Cyclotron

7.1 Learning outcomes

Mathematics and Physics

- Second order ordinary differential equations.
- Physical Modelling
- Lorentz Force

Computing and Python

- Optimizing python code

7.2 Project Outline

So far we have studied systems for which little mathematical information was needed to derive a mathematical model. These models were derived from purely qualitative information. In this section we will study the cyclotron, a device invented by physicists to make charged particles travel at very high speeds. To model the system we will need the so called Lorentz force which describes the force an electro-magnetic field exerts on a charged particle which, once inserted in Newton's dynamic equation, will allow us to describe mathematically the trajectory of a proton.

We will start by describing the cyclotron and how it works. We will then write Newton's equation for a proton inside the cyclotron. Using well known parameters such as the proton mass and charge as well as the magnetic fields that can be created in a laboratory, we will be able to decide on the size of the cyclotron.

We will then solve the equation for the proton trajectory numerically to test that the cyclotron design works.

7.3 Introduction

The first cyclotron was created by E.O. Lawrence in 1931 to accelerate charged particles to high energies, *i.e.* high speed. (see `https://www.youtube.com/watch?v=D-TZKufqTg8`) At the time this was used to send heavy ions onto material to study the structure of atoms. Cyclotrons can only accelerate charged particles up to a few MeV (10^6eV), unlike modern particle accelerators such as the LHC at CERN which can accelerate protons up to 6.5 TeV (6.5×10^{12}eV). Cyclotrons are still used as the first acceleration stage for larger accelerators such as the LHC but they are also widely used in hospitals to create radioactive fluorine which is then injected in patients as an active organic compound. (see `https://www.youtube.com/watch?v=XpqdmGRqmw0`) The gamma rays emitted by the fluorine are then used to perform medical imaging to detect pathologies such as cancer.

7.4 Dynamics of a Charged Particle

The principle of the cyclotron is quite simple but we must first understand how a charged particle moves in an electro-magnetic field under the influence of the so called Lorentz force.

- The electric field exerts on a charged particle a force parallel to the direction of the field. The charged particle, if it is free, will then accelerate in that direction.
- The force due to the magnetic field is more complex. First of all, it does not exert any force on a charged particle at rest. When a charged particle moves at speed v in the magnetic field, it will be subject to a force which is perpendicular to both the speed and the magnetic field. The amplitude of the force is $vB\sin(\theta)$ where θ is the angle between v and B. In the setup above, where the electron is confined to a plane perpendicular to the magnetic filed, $\theta = 90°$. The net effect is that the charged particle will be deflected and, if B and $|v|$ do not vary in time, the particle will have a circular trajectory.

In what follows, we assume that the magnetic field, B, is parallel to the z axis while the electric field lies in the x, y plane. The equation of motion for a charged particle moving in that plane is then given by

$$m\ddot{x} = q(E_x + \dot{y}B_z)\,, \tag{7.1}$$

$$m\ddot{y} = q(E_y - \dot{x}B_z)\,, \tag{7.2}$$

where m and q are respectively the mass and the charge of the particle. The right hand side of the equation is the Lorentz force. We also use the *dot* to denote derivative with respect to time.

7.5 Charged Particle in a Constant Magnetic Field

If there is no electric field, Newton's equation for the particle reduces to

$$m\,\ddot{x} = q\,\dot{y}\,B_z\,, \tag{7.3}$$

$$m\,\ddot{y} = -q\,\dot{x}\,B_z\,. \tag{7.4}$$

If we multiply (7.3) by $\dot{x}$, (7.4) by $\dot{y}$ and add the 2 together we see that

$$\frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2\right) = \dot{x}\ddot{x} + \dot{y}\ddot{y} = 0. \tag{7.5}$$

This means that the norm of the speed is constant and that the only change is the direction of the speed, or in other words that the charged particle will move along a circular trajectory:

$$x = R\sin(2\pi\nu t)\,, \tag{7.6}$$

$$y = R\cos(2\pi\nu t)\,, \tag{7.7}$$

where ν is the frequency of rotation and R the radius of the trajectory. This implies that

$$\dot{x} = V\cos(2\pi\nu t)\,, \tag{7.8}$$

$$\dot{y} = -V\sin(2\pi\nu t)\,, \tag{7.9}$$

where $V = 2\pi\,R\,\nu$ is the norm of the speed vector.

Inserting (7.8) and (7.9) into (7.3) and (7.4) we obtain

$$-m2\pi\nu V\begin{pmatrix}\sin(2\pi\nu t)\\ \cos(2\pi\nu t)\end{pmatrix} = -Vq\begin{pmatrix}\sin(2\pi\nu t)\\ \cos(2\pi\nu t)\end{pmatrix}B_z\,, \tag{7.10}$$

and after cancelling out the common factors, we see that

$$\nu = \frac{q\,B_z}{2\pi m}\,, \tag{7.11}$$

which is called the cyclotron frequency. Notice that it does not depend on the speed of the charge particle and this is precisely what is exploited in the cyclotron. (See `http://cds.cern.ch/record/2315176/files/1804.08541.pdf` for a more detailed description.)

