4月7日作业

1 **1.** Compute a few low degree Padé approximants of $f(x) = (1+x)^{-1/2}$ and visualize the approximation errors over $[0, 2]$.

解：首先计算2种Padé逼近

(1) $\dfrac{a+bx}{1+dx} \approx (1+x)^{-1/2}$

$\Rightarrow (a+bx)(1+x)^{1/2} \approx 1+dx$

其中 $(1+x)^{1/2} = 1+\frac{1}{2}x - \frac{1}{4}x^2 + o(x^2)$

$\Rightarrow (a+bx)\left(1+\frac{1}{2}x - \frac{1}{4}x^2 + o(x^2)\right) \approx 1+dx$

$\Rightarrow a+(\frac{1}{2}a+b)x + (\frac{1}{2}b-\frac{1}{4}a)x^2 + o(x^2) \approx 1+dx$

$\Rightarrow \begin{cases} a=1 \\ \frac{1}{2}a+b=d \\ \frac{1}{2}b-\frac{1}{4}a=0 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=\frac{1}{2} \\ d=1 \end{cases}$

则Padé逼近为 $R_{1,1}(x) = \dfrac{1+\frac{1}{2}x}{1+x}$

(2) $\dfrac{a+bx}{1+dx+ex^2} \approx (1+x)^{-1/2}$

$\Rightarrow (a+bx)(1+x^{-1/2}) \approx 1+dx+ex^2$

$\Rightarrow (a+bx)\left(1+\frac{1}{2}x - \frac{1}{4}x^2 + \frac{3}{8}x^3 + o(x^3)\right) \approx 1+dx+ex^2$

$$a+\left(\tfrac{1}{2}a+b\right)x+\left(\tfrac{1}{2}b-\tfrac{1}{4}a\right)x^2+\left(\tfrac{3}{8}a-\tfrac{1}{4}b\right)x^3+o(x^3)\approx 1+dx+ex^2$$

$$\Rightarrow \begin{cases} a=1 \\ \tfrac{1}{2}a+b=d \\ \tfrac{1}{2}b-\tfrac{1}{4}a=e \\ \tfrac{3}{8}a-\tfrac{1}{4}b=0 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=\tfrac{3}{2} \\ d=2 \\ e=\tfrac{1}{2} \end{cases}$$

则 Padé 逼近为 $R_{1/2}=\dfrac{1+\frac{3}{2}x}{1+2x+\frac{1}{2}x^2}$

(3) $$\frac{a+bx+cx^2}{1+dx}\approx(1+x)^{-1/2}$$

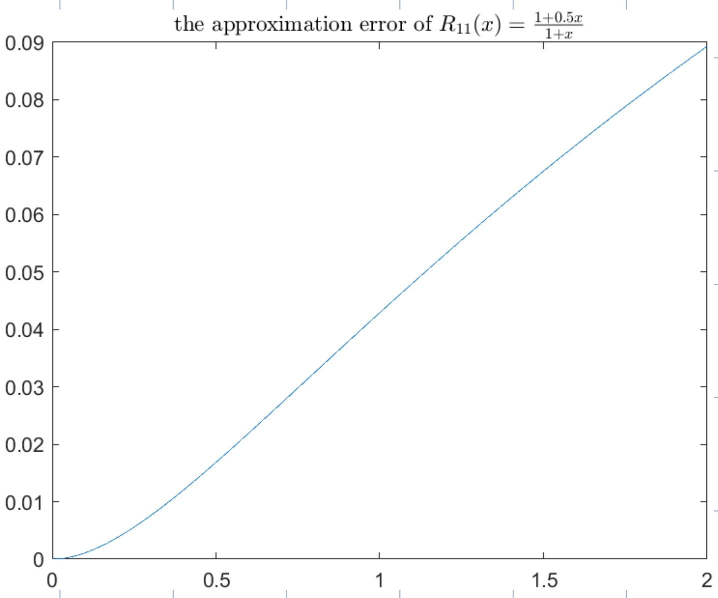
$$\Rightarrow (a+bx+cx^2)\left(1+\tfrac{1}{2}x-\tfrac{1}{4}x^2+\tfrac{3}{8}x^3+o(x^3)\right)\approx 1+dx$$

