5月29日作业

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1. Develop a quadrature rule for the integral $\int_a^b \cos(mx) f(x)\, \mathrm{d}x$ such that it provides exact results for polynomials of degree up to three.

解：把 e^{imx} 视为权函数，再在 $[a,b]$ 上对 $f(x)$ 进行四个点插值，

得到系数 A_k，再取实部即可

$$\int_a^b e^{imx} f(x)\, dx \approx \int_a^b e^{imx} \sum_{k=0}^{3} f(x_k)\cdot l_k(x)\, dx$$

$$= \sum_{k=0}^{3} A_k f(x_k)$$

其中 $l_k(x)$ 为 Lagrange 插值多项式

$$A_0 = \int_a^b e^{imx} l_0(x)$$

$$= \int_a^b \frac{\exp(i\,m\,x)\left(\left(x-\frac{1}{3}(2a+b)\right)\left(x-\frac{1}{3}(a+2b)\right)(x-b)\right)}{\left(a-\frac{1}{3}(2a+b)\right)\left(a-\frac{1}{3}(a+2b)\right)(a-b)}\,dx = \frac{1}{2m^4(a-b)^3}\left(2e^{ibm}(-27+m(a-b)(am-bm-9i))+e^{iam}(54+m(a-b)(m(a-b)(-11+2im(a-b))-36i))\right)$$

类似的还有：

$$A_1 = \int_a^b e^{imx} l_1(x)$$

$$= \int_a^b \frac{\exp(i\,m\,x)\left((x-a)\left(x-\frac{1}{3}(a+2b)\right)(x-b)\right)}{\left(\frac{1}{3}(2a+b)-a\right)\left(\frac{1}{3}(2a+b)-\frac{1}{3}(a+2b)\right)\left(\frac{1}{3}(2a+b)-b\right)}\,dx = \frac{1}{2m^4(a-b)^3}\left(18e^{iam}(-9+m(a-b)(am-bm+5i))-9e^{ibm}(-18+m(a-b)(am-bm-8i))\right)$$

$$A_2 = \int_a^b e^{imx} l_2(x)$$

$$= \int_a^b \frac{\exp(i\,m\,x)\left((x-a)\left(x-\frac{1}{3}(2a+b)\right)(x-b)\right)}{\left(\frac{1}{3}(a+2b)-a\right)\left(\left(\frac{1}{3}(a+2b)-\frac{1}{3}(2a+b)\right)\left(\frac{1}{3}(a+2b)-b\right)\right)}\,dx = \frac{1}{2m^4(a-b)^3}\left(18e^{ibm}(-9+m(a-b)(am-bm-5i))-9e^{iam}(-18+m(a-b)(am-bm+8i))\right)$$

$$A_3 = \int_a^b e^{imx}\, l_3(x)$$

$$= \int_a^b \frac{\exp(i\,m\,x)\left((x-a)\left(x-\frac{1}{3}(2a+b)\right)\left(x-\frac{1}{3}(a+2b)\right)\right)}{(b-a)\left(b-\frac{1}{3}(2a+b)\right)\left(b-\frac{1}{3}(a+2b)\right)}\,dx =$$

$$\frac{1}{2m^4(a-b)^3}\left(2e^{iam}(-27+m(a-b)(am-bm+9i))+e^{ibm}(54+m(a-b)(m(a-b)(-11-2im(a-b))+36i))\right)$$

$$\Rightarrow \int_a^b \cos mx\, f(x) \approx \sum_{k=0}^{3} \mathrm{Re}(A_k)\cdot f(x_k) \qquad (*)$$

由构造过程可知，上式 (*) 对不超过 3 次的多项式精确成立。 $\square$

2. Determine the degree of exactness of the following 2-D quadrature rule:

$$\int_0^1\int_0^{1-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{6}\left(f\left(\frac{2}{3},\frac{1}{6}\right)+f\left(\frac{1}{6},\frac{2}{3}\right)+f\left(\frac{1}{6},\frac{1}{6}\right)\right).$$

Hint: Check whether the quadrature rules provide exact results for bivariate polynomials 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, ...

