5月8日作业

1 **1.** Derive the circular convolution theorem based on the following convention of DFT:

$$\widehat{u}_k = \sum_{j=0}^{n-1} \exp\Big(-\frac{2jk\pi\mathrm{i}}{n}\Big) u_j, \qquad (u \in \mathbb{C}^n).$$

解：设周期为 n，则 $u_{k+rn} = u_k, \quad r = \pm 1, \pm 2, \cdots$

$$v_{k+rn} = v_k, \quad r = \pm 1, \pm 2, \cdots$$

$$\exp\Big(\frac{-2(j+rn)k\pi i}{n}\Big) = \exp\Big(\frac{-2jk\pi i}{n}\Big), \quad r = \pm 1, \pm 2 \cdots$$

$$(u * v)_j = \sum_{p+q=j} u_p \cdot v_q$$

$$= \sum_{p=0}^{n-1} u_p \cdot v_{j-p}$$

（"∧"表示Fourier变换
"∨"表示逆Fourier变换）

$$\widehat{(u * v)}_k = \sum_{j=0}^{n-1} \exp\Big(\frac{-2jk\pi i}{n}\Big) \cdot (u * v)_j$$

$$= \sum_{j=0}^{n-1} \exp\Big(\frac{-2jk\pi i}{n}\Big) \cdot \sum_{p=0}^{n-1} u_p \cdot v_{j-p}$$

$$= \sum_{p=0}^{n-1} \sum_{j=0}^{n-1} \exp\Big(\frac{-2(j-p)k\pi i}{n}\Big) \cdot v_{j-p} \cdot \exp\Big(\frac{-2pk\pi i}{n}\Big) \cdot u_p$$

注意到：$\displaystyle\sum_{j=0}^{n-1} \exp\Big(\frac{-2(j-p)k\pi i}{n}\Big) \cdot v_{j-p}$

$$= \left(\sum_{j=0}^{p-1} + \sum_{j=p}^{n-1} \right) \exp\left(\frac{-2(j-p)k\pi i}{n}\right) \cdot v_{j-p}$$

$$= \left(\sum_{j=n}^{p-1+n} + \sum_{j=p}^{n-1} \right) \exp\left(\frac{-2(j-p)k\pi i}{n}\right) \cdot v_{j-p}$$

$$= \sum_{s=0}^{n-1} \cdot \exp \cdot \left(\frac{-2sk\pi i}{n}\right) \cdot v_s$$

则：

$$(\widehat{u * v})_k = \sum_{p=0}^{n-1} \cdot \sum_{j=0}^{n-1} \cdot \exp\left(\frac{-2(j-p)k\pi i}{n}\right) \cdot v_{j-p} \cdot \exp\left(\frac{-2pk\pi i}{n}\right) \cdot u_p$$

$$= \sum_{p=0}^{n-1} \cdot \sum_{s=0}^{n-1} \cdot \exp\left(\frac{-2sk\pi i}{n}\right) \cdot v_s \cdot \exp\left(\frac{-2pk\pi i}{n}\right) u_p$$

$$= \sum_{s=0}^{n-1} \cdot \exp\left(\frac{-2sk\pi i}{n}\right) \cdot v_s \cdot \sum_{p=0}^{n-1} \cdot \exp\left(\frac{-2pk\pi i}{n}\right) \cdot u_p$$

$$= \hat{v}_k \cdot \hat{u}_k$$

反过来：若 $\hat{u}$ 与 $\hat{v}$ 做卷积：

$$(\hat{u} * \hat{v})_k$$

$$= \sum_{p+q=k} \hat{u}_p \cdot \hat{v}_q$$

$$= \sum_{p=0}^{n-1} \hat{u}_p \cdot \hat{v}_{k-p}$$

$$= \sum_{p=0}^{n-1} \cdot \left[\sum_{j=0}^{n-1} \exp\left(\frac{-2jp\pi i}{n}\right) \cdot u_j \right] \cdot \left[\sum_{s=0}^{n-1} \cdot \exp\left(\frac{-2s(k-p)\pi i}{n}\right) v_s \right]$$

$$= \sum_{j=0}^{n-1} \sum_{s=0}^{n-1} \sum_{p=0}^{n-1} u_j \cdot v_s \cdot \exp\left(\frac{-2sk\pi i}{n}\right) \cdot \exp\left(\frac{-2(j-s)p\pi i}{n}\right)$$

$$= \sum_{j=0}^{n-1} \sum_{s=0}^{n-1} u_j \cdot v_s \cdot \exp\left(\frac{-2sk\pi i}{n}\right) \cdot \sum_{p=0}^{n-1} \cdot \exp \cdot \left(\frac{-2(j-s)p\pi i}{n}\right)$$

