

# Primer - Potential outside a Oblate Spheroid

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Notes on a satellite orbiting a non-rotating uniform spheroid.

## Contents

<b>1</b>	<b>Gravitational Potential <math>V</math> in terms of <math>r</math> and <math>\phi</math></b>	<b>3</b>
1.1	Solution for a Spheroid . . . . .	3
1.2	Value of $C_{20}$ constant . . . . .	4
1.3	$V_{20}$ Zonal Term . . . . .	4
<b>2</b>	<b>Inclination Functions <math>F_{lmp}(i)</math></b>	<b>7</b>
2.1	Prerequisites . . . . .	7
2.2	$V_{20}$ Inclination Functions . . . . .	7
2.3	Inclination function for $p = 0$ . $F_{200}$ . . . . .	8
2.4	Inclination function for $p = 1$ . $F_{201}$ . . . . .	8
2.5	General Inclination Functions $F_{lmp}$ . . . . .	9
<b>3</b>	<b>Eccentricity Functions <math>G_{lpq}</math></b>	<b>10</b>
3.1	Prerequisites . . . . .	10
3.2	Eccentricity Functions for $p = 1$ $G_{21q}$ . . . . .	11
3.2.1	Eccentricity Function for $G_{210}$ $p = 1, q = 0$ . . . . .	11
3.2.2	Eccentricity Function for $G_{211}$ $p = 1, q = 1$ . . . . .	12
3.3	Eccentricity Functions for $p \neq 1$ $G_{2pq}$ . . . . .	14
3.3.1	Prerequisites . . . . .	15
3.3.2	Expansion of $\cos((2 - 2p)(\omega + f))$ . . . . .	15
3.3.3	Relate $c_k$ terms to Hansen Coefficients $X_k^{n,m}$ . . . . .	18
3.3.4	Relate Hansen Coefficients $X_k^{n,m}$ to Kaula's eccentricity functions $G_{lpq}$ . . . . .	19
3.4	General $G_{2pq}$ terms . . . . .	20
3.5	Conclusion . . . . .	20
<b>4</b>	<b>Forces due to <math>V_{20}</math></b>	<b>21</b>
4.1	Lagrange Bracket . . . . .	21
4.1.1	Partial Derivative Meaning e.g. $\frac{\partial x}{\partial e}, \frac{\partial y}{\partial e}, \frac{\partial z}{\partial e}$ . . . . .	23
4.1.2	$[a, \omega]$ Lagrange Bracket . . . . .	23
4.1.3	All Lagrange Brackets . . . . .	26
4.2	Rationale for Lagrange Brackets . . . . .	26
4.3	Solutions of $\frac{ds_i}{dt}$ . . . . .	27
4.4	Disturbing Function $R$ . . . . .	28
4.5	Apsidal Precession $\frac{d\omega}{dt}$ using orbital elements . . . . .	28
4.5.1	Inclination $i = 0$ . . . . .	28
4.6	Apsidal Precession using Inertial Frame . . . . .	29
4.6.1	Prerequisites . . . . .	30
4.6.2	Period $T_u$ . . . . .	30
4.7	Apparent Discrepancy in Orbital and Inertial Calculations . . . . .	32
4.8	Mercury Precession due to Sun Oblateness . . . . .	33
4.9	Mercury Precession due to Relativity . . . . .	33
4.10	Newton's Theorem of Revolving Orbits . . . . .	34
4.10.1	Semi-Latus Rectum Change . . . . .	35

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<b>5 Mercury Spin-Orbit Coupling.</b>	<b>36</b>
5.1 Prerequisites . . . . .	37
5.2 Expansion of $\frac{a^3}{r^3} \sin(2\gamma + 3M - 2f)$ . . . . .	37
<b>A Asaph Hall Apsidal Precession derivation</b>	<b>41</b>
<b>B True to Eccentric Anomaly</b>	<b>42</b>
<b>C Program to Calculate Coefficients of cosine term</b>	<b>43</b>
<b>D Radius of a Spheroid (ellipse)</b>	<b>43</b>
<b>E Potential due to a Uniform Spheroid</b>	<b>45</b>
E.1 $J_2$ term . . . . .	46
<b>F Mean Radius</b>	<b>46</b>
<b>G Semi-latus rectum</b>	<b>46</b>
<b>H Notation</b>	<b>47</b>

# 1 Gravitational Potential $V$ in terms of $r$ and $\phi$

Outside the spheroid, the gravitational potential satisfies Laplace's equation  $\nabla^2 V = 0$ . The solution to this is found by separation of variables - looking for a solution, in spherical coordinates, of the form  $V = R(r)\Phi(\phi)\Lambda(\lambda)$ .

Note the use of geodetic notation for coordinates  $[r, \phi, \lambda]$  as opposed to the usual Physics convention of  $[r, \theta, \phi]$ . The solution can be written as below [Kaula]

$$V = \sum_{l=0}^{\infty} \sum_{m=0}^l \frac{1}{r^{l+1}} P_{lm}(\sin \phi) [C_{lm} \cos m\lambda + S_{lm} \sin m\lambda] \quad (1)$$

where  $P_{lm}$  are Legendre associated functions and  $C_{lm}$  and  $S_{lm}$  are constants. Note the  $\sin \phi$  term rather than the usual  $\cos \theta$  we normally see in Physics books when discussing Legendre polynomials in spherical coordinates. This is because  $\phi$  is the latitude rather than the usual co-latitude  $\theta$  we use in spherical coordinates.

$$\sin \phi = \sin \left( \frac{\pi}{2} - \theta \right) = \cos \theta$$

Terms involving  $m = 0$  i.e.  $P_{l0}$  are called zonal terms. Terms involving  $m \geq 1$  are called tesseral terms [GeoPotential Model]. I'm interested in the uniform spheroid case, only considering terms  $l \leq 2$ , where the potential has no  $\lambda$  dependency i.e.  $m = 0$ . For  $m = 0$ , the Legendre associated functions  $P_{lm}(\sin \phi)$  become the Legendre Polynomials  $P_l(\sin \phi)$ . The above reduces to (where Legendre Polynomial  $P_0(\sin \phi) = 1$ )

$$V = C_{00} \frac{1}{r} + C_{20} \frac{1}{r^3} P_2(\sin \phi) \quad (2)$$

Note there is no  $l = 1$  term as expected, i.e.  $P_1$ , due to symmetry of the spheroid around the equator (see Appendix E).

## 1.1 Solution for a Spheroid

[Potential Uniform Spheroid] (also see brief summary in Appendix E) gives the potential for a uniform spheroid as below.  $\epsilon$  - the ellipticity (see Appendix H) - is nothing to do with an orbiting satellite path/eccentricity - it is related to the oblateness/flattening of the spheroid.

$$V(r, \phi) = -\frac{GM}{r} + \frac{2}{5} \epsilon \frac{GM R_\mu^2}{r^3} P_2(\sin \phi) + O(\epsilon^2)$$

( $P_2$  is the Legendre polynomial of order 2 i.e.  $P_2 = \frac{1}{2} [3 \sin^2 \phi - 1]$ ).  $R_\mu$  is the mean radius of the spheroid (see Appendix F)

The derivation in [Potential Uniform Spheroid] uses the typical Physics convention for the potential as (note the -ve sign)

$$V(\mathbf{r}) = -\frac{GM}{r}$$

and the acceleration is (again note the -ve sign)

$$\mathbf{a} = -\nabla V(\mathbf{r})$$

However [Kaula] uses

$$V(\mathbf{r}) = \frac{GM}{r}$$

and

$$\mathbf{a} = \nabla V(\mathbf{r})$$

and states “ $V$  is shown as a positive quantity which is consistent with the sign convention of astronomy and geodesy. In Physics  $V$  is conventionally taken to be negative”

I'll do the same so

$$V(r, \phi) = \frac{GM}{r} - \frac{2}{5} \epsilon \frac{GM R_\mu^2}{r^3} P_2(\sin \phi) + O(\epsilon^2)$$

and the  $V_{20}$  term here is

$$V_{20} = -\frac{2}{5} \epsilon GM R_\mu^2 \frac{1}{r^3} P_2(\sin \phi) \quad (3)$$

## 1.2 Value of $C_{20}$ constant

Comparing 2 with 3 we can see

$$C_{20} = -\frac{2}{5}\epsilon GMR_\mu^2$$

Typically we make  $C_{20}$  nondimensional and introduce a factor  $GMR_\mu^2$  i.e. make

$$C_{20} = -\frac{2}{5}\epsilon$$

and rewrite 3 as

$$V_{20} = C_{20}GMR_\mu^2 \frac{1}{r^3} P_2(\sin \phi) \quad (4)$$

However for below, unless stated otherwise,  $C_{20}$  is the non factored value

## 1.3 $V_{20}$ Zonal Term

$P_{lm}$  can be written as a series expansion [Kaula]. For  $P_{20}$  this expansion gives

$$P_{20}(\sin \phi) = \sum_{t=0}^1 [T_{20t} \sin^{2-2t} \phi] \quad (5)$$

where

$$T_{lmt} = \frac{(-1)^t (2l - 2t)!}{2^l t! (l - t)! (l - m - 2t)!}$$

In our case,  $l = 2$  and  $m = 0$  this reduces to

$$T_{20t} = \frac{(-1)^t (4 - 2t)!}{4t! (2 - t)! (2 - 2t)!}$$

$t$  ranges from 0 to 1 so we have

$$T_{200} = \frac{3}{2} \quad (6)$$

and

$$T_{201} = -\frac{1}{2} \quad (7)$$

so from 5

$$P_{20} = \frac{3}{2} \sin^2 \phi - \frac{1}{2}$$

or (as expected)

$$P_{20} = \frac{1}{2} [3 \sin^2 \phi - 1]$$

The  $l = 2$  term in 2 can now be re-written as

$$V_{20} = \frac{C_{20}}{r^3} \sum_{t=0}^1 T_{20t} \sin^{2-2t} \phi \quad (8)$$

Ultimately I am looking at the potential  $V_{20}$  at the points along an ellipse that a satellite is travelling along. The ellipse might be at an angle to the x-y plane of the spheroid so using  $r$  and  $\phi$  isn't the most convenient. I want to express  $V_{20}$  in terms of orbital elements  $\{a, e, i, M, \omega, \Omega\}$  i.e. semi-major axis of the ellipse  $a$ , eccentricity  $e$ , inclination  $i$ , mean anomaly  $M$ , argument of the perigee  $\omega$  and longitude of node  $\Omega$ . I want to replace  $\phi$  and  $r$  in 8.

Starting with the  $\sin \phi$  term. We can use 3-D trigonometry relationships which states that for a triangle on a sphere (see Figure 1)

$$\frac{\sin(side)}{\sin(angle)} = const$$

gives

$$\frac{\sin(\omega + f)}{\sin\left(\frac{\pi}{2}\right)} = \frac{\sin \phi}{\sin i}$$



Figure 1: Orbit-equator-meridian triangle (from Fig 4 Kaula)



FIGURE 3. Orbital orientation.

Figure 2: Orbital Orientation (from Fig 3 Kaula)

$$\sin \phi = \sin i \sin (\omega + f)$$

where  $f$  is the true anomaly.

If the satellite is moving in an ellipse around a central body, then there will be points where it is closest and furthest away. The perigee is the position in the elliptic path where the satellite is closest to the central body. The apogee is where the satellite is furthest away from the central body.  $\omega$  is the argument of the perigee (see Figure 2). If  $\omega = 0$  then the perigee is at the point where the elliptical path crosses the equatorial plane of the central body.

Eq 8 becomes

$$V_{20} = \frac{C_{20}}{r^3} \sum_{t=0}^1 T_{20t} \sin^{2-2t} i \sin^{2-2t} (\omega + f) \quad (9)$$

## 2 Inclination Functions $F_{lmp}(i)$

We are now going to introduce some of the inclination functions that are seen in [Kaula]. These are functions of  $i$  i.e. the inclination of the satellite ellipse w.r.t. the spheroid x-y (equatorial) plane. (Imagine the satellite is orbiting in an ellipse in the same plane as the spheroid x-y plane and then tilt the satellite ellipse by an angle  $i$ . See Figure 2).

### 2.1 Prerequisites

#### Binomial Expansion (positive $n$ )

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \quad (10)$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Some special cases

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1$$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = 1$$

#### Sine Power Expansion

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

multiply top and bottom by  $i$  and raise to the power of  $n$

$$\begin{aligned} \sin^n \theta &= \left[ \frac{-i}{2} (e^{i\theta} - e^{-i\theta}) \right]^n \\ &= \frac{(-i)^n}{2^n} (e^{i\theta} - e^{-i\theta})^n \end{aligned}$$

using 10

$$\begin{aligned} &= \frac{(-i)^n}{2^n} \sum_{k=0}^n \binom{n}{k} e^{i\theta(n-k)} e^{-i\theta k} (-1)^k \\ &= \frac{(-i)^n}{2^n} \sum_{k=0}^n \binom{n}{k} (-1)^k e^{i\theta(n-2k)} \\ \sin^n \theta &= \frac{(-i)^n}{2^n} \sum_{k=0}^n \binom{n}{k} (-1)^k [\cos(n-2k)\theta + i \sin(n-2k)\theta] \end{aligned}$$

### 2.2 $V_{20}$ Inclination Functions

Lets expand the term  $\sin^{2-2t}(\omega + f)$  in 9

$$\sin^{2-2t}(\omega + f) = \frac{(-i)^{2-2t}}{2^{2-2t}} \sum_{k=0}^{2-2t} \binom{2-2t}{k} (-1)^k [\cos(2-2t-2k)(\omega + f) + i \sin(2-2t-2k)(\omega + f)] \quad (11)$$

From 9,  $t$  can be either 0 or 1 for  $V_{20}$ . For  $t = 0$ ,  $k$  can have the values 0,1,2. For  $t = 1$ ,  $k$  can have the value 0. So writing the permutations gives the table below. (The table also shows a variable  $p$  which will be used later)

$t$	$k$	$2 - 2t - 2k$	$\sin(2 - 2t - 2k)(\omega + f)$	$p = t + k$
0	0	2	$\sin 2(\omega + f)$	0
0	1	0	0	1
0	2	-2	$-\sin 2(\omega + f)$	2
1	0	0	0	1

From the table above, we can see that when  $t = 0$  then the  $k = 0$  and  $k = 2$  cancel out so we can ignore the sin terms. Also

$(-i)^{2-2t}$  for the two cases we have,  $t = 0 \Rightarrow -1$  and  $t = 1 \Rightarrow 1$ , so lets rewrite  $(-i)^{2-2t}$  as  $(-1)^{t+1}$ . 11 becomes

$$\sin^{2-2t}(\omega + f) = \frac{(-1)^{t+1}}{2^{2-2t}} \sum_{k=0}^{2-2t} \binom{2-2t}{k} (-1)^k [\cos(2-2t-2k)(\omega + f)] \quad (12)$$

Let's introduce a variable  $p = t + k$ . 12 becomes

$$\sin^{2-2t}(\omega + f) = \frac{(-1)^{t+1}}{2^{2-2t}} \sum_{k=0}^{2-2t} \binom{2-2t}{k} (-1)^k [\cos(2-2p)(\omega + f)] \quad (13)$$

Equation 9 now becomes

$$V_{20} = \frac{C_{20}}{r^3} \sum_{t=0}^1 T_{20t} \sin^{2-2t} i \frac{(-1)^{t+1}}{2^{2-2t}} \sum_{k=0}^{2-2t} \binom{2-2t}{k} (-1)^k [\cos(2-2p)(\omega + f)] \quad (14)$$

From the table above,  $p$  can have the values 0, 1, 2.

### 2.3 Inclination function for $p = 0$ . $F_{200}$

Just looking at the  $p = 0$  term, there is only one  $t$  and  $k$  that gives  $p = 0$  i.e.  $t = 0$  and  $k = 0$ . So the  $p = 0$  term in 14 becomes

$$\frac{C_{20}}{r^3} T_{200} \sin^2 i \frac{-1}{4} \binom{2}{0} (1) [\cos(2-2p)(\omega + f)]$$

From 6

$$\begin{aligned} & \frac{C_{20}}{r^3} \frac{3}{2} \frac{-1}{4} \sin^2 i [\cos(2-2p)(\omega + f)] \\ & \frac{C_{20}}{r^3} \frac{-3}{8} \sin^2 i [\cos(2-2p)(\omega + f)] \end{aligned}$$

So let  $F_{lmp}$  be the inclination function. In this case  $l = 2$ ,  $m = 0$  and  $p = 0$  so

$$F_{200} = \frac{-3}{8} \sin^2 i$$

and the  $p = 0$  term is

$$\frac{C_{20}}{r^3} F_{200}(i) [\cos(2-2p)(\omega + f)]$$

### 2.4 Inclination function for $p = 1$ . $F_{201}$

Looking at the  $p = 1$  term, there are two  $t$  and  $k$  permutations that give  $p = 1$  i.e.  $t = 0, k = 1$  and  $t = 1, k = 0$ . So the  $p = 1$  term in 14 becomes

$$\begin{aligned} & \frac{C_{20}}{r^3} [\cos(2-2p)(\omega + f)] \left[ T_{200} \sin^2 i \frac{-1}{4} \binom{2}{1} (-1) + T_{201} \right] \\ & \frac{C_{20}}{r^3} [\cos(2-2p)(\omega + f)] \left[ \frac{3}{2} \frac{2}{4} \sin^2 i - \frac{1}{2} \right] \\ & \frac{C_{20}}{r^3} [\cos(2-2p)(\omega + f)] \left[ \frac{3}{4} \sin^2 i - \frac{1}{2} \right] \end{aligned}$$



$$F_{201} = \frac{3}{4} \sin^2 i - \frac{1}{2}$$

and the  $p = 1$  term becomes

$$\frac{C_{20}}{r^3} F_{201} [\cos(2 - 2p)(\omega + f)]$$

## 2.5 General Inclination Functions $F_{lmp}$

Generally, we can write  $V_{20}$  as

$$V_{20} = \frac{C_{20}}{r^3} \sum_{p=0}^2 F_{20p}(i) [\cos(2 - 2p)(\omega + f)] \quad (15)$$

i.e. we are now writing  $V_{20}$  as a sum over  $p$  which contains a function of the inclination  $i$ . So for an elliptic path, this is easier to work with. However the terms with  $f$  and  $r$  are still tricky to work with - so we will look to substitute those. We will attempt to use  $e$  and  $M$  instead i.e. the ellipticity and mean anomaly.