解：下面记 $I[f] = \int_0^1\int_0^{1-y} f(x,y)\,dx\,dy$

$$I_n[f] = \frac{1}{6}\left(f\left(\frac{2}{3},\frac{1}{6}\right)+f\left(\frac{1}{6},\frac{2}{3}\right)+f\left(\frac{1}{6},\frac{1}{6}\right)\right)$$

$f(x,y) = 1$ 时

$$I[f] = I_n[f] = \frac{1}{2}$$

$f(x,y) = x$ 时

$$I[f] = I_n[f] = \frac{1}{6}$$

由对称性，$f(x,y)=y$ 时也有 $I[f]=I_n[f]$

$f(x,y)=x^2$ 时

$I[f]=I_n[f]=\frac{1}{12}$

由对称性，$f(x,y)=y^2$ 时也有 $I[f]=I_n[f]=\frac{1}{12}$

$f(x,y)=xy$ 时

$I[f]=I_n[f]=\frac{1}{24}$

$f(x,y)=x^3$ 时

$$\left.\begin{aligned} I[f] &= \frac{1}{20} \\ I_n[f] &= \frac{11}{216} \end{aligned}\right\} \quad I[f]\neq I_n[f]$$

$\Rightarrow$ 代数精度为2 $\square$

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3. Let $\mathcal{D}=\{(x,y)\in\mathbb{R}^2\colon x+y\le 1,\ x\ge 0,\ y\ge 0\}$. Estimate

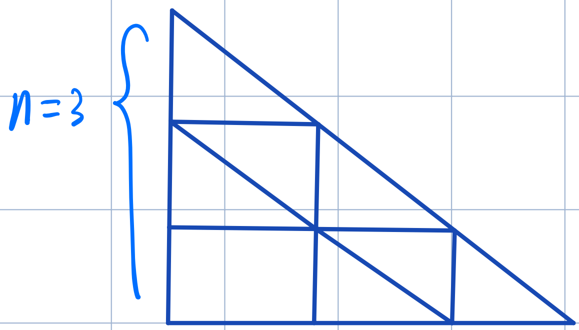
$$\iint_{\mathcal{D}} e^x \sin y \,dx\,dy$$

by partitioning $\mathcal{D}$ with a triangular mesh and applying a composite quadrature rule. Compare your result with the exact one.

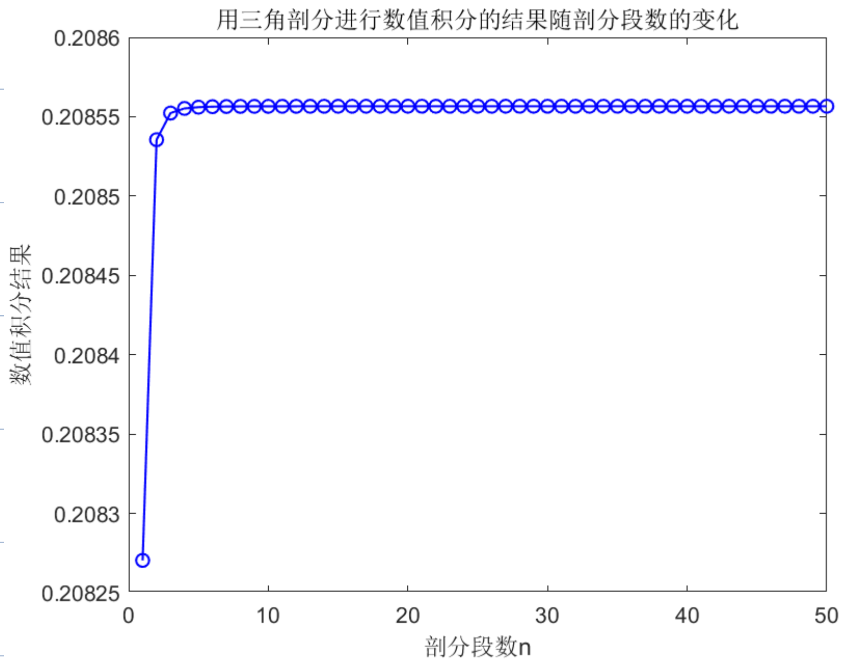
解：精确解：

$$\int_0^1\int_0^{1-y} e^x \sin y \,dx\,dy = \frac{1}{2}(-2+e-\sin 1+\cos 1)$$
$$\approx 0.20856$$

