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1. Let $A \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$. Suppose that $X = [x, Ax, \ldots, A^{n-1}x]$ is nonsingular. Show that $X^{-1}AX$ is upper Hessenberg.

解: $A\underline{X} = A[x, Ax, \cdots A^{n-1}x]$

$= [Ax, A^2x, \cdots A^n x]$

设 $A\underline{X} = XH$

则: $[Ax, A^2x, \cdots A^n x] = [x, Ax, \cdots A^{n-1}x][H_1, H_2 \cdots H_n]$

记 $e_i = (0, \cdots 0, 1, 0, \cdots 0)^T$ 第i位为1, 其它是0

则观察可知:

$H_1 = e_2 \quad H_2 = e_3 \quad \cdots \quad H_{n-1} = e_n$

又由于 $[x, Ax, \cdots A^{n-1}x]$ 是 $\mathbb{C}^{n\times n}$ 的一组基

则必有 $A^n x = c_1 x + c_2 Ax + \cdots + c_n A^{n-1}x$

记 $H_n = (c_1, c_2, \cdots c_n)$

则 $H = [H_1, H_2, \cdots H_n]$

$$= \begin{bmatrix} 0 & & & & c_1 \\ 1 & 0 & & \text{O} & c_2 \\ & 1 & \ddots & & \vdots \\ & \text{O} & \ddots & 0 & c_{n-1} \\ & & & 1 & c_n \end{bmatrix}$$

(是上Hessenberg阵)

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2. Let $A_0 \in \mathbb{C}^{n\times n}$, $\mu_0, \mu_1, \ldots, \mu_m \in \mathbb{C}$. Define $A_1, A_2, \ldots, A_{m+1}$ by

$$A_k - \mu_k I = Q_k R_k, \qquad A_{k+1} = R_k Q_k + \mu_k I,$$

for $k \in \{0, 1, \ldots, m\}$, where Q_k's are unitary matrices. Show that

$$(A_0 - \mu_0 I)(A_0 - \mu_1 I)\cdots(A_0 - \mu_m I) = (Q_0 Q_1 \cdots Q_m)(R_m \cdots R_1 R_0).$$

解：先证明：$A_{m+1} R_m \cdots R_1 R_0 = R_m \cdots R_1 R_0 A_0$

$k=0$ 时：

由于 $A_0 - \mu_0 I = Q_0 R_0$

$A_1 = R_0 Q_0 + \mu_0 I$

则 $A_1 R_0 = (R_0 Q_0 + \mu_0 I) R_0$

$= R_0 \cdot (A_0 - \mu_0 I) + \mu_0 R_0$

$= R_0 \cdot A_0$

$\therefore$ $k=0$ 时成立

现假设 $k=m$ 时成立. 当 $k=m+1$ 时:

$A_{m+2} R_{m+1} \cdots R_1 R_0$

$= (R_{m+1} Q_{m+1} + \mu_{m+1} I) R_{m+1} \cdots R_1 R_0$

$= (R_{m+1} \cdot (A_{m+1} - \mu_{m+1}) + \mu_{m+1} R_{m+1}) \cdot R_m \cdots R_1 R_0$

$= R_{m+1} \cdot A_{m+1} \cdot R_m \cdots R_1 R_0$

由于 $A_{m+1} R_m \cdots R_1 R_0 = R_m \cdots R_1 R_0 A$

$\therefore$ $R_{m+1} A_{m+1} R_m \cdots R_1 R_0 = R_{m+1} R_m \cdots R_1 R_0 A$

$\therefore$ $k=m+1$ 时也成立

对于 $(A_0-\mu_0 I)\cdots(A_0-\mu_m I)=Q_0\cdots Q_m R_m\cdots R_0$

$k=0$ 时 $A_0-\mu_0 I=Q_0R_0$ 显然成立

假设 $k=m$ 时成立，当 $k=m+1$ 时：

$$(A_0-\mu_0 I)\cdots(A_0-\mu_m I)(A_0-\mu_{m+1}I)$$

$$=Q_0\cdots Q_m R_m\cdots R_0(A_0-\mu_{m+1}I)$$

$$=Q_0\cdots Q_m A_{m+1} R_m\cdots R_0-\mu_{m+1}Q_0\cdots Q_m R_m\cdots R_0$$

$$=Q_0\cdots Q_m(Q_{m+1}R_{m+1}+\mu_{m+1}I)R_m\cdots R_0-\mu_{m+1}Q_0\cdots Q_m R_m\cdots R_0$$

$$=Q_0\cdots Q_m Q_{m+1}R_{m+1}\cdots R_1R_0$$

$\therefore$ $k=m+1$ 也成立

$\therefore$ 对所有 $k\geq 0$，$k\in \mathbb{Z}^+$ 都有：

$$(A-\mu_0 I)\cdots(A-\mu_m I)=(Q_0\cdots Q_m)\cdot(R_m\cdots R_0)$$

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3. Let

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Design an algorithm to compute an orthogonal matrix $Q \in \mathbb{R}^{2\times 2}$ such that