So a charge particle moving in a constant magnetic field will follow a circular trajectory with a constant period of rotation. So if the speed increases the radius of the trajectory will increase but the period will not be affected.

7.6 The Cyclotron

To accelerate the electron, the idea is to use an electric field. The cyclotron is made out of 2 metallic hollow half-circle plates, called 'D' because of their shape (see fig 7.1). They are placed in a magnetic field perpendicular to the plane of the Ds and each D is connected to the opposite ends of an oscillator. The oscillator generates a periodic electric potential difference at exactly the cyclotron frequency. This results in an electric field in the small region between the two Ds. Because the electric potential alternate, the electric field will always been oriented so that it pushes the electron in its travelling direction hence increasing its speed. This is where the cyclotron frequency is used: if it did depend on the speed of the charge particle, the frequency of the oscillator would have to be tuned to the frequency of the electron, a somewhat difficult tasks. The fact that it is constant makes the cyclotron design much simpler.

In our setup, the magnetic field, B, is constant inside the Ds as well as in the thin space between the Ds but it is zero elsewhere.

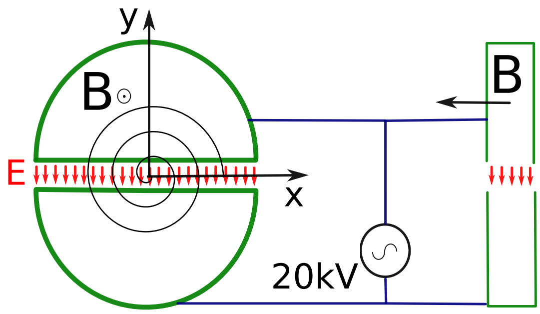


Figure 7.1: Cyclotron, (left: top view; rigght: side view) . 2 hollow Ds (green). Magnetic field B perpendicular to the plane of the Ds and Electric field E non zero only in the thin space between the Ds, zero elsewhere. A 20kV alternator creates an alternating electric potential on the two Ds, inducting the electric field between them.

To model a charged particle inside the cyclotron, we must solve equations (7.1) and (7.2) where E_x and E_y will be varying with time and be non-zero only in the small region separating the 2 Ds, while B will be constant and non-zero inside the 2 Ds and in the small space between them.

The electric field changes signs every half period, but to model the cyclotron we can just use an electric field that points downward when x is positive and upwards when it is negative. We effectively set the electric field so that it pushes the charged particle in each region.

So the magnetic and electric fields will be given by $E_x = E_z = B_x = B_y = 0$

$$B_z = \begin{cases} 0 & \text{if} \quad x^2 + y^2 > R_c^2 \\ B & \text{if} \quad x^2 + y^2 \leq R_c^2 \end{cases} \tag{7.12}$$

$$E_y = \begin{cases} -E & \text{if} \quad 0 < x < R_c, \quad -\frac{w}{2} < y < \frac{w}{2} \\ E & \text{if} \quad -R_c < x < 0, \quad -\frac{w}{2} < y < \frac{w}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{7.13}$$

$$\tag{7.14}$$

where R_c is the radius of the cyclotron and w the width of the small region between the 2 Ds.

The electron must enter the cyclotron at the point

$$x = -R_c + O_x, \quad y = 0, \quad v_x = 0, \quad v_y = 2\pi R_c \nu, \tag{7.15}$$

where O_x is an offset which we will use to better centre the spiral trajectory.

Problem 7.1:

Assuming that the charged particle is a proton

$$\begin{aligned} q &= 1.602 \times 10^{-19} C \\ m &= 1.672 \times 10^{-27} kg, \end{aligned} \tag{7.16}$$

compute the:

- The cyclotron frequency ν (7.11), assuming $B = 1$ Tesla.
- The speed of the proton if it has an initial energy of 100keV. The kinetic energy, in Joules, is given by $\mathcal{E} = mv^2/2 = 10^5\text{eV} \times q$ where q is the charge of the proton.
- What is the radius of the proton trajectory at that speed.

In SI units, the energy is measured in Joules, but for charged particles it is usually measured in electron-Volt (eV), This is the energy that an electron acquires when going across a potential difference of 1 Volt. To convert an energy from eV to Joules, one must multiply it by the electric charge of a proton (expressed in Coulomb).

7.7 Systems of ODEs

The differential equation we have to solve is second order, but we only know how to solve systems of first order differential equations. We must thus convert the system of 2 second order differential equations (7.1) and (7.2) into a system of 4 first order differential equations.

