Assignment 5

Math Fundamentals for Robotics 16-811, Fall 2018

DUE: Thursday, November 8, 2018

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1. Consider a plane curve $y(x)$ over the interval $[x_0, x_1]$, with specified endpoints $y_0 = y(x_0)$ and $y_1 = y(x_1)$. Assume that $y_0 > 0$ and $y_1 > 0$ and that $y(x) \geq 0$ for $x_0 \leq x \leq x_1$. Now imagine rotating the curve about the x-axis to obtain a surface of revolution. Find the C^2 curve $y(x)$ with specified endpoints that minimizes the surface area of this surface of revolution.
[Hint: This problem explores further some of the limitations of the Calculus of Variations. Depending on the endpoint conditions there may or may not be a C^2 solution. What does the optimal "curve" look like when it is not C^2? Can you say how the endpoint conditions matter? Be aware: There are many subtleties; don't expect to cover all, but explore what you can.]

SOLUTION:

For this problem, we are trying to minimize the surface area of revolution. That is

$$\begin{aligned} A(y) &= \int_{x_0}^{x_1} 2\pi y\sqrt{1+(y')^2}dx \\ &= 2\pi \int_{x_0}^{x_1} y\sqrt{1+(y')^2}dx \end{aligned}$$

So this problem can be described as to solve

$$\min J(y) = \int_{x_0}^{x_1} F(x, y, y')dx$$

with $F(x, y, y') = y\sqrt{1+(y')^2}$, x_0, x_1, $y_0 = y(x_0)$ and $y_1 = y(x_1)$ specified.

According to Euler-Lagrange Equation, we have

$$F_y - \frac{d}{dx}F_{y'} = 0$$

The derivative of F respect to x is:

$$\frac{dF}{dx} = F_x + F_y y' + F_{y'}y''$$

So we have

$$F_y y' = \frac{dF}{dx} - F_x - F_{y'}y'' = y'\frac{d}{dx}F_{y'}$$

$$\rightarrow -F_x + \frac{d}{dx}(F - F_{y'}y') = 0$$

Since $F_x = 0$, so we have:

$$\frac{d}{dx}(F - F_{y'}y') = 0$$
$$\rightarrow F - F_{y'}y' = c$$

where c is a constant. So

$$y\sqrt{1+(y')^2} - \frac{y(y')^2}{\sqrt{1+(y')^2}} = c$$

$$\rightarrow y(1+(y')^2) - y(y')^2 = c\sqrt{1+(y')^2}$$
$$\rightarrow \quad y = c\sqrt{1+(y')^2}$$
$$\rightarrow \quad (y')^2 = \frac{y^2-c^2}{c^2}$$
$$\rightarrow \quad \frac{dx}{dy} = \frac{1}{y'} = \frac{c}{\sqrt{y^2-c^2}}$$

So we get that:

$$x = c\int \frac{1}{\sqrt{y^2-c^2}}dy = c\cosh^{-1}\frac{y}{c} + b$$
$$y = c\cosh\left(\frac{x-b}{c}\right)$$

where b is a constant. Constants c and b are constrained with:

$$y_0 = y(x_0) = c\cosh\left(\frac{x_0-b}{c}\right)$$
$$y_1 = y(x_1) = c\cosh\left(\frac{x_1-b}{c}\right)$$

When the optimal "curve" is not C^2, then it would just becomes:

$$y = \begin{cases} y_0, & x = x_0 \\ 0, & x_0 < x < x_1 \\ y_1, & x = x_1 \end{cases}$$

In this case, the formed surface areas are two ring areas.

The endpoint conditions would matter that whether there is a C^2 solution. When the value of $x_1 - x_0$ becomes larger, it tends to not have a C^2 solution. And when the difference between y_0 and y_1 becomes larger, it also tends to not have a C^2 solution.

2. Using Calculus of Variations, show that the shortest curve between two points on a sphere is an arc of a great circle. [Hints: Use spherical (u, v) coordinates, where $x = R \sin v \cos u$, $y = R \sin v \sin u$, $z = R \cos v$, with R the radius of the sphere. Cast 3D arclength $\sqrt{dx^2 + dy^2 + dz^2}$ into (u, v) space, and parametrize the curve in terms of the coordinate u.
Observe that u does not appear directly in the integrand in the expression for arclength. You may find the following identity useful:

$$\int \frac{a\ dw}{\sqrt{\sin^4 w - a^2 \sin^2 w}} = -\sin^{-1}\left(\frac{\cot w}{\sqrt{\frac{1}{a^2} - 1}}\right) + k$$

where a and k are appropriate constants.]

