4月23日作业

1 **1.** Show that the DFT matrix F_n diagonalizes

$$J_n = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix},$$

i.e., $F_n^{-1} J_n F_n$ is diagonal.

解：设 $F_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{n-1} \\ \vdots & & & \\ 1 & w^{n-1} & \cdots & w^{(n-1)^2} \end{bmatrix}$

或者：$(F_n)_{ij} = w^{ij} \qquad i,j = 0,1,\cdots,n-1$

$$J_n \cdot F_n = \begin{bmatrix} 1 & w & \cdots & w^{n-1} \\ 1 & w^2 & \cdots & w^{2(n-1)} \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

$(J_n F_n) = w^{(i+1)\cdot j} \qquad i,j = 0,1,\cdots,n-1$

$F_n^{-1} = \frac{1}{n} \cdot F_n^* = \frac{1}{n} \cdot \overline{F_n}$, $(F_n^{-1})_{ij} = \frac{1}{n} \cdot \bar{w}^{ij}$

即 $F_n^{-1} = \frac{1}{n} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \bar{w} & \cdots & \bar{w}^{n-1} \\ \vdots & \vdots & & \\ 1 & \bar{w}^{n-1} & \cdots & \bar{w}^{(n-1)^2} \end{bmatrix}$

$(F_n^{-1} \cdot J_n \cdot F_n)_{ij}$

$$= \sum_{k=0}^{n-1} (F_n^{-1})_{ik} \cdot (J_n F_n)_{k\cdot j}$$

$$= \sum_{k=0}^{n-1} \cdot \frac{1}{n} \bar{\omega}^{ik} \cdot \omega^{(k+1)j}$$

$$= \frac{1}{n} \cdot \omega^j \sum_{k=0}^{n-1} (\bar{\omega}^i \cdot \omega^j)^k$$

$$= \frac{1}{n} \cdot \omega^j \sum_{k=0}^{n-1} \cdot (\omega^{j-i})^k$$

当 $j = i$ 时，$(F_n^{-1} J_n F_n)_{ij} = \frac{1}{n} \cdot \omega^j \cdot n = \omega^j$

当 $j \neq i$ 时，$\sum_{k=0}^{n-1} (\omega^{j-i})^k = 0$，则 $(F_n^{-1} J_n F_n)_{ij} = 0$

$\Rightarrow$ $F_n^{-1} J_n F_n$ 是对角阵 $\square$

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2. For $x \in \mathbb{C}^n$, its Fourier transform $X = F_n x$ is in general a complex vector. When is X real? Can you find a sufficient and necessary condition? What about 2-D FFT?

解：(1) 首先讨论一维情况

若 $X = F_n x \in \mathbb{R}^n$

则 $\bar{X} = X$

也即 $\bar{F}_n \bar{x} = F_n x$

$\Rightarrow x = F_n^{-1} \cdot \bar{F}_n \bar{x}$

其中 $F_n^{-1}\cdot\bar{F_n}$

$$= \frac{1}{n}\cdot(\bar{F_n})^2$$

设 $\bar{F_n} = \begin{bmatrix} f_1^T \\ \vdots \\ f_n^T \end{bmatrix}$ 其中 $f_i = 1, \bar{\omega}^i, \cdots, \bar{\omega}^{i(n-1)}$

$$(\bar{F_n})_{ij} = f_i^T\cdot f_j$$

$$= \sum_{k=0}^{n-1}\left(\bar{\omega}^{i+j}\right)^k$$

当 $i+j=0$ 或 $i+j=n$ 时 $(\bar{F_n})_{ij} = n$

当 $i+j\neq 0$ 且 $i+j\neq n$ 时，$(\bar{F_n})_{ij} = 0$

则 $F_n^{-1}\cdot\bar{F_n} = \begin{bmatrix} 1 & & & 0 \\ & & & 1 \\ & & \iddots & \\ 0 & 1 & & \mathbf{0} \end{bmatrix}$