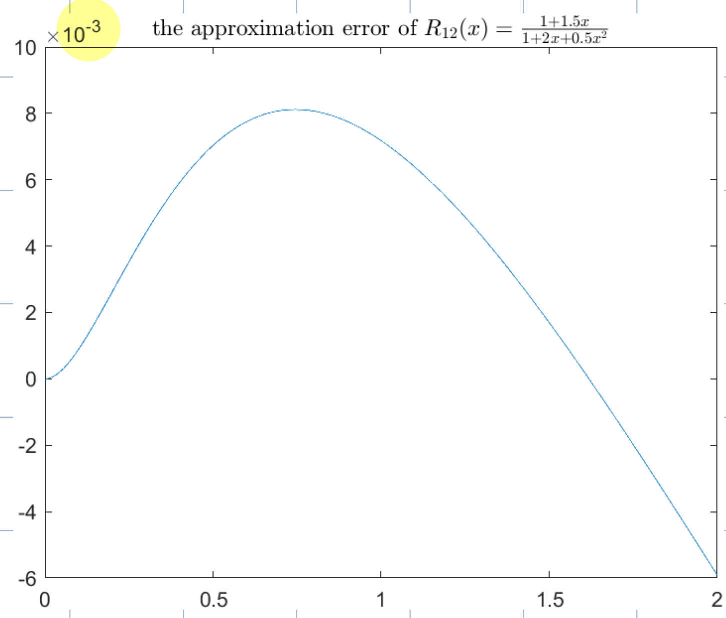
$$\Rightarrow a+\left(\tfrac{1}{2}a+b\right)x+\left(-\tfrac{1}{4}a+\tfrac{1}{2}b\right)x^2+\left(\tfrac{3}{8}a-\tfrac{1}{4}b+\tfrac{1}{2}c\right)x^3+o(x^3)$$
$$\approx 1+dx$$

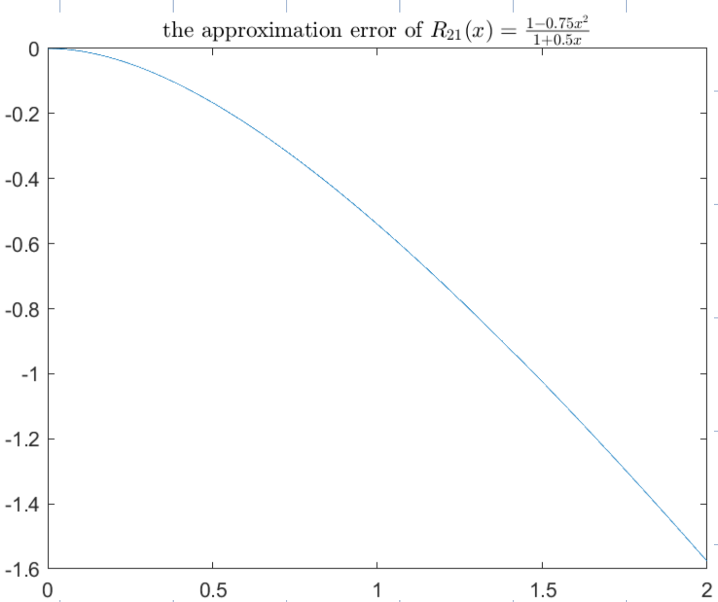
$$\Rightarrow \begin{cases} a=1 \\ \tfrac{1}{2}a+b=d \\ -\tfrac{1}{4}a+\tfrac{1}{2}d=0 \\ \tfrac{3}{8}a-\tfrac{1}{4}b+\tfrac{1}{2}c=0 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=0 \\ c=-\tfrac{3}{4} \\ d=\tfrac{1}{2} \end{cases}$$

则 Padé 逼近为 $R_{2/1}=\dfrac{1-\frac{3}{4}x^2}{1+\frac{1}{2}x}$

三种 Padé 逼近的误差如下:







对比可知, $R_{1/2}(x)$ 的逼近效果最好。 □

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2. Construct a concrete example to show that Padé approximation is not always a well-posed problem.