SOLUTION:

The infinitesimal distance between two points on a sphere can be described as:

$$dl = \sqrt{dx^2 + dy^2 + dz^2}$$

Since

$$x = R \sin v \cos u, \quad y = R \sin v \sin u, \quad z = R \cos v$$

We have:

$$\begin{aligned} dx &= R \cos v \cos u dv - R \sin v \sin u du \\ dy &= R \cos v \sin u dv + R \sin v \cos u du \\ dz &= -R \sin v dv \end{aligned}$$

So,

$$\begin{aligned} dl &= \sqrt{dx^2 + dy^2 + dz^2} \\ &= R\sqrt{dv^2 + \sin^2 v du^2} \end{aligned}$$

So the total distance between two points $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ is:

$$\begin{aligned} L(u) &= \int_{x_0}^{x_1} R\sqrt{dv^2 + \sin^2 v du^2} \\ &= \int_{x_0}^{x_1} R\sqrt{1 + \sin^2 v (u')^2} dv \end{aligned}$$

where $u' = \frac{du}{dv}$
So this problem is to solve:

$$\min \int_{x_0}^{x_1} F(v, u, u') dv$$

where $F(v, u, u') = R\sqrt{1 + \sin^2 v (u')^2}$

According to the Euler-Lagrange Equation, we can get that

$$F_u - \frac{d}{dv}F_{u'} = 0$$

$$\rightarrow \quad \frac{d}{dv}\left(\frac{R\sin^2 vu'}{\sqrt{1+\sin^2 v(u')^2}}\right) = 0$$

$$\rightarrow \quad \frac{\sin^2 vu'}{\sqrt{1+\sin^2 v(u')^2}} = c$$

where c is a constant. Solve for u' we can get

$$\frac{du}{dv} = u' = \frac{c}{\sqrt{\sin^4 v - c^2\sin^2 v}}$$

So,

$$\begin{aligned} u &= \int \frac{c}{\sqrt{\sin^4 v - c^2\sin^2 v}}dv \\ &= -\sin^{-1}\left(\frac{\cot v}{\sqrt{\frac{1}{c^2}-1}}\right) + k \\ &= -\sin^{-1}(b\cot v) + k \end{aligned}$$

where k and $b = \frac{1}{\sqrt{\frac{1}{c^2}-1}}$ are constants. Then we can get

$$b\cot v = \sin(k-u) = \sin k\cos u - \cos k\sin u$$

$$\rightarrow \quad b\cos v = \sin k\sin v\cos u - \cos k\sin v\sin u$$

$$\rightarrow \quad bz = (\sin k)x - (\cos k)y$$

This is a plane that passes through the origin, which forms a great circle by cutting the sphere. So the shortest curve between two points on a sphere is an arc of a great circle.

3. In the brachistochrone problem, suppose the right endpoint is constrained only to touch some curve given implicitly by an equation of the form $g(x, y) = 0$. Show that the optimizing curve $y(x)$ must intersect the iso-contour $g(x, y) = 0$ orthogonally. [Hint: Use an equation from lecture.]

SOLUTION:

$$F_{y'} - \frac{g_y F}{g_x + g_y y'} = 0$$

Since we have

$$F = \sqrt{\frac{1+(y')^2}{y_0 - y}}$$

$$F_{y'} = \frac{y'}{\sqrt{(1+(y')^2)(y_0 - y)}}$$

Plug these two equations into the first equation, we can get that

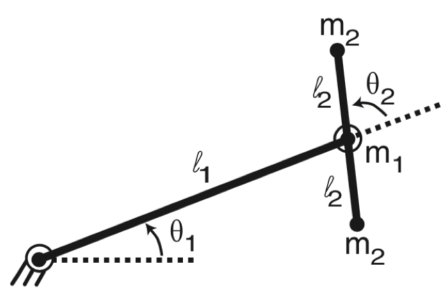
$$y' - \frac{g_y}{g_x + g_y y'}(1+(y')^2) = 0$$

$$\rightarrow \quad (g_x + g_y y')y' = g_y(1+(y')^2)$$

$$\rightarrow \quad y' = \frac{g_y}{g_x}$$

So the optimizing curve $y(x)$ must intersect the iso-contour $g(x,y) = 0$ orthogonally.

4. (a) Using Lagrangian Dynamics, derive the relationship between joint torques and the angular state (angles, velocities, and accelerations) of the following balanced manipulator:



There is no gravity (in practice, gravity is perpendicular to the sheet of the paper). Legend: All of link #1's mass, m_1, is concentrated at distance ℓ_1 from its rotational joint (which is attached to the ground). In turn, link #2 rotates around this distal point, with two masses, m_2, located symmetrically, each at distance ℓ_2, from the joint. In practice, these two masses might constitute one counter-balanced end-effector or two different but equally weighted end-effectors. – This is a variation of a basic Scara-type robot arm, often used in industrial assembly, for instance by SONY.