$$= \sum_{j=0}^{n-1} \cdot u_j \cdot v_j \cdot \exp\left(\frac{-2jk\pi i}{n}\right) \cdot n$$

$$= n \cdot \widehat{(u \cdot v)}$$

也即：

$$\frac{\widecheck{(\hat{u} * \hat{v})}}{n} = u \cdot v \qquad \square$$

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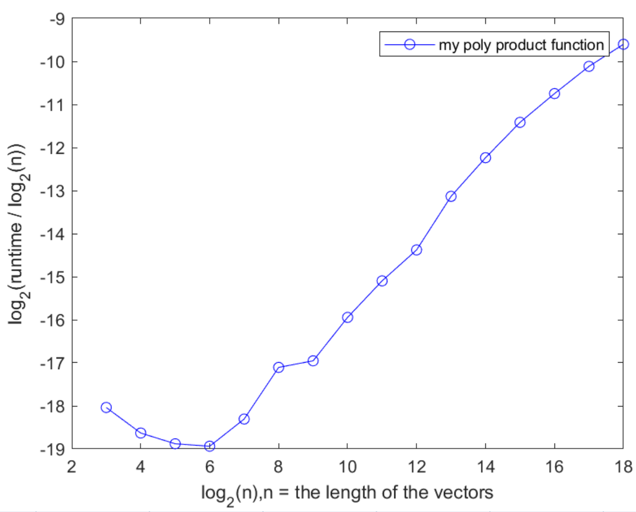
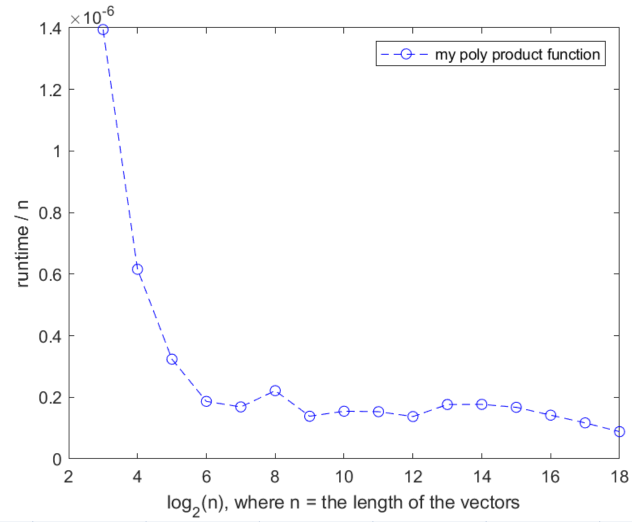
2. Write a program to compute the product of two complex polynomials using fast convolution algorithms. You can make use of the MATLAB/Octave function `conv` to check the correctness of your implementation. Make a plot to demonstrate that the complexity of your implementation is $\Theta(n \log n)$. What is the complexity of `conv`?

我写的多项式乘法函数:





