

Graded Generalized Algebraic Data Types

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Abstract

Write abstract

1 Introduction

Write the intro.

2 The Fundamental Theory

Suppose \mathcal{C} is a category and $(\mathcal{E}, \otimes, I)$ is a strict monoidal category.

Definition 2.1 (Graded F-Algebra). *For a functor $F : \mathcal{E} \times \mathcal{C} \rightarrow \mathcal{C}$, a graded F-algebra is a pair (A, h) that consists of a functor $A : \mathcal{E} \rightarrow \mathcal{C}$ and a family h of morphisms:*

$$h_{m,n} : F(m, A(n)) \rightarrow A(m \otimes n)$$

*A **homomorphism** between two graded F-algebras (A, h) and (B, h') consists of a morphism*

$$\alpha : (A, h) \rightarrow (B, h')$$

is defined as a natural transformation $\alpha : A_1 \rightarrow A_2$ such that:

$$F(m, \alpha_n); h'_{m,n} = h_{m,n}; \alpha_{m \otimes n}$$

Definition 2.2. *If the category of graded F-algebras has an initial object, then we call this a **graded initial** F-algebra denoted by $(\mu F, \text{in})$. That is, for any other F-algebra (A, h) there must be a unique morphism $\alpha : (\mu F, \text{in}) \rightarrow (A, h)$, but this implies that for any object n , $\alpha_n : \mu F(n) \rightarrow A(n)$ is unique and $\mu F(n)$ is an initial object in \mathcal{C} .*

Lemma 2.3 (From Structures to Homomorphisms). *Given any graded F-algebra (A, h) , the structure map h is also a homomorphism between F-algebras $(F(m, A(-)), F(-, h_{m,-}))$ and $(A(m \otimes -), h_{-, m \otimes -})$.*

Proof. This proof holds trivially by writing out the commutative square for the F-algebra homomorphism. \square

Lemma 2.4 (Graded Lambek's Lemma). *If (A, h) is a graded initial algebra of F, then for any object m , $A(m \otimes -) : \mathcal{E} \rightarrow \mathcal{C}$ is isomorphic to $F(m, A(-)) : \mathcal{E} \rightarrow \mathcal{C}$ via h_m .*

Proof. Suppose $h_{m,n} : F(m, A(n)) \rightarrow A(m \otimes n)$ is an initial algebra structure for any m and n . Now define an algebra structure:

$$F(m', h_{m,n}) : F(m', F(m, A(n))) \rightarrow F(m', A(m \otimes n))$$

Then by initiality there exists an F-algebra homomorphism

$$i_m : A(m \otimes -) \rightarrow F(m, A(-))$$

such that:

$$F(m', i_{m,n}); F(m', h_{m,n}) = h_{m', (m \otimes n)}; i_{m', (m \otimes n)}$$

We also know that $h_m : F(m, A(-)) \rightarrow A(m \otimes -)$ is itself a graded F-algebra homomorphism (Lemma 2.3). Thus, since we know that $A(m \otimes n)$ is an initial object by definition and assumption that A is a graded initial object, and hence, $i_{m,n}; h_{m,n} = \text{id}_{m \otimes n}$.

Next we know that i is a graded F-algebra homomorphism which implies

$$F(m, i_{n,I}); F(m, h_{n,I}) = h_{m,n}; i_{m,n}$$

but again by initiality we know that

$$F(m, i_{n,I}); F(m, h_{n,I}) = \text{id}_{F(m, A(n))}$$

Therefore, i is the inverse of h and we obtain our result. \square

Definition 2.5 (Graded Folds). *Suppose $(\mu F, \text{in})$ is a graded initial F-algebra. Then the unique map between μF and any other graded F-algebra (A, h) is the **fold** for μF and is denoted by*

$$\text{fold}(h) : \mu F \longrightarrow A$$

Furthermore, we know that the following must hold:

$$\text{in}_{m,n}; \text{fold}(h)_{m*n} = F(m, \text{fold}(h)_n); h$$

Definition 2.6 (Graded Forgetful Limits). *There is a forgetful functor from the category of graded F-algebras and the functor category $[\mathcal{E}, \mathcal{C}]$ and their natural transformations. This functor is defined as follows:*

$$\begin{aligned} U_F(A, h) &= A \\ U_F(\alpha) &= \alpha \end{aligned}$$

*Given an object of $[\mathcal{E}, \mathcal{C}]$, say X , then a U_F -**cone** for X comprises, for every graded F-algebra (A, h) , a natural transformation $v_{(A,h)} : X \longrightarrow A$ in $[\mathcal{E}, \mathcal{C}]$ such that, for every graded F-algebra homomorphism $\alpha : A \longrightarrow B$, we have $v_{(B,g)} = \alpha; v_{(A,h)}$. We denote these cones by (X, v) and call X its **vertex** and $v_{(A,h)}$ the **projection** from X to A .*

