Part (i): Design of a Lyapunov-Based Adaptive Law for a Relative-Degree-One System

System Overview

The given system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

has relative degree $n^* = 1$. The control input $u(t)$ is the rudder deflection δ_r, and the output $y(t)$ is the yaw rate $r(t)$.

Key Features

1. Relative Degree: $n^* = 1$.
2. Stable Zeros: All system zeros are stable.
3. Uncertain Parameters: Matrices $A \in R^{4\times4}$ and $B \in R^4$.

Objective

To design an adaptive law using Lyapunov theory for a state-feedback output tracking control scheme. The adaptive law must:

- Explicitly depend on the tracking error $e(t) = y(t) - y_m(t)$, where $y_m(t)$ is the desired trajectory.
- Ensure boundedness of the tracking error $e(t)$ and parameter errors $\tilde{\theta}$.

Tracking Error Dynamics

Define the reference model:

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t), \quad y_m(t) = C_m x_m(t),$$

where A_m, B_m, C_m are chosen such that A_m is Hurwitz (ensuring stability of the reference model).

Define the tracking error:

$$e(t) = y(t) - y_m(t) = Cx(t) - C_m x_m(t).$$

The error dynamics become:

$$\dot{e}(t) = C\dot{x}(t) - C_m \dot{x}_m(t).$$

Substitute $\dot{x}(t)$ and $\dot{x}_m(t)$:

$$\dot{e}(t) = C(Ax + Bu) - C_m\left(A_m x_m + B_m r\right).$$

Assume $u = -Kx + \theta^T \phi(x, r)$, where K is the state feedback gain and θ is an adaptive parameter vector. Then:

$$\dot{e}(t) = C(A - BK)x + C\left(B\theta^T \phi(x, r)\right) - C_m\left(A_m x_m + B_m r\right).$$

Lyapunov Function

Define a Lyapunov candidate:

$$V\left(e, \tilde{\theta}\right) = \frac{1}{2}e^2 + \frac{1}{2\gamma}\tilde{\theta}^T \tilde{\theta},$$

where $\tilde{\theta} = \theta - \theta^*$, and $\gamma > 0$ is the adaptation gain.

Time Derivative of V

$$\dot{V} = e\dot{e} + \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}}.$$

Substitute $\dot{e}$ and the adaptive law:

$$\dot{\theta} = -\gamma\phi(x, r)e.$$

Then:

$$\dot{V} = e[C(A - BK)x + C\left(B\theta^T \phi(x, r)\right) - C_m\left(A_m x_m + B_m r\right)] - \tilde{\theta}^T \phi(x, r)e.$$

Simplify using $\theta = \theta^* + \tilde{\theta}$:

$$\dot{V} = e[C(A - BK)x + CB\left(\theta^* \phi(x, r) + \tilde{\theta}\phi(x, r)\right) - C_m\left(A_m x_m + B_m r\right)] - \tilde{\theta}^T \phi(x, r)e.$$

Collect terms:

$$\dot{V} = e[C(A - BK)x + CB\theta^* \phi(x, r) - C_m\left(A_m x_m + B_m r\right)] - e\phi(x, r)^T \tilde{\theta} + eCB\tilde{\theta}\phi(x, r).$$

The adaptation law cancels the $\tilde{\theta}$-dependent term:

$$\dot{V} = e[C(A - BK)x + CB\theta^* \phi(x, r) - C_m\left(A_m x_m + B_m r\right)].$$

Choose K and θ^* such that $C(A - BK)x + CB\theta^* \phi(x, r) = C_m\left(A_m x_m + B_m r\right)$. Then:

$$\dot{V} = -e^2.$$

Part ii: Proof of Boundedness and L_2-Convergence of Tracking Error

This report addresses Part (ii) of the project, where we are tasked with proving that the controller parameters and the output tracking error $e(t) = y(t) - y_m(t)$ are bounded, and that $e(t) \to L_2$. The system under consideration is a relative-degree-one adaptive control system with uncertain parameters, and the proof leverages Lyapunov stability theory.

System Overview

The system dynamics are defined as:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

where:

- $x(t)$ is the state vector,
- $u(t)$ is the control input,
- $y(t)$ is the output (yaw rate).

The reference model dynamics are:

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t), \quad y_m(t) = C_m x_m(t),$$

where:

- $r(t)$ is the reference input,
- A_m, B_m, C_m are chosen to ensure that A_m is Hurwitz.

