Mar. 6, 2023 (Due: 08:00 Mar. 13, 2023)

1. Use bisection and *regula falsi*, respectively, over the interval $[0, 1]$ to find the root of $x^{64} - 0.1 = 0$ with absolute accuracy 10^{-12}. Visualize the convergence history of these methods in one figure.
2. Solve Exercise 3 from last week's homework again, using Broyden's method.
3. Polynomial interpolation provides one way to approximate a given function. For instance, let $0 = x_1 < x_2 < \cdots < x_n = 2\pi$ be equally spaced interpolation nodes. The interpolation polynomial passing through all $(x_i, \sin x_i)$'s can be used to approximate the sine function $f(x) = \sin x$. Try to visualize the difference between the interpolation polynomial and the sine function for a few different choices of n.

What happens if the same technique is applied to approximate the rational function $f(x) = (1 + 25x^2)^{-1}$ over $[-1, 1]$ (using equally spaced interpolation nodes)?
4. Polynomial interpolation are useful not only in modeling, but also in pure mathematics. For instance, the Chinese Remainder Theorem can be derived from Lagrange's approach. This exercise is another example.

Let $f(x)$ be a real polynomial of degree n. Suppose that there exists $i \in \mathbb{Z}$ such that $f(i)$, $f(i+1)$, ..., $f(i+n)$ are all integers. Show that $f(k) \in \mathbb{Z}$ for any $k \in \mathbb{Z}$. Is it also true that $f(x) \in \mathbb{Z}[x]$, i.e., $f(x)$ has integer coefficients?
5. (optional) Let

$$a_1, a_2, \ldots, a_n, a_1, a_2, \ldots, a_n, \ldots, a_1, a_2, \ldots, a_n, \ldots$$

be a periodic sequence. Can you find a smooth function to interpolate this infinite sequence?
6. (optional) In the divide and conquer algorithm for solving symmetric eigenvalue problems the eigenvalues of a *diagonal plus rank one* matrix needs to be computed by solving a rational function. Write a program to solve this type of rational equations.