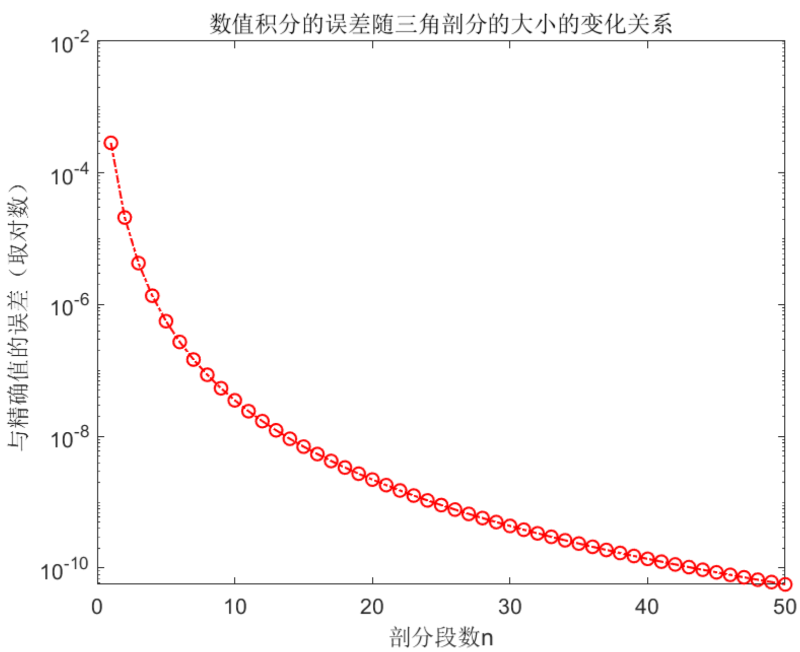
用如下方式进行三角剖分，记剖分的段数为n



结果如下：



误差为：



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4. Let us consider the following family of methods

$$u_{k+1} = u_k + h\big(\theta f(t_{k+1}, u_{k+1}) + (1-\theta) f(t_k, u_k)\big)$$

to approximate the solution of the IVP

$$\begin{cases} u'(t) = f\big(t, u(t)\big), & (t > 0), \\ u(0) = u_0, \end{cases}$$

where $\theta \in [0,1]$ is a parameter to be chosen. When this family of methods is applied to solve the model problem

$$\begin{cases} u'(t) = -\lambda u(t), & t > 0, \\ u(0) = u_0, \end{cases}$$

where λ is a given positive real number, the step size h needs to be chosen inside a certain region (the so-called *stable region*) to preserve the decay property of the solution. Determine (in terms of θ), the values of h for which the computed solution converges to 0 as $k \to \infty$. Note that by default h is a positive real number in practice.

解: $$u_{k+1} = u_k + h\big(\theta \cdot (-\lambda u_{k+1}) + (1-\theta) \cdot (-\lambda \cdot u_k)\big)$$

$$\Rightarrow \quad u_{k+1} = \frac{1 - \lambda h (1-\theta)}{1 + \lambda h \theta} \cdot u_k$$

令: $$\left| \frac{1 - \lambda h (1-\theta)}{1 + \lambda h \theta} \right| < 1$$

$$\Rightarrow \quad |1 - \lambda h (1-\theta)| < |1 + \lambda h \theta|$$

由于 $\lambda > 0$, $\theta \geq 0$, 则 $1 + \lambda h \theta > 0$

$$\Rightarrow \quad |1 - h \cdot \lambda (1-\theta)| < 1 + \lambda h \theta$$

若 $1 - h\lambda(1-\theta) \geq 0$

即 $h \leq \dfrac{1}{\lambda(1-\theta)}$ $\quad (\theta \neq 1)$

$\Rightarrow \quad 1 - h\lambda(1-\theta) \leq 1 + \lambda h \theta$ 显然成立

若 $\quad 1 - h\lambda(1-\theta) < 0$

即 $h > \dfrac{1}{\lambda(1-\theta)}$ $\quad (\theta \neq 1)$

$\Rightarrow \quad -1 + h\lambda(1-\theta) \leq 1 + \lambda h \theta$

$\Rightarrow \quad h > 0 \quad$ if $\quad \theta \geq \frac{1}{2}$

$h \leq \dfrac{2}{\lambda(1-2\theta)} \quad \theta < \frac{1}{2}$

另外还有 $\theta = 1$ 时 $|1 + \lambda h \theta| > 1 \Rightarrow h > 0$

综上: $\theta \in [0, \frac{1}{2})$, $\quad h \in \left(0, \dfrac{2}{\lambda(1-2\theta)}\right)$