$$Q^\top A Q = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}.$$

解: A 的特征值为 $\lambda = a$ or $\lambda = b$

$\lambda = b$ 时 $A - bI = \begin{bmatrix} a-b & c \\ 0 & 0 \end{bmatrix}$

对应的特征向量为 $x_b = \begin{pmatrix} c \\ b-a \end{pmatrix}$

若 $a=b$, 取 $Q = I$ 即可

若 $a \neq b$, 则对 x_b 进行单位化, 得到

$$\tilde{x}_b = \frac{x_b}{\|x_b\|_2} = \frac{1}{\sqrt{c^2+(a-b)^2}} \cdot \begin{pmatrix} c \\ b-a \end{pmatrix}$$

设: $Q = \begin{pmatrix} s & -t \\ t & s \end{pmatrix}$ 　　则 $Q^T = \begin{pmatrix} s & t \\ -t & s \end{pmatrix}$

令 $Q^T x = \|\tilde{x}_b\|_2 e_1 = e_1$, $\Rightarrow Qe_1 = x$

t 与 s 满足:

$$\begin{cases} sc + t(b-a) = \sqrt{c^2+(a-b)^2} \\ -tc + s(b-a) = 0 \\ s^2 + t^2 = 1 \end{cases}$$

$$\Rightarrow \quad s = \frac{c}{\sqrt{c^2+(a-b)^2}} \qquad t = \frac{b-a}{\sqrt{c^2+(a-b)^2}}$$

注意到 $Q^T A Q e_1 = Q^T A \tilde{x}_b = Q^T b \tilde{x}_b$

$$= b \cdot e_1$$

再由 $Q^T A Q e_2$

$$= Q^T A \cdot \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= Q^T A \cdot \begin{pmatrix} -t \\ s \end{pmatrix}$$

$$= \begin{bmatrix} s & t \\ -t & s \end{bmatrix} \cdot \begin{bmatrix} -at + cs \\ bs \end{bmatrix}$$

$$= \begin{bmatrix} -sat + cs^2 + bst \\ at^2 - cst + bs^2 \end{bmatrix}$$

代入 $s = \frac{c}{\sqrt{c^2+(a-b)^2}} \qquad t = \frac{b-a}{\sqrt{c^2+(a-b)^2}}$

得: $Q^T A Q e_2 = \begin{bmatrix} c \\ a \end{bmatrix}$

$\therefore Q^T A Q(e_1, e_2) = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}$

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4. Use the singular value decomposition of $A \in \mathbb{C}^{m\times n}$ to find the spectral decomposition of

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}.$$

解：设 $A = U\Sigma V^*$ $\quad A^* = V\Sigma^T U^*$

$$\begin{bmatrix} U^* & \\ & V^* \end{bmatrix}\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\begin{bmatrix} U & \\ & V \end{bmatrix}$$

$$= \begin{bmatrix} U^* & \\ & V^* \end{bmatrix}\begin{bmatrix} 0 & U\Sigma \\ V\Sigma^T & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \Sigma \\ \Sigma^T & 0 \end{bmatrix}_{m+n}$$

① 先证明 $m=n$ 的情况，此时 $\Sigma = \Sigma^T$

令 $W = \frac{1}{\sqrt{2}}\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}$

且有：$W^T = W \quad W\cdot W = I$，$W^* = W = W^{-1}$

$$W\begin{bmatrix} 0 & \Sigma \\ \Sigma^T & 0 \end{bmatrix} W^{-1}$$

$$= W\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} W^{-1}$$

$$= \frac{1}{2}\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix}\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \Sigma & \Sigma \\ -\Sigma & \Sigma \end{bmatrix}\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}$$

则有：$W^{-1}\begin{bmatrix} U^* & \\ & V^* \end{bmatrix}\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\begin{bmatrix} U & \\ & V \end{bmatrix} W$

$$= \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}$$

$\therefore \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ 的谱分解为：

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\begin{bmatrix} I_m/\sqrt{2} & I_m/\sqrt{2} \\ I_m/\sqrt{2} & -I_m/\sqrt{2} \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}\begin{bmatrix} I_m/\sqrt{2} & I_m/\sqrt{2} \\ I_m/\sqrt{2} & -I_m/\sqrt{2} \end{bmatrix}\begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix}$$