解: 用 $R_{1/1} = \dfrac{a+bx}{1+dx}$ 去逼近 $(1+x^2)$

$$\frac{a+bx}{1+dx} \approx 1+x^2$$

$$a+bx \approx (1+dx)(1+x^2)$$

$$\Rightarrow a+bx \approx 1+dx+x^2+dx^3$$

$$\Rightarrow \begin{cases} a=1 \\ b=d \end{cases}$$

这个方程组是正定的，而非适定的。 □

3. **3.** Show that a quadratic Bézier curve is part of a parabola.

解：2次 Bézier 曲线需要三个控制点，分别设为：

$P_0=(x_0,y_0)$, $P_1=(x_1,y_1)$. $P_2(x_2,y_2)$

抛物线的形状在平移后保持不变，可以把 P_0 移到原点处，因此可设 $P_0=(0,0)$

类似的，围绕原点进行旋转也不改变抛物线的形状

设：
$$\tilde{P}_1 = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} cx_1+sy_1 \\ -sx_1+cy_1 \end{pmatrix}$$

$$\tilde{P}_2 = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} cx_2+sy_2 \\ -sx_2+cy_2 \end{pmatrix}$$

这里对 c 和 s 提两个约束：

$$\begin{cases} c^2+s^2=1 \\ 2(cx_1+sy_1)-cx_2+sy_2=0 \end{cases}$$

即：$\begin{cases} c^2+s^2=1 \\ (2x_1+x_2)\cdot c-(2y_1+y_2)\cdot s=0 \end{cases}$ (*)

方程组(*)在 (1) $2x_1+x_2=0$, $2y_1+y_2=0$；(2) $2x_1+x_2=0$, $2y_1+y_2\neq 0$；(3) $2x_1+x_2\neq 0$, $2y_1+y_2=0$；(4) $2x_1+x_2\neq 0$, $2y_1+y_2\neq 0$ 情况下都有解，则(*)总是有解

记旋转后的点为 $\widetilde{P_1}=(\widetilde{x_1},\widetilde{y_1})$, $\widetilde{P_2}(\widetilde{x_2},\widetilde{y_2})$

且满足 $2\widetilde{x_1}-\widetilde{x_2}=0$

$$x(t)=0\cdot(1-t)^2+2\cdot\widetilde{x_1}\,t\cdot(1-t)+\widetilde{x_2}\,t^2$$
$$=2\widetilde{x_1}t-2\widetilde{x_1}t^2+\widetilde{x_2}t^2$$
$$=2\widetilde{x_1}t$$

$$y(t)=0\cdot(1-t)^2+2\widetilde{y_1}\,t(1-t)+\widetilde{y_2}\,t^2$$
$$=(\widetilde{y_2}-2\widetilde{y_1})t^2+2\widetilde{y_1}t$$

当 $\widetilde{x_1}\neq 0$, $\widetilde{y_2}-2\widetilde{y_1}\neq 0$ 时，$(x(t),y(t))$ 是抛物线一部分

当 $\widetilde{x_1}\neq 0$, $\widetilde{y_2}-2\widetilde{y_1}=0$ 时，$(x(t),y(t))$ 退化为线段