$\theta \in [\frac{1}{2}, 1]$, $\quad h \in (0, +\infty)$

$\square$

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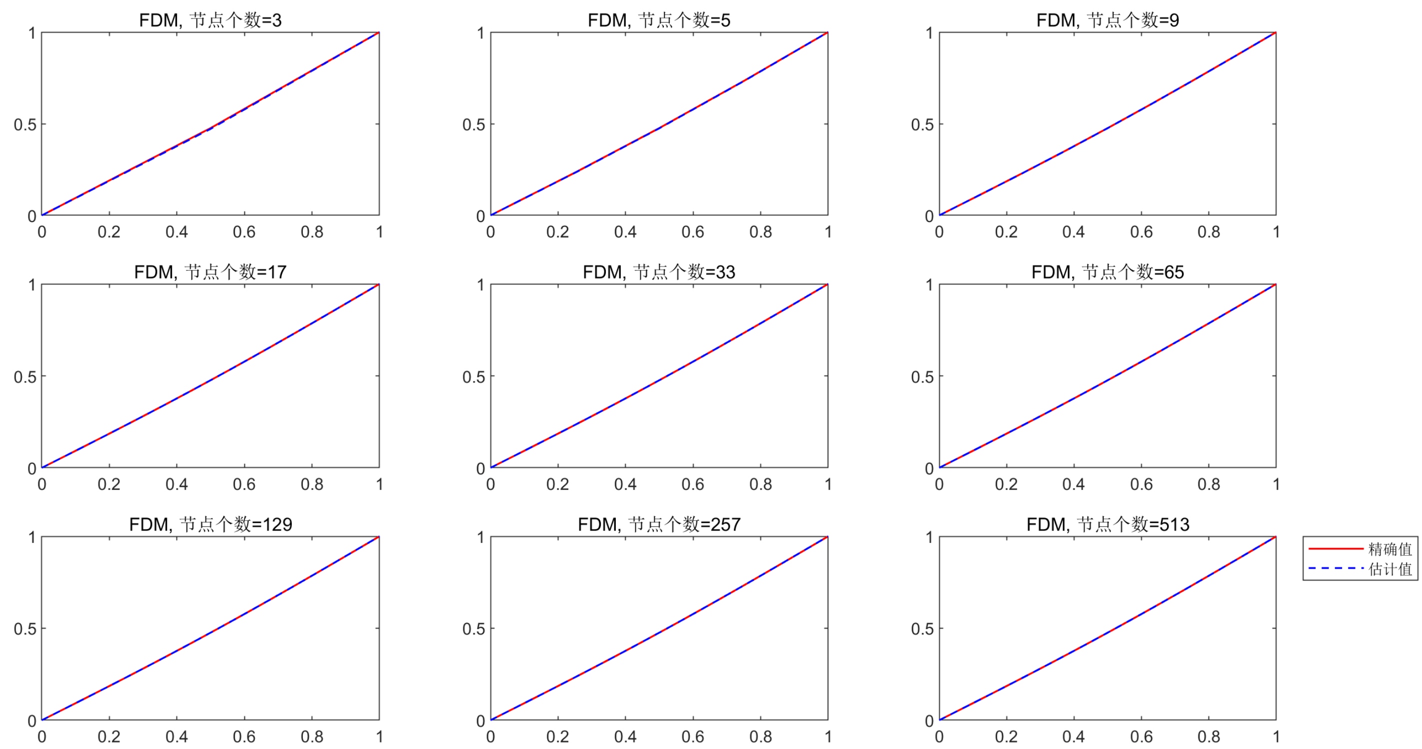
5. Use the finite difference method and the finite element method (with linear elements), both on $n+1$ equispaced nodes, to solve the boundary value problem