### 3 Eccentricity Functions $G_{lpq}$

We are going to introduce a number of functions of  $e$  called the eccentricity functions i.e.  $G_{lpq}(e)$  where  $l, p$  are as before and  $q$  is a new index. These functions will be different depending on the values of  $l, p, q$  but will still be functions of  $e$  alone. The aim is to produce the same  $G_{lpq}(e)$  as listed in [Kaula]. Initially I'll also set the argument of the perigee  $\omega = 0$ .

When expressing  $V_{20}$  in orbital elements [Kaula] gives the below - where he uses  $a_e$  the mean equatorial radius of the earth, rather than  $R_\mu$ . Note that  $C_{20}$  here is the nondimensional value discussed in 1.2.

$$V_{20} = \frac{C_{20}GMa_e^2}{a^3} \sum_{p=0}^2 F_{20p}(i) \sum_{q=-\infty}^{\infty} G_{2pq}(e) \cos[(2-2p+q)M] \quad (16)$$

Note in [Kaula], and the above,  $q$  ranges from  $-\infty$  not 0.

#### 3.1 Prerequisites

$e$  is the eccentricity (unless otherwise stated  $\exp()$  is used for exponential).

$$e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

$$E - e \sin E = M \quad (17)$$

where  $E$  is the eccentric anomaly,  $M$  is the mean anomaly.

$$r = a(1 - e \cos E) \quad (18)$$

$$\frac{dM}{dE} = 1 - e \cos E$$

From 18

$$\frac{dM}{dE} = \frac{r}{a}$$

or

$$\boxed{dE = \frac{a}{r} dM} \quad (19)$$

$$\frac{r}{a} = 1 - e \cos E$$

$$\frac{r}{a} = 1 - e \left[ \frac{\exp(iE) + \exp(-iE)}{2} \right]$$

Let  $e = \frac{2\beta}{(1+\beta^2)}$

$$\frac{r}{a} = 1 - \frac{2\beta}{(1+\beta^2)} \left[ \frac{\exp(iE) + \exp(-iE)}{2} \right]$$

$$= 1 - \frac{\beta}{(1+\beta^2)} [\exp(iE) + \exp(-iE)]$$

$$\boxed{\frac{r}{a} = \frac{1}{(1+\beta^2)} [(1 - \beta \exp(iE))(1 - \beta \exp(-iE))]} \quad (20)$$

### 3.2 Eccentricity Functions for $p = 1$ $G_{21q}$

We have from 15

$$V_{20} = \frac{C_{20}}{r^3} \sum_{p=0}^2 F_{20p}(i) [\cos(2-2p)(\omega + f)]$$

Let's look at the  $p = 1$  term initially. We'll use the notation  $V_{20p}$  to represent the potential due to the  $p^{th}$  term.

$$V_{20p} = \frac{C_{20}}{r^3} F_{20p}(i) [\cos(2-2p)(\omega + f)]$$

The cos term = 1 for  $p = 1$  so

$$V_{201} = \frac{C_{20}}{r^3} F_{201}(i) \quad (21)$$

We would like to remove the reference to  $r$ . Lets introduce  $a$  the semi-major axis of the ellipse into 21

$$V_{201} = \frac{C_{20}}{a^3} \frac{a^3}{r^3} F_{201}(i)$$

$\frac{a^3}{r^3}$  is a periodic function between 0 and  $2\pi$  assuming that the satellite is moving in a closed ellipse. Let us expand  $\frac{a^3}{r^3}$  in an exponential Fourier series w.r.t. the mean anomaly  $M$ .

$$\frac{a^3}{r^3} = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

the Fourier coefficients  $c_k$  are got by

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \exp(-ikM) dM \quad (22)$$

so

$$V_{201} = \frac{C_{20}}{a^3} F_{201}(i) \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

and the  $k$  term is

$$V_{201k} = \frac{C_{20}}{a^3} F_{201}(i) c_k \exp ikM \quad (23)$$

#### 3.2.1 Eccentricity Function for $G_{210}$ $p = 1, q = 0$

For  $k = 0$ , 22 becomes

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} dM \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2}{r^2} \frac{a}{r} dM \end{aligned}$$

From 19

$$\begin{aligned} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2}{r^2} dE \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^{-2} dE \end{aligned}$$

From 20

$$\begin{aligned} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 + \beta^2)^2}{[(1 - \beta \exp(iE))(1 - \beta \exp(-iE))]^2} dE \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 + \beta^2)^2}{[1 + \beta^2 - \beta(\exp(iE) + \exp(-iE))]^2} dE \end{aligned}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 + \beta^2)^2}{[1 + \beta^2 - 2\beta \cos E]^2} dE$$

dividing by  $(1 + \beta^2)^2$

$$\begin{aligned} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\left[1 - \frac{2\beta}{1+\beta^2} \cos E\right]^2} dE \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{[1 - e \cos E]^2} dE \end{aligned}$$

This isn't a straightforward integral to solve but can look it up (e.g. Wolfram) and the answer is

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \frac{2\pi}{(1 - e^2)^{\frac{3}{2}}} \\ c_0 &= \frac{1}{(1 - e^2)^{\frac{3}{2}}} \end{aligned}$$

Compare 16 with 23 implies

$$G_{210} = (1 - e^2)^{-\frac{3}{2}}$$

This is the same as we see in [Kaula].

### 3.2.2 Eccentricity Function for $G_{211}$ $p = 1$ , $q = 1$

Let us look at  $k = 1$ , so we have

$$\begin{aligned} c_1 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \exp(-iM) dM \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2}{r^2} \exp(-iM) \frac{a}{r} dM \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 + \beta^2)^2}{[(1 - \beta \exp(iE))(1 - \beta \exp(-iE))]^2} \exp(-iM) dE \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 + \beta^2)^2}{[(1 - \beta \exp(iE))(1 - \beta \exp(-iE))]^2} \exp(-i(E - e \sin E)) dE \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 + \beta^2)^2}{[(1 - \beta \exp(iE))(1 - \beta \exp(-iE))]^2} \exp(i(e \sin E - E)) dE \\ &= \frac{(1 + \beta^2)^2}{2\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \exp(i(e \sin E - E)) dE \end{aligned}$$

Binomial expansion works for negative powers as well but, instead of ending up with a polynomial with a finite  $x^n$  term, we end up with an infinite series.

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

The binomial coefficient is given by

$$\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}$$

Special case.

$$\binom{k}{0} = 1$$

The cases we will be interested in are

$$\binom{-2}{1} = \frac{(-2)}{1!} = -2$$

$$\binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$$

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4$$

binomially expand

$$c_1 = \frac{(1 + \beta^2)^2}{2\pi} \int_0^{2\pi} \sum_{b=0}^{\infty} \binom{-2}{b} (-1)^b \beta^b \exp(iEb) \sum_{d=0}^{\infty} \binom{-2}{d} (-1)^d \beta^d \exp(-iEd) \exp(i(e \sin E - E)) dE$$

rearrange

$$c_1 = \frac{(1 + \beta^2)^2}{2\pi} \int_0^{2\pi} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-2}{b} \binom{-2}{d} (-1)^{b+d} \beta^{b+d} \exp(iEb) \exp(-iEd) \exp(i(e \sin E - E)) dE$$

combine exp terms

$$c_1 = \frac{(1 + \beta^2)^2}{2\pi} \int_0^{2\pi} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-2}{b} \binom{-2}{d} (-1)^{b+d} \beta^{b+d} \exp(i(e \sin E - (1 - b + d) E)) dE$$

take summations outside integral

$$c_1 = \frac{(1 + \beta^2)^2}{2\pi} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-2}{b} \binom{-2}{d} (-1)^{b+d} \beta^{b+d} \int_0^{2\pi} \exp(i(e \sin E - (1 - b + d) E)) dE \quad (24)$$

The integral is of the form

$$\frac{1}{2\pi} \int_0^{2\pi} \exp(i(e \sin E - nE)) dE$$

where  $n$  is an integer. The solutions to these integrals have a dependency on  $n$  and  $e$  are called Bessel Functions  $J_n(e)$ .

$$c_1 = (1 + \beta^2)^2 \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-2}{b} \binom{-2}{d} (-1)^{b+d} \beta^{b+d} J_{1-b+d}(e)$$

Note that in the above for  $c_1$ , it shouldn't matter if we interchange  $b$  and  $d$  (we'll check this later). so we could also write

$$c_1 = (1 + \beta^2)^2 \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-2}{b} \binom{-2}{d} (-1)^{b+d} \beta^{b+d} J_{1+b-d}(e)$$

The Bessel function can be written as a series solution

$$J_n(e) = \left(\frac{e}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{e}{2}\right)^{2m}}{m!(n+m)!}$$

The first few Bessel functions are below. The eccentricity functions are power series in  $e$ . I'll only expand as far as  $e^3$

$$J_0(e) = 1 - \frac{e^2}{4} + O(e^4)$$

$$J_1(e) = \frac{e}{2} - \frac{e^3}{16} + O(e^5)$$

$$J_2(e) = \frac{e^2}{8} + O(e^4)$$

$$J_3(e) = \frac{e^3}{48} + O(e^5)$$

In 24  $\beta$  is defined as

$$\beta = \frac{\left(1 - (1 - e^2)^{\frac{1}{2}}\right)}{e}$$

For small  $e$ , binomially expand and keep terms up to  $e^3$  as that is as much as I'm going to calculate.

$$\beta \simeq \frac{1 - \left(1 - \frac{e^2}{2} - \frac{e^4}{8}\right)}{e} = \frac{e}{2} + \frac{e^3}{8}$$

Similarly

$$(1 + \beta^2)^2 \simeq \left(1 + \frac{e^2}{4}\right)^2 \simeq 1 + \frac{e^2}{2}$$

In 24 there is a term  $\beta^{b+d}$  so  $b + d \leq 3$  otherwise we will have  $O(e^4)$  terms. Below are the possible combinations.

Row No	b	d	b+d	$\binom{-2}{b}$	$\binom{-2}{d}$	$(-1)^{b+d}$	$(\beta)^{b+d}$	$1+b-d$	$J_{1+b-d}(e)$	$(\beta)^{b+d} * J_{1+b-d}(e)$ where $O(e^3)$ and less	$\binom{-2}{b} * \binom{-2}{d} * (-1)^{b+d}$	Multiply by $\binom{-2}{b} * \binom{-2}{d} * (-1)^{b+d}$
a	0	0	0	1	1	1	1	1	$\frac{e}{2} - \frac{e^3}{16}$	$\frac{e}{2} - \frac{e^3}{16}$	1	$\frac{e}{2} - \frac{e^3}{16}$
b	0	1	1	1	-2	-1	$\frac{e}{2} + \frac{e^3}{8}$	0	$1 - \frac{e^2}{4}$	$\frac{e}{2} + \frac{e^3}{8} - \frac{e^3}{8} = \frac{e}{2}$	2	$e$
c	0	2	2	1	3	1	$\frac{e^2}{4}$	-1	$-\frac{e}{2} + \frac{e^3}{16}$	$-\frac{e^2}{8}$	3	$-\frac{6e^3}{16}$
d	0	3	3	1	-4	-1	$\frac{e^3}{8}$	-2	$\frac{e^2}{8}$	None		
e	1	0	1	-2	1	-1	$\frac{e}{2} + \frac{e^3}{8}$	2	$\frac{e^2}{8}$	$\frac{e^3}{16}$	2	$\frac{2e^3}{16}$
f	1	1	2	-2	-2	1	$\frac{e^2}{4}$	1	$\frac{e}{2} - \frac{e^3}{16}$	$\frac{e^3}{8}$	4	$\frac{8e^3}{16}$
g	1	2	3	-2	3	-1	$\frac{e^3}{8}$	0	$1 - \frac{e^2}{4}$	$\frac{e^3}{8}$	6	$\frac{12e^3}{16}$
h	2	0	2	3	1	1	$\frac{e^2}{4}$	3	$\frac{e^3}{16}$	None		
i	2	1	3	3	-2	-1	$\frac{e^3}{8}$	2	$\frac{e^2}{8}$	None		
j	3	0	3	-4	1	-1	$\frac{e^3}{8}$	4	$O(e^4)$	None		

From inspection we have a  $\frac{3e}{2}$  term.

For the  $e^3$  term we have

$$\frac{e^3}{16} [-1 - 6 + 2 + 8 + 12] = \frac{15e^3}{16}$$

However, because of the  $\frac{e^2}{2}$  term in  $\left(1 + \frac{e^2}{2}\right)$  there are also  $\frac{e^3}{4}$  and  $\frac{e^3}{2}$  terms for rows a and b respectively when we multiply through by  $\frac{e^2}{2}$  i.e. total of  $\frac{12e^3}{16}$ .

In total  $\frac{27e^3}{16}$ . Therefore

$$c_1 = \frac{3e}{2} + \frac{27e^3}{16} + \dots$$

This is what [Kaula] lists for  $G_{211}$ . Note that in [Kaula] in the case of  $p = 1$  his equation reduces to a sum of cosine terms  $\cos qM$  and you'd expect the Fourier cosine coefficient  $a_k$  to be related to the Fourier complex coefficient  $c_k$  by  $a_k = c_k + c_{-k}$ . Also you'd expect the summation to be over  $[0, \infty]$  not  $[-\infty, \infty]$ . However in [Kaula], the summation of the cosine series is over the range  $[-\infty, \infty]$  so he is using  $c_k$  and  $c_{-k}$ . It'd imply that  $G_{211} = G_{21-1}$ , which is the case.

The coefficients of a Fourier series are related to the coefficients of the exponential expansion as follows

$$a_0 = c_0$$

$$a_k = c_k + c_{-k}$$

$$b_k = i(c_k - c_{-k})$$

### 3.3 Eccentricity Functions for $p <> 1$ $G_{2pq}$

For  $p <> 1$ , we have to deal with the  $\cos(2 - 2p)(\omega + f)$  term as well (which disappeared in the  $p = 1$  case).

### 3.3.1 Prerequisites

see Appendix B

$$\boxed{e^{if} = \frac{e^{iE} (1 - \beta e^{-iE})}{1 - \beta e^{iE}}} \quad (25)$$

### 3.3.2 Expansion of $\cos((2-2p)(\omega + f))$

Let's take the case  $\omega = 0$  to simplify the maths initially.

$$2 \cos((2-2p)f) = \exp(i(2-2p)f) + \exp(-i(2-2p)f)$$

From 25

$$\exp(i(2-2p)f) = \frac{\exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)}}{(1 - \beta \exp(iE))^{(2-2p)}}$$

$$\begin{aligned} 2 \cos((2-2p)f) &= \frac{\exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)}}{(1 - \beta \exp(iE))^{(2-2p)}} + \frac{\exp(-i(2-2p)E) (1 - \beta \exp(-iE))^{-(2-2p)}}{(1 - \beta \exp(iE))^{-(2-2p)}} \\ &= \exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)} (1 - \beta \exp(iE))^{-(2-2p)} \\ &\quad + \exp(-i(2-2p)E) (1 - \beta \exp(-iE))^{-(2-2p)} (1 - \beta \exp(iE))^{(2-2p)} \end{aligned}$$

It follows from the above that the value for  $p = 0$  is the same as  $p = 2$  i.e.  $\cos(i2f) = \cos(-i2f)$  as we'd expect. The first and second terms would just interchange.

The potential term is given by

$$V_{20p} = \frac{C_{20}}{a^3} \frac{a^3}{r^3} F_{20p}(i) [\cos(2-2p)f]$$

Let's expand the  $r, f$  term in a Fourier series - but use the exponential version of the Fourier expansion - note that sum is from  $-\infty$  not 0 i.e.

$$function = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

$$\frac{a^3}{r^3} [\cos(2-2p)f] = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} [\cos(2-2p)f] \exp -ikM dM$$

(Aside - if, instead of expanding  $\frac{a^3}{r^3} [\cos(2-2p)f]$ , we expand  $\frac{a^3}{r^3} [\exp(2-2p)f]$ , we would get the Hansen coefficients. If we let  $m = 2-2p$ , we would get the Hansen coefficients  $X_k^{-3,m}$ .)

$$\begin{aligned} c_k &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \left[ \frac{\exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)} (1 - \beta \exp(iE))^{-(2-2p)}}{2} \right] \\ &\quad + \left[ \frac{\exp(-i(2-2p)E) (1 - \beta \exp(-iE))^{-(2-2p)} (1 - \beta \exp(iE))^{(2-2p)}}{2} \right] \exp -ikM dM \end{aligned}$$

(Note the  $\square$  term is an even function of  $E$  i.e.  $f(E) = f(-E)$ )

This is the integral of a sum, so let's look at the first term first

$$c_{k1} = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \left[ \frac{\exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)} (1 - \beta \exp(iE))^{-(2-2p)}}{2} \right] \exp -ikM dM$$

From previously, we can substitute  $\frac{a^3}{r^3}$  with  $\frac{a^2}{r^2} \frac{a}{r} dM$  or  $\frac{a^2}{r^2} dE$  and take the factor  $\frac{1}{2}$  out

$$\frac{1}{4\pi} \int_0^{2\pi} \frac{a^2}{r^2} \left[ \exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)} (1 - \beta \exp(iE))^{-(2-2p)} \right] \exp(-ikM) dE$$

substitute  $\frac{a^2}{r^2}$

$$= \frac{1}{4\pi} \int_0^{2\pi} \frac{(1 + \beta^2)^2}{[(1 - \beta \exp(iE))(1 - \beta \exp(-iE))]^2} \\ * \left[ \exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)} (1 - \beta \exp(iE))^{-(2-2p)} \right] \exp(-ikM) dE$$

(Note that the  $\frac{a^2}{r^2}$  term is also an even function of  $E$ .)

