11月15日作业 林子开 21307110161 第1-3题

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1. Let (λ, x) be a normalized eigenpair of a Hermitian matrix A. Suppose that $\hat{x}$ is an approximate eigenvector satisfying $\|\hat{x} - x\|_2 = O(\epsilon)$. Show that

$$\frac{\hat{x}^* A \hat{x}}{\hat{x}^* \hat{x}} - \lambda = O(\epsilon^2).$$

解：记 $\hat{x} = x + e$，其中 $\|e\|_2 = O(\varepsilon)$

$$\hat{x}^* A \hat{x} = (x+e)^* \cdot A \cdot (x+e)$$

$$= x^* A x + e^* A x + x^* A e + e^* A e$$

由于 A 是 Hermite 阵，则 $\lambda \in \mathbb{R}$

$$\therefore \text{上式} = \lambda x^* x + e^* x \lambda + \lambda x^* e + e^* A e$$

$$= \lambda x^* x + 2\lambda \cdot \mathrm{Re}(e^* x) + e^* A e \quad \cdots\cdots (1)$$

$$\hat{x}^* \hat{x} = (x+e)^* (x+e)$$

$$= x^* x + x^* e + e^* x + e^* e$$

$$= x^* x + 2 \cdot \mathrm{Re}(e^* x) + e^* e \quad \cdots\cdots (2)$$

比较(1),(2)式 惊奇地发现：

$$\hat{x}^* A \hat{x} = \lambda \cdot \hat{x}^* \hat{x} - \lambda e^* e + e^* A e$$

$$\Rightarrow \frac{\hat{x}^* A \hat{x}}{\hat{x}^* \hat{x}} = \lambda + \frac{-\lambda e^* e + e^* A e}{\hat{x}^* \hat{x}}$$

$$\Rightarrow \quad \frac{\hat{x}^* A \hat{x}}{\hat{x}^* \hat{x}} - \lambda = \frac{e^* A e - \lambda e^* e}{\hat{x}^* \hat{x}}$$

$$\left| \frac{e^* A e - \lambda e^* e}{\hat{x}^* \hat{x}} \right| \leq \frac{\|e^*\|_2 \|A\|_2 \|e\|_2 + |\lambda| \|e^*\|_2 \|e\|_2}{\hat{x}^* \hat{x}}$$

$$= \left[\frac{\|A\|_2 + |\lambda|}{\hat{x}^* \hat{x}} \right] \cdot \|e^*\|_2 \|e\|_2$$

由于 $\|e\|_2 = O(\varepsilon)$

则 $\frac{\hat{x}^* A \hat{x}}{\hat{x}^* \hat{x}} - \lambda = O(\varepsilon^2)$

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2. Given $x, y \in \mathbb{R}^n$. Describe in detail how to construct a rotation matrix Q such that the columns of $[x, y]Q$ are orthogonal to each other.

解: 设 $Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$

$$[x, y] \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = [cx - sy, \ \ sx + cy]$$

由题意:

$$(cx - sy)^T (sx + cy)$$

$$= cs x^T x + c^2 x^T y - s^2 y^T x - sc y^T y = 0$$

$$\Rightarrow \quad s^2 x^T y + (y^T y - x^T x) cs - x^T y c^2 = 0$$

若 x 与 y 原本已正交，则令 $c=0$，$s=1$ 即可

故只需考虑 $c\neq 0$ 的情况

令 $t=\frac{s}{c}$ $(c\neq 0)$

$\Rightarrow (x^Ty)t^2+(y^Ty-x^Tx)t-x^Ty=0$

记 $x^Ty=A\neq 0$，$y^Ty-x^Tx=B$，$-x^Ty=C$

$$\Rightarrow At^2+Bt+C=0$$

$$t=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

$\Delta=B^2-4AC$ 可能小于零，说明此时的 c, s 在复数域上，具有 $\cos(a+bi)$，$\sin(a+bi)$ 的形式

当 $\Delta\leq 0$ 时，取 "+, −" 都无所谓

当 $\Delta>0$ 时，则取 $t=\frac{-B-\mathrm{sign}(B)\cdot\sqrt{B^2-4AC}}{2A}$

（避免舍入误差）

得到 t 后，由于 $t=\frac{s}{c}$，$c^2+s^2=1$

得到 $c=\frac{1}{\sqrt{1+t^2}}$ $s=ct$

这样我们得到满足题意的 $Q=\begin{bmatrix} c & s \\ -s & c\end{bmatrix}$

留言：

美丽的助教学姐/帅气的助教学长，第2题我有一处不解，正交变换保持内积，长度，夹角不变，那为什么却能得到旋转阵Q使得原本不正交的 x, y 变得正交呢？太奇怪了！如果学姐/学长有时间的话，还烦请在下方留言一下，谢谢！

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3. You are given a matrix $A \in \mathbb{R}^{m\times n}$ and unit vectors $\hat{u} \in \mathbb{R}^m$, $\hat{v} \in \mathbb{R}^n$. Suppose that $\hat{u}$ and $\hat{v}$, respectively, are approximate left and right singular vectors corresponding to one singular value σ of A. How to find a "good" approximation $\hat{\sigma}$ to this singular value? Can you find a perturbation ΔA such that $(\hat{\sigma}, \hat{u}, \hat{v})$ is an exact singular triplet of $A + \Delta A$?

解：若 u, v 都是 σ "真正"的左、右奇异特征向量

则 $A = U\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}V^T \cdot v$

$$= \begin{pmatrix} * & u & * \end{pmatrix}\begin{pmatrix} *\ \sigma & 0 \\ \ * & \\ 0 & 0\end{pmatrix}\begin{pmatrix} * \\ v^T \\ * \end{pmatrix} v$$

$$= \begin{pmatrix} * & u & * \end{pmatrix}\begin{pmatrix} *\ \sigma & 0 \\ \ * & \\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$= (* \ u *) \begin{pmatrix} 0 \\ \delta \\ \vdots \\ 0 \end{pmatrix}$$

$$= u\delta$$

即本应该有 $Av = u\delta$

现在的情况是:

$$\hat{u}\delta \approx A\hat{v}$$

使用最小二乘法:

$$\hat{u}^T\hat{u}\hat{\delta} = \hat{u}^T A\hat{v}$$

$$\Rightarrow \hat{\delta} = \frac{\hat{u}^T A \hat{v}}{\hat{u}^T\hat{u}} = \hat{u}^T A \hat{v}$$

此时的 $\hat{\delta}$ 就是对 δ 一个较好的近似

现在要找 $A+\Delta A$，使得

$(A+\Delta A)\hat{v} = \hat{u}\hat{\delta}$ 成立

即 $(A+\Delta A)\hat{v} = \hat{u}\cdot\hat{u}^T A\hat{v}$

即寻找 ΔA 使得

$$A+\Delta A = \hat{u}\hat{u}^T A \Rightarrow \Delta A = \hat{u}\hat{u}^T A - A$$

$$\Rightarrow \Delta A = (\hat{u}\hat{u}^T - I)A$$

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4. Implement the Jacobi diagonalization algorithm for real symmetric matrices. Visualize the performance for a few matrices with different sizes. Visualize the convergence history for one example. You are encouraged to make observations on *entry-wise* convergence.

You are also encouraged to try the "wrong" choice of Jacobi rotations for the cyclic Jacobi algorithm.

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