3月20日作业

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1. Find the polynomial $p(x)$ with lowest degree that satisfies $f(1) = f'(1) = f''(1) = 3$, $f(2) = f'(2) = f''(2) = 1$, and $f(3) = f'(3) = 2$.

解：做对应的差分表如下：

x	1	1	1	2	2	2	3	3
y	3	3	3	1	1	1	2	2
	3	3	-2	1	1	1	2	
	3	-5	3	1	0	1		
	-8	8	-2	-1	1			
	16	-10	$\frac{1}{2}$	2				
	-26	$\frac{21}{4}$	$\frac{3}{4}$					
	$\frac{125}{8}$	$-\frac{9}{4}$						
	$-\frac{143}{16}$							

（注：Hermite插值多项式的系数也可以由对应的 exercise_1_coef.m 文件自动生成）

$$\Rightarrow p(x) = 3 + 3(x-1) + 3(x-1)^2 - 8(x-1)^3 + 16(x-1)^3(x-2)$$
$$- 26(x-1)^3(x-2)^2 + \frac{125}{8}(x-1)^3(x-2)^3 - \frac{143}{16}(x-1)^3(x-2)^3(x-3)$$

2. Similar to complete, natural, and periodic cubic splines, when the "not-a-knot" condition is used in cubic spline interpolation, the computational kernel is also to solve a sparse linear system. Try to derive the corresponding linear system.

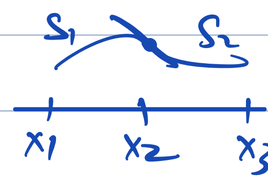
解：若记 $\lambda_i = \dfrac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}$，$\mu_i = \dfrac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}}$

再记 k_i 为曲线在 x_i 处的一阶导，则有：

$$\lambda_i k_{i-1} + 2k_i + \mu_i k_{i+1} = 3\left(\mu_i \frac{y_{i+1} - y_i}{x_{i+1} - x_i} + \lambda_i \frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right)$$

$(i = 3, 4, \cdots, n-2)$

对于 "not-a-knot" 的情况，



$$S_1'''(x_2) = S_2'''(x_2)$$

注意到 $S_i(x)$ 在 $[x_i, x_{i+1}]$ 上的 Hermite 插值为：

$$S_i(x) = y_i + k_i(x - x_i) + \delta(x_i, x_i, x_{i+1})(x - x_i)^2 + \delta(x_i, x_i, x_{i+1}, x_{i+1})(x - x_i)^2(x - x_{i+1})$$

$$\Rightarrow S_i'''(x) = 6\delta(x_i, x_i, x_{i+1}, x_{i+1})$$

$$\begin{array}{ccccccc}
x_i, & & x_i, & & x_{i+1}, & & x_{i+1} \\
y_i & & y_i & & y_{i+1} & & y_{i+1} \\
 & k_i & & \dfrac{y_{i+1} - y_i}{x_{i+1} - x_i} & & k_{i+1} & \\
 & & \dfrac{\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - k_i}{x_{i+1} - x_i} & & \dfrac{k_{i+1} - \frac{y_{i+1} - y_i}{x_{i+1} - x_i}}{x_{i+1} - x_i} & &
\end{array}$$

$$\frac{k_{i+1} - 2\frac{y_{i+1}-y_i}{x_{i+1}-x_i} + k_i}{(x_{i+1}-x_i)^2}$$

$$\Rightarrow S(x_i, x_i, x_{i+1}, x_{i+1}) = \left(k_{i+1} - 2\frac{y_{i+1}-y_i}{x_{i+1}-x_i} + k_i\right) \Big/ (x_{i+1}-x_i)^2$$

则对于 $S_1'''(x_2) = S_2'''(x_2)$ 有：

$$\frac{\left(k_2 - 2\cdot\frac{y_2-y_1}{x_2-x_1} + k_1\right)}{(x_2-x_1)^2} = \frac{\left(k_3 - 2\cdot\frac{y_3-y_2}{x_3-x_2} + k_2\right)}{(x_3-x_2)^2}$$

$$\Rightarrow$$

$$(x_3-x_2)^2 k_1 + \left[(x_3-x_2)^2 - (x_2-x_1)^2\right] k_2 - (x_2-x_1)^2 k_3$$

$$= 2(x_3-x_2)^2 \frac{y_2-y_1}{x_2-x_1} - 2(x_2-x_1)^2 \frac{y_3-y_2}{x_3-x_2}$$