Rearrange  $\beta$  terms

$$= \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \\ * \left[ \exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)} (1 - \beta \exp(iE))^{-(2-2p)} \right] \exp(-ikM) dE$$

substitute  $M$

$$= \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \\ * \exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)} (1 - \beta \exp(iE))^{-(2-2p)} \exp(-ik(E - e \sin E)) dE$$

take minus sign inside bracket

$$= \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \\ * \exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)} (1 - \beta \exp(iE))^{-(2-2p)} \exp(ik(e \sin E - E)) dE$$

take  $k$  inside bracket

$$= \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \\ * \exp(i(2-2p)E) (1 - \beta \exp(-iE))^{(2-2p)} (1 - \beta \exp(iE))^{-(2-2p)} \exp(i[ke \sin E - kE]) dE$$

rearrange

$$= \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(iE))^{-(2-2p)} \\ * (1 - \beta \exp(-iE))^{-2} (1 - \beta \exp(-iE))^{(2-2p)} \exp(i(2-2p)E) \exp(i[ke \sin E - kE]) dE$$

so the first term is

$$c_{k_1} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-(4-2p)} (1 - \beta \exp(-iE))^{-2p} \exp(i(2-2p)E) \exp(i[ke \sin E - kE]) dE$$

The 2nd term in the integral is

$$c_{k_2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \left[ \frac{\exp(-i(2-2p)E) (1 - \beta \exp(-iE))^{-(2-2p)} (1 - \beta \exp(iE))^{(2-2p)}}{2} \right] \exp(-ikM) dM$$

omitting a few steps for brevity leads to



$$= \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} (1 - \beta \exp(-iE))^{-(2-2p)} (1 - \beta \exp(iE))^{(2-2p)} \\ * \exp(-i(2-2p)E) \exp(i(ke \sin E - kE)) dE$$

$$= \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(iE))^{(2-2p)} (1 - \beta \exp(-iE))^{-2} (1 - \beta \exp(-iE))^{-(2-2p)} \\ * \exp(-i(2-2p)E) \exp(i(ke \sin E - kE)) dE$$

$$c_{k_2} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2p} (1 - \beta \exp(-iE))^{-(4-2p)} \exp(-i(2-2p)E) \exp(i[ke \sin E - kE]) dE$$

Sanity Check 1:

Let  $p = 1$  and look at the  $k = 1$  term

$$c_{1_1} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \exp(i(e \sin E - E)) dE$$

$$c_{1_2} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \exp(i(e \sin E - E)) dE$$

so they are both the same.  $c_1$  below is the same as in the previous section.

$$c_1 = c_{1_1} + c_{1_2} = \frac{(1 + \beta^2)^2}{2\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \exp(i(e \sin E - E)) dE$$

Sanity Check 2: For an exponential Fourier expansion of a real function, the coefficient for  $k = -1$  should be the complex conjugate of  $k = 1$ .

$$c_{-1_1} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \exp(i(-e \sin E + E)) dE$$

$$c_{-1_2} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \exp(i(-e \sin E + E)) dE$$

which gives  $c_1$  equal to the complex conjugate as expected

$$c_{-1} = c_{-1_1} + c_{-1_2} = \frac{(1 + \beta^2)^2}{2\pi} \int_0^{2\pi} (1 - \beta \exp(-iE))^{-2} (1 - \beta \exp(iE))^{-2} \exp(-i(e \sin E - E)) dE$$

So continuing with the first term, binomially expand terms

$$c_{k_1} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-(4-2p)} (1 - \beta \exp(-iE))^{-2p} \\ * \exp(i(2-2p)E) \exp(i(ke \sin E - kE)) dE$$

$$c_{k_1} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} \sum_{b=0}^{\infty} \binom{-(4-2p)}{b} (-1)^b \beta^b \exp(iEb) \sum_{d=0}^{\infty} \binom{-2p}{d} (-1)^d \beta^d \exp(-iEd) \\ * \exp(i(ke \sin E - (k-2+2p)E)) dE$$

$$c_{k_1} = \frac{(1 + \beta^2)^2}{4\pi} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-(4-2p)}{b} \binom{-2p}{d} (-1)^{b+d} \beta^{b+d} \int_0^{2\pi} \exp(i(ke \sin E - (k-2+2p-b+d)E)) dE$$

The 2nd term similarly is

$$c_{k_2} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2p} (1 - \beta \exp(-iE))^{-(4-2p)} \exp(-i(2-2p)E) \exp(i[ke \sin E - kE]) dE$$

$$c_{k_2} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} (1 - \beta \exp(iE))^{-2p} (1 - \beta \exp(-iE))^{-(4-2p)} \exp(i[ke \sin E - (k+2-2p)E]) dE$$

swapping indices  $b$  and  $d$  around so we have the same range

$$c_{k_2} = \frac{(1 + \beta^2)^2}{4\pi} \int_0^{2\pi} \sum_{d=0}^{\infty} \binom{-2p}{d} (-1)^d \beta^d \exp(iEd) \sum_{b=0}^{\infty} \binom{-(4-2p)}{b} (-1)^b \beta^b \exp(-iEb) \\ * \exp(i[ke \sin E - (k+2-2p)E]) dE$$

$$c_{k_2} = \frac{(1 + \beta^2)^2}{4\pi} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-(4-2p)}{b} \binom{-2p}{d} (-1)^{b+d} \beta^{b+d} \int_0^{2\pi} \exp(i[ke \sin E - (k+2-2p+b-d)E]) dE$$

So we have

$$c_{k_1} = \frac{(1 + \beta^2)^2}{2} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-(4-2p)}{b} \binom{-2p}{d} (-1)^{b+d} \beta^{b+d} J_{k-2+2p-b+d}(ke) \\ c_{k_2} = \frac{(1 + \beta^2)^2}{2} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-(4-2p)}{b} \binom{-2p}{d} (-1)^{b+d} \beta^{b+d} J_{k+2-2p+b-d}(ke)$$

### 3.3.3 Relate $c_k$ terms to Hansen Coefficients $X_k^{n,m}$

In the above we expanded as follows

$$\frac{a^3}{r^3} [\cos(2-2p)f] = \sum_{k=-\infty}^{\infty} c_k \exp(ikM)$$


Let us put  $m = 2 - 2p$

$$\frac{a^3}{r^3} [\cos mf] = \sum_{k=-\infty}^{\infty} c_k \exp(ikM)$$

This can also be written as

$$\frac{1}{2} \frac{a^3}{r^3} [\exp(imf) + \exp(-imf)] = \sum_{k=-\infty}^{\infty} c_k \exp(ikM) \quad (26)$$

In the above we calculate  $c_k$  as  $c_{k_1} + c_{k_2}$  corresponding to the  $\exp(imf)$  and  $\exp(-imf)$  term respectively. For  $p = 0$  and  $k = 1$ , we find

 file:///C:/dev/SpheroidCode/spheroid/bin/Debug/spheroid.EXE

**k=1 1st:-1/4e^1 + 1/32e^3, 2nd:1/96e^3**

$$c_1 = c_{1_1} + c_{1_2} = \left(-\frac{1}{4}e + \frac{1}{32}e^3\right) + \left(\frac{1}{96}e^3\right) = -\frac{1}{4}e + \frac{1}{24}e^3$$

To relate this to Hansen coefficients, we use the following definition of the Hansen coefficient [Hansen Coefficients]

$$\left(\frac{r}{a}\right)^n \exp(imf) = \sum_k X_k^{n,m}(e) \exp(ikM) \quad (27)$$

Comparing 26 with 27 for  $p = 0$  and  $k = 1$  we see that  $n = -3$  and

$$X_1^{-3,2} = 2 * c_{1_1} = -\frac{1}{2}e + \frac{1}{16}e^3$$

and

$$X_1^{-3,-2} = 2 * c_{1_2} = \frac{1}{48}e^3$$

### 3.3.4 Relate Hansen Coefficients $X_k^{n,m}$ to Kaula's eccentricity functions $G_{lpq}$

Let us consider the  $p = 0$  and  $k = 1$  term

Using Appendix C we calculate  $c_1$  in the expansion of  $(p = 0)$  up to  $O(e^3)$

$$\frac{a^3}{r^3} [\cos 2f]$$

as

$$c_1 = -\frac{1}{4}e + \frac{1}{24}e^3$$

and

$$c_{-1} = -\frac{1}{4}e + \frac{1}{24}e^3$$

This is as expected for complex Fourier coefficients, because  $\frac{a^3}{r^3} [\cos 2f]$  is real so  $c_k = \bar{c}_{-k}$  i.e. the complex conjugate, and because  $\frac{a^3}{r^3} [\cos 2f]$  is also even,  $c_k$  is real, so  $c_k = c_{-k}$ .

So the cosine coefficient  $a_1 = c_1 + c_{-1}$  is

$$a_1 = -\frac{1}{2}e + \frac{1}{12}e^3$$

This is what we'd expect for the cosine coefficient in the Fourier expansion. Revisiting Kaula's formula 16

$$V_{20} = \frac{C_{20}GMa_e^2}{a^3} \sum_{p=0}^2 F_{20p}(i) \sum_{q=-\infty}^{\infty} G_{2pq}(e) \cos[(2 - 2p + q)M]$$

we can see that  $G_{2pq}$  are cosine coefficients. However his sum is over  $[-\infty, \infty]$  whereas we'd expect  $[0, \infty]$  for a Fourier cosine expansion. The terms that contribute to the  $k = 1$  cosine term are  $q = -1$  and  $q = -3$  i.e.  $G_{20-1}$  and  $G_{20-3}$ . So we must have for the cosine  $k = 1$  term

$$a_1 = G_{20-1} + G_{20-3} = -\frac{1}{2}e + \frac{1}{12}e^3$$

Aside - in Reference [Hansen Coefficients] the following definition is given

$$G_{lpq} = X_{l-2p+q}^{-l-1, l-2p}$$

Which implies

$$G_{20-1} = X_1^{-3,2} = -\frac{1}{2}e + \frac{1}{16}e^3 \text{ which is what we had above and also what Kaula lists in his book}$$

This also implies

$$G_{20-3} = X_{-1}^{-3,2} = \frac{1}{48}e^3 \text{ Kaula doesn't list this term in his book}$$

$$G_{20-1} + G_{20-3} = X_1^{-3,2} + X_{-1}^{-3,2} = -\frac{1}{2}e + \frac{4}{48}e^3 = -\frac{1}{2}e + \frac{1}{12}e^3$$

### 3.4 General $G_{2pq}$ terms

So we have expanded the original in terms of  $c_k \exp(ikM)$  and we can consequently also write this as a cosine expansion  $(c_k + c_{-k}) \cos kM$ . However, as we saw from the above, this doesn't necessarily mean that  $c_k$  is  $G_{lpq}$  and  $c_{-k}$  is  $G_{lp-q}$ . Going back to the Kaula notation and let  $p = 0$

$$V_{20} = \frac{C_{20}}{a^3} \sum_{p=0}^2 F_{20p}(i) \sum_{q=-\infty}^{\infty} G_{2pq}(e) \cos((2 - 2p + q)M)$$

$$V_{200} = \frac{C_{20}}{a^3} F_{200}(i) \sum_{q=-\infty}^{\infty} G_{20q}(e) \cos((2 - 2 * 0 + q)M)$$

$$V_{200} = \frac{C_{20}}{a^3} F_{200}(i) \sum_{q=-\infty}^{\infty} G_{20q}(e) \cos((2 + q)M)$$

If I look at the  $k = 0$  term,  $c_0$ , this would imply  $q = -2$ . So this would imply that  $c_0 = G_{20-2}$ . If we calculate  $c_0$  we get a  $J_n(ke) = J_n(0)$  term because  $k = 0$  and  $n < 0$

$$c_0 = \frac{(1 + \beta^2)^2}{2\pi} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-4}{b} \binom{0}{d} (-1)^{b+d} \beta^{b+d} [J_{-2-b+d}(0) + J_{2+b-d}(0)]$$

For a Bessel function,  $J_n(0) = 0$  so

$$G_{20-2} = c_0 = 0$$

$G_{20-2} = 0$  agrees with [Kaula]

Similarly if we let  $p = 2$

$$V_{202} = \frac{C_{20}}{a^3} F_{202}(i) \sum_{q=-\infty}^{\infty} G_{20q}(e) \cos((2 - 2 * 2 + q)M)$$

$$V_{202} = \frac{C_{20}}{a^3} F_{202}(i) \sum_{q=-\infty}^{\infty} G_{20q}(e) \cos((-2 + q)M)$$

If I look at the  $k = 0$  term,  $c_0$ , this would imply  $q = 2$ . So this would imply that  $c_0 = G_{222}$ . If we calculate  $c_0$  we similarly have a  $J_n(0)$  term so  $c_0 = 0$ .

$G_{222} = 0$  also agrees with [Kaula]

### 3.5 Conclusion

Ultimately, we have written the potential term  $V_{20}$  at points on the ellipse in terms of the orbital elements  $\{a, e, i, M, \omega, \Omega\}$  - albeit simplified because we have omitted  $\omega, \Omega$ . We have also shown the link between Kaula's formula and our derivation. To recap, here is Kaula's formula.

$$V_{20} = \frac{C_{20}}{a^3} \sum_{p=0}^2 F_{20p}(i) \sum_{q=-\infty}^{\infty} G_{2pq}(e) \cos[(2 - 2p + q)M]$$

Using the nondimensional version of  $C_{20}$  we have

$$V_{20} = \frac{GMR_{\mu}^2 C_{20}}{a^3} \sum_{p=0}^2 F_{20p}(i) \sum_{q=-\infty}^{\infty} G_{2pq}(e) \cos[(2 - 2p + q)M]$$

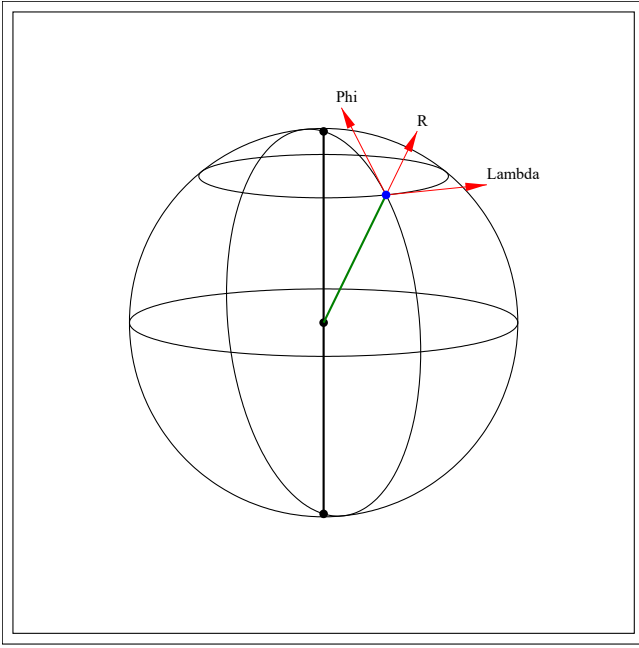


Figure 3: Longitudinal Force Component

## 4 Forces due to $V_{20}$

$$V_{20} = C_{20} \frac{1}{r^3} P_2(\sin \phi)$$

$$V_{20} = C_{20} \frac{1}{2r^3} (3 \sin^2 \phi - 1)$$

$$F_\phi = \frac{1}{r} \frac{\partial V_{20}}{\partial \phi} = C_{20} \frac{1}{r^4} 3 \sin \phi \cos \phi$$

This force is longitudinal. As  $C_{20}$  is negative, it is towards the equator, away from the poles (see Figure 3). The following is from [Orbital Perturbation Analysis].

When describing the motion of a satellite, we can see two frames of reference in Figure 4.

1. **x** coordinates - based on the equatorial plane of the spheroid  
There is an inertial reference frame **a, b, n** with **a** and **b** in the equatorial plane of the spheroid. Additionally let **a** point to the ascending node of the satellite orbit (the blue dot i.e. the point at which the satellite orbit crosses the equatorial plane of the spheroid)

2. **q** coordinates - based on the orbiting satellite Keplerian ellipse.  
We can see that there is a reference frame **r, t, z** with **r** and **t** in the plane of the satellite orbital Keplerian ellipse.

See Figure 5

Note that we need to be a bit more precise when we include the  $V_{20}$  potential. Without the  $V_{20}$  term then we have a purely inverse square force and the path of the satellite is an ellipse that closes in on itself e.g. it doesn't precess. In this case the meaning of the orbital elements like the semi-major axis  $a$  are clear as the ellipse is fixed in space. The satellite follows a path along this ellipse. When we include  $V_{20}$  the meaning of  $a$  is not so clear. Essentially what we have to do is adjust the ellipse shape at each point in time to reflect how the satellite is actually moving in the inertial **x** frame i.e. the satellite is moving on an time dependant ellipse  $\xi_t(a, e, i, \omega, \Omega)$  at time  $t$ .

We can describe the motion of the satellite in the inertial frame by position  $\{x, y, z\}$  and velocity  $\{\dot{x}, \dot{y}, \dot{z}\}$ . Alternatively we could describe the motion of the satellite in terms of what are called the Keplerian orbital elements  $\{a, e, i, M, \omega, \Omega\}$ .