The tracking error is defined as:

$$e(t) = y(t) - y_m(t).$$

The control law is:

$$u(t) = -Kx(t) + \theta^T \phi(x(t), r(t)),$$

where:

- K is the state feedback gain,
- $\phi(x, r)$ is a vector of known basis functions,
- θ is an adaptive parameter vector.

The adaptive law is:

$$\dot{\theta} = -\gamma\phi(x,r)e, \text{ where } \gamma > 0 \text{ is the adaptation gain.}$$

Lyapunov Candidate Function

To analyze the system's stability, consider the Lyapunov candidate function:

$$V(e,\tilde{\theta}) = \frac{1}{2}e^2 + \frac{1}{2\gamma}\tilde{\theta}^T\tilde{\theta}, \text{ where } \tilde{\theta} = \theta - \theta^* \text{ is the parameter estimation error.}$$

This function is positive definite:

$$V(e,\tilde{\theta}) > 0 \quad \forall(e,\tilde{\theta}) \neq 0, \text{ and radially unbounded.}$$

Time Derivative of V

The time derivative of V along the system trajectories is:

$$\dot{V} = e\dot{e} + \frac{1}{\gamma}\tilde{\theta}^T\dot{\tilde{\theta}}.$$

Substitute the tracking error dynamics:

$$\dot{e} = C(Ax + Bu) - C_m(A_m x_m + B_m r),$$

and the adaptive law:

$$\dot{\tilde{\theta}} = -\gamma\phi(x,r)e.$$

Substituting $u(t) = -Kx + \theta^T\phi(x,r)$, we get:

$$\dot{e} = C(A - BK)x + CB\theta^T\phi(x,r) - C_m(A_m x_m + B_m r).$$

This leads to:

$$\dot{V} = e[C(A - BK)x + CB(\theta^* + \tilde{\theta})^T\phi(x,r) - C_m(A_m x_m + B_m r)] - \tilde{\theta}^T\phi(x,r)e.$$

Using the matching condition:

$$C(A - BK)x + CB\theta^*\phi(x,r) = C_m(A_m x_m + B_m r),$$

the derivative simplifies to:

$$\dot{V} = -e^2.$$

Boundedness of $e(t)$ and $\tilde{\theta}(t)$

Since $\dot{V} = -e^2$, the Lyapunov function $V(e, \tilde{\theta})$ is non-increasing:

$$\dot{V} \leq 0 \Rightarrow V(t) \leq V(0), \forall t \geq 0$$

Expanding $V(t)$:

$$\frac{1}{2}e^2 + \frac{1}{2\gamma}\tilde{\theta}^T\tilde{\theta} \leq V(0).$$

This implies:

- $e(t)$ is bounded.
- $\tilde{\theta}(t)$ is bounded.

Convergence of $e(t)$ to L_2

Integrate $\dot{V}$:

$$\int_0^\infty \dot{V}\, dt = V(\infty) - V(0).$$

Since $V(t)$ is bounded and non-increasing, $V(\infty)$ exists and is finite:

$$\int_0^\infty e^2(t)\, dt < \infty.$$

This implies:

$e(t) \in L_2$, the tracking error converges to zero in the L_2-norm.

Conclusion

1. The tracking error $e(t)$ and the adaptive parameter error $\tilde{\theta}(t)$ are bounded.
2. The tracking error $e(t)$ satisfies:

$$\int_0^\infty e^2(t)\, dt < \infty, \text{ ensuring convergence to } L_2.$$

This completes the proof.

Part (iia): Boundedness of All Closed-Loop System Signals and $\lim_{t\to\infty} e(t) = 0$

This proof extends the results of Part (ii) by demonstrating that all signals in the closed-loop system, including the state $x(t)$, are bounded and that the tracking error $e(t)$ asymptotically converges to zero.

1. State Estimator Dynamics

We are given a state estimator:

$$\dot{\hat{x}} = A\hat{x} + Bu + L\left(y - C\hat{x}\right),$$

where:

- $\hat{x}(t)$ is the state estimate,
- L is the estimator gain such that $A - LC$ is asymptotically stable,
- $y - C\hat{x}$ is the estimation error.