当 $\widetilde{x_1}=0$ 时，可知 $\widetilde{x_2}=0$，此时 $(x(t),y(t))$ 退化为线段

Q：似乎题目中忽略了可能退化为线段的情况？

□

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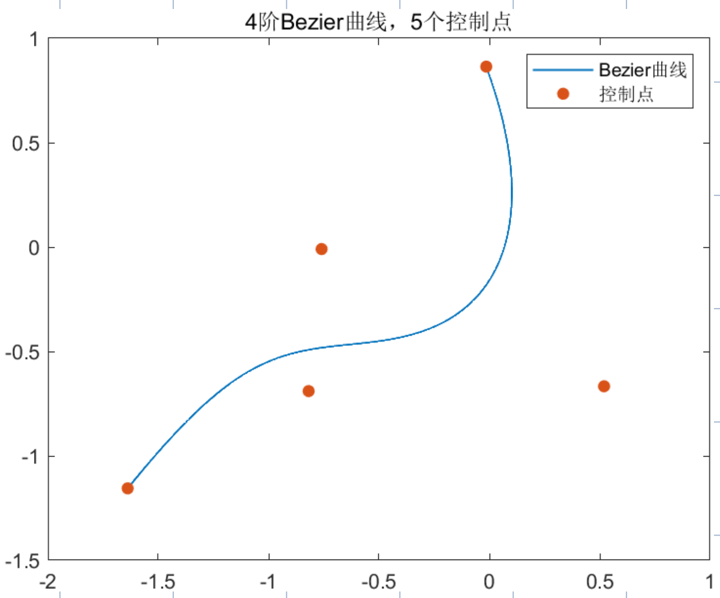
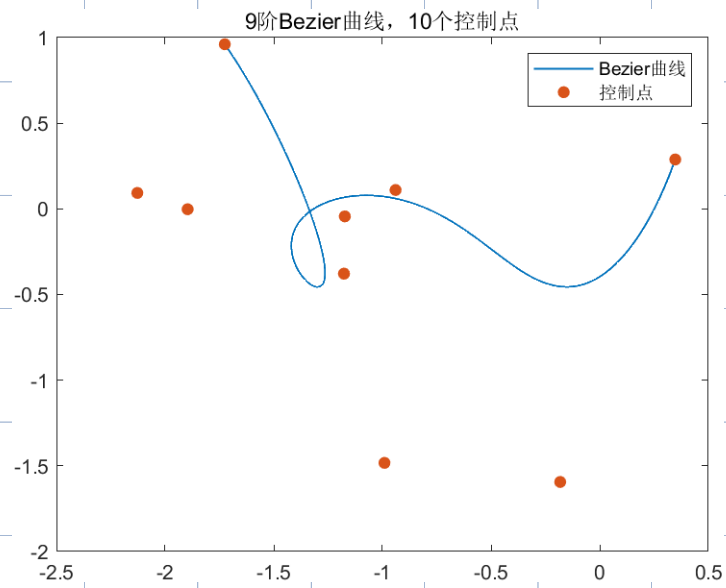
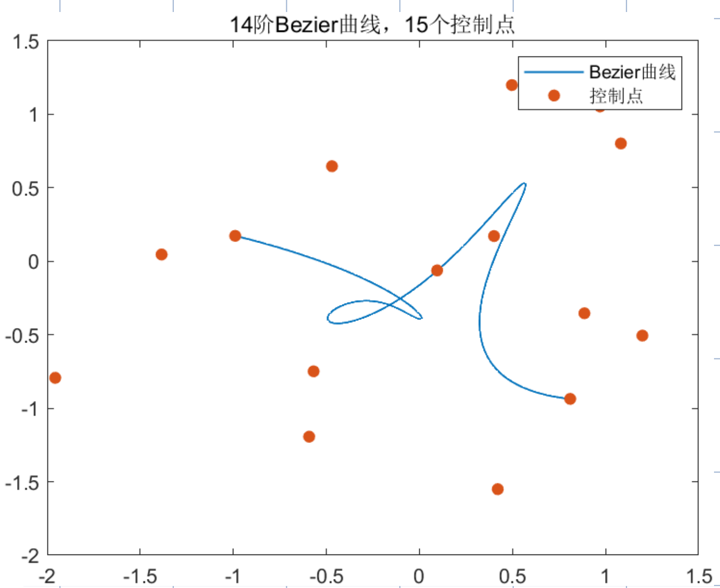
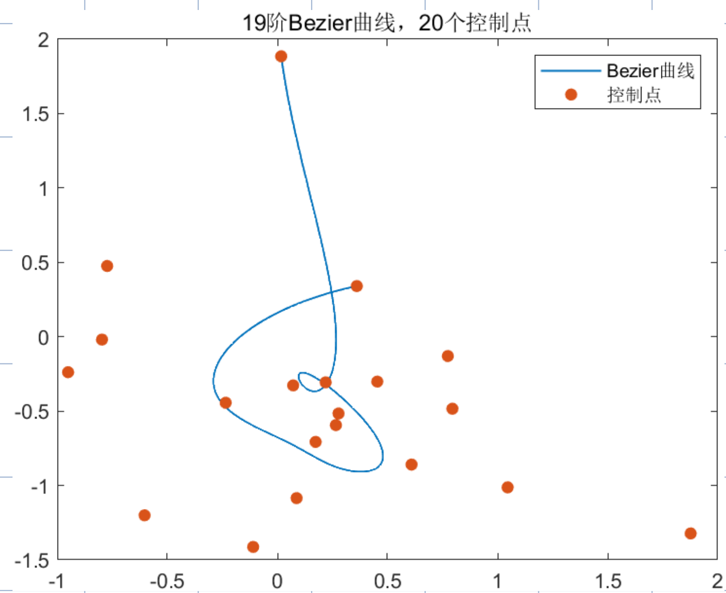
4. Write a program to plot the Bézier curve with arbitrary set of control points. Visualize a few examples.