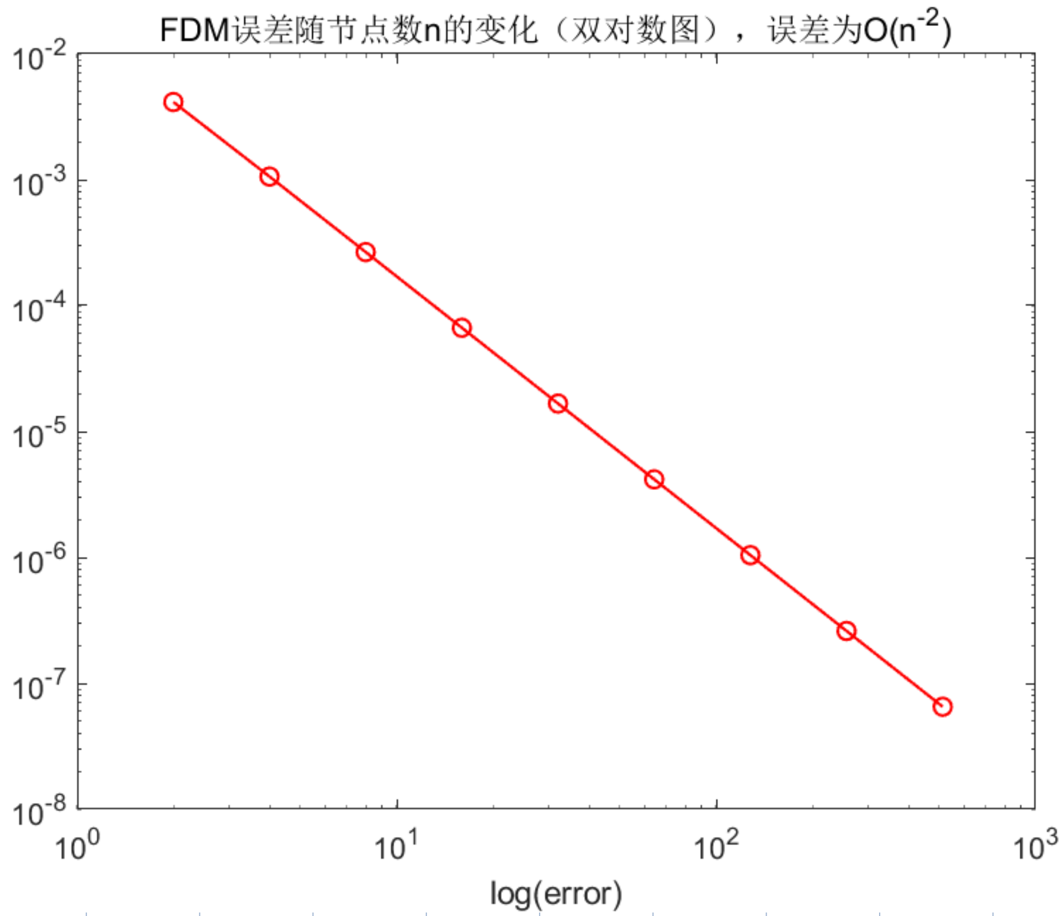
$$\begin{cases} -u''(x) + u(x) = x^2, & (0 < x < 1) \\ u(0) = 0, \ u(1) = 1. \end{cases}$$

Try a few different values of n and compare your solutions with the exact one.