因为 $J_n = F_n^{-1}\cdot\bar{F_n}$

则 $\bar{x}\in R^n$ 的必要条件是 $x = J_n\cdot\bar{x}$

也即: $x_i = \begin{cases} \bar{x}_i & \text{if } i=0 \\ \bar{x}_{n-i} & \text{if } 0<i\le n-1 \end{cases}$

反过来，若 $x = J_n \cdot \bar{x} = F_n^{-1} \cdot \bar{F_n} \cdot \bar{x}$

$\Rightarrow \quad F_n \cdot x = \bar{F_n} \cdot \bar{x}$

记 $X = F_n \cdot x$ 则 $\bar{X} = \bar{F_n} \cdot \bar{x}$

$\Rightarrow \quad X = \bar{X}$

$\Rightarrow \quad X \in \mathbb{R}^n$

故 $x = J_n \cdot \bar{x}$ 也是充分条件

(2) 考虑二维情况

设 $Z \in \mathbb{C}^{m \times n}$，则 2-D FFT 可写作：

$F_m \cdot Z \cdot F_n$

类似一维情况的讨论，必要条件为：

$F_m \cdot Z \cdot F_n = \bar{F_m} \cdot \bar{Z} \cdot \bar{F_n}$

即 $Z = (F_m^{-1} \cdot \bar{F_m}) \cdot \bar{Z} \cdot (\bar{F_n} \cdot F_n^{-1})$

由于 J 是一个对称阵

则 $Z = J_m \cdot \bar{Z} \cdot J_n$

类似一维的情况，也可知 $Z = J_m \cdot \bar{Z} \cdot J_n$ 是充分条件 □

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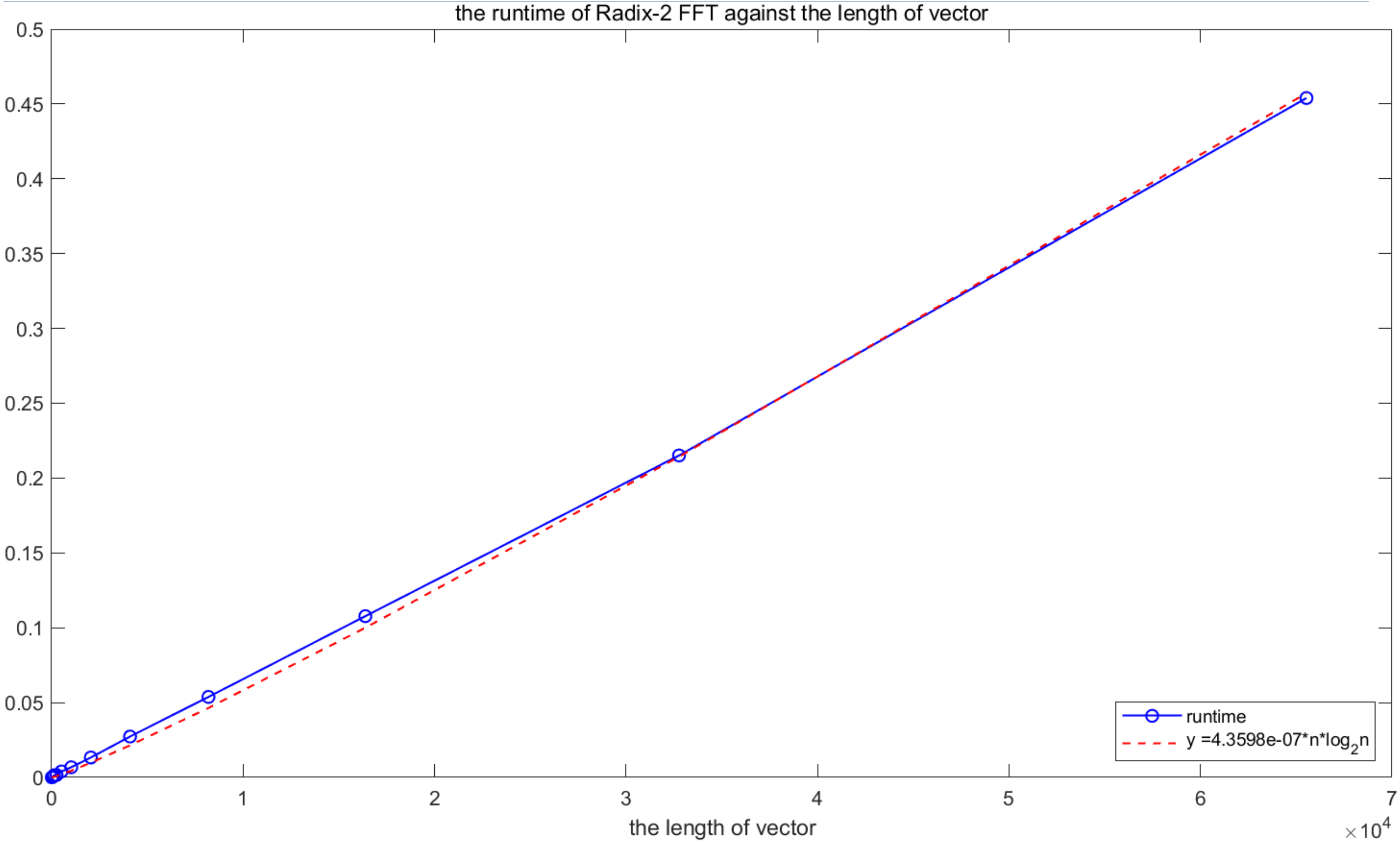
3. Implement Radix-2 FFT. (A non-recursive implementation is preferred.) Test your implementation with vectors of various lengths to make sure your implementations have complexity $\Theta(n \log n)$.