### 4.1 Lagrange Bracket

First we will define and calculate a Lagrange bracket and then show why they are useful.

The rotation matrix that relates **x** and **q** coordinates i.e.  $\mathbf{x} = R\mathbf{q}$  is

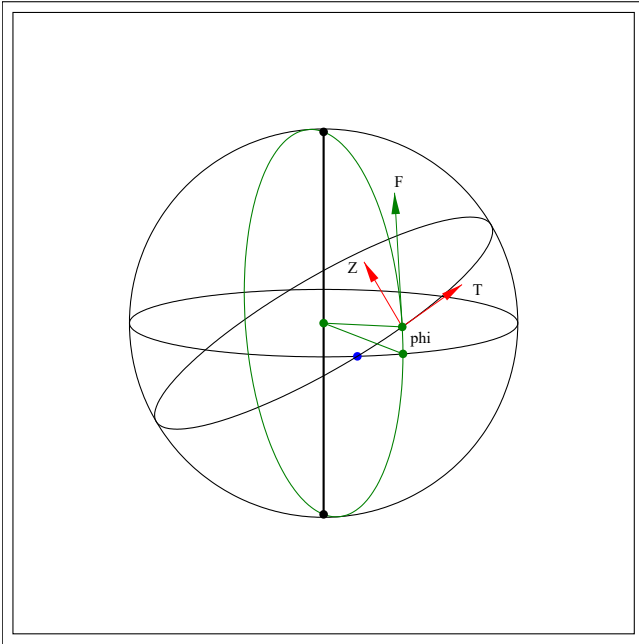


Figure 4: Resolved Longitudinal Force Components along Satellite Ellipse

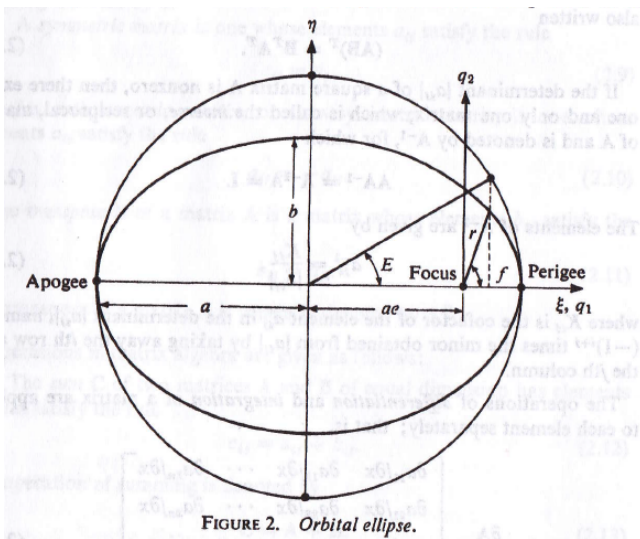


FIGURE 2. *Orbital ellipse.*

Figure 5: Keplerian Orbital Ellipse (from [Kaula])

$$R = \begin{bmatrix} (\cos \Omega \cos \omega - \sin \Omega \cos i \sin \omega) & (-\cos \Omega \sin \omega - \sin \Omega \cos i \cos \omega) & \sin \Omega \sin i \\ (\sin \Omega \cos \omega + \cos \Omega \cos i \sin \omega) & (-\sin \Omega \sin \omega + \cos \Omega \cos i \cos \omega) & -\cos \Omega \sin i \\ \sin i \sin \omega & \sin i \cos \omega & \cos i \end{bmatrix}$$

The  $\mathbf{q}$  coordinates are given by

$$q = \begin{bmatrix} a(\cos E - e) \\ a\sqrt{1-e^2} \sin E \\ 0 \end{bmatrix} = \begin{bmatrix} r \cos f \\ r \sin f \\ 0 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} -\sin E \\ \sqrt{1-e^2} \cos E \\ 0 \end{bmatrix} \frac{na}{1-e \cos E}$$

The Lagrange bracket is defined as below where  $s_l, s_k$  can be any of the Keplerian orbital elements. We sum over the  $i$  terms i.e.  $i$  has values  $[1, 2, 3]$ .

$$[s_l, s_k] = \frac{\partial x_i}{\partial s_l} \frac{\partial \dot{x}_i}{\partial s_k} - \frac{\partial \dot{x}_i}{\partial s_l} \frac{\partial x_i}{\partial s_k}$$

#### 4.1.1 Partial Derivative Meaning e.g. $\frac{\partial x}{\partial e}, \frac{\partial y}{\partial e}, \frac{\partial z}{\partial e}$

The partial derivatives relate changes to coordinates in the inertial frame with changes in the orbital elements e.g.  $\frac{\partial x}{\partial e}$  determines how much the  $x$  coordinate changes if the eccentricity  $e$  changes and we keep all the other orbital elements unchanged.

So, as an example, for the  $x$  coordinate, let  $\Omega = \omega = i = 0$  so we have the elliptic orbit in the  $x, y$  plane of the inertial frame with the perigee lined up in the direction of the  $x$  axis. An increase in the eccentricity  $\partial e$  means that the ellipse becomes flatter ( $a$  is unchanged) and in effect the focus of the ellipse which was at  $ae$  is now at  $a(e + \partial e)$ . From the point of view of the origin (i.e. the focus) the perigee - which was at  $a(1 - e)$  is now at  $a(1 - (e + \partial e))$  so the change in the  $x$  coordinate is  $-a\partial e$ .

$$\frac{\partial x}{\partial e} = \frac{\partial (a(\cos E - e))}{\partial e} = -a$$

$$\partial x = -a\partial e$$

For the  $y$  coordinate, we have

$$\frac{\partial y}{\partial e} = \frac{\partial (a\sqrt{1-e^2} \sin E)}{\partial e} = a \sin E \frac{1}{2} (1-e^2)^{-\frac{1}{2}} (-2e) = -a \sin E \frac{e}{\sqrt{1-e^2}}$$

$$\left. \frac{\partial y}{\partial e} \right|_{E=0} = 0$$

$$\left. \frac{\partial y}{\partial e} \right|_{E=\frac{\pi}{2}} = -a \frac{e}{\sqrt{1-e^2}}$$

so in the second case,  $y$  decreases when  $E = 90^\circ$  because if  $e$  increases then the ellipse becomes flatter.

#### 4.1.2 $[a, \omega]$ Lagrange Bracket

So we can see from  $\mathbf{x} = R\mathbf{q}$  that  $x$  and it's derivatives can be written in terms of the orbital elements  $\{a, e, i, M, \omega, \Omega\}$ .

As an example, we'll calculate the  $[a, \omega]$  Lagrange Bracket.

$$[a, \omega] = \frac{\partial x_i}{\partial a} \frac{\partial \dot{x}_i}{\partial \omega} - \frac{\partial \dot{x}_i}{\partial a} \frac{\partial x_i}{\partial \omega}$$

$$[\omega, a] = \frac{\partial x_i}{\partial \omega} \frac{\partial \dot{x}_i}{\partial a} - \frac{\partial \dot{x}_i}{\partial \omega} \frac{\partial x_i}{\partial a}$$

so we can see that  $[s_l, s_k] = -[s_k, s_l]$ . We can also see that  $[s_k, s_k] = 0$ . In the Lagrange Bracket, we sum over the  $i$  terms i.e.

$$[a, \omega] = \left[ \frac{\partial x}{\partial a} \frac{\partial \dot{x}}{\partial \omega} - \frac{\partial \dot{x}}{\partial a} \frac{\partial x}{\partial \omega} \right] + \left[ \frac{\partial y}{\partial a} \frac{\partial \dot{y}}{\partial \omega} - \frac{\partial \dot{y}}{\partial a} \frac{\partial y}{\partial \omega} \right] + \left[ \frac{\partial z}{\partial a} \frac{\partial \dot{z}}{\partial \omega} - \frac{\partial \dot{z}}{\partial a} \frac{\partial z}{\partial \omega} \right]$$

Looking at the  $x_1$  or  $x$  term, the initial position  $x$  and velocity  $\dot{x}$  is given by the below. However the *change* in position and velocity can be due to *any* of the orbital elements  $\{a, e, i, M, \omega, \Omega\}$ . changing

$$x = (\cos \Omega \cos \omega - \sin \Omega \cos i \sin \omega) a (\cos E - e) - (\cos \Omega \sin \omega + \sin \Omega \cos i \cos \omega) a \sqrt{1 - e^2} \sin E$$

$$\dot{x} = \left[ (\cos \Omega \cos \omega - \sin \Omega \cos i \sin \omega) (-\sin E) - (\cos \Omega \sin \omega + \sin \Omega \cos i \cos \omega) \sqrt{1 - e^2} \cos E \right] \frac{na}{1 - e \cos E}$$

Lagrange brackets are time invariant [Kaula] so we calculate the Lagrange bracket at the perigee where  $E = 0$

$$q = \begin{bmatrix} a(1 - e) \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} 0 \\ \sqrt{1 - e^2} \\ 0 \end{bmatrix} \frac{na}{1 - e} = \frac{na\sqrt{1 - e^2}}{1 - e}$$

so just keeping non-zero terms

$$x = (\cos \Omega \cos \omega - \sin \Omega \cos i \sin \omega) a (1 - e)$$

$$\dot{x} = (-\cos \Omega \sin \omega - \sin \Omega \cos i \cos \omega) \frac{na\sqrt{1 - e^2}}{1 - e}$$

$$\frac{\partial x}{\partial a} = (\cos \Omega \cos \omega - \sin \Omega \cos i \sin \omega) (1 - e)$$

In the next step be careful with the derivative of  $na$  wrt  $a$  because  $n = \sqrt{\frac{\mu}{a^3}}$  so  $na = \left(\frac{\mu}{a}\right)^{\frac{1}{2}}$  and  $\frac{\partial(na)}{\partial a} = \frac{\mu^{\frac{1}{2}} \partial(a)^{-\frac{1}{2}}}{\partial a} = -\frac{1}{2} \mu^{\frac{1}{2}} a^{-\frac{3}{2}} = -\frac{1}{2} n$

$$\frac{\partial \dot{x}}{\partial a} = (-\cos \Omega \sin \omega - \sin \Omega \cos i \cos \omega) \left(-\frac{n}{2}\right) \frac{\sqrt{1 - e^2}}{1 - e}$$

$$\frac{\partial \dot{x}}{\partial \omega} = (-\cos \Omega \sin \omega - \sin \Omega \cos i \cos \omega) a (1 - e)$$

$$\frac{\partial \dot{x}}{\partial \omega} = (-\cos \Omega \cos \omega + \sin \Omega \cos i \sin \omega) \frac{na\sqrt{1 - e^2}}{1 - e}$$

Similarly for  $y$

$$y = (\sin \Omega \cos \omega + \cos \Omega \cos i \sin \omega) a (1 - e)$$

$$\dot{y} = (-\sin \Omega \sin \omega + \cos \Omega \cos i \cos \omega) \frac{na\sqrt{1 - e^2}}{1 - e}$$

$$\frac{\partial y}{\partial a} = (\sin \Omega \cos \omega + \cos \Omega \cos i \sin \omega) (1 - e)$$

$$\frac{\partial \dot{y}}{\partial a} = (-\sin \Omega \sin \omega + \cos \Omega \cos i \cos \omega) \left(-\frac{n}{2}\right) \frac{\sqrt{1 - e^2}}{1 - e}$$

$$\frac{\partial \dot{y}}{\partial \omega} = (-\sin \Omega \sin \omega + \cos \Omega \cos i \cos \omega) a (1 - e)$$



$$\frac{\partial y}{\partial \omega} = (-\sin \Omega \cos \omega - \cos \Omega \cos i \sin \omega) \frac{na\sqrt{1-e^2}}{1-e}$$

and for z

$$z = \sin i \sin \omega a (1-e)$$

$$\dot{z} = \sin i \cos \omega \frac{na\sqrt{1-e^2}}{1-e}$$

$$\frac{\partial z}{\partial a} = \sin i \sin \omega (1-e)$$

$$\frac{\partial \dot{z}}{\partial a} = \sin i \cos \omega \left(-\frac{n}{2}\right) \frac{\sqrt{1-e^2}}{1-e}$$

$$\frac{\partial z}{\partial \omega} = \sin i \cos \omega a (1-e)$$

$$\frac{\partial \dot{z}}{\partial \omega} = -\sin i \sin \omega \frac{na\sqrt{1-e^2}}{1-e}$$

Putting this all together

$$[a, \omega] = \left[ \frac{\partial x}{\partial a} \frac{\partial \dot{x}}{\partial \omega} - \frac{\partial \dot{x}}{\partial a} \frac{\partial x}{\partial \omega} \right] + \left[ \frac{\partial y}{\partial a} \frac{\partial \dot{y}}{\partial \omega} - \frac{\partial \dot{y}}{\partial a} \frac{\partial y}{\partial \omega} \right] + \left[ \frac{\partial z}{\partial a} \frac{\partial \dot{z}}{\partial \omega} - \frac{\partial \dot{z}}{\partial a} \frac{\partial z}{\partial \omega} \right]$$

Look at  $na\sqrt{1-e^2}$  terms first  $\frac{\partial x}{\partial a} \frac{\partial \dot{x}}{\partial \omega} + \frac{\partial y}{\partial a} \frac{\partial \dot{y}}{\partial \omega} + \frac{\partial z}{\partial a} \frac{\partial \dot{z}}{\partial \omega}$  i.e. the terms that don't have  $\frac{\partial \dot{x}_i}{\partial a}$ .

$$\begin{aligned} [a, \omega]_1 &= na\sqrt{1-e^2} \\ &\quad (\cos \Omega \cos \omega - \sin \Omega \cos i \sin \omega) (-\cos \Omega \cos \omega + \sin \Omega \cos i \sin \omega) \\ &\quad + (\sin \Omega \cos \omega + \cos \Omega \cos i \sin \omega) (-\sin \Omega \cos \omega - \cos \Omega \cos i \sin \omega) \\ &\quad - (\sin^2 i \sin^2 \omega) \end{aligned}$$

$$\begin{aligned} &= na\sqrt{1-e^2} \\ &\quad - (\cos \Omega \cos \omega)^2 - (\sin \Omega \cos i \sin \omega)^2 + \cos \Omega \cos \omega \sin \Omega \cos i \sin \omega + \sin \Omega \cos i \sin \omega \cos \Omega \cos \omega \\ &\quad - (\sin \Omega \cos \omega)^2 - (\cos \Omega \cos i \sin \omega)^2 - \sin \Omega \cos \omega \cos \Omega \cos i \sin \omega - \cos \Omega \cos i \sin \omega \sin \Omega \cos \omega \\ &\quad - (\sin^2 i \sin^2 \omega) \end{aligned}$$

$$= na\sqrt{1-e^2} \cdot (-\cos^2 \omega - \cos^2 i \sin^2 \omega - \sin^2 i \sin^2 \omega)$$

$$[a, \omega]_1 = -na\sqrt{1-e^2}$$

Now look at  $\left(-\frac{na}{2}\right)\sqrt{1-e^2}$  terms  $-\frac{\partial \dot{x}}{\partial a} \frac{\partial x}{\partial \omega} - \frac{\partial \dot{y}}{\partial a} \frac{\partial y}{\partial \omega} - \frac{\partial \dot{z}}{\partial a} \frac{\partial z}{\partial \omega}$

$$\begin{aligned} [a, \omega]_2 &= \left(-\frac{na}{2}\right) \sqrt{1-e^2} \\ &\quad - (-\cos \Omega \sin \omega - \sin \Omega \cos i \cos \omega) (-\cos \Omega \sin \omega - \sin \Omega \cos i \cos \omega) \\ &\quad - (-\sin \Omega \sin \omega + \cos \Omega \cos i \cos \omega) (-\sin \Omega \sin \omega + \cos \Omega \cos i \cos \omega) \\ &\quad - (\sin i \cos \omega \sin i \cos \omega) \end{aligned}$$

$$\begin{aligned} &= \left(-\frac{na}{2}\right) \sqrt{1-e^2} \\ &\quad - (\cos \Omega \sin \omega)^2 - (\sin \Omega \cos i \cos \omega)^2 - 2 \sin \Omega \cos i \cos \omega \cos \Omega \sin \omega \\ &\quad - (\sin \Omega \sin \omega)^2 - (\cos \Omega \cos i \cos \omega)^2 + 2 \sin \Omega \sin \omega \cos \Omega \cos i \cos \omega \\ &\quad - (\sin^2 i \cos^2 \omega) \end{aligned}$$

$$\begin{aligned}
&= \left(-\frac{na}{2}\right) \sqrt{1-e^2} \\
&\quad - \sin^2 \omega - \cos^2 i \cos^2 \omega \\
&\quad - \sin^2 i \cos^2 \omega
\end{aligned}$$

$$[a, \omega]_2 = \frac{na}{2} \sqrt{1-e^2}$$

So putting it together

$$[a, \omega] = [a, \omega]_1 + [a, \omega]_2 = -\frac{na}{2} \sqrt{1-e^2}$$

#### 4.1.3 All Lagrange Brackets

The full list of non zero Lagrange Brackets are

$$\begin{aligned}
[\Omega, i] &= -[i, \Omega] = -na^2 (1-e^2)^{\frac{1}{2}} \sin i \\
[\Omega, a] &= -[a, \Omega] = (1-e^2)^{\frac{1}{2}} \frac{na}{2} \cos i \\
[\Omega, e] &= -[e, \Omega] = -\frac{na^2 e \cos i}{(1-e^2)^{\frac{1}{2}}} \\
[\omega, a] &= -[a, \omega] = \frac{na}{2} (1-e^2)^{\frac{1}{2}} \\
[\omega, e] &= -[e, \omega] = -\frac{na^2 e}{(1-e^2)^{\frac{1}{2}}} \\
[a, M] &= -[M, a] = -\frac{na}{2}
\end{aligned}$$