② 当 $m \neq n$ 时，不妨设 $m > n$

$$\Sigma_{m\times n} = \begin{bmatrix} \Lambda_{n\times n} \\ O_{(m-n)\times n} \end{bmatrix}_{m\times n} \quad \Sigma^T = \begin{bmatrix} \Lambda_{n\times n} & O_{n\times(m-n)} \end{bmatrix}_{n\times m}$$

$$\begin{bmatrix} 0 & \Sigma \\ \Sigma^T & 0 \end{bmatrix} = \begin{bmatrix} O_{m\times m} & \begin{pmatrix} \Lambda_n \\ 0 \end{pmatrix} \\ (\Lambda_n, 0) & O_{n\times n} \end{bmatrix}$$

$$\begin{bmatrix} 0 & \Sigma \\ \Sigma^T & 0 \end{bmatrix} = \left[\begin{array}{c:c|c} 0 & 0 & \Lambda \\ \hdashline 0 & 0 & 0 \\ \hline \Lambda & 0 & 0 \end{array}\right]$$ (示意图)

取 $W = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & 0 & I_n \\ 0 & \sqrt{2} I_{m-n} & 0 \\ I_n & 0 & -I_n \end{bmatrix}$

注意到仍有 $W^T = W$ $\quad W \cdot W = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_{m-n} & 0 \\ 0 & 0 & I_n \end{bmatrix} = I_{m+n}$

则 $W = W^{-1}$

$$W^{-1} \begin{bmatrix} 0 & \Sigma \\ \Sigma^T & 0 \end{bmatrix} W$$

$$= \frac{1}{2} W^{-1} \begin{bmatrix} \Lambda & 0 & -\Lambda \\ 0 & 0 & 0 \\ \Lambda & 0 & \Lambda \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2\Lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\Lambda \end{bmatrix}$$

$$= \begin{bmatrix} \Lambda & & \\ & 0 & \\ & & -\Lambda \end{bmatrix}$$ (是一个对角阵)

$$\therefore \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} W \begin{pmatrix} \Lambda & & \\ & 0 & \\ & & -\Lambda \end{pmatrix} W^{-1} \begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix}$$

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5. Let

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_2 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-1} & a_{n-1} & b_{n-1} \\ & & & c_n & a_n \end{bmatrix} \in \mathbb{R}^{n\times n},$$

with $b_i c_{i+1} > 0$ for $i \in \{1, 2, \ldots, n-1\}$. Show that A is diagonalizable, and has real spectrum.

解：记 $p_n(\lambda) = |\lambda I - A|_n$

$$|\lambda I - A|_n = \begin{vmatrix} \lambda - a_1 & -b_1 & & & & \\ -c_2 & \lambda - a_2 & -b_2 & & & \\ & -c_3 & \lambda - a_3 & -b_3 & & \\ & & & \ddots & & \\ & & & -c_{n-1} & \lambda - a_{n-1} & -b_{n-1} \\ & & & & -c_n & \lambda - a_n \end{vmatrix}_n$$

$$= (\lambda - a_n)\cdot p_{n-1}(\lambda) - (b_{n-1} c_{n-1})\cdot p_{n-2}(\lambda)$$

$$= (\lambda - a_n)\, p_{n-1}(\lambda) - \left(\sqrt{b_{n-1} c_{n-1}}\right)^2 \cdot p_{n-2}(\lambda)$$

现在引入：

$$B = \begin{vmatrix} a_1 & \sqrt{b_1 c_2} & & & \\ \sqrt{b_1 c_2} & a_2 & \sqrt{b_2 c_3} & & \\ & & \ddots & & \\ & & & a_{n-1} & \sqrt{b_{n-1} c_n} \\ & & & \sqrt{b_{n-1} c_n} & a_n \end{vmatrix}_n$$

再记 $p_n'(\lambda) = |\lambda I - B|_n$

下面证明 $p_n(\lambda) = p_n'(\lambda)$

$n=2$ 时

$$\begin{vmatrix} a_1 & b_1 \\ c_2 & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & \sqrt{b_1 c_2} \\ \sqrt{b_1 c_2} & a_2 \end{vmatrix} = a_1 a_2 - b_1 c_2$$

假设 $n \le k$ 时成立

当 $n = k+1$ 时：

$$p_{k+1}(\lambda) = (\lambda - a_{k+1}) p_k(\lambda) - b_k c_{k+1} p_{k-1}(\lambda)$$

$$p'_{k+1}(\lambda) = (\lambda - a_{k+1}) p_k(\lambda) - \left(\sqrt{b_k c_{k+1}}\right)^2 p'_{k-1}(\lambda)$$

由归纳假设，$p_{k+1}(\lambda) = p'_{k+1}(\lambda)$ 成立！

$\therefore |\lambda I - A| = |\lambda I - B|$

$\therefore$ A 与 B 具有相同特征值

$\because$ B 是 Hermite 阵，则 $\lambda(B)$ 互不相同且均为实数

$\therefore$ $\lambda(A)$ 也互不相同且均为实数