Mar. 13, 2023 (Due: 08:00 Mar. 20, 2023)

1. Interpolate the rational function $f(x) = (1 + 25x^2)^{-1}$ over $[-1, 1]$ using polynomials. Estimate the interpolation error. Use equispaced nodes and Chebyshev nodes and plot the results.

Note: Chebyshev nodes are the roots of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$.

2. Try to simplify Newton's interpolation polynomial for equally spaced interpolation nodes $x_1 < x_2 < \cdots < x_n$ (with $x_i = x_1 + (i-1)h$).

3. Approximate the sine function over the closed interval $[0, 2\pi]$ using piecewise cubic Hermite interpolation, and visualize your result. You are recommended to partition the interval with n equally spaced interpolation nodes for $n = 2, 3, 5, 9$.

4. Let x_0, x_1, …, x_n be distinct real numbers. Show that

$$f(x_0, x_1, \ldots, x_n) = \int \cdots \int_{\mathcal{T}_n} f^{(n)}(t_0 x_0 + t_1 x_1 + \cdots + t_n x_n)\,\mathrm{d}t_1\,\mathrm{d}t_2 \cdots \mathrm{d}t_n$$

for any sufficiently smooth function f, where

$$t_0 = 1 - (t_1 + t_2 + \cdots + t_n),$$
$$\mathcal{T}_n = \{(t_1, t_2, \ldots, t_n) \colon t_i \geq 0,\ t_1 + t_2 + \cdots + t_n \leq 1\}.$$

5. (optional) This exercise is about an atypical approach for two-dimensional interpolation.

Interpolating a data set $\{(x_i, y_i, z_i)\}_{i=1}^n \subset \mathbb{R}^3$ can be understood as interpolating $\{(x_i + \mathrm{i}y_i, z_i)\}_{i=1}^n \subset \mathbb{C} \times \mathbb{R}$, where the interpolation nodes $x_i + \mathrm{i}y_i$'s are complex numbers. The polynomial interpolation techniques we have learned from this course theoretically carry over to complex inputs, while the resulting interpolation polynomial is in general complex-valued. Nevertheless, we can take the real part of the output.

Use this approach to interpolate the following data set over the unit disk and visualize the result.

x_i	y_i	z_i
1.00000	0.00000	−1.0000
0.80902	0.58779	−2.6807
0.30902	0.95106	5.6161
−0.30902	0.95106	5.6161
−0.80902	0.58779	−2.6807
−1.00000	0.00000	−1.0000
−0.80902	−0.58779	−2.6807
−0.30902	−0.95106	5.6161
0.30902	−0.95106	5.6161
0.80902	−0.58779	−2.6807

(If you use MATLAB/Octave, the functions `imagesc` and `colorbar` are useful for visualizing a bivariate function.)