## 4.2 Rationale for Lagrange Brackets

In the inertial frame we have

$$\frac{dx_i}{dt} = \dot{x}_i$$

$$\frac{d\dot{x}_i}{dt} = \frac{\partial V}{\partial x_i}$$

We want to express this in terms of Keplerian orbital elements so we can write

$$\sum_{k=1}^6 \frac{\partial x_i}{\partial s_k} \frac{ds_k}{dt} = \dot{x}_i$$

and

$$\sum_{k=1}^6 \frac{\partial \dot{x}_i}{\partial s_k} \frac{ds_k}{dt} = \frac{\partial V}{\partial x_i}$$

On the RHS we still have inertial parameters i.e.  $\dot{x}_i$  and  $\frac{\partial V}{\partial x_i}$ . The trick is to multiply the first equation by  $-\frac{\partial \dot{x}_i}{\partial s_l}$  and the second equation by  $\frac{\partial x_i}{\partial s_l}$  giving the below

$$\sum_{k=1}^6 -\frac{\partial \dot{x}_i}{\partial s_l} \frac{\partial x_i}{\partial s_k} \frac{ds_k}{dt} = -\frac{\partial \dot{x}_i}{\partial s_l} \dot{x}_i = -\frac{1}{2} \frac{\partial (\dot{x}_i \dot{x}_i)}{\partial s_l}$$

$$\sum_{k=1}^6 \frac{\partial x_i}{\partial s_l} \frac{\partial \dot{x}_i}{\partial s_k} \frac{ds_k}{dt} = \frac{\partial x_i}{\partial s_l} \frac{\partial V}{\partial x_i}$$

In the above, let  $i$  range over range  $[1, 2, 3]$  i.e.  $x, y, z$  and add the two equations together

$$\begin{aligned} \sum_{k=1}^6 \left[ \left[ \frac{\partial x}{\partial s_l} \frac{\partial \dot{x}}{\partial s_k} - \frac{\partial \dot{x}}{\partial s_l} \frac{\partial x}{\partial s_k} \right] + \left[ \frac{\partial y}{\partial s_l} \frac{\partial \dot{y}}{\partial s_k} - \frac{\partial \dot{y}}{\partial s_l} \frac{\partial y}{\partial s_k} \right] + \left[ \frac{\partial z}{\partial s_l} \frac{\partial \dot{z}}{\partial s_k} - \frac{\partial \dot{z}}{\partial s_l} \frac{\partial z}{\partial s_k} \right] \right] \frac{ds_k}{dt} \\ = -\frac{1}{2} \frac{\partial (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\partial s_l} + \frac{\partial x}{\partial s_l} \frac{\partial V}{\partial x} + \frac{\partial y}{\partial s_l} \frac{\partial V}{\partial y} + \frac{\partial z}{\partial s_l} \frac{\partial V}{\partial z} \end{aligned}$$

Let  $T = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\sum_{k=1}^6 \left[ \left[ \frac{\partial x}{\partial s_l} \frac{\partial \dot{x}}{\partial s_k} - \frac{\partial \dot{x}}{\partial s_l} \frac{\partial x}{\partial s_k} \right] + \left[ \frac{\partial y}{\partial s_l} \frac{\partial \dot{y}}{\partial s_k} - \frac{\partial \dot{y}}{\partial s_l} \frac{\partial y}{\partial s_k} \right] + \left[ \frac{\partial z}{\partial s_l} \frac{\partial \dot{z}}{\partial s_k} - \frac{\partial \dot{z}}{\partial s_l} \frac{\partial z}{\partial s_k} \right] \right] \frac{ds_k}{dt} = \frac{\partial [V - T]}{\partial s_l}$$

Use convention that summation happens over the repeated subscript  $i$ .

$$\sum_{k=1}^6 \left[ \frac{\partial x_i}{\partial s_l} \frac{\partial \dot{x}_i}{\partial s_k} - \frac{\partial \dot{x}_i}{\partial s_l} \frac{\partial x_i}{\partial s_k} \right] \frac{ds_k}{dt} = \frac{\partial [V - T]}{\partial s_l}$$

and introducing  $[s_l, s_k]$  for the Lagrange Bracket (using the convention the we sum over the repeated  $k$  subscript), we finally have

$$[s_l, s_k] \frac{ds_k}{dt} = \frac{\partial [V - T]}{\partial s_l}$$

If we let  $H = V - T$

$$[s_l, s_k] \frac{ds_k}{dt} = \frac{\partial H}{\partial s_l}$$

The key thing in this equation is  $\frac{ds_k}{dt}$ . We have the possibility now to calculate the change in orbital elements i.e  $\frac{d\omega}{dt}$ ,  $\frac{dM}{dt}$ ,  $\frac{d\Omega}{dt}$ , etc if we know (1) the values of the Lagrange Brackets and (2) how  $(V - T)$  depends on the orbital elements. We have 6 equations and we can solve for  $\frac{ds_l}{dt}$  e.g.

$$\begin{aligned} [a, a] \frac{da}{dt} + [a, e] \frac{de}{dt} + [a, i] \frac{di}{dt} + [a, \omega] \frac{d\omega}{dt} + [a, \Omega] \frac{d\Omega}{dt} + [a, M] \frac{dM}{dt} &= \frac{\partial H}{\partial a} \\ [i, a] \frac{da}{dt} + [i, e] \frac{de}{dt} + [i, i] \frac{di}{dt} + [i, \omega] \frac{d\omega}{dt} + [i, \Omega] \frac{d\Omega}{dt} + [i, M] \frac{dM}{dt} &= \frac{\partial H}{\partial i} \end{aligned}$$

etc

### 4.3 Solutions of $\frac{ds_l}{dt}$

I shall just quote them here

$$\begin{aligned} \frac{da}{dt} &= \frac{2}{na} \frac{\partial H}{\partial M} \\ \frac{de}{dt} &= \frac{1 - e^2}{na^2 e} \frac{\partial H}{\partial M} - \frac{(1 - e^2)^{\frac{1}{2}}}{na^2 e} \frac{\partial H}{\partial \omega} \\ \frac{d\omega}{dt} &= -\frac{\cos i}{na^2 (1 - e^2)^{\frac{1}{2}} \sin i} \frac{\partial H}{\partial i} + \frac{(1 - e^2)^{\frac{1}{2}}}{na^2 e} \frac{\partial H}{\partial e} \\ \frac{di}{dt} &= \frac{\cos i}{na^2 (1 - e^2)^{\frac{1}{2}} \sin i} \frac{\partial H}{\partial \omega} - \frac{1}{na^2 (1 - e^2)^{\frac{1}{2}} \sin i} \frac{\partial H}{\partial \Omega} \\ \frac{d\Omega}{dt} &= \frac{1}{na^2 (1 - e^2)^{\frac{1}{2}} \sin i} \frac{\partial H}{\partial i} \\ \frac{dM}{dt} &= -\frac{1 - e^2}{na^2 e} \frac{\partial H}{\partial e} - \frac{2}{na} \frac{\partial H}{\partial a} \end{aligned}$$

## 4.4 Disturbing Function $R$

Let us write the potential  $V$  as

$$V = \frac{GM}{r} + R$$

i.e. the central term  $\frac{GM}{r}$  plus  $R$  (the disturbing function) which is the other terms, including  $V_{20}$  - which is the only one we are considering, then

$$H = \frac{GM}{r} + R - T$$

. However  $-\frac{GM}{r} + T$  is the total energy of a unit mass orbiting in an ellipse. This is equal to  $-\frac{GM}{2a}$  so

$$H = \frac{GM}{2a} + R$$

so in the solutions above to  $\frac{ds_l}{dt}$  we can replace  $\frac{\partial H}{\partial s_l}$  by  $\frac{\partial R}{\partial s_l}$  except for  $\frac{\partial H}{\partial a}$  where

$$\frac{2}{na} \frac{\partial H}{\partial a} = \frac{2}{na} \frac{GM}{2} \frac{\partial}{\partial a} \left( \frac{1}{a} \right) + \frac{2}{na} \frac{\partial R}{\partial a} = -\frac{2}{na} \frac{GM}{2a^2} + \frac{2}{na} \frac{\partial R}{\partial a} = -\frac{1}{n} \frac{GM}{a^3} + \frac{2}{na} \frac{\partial R}{\partial a}$$

However using Keplers law  $n^2 a^3 = GM$  or  $n^2 = \frac{GM}{a^3}$  gives

$$\frac{2}{na} \frac{\partial H}{\partial a} = -n + \frac{2}{na} \frac{\partial R}{\partial a}$$

so

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial e} - \frac{2}{na} \frac{\partial R}{\partial a}$$

## 4.5 Apisidal Precession $\frac{d\omega}{dt}$ using orbital elements

Intuitively, if  $i = 0$  then the satellite orbits around the equator of the spheroid and due to symmetry we might expect no change in the orientation of the ellipse. However lets do the calculation

### 4.5.1 Inclination $i = 0$

The full  $V_{20}$  term is

$$V_{20} = \frac{GMR_\mu^2 C_{20}}{a^3} \sum_{p=0}^2 F_{20p}(i) \sum_{q=-\infty}^{\infty} G_{2pq}(e) \cos[(2-2p+q)M]$$

The only term in the above that doesn't have a  $\cos M$  dependency is  $G_{210}$ . Terms  $G_{20-2} = G_{222} = 0$ . The remaining terms have a cosine dependency so over a full orbit will average to zero. Lets consider only the  $p = 1, q = 0$  term.

$$R = \frac{GMR_\mu^2 C_{20}}{a^3} F_{201}(i) G_{210}(e)$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2 (1-e^2)^{\frac{1}{2}} \sin i} \frac{\partial R}{\partial i} + \frac{(1-e^2)^{\frac{1}{2}}}{na^2 e} \frac{\partial R}{\partial e}$$

Calculating  $\frac{\partial F_{201}}{\partial i}$  term when  $i = 0$

$$\begin{aligned} \frac{\partial F_{201}}{\partial i} &= \frac{\partial \left( \frac{3}{4} \sin^2 i - \frac{1}{2} \right)}{\partial i} = \frac{3}{4} 2 \sin i \cos i \\ -\frac{\cos i}{na^2 (1-e^2)^{\frac{1}{2}} \sin i} G_{210} \frac{\partial F_{201}}{\partial i} &= -\frac{\cos i}{na^2 (1-e^2)^{\frac{1}{2}} \sin i} (1-e^2)^{-\frac{3}{2}} \frac{3}{2} \sin i \cos i \\ &= -\frac{\cos^2 i}{na^2 (1-e^2)^2} \frac{3}{2} \\ \frac{GMR_\mu^2 C_{20}}{a^3} \left( -\frac{\cos i}{na^2 (1-e^2)^{\frac{1}{2}} \sin i} \right) G_{210} \frac{\partial F_{201}}{\partial i} &= \frac{GMR_\mu^2 C_{20}}{a^3} \left( -\frac{\cos^2 i}{na^2 (1-e^2)^2} \frac{3}{2} \right) \end{aligned}$$

$$= n^2 R_\mu^2 C_{20} \left( -\frac{\cos^2 i}{na^2 (1-e^2)^2} \frac{3}{2} \right)$$

evaluating at  $i = 0$

$$= -\frac{3nR_\mu^2 C_{20}}{2a^2 (1-e^2)^2}$$

Calculating  $\frac{\partial G_{210}}{\partial e}$  term when  $i = 0$

$$\begin{aligned} \frac{\partial G_{210}}{\partial e} &= \frac{\partial (1-e^2)^{-\frac{3}{2}}}{\partial e} = -\frac{3}{2} (1-e^2)^{-\frac{5}{2}} (-2e) \\ F_{201} \frac{\partial G_{210}}{\partial e} &= \left( \frac{3}{4} \sin^2 i - \frac{1}{2} \right) 3 (1-e^2)^{-\frac{5}{2}} e \\ F_{201} \frac{\partial G_{210}}{\partial e} &= \left( -\frac{1}{2} \right) 3 (1-e^2)^{-\frac{5}{2}} e \\ \frac{(1-e^2)^{\frac{1}{2}}}{na^2 e} F_{201} \frac{\partial G_{210}}{\partial e} &= \frac{(1-e^2)^{\frac{1}{2}}}{na^2 e} \left( -\frac{1}{2} \right) 3 (1-e^2)^{-\frac{5}{2}} e \\ &= -\frac{3}{2 (1-e^2)^2 na^2} \\ \frac{GMR_\mu^2 C_{20}}{a^3} \frac{(1-e^2)^{\frac{1}{2}}}{na^2 e} F_{201} \frac{\partial G_{210}}{\partial e} &= -\frac{GMR_\mu^2 C_{20}}{a^3} \frac{3}{2 (1-e^2)^2 na^2} \end{aligned}$$

Using  $n^2 = \frac{GM}{a^3}$

$$= -\frac{3nR_\mu^2 C_{20}}{2a^2 (1-e^2)^2}$$

Summing both terms together

$$\frac{d\omega}{dt} = -\frac{3nR_\mu^2 C_{20}}{(1-e^2)^2 a^2}$$

So we see there is apsidal precession around a spheroid object (but not around a spherical object because  $C_{20} = 0$ ).

The precession over one period is

$$\begin{aligned} \frac{d\omega}{dt} T_0 &= \frac{d\omega}{dt} \cdot \frac{2\pi}{n} = -\frac{3nR_\mu^2 C_{20}}{(1-e^2)^2 a^2} \cdot \frac{2\pi}{n} = -\frac{6\pi R_\mu^2 C_{20}}{(1-e^2)^2 a^2} \\ \Delta\omega_p &= -\frac{6\pi R_\mu^2 C_{20}}{(1-e^2)^2 a^2} \end{aligned} \tag{28}$$

## 4.6 Apsidal Precession using Inertial Frame

Because we are looking at  $i = 0$  lets see what this looks like in the inertial frame. To recap, we are putting  $\phi = 0$  and  $C_{20}$  is negative. We'll use the dimensionless  $C_{20}$  in what follows. From 4

$$\begin{aligned} V_{20} &= GMR_\mu^2 C_{20} \frac{1}{r^3} P_2(\sin \phi) \\ V_{20} &= GMR_\mu^2 C_{20} \frac{1}{r^3} \frac{1}{2} (3 \sin^2 \phi - 1) \\ F_r &= \frac{\partial V_{20}}{\partial r} = GMR_\mu^2 C_{20} \frac{-3}{r^4} \frac{1}{2} (3 \sin^2 \phi - 1) = \frac{3}{2} GMR_\mu^2 C_{20} \frac{1}{r^4} \\ F_\phi &= \frac{1}{r} \frac{\partial V_{20}}{\partial \phi} = GMR_\mu^2 C_{20} \frac{1}{r^4} 3 \sin \phi \cos \phi = 0 \end{aligned}$$

So we have an extra inward radial acceleration at the equator due to the negative  $F_r$  that we wouldn't have had if the satellite were orbiting a spherical object.