The system of equations we need to solve is of the form

$$\frac{\mathrm{d}^2\mathbf{f}(t)}{\mathrm{d}t^2} = \mathbf{F}\left(t, \mathbf{f}(t), \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t}\right), \tag{7.17}$$

where $\mathbf{f}$ and $\mathbf{F}$ are vectors. To solve this system of equations numerically, we must first convert it to a system of first order differential equations counting twice as many equations.

This can be done by defining an auxiliary function

$$\mathbf{g}(t) = \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} \tag{7.18}$$

and then rewriting (7.17) as

$$\frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} = \mathbf{g}(t), \tag{7.19}$$

$$\frac{\mathrm{d}\mathbf{g}(t)}{\mathrm{d}t} = \mathbf{F}(t, \mathbf{f}(t), \mathbf{g}(t)). \tag{7.20}$$

Notice that (7.19,7.20) and (7.17) are totally equivalent. Notice also that the function $\mathbf{F}$ depends on t, $f(t)$ and $g(t)$ but not on $df(t)/dt$.

For example if we consider the equation of a damped pendulum,

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\sin(x) - \Gamma\frac{\mathrm{d}x(t)}{\mathrm{d}t} \tag{7.21}$$

we define $v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$ and rewrite (7.21) as the following pair of equations:

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= v \\ \frac{\mathrm{d}v(t)}{\mathrm{d}t} &= \frac{1}{m}\left(\sin(x(t)) - \Gamma v(t)\right), \end{aligned} \tag{7.22}$$

where $x(t)$ and $v(t)$ are the functions we try to determine. Notice that there are no derivatives on the right hand side of the equation. Moreover we have moved the mass term, m, on the right hand side so that we only have derivative terms on the left hand side of the equations. This is required if we wish to solve the equation numerically.

Problem 7.2:

- Rewrite (7.1), (7.2) as a system of 4 first-order differential equations. Hint : **f** has 2 components: x and y, and so does **g**: $v_x = \dot{x}$ and $v_y = \dot{y}$.

Check Point 7.1

We will now solve the motion equation for a proton in the cyclotron. We will assume that the cyclotron has a radius of 1m and that the distance between the 2 Ds is $w = 2cm$. We will also assume that the magnetic field $B = 1T$ (T stands for Tesla, the magnetic field unit in SI units).

Problem 7.3:

- `cyclotron.py`: The module `cyclotron.py` defines the class `CyclotronRK4` to solve the proton equation inside the cyclotron. The class `CyclotronRK4` is a subclass of the class `ODE_RK4` used to solve system of ODEs using the 4th order Runge-Kutta method.

  The class `CyclotronRK4` is made out of a few functions:

  - `__init__()` : sets the initial conditions for the system of equations. It also defines a few class variables for the properties of the proton and the dimensions of the cyclotron.
  - `F()`: the function computes the right hand side of the equation. In this case it first computes the distance of the proton from the origin and checks if the proton is still inside the cyclotron. If it is, B is set to the value `self.B`, otherwise it is set to 0. Notice that you can't modify `self.B` as it corresponds to the value of the magnetic field where it is non zero.

    The 4 equations for the variable x, y, v_x and v_y must then be computed.
  - `draw_Ds(()`: draws the cyclotron Ds. You do not need to understand its details. It is used in the program `run_cyclotron.py` described below.

  Using the answer to problem 7.2 complete the definition of the proton equation in the class function `F()`.

- `run_cyclotron.py`:

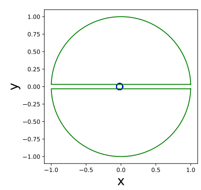
  The program `run_cyclotron.py` integrates the proton equation using the class `CyclotronRK4` defined in the module `cyclotron.py`. The initial conditions are set to correspond to a proton with an initial energy of 100keV, starting at the position $y = 0$ and $x = -r_0 + O_x$, where r_0 is the radius of the orbit of a 100keV proton in a 1 Tesla magnetic field. The initial speed is set in the y direction so that the circular or spiral orbit is more or less centred at the centre of the cyclotron. The integration time step is chosen to be much smaller than the cyclotron period and `tmax` is chosen to correspond to a few rotations. For the time being, we set `dV=0` so that there is no electric field.

  Modify the program `run_cyclotron.py` and on line 7 and 10 and write the parameter values you have computed for problem 7.1 so that the correct initial parameters are set.

  - Replace `__XXX__` by the obtained radius value.
  - Replace `__VVV__` by the obtained initial speed.

  As the system is very sensitive to the initlal conditions, you must enter values with at least 6 decimal places for x and v.

- Test : run the program `run_cyclotron.py`. It should generate a figure similar to the one in the margin. Notice the small circle at the centre: this is the trajectory of the proton. It is not yet accelerated.



