_M + |A|^2) \le \int_\Sigma K_\Sigma = 2\pi\chi(\Sigma) = 0$$

However, with $\mathrm{Scal}_{\mathbb{T}^3} > 0$, the left hand side is clearly positive, and this is a contradiction. □

Now, we would look at a recent rigidity result of Daniel Stern that provides a way to prove the same theorem (and in fact does much more) merely using harmonic maps and the Bochner formula. The setting is this: we examine harmonic maps $u : M^3 \to S^1$. It is convenient to define harmonicity in the following sense:

Definition 5 (Harmonicity for circle-valued maps)**.** *Given $u : M^n \to S^1$, u is harmonic in the Hodge sense if the gradient 1-form $\omega_u = u^*(d\theta)$ satsifies*

$$d\omega_u = 0, d^*\omega_u = 0$$

An equivalent definition is that u energizes the Dirichlet energy in its homotopy class $[u] \in [M : S^1]$ (why?).

Now the theorem of Stern:

Theorem 6 (Stern 2019)**.** *Let (M^3, g) be closed and oriented, and $u : M^3 \to S^1$ a (nontrivial) harmonic map. Then*

$$2\pi \int_{\theta\in S^1} \chi(\Sigma_\theta) \geq \frac{1}{2}\int_{\theta\in S^1}\int_{\Sigma_\theta} (|du|^{-2}|Hess(u)|^2 + R_M)$$

where $\Sigma_\theta = u^{-1}\{\theta\}$ are the level sets of u.

Proof. Let $u : M \to S^1 = \mathbb{R}\setminus\mathbb{Z}$ be harmonic, that is, $h := u^*d\theta$ is a harmonic form. The standard Bochner identity for h is then

$$\Delta\frac{1}{2}|h|^2 = |Dh|^2 + \mathrm{Ric}(h, h)$$

and set $\varphi_\delta := (|h|^2 + \delta)^{1/2}$, and for any $\delta > 0$, we can verify

$$\Delta\varphi_\delta = \frac{1}{\varphi_\delta}[\frac{1}{2}\Delta|h|^2 - \frac{|h|^2}{\varphi_\delta^2}|d|h||^2] \geq \frac{1}{\varphi_\delta}[|Dh|^2 - |d|h||^2 + \mathrm{Ric}(h, h)]$$

the regular level sets Σ_ℓ are co-dimension one hypersurfaces in M and so the unit normal vector corresponds to the pullback of u, that is

$$\nu := \frac{h}{|h|}$$

Now by the traced Gauss equation

$$\mathrm{Ric}(\nu, \nu) = \frac{1}{2}(R_M - R_\Sigma + H_\Sigma^2 - |A_\Sigma|^2)$$

where A_Σ is the second fundamental form on Σ. Now from the fact that $\mathrm{Ric}(\nu, \nu) = \frac{1}{|h|^2}\mathrm{Ric}(h, h)$ we can rewrite

$$\mathrm{Ric}(h, h) = |h|^2\,\mathrm{Ric}(\nu, \nu)$$

We have in fact that second fundamental form is given by the normalized Hessian

$$A_\Sigma = (|h|^{-1}Dh)|_\Sigma$$

and thus

$$|h|^2|A_\Sigma|^2 = |Dh|^2 - 2|d|h||^2 + Dh(\nu, \nu)^2$$

and the mean curvature $H_\Sigma := \mathrm{Trace}_\Sigma(A_\Sigma)$ gives

$$|h|H_\Sigma = \mathrm{Trace}_M(Dh) - Dh(\nu, \nu) = -Dh(\nu, \nu)$$

since h is harmonic on M and hence $\mathrm{Trace}_M(Dh) = 0$. Moreover,

$$|h|^2(H_\Sigma^2 - |A_\Sigma|^2) = 2|d|h||^2 - |dh|^2$$

and thus

$$\operatorname{Ric}(h,h) = |h|^2 \operatorname{Ric}(\nu,\nu) = \frac{1}{2}|h|^2(R_M - R_\Sigma) + \frac{1}{2}(2|d|h||^2 - |Dh|^2)$$

Now substituting this is for the Ricci term from before and writing $\operatorname{Hess}(u) = Dh$, we have

$$\Delta\varphi_\delta \geq \frac{1}{2\varphi_\delta}[|\operatorname{Hess}(u)|^2 + |du|^2(R_M - R_\Sigma)]$$

Now, define $\mathcal{C} = \{\text{critical values of u}\}$ and let $\mathcal{R} \subset \operatorname{Reg}(u)$ be the complementary closed set of regular values. From the formula above, we have that

$$\int_{u^{-1}(\mathcal{R})} \frac{1}{2\varphi_\delta}[|\operatorname{Hess}(u)|^2 + |du|^2(R_M - R_\Sigma)] \leq -\int_{u^{-1}(\mathcal{C})} \Delta\varphi_\delta$$

and moreover globally over M that

$$-\int_{u^{-1}(\mathcal{C})} \Delta\varphi_\delta \leq C_M \int_{u^{-1}(\mathcal{C})} |h| = C_M \int_{\mathcal{C}} \operatorname{Area}(\Sigma_\theta)$$

by the coarea formula, and the fact that $|h| > 0$ is bounded away from 0 on $u^{-1}(\mathcal{R})$ and letting $\delta \to 0$ we have that

$$\int_{u^{-1}(\mathcal{R})} \frac{|du|}{2}\left(\frac{|\operatorname{Hess}(u)|^2}{|du|^2} + (R_M - R_\Sigma)\right) = \frac{1}{2}\int_{\mathcal{R}}\int_{\Sigma_\theta}(|du|^{-2}|\operatorname{Hess}(u)|^2 + R_M) - \int_{\mathcal{R}} 2\pi\chi(\Sigma_\theta)$$

So the estimate now becomes

$$\frac{1}{2}\int_{\mathcal{R}}\int_{\Sigma_\theta}(|du|^{-2}|\operatorname{Hess}(u)|^2 + R_M) \leq 2\pi\int_B \chi(\Sigma_\theta) + C\int_A \operatorname{Area}(\Sigma_\theta)$$

By Sard's theorem, we can take $\mu(\mathcal{C})$ to be arbitrarily small, and as $\theta \to \operatorname{Area}(\Sigma_\theta)$ is integrable over S^1, taking $\mu(A) \to 0$ yields the identity □

It is clear now to see how this rigidity result beautifully provides and alternate proof of the previous theorem.

Alternate proof of Theorem 5. Let $M^3 = \mathbb{T}^3$, and suppose there exists a Riemannian metric g such that $\operatorname{Scal}_M > 0$ everywhere. We see that the level sets $\Sigma_\theta = u^{-1}\{\theta\}$ correspond to 2-tori inside $\mathbb{T}^3$, and so $\chi(\Sigma_\theta) = 0$. Hence, by Stern's theorem,

$$0 \geq \frac{1}{2}\int_{\theta\in S^1}\int_{\Sigma_\theta}(|du|^{-2}|\operatorname{Hess}(u)|^2 + R_M) > 0$$

since surely $|du|^{-2}|\operatorname{Hess}(u)|^2 > 0$ and we assumed $R_M > 0$, which is a contradiction. □