Nov 8, 2022 (Due: 08:00 Nov 17, 2022)

1. Let $A \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$. Suppose that $X = [x, Ax, \dots, A^{n-1}x]$ is nonsingular. Show that $X^{-1}AX$ is upper Hessenberg.

2. Let $A_0 \in \mathbb{C}^{n\times n}$, μ_0, μ_1, …, $\mu_m \in \mathbb{C}$. Define A_1, A_2, …, A_{m+1} by

$$A_k - \mu_k I = Q_k R_k, \qquad A_{k+1} = R_k Q_k + \mu_k I,$$

for $k \in \{0, 1, \dots, m\}$, where Q_k's are unitary matrices. Show that

$$(A_0 - \mu_0 I)(A_0 - \mu_1 I) \cdots (A_0 - \mu_m I) = (Q_0 Q_1 \cdots Q_m)(R_m \cdots R_1 R_0).$$

3. Let

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Design an algorithm to compute an orthogonal matrix $Q \in \mathbb{R}^{2\times 2}$ such that

$$Q^\top A Q = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}.$$

4. Use the singular value decomposition of $A \in \mathbb{C}^{m\times n}$ to find the spectral decomposition of

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}.$$

5. Let

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_2 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-1} & a_{n-1} & b_{n-1} \\ & & & c_n & a_n \end{bmatrix} \in \mathbb{R}^{n\times n},$$

with $b_i c_{i+1} > 0$ for $i \in \{1, 2, \dots, n-1\}$. Show that A is diagonalizable, and has real spectrum.

6. (optional) When bidiagonalizing an $m \times n$ matrix with $m > n$, there are two common options: bidiagonalization after QR factorization vs. direct bidiagonalization. Suppose that both left and right orthogonal transformations need to be accumulated. Calculate the cost in terms of number of floating-point operations for these options, and determine the crossover point.

Will the crossover point change if orthogonal transformations are not accumulated?

7. (H) Implement the Arnoldi process using MGS. Observe the residual norm and the orthogonality by some concrete test matrices.

(optional) Implement the Arnoldi process using Householder transformations.