#### 4.6.1 Prerequisites

From  $\mathbf{r} = r\mathbf{e}_r$  in polar coordinates and differentiating (see [Newtonian Mechanics]), we have

$$a_r = \frac{d^2 r}{dt^2} - r \left( \frac{d\lambda}{dt} \right)^2$$

$$a_\lambda = r \frac{d^2 \lambda}{dt^2} + 2 \frac{dr}{dt} \frac{d\lambda}{dt}$$

For a central force  $a_\lambda = 0$

$$r \frac{d^2 \lambda}{dt^2} + 2 \frac{dr}{dt} \frac{d\lambda}{dt} = 0$$

Multiply through by  $r$

$$r^2 \frac{d^2 \lambda}{dt^2} + 2r \frac{dr}{dt} \frac{d\lambda}{dt} = \frac{d}{dt} \left( r^2 \frac{d\lambda}{dt} \right) = 0$$

$$r^2 \frac{d\lambda}{dt} = \text{const} = C = \frac{L}{m} = 2 \frac{dA}{dt}$$

where  $L$  is the angular momentum of the mass  $m$

$$\frac{d\lambda}{dt} = \frac{C}{r^2}$$

Let  $r = \frac{1}{u}$

$$\frac{d\lambda}{dt} = Cu^2$$

$$\frac{dr}{dt} = -\frac{1}{u^2} \frac{du}{dt} = -\frac{1}{u^2} \frac{du}{d\lambda} \frac{d\lambda}{dt} = -\frac{1}{u^2} \frac{du}{d\lambda} Cu^2 = -C \frac{du}{d\lambda}$$

$$\frac{d^2 r}{dt^2} = -C \frac{d^2 u}{d\lambda^2} \frac{d\lambda}{dt} = -C \frac{d^2 u}{d\lambda^2} Cu^2 = -C^2 u^2 \frac{d^2 u}{d\lambda^2}$$

Putting this all together

$$a_r = \frac{d^2 r}{dt^2} - r \left( \frac{d\lambda}{dt} \right)^2 = -C^2 u^2 \frac{d^2 u}{d\lambda^2} - \frac{1}{u} C^2 u^4 = -C^2 u^2 \left( \frac{d^2 u}{d\lambda^2} + u \right)$$

$$F_r = ma_r = -mC^2 u^2 \left( \frac{d^2 u}{d\lambda^2} + u \right)$$

#### 4.6.2 Period $T_u$

$$F_r = ma_r = -mC^2 u^2 \left( \frac{d^2 u}{d\lambda^2} + u \right)$$

where  $C = \frac{L}{m}$

$$F_r = \frac{-mL^2 u^2}{m^2} \left( \frac{d^2 u}{d\lambda^2} + u \right)$$

$$\frac{d^2 u}{d\lambda^2} + u = -\frac{m}{L^2 u^2} F(u) \tag{29}$$

Aside - from the above we can see that for an inverse square force  $-\frac{GMm}{r^2} = -GMmu^2$ , we have  $\frac{d^2 u}{d\lambda^2} + u = \frac{GMm^2}{L^2}$ . The RHS is a constant with units  $m^{-1}$  as expected.

$$\frac{d^2 u}{d\lambda^2} + u = -\frac{m}{L^2 u^2} \left[ -GMmu^2 + \frac{3}{2} GMmR_\mu^2 C_{20} \frac{1}{r^4} \right]$$

$$\frac{d^2 u}{d\lambda^2} + u = -\frac{m}{L^2 u^2} \left[ -GMmu^2 + \frac{3}{2} GMmR_\mu^2 C_{20} u^4 \right]$$

$$\frac{d^2 u}{d\lambda^2} + u = \frac{GMm^2}{L^2} - \frac{3GMm^2}{2L^2} R_\mu^2 C_{20} u^2$$

The following is from [Orbital Precession Central Force]. Let

$$g(u) = \frac{3GMm^2}{2} R_\mu^2 C_{20} u^2$$

so

$$\frac{d^2 u}{d\lambda^2} + u = \frac{GMm^2}{L^2} - \frac{g(u)}{L^2}$$

Let  $\eta = \frac{GMm^2}{L^2}$ . The unperturbed solution is

$$u_0 = \eta (1 + e \cos \lambda)$$

Let the perturbed solution  $u$  be in the vicinity of  $\eta$ . Let us expand  $g(u)$  around  $u = \eta$ .

$$g_0 = g(\eta) = \frac{3GMm^2}{2} R_\mu^2 C_{20} \eta^2$$

$$g_1 = \left. \frac{dg}{du} \right|_{u=\eta} = 3GMm^2 R_\mu^2 C_{20} \eta$$

$$\Delta u = u - \eta$$

$$g(u) = g_0 + g_1 \Delta u = \frac{3GMm^2}{2} R_\mu^2 C_{20} \eta^2 + 3GMm^2 R_\mu^2 C_{20} \eta (u - \eta)$$

So

$$\frac{d^2 u}{d\lambda^2} + u = \frac{GMm^2}{L^2} - \frac{3GMm^2}{2L^2} R_\mu^2 C_{20} \eta^2 - \frac{3GMm^2 R_\mu^2 C_{20} \eta}{L^2} (u - \eta)$$

$$\frac{d^2 u}{d\lambda^2} + \left(1 + \frac{3GMm^2 R_\mu^2 C_{20} \eta}{L^2}\right) u = \frac{GMm^2}{L^2} - \frac{3GMm^2}{2L^2} R_\mu^2 C_{20} \eta^2 + \frac{3GMm^2 R_\mu^2 C_{20} \eta^2}{L^2}$$

$$\frac{d^2 u}{d\lambda^2} + \left(1 + \frac{3G^2 M^2 m^4 R_\mu^2 C_{20}}{L^4}\right) u = \frac{GMm^2}{L^2} + \frac{3GMm^2}{2L^2} R_\mu^2 C_{20} \eta^2$$

$$\frac{d^2 u}{d\lambda^2} + \left(1 + \frac{3R_\mu^2 C_{20}}{a^2 (1 - e_0^2)^2}\right) u = \frac{GMm^2}{L^2} + \frac{3GMm^2}{2L^2} R_\mu^2 C_{20} \eta^2$$

This is an equation of the form  $\frac{d^2 u}{d\lambda^2} + k^2 u = \text{const}$  so  $u$  is a periodic function  $\cos k\lambda$

$$k = \sqrt{1 + \frac{3R_\mu^2 C_{20}}{a^2 (1 - e_0^2)^2}} \simeq 1 + \frac{3R_\mu^2 C_{20}}{2a^2 (1 - e_0^2)^2}$$

so  $u(\lambda)$  reaches it's maximum at

$$\lambda = \frac{2\pi}{\left(1 + \frac{3R_\mu^2 C_{20}}{2a^2 (1 - e_0^2)^2}\right)} = 2\pi \left(1 - \frac{3R_\mu^2 C_{20}}{2a^2 (1 - e_0^2)^2}\right)$$

a increased angle of

$$\Delta \lambda_p = -\frac{3\pi R_\mu^2 C_{20}}{a^2 (1 - e_0^2)^2} \quad (30)$$

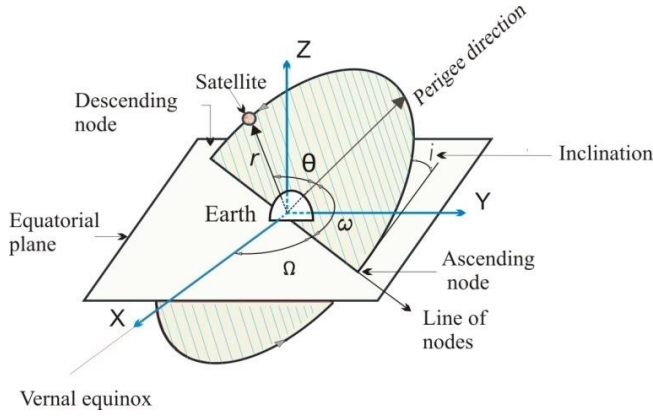


Figure 6: Space orbital parameters (ref [Apsidal Precession Low Earth])

#### 4.7 Apparent Discrepancy in Orbital and Inertial Calculations

Comparing 28 and 30 we seem to have a discrepancy as they differ by a factor 2.

$$\Delta\omega_p = -\frac{6\pi R_\mu^2 C_{20}}{(1-e^2)^2 a^2}$$

$$\Delta\lambda_p = -\frac{3\pi R_\mu^2 C_{20}}{a^2 (1-e_0^2)^2}$$

To see why this is, we need to revisit the calculation using orbital elements. Instead of using  $\Omega = \omega = i = 0$ , let us instead use  $\Omega = \omega = 0$  but have  $i \approx 0$  i.e. very small but not 0. In this case, we have a well defined ascending node  $\Omega$ . However  $\Omega$  also precesses in the following way

$$\frac{d\Omega}{dt} = \frac{3nR_e^2 C_{20}}{2a^2 (1-e^2)^2} \cos i$$

For small  $i$ , in one period

$$\Delta\Omega_p = \frac{d\Omega}{dt} \cdot \frac{2\pi}{n} = \frac{3\pi R_e^2 C_{20}}{a^2 (1-e^2)^2}$$

$$\Delta\Omega_p + \Delta\omega_p = -\frac{3\pi R_e^2 C_{20}}{a^2 (1-e^2)^2}$$

i.e.

$$\Delta\lambda_p = \Delta\Omega_p + \Delta\omega_p$$

See also Figure 6



#### 4.8 Mercury Precession due to Sun Oblateness

For Mercury we have the following (see [Mercury: Planet and Orbit]).

Sun Radius (km) $R_{\odot}$	695,700
Mercury $e$	0.2056
Mercury semi-major axis (km) $a$	57.91E6
Orbital Period (days) $T$	87.969
Mean angular velocity $n$	$\frac{2\pi}{T}$
Inclination $i$	$7^\circ$
Sun $J_2$ (or $-C_{20}$ )	2.0E-7

$$\omega_{prec} = \frac{3nR_{\odot}^2 J_2}{2a^2 (1 - e^2)^2} = \frac{3nR_{\odot}^2 (10^{-7})}{2a^2 (1 - e^2)^2} \left( \frac{J_2}{10^{-7}} \right)$$

$$\omega_{prec} = 1.68575E - 12 \left( \frac{J_2}{10^{-7}} \right) \text{ radians per day}$$

$$\omega_{prec} = 6.15299E - 08 \left( \frac{J_2}{10^{-7}} \right) \text{ radians per century}$$

$$\omega_{prec} = 3.5254E - 06 \left( \frac{J_2}{10^{-7}} \right) \text{ degrees per century}$$

$$\omega_{prec} = 3.5254E - 06 \left( \frac{J_2}{10^{-7}} \right) * 60 * 60 \approx 0.012 \left( \frac{J_2}{10^{-7}} \right) \text{ arcsec per century}$$

If we choose  $J_2 = 2E - 7$

$$\omega_{prec} \approx 0.024 \text{ arcsec per century}$$

Note General Relativity contributes  $\approx 43$  arcsec per century. See next section.

#### 4.9 Mercury Precession due to Relativity

The formula for Mercury precession due to Relativity is (see [Mercury: Planet and Orbit])

$$\omega_{prec} = \frac{3GM_{sun}n}{a(1 - e^2)c^2}$$

$GM_{sun} \left( \frac{km^3}{s^2} \right)$	132,712E6
Mercury $e$	0.2056
Mercury semi-major axis (km) $a$	57.91E6
Orbital Period (days) $T_d$	87.969
Orbital Period (sec) $T_s$	7,600,522
Mean angular velocity (per sec) $n$	8.27E-07
Speed of light $c \frac{km}{s}$	299,792

$$\omega_{prec} = 6.6 (10^{-14}) \text{ radians per second}$$

$$\omega_{prec} = 2.08 (10^{-4}) \text{ radians per century}$$

$$\omega_{prec} = 0.01193 \text{ degrees per century}$$

$$\omega_{prec} = 42.95 \text{ arcsec per century}$$

#### 4.10 Newton's Theorem of Revolving Orbits

We could also account for Mercury's precession by making a small amendment  $\delta$  to the inverse square law [Newton Apsidal]. Let

$$F(r) = -\frac{GMm}{r^{(2+\delta)}}$$

Let  $u = \frac{1}{r}$

$$F(u) = -GMmu^{(2+\delta)} = -GMmu^2u^\delta$$

Using 29

$$\frac{d^2u}{d\lambda^2} + u = -\frac{m}{L^2u^2}F(u) = \left(-\frac{m}{L^2u^2}\right)(-GMmu^2u^\delta) = \frac{GMm^2u^\delta}{L^2}$$

So we have the following (where if  $\delta = 0$  we end up with the standard inverse square law equation).

$$\frac{d^2u}{d\lambda^2} + u = \frac{GMm^2u^\delta}{L^2}$$

$$\frac{d^2u}{d\lambda^2} + u = \eta u^\delta$$

$u^\delta = \exp(\ln u^\delta)$ . Since  $\ln u^\delta = \delta \ln u$  then  $u^\delta = \exp(\delta \ln u)$ . Expanding this to the first order in  $\delta$  gives

$$u^\delta = 1 + \delta \ln u$$

As previously, let us assume that the solution  $u$  is in the region of  $\eta$  (so this derivation is only working for small  $e$ ). Let

$$g(u) = u^\delta = 1 + \delta \ln u$$

Expand around  $\eta$

$$g(u) = g_0(\eta) + g_1(\eta) \Delta u$$

$$g_0 = g(\eta) = 1 + \delta \ln \eta$$

$$g_1 = \left. \frac{dg}{du} \right|_{u=\eta} = \frac{\delta}{\eta}$$

$$\Delta u = u - \eta$$

$$g(u) = g_0 + g_1 \Delta u = (1 + \delta \ln \eta) + \frac{\delta}{\eta} (u - \eta)$$

$$\frac{d^2u}{d\lambda^2} + u = \eta g(u) = \eta(1 + \delta \ln \eta) + \delta u - \delta \eta$$

$$\frac{d^2u}{d\lambda^2} + u = \eta(1 - \delta + \delta \ln \eta) + \delta u$$

$$\frac{d^2u}{d\lambda^2} + (1 - \delta)u = \eta(1 - \delta + \delta \ln \eta) \quad (31)$$

This is of the form  $\frac{d^2u}{d\lambda^2} + k^2u = \text{const}$  so is a periodic function  $\cos k\lambda$ .  $k = \sqrt{1 - \delta}$ . The maximum is at  $\lambda = \frac{2\pi}{\sqrt{1 - \delta}} = 2\pi(1 - \delta)^{-\frac{1}{2}} \approx 2\pi(1 + \frac{\delta}{2}) = 2\pi + \delta\pi$

So the increase is

$$\delta\pi \text{ per orbit}$$

For Mercury, letting  $\delta = 0.00000016$

$$0.00000016\pi \text{ radians per } 87.969 \text{ days}$$

$$6.6387023\pi (10^{-5}) \text{ radians per century}$$

$$0.003803696\pi \text{ degrees per century}$$

$$\approx 43 \text{ arcsec per century}$$

Aside 1: Newton's derivation (Proposition 45 in Book I of the Principia) relates the angle between apocentre and pericentre to be  $\frac{\pi}{\sqrt{n}}$  where the centripetal force is of magnitude  $\mu r^{n-3}$ . Relating this to the above we have  $n - 3 = -(2 + \delta)$  or  $n = (1 - \delta)$  so the angle is  $\frac{\pi}{\sqrt{1-\delta}}$ . Therefore the angle from apocentre to apocentre is twice this i.e.  $\frac{2\pi}{\sqrt{1-\delta}}$  (as above).

#### 4.10.1 Semi-Latus Rectum Change

I'm still trying to figure out perturbation theory/precessing ellipses so I need to check the following. Also refer to G.

The above 31 means that the ellipse, besides precessing, has the following particular solution

$$u_p = \eta \frac{(1 - \delta + \delta \ln \eta)}{(1 - \delta)}$$

So it also has a increased semi-latus rectum  $l_{new} = \frac{1}{\eta_{new}}$  i.e.

$$\eta_{new} = \eta \left( \frac{(1 - \delta + \delta \ln \eta)}{(1 - \delta)} \right) = \left( 1 + \frac{\delta \ln \eta}{1 - \delta} \right) \eta$$

I haven't found any literature on this consequence of changes to the inverse-square force and how significant/observable it is.

In the case of Mercury

$$\eta_{new} = \left( 1 + \frac{(0.00000016) \ln(1.803(10^{-8}))}{(1 - 0.00000016)} \right) \eta = 0.9999971467\eta$$

And the semi-latus rectum is

$$l_{new} = \frac{1}{\eta_{new}} = \frac{1}{0.9999971467\eta}$$

semi-latus rectum goes from 55,462,066 to 55,462,224 km. So an increase of  $\sim 150$ km.

## 5 Mercury Spin-Orbit Coupling.

Mercury has a 3:2 spin orbit coupling i.e. it rotates on it's axis 3 times in the time it takes to orbit twice around the sun. The following references explain it well [Spin Orbit coupling], [Mercury: Planet and Orbit], [Tidal Torques].

The 3:2 spin orbit coupling involves the effects of 1) tidal torques and 2) triaxial torques due to permanent asymmetry in Mercury's equatorial plane i.e.  $a > b > c$ . Note - the tidal torque does not require a permanent asymmetry in the axis i.e. we could have tidal torque if Mercury was spherical. The tidal torque on it's own would act to slow down Mercury's rotation until it reaches a 1:1 coupling i.e. Mercury would take the same time to rotate on it's axis as it does to orbit the sun. Note the moon is in a 1:1 spin orbit coupling which is why it always presents the same face to the earth. However Mercury hasn't reached the 1:1 state - it spins faster than it orbits and is captured in a 3:2 state.

Lets consider the triaxial torque due to the *permanent* asymmetry of Mercury where the moments of inertia around the 3 principal axis are  $C > B > A$ . [Spin Orbit coupling] gives a good account of the torques in play and the derivation of subsequent equations(see Fig 7).

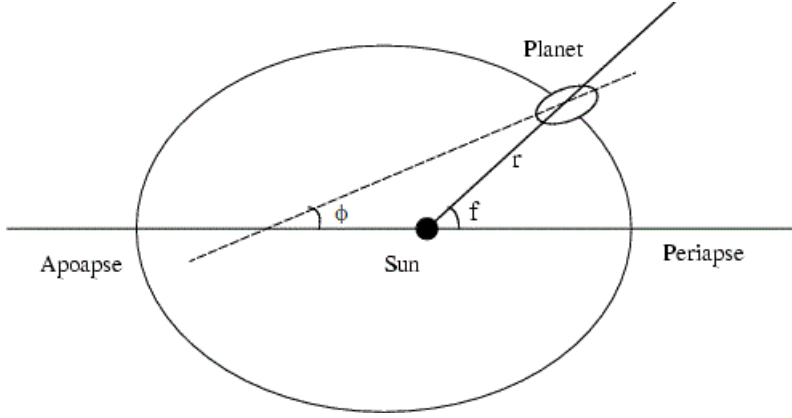


Figure 7: Geometry of spin-orbit coupling (ref [Mercury Main librations]).

Using MacCullagh's formula for the gravitational potential due to Mercury leads to the following formula that describes  $\phi$ . Note  $\dot{\phi}$  describes Mercury's spin.

$$\ddot{\phi} + \frac{3}{2}n^2 \left( \frac{B-A}{C} \right) \left( \frac{a}{r} \right)^3 \sin[2(\phi - f)] = 0 \quad (32)$$

Now 32 is not straightforward to solve. Lets consider the situation where Mercury has been spun down by tidal torques until it is approaching the 3:2 resonance. Let us define

$$\gamma = \phi - \frac{3}{2}M$$

where  $M$  is the mean anomaly. So

$$\phi = \gamma + \frac{3}{2}M$$

$$\dot{\phi} = \dot{\gamma} + \frac{3}{2}n$$

$$\ddot{\phi} = \ddot{\gamma}$$

So 32 becomes

$$\ddot{\gamma} + \frac{3}{2}n^2 \left( \frac{B-A}{C} \right) \left( \frac{a}{r} \right)^3 \sin \left[ 2 \left( \gamma + \frac{3}{2}M - f \right) \right] = 0$$

$$\ddot{\gamma} + \frac{3}{2}n^2 \left( \frac{B-A}{C} \right) \left( \frac{a}{r} \right)^3 \sin[2\gamma + 3M - 2f] = 0$$

The following term is similar to what we solved in 3.3.

$$\left( \frac{a}{r} \right)^3 \sin[2\gamma + 3M - 2f]$$

## 5.1 Prerequisites

Revisiting the complex Fourier expansion of a function we have

$$f = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

If  $f$  is real then  $c_k = \bar{c}_{-k}$  i.e. the complex conjugate.