4. Use trigonometric polynomials up to degree d to interpolate the periodic square wave

$$f(x)=\begin{cases}1, & x\in(2n,2n+1)\\ -1, & x\in(2n-1,2n)\\ 0, & x \text{ is an integer}\end{cases}$$

with equispaced interpolation nodes. Make plots for a few different values of d. What happens if d becomes large?

用"exercise_4_v1.m"文件画出来的不对，尽管是按照教材中以下公式写的：

因此，$f(x)$在 N 个点$\left\{x_j=\frac{2\pi}{N}j, j=0,1,\cdots,N-1\right\}$上的最小二乘傅里叶逼近为

$$S(x)=\sum_{k=0}^{n-1}c_k e^{ikx},\quad n\leqslant N, \tag{6.6}$$

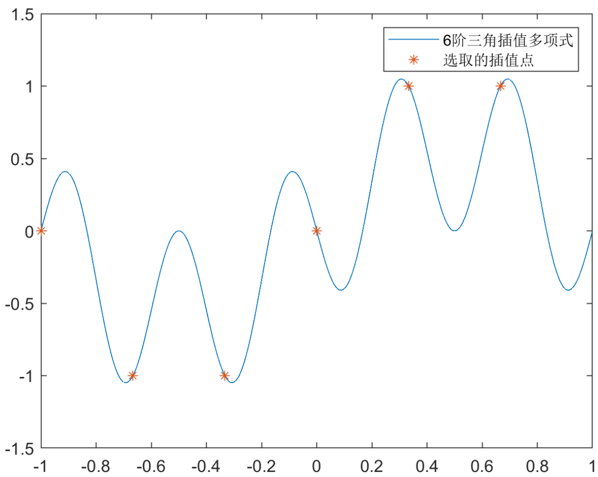
其中

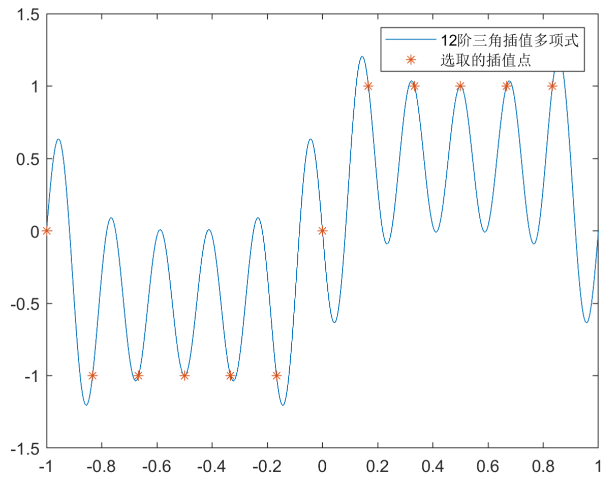
$$c_k=\frac{1}{N}\sum_{j=0}^{N-1}f_j e^{-ikj\frac{2\pi}{N}},\quad k=0,1,\cdots,n-1. \tag{6.7}$$