沿用上面关于 λ_i 与 μ_i 的记号 则有：

$$\lambda_2^2 k_1 + (\lambda_2^2 - \mu_2^2) k_2 - \mu_2^2 k_3 = 2\lambda_2^2 \frac{y_2-y_1}{x_2-x_1} - 2\mu_2^2 \frac{y_3-y_2}{x_3-x_2}$$

对于 $S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$ 也类似有：

$$\lambda_{n-1}^2 k_{n-2} + (\lambda_{n-1}^2 - \mu_{n-1}^2) k_{n-1} - \mu_{n-1}^2 k_n = 2\lambda_{n-1}^2 \frac{y_{n-1}-y_{n-2}}{x_{n-1}-x_{n-2}} - 2\mu_{n-1}^2 \frac{y_n-y_{n-1}}{x_n-x_{n-1}}$$

那么，这个线性系统最终变成：

$$\begin{cases} \lambda_2^2 k_1 + (\lambda_2^2 - \mu_2^2) k_2 - \mu_2^2 k_3 = 2\lambda_2^2 \dfrac{y_2 - y_1}{x_2 - x_1} - 2\mu_2^2 \dfrac{y_3 - y_2}{x_3 - x_2} \\ \lambda_i k_{i-1} + 2k_i + \mu_i k_{i+1} = 3\left(\mu_i \dfrac{y_{i+1} - y_i}{x_{i+1} - x_i} + \lambda_i \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}\right) \\ (i = 2, 3, \cdots, n-1) \\ \lambda_{n-1}^2 k_{n-2} + (\lambda_{n-1}^2 - \mu_{n-1}^2) k_{n-1} - \mu_{n-1}^2 k_n = 2\lambda_{n-1}^2 \dfrac{y_{n-1} - y_{n-2}}{x_{n-1} - x_{n-2}} - 2\mu_{n-1}^2 \dfrac{y_n - y_{n-1}}{x_n - x_{n-1}} \end{cases}$$

若写成矩阵形式: $Ax = b$

其中: $A = \begin{bmatrix} \lambda_2^2 & \lambda_2^2 - \mu_2^2 & -\mu_2^2 & & & \\ \lambda_2 & 2 & \mu_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \lambda_{n-1} & 2 & \mu_{n-1} & \\ & & & \lambda_{n-1}^2 & \lambda_{n-1}^2 - \mu_{n-1}^2 & -\mu_{n-1}^2 \end{bmatrix}$