If  $f$  is even,  $c_k$  is real, so  $c_k = c_{-k}$ .

If  $f$  is odd,  $c_k$  is imaginary.

The relation between cosine and sin coefficients to the complex coefficients are

$$a_0 = c_0$$

$$a_k = c_k + c_{-k}$$

$$b_k = i(c_k - c_{-k})$$

## 5.2 Expansion of $\frac{a^3}{r^3} \sin(2\gamma + 3M - 2f)$

$$\sin(2\gamma + 3M - 2f) = \frac{\exp(i(2\gamma + 3M - 2f)) - \exp(-i(2\gamma + 3M - 2f))}{2i}$$

From 25

$$\exp(i(2\gamma + 3M - 2f)) = \exp(i(2\gamma + 3M)) \exp(-i2f) = \exp(i(2\gamma + 3M)) \frac{\exp(-i2E)(1 - \beta \exp(-iE))^{-2}}{(1 - \beta \exp(iE))^{-2}}$$

$$\begin{aligned} 2i \sin(2\gamma + 3M - 2f) &= \exp(i(2\gamma + 3M)) \frac{\exp(-i2E)(1 - \beta \exp(-iE))^{-2}}{(1 - \beta \exp(iE))^{-2}} \\ &\quad - \exp(-i(2\gamma + 3M)) \frac{\exp(i2E)(1 - \beta \exp(-iE))^2}{(1 - \beta \exp(iE))^2} \end{aligned}$$

$$\begin{aligned} &= \exp(i(2\gamma + 3M)) \exp(-i2E)(1 - \beta \exp(-iE))^{-2} (1 - \beta \exp(iE))^2 \\ &\quad - \exp(-i(2\gamma + 3M)) \exp(i2E)(1 - \beta \exp(-iE))^2 (1 - \beta \exp(iE))^{-2} \end{aligned}$$

Let's expand the r, f term in a Fourier series as before

$$function = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

$$\frac{a^3}{r^3} \sin(2\gamma + 3M - 2f) = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} [\sin(2\gamma + 3M - 2f)] \exp -ikM dM$$

Let's look at the first term first

$$c_{k_1} = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \left[ \frac{\exp(i(2\gamma + 3M)) \exp(-i2E)(1 - \beta \exp(-iE))^{-2} (1 - \beta \exp(iE))^2}{2i} \right] \exp -ikM dM$$

From previously, we can substitute  $\frac{a^3}{r^3}$  with  $\frac{a^2}{r^2} \frac{a}{r} dM$  or  $\frac{a^2}{r^2} dE$  and take constant terms out of the integral

$$c_{k_1} = \frac{1}{4\pi i} \exp(i2\gamma) \int_0^{2\pi} \frac{a^2}{r^2} \left[ \exp(i3M) \exp(-i2E)(1 - \beta \exp(-iE))^{-2} (1 - \beta \exp(iE))^2 \right] \exp -ikM dE$$

substitute  $\frac{a^2}{r^2}$

$$= \frac{1}{4\pi i} \exp(i2\gamma) \int_0^{2\pi} \frac{(1 + \beta^2)^2}{[(1 - \beta \exp(iE))(1 - \beta \exp(-iE))]^2} \\ * \left[ \exp(i3M) \exp(-i2E) (1 - \beta \exp(-iE))^{-2} (1 - \beta \exp(iE))^2 \right] \exp -ikM dE$$

Rearrange terms

$$= \frac{(1 + \beta^2)^2}{4\pi i} \exp(i2\gamma) \int_0^{2\pi} (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \\ * \left[ \exp(-i2E) (1 - \beta \exp(-iE))^{-2} (1 - \beta \exp(iE))^2 \right] \exp -i(k-3)M dE$$

substitute  $M$  and combine  $iE$  terms

$$= \frac{(1 + \beta^2)^2}{4\pi i} \exp(i2\gamma) \int_0^{2\pi} (1 - \beta \exp(-iE))^{-4} \\ * \exp(-i2E) \exp(i[(k-3)e \sin E - (k-3)E]) dE$$

so the first term is

$$\boxed{c_{k_1} = \frac{(1 + \beta^2)^2}{4\pi i} \exp(i2\gamma) \int_0^{2\pi} (1 - \beta \exp(-iE))^{-4} \exp(i[(k-3)e \sin E - (k-1)E]) dE}$$

$$c_{k_1} = \frac{(1 + \beta^2)^2}{4\pi i} \exp(i2\gamma) \int_0^{2\pi} \sum_{b=0}^{\infty} \binom{-4}{b} (-1)^b \beta^b \exp(-iEb) \exp(i[(k-3)e \sin E - (k-1)E]) dE$$

$$c_{k_1} = \frac{(1 + \beta^2)^2}{4\pi i} \exp(i2\gamma) \int_0^{2\pi} \sum_{b=0}^{\infty} \binom{-4}{b} (-1)^b \beta^b \exp(i[(k-3)e \sin E - (k-1+b)E]) dE$$

$$c_{k_1} = \frac{\exp(i2\gamma)}{2i} \left[ (1 + \beta^2)^2 \sum_{b=0}^{\infty} \binom{-4}{b} (-1)^b \beta^b J_{k-1+b}((k-3)e) \right]$$

For the second term, and omitting some steps

$$c_{k_2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \left[ \frac{\exp(-i(2\gamma + 3M)) \exp(i2E) (1 - \beta \exp(-iE))^2 (1 - \beta \exp(iE))^{-2}}{2i} \right] \exp -ikM dM$$

$$= \frac{1}{4\pi i} \exp(-i2\gamma) \int_0^{2\pi} (1 + \beta^2)^2 (1 - \beta \exp(iE))^{-2} (1 - \beta \exp(-iE))^{-2} \\ * \left[ \exp(-i3M) \exp(i2E) (1 - \beta \exp(-iE))^2 (1 - \beta \exp(iE))^{-2} \right] \exp -ikM dE$$

$$= \frac{(1 + \beta^2)^2}{4\pi i} \exp(-i2\gamma) \int_0^{2\pi} (1 - \beta \exp(iE))^{-4} \exp(i((k+3)e \sin E - (k+1)E)) dE$$

$$c_{k_2} = \frac{\exp(-i2\gamma)}{2i} \left[ (1 + \beta^2)^2 \sum_{b=0}^{\infty} \binom{-4}{b} (-1)^b \beta^b J_{k+1-b}((k+3)e) \right]$$

$$c_k = c_{k_1} - c_{k_2}$$

Using a program, we can calculate the bit in the bracket for both  $c_{k_1}$  and  $c_{k_2}$  which will be a function of  $k$  and  $e$ . This is basically the same way we did manually in section 3.2.2).

Here are the results.

```

file:///C:/dev/SpheroidCode/spheroid/bin/Debug/spheroid.EXE
k=0 1st:7/2e^1 + -123/16e^3, 2nd:7/2e^1 + -123/16e^3
k=1 1st:1/1e^0 + -5/2e^2, 2nd:17/2e^2
k=-1 1st:17/2e^2, 2nd:1/1e^0 + -5/2e^2
k=2 1st:-1/2e^1 + 1/16e^3, 2nd:845/48e^3
k=-2 1st:845/48e^3, 2nd:-1/2e^1 + 1/16e^3

```

Figure 8: Calculation of  $e$  dependency

Calculate the  $k = 0$  term -  $c_0$  - keeping only as high as  $O(e)$ .

$$c_0 \exp i0M = c_{0_1} \cdot 1 - c_{0_2} \cdot 1 = \frac{7e \exp(i2\gamma)}{2} \frac{1}{2i} - \frac{7e \exp(-i2\gamma)}{2} \frac{1}{2i}$$

$$c_0 = \frac{7e}{2} \left[ \frac{\exp(i2\gamma) - \exp(-i2\gamma)}{2i} \right]$$

$$c_0 = \frac{7e}{2} \sin(2\gamma)$$

Calculate the  $k = 1$  and  $k = -1$  terms,  $c_1$  and  $c_{-1}$  keeping only as high as  $O(e)$ .

$$c_1 \exp iM = c_{1_1} \exp iM - c_{1_2} \exp iM = 1 \cdot \frac{\exp(i2\gamma) \exp iM}{2i} - O(e^2) = \frac{\exp(i(2\gamma + M))}{2i}$$

$$c_{-1} \exp -iM = c_{-1_1} \exp -iM - c_{-1_2} \exp -iM = O(e^2) - 1 \cdot \frac{\exp(-i2\gamma) \exp -iM}{2i} = -\frac{\exp(-i(2\gamma + M))}{2i}$$

$$c_1 \exp iM + c_{-1} \exp -iM = \sin(2\gamma + M)$$

Calculate the  $k = 2$  and  $k = -2$  terms,  $c_2$  and  $c_{-2}$  keeping only as high as  $O(e)$ .

$$c_2 \exp i2M = c_{2_1} \exp i2M - c_{2_2} \exp i2M = -\frac{e \exp(i2\gamma) \exp i2M}{2} \frac{1}{2i} + O(e^3) = -\frac{e \exp(i(2\gamma + 2M))}{2}$$

$$c_{-2} \exp -i2M = c_{-2_1} \exp -i2M - c_{-2_2} \exp -i2M = O(e^3) + \frac{e \exp(-i2\gamma) \exp -i2M}{2} \frac{1}{2i} = +\frac{e \exp(-i(2\gamma + 2M))}{2}$$

$$c_2 \exp i2M + c_{-2} \exp -i2M = -\frac{e}{2} \sin(2\gamma + 2M)$$

$$\frac{a^3}{r^3} \sin(2\gamma + 3M - 2f) = \frac{7e}{2} \sin(2\gamma) + \sin(2\gamma + M) - \frac{e}{2} \sin(2\gamma + 2M) + \dots$$

$$\ddot{\gamma} + \frac{3}{2} n^2 \left( \frac{B-A}{C} \right) \left( \frac{a}{r} \right)^3 \sin \left[ 2 \left( \gamma + \frac{3}{2} M - f \right) \right] = 0$$

$$\ddot{\gamma} + \frac{3}{2} n^2 \left( \frac{B-A}{C} \right) \left[ \frac{7e}{2} \sin(2\gamma) + \sin(2\gamma + M) - \frac{e}{2} \sin(2\gamma + 2M) \right] = 0 \quad (33)$$

$\gamma = \phi - \frac{3}{2}M$  so  $\gamma$  is approximately constant near the 3:2 resonance. The sine terms with a dependency on  $M$  in 33 average to zero so we have

$$\ddot{\gamma} + \frac{3}{2} n^2 \left( \frac{B-A}{C} \right) \frac{7e}{2} \sin(2\gamma) = 0$$

$$\ddot{\gamma} + \frac{21}{4} e n^2 \left( \frac{B-A}{C} \right) \sin(2\gamma) = 0 \quad (34)$$

Equation 34 is a pendulum equation. For the 3:2 reference,  $\gamma$  as defined is small so  $\sin(2\gamma) \approx 2\gamma$

$$\ddot{\gamma} + \frac{21}{4}en^2 \left( \frac{B-A}{C} \right) 2\gamma = 0 \quad (35)$$

This is of the form  $\ddot{\gamma} + k^2\gamma = 0$  so the orientation of Mercury's longest axis points towards the Sun at the perihelion and librates with a frequency

$$k = \sqrt{\frac{21}{2}en^2 \left( \frac{B-A}{C} \right)} = n\sqrt{\frac{21}{2}e \left( \frac{B-A}{C} \right)} \approx n\sqrt{10.5 * 0.2 * 0.00012} = 0.016n$$

So the period of the librations compared to the orbital period  $T_M$  is

$$\frac{\frac{2\pi}{0.016n}}{\frac{2\pi}{n}} = \frac{1}{0.016} \approx 62T_M$$



## A Asaph Hall Apsidal Precession derivation

Precession due to a force law differing slightly from Inverse-square (this was the hypothesis made by Asaph Hall in 1894). Derivation from [Newton Apsidal]

$$\frac{d^2 u}{d\lambda^2} + u = \eta u^\delta \quad (36)$$

Let the solution of the above be

$$u = c_h (1 + e \cos k_h \lambda) + \delta u' \quad (37)$$

where  $c_h$  is a constant and  $k_h$  represents a rotating ellipse. From [Newton Apsidal]

. Where  $e$  is larger, we cannot simply identify a rotating ellipse with the orbit of interest. Instead, we shall employ a rotating ellipse *plus* additional terms; this will allow for deformation of the shape as well as motion of the apse. The additional terms will be given by the perturbation  $\delta u'$ . We shall restrict this perturbation so as not to contain terms proportional to  $\cos \lambda \theta$ ; thus it will not be implicated in the determination of the constants of the rotating ellipse.

Substituting this into  $u^\delta$  and keeping  $O(\delta)$  only

$$u^\delta = c_h^\delta (1 + e \cos k_h \lambda)^\delta$$

$$\ln u^\delta = \ln c_h^\delta + \delta \ln (1 + e \cos k_h \lambda)$$

Taking exponential

$$u^\delta = c_h^\delta \exp(\delta \ln (1 + e \cos k_h \lambda))$$

Expanding exponential to  $O(\delta)$

$$u^\delta = c_h^\delta (1 + \delta \ln (1 + e \cos k_h \lambda))$$

Expanding the  $\ln$

$$u^\delta = c_h^\delta \left( 1 + \delta e \left( \cos k_h \lambda - \frac{e}{2} \cos^2 k_h \lambda + \frac{e^2}{3} \cos^3 k_h \lambda \right) \right)$$

Using the identities

$$\cos 2\theta = 2 \cos^2 \theta - 1 \implies \cos^2 k_h \lambda = \frac{1}{2} + \frac{1}{2} \cos 2k_h \lambda$$

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \implies \cos^3 k_h \lambda = \frac{3}{4} \cos k_h \lambda + \frac{1}{4} \cos 3k_h \lambda$$

We get

$$\begin{aligned} u^\delta &= c_h^\delta \left( 1 + \delta e \left( \cos k_h \lambda - \frac{e}{2} \left( \frac{1}{2} + \frac{1}{2} \cos 2k_h \lambda \right) + \frac{e^2}{3} \left( \frac{3}{4} \cos k_h \lambda + \frac{1}{4} \cos 3k_h \lambda \right) \right) \right) \\ u^\delta &= c_h^\delta \left( 1 + \delta e \left( -\frac{e}{4} + \cos k_h \lambda - \frac{e}{4} \cos 2k_h \lambda + \frac{e^2}{4} \cos k_h \lambda + \frac{e^2}{12} \cos 3k_h \lambda \right) \right) \\ u^\delta &= c_h^\delta \left( 1 + \delta e \left( -\frac{e}{4} + \left( 1 + \frac{e^2}{4} \right) \cos k_h \lambda - \frac{e}{4} \cos 2k_h \lambda + \frac{e^2}{12} \cos 3k_h \lambda \right) \right) \end{aligned} \quad (38)$$

Now insert 37 into the left side of 36

$$\begin{aligned} \delta \left( \frac{d^2 u'}{d\lambda^2} + u' \right) - c_h e k_h^2 \cos k_h \lambda + c_h + c_h e \cos k_h \lambda \\ \delta \left( \frac{d^2 u'}{d\lambda^2} + u' \right) + c_h + c_h e (1 - k_h^2) \cos k_h \lambda \end{aligned} \quad (39)$$

From 36 and using 39 and 38

$$\delta \left( \frac{d^2 u'}{d\lambda^2} + u' \right) + c_h + c_h e (1 - k_h^2) \cos k_h \lambda = \eta c_h^\delta \left( 1 + \delta e \left( -\frac{e}{4} + \left( 1 + \frac{e^2}{4} \right) \cos k_h \lambda - \frac{e}{4} \cos 2k_h \lambda + \frac{e^2}{12} \cos 3k_h \lambda \right) \right)$$

Equating constant terms

$$c_h = \eta c_h^\delta \left(1 - \frac{\delta e^2}{4}\right)$$

$$\eta c_h^{\delta-1} = \left(1 - \frac{\delta e^2}{4}\right)^{-1} \approx \left(1 + \frac{\delta e^2}{4}\right)$$

The coefficients of  $\cos k_h \lambda$  must also be equal so

$$c_h e (1 - k_h^2) = \eta c_h^\delta \delta e \left(1 + \frac{e^2}{4}\right)$$

$$1 - k_h^2 = \delta \eta c_h^{\delta-1} \left(1 + \frac{e^2}{4}\right)$$

$$1 - k_h^2 = \delta \left(1 + \frac{\delta e^2}{4}\right) \left(1 + \frac{e^2}{4}\right) \approx \delta \left(1 + \frac{e^2}{4}\right)$$

$$k_h = \left(1 - \delta \left(1 + \frac{e^2}{4}\right)\right)^{\frac{1}{2}}$$

$$k_h \approx 1 - \frac{\delta}{2} \left(1 + \frac{e^2}{4}\right)$$

The precession per orbit is

$$\Delta\lambda = \frac{2\pi}{k_h} = 2\pi \left(1 - \frac{\delta}{2} \left(1 + \frac{e^2}{4}\right)\right)^{-1} - 2\pi$$

$$\Delta\lambda \approx 2\pi \left(1 + \frac{\delta}{2} \left(1 + \frac{e^2}{4}\right)\right) - 2\pi$$

$$\Delta\lambda = \frac{2\pi}{k_h} = \delta\pi \left(1 + \frac{e^2}{4}\right)$$

so the precession increases as the eccentricity increases. The difference is small though in the case of Mercury.