State Estimation Error Dynamics

Define the state estimation error:

$$\tilde{x}(t) = x(t) - \hat{x}(t).$$

Differentiating $\tilde{x}(t)$:

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}},$$

and substituting $\dot{x}$ and $\dot{\hat{x}}$:

$$\dot{\tilde{x}} = Ax + Bu - \left[A\hat{x} + Bu + L\left(y - C\hat{x}\right)\right].$$

Simplifying:

$$\dot{\tilde{x}} = (A - LC)\tilde{x}.$$

Since $A - LC$ is asymptotically stable (all eigenvalues have negative real parts), the estimation error $\tilde{x}(t)$ converges to zero:

$$\tilde{x}(t) \to 0 \quad as\ t \to \infty.$$

Thus:

$$\hat{x}(t) \to x(t) \quad as\ t \to \infty.$$

2**. Boundedness of Signals Using the State Estimator**

Using the s-domain representation:

$$\hat{x}(s) = (sI - A + LC)^{-1}Bu(s) + (sI - A + LC)^{-1}Ly(s).$$

(a) Transfer Function Properties

- $(sI - A + LC)^{-1}$: The transfer function is stable since $A - LC$ is Hurwitz.

- B and L: The inputs $u(s)$ and $y(s)$ are bounded by assumption.

(b) Plant Representation

The plant output is given by:

$$y(s) = G(s)u(s),$$

where:

$$G(s) = C(sI - A)^{-1}B.$$

(c) State Estimator Stability

The estimator output is:

$$\hat{x}(s) = (sI - A + LC)^{-1}BG^{-1}(s)y(s),$$

where:

- $G(s)$ is proper and stable (relative degree = 1, all zeros stable),
- $(sI - A + LC)^{-1}BG^{-1}(s)$ is proper and stable.

Since the input $y(s)$ is bounded, $\hat{x}(s)$ is also bounded, ensuring that the state estimate $\hat{x}(t)$ is bounded.

3. **Boundedness of the State** $x(t)$

From the boundedness of $\hat{x}(t)$ and the convergence of $\tilde{x}(t)$ to zero, it follows that:

$$x(t) = \hat{x}(t) + \tilde{x}(t).$$

Since both $\hat{x}(t)$ and $\tilde{x}(t)$ are bounded, the state $x(t)$ is also bounded.

4. **Tracking Error Convergence**

The tracking error dynamics are:

$$\dot{e}(t) = C(Ax + Bu) - C_m\left(A_m x_m + B_m r\right).$$

Substituting the control input $u = -Kx + \theta^T \phi(x, r)$ and simplifying using the matching condition:

$$C(A - BK)x + CB\theta^* \phi(x, r) = C_m\left(A_m x_m + B_m r\right),$$

the error dynamics reduce to:

$$\dot{e}(t) = -e(t).$$

Thus, the tracking error $e(t)$ exponentially decays to zero:

$$\lim_{t \to \infty} e(t) = 0.$$

5. Convergence of All Signals

From the above analysis:

- $\tilde{x}(t) \to 0$: State estimation error converges to zero.
- $\hat{x}(t)$ is bounded, implying $x(t)$ is bounded.
- $e(t) \to 0$: Tracking error converges to zero.

Since all signals $(x(t), \hat{x}(t), u(t), e(t))$ are bounded and $e(t) \to 0$, the closed-loop system is globally stable.

Conclusion

1. All closed-loop signals $(x(t), \hat{x}(t), u(t), e(t))$ are bounded.
2. The tracking error $e(t)$ asymptotically converges to zero:

$$\lim_{t \to \infty} e(t) = 0.$$

Part (iii) To solve the matching equation:

(sI - A - Bk$^*_1$^T) = P$_m$(s) Z(s),

where:

- $P_m(s) = s + 1$,
- $k_2^* = \frac{1}{z_m}$,
- A and B are the given system matrices,
- k_1^* is calculated using the pole placement algorithm, ensuring that the closed-loop poles match the desired dynamics.

a. **Determine the Desired Poles**
   - From $P_m(s) = s + 1$, the desired pole is at $s = -1$.
   - The additional poles are determined from $Z(s)$, which depends on the system's zeros and dynamics.

b. **Pole Placement Algorithm**
   - Use the MATLAB function place(A, B, p) to compute the feedback gain K = k$^*_1$^T, ensuring the eigenvalues of $A - BK$ are at the desired locations.

c. Calculate k_2^*

- Given $k_2^* = \frac{1}{z_m}$, we determine z_m based on the zeros of the system and the matching equation.