在(6.6)式中若 $n=N$，则 $S(x)$为 $f(x)$在点 $x_j(j=0,1,\cdots,N-1)$上的插值函数，即 $S(x_j)=f(x_j)$，于是由(6.6)式得

$$f_j=\sum_{k=0}^{N-1}c_k e^{ik\frac{2\pi}{N}j},\quad j=0,1,\cdots,N-1. \tag{6.8}$$

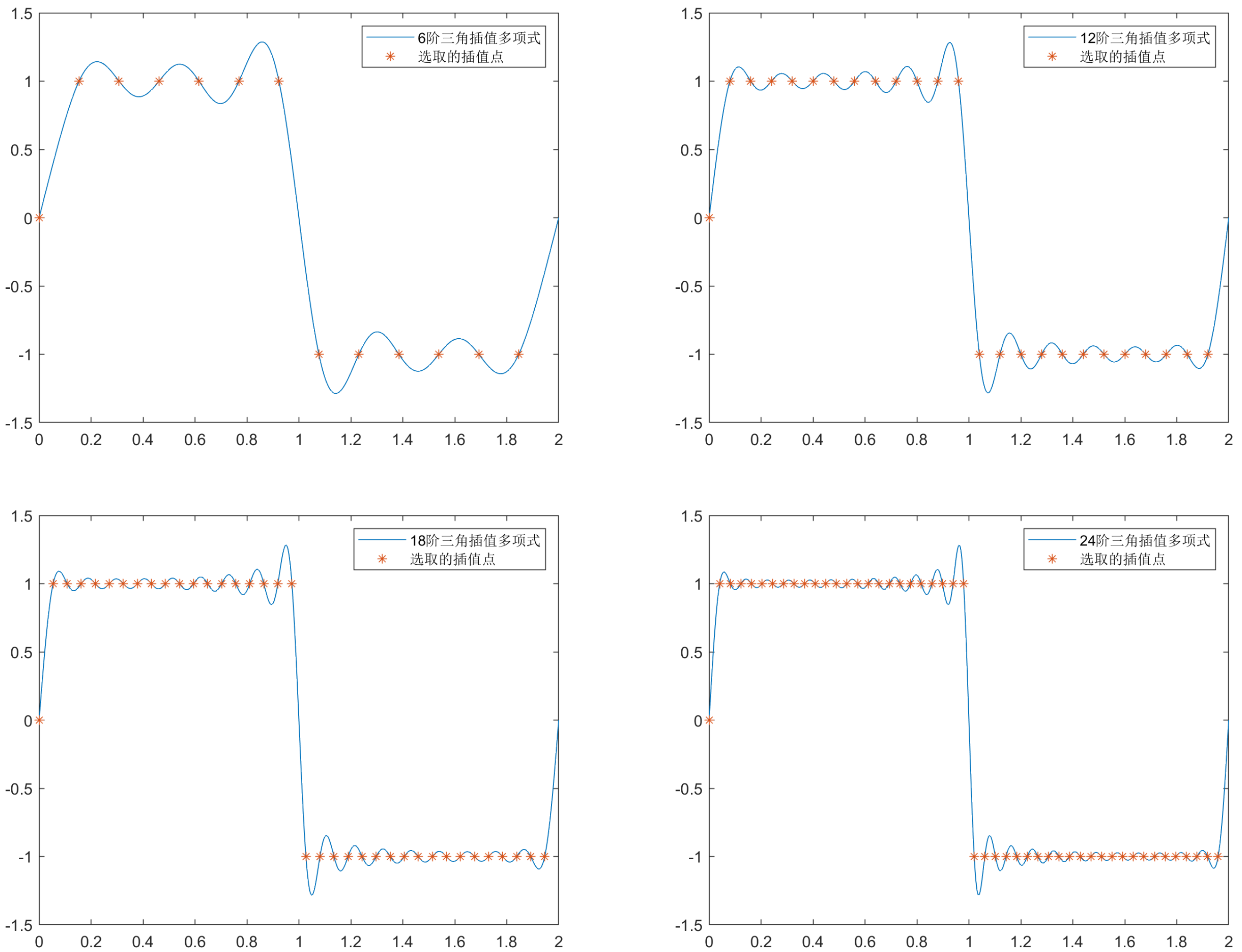
错误的图像如下：





Q：实在不明白为什么不对呀？我看了笔记，老师上课也是按照 $S(x)=\sum_{k=0}^{n-1}c_k\cdot e^{ikx}$ 的公式在讲呀！

然后在网上又查了一下，改用"exercise_4_v2.m"文件中的方法，便能得到预期结果，如下：



随着阶数增大，三角插值多项式越来越逼近原函数 □

5

5. You are given an audio file `DTMF_dialing.ogg`, which contains 80 touch tones from a DTMF keyboard. Try to determine the keys corresponding to the tones according to the following table.

	1209 Hz	1336 Hz	1477 Hz	1633 Hz
697 Hz	1	2	3	A
770 Hz	4	5	6	B
852 Hz	7	8	9	C
941 Hz	*	0	#	D

第1段数字为：	0	6	9	6	6	7	5	3	5	6
第2段数字为：	4	6	4	6	4	1	5	1	8	0
第3段数字为：	2	3	3	6	7	3	1	4	1	6
第4段数字为：	3	6	0	8	3	3	8	1	6	0
第5段数字为：	4	4	0	0	8	2	6	1	4	6
第6段数字为：	6	2	5	3	6	8	9	6	3	8
第7段数字为：	8	4	8	2	1	3	8	1	7	8
第8段数字为：	5	0	7	3	6	4	3	3	9	9

提示："exercise_5.m"文件会播放声音，请小心。□

6

6. (optional) Show that all eigenvalues of the unitary DFT matrix $n^{-1/2}F_n$ belong to the set $\{1, -1, \mathrm{i}, -\mathrm{i}\}$.