左图相当于对 $\log(t/\log n) \sim \log n$ 作图

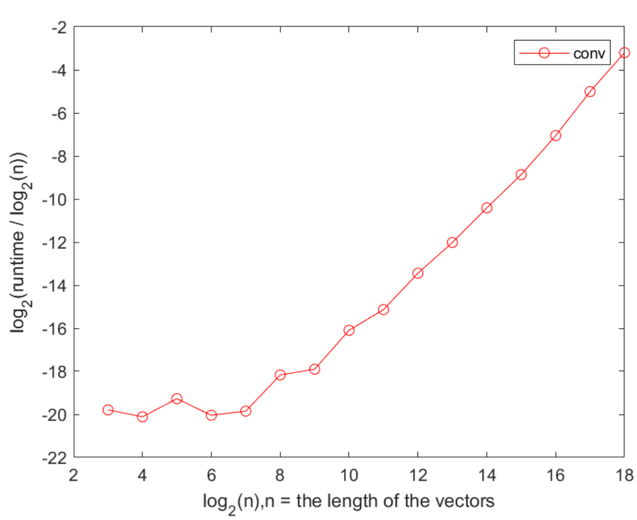
大致是线性的，斜率大致为1，说明复杂度至少是 $\Theta(n)$

右图相当于对 $t/n \sim \log n$ 作图，预期会出现一条

斜率大于0的直线，但结果与预期不符，不知什么原因？

Q: 我舍友做出来也是！难道比 $\Theta(n\log n)$ 还快？

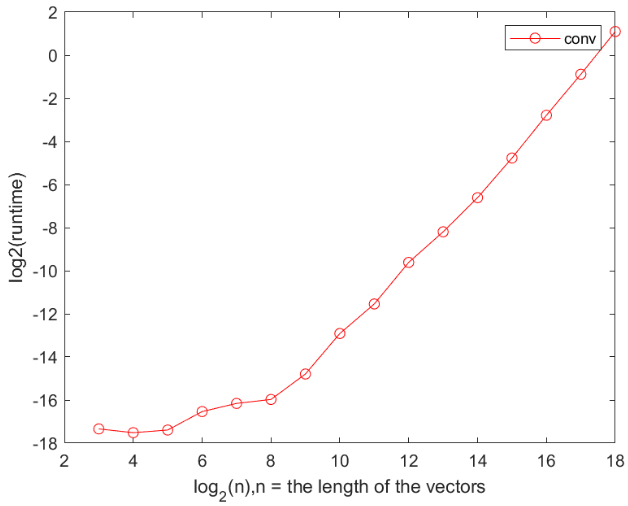
Matlab 的内置 conv:



注意到对 $\log(t/(\log n)) \sim \log n$ 作图 斜率大致为 2

说明复杂度高于 $\Theta(n)$

再作 $\log(t) \sim \log(n)$ 如下:



斜率仍大致为 2，故说明 conv 的复杂度在 $\Theta(n^2)$

也即 Matlab 的 conv 应该是直接一项一项地暴力展开的 □

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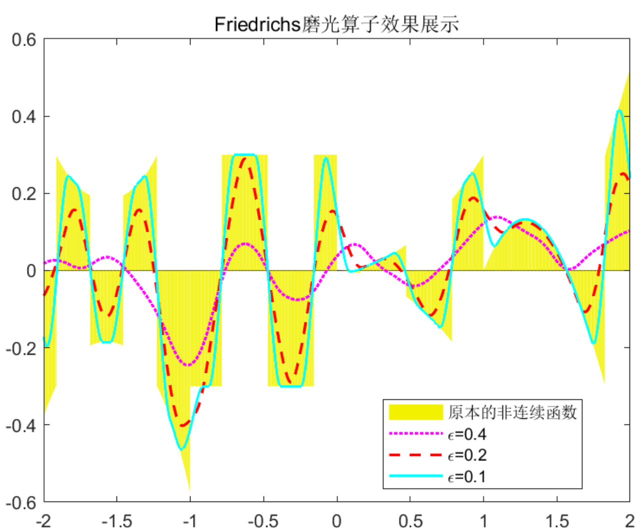
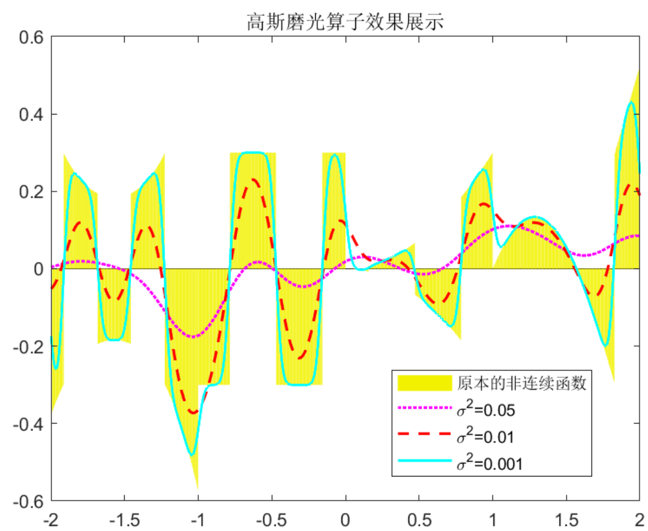
3. Create a discontinuous function and smoothen it by convolving with Gaussian functions and Friedrichs mollifiers. Make plots to visualize the results.

Note: Friedrichs mollifiers are functions of the form $\eta_\epsilon(x) = \eta(x/\epsilon)$ with $\epsilon > 0$, where

$$\eta(x) = \begin{cases} \exp\Big(\dfrac{1}{x^2 - 1}\Big), & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1. \end{cases}$$





4. Compute all eigenvalues and eigenvectors of the following $n \times n$ matrix:

$$T_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

解：三对角矩阵的行列式为

$$\det\begin{bmatrix} a & b & & \\ c & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & c & a \end{bmatrix}_{n\times n} = \begin{cases} \dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} & \text{时 } a^2 \neq 4bc \\ (n+1)\cdot\left(\dfrac{a}{2}\right)^n & \text{时 } a^2 = 4bc \end{cases}$$

其中 α, β 分别是

$$x^2 - ax + bc = 0 \text{ 的两根}$$

对于 T_n，只需要求 $T_n - 2I$ 的特征值与特征向量即可

$$T_n - 2I = \begin{bmatrix} 0 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 0 \end{bmatrix}$$