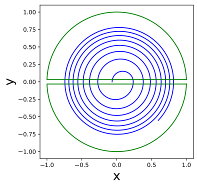
Cyclotron trajectory $V = 0$

7.8 A Realistic Cyclotron

Problem 7.4:

In real cyclotrons, the potential difference between the 2 Ds is typically 20kV, but to make the program faster and to make the trajectory easier to analyse we will take a value 100 times larger: $V = 2MeV$ So far we have set the electric field to zero to test the code.

- Inside the cyclotron, the electric field is given by (7.13). Modify the function `F()` in the class `CyclotronRK4` and add the terms corresponding to the electric field in (7.1) and (7.2). The electric field value is computed by `__init__` and stored in `self.E`. In the function `F`, you must test where the electron is and use the correct value for E_y according to (7.13). Notice that you can't modify `self.E` as it corresponds to the value of the electric field between the 2 Ds.
- In `run_cyclotron.py` line 15, set the variable `dV` to 2×10^6 (remember this is `2e6` in python).
- Run `run_cyclotron.py` and check that you get a figure similar to the one in the margin.



Cyclotron trajectory
$V = 2MeV$

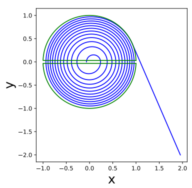
Check Point 7.2

Problem 7.5:

So far we have integrated the equation for a fixed length of time. What we would like to do is to stop the integration when the proton has moved outside a square box centred on the centre of the cyclotron and of edge size twice the cyclotron diameter.

For this, we need to replace the function `iterate` by a function called `iterate_max` which will stop the integration when the proton has moved outside the box. To do this proceed as follows:

- Cut and paste the function `iterate` from the file `ode_rk4.py` into the file `cyclotron.py` and rename the function `iterate_max`. It must also take only 1 argument: `fig_dt`, so you must remove the argument `tmax`.
- In the while loop, replace the test on `self.t` with a test on the position of the electron. Remember that x and y are available as `self.V[0]` and `self.V[1]` respectively. The radius of the cyclotron is available as `self.r_D`. Moreover pay attention to the fact that the electron can exit the cyclotron anywhere around the Ds, not just in the top right corner.
  All the tests with `fig_dt` and `next_fig_t` are there to store data values for the figures and need to be kept as they are.



Cyclotron trajectory
$V = 2MeV$

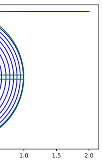
You can now make a copy of `run_cyclotron.py` and call it `run_cyclotron_max.py`. In that file, instead of calling the function `iterate(tmax)`, call `iterate_max()`. When you execute `run_cyclotron_max.py` it should generate a figure like the one in the margin.

Check Point 7.3

Problem 7.6:

At the moment, the potential difference between the 2 Ds is kept fixed at 2MeV and has a result we do no control where the proton comes out of the cyclotron. Cyclotron are shaped to allow the proton to exit at a specific place (see the diagram on https://en.wikipedia.org/wiki/Cyclotron). We will assume the exit is tangent to the cyclotron at the top, meaning, as our cyclotron has a 1m radius, that the proton should hit the point $(x = 2, y = 1)$ after exiting the cyclotron.

- Make a copy of `run_cyclotron_max.py` and call it `run_cyclotron_target.py`. Then modify it so that after integrating the equations it prints the value of the potential between the Ds (`dV`) as well as the position of the proton (V[0] and V[1] from the `ODE_RK4` class).



trajectory

- Add a loop to increase dV from $2.116 \times 10^6 eV$ by steps of 100eV until the proton nearly hits the target point $(2, 1)$. Take note of the value of dV which gets the proton the nearest to the target point.
- At the bottom of the file add some code to compute the proton trajectory using the best value of dV you have found and plot the trajectory of that proton.
- The output of run_cyclotron_target.py must then be the final proton coordinate for the different values of dV and then generate a figure but the best trajectory.

Check Point 7.4

Problem 7.7:

- Make a copy of run_cyclotron_max.py and call it run_cyclotron_max_b.py.
- Replace the value of dV with the value you have found from problem 7.6.
- At the bottom of the program print the initial speed of the proton.
- On a separate line, print the total speed of the proton when it reach the end point. (Remember the speed is available as self.V[2] and self.V[3] in the class CyclotronRK4).
- The initial energy of the proton was $0.1 MeV$. What is its final kinetic energy given that it is proportional to v^2? Print the value in MeV as an extra output.

Problem 7.8:

The reason cyclotron were build by particle physicists was to send charged particles on atoms to break them apart ans analyse the by product of the collisions. The more kinetic energy the particle has, the better.

So far we have considered only a proton but one can also accelerate charged ions which will always be more massive. Assuming an ion has the same charge as a proton and that it is accelerated by the same cyclotron, would it have a larger, smaller or identical kinetic energy as an accelerated proton?