## B True to Eccentric Anomaly

For this section, I'll use  $e$  instead of  $\exp$  as there is no confusing  $e$  with eccentricity.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\tan \frac{\theta}{2} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}$$

$$\tan \frac{\theta}{2} = \frac{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}{i(e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}})}$$

times through by  $e^{\frac{i\theta}{2}}$

$$\tan \frac{\theta}{2} = \frac{e^{i\theta} - 1}{i(e^{i\theta} + 1)}$$

The relationship between true and eccentric anomaly is given by

$$\tan \left(\frac{f}{2}\right) = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan \left(\frac{E}{2}\right)$$

Let  $p = \sqrt{\frac{1+e}{1-e}}$

$$\tan\left(\frac{f}{2}\right) = p \tan\left(\frac{E}{2}\right)$$

$$\frac{e^{if} - 1}{i(e^{if} + 1)} = p \frac{(e^{iE} - 1)}{i(e^{iE} + 1)}$$

$$e^{if} - 1 = p \frac{(e^{iE} - 1)}{(e^{iE} + 1)} (e^{if} + 1)$$

$$e^{if} - 1 = p \frac{(e^{iE} e^{if} + e^{iE} - e^{if} - 1)}{(e^{iE} + 1)}$$

$$e^{if} - 1 = \frac{pe^{iE} e^{if} + pe^{iE} - pe^{if} - p}{(e^{iE} + 1)}$$

$$e^{if} - 1 = \frac{e^{if} (pe^{iE} - p) + pe^{iE} - p}{(e^{iE} + 1)}$$

$$(e^{if} - 1) (e^{iE} + 1) = e^{if} (pe^{iE} - p) + pe^{iE} - p$$

$$e^{if} e^{iE} + e^{if} - e^{iE} - 1 = e^{if} (pe^{iE} - p) + pe^{iE} - p$$

$$e^{if} (e^{iE} + 1) - e^{iE} - 1 = e^{if} (pe^{iE} - p) + pe^{iE} - p$$

Bring all  $e^{if}$  terms to the left

$$e^{if} (e^{iE} + 1 + p - pe^{iE}) = pe^{iE} - p + e^{iE} + 1$$

$$e^{if} ((1 + p) + (1 - p) e^{iE}) = (1 - p) + (1 + p) e^{iE}$$

$$e^{if} = \frac{(1 - p) + (1 + p) e^{iE}}{(1 + p) + (1 - p) e^{iE}}$$

If we let  $\beta = \frac{p-1}{p+1}$  and divide the above by  $(1 + p)$  we get

$$e^{if} = \frac{-\beta + e^{iE}}{1 - \beta e^{iE}}$$

$$e^{if} = \frac{e^{iE} (1 - \beta e^{-iE})}{1 - \beta e^{iE}}$$

## C Program to Calculate Coefficients of cosine term

To be included

## D Radius of a Spheroid (ellipse)

Because we are considering a spheroid which is got by rotating an ellipse around the  $z$  axis, there is no azimuthal  $\lambda$  dependency for  $r$  i.e.  $r(\phi)$

$$r = \frac{ab}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}}$$

$$r = \frac{ab}{\sqrt{a^2 \sin^2 \phi + b^2 (1 - \sin^2 \phi)}}$$

$$r = \frac{ab}{\sqrt{(a^2 - b^2) \sin^2 \phi + b^2}}$$

$$\begin{aligned}
r &= \frac{ab}{b\sqrt{\frac{(a^2-b^2)}{b^2} \sin^2 \phi + 1}} \\
r &= \frac{a}{\sqrt{\frac{(a^2-b^2)}{b^2} \sin^2 \phi + 1}} \\
r &= \frac{a}{\sqrt{1 + \frac{(a^2-b^2)}{b^2} \sin^2 \phi}} \\
r &= a \left[ 1 + \frac{(a^2-b^2)}{b^2} \sin^2 \phi \right]^{-\frac{1}{2}} \\
r &= a \left[ 1 + \frac{(a+b)(a-b)}{b^2} \sin^2 \phi \right]^{-\frac{1}{2}}
\end{aligned}$$

using  $\epsilon = \frac{a-b}{R_\mu}$

$$r = a \left[ 1 + \frac{(a+b)\epsilon R_\mu}{b^2} \sin^2 \phi \right]^{-\frac{1}{2}}$$

using  $a+b = (a-b) + 2b = \epsilon R_\mu + 2b$

$$\begin{aligned}
r &= a \left[ 1 + \frac{(\epsilon R_\mu + 2b)\epsilon R_\mu}{b^2} \sin^2 \phi \right]^{-\frac{1}{2}} \\
r &= a \left[ 1 + \left( \frac{2b\epsilon R_\mu}{b^2} \sin^2 \phi + \frac{\epsilon^2 R_\mu^2}{b^2} \sin^2 \phi \right) \right]^{-\frac{1}{2}}
\end{aligned}$$

Binomial expansion of  $(1+x)^{-\frac{1}{2}}$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2$$

So binomially expand to  $x$  term

$$r \simeq a \left[ 1 - \frac{2b\epsilon R_\mu}{2b^2} \sin^2 \phi - \frac{\epsilon^2 R_\mu^2}{2b^2} \sin^2 \phi \right]$$

Aside.  $\sin^2 \phi$  can be expressed in terms of Legendre Polynomials i.e.  $P_2 = \frac{3}{2} \sin^2 \phi - \frac{1}{2} = \frac{3}{2} \sin^2 \phi - \frac{1}{2} P_0 \Rightarrow \frac{2}{3} P_2 = \sin^2 \phi - \frac{1}{3} P_0 \Rightarrow \sin^2 \phi = \frac{2}{3} P_2 + \frac{1}{3} P_0$

We will be expressing  $r$  as a Legendre Polynomial expansion, however we are just keeping  $O(\epsilon)$  terms in our calculation of  $r$

$$\begin{aligned}
r &\simeq a \left[ 1 - \frac{2\epsilon R_\mu}{2b} \sin^2 \phi \right] \\
r &\simeq a - \frac{a\epsilon R_\mu}{b} \sin^2 \phi \\
r &\simeq (2a+b) - \frac{a\epsilon R_\mu}{b} \sin^2 \phi - (a+b)
\end{aligned}$$

using  $R_\mu = \frac{2a+b}{3}$

$$r \simeq 3R_\mu - \frac{a\epsilon R_\mu}{b} \sin^2 \phi - (a+b)$$

From the definition of flattening  $f$  Appendix H,  $a/b \simeq 1 + f \simeq 1 + \epsilon$

$$r \simeq 3R_\mu - \epsilon R_\mu \sin^2 \phi - (a+b)$$

$$r \simeq 3R_\mu - \frac{2}{3}\epsilon R_\mu \frac{1}{2} 3 \sin^2 \phi - (a+b)$$

Need  $(a+b)$  in terms of  $2R_\mu$

$$(a + b) = 2R_\mu + ?$$

$$? = (a + b) - 2R_\mu = (a + b) - 2 \frac{(2a + b)}{3} = \frac{3(a + b) - 2(2a + b)}{3} = \frac{-(a - b)}{3} = \frac{-\epsilon R_\mu}{3}$$

so

$$(a + b) = 2R_\mu - \frac{\epsilon R_\mu}{3}$$

$$r \simeq 3R_\mu - \frac{2}{3}\epsilon R_\mu \frac{1}{2} 3 \sin^2 \phi - 2R_\mu + \frac{\epsilon R_\mu}{3}$$

$$r \simeq R_\mu - \frac{2}{3}\epsilon R_\mu \frac{1}{2} 3 \sin^2 \phi + \frac{1}{2} \frac{2}{3} \epsilon R_\mu$$

$$r \simeq R_\mu - \frac{2}{3}\epsilon R_\mu \left[ \frac{1}{2} (3 \sin^2 \phi - 1) \right]$$

$$r \simeq R_\mu \left[ 1 - \frac{2}{3} \epsilon P_2(\sin \phi) \right]$$

## E Potential due to a Uniform Spheroid

The below is a summary from [Potential Uniform Spheroid] - he gives a full derivation for the below.

NOTATION: Note for this section, I'll follow his notation of using standard Physics spherical coordinates  $r, \theta, \phi$  not  $r, \phi, \lambda$

$$\Phi(\mathbf{r}) = \Phi(r, \theta, \phi) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} d^3 \mathbf{r}'$$

In the case of an axially symmetric body there is no azimuthal dependency  $\phi$ . So we have  $\Phi(r, \theta)$

$$\Phi(r, \theta) = \sum_{n=0}^{\infty} \Phi_n(r) P_n(\cos \theta)$$

where

$$\Phi_n(r) = -\frac{2\pi G}{r^{n+1}} \int_0^r \int_0^\pi r'^{n+2} \rho(r', \theta') P_n(\cos \theta') \sin \theta' d\theta' dr'$$

This can be rewritten as below where (we have basically multiplied top and bottom by  $R_\mu^3$  and  $M$  terms.)

$$\Phi(r, \theta) = \frac{GM}{R_\mu} \sum_{n=0}^{\infty} J_n \left( \frac{R_\mu}{r} \right)^{n+1} P_n(\cos \theta)$$

$$J_n = -\frac{2\pi R_\mu^3}{M} \int_0^r \int_0^\pi \left[ \frac{r'^{n+2}}{R_\mu^{n+3}} dr' \right] \rho(r', \theta') P_n(\cos \theta') \sin \theta' d\theta'$$

If the spheroid is uniform density then  $\rho(r, \theta)$  is a const. Let call it  $\gamma$ . In what follows, refer to Appendix H for definitions of  $\epsilon$  and  $R_\mu$ . The mass of a uniform spheroid is  $M = \left( \frac{4\pi}{3} \right) \gamma R_\mu^3$ . The reasoning for this is that  $a = R_e = R_\mu \left( 1 + \frac{\epsilon}{3} \right)$  for the equator radius and  $b = R_p = R_\mu \left( 1 - \frac{2\epsilon}{3} \right)$  for polar radius. The equation for the volume of an oblate spheroid is  $V = \frac{4}{3} \pi a^2 b = \frac{4}{3} \pi R_e^2 R_p = R_\mu^2 \left( 1 + \frac{\epsilon}{3} \right)^2 R_\mu \left( 1 - \frac{2\epsilon}{3} \right) \simeq R_\mu^2 \left( 1 + \frac{2\epsilon}{3} \right) R_\mu \left( 1 - \frac{2\epsilon}{3} \right) = R_\mu^3 + O(\epsilon^2)$ .

So using the above we now have

$$J_n = -\frac{2\pi R_\mu^3 \gamma}{M} \int_0^r \int_0^\pi \left[ \frac{r'^{n+2}}{R_\mu^{n+3}} dr' \right] P_n(\cos \theta') \sin \theta' d\theta'$$

$$J_n = -\frac{3}{2} \int_0^\pi \int_0^{R(\theta)} \left[ \frac{r'^{n+2}}{R_\mu^{n+3}} dr' \right] P_n(\cos \theta') \sin \theta' d\theta'$$

Using Appendix D, we have

$$R(\theta) \simeq R_\mu \left[ 1 - \frac{2}{3} \epsilon P_2(\cos \theta) \right]$$

The calculation in [Potential Uniform Spheroid] gives

$$\Phi(r, \theta) = -\frac{GM}{r} + \frac{2}{5} \epsilon \frac{GM R_\mu^2}{r^3} P_2(\cos \theta) + O(\epsilon^2)$$

equivalently - reverting back to geodetic notation

$$\Phi(r, \phi) = -\frac{GM}{r} + \frac{2}{5} \epsilon \frac{GM R_\mu^2}{r^3} P_2(\sin \phi) + O(\epsilon^2)$$

Note there is no  $P_1(\sin \phi)$  term which is to be expected as  $P_1(\sin \phi) = \sin \phi$  and this would give us a difference in  $\Phi$  at  $+\phi$  and  $-\phi$  which wouldn't fit with the symmetry of the spheroid around the equator.

### E.1 $J_2$ term

We have an oblate (flattened) spheroid obtained by rotating an ellipse with semi-major axis  $a_e$  and semi-minor axis  $b_e$  around around it's minor axis b.

B=The moment of Inertia around b is  $\frac{2}{5} M a_e^2$

A=The moment of Inertia around a is  $\frac{1}{5} M (a_e^2 + b_e^2)$

Note also that  $J_2$  in [GeoPotential Model] is  $\frac{(B-A)}{M a^2}$  (see [GRS80]) where A and B are the principal moments of inertia of the ellipsoid. Note also that  $J_2$  is nondimensional.

$$B - A = \frac{2}{5} M a_e^2 - \frac{1}{5} M (a_e^2 + b_e^2) = \frac{1}{5} M (a_e^2 - b_e^2)$$

$$J_2 = \frac{1}{5 a_e^2} (a_e^2 - b_e^2) = \frac{1}{5 a_e^2} (a_e - b_e) (a_e + b_e) = \frac{1}{5 a_e^2} \epsilon R_\mu (a_e + b_e) = \frac{1}{5 a_e^2} \epsilon R_\mu (2 R_\mu - \epsilon R_\mu) = \frac{2}{5 a_e^2} \epsilon R_\mu^2 + O(\epsilon^2) \simeq \frac{2}{5} \epsilon$$

## F Mean Radius

This text uses  $R_\mu$  - the "mean radius" of an ellipsoid (note - not ellipse). It's not obvious what the definition of "mean radius" is. From [Mean Radius Wikipedia]

"In geophysics, the International Union of Geodesy and Geophysics (IUGG) defines the mean radius to be  $R_\mu = \frac{2a+b}{3}$ "

This is the arithmetic mean of the radius i.e.  $(a + a + b)/3$ .

## G Semi-latus rectum

Since the solution for the the orbital motion due to the inverse square law is

$$\frac{1}{r} = u = \eta(1 + e \cos \lambda) \quad (40)$$

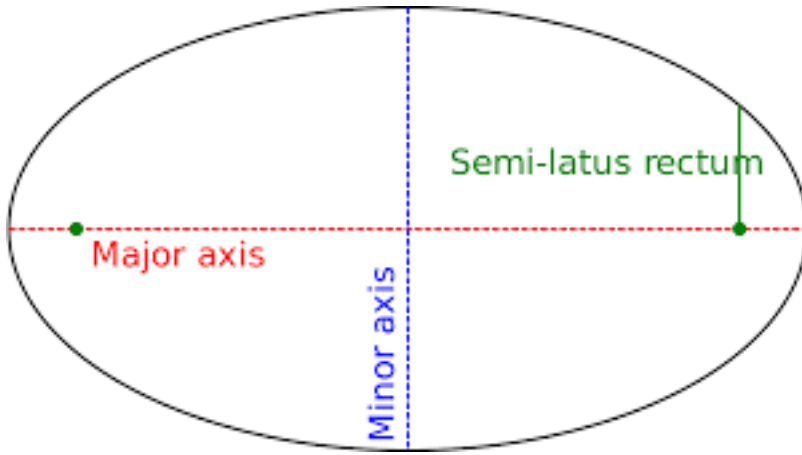
where  $\eta = \frac{GMm^2}{L^2}$ , then for  $\lambda = 0$  (the periapse),  $u = \eta(1 + e)$ . But we know at the periapse  $r = \frac{1}{u} = a(1 - e)$  i.e.  $u = \frac{1}{a(1-e)}$ . Therefore

$$\begin{aligned} \eta(1 + e) &= \frac{1}{a(1 - e)} \\ \eta &= \frac{1}{a(1 + e)(1 - e)} = \frac{1}{a(1 - e^2)} \end{aligned}$$

i.e.

$$\eta = \frac{GMm^2}{L^2} = \frac{1}{a(1 - e^2)}$$

Note the semi-latus rectum of the ellipse is given by  $l = \frac{1}{\eta} = a(1 - e^2)$ . We can see that from 40 that this is  $r(\lambda = \pi)$ .



For Mercury we have

$$\eta = \frac{1}{57,910,000 (1 - 0.2056^2)}$$

$$\eta = \frac{1}{55,462,066} = 1.803 (10^{-8}) \text{ km}^{-1}$$

## H Notation

The notation used varies across different authors/texts. Below is listed some definitions.

$a$  semi-major axis.

$b$  semi-minor axis.

Mean Radius of an ellipsoid (not ellipse)  $R_\mu = \frac{2a+b}{3}$ . see Appendix F

Eccentricity  $e = \sqrt{\frac{a^2-b^2}{a^2}}$ . This is a property of the orbiting satellite.

Flattening  $f = \frac{(a-b)}{a}$ . This is a property of the central body i.e. spheroid.

Ellipticity  $\epsilon = \frac{(a-b)}{R_\mu}$ . This is a property of the central body i.e. spheroid.

Note the previous two (flattening and ellipticity) are very similar and are equivalent to  $O(\epsilon^2)$  or  $O(f^2)$  i.e.

$$\epsilon = \frac{3(a-b)}{2a+b}$$

using  $b = a(1-f)$

$$\epsilon = \frac{3(a - (a - af))}{2a + (a - af)} = \frac{3af}{3a - af}$$

Rearrange

$$\frac{f}{\epsilon} = \frac{3a - af}{3a} = 1 - \frac{f}{3}$$

$$\epsilon = f \left(1 - \frac{f}{3}\right)^{-1} \simeq f + O(f^2)$$

The following  $r_0$  is seen in [Kaula].  $r_0 = a \left(1 - \frac{f}{3} + O(f^2)\right)$  which gives the same as  $R_\mu$

The following is used in [Kaula].  $a_e$  mean equatorial radius of the earth.

Gravitational constant  $k$  or  $G$

The following is used in [Kaula].  $\mu = kM$

## References

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