11月22日作业　　林子开　　21307110161

1

1. Let $D \in \mathbb{R}^{n\times n}$ be diagonal with distinct eigenvalues, $z \in \mathbb{R}^n$ be a vector with no zero entries, and $\rho \in \mathbb{R}\setminus\{0\}$. Suppose that (λ, u) is an eigenpair of $D+\rho zz^\top$. Show that $\lambda I - D$ is nonsingular, and $z^\top u \neq 0$.

解：λ, u 满足 $(D+\rho zz^T)u = u\lambda$

若 $z^Tu=0$　则 $Du=u\lambda$

即 λ 也是 D 的特征值

由于 D 是对角阵，则 λ 必为 D 的某一对角元 d_i

即 $Du = u\cdot d_i$

又由于 D 的对角元互不相同，则 u 在不是 i 的位置上的分量只能为零；而 u 是非零向量，故 i 分量不为零

$\therefore u = k\cdot e_i$　其中 $k\neq 0$

$\therefore 0 = z^T u = k z^T e_i$

而 z 的所有分量均不为零，则 $kz^Te_i\neq 0$

矛盾！　$\therefore z^Tu\neq 0$

下证 $D-\lambda I$ 非奇异

假设 $D-\lambda I$ 奇异，由于 $D-\lambda I$ 是对角阵，则由 $\det(D-\lambda I)=0$ 可知 $D-\lambda I$ 必有一个零对角元

不妨设 $d_i-\lambda=0$ 即 $D-\lambda I$ 的第 i 个对角元为 0

$\Rightarrow \quad e^i(D-\lambda I)=0$

$\Rightarrow 0=e^i(D-\lambda I)u=e^i(-\rho zz^Tu)$

其中，e^iz 与 z^Tu 均为标量，故还可以写成

$-\rho z^Tu e^iz=0$

注意到 $z^Tu\neq 0$，且 $e^iz\neq 0$，$\rho\neq 0$

则 $-\rho z^Tu e^iz\neq 0$ 矛盾！

$\therefore$ $D-\lambda I$ 可逆，也即 $\lambda I-D$ 可逆

2

2. Let $D = \mathrm{diag}\{d_1, \ldots, d_n\}$ be a real diagonal matrix. Let $\alpha_1, \ldots, \alpha_n$ be real scalars that satisfy

$$d_n < \alpha_n < \cdots < d_i < \alpha_i < d_{i-1} < \alpha_{i-1} < \cdots < d_1 < \alpha_1.$$

Show that α_i's are exact eigenvalues of $D + uu^\top$, where entries of the real vector u are defined by

$$u_i = \left(\frac{\prod_{1\le j\le n}(\alpha_j - d_i)}{\prod_{1\le j\le n, j\ne i}(d_j - d_i)} \right)^{1/2}, \qquad (1 \le i \le n).$$

$D+uu^T$ 的特征多项式可以写成

$$\det(D+uu^T-\lambda I) = \prod_{j=1} (\alpha_j - \lambda) \quad (1)$$

又由 Schur 降阶公式:

$$\left[\begin{array}{c|c} D-\lambda I & u \\ \hline u^T & -1 \end{array}\right]$$

$$(-1)\det(D-\lambda I - u(-1)u^T) = \det(D-\lambda I)\det(-1-u^T(D-\lambda I)^{-1}u)$$

$$\Rightarrow \det(D-\lambda I+uu^T) = \det(D-\lambda I)\cdot(1+u^T(D-\lambda I)^{-1}u)$$

$$\Rightarrow \det(D-\lambda I+uu^T) = \left(\prod_{j=1}^n (d_j-\lambda)\right)\cdot\left(1+\sum_{j=1}^n \frac{|u_j|^2}{(d_j-\lambda)}\right)$$

$$= \left(\prod_{j=1}^n (d_j-\lambda)\right)\left(1+\sum_{\substack{j=1\\ j\ne i}}^n \frac{|u_j|^2}{(d_j-\lambda)}\right) + \prod_{\substack{j=1\\ j\ne i}}^n (d_j-\lambda)\,|u_i|^2 \quad \cdots\cdots (2)$$

令 $\lambda = d_i$ 得到:

$$\prod_{\substack{j=1\\ j\ne i}}^n (d_j - d_i)\,|u_i|^2 \quad \cdots\cdots (3)$$