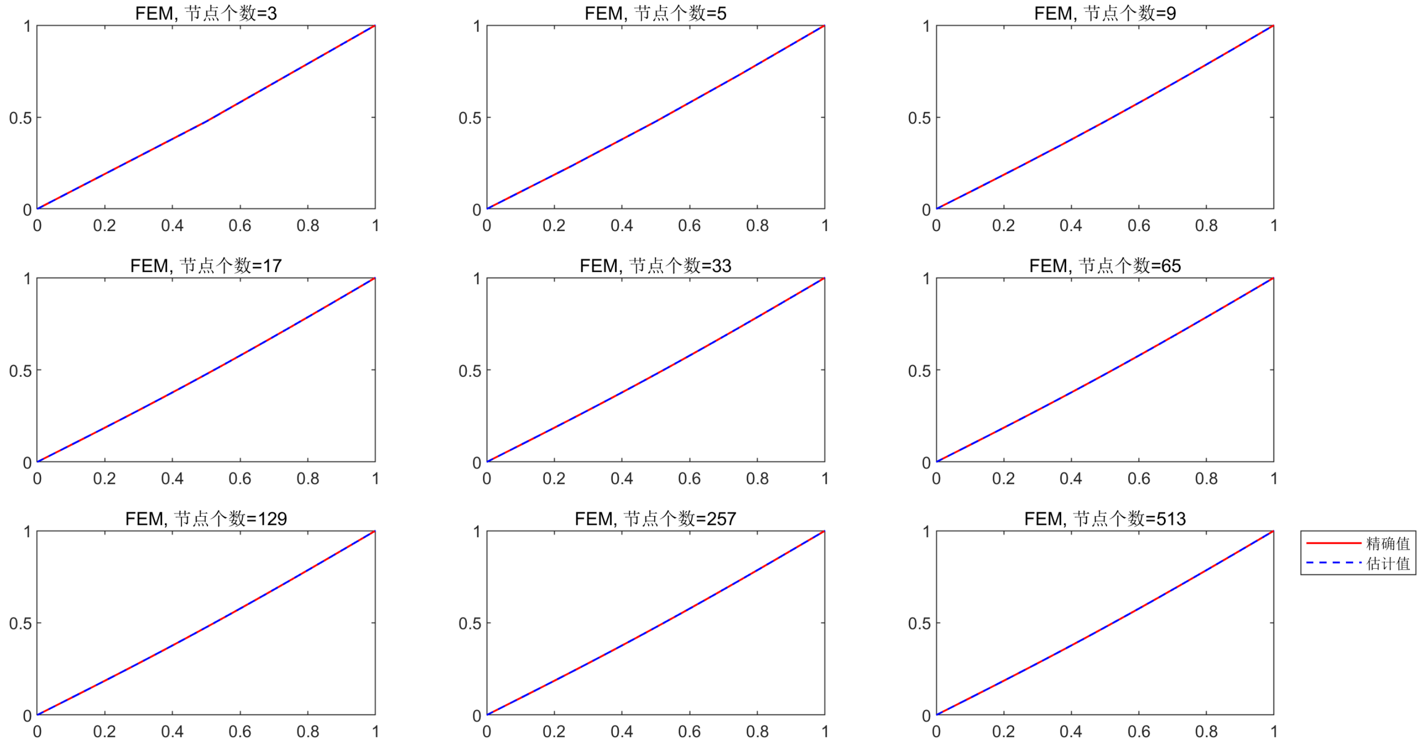
有限差分法效果如下：



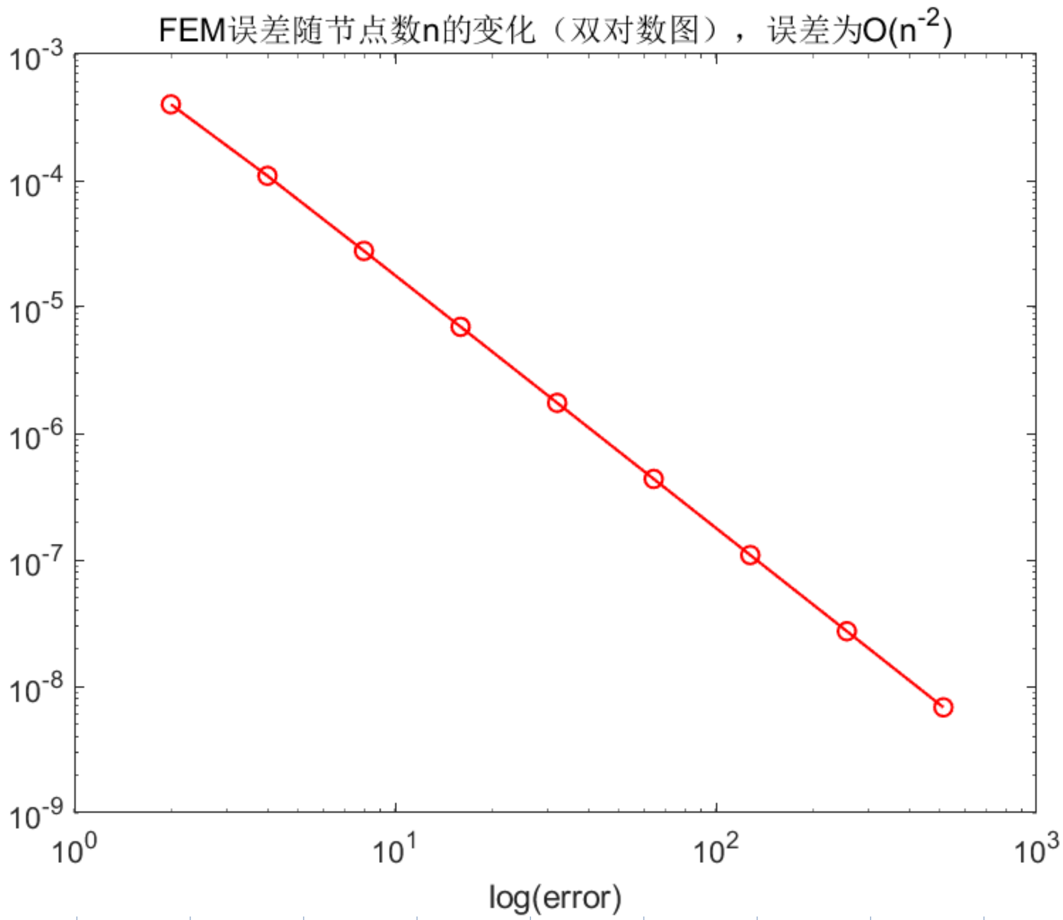
误差如下，误差为 $O(n^{-2})$，或者 $O(h^2)$，其中 $h=\frac{1}{n+1}$