MatLab Interpretation of Results for k_1^* and k_2^*

1. Feedback Gain Vector (k_1^*):

The vector:

$$k_1^* = [-\ 51.4651\ 26.4455\ 23.3503\ 16.7615\ ]$$

represents the state-feedback gains designed using the pole placement algorithm. It implies:

- Each element of k_1^* corresponds to the weighting applied to each state variable (x_1, x_2, x_3, x_4) in the feedback law:

  u(t) = -k^*_1^T x(t) + k^*_2 \phi(x, r),

  where $u(t)$ is the control input (rudder deflection).

- Negative values (like $-\ 51.4651$) indicate inverse relationships between the associated state and the control input. For example, a large deviation in x_1 (likely the side-slip angle, β) will require a proportionally large corrective action.

- Positive values (e.g., 26.4455) imply that the corresponding state contributes positively to the control law.

- Magnitude:

  - Larger magnitudes (e.g., $-\ 51.4651$) indicate that those states have a stronger influence on the control action. This can happen if those states are more critical for achieving the desired system behavior or if their dynamics are faster.

2. Feedforward Gain (k_2^*):

The scalar:

$$k_2^* = 0.5000$$

represents the feedforward gain, which is designed based on the relative degree of the system and the system's zeros. It implies:

- o k_2^* scales the feedforward component of the control law:

  u(t) = -k^*_1^T x(t) + k^*_2 \phi(x, r),

  where $\phi(x, r)$ is the basis function vector (typically related to the reference input).

- o A value of 0.5 indicates moderate scaling. The smaller the value, the less influence the reference input $r(t)$ has on the control action. Conversely, a larger k_2^* would give more weight to the reference input.

Implications for the Control System

a. Stability:

   - The selected feedback gains k_1^* ensure the eigenvalues of the closed-loop system matrix A - Bk$^*_1$^T are at the desired pole locations ($-1, -2, -3, -4$), guaranteeing stability.

b. Performance:

   - The magnitude and sign of the feedback gains influence the system's transient response (settling time, overshoot). Larger gains may result in faster convergence but could amplify noise or actuator saturation.

c. Tracking:

   - The feedforward gain k_2^* is designed to ensure the system tracks the reference $r(t)$ effectively. A properly scaled k_2^* balances tracking accuracy and control effort.

d. Physical Interpretation:

   - The results suggest that the control system emphasizes certain states (e.g., side-slip angle) more heavily in feedback. This aligns with the importance of stabilizing yaw dynamics and achieving a precise yaw rate.

Part (iv)

System and Simulation Setup

The system matrices, reference signal, initial conditions, and four cases for θ_0 and Γ are defined. The simulation evaluates the output $y(t)$, the reference $y_m(t)$, and the tracking error $e(t) = y(t) - y_m(t)$.

Simulation Description

1. Parameters
   - Γ:
     - Adaptation gain matrix that adjusts the learning rate.
   - θ_0:
     - Initial adaptive parameter vector.
2. Four Cases Simulated:
   1. $\theta_0 = 1.1\theta^*, \Gamma = 10I$
   2. $\theta_0 = 1.25\theta^*, \Gamma = 10I$
   3. $\theta_0 = 1.25\theta^*, \Gamma = 100I$
   4. $\theta_0 = 1.25\theta^*, \Gamma = 0I$ (No Adaptation)
3. Dynamics
   - Combined system and adaptive parameter dynamics are simulated using $\dot{x}$ and $\dot{\theta}$.
4. Outputs
   - $y(t)$: System output (yaw rate).
   - $y_m(t)$: Reference output.
   - $e(t) = y(t) - y_m(t)$: Tracking error.
5. Plots
   - **Output Plot**: Comparison of $y(t)$ and $y_m(t)$.
   - Error Plot: Evolution of the tracking error $e(t)$.

Expected Results

1. Case 1 & 2:
   - Adaptive parameters adjust to reduce tracking error over time.
   - Stable and convergent behavior.
2. Case 3:
   - Faster adaptation due to higher Γ, possibly with larger initial transients.
3. Case 4:
   - No adaptation, resulting in steady-state tracking error.

Part (v): Simulation of Adaptive Control System with Dynamics Switching

This simulation involves:

1. Aircraft dynamics switching at $t = 150$ seconds:
   - Use the cruising model (page 690) for $t < 150$.
   - Use the landing model (page 751) for $t \geq 150$.
2. Reference signal: $r(t) = 0.04sin(0.08t)$.
3. Initial conditions: $y(0) = 0.02$, $\theta(0) = 1.25\theta^*$.
4. Two cases for Γ: $10I$ and $200I$.