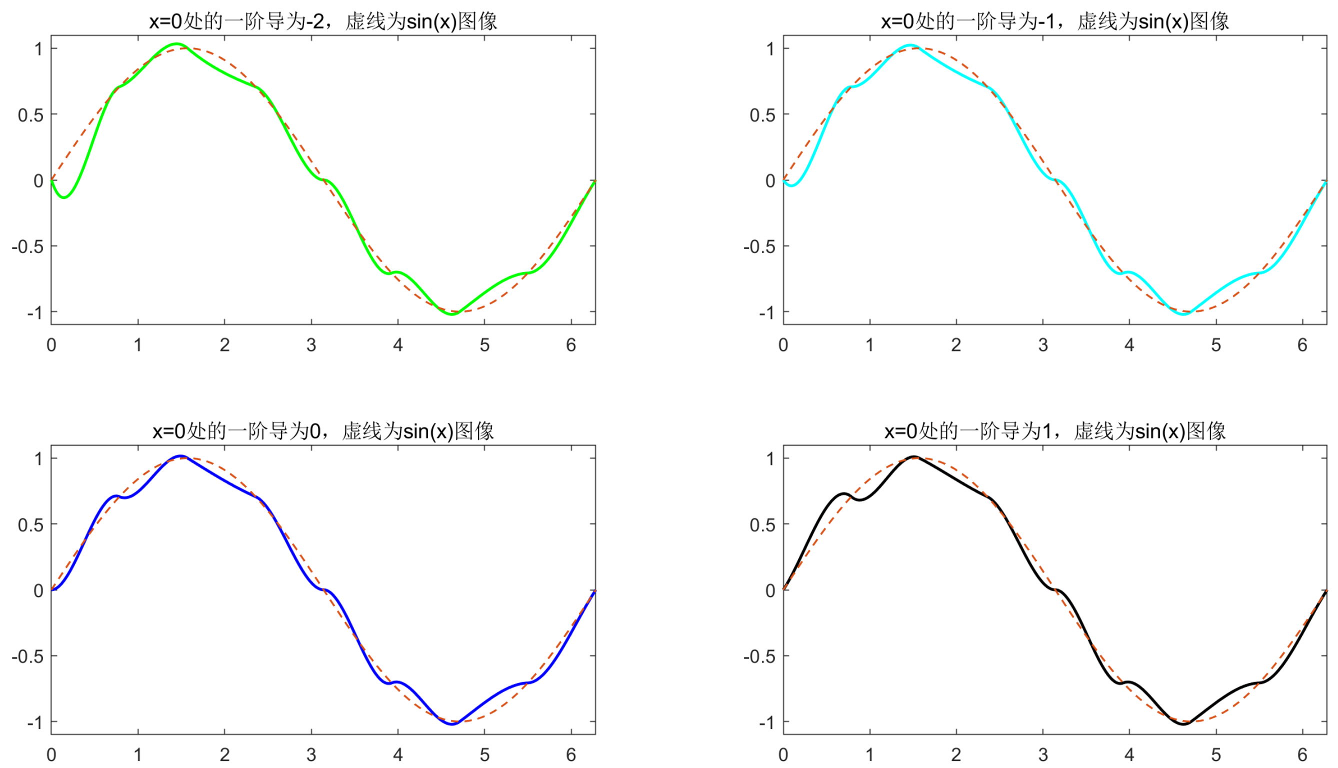
$x = (k_1, k_2, \cdots, k_n)^T$

$$b = \begin{bmatrix} 2\lambda_2^2 \dfrac{y_2 - y_1}{x_2 - x_1} - 2\mu_2^2 \dfrac{y_3 - y_2}{x_3 - x_2} \\ \vdots \\ 3\left(\mu_i \dfrac{y_{i+1} - y_i}{x_{i+1} - x_i} + \lambda_i \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}\right) \\ \vdots \\ 2\lambda_{n-1}^2 \dfrac{y_{n-1} - y_{n-2}}{x_{n-1} - x_{n-2}} - 2\mu_{n-1}^2 \dfrac{y_n - y_{n-1}}{x_n - x_{n-1}} \end{bmatrix} \quad (i = 2, \cdots, n-1)$$

$\square$

3. When performing interpolation with a complete cubic spline, the choice of derivatives on the boundary is important. Suppose that Bob wants to interpolate the sine function $f(x) = \sin x$ at nine equispaced nodes over $[0, 2\pi]$, with $f'(0) = f'(2\pi) = 1$. Unfortunately, he made a typo on $f'(0)$ in his program and observed some strange results. Try to reproduce Bob's result with a few different values of $f'(0)$. For instance, $f'(0) = 0$, $f'(0) = -1$, etc.

解：实验结果如下：



观察发现，三次样条插值对边界条件的变化不敏感，当 $f'(0)$ 从 (-2) 变化至 $(+1)$ 时，插值的图形变化很小，这是其相较于多项式插值（局部变化引发整体显著扭动）的一个优势。□

4. The temperature in the human body is not a constant, but rather follows a daily rhythm driven by an internal biological clock. The following table lists 20 averaged values of temperature measurements taken from 70 English sailors in an experiment done in 1971.

Interpolate the data with a periodic cubic spline and plot your solution for a two-day-period.

解：本题给的数据是从 1:00～23:00 的体温，不足一个 24h（一个周期），为完成周期样条插值，还需要自己补充 $x_{n+1}=25h$ 处的数据 $y_{n+1}=y_1=36.37°C$，下面是在两个周期（48h）中的插值效果图：