解：以下是四个实验结果：









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5. (optional) Linear and quadratic Bézier curves can exactly fit line segments and parabolas, respectively. However, many computer programs only support cubic B'ezier curves. Suppose you want to plot a segment of parabola of the form $y = ax^2 + bx + c$ for $x \in [x_1, x_2]$. How to construct four control points for this purpose?

解：为了书写方便，仍记 $x \in [\hat{x}_1, \hat{x}_2]$，其中 $\hat{x}_1 \neq \hat{x}_2$

再记四个控制点分别为：

$P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$

$$x(t) = x_0(1-t)^3 + 3x_1 t(1-t)^2 + 3x_2 t^2(1-t) + x_3 t^3$$

$$= C_3 t^3 + C_2 t^2 + C_1 t + C_0$$

其中 C_j 是 t^j 的系数，是 x_0, x_1, x_2, x_3 的线性组合

类似的：

$$y(t) = y_0(1-t)^3 + 3y_1 t(1-t)^2 + 3y_2 t^2(1-t) + y_3 t^3$$

$$= D_3 t^3 + D_2 t^2 + D_1 t + D_0$$

由于最终希望 $(x(t), y(t))$ 具有 $y = ax^2 + bx + c$, $x \in [\hat{x}_1, \hat{x}_2]$ 的形式，那么对控制点有如下四个约束：

$$\begin{cases} x_0 = \hat{x}_1, \quad x_3 = \hat{x}_2 & (\text{首尾控制点}) \\ C_3 = 0, \quad C_2 = 0 & (x(t) \text{ 关于 } t \text{ 是线性的}) \end{cases}$$

以上四个约束确定了 x_0, x_1, x_2, x_3，且易验证方程组适定

并且，由于 $\hat{x}_1 \neq \hat{x}_2$，则 $(x(t), y(t))$ 不会退化为平行于 y 轴的线段，则 $C_1 \neq 0$

$$x(t) = C_1 t + C_0 \Rightarrow t = \frac{x - C_0}{C_1}$$

$$\Rightarrow y(t) = D_3\left(\frac{x-C_0}{C_1}\right)^3 + D_2\left(\frac{x-C_0}{C_1}\right)^2$$

$$+ D_1\left(\frac{x-C_0}{C_1}\right) + D_0$$

$= E_3 \cdot x^3 + E_2 \cdot x^2 + E_1 \cdot x + E_0$

其中 E_j 分别是 D_3, D_2, D_1, D_0 的线性组合

满足下面四个约束：

$E_3 = 0, \quad E_2 = a, \quad E_1 = b, \quad E_0 = c$

四个约束确定了 D_3, D_2, D_1, D_0，而 D_j 都是 y_3, y_2, y_1, y_0

的线性组合，则也同时确定了 y_3, y_2, y_1, y_0

这样，就确定了四个控制点 P_0, P_1, P_2, P_3 $\square$