HOMEWORK 2

1. 证明题 (请提交 PDF 格式)

(1) Prove that the OLS estimator $\widehat{\beta}$ is the same as the maximum likelihood estimator.

(2) Prove the Gauss-Markov Theorem.

(3) Prove $E(\widehat{\sigma}^2) = \sigma^2$.

(4) Given conditions:

(A1) The relationship between response ($\mathbf{y}$) and covariates ($\mathbf{X}$) is linear;

(A2) $\mathbf{X}$ is a non-stochastic matrix and $\text{rank}(\mathbf{X}) = p$;

(A3) $E(\varepsilon) = \mathbf{0}$. This implies $E(\mathbf{y}) = \mathbf{X}\beta$;

(A4) $\text{cov}(\varepsilon) = E(\varepsilon\varepsilon^\top) = \sigma^2 I_N$; (Homoscedasticity)

(A5) ε follows multivariate normal distribution $N(\mathbf{0}, \sigma^2 I_N)$ (Normality)

Prove the following results:

$$\widehat{\beta} \sim N(\beta, \sigma^2(\mathbf{X}^\top\mathbf{X})^{-1}) \tag{0.1}$$

$$(N-p)\widehat{\sigma}^2 \sim \sigma^2\chi^2_{N-p} \tag{0.2}$$

(5) Suppose y follows the log-linear regression relationship with $x \in \mathbb{R}^p$, i.e.,

$$\log(y) = x^\top\beta + \epsilon, \tag{0.3}$$

where ϵ follows normal distribution $N(0, \sigma^2)$. Please calculate $E(y)$.

(6) Define $\widehat{y}_i = x_i^\top\beta$. Let the intercept be included in the regression model. Define the

total sum of squares (TSS) and explained sum of squares (ESS) as follows

$$\mathrm{TSS} = \sum_i (y_i - \overline{y})^2, \quad \mathrm{ESS} = \sum_i (\widehat{y}_i - \overline{y})^2.$$

Please prove:

$$\mathrm{TSS} = \mathrm{ESS} + \mathrm{RSS}.$$

提交时间：10 月 2 日 20:00 之前。请预留一定的时间，迟交作业扣 3 分，作业抄袭 0 分。作业作答使用中英文皆可。