SOLUTION:

For this problem, the potential energy is zero. The kinetic energy is:

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_{21}^2 + \frac{1}{2}m_2 v_{22}^2$$

where $v_1 = l_1\dot{\theta}_1$.

We will use (θ_1, θ_2) as generalized coordinates, (τ_1, τ_2) as the generalized force. Here we use the writation that c_1 means $\cos\theta_1$, c_2 means $\cos\theta_2$, c_{12} means $\cos(\theta_1+\theta_2)$, s_1 means $\sin\theta_1$, s_2 means $\sin\theta_2$ and s_{12} means $\sin(\theta_1+\theta_2)$

For the end point of link #2 that locates above the joint, we have

$$x_1 = l_1 c_1 + l_2 c_{12}$$

$$y_1 = l_1 s_1 + l_2 s_{12}$$

For the end point of link #2 that locates below the joint, we have

$$x_2 = l_1 c_1 - l_2 c_{12}$$

$$y_2 = l_1 s_1 - l_2 s_{12}$$

Then we can get

$$\begin{aligned} \dot{x_1} &= -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{y_1} &= l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{x_2} &= -l_1 s_1 \dot{\theta}_1 + l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{y_2} &= l_1 c_1 \dot{\theta}_1 - l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \end{aligned}$$

So,

$$\begin{aligned} v_{21}^2 &= \dot{x_1}^2 + \dot{y_1}^2 = l_1^2 \dot{\theta_1}^2 + l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 l_2 c_2 (\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 \\ v_{22}^2 &= \dot{x_1}^2 + \dot{y_1}^2 = l_1^2 \dot{\theta_1}^2 + l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 - 2 l_1 l_2 c_2 (\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 \end{aligned}$$

$$L = T - V = (\frac{1}{2}m_1 + m_2) l_1^2 \dot{\theta_1}^2 + m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$\begin{aligned} &\frac{\partial L}{\partial \theta_1} = 0 \\ &\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + 2m_2) l_1^2 \dot{\theta}_1 + 2 m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2) \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = [(m_1 + 2m_2) l_1^2 + 2 m_2 l_2^2)\ddot{\theta}_1 + 2 m_2 l_2^2 \ddot{\theta}_2 \end{aligned}$$

$$\begin{aligned} &\frac{\partial L}{\partial \theta_2} = 0 \\ &\frac{\partial L}{\partial \dot{\theta}_2} = 2 m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2) \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = 2 m_2 l_2^2 \ddot{\theta}_1 + 2 m_2 l_2^2 \ddot{\theta}_2 \end{aligned}$$

Since

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = \tau_i$$

Then we can get

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} (m_1 + 2m_2)l_1^2 + 2m_2 l_2^2 & 2m_2 l_2^2 \\ 2m_2 l_2^2 & 2m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

(b) When $\ddot{\theta}_2 = 0$, explain the terms relating $\ddot{\theta}_1$ to τ_1.

SOLUTION:

When $\ddot{\theta}_2 = 0$, according to the result obtained in (a), we can get

$$\begin{aligned} \tau_1 &= [(m_1 + 2m_2)l_1^2 + 2m_2 l_2^2]\ddot{\theta}_1 \\ &= m_1 l_1^2 \ddot{\theta}_1 + 2m_2(l_1^2 + l_2^2)\ddot{\theta}_1 \end{aligned}$$

We know that when $\ddot{\theta}_2 = 0$, it means that there is no acceleration on the two masses m_2 respect to the joint of link #2 and link #1. So we can regards the whole system as an entity with force τ_1 on link #1.

Then the angular momentum of mass m_1 is:

$$L_1 = m_1 l_1^2 w = m_1 l_1 \dot{\theta}_1^2$$

The angular momentum of two masses m_2 is:

$$L_2 = m_2 d_1^2 w + m_2 d_2^2 w$$

where d_1 and d_2 are the distance between the masses and the ground joint, and we can calculate that

$$\begin{aligned} d_1^2 &= l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta_2 \\ d_2^2 &= l_1^2 + l_2^2 - 2l_1 l_2 \cos\theta_2 \end{aligned}$$

Plug these into the previous equation, we can get

$$\begin{aligned} L_2 &= 2m_2(l_1^2 + l_2^2)w \\ &= 2m_2(l_1^2 + l_2^2)\dot{\theta}_1 \end{aligned}$$

So,

$$\begin{aligned} \tau_1 &= \frac{d(L_1 + L_2)}{dt} \\ &= \frac{d(m_1 l_1^2 \dot{\theta}_1 + 2m_2(l_1^2 + l_2^2)\dot{\theta}_1)}{dt} \\ &= m_1 l_1^2 \ddot{\theta}_1 + 2m_2(l_1^2 + l_2^2)\ddot{\theta}_1 \end{aligned}$$