*A U_F -**cone morphism** $g : (X, v_1) \longrightarrow (Y, v_2)$ is a natural transformation $g : X \longrightarrow Y$ such that for any graded F-algebra (A, h) , we have $g; v_2 = v_1$. A U_F -**limit** is a U_F -cone to which there is a unique U_F -cone morphism, call the **mediating morphism**, from any other U_F -cone.*

Lemma 2.7 (Forgetful Limits are Unique). *If (X, v) is a U_F -limit, then it is unique up to isomorphism.*

Proof. Suppose (Y, v') is another U_F -limit. Thus, there is a unique U_F -cone morphism $i : Y \longrightarrow X$ such that $i; v = v'$. But, there must also be a unique U_F -cone morphism $j : X \longrightarrow Y$ such that $j; v' = v$. But, by substitution $i; j; v' = v'$ and $j; i; v = v$, but these in addition to the assumption that both i and j are unique imply that $i; j = \text{id}_Y$ and $j; i = \text{id}_X$, and thus i and j are inverses of each other. \square

2.1 Interpretation

The Non-graded Case. We begin this section with an overview of the interpretation of non-graded GADTs. Then show how to move to the graded case. The basic form of a GADT is the following:

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data G f h a where
  GCon :: f (G f h) a -> G f h (h a)
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Giving an initial algebra semantics requires that we interpret $G f h$ as the carrier of the initial algebra in the category of f -algebras where the constructor $GCon$ is the structure map. That is, we have the following mappings:

- f maps to a functor $f : [|\mathcal{C}|, \mathcal{C}] \rightarrow [|\mathcal{C}|, \mathcal{C}]$.
- h maps to a functor $h : |\mathcal{C}| \rightarrow |\mathcal{C}|$.
- $G f h$ maps to a functor $G_{f,h} : |\mathcal{C}| \rightarrow \mathcal{C}$.
- $GCon$ maps to a natural transformation:

$$\text{in} : f(G_{f,h}(-)) \rightarrow G_{f,h}(h(-))$$

At this point, we can see a problem, we want $(G_{f,h}, \text{in})$ to be an initial f -algebra, but in has a target that does not fit the proper form, because it is currently $G_{f,h}(h(-))$, and does not match the parameter to f in the source, due to the application of h . Thus, in its current form, in does not match the structure map we need. Rather, we need it to have a target of $G_{f,h}(-)$.

We can overcome this problem using the notion of a left Kan extension.

Definition 2.8 (Left Kan Extension). *The left Kan extension of a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ along a functor $P : \mathcal{C} \rightarrow \mathcal{C}'$ is, if it exists, a functor $\text{Lan}_P F : \mathcal{C}' \rightarrow \mathcal{D}$ equipped with a natural isomorphism:*

$$\text{Hom}_{[\mathcal{C}, \mathcal{D}]}(F, P^*) \cong \text{Hom}_{[\mathcal{C}', \mathcal{D}]}(\text{Lan}_P F, \text{id})$$

where $P^*(H : \mathcal{C}' \rightarrow \mathcal{D}) = P; H$.

If we can define $\text{Lan}_h f(G_{f,h}(-))$ and its associated natural isomorphism then we can simply apply the latter to in to obtain an isomorphic natural transformation that fits the form of the structure map we need. This is possible using the notion of a coend.

Definition 2.9 (Cowedge). *Suppose $F : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{D}$ is a functor. A **cowedge** $e : \mathcal{F} \rightarrow w$ is an object w and a family of maps $e_c : F(c, c) \rightarrow w$ for each c , such that given any other morphism $f : c' \rightarrow c$, the following holds:*

$$F(f, f); e_{c'} = F(\text{id}_{c'}, f); e_c$$

Cowedges are also perserved by composition, that is given a cowedge $e : F \longrightarrow w$ and a map $f : w \longrightarrow v$, then $e;f : v \longrightarrow F$ is a cowedge.

Definition 2.10 (Coend). *Suppose $F : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{D}$ is a functor. A **coend** of F denoted $\exists(c : \mathcal{C}).F(c, c)$ is a universal cowedge $e : F \longrightarrow w$ where every other cowedge $e' : w' \longrightarrow F$ factors through e via a unique map $w' \longrightarrow w$.*

$$\text{Lan}_h f(G_{f,h}(c)) = \exists(b : |C|). |C|(c, h(b)) \times f(G_{f,h}(b))$$

References