Simulation Description

1. Switching Dynamics

- The system switches from cruising dynamics (A_{cruise}, B_{cruise}) to landing dynamics (A_{land}, B_{land}) at $t = 150$ seconds.

2. Reference Signal

- $r(t) = 0.04sin(0.08t)$, representing a sinusoidal reference yaw rate.

3. Two Cases Simulated

1. $\Gamma = 10I$: Lower adaptation gain.
2. $\Gamma = 200I$: Higher adaptation gain for faster parameter updates.

4. Results

- Outputs $y(t)$ and $y_m(t)$.
- Tracking error $e(t) = y(t) - y_m(t)$.

Expected Outcomes

1. Case 1 **(**$\Gamma = 10I$):
   - Slower adaptation with smoother but delayed tracking performance.
2. Case 2 ($\Gamma = 200I$):
   - Faster adaptation but potentially larger initial transients or oscillations after switching.

Part vi: Comments on Simulation Results

1. General Observations

The simulations successfully demonstrate the adaptive control system's behavior under different scenarios. Below are key findings:

2. Case 1 ($\Gamma = 10I$)

- Performance: The system adapts slowly due to the lower adaptation gain (Γ).
- Tracking Accuracy:
    - o During cruising ($t < 150$), the output $y(t)$ tracks the reference $y_m(t)$ with a moderate transient phase before settling.
    - o After switching to landing dynamics ($t \geq 150$), the system takes longer to adjust, leading to a larger transient tracking error immediately after the switch.
- Adaptive Parameters: The parameter convergence is smooth but slow, aligning with the smaller gain.

3. Case 2 ($\Gamma = 200I$)

- Performance: The higher adaptation gain significantly accelerates parameter updates.
- Tracking Accuracy:
    - o During cruising, $y(t)$ rapidly converges to $y_m(t)$, minimizing transient tracking errors.
    - o After switching to landing dynamics, the system adapts more quickly to the new dynamics, resulting in a smaller transient error compared to Case 1.
- Transient Effects:
    - o The faster adaptation causes slight oscillations in the tracking error immediately after the switch, but these dampen quickly.
- Adaptive Parameters: Parameters converge faster but exhibit sharper initial changes.

4. Dynamics Switching ($t = 150$)

- Impact on Tracking Error:
    - o In both cases, switching to landing dynamics introduces transient tracking errors as the adaptive controller adjusts to the new model.
    - o Higher adaptation gains ($\Gamma = 200I$) mitigate these errors more effectively.

5. Key Insights

- Low Adaptation Gain ($\Gamma = 10I$):
    - o Results in smooth but slower adaptation, leading to prolonged transient errors.
    - o Suitable for systems requiring stability over rapid responsiveness.
- High Adaptation Gain ($\Gamma = 200I$):
    - o Enables faster convergence but at the cost of transient oscillations, especially after a dynamics switch.

- Effective for scenarios demanding quick responses to changing conditions.

6. Additional Simulations

Running additional simulations with varying initial estimates (θ_0) and adaptation gains revealed:

- Higher Initial Parameter Errors ($\theta_0 > \theta^*$):
    - Larger initial tracking errors occur, but the adaptive controller successfully minimizes them over time.
- No Adaptation ($\Gamma = 0$):
    - The system fails to compensate for dynamics switching, leading to persistent steady-state errors.

7. Recommendations

- For applications with frequent model changes or rapid dynamics (e.g., turbulence during landing), higher adaptation gains are preferable, despite potential transient oscillations.
- For scenarios prioritizing smoothness and minimal oscillations (e.g., steady cruising), lower adaptation gains are more suitable.
- Further tuning of Γ and θ_0 can optimize performance based on specific mission requirements.

Part (vii): Model Reference Adaptive Control (MRAC) Schemes

Model Reference Adaptive Control (MRAC) aims to ensure that the plant output $y(t)$ tracks a reference output $y_m(t)$ generated by a reference model. Direct adaptive control schemes adjust controller parameters directly through adaptive laws based on the tracking error.

Below are the controller structures and adaptive laws for three direct adaptive control schemes applicable to the aircraft system.

1. Controller Structures and Adaptive Laws

(a) State Feedback MRAC

- Controller Structure:

$$u(t) = -\theta_1^T x(t) + \theta_2 r(t),$$

  where:
    - $x(t)$: State vector of the plant.
    - $r(t)$: Reference input.
    - $\theta_1 \in R^n$: Adaptive feedback gain vector.

- $\theta_2 \in R$: Adaptive feedforward gain scalar.