解：由第2题可知：$(n^{-1/2}F_n)\cdot(n^{-1/2}F_n) = J_n$

其中 $J_n = \begin{bmatrix} 1 & & & 0 \\ & & & 1 \\ & & \iddots & \\ 0 & 1 & & 0 \end{bmatrix}$

设λ是 $n^{-1/2}F_n$ 的特征值，x是对应特征向量：

（为方便，下面的下标编号从0开始，到 $n-1$ 为止）

$$n^{-1/2} F_n x = \lambda x$$

$$\Rightarrow \frac{1}{n} F_n \cdot F_n x = \lambda^2 x$$

$$\Rightarrow J_n \cdot x = \lambda^2 x$$

记 $x = (x_0, x_1, \cdots, x_{n-1})^T$

若 $x_0 \neq 0$，则 $x_0 = \lambda^2 x_0 \Rightarrow \lambda \in \{1, -1\}$

若 $x_0 = 0$，由于 x 为非零向量，必有某个分量

$x_k \neq 0$ $(k>0)$，那么：

$$\begin{cases} x_{n-k} = \lambda^2 x_k & (1) \\ x_k = \lambda^2 x_{n-k} & (2) \end{cases}$$

(1) 代入 (2) 得 $x_k = \lambda^2 \cdot \lambda^2 \cdot x_k$

$$\Rightarrow x_k = \lambda^4 x_k$$

由于 $x_k \neq 0 \Rightarrow \lambda \in \{1, -1, i, -i\}$

综上所述，$\lambda \in \{1, -1, i, -i\}$ $\square$

7. (optional) Implement Radix-3 FFT. Make sure your implementations have complexity $\Theta(n \log n)$.

You may find the MATLAB/Octave function `fft` helpful for debugging purpose.