有限元法效果如下:



误差情况如下, 为 $O(n^{-2})$, 或 $O(h^2)$, 其中 $h=\frac{1}{n+1}$



□

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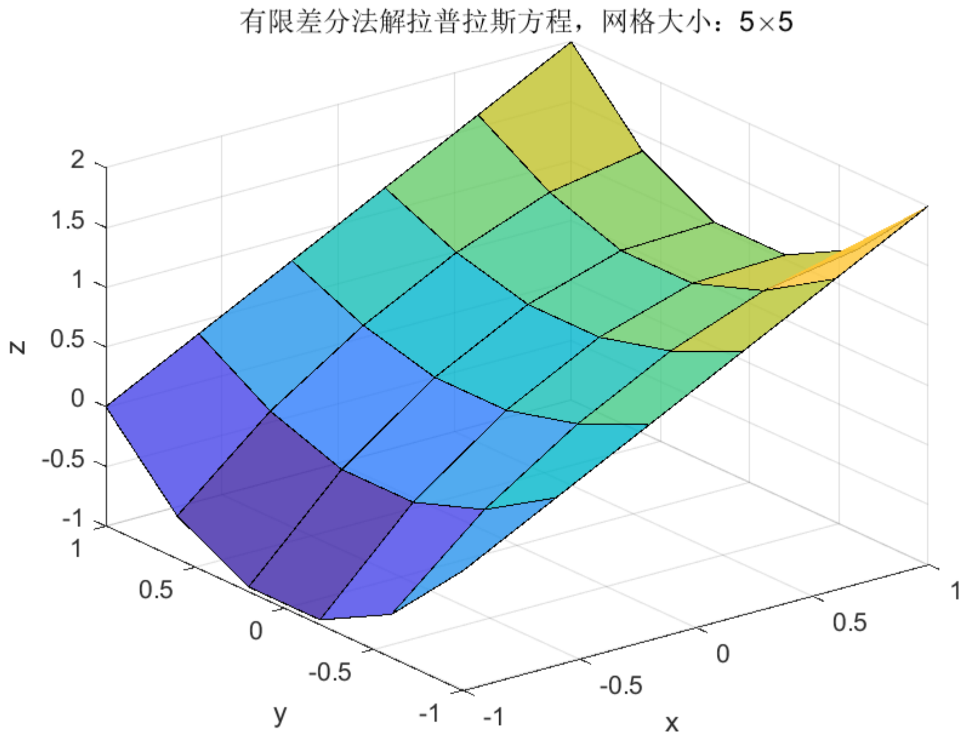
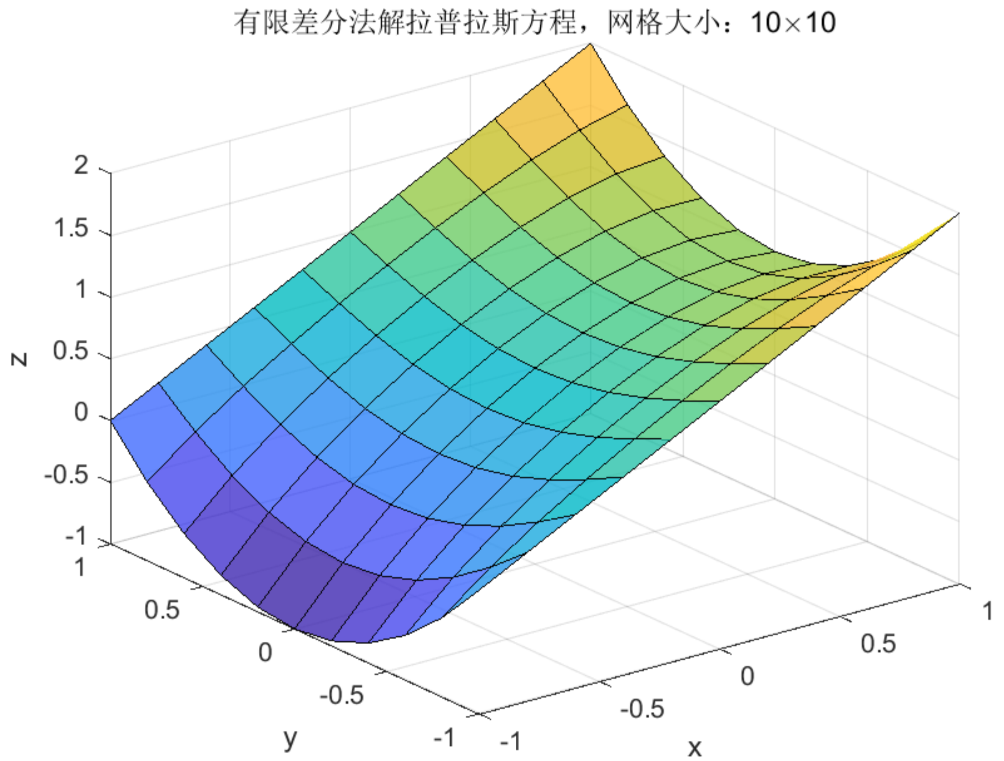
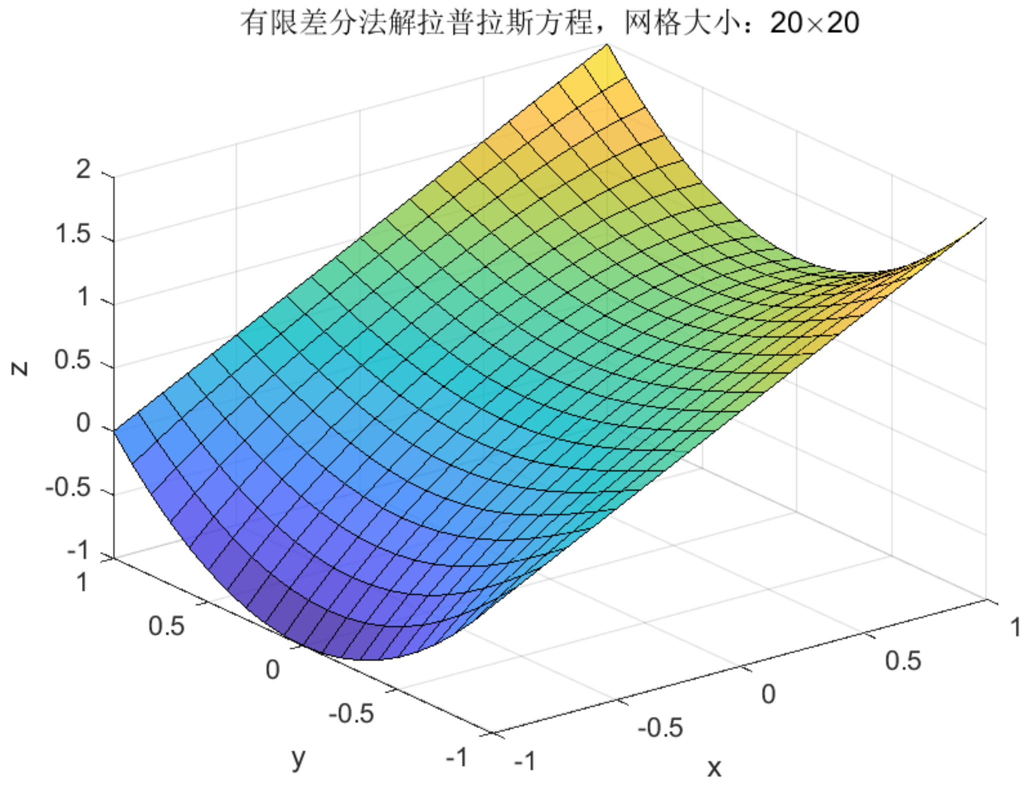
6. Solve the partial differential equation

$$\begin{cases} \dfrac{\partial^2 u(x,y)}{\partial x^2} + \dfrac{\partial^2 u(x,y)}{\partial y^2} = 0, & (-1 < x < 1,\ -1 < y < 1) \\ u(x,-1) = u(x,1) = x + 1, & (-1 < x < 1) \\ u(-1,y) = y^2 - 1,\ \ u(1,y) = y^2 + 1, & (-1 < y < 1) \end{cases}$$

using the finite difference method. Visualize your solution.






有限差分法解拉普拉斯方程，网格大小：20×20
z
2
1.5
1
0.5
0
-0.5
-1
y
1
0.5
0
-0.5
-1
x
-1
-0.5
0
0.5
1