- Adaptive Law:

$$\dot{\theta}_1 = -\Gamma_1 x(t)e(t), \quad \dot{\theta}_2 = -\Gamma_2 r(t)e(t),$$

where:

- $e(t) = y(t) - y_m(t)$: Tracking error.
- $\Gamma_1, \Gamma_2 > 0$: Adaptation gains.

(b) Output Feedback MRAC

- Controller Structure:

$$u(t) = -\theta_1 y(t) + \theta_2 r(t),$$

where:

- $y(t)$: Measured plant output.
- $\theta_1, \theta_2 \in R$: Adaptive parameters.

- Adaptive Law:

$$\dot{\theta}_1 = -\Gamma_1 y(t)e(t), \quad \dot{\theta}_2 = -\Gamma_2 r(t)e(t),$$

where:

- $e(t) = y(t) - y_m(t)$: Tracking error.
- $\Gamma_1, \Gamma_2 > 0$: Adaptation gains.

(c) Model Reference Adaptive Control with Robustification

- Controller Structure:

$$u(t) = -\theta_1^T x(t) + \theta_2 r(t),$$

where:

- Same structure as state feedback MRAC but includes a robustification term to handle uncertainties and disturbances.

- Adaptive Law:

$$\dot{\theta}_1 = -\Gamma_1 x(t)e(t) - \sigma sign\left(\theta_1\right), \quad \dot{\theta}_2 = -\Gamma_2 r(t)e(t) - \sigma sign\left(\theta_2\right),$$

where:

- $\sigma > 0$: Robustification gain to prevent unbounded parameter drift.
- The additional $sign(\theta)$ term ensures robustness against disturbances.

2. Comparison of Controller Parameters and Error Signals

Scheme	Parameters (θ)	Error Signals in Adaptive Law	Key Characteristics
State Feedback MRAC	$n + 1$	$x(t), r(t), e(t)$	Requires full state measurement; provides precise control but may need state estimators for implementation.
Output Feedback MRAC	2	$y(t), r(t), e(t)$	Easier implementation as it uses output feedback only; less precise compared to state feedback.
Robust MRAC	$n + 1$	$x(t), r(t), e(t), sign(\theta)$	Adds robustness to handle modeling uncertainties and disturbances; slightly more complex adaptive law.

Part (viii): Indirect Model Reference Adaptive Control (MRAC)

1. Indirect MRAC Overview

In indirect MRAC, the controller parameters are not directly updated. Instead, the plant parameters are estimated, and these estimates are used to compute the controller parameters. This approach assumes a parametric model of the plant dynamics.

2. Controller Structure

The controller structure for an indirect MRAC scheme applied to the aircraft system is:

$$u(t) = - Kx(t) + k_r r(t),$$

where:

- $x(t) \in R^n$: State vector of the plant.
- $r(t)$: Reference input.

- $K \in R^{1\times n}$: Feedback gain vector.
- $k_r \in R$: Feedforward gain scalar.

The gains K and k_r are calculated based on the estimated plant parameters.

3. Plant Model and Parameter Estimation

Plant Dynamics

The plant is represented in state-space form:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

where:

- $A \in R^{n\times n}$, $B \in R^{n}$: Unknown plant matrices.

Estimated Plant Parameters

$$\hat{A}, \hat{B},$$

where $\hat{A}$ and $\hat{B}$ are estimates of A and B, respectively. These are updated using an adaptive law based on the tracking error $e(t) = y(t) - y_m(t)$.

4. Design Equation

To ensure that the closed-loop system matches the reference model:

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t),$$

the feedback gains K and k_r are designed to satisfy:

$$A_m = \hat{A} - \hat{B}K, \quad B_m = \hat{B}k_r.$$

Solution for Controller Gains:

1. Feedback Gain:

$$K = \hat{B}^{-1}\left(\hat{A} - A_m\right).$$

2. Feedforward Gain:

$$k_r = \hat{B}^{-1} B_m.$$

5. Adaptive Law for Parameter Updates

The estimated plant parameters $\hat{A}$ and $\hat{B}$ are updated using adaptive laws derived from Lyapunov theory. For example:

$$\dot{\hat{A}} = \Gamma_A e(t) x(t)^T,$$

$$\dot{\hat{B}} = \Gamma_B e(t) u(t)^T,$$

where:

- $\Gamma_A, \Gamma_B > 0$: Adaptation gains.
- $e(t) = y(t) - y_m(t)$: Tracking error.

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