在(1)式中同样令 $a = d_i$ 得

$$\prod_{j=1}^{n} (a_j - d_i) \quad \cdots\cdots (4)$$

由 (3) = (4) 可知

$$|u_i|^2 \cdot \prod_{\substack{j=1 \\ j\neq i}}^{n} (d_j - d_i) = \prod_{j=1}^{n} (a_j - d_i)$$

$$\Rightarrow \quad |u_i|^2 = \frac{\prod_{j=1}^{n} (a_j - d_i)}{\prod_{\substack{j=1 \\ j\neq i}}^{n} (d_j - d_i)} \quad \cdots\cdots (5)$$

由于 $d_n < a_n < d_{n-1} < a_{n-1} < \cdots < d_1 < a_1$

则 $\prod_{j=1}^{n} (a_j - d_i)$ 中有 i 个正项

有 $(n-i)$ 个负项

$\prod_{\substack{j=1 \\ j\neq i}}^{n} (d_j - d_i)$ 中有 $(i-1)$ 个正项

有 $(n-i)$ 个负项

∴ (5)式右端必为正

$$\therefore \quad |u_i| = \left[\frac{\prod_{1\le j\le n} (a_j - d_i)}{\prod_{\substack{1\le j\le n \\ j\ne i}} (d_j - d_i)} \right]^{1/2}$$

3

3. Let A and E be Hermitian matrices with $AE = EA$. Try to give an upper bound on

$$\|\exp(A+E) - \exp(A)\|_2.$$

Make sure your upper bound tends to zero when $\|E\|_2 \to 0$.

解：A 与 E 对角化：

$A = Q_1 \Lambda_1 Q_1^*$ $\quad E = Q_2 \Lambda_2 Q_2^*$

且 $\|E\|_2 = \|\Lambda_2\|$

则 $\|E\|_2 \to 0$ 时 $\|\Lambda_2\| \to 0$

此时 $e^{\Lambda_2} = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix} \to \begin{bmatrix} e^0 & & & \\ & e^0 & & \\ & & \ddots & \\ & & & e^0 \end{bmatrix}$

即 $\|E\|_2 \to 0 \Rightarrow e^{\Lambda_2} \to I$

$\|e^{A+E} - e^A\|_2 = \|e^A(e^E - I)\|_2$

$$= \| Q_1 e^{\Lambda_1} Q_1^* \cdot (Q_2 e^{\Lambda_2} Q_2^* - I) \|_2$$

$$= \| Q_1 e^{\Lambda_1} Q_1^* \cdot Q_2 (e^{\Lambda_2} - I) Q_2^* \|_2$$

$$\underline{\le \| e^{\Lambda_1} \|_2 \cdot \| e^{\Lambda_2} - I \|_2}$$

当 $\|E\|_2 \to 0$, $e^{\Lambda_2} - I \to 0 \Rightarrow \| e^{\Lambda_2} - I \|_2 \to 0$

即 $\| e^{\Lambda_1} \|_2 \| e^{\Lambda_2} - I \|_2 \to 0$

4

4. Implement the scaling-and-squaring algorithm (combined with truncated Taylor series) for computing the matrix exponential. Test the accuracy of your algorithm by a few diagonalizable matrices with known spectral decomposition.

(optional) Implement the Schur–Parlett algorithm and compare the accuracy.

第四题

Matlab 源码

```
%% test the algorithm
max_n=50;
depature = zeros(1,max_n);
for n = 1:max_n
    U = orth(rand(n,n));
    A = U*diag(10*rand(1,n))*U';
    exp_A_by_function = matrix_exponential(A);
    exp_A = U*exp(diag(1:n))*U';
    depature(1,n) = norm(exp_A_by_function-exp_A,'fro');
end
figure()
plot([1:max_n],log10(depature),'--o');
xlabel("矩阵阶数")
ylabel("log10(depature)")
[t,s] = title('误差（取对数）','矩阵阶数：1~50')
%% implement the scaling_and_squaring algorithm for computing the matrix
exponential
% (combined with truncated Taylor series)
function exp_M = matrix_exponential(M)
n = size(M);
n = n(1,1);
e = eig(M);
max_eigenvalue = max(e);
k = ceil(log2(max_eigenvalue/0.001));
M_scaled = M/(2^k);
exp_M_scaled = eye(n) + M_scaled + (1/2)*M_scaled*M_scaled +
(1/6)*M_scaled*M_scaled*M_scaled +
(1/24)*M_scaled*M_scaled*M_scaled*M_scaled;% truncated Taylor series
exp_M = exp_M_scaled;
for i = 1:k
    exp_M = exp_M*exp_M;
end
end
```

试验结果