那么：对 $|\lambda I - T_n - 2I|$：

$$x^2 - \lambda x + 1 = 0$$

两根分别为：

$$\alpha = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \qquad \beta = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}$$

若 $\alpha = \beta$，即 $\lambda = 2$ or $\lambda = -2$ 时，显然 $|\lambda I - T_n - 2I| \neq 0$

也即 $\lambda = 2$ or $\lambda = -2$ 不是 $T_n - 2I$ 特征值

若 $\alpha \neq \beta$，则 $|\lambda I - T_n + 2I| = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$

$\Rightarrow \alpha^{n+1} - \beta^{n+1} = 0$

又有 $\alpha \cdot \beta = 1$

$\Rightarrow (\alpha \cdot \beta)^{n+1} = \alpha^{2n+2} = 1$

$\Rightarrow \alpha = e^{\frac{2\pi i k}{2n+2}} = e^{\frac{k\pi i}{n+1}}$, $k = 0, \pm 1, \cdots, \pm n$

又注意到 $\alpha = \bar{\beta}$ 并且 $\alpha \neq \beta$

则 $\lambda = \alpha + \beta = 2\cos\frac{k\pi}{n+1}$, $k = 1, 2, \cdots, n$

$\Rightarrow$ T_n 的特征值为 $\lambda + 2$

也即 $2\cos\frac{k\pi}{n+1} + 2$, $k = 1, 2, \cdots, n$

再看特征向量的情况: $T_n - 2I$ 与 T_n 的特征向量是相同的

若 $\lambda = 2\cos\frac{k\pi}{n+1}$

取 $X = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \cdots\cdots, \sin\frac{nk\pi}{n+1}\right)^T$

注意到

$2\cos\frac{k\pi}{n+1} \cdot \sin\frac{k\pi}{n+1} - \sin\frac{2k\pi}{n+1} = 0$

$$\begin{cases} 2\cos\frac{k\pi}{n+1} \cdot \sin\frac{jk\pi}{n+1} - \sin\frac{(j-1)k\pi}{n+1} - \sin\frac{(j+1)k\pi}{n+1} = 0 \\ -\sin\frac{(n-1)k\pi}{n+1} + 2\cos\frac{k\pi}{n+1} \cdot \sin\frac{n \cdot k\pi}{n+1} = 0 \end{cases}$$

(通过和差化积公式易验证)

则：

$$\begin{bmatrix} \lambda & -1 & & \\ -1 & \lambda & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & \lambda \end{bmatrix} \cdot x = 0$$

$\Rightarrow$ x 是 $-\lambda$，也即 $\lambda' = 2\cos\frac{(n+1-k)\pi}{n+1}$ 的特征向量

$\therefore$ T_n 的特征向量为：

$$x = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \cdots\cdots, \sin\frac{nk\pi}{n+1}\right)^T$$

其中 $k = 1, 2, \cdots, n$

Q：这题做得要吐血！有无简单妙法？

$\square$

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5. (optional) Prove that

$$\delta(x) = \frac{1}{\pi} \lim_{\eta \to 0+} \frac{\eta}{x^2 + \eta^2}.$$

It suffices to show

$$\lim_{\eta \to 0+} \int_{-1}^{1} \frac{\eta f(x)}{x^2 + \eta^2} \mathrm{d}x = \pi f(0)$$

Q：这个证不出来！

for any continuous function $f(x)$.

FYI. Other frequently used approximations to $\delta(x)$ include

$$\delta(x) = \lim_{\sigma \to 0+} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) = \lim_{\eta \to 0+} \frac{1}{\pi x} \sin\frac{x}{\eta}.$$

解：首先讨论 $\delta(x) = \frac{1}{\pi} \cdot \lim\limits_{\eta \to 0^+} \frac{\eta}{x^2+\eta^2}$

若 $x \neq 0$，

$$\lim_{\eta \to 0^+} \frac{\eta}{x^2+\eta^2} = \frac{0}{x^2+0} = 0$$

若 $x=0$

$$\lim_{\eta\to0^+}\frac{\eta}{x^2+\eta^2}=\lim_{\eta\to0^+}\frac{1}{\eta}=+\infty$$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\eta}{x^2+\eta^2}\,dx=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\eta^2}{\eta^2((\frac{x}{\eta})^2+1)}\,d\frac{x}{\eta}$$

$$\underline{\underline{s=\frac{x}{\eta}}}\quad\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{ds}{s^2+1}=\frac{1}{\pi}\arctan s\Big|_{-\infty}^{+\infty}=1$$

则 $\delta(x)$ 满足：

(1) $\delta(x)=\begin{cases}+\infty & x=0\\ 0 & x\neq0\end{cases}$ (2) $\int_{-\infty}^{+\infty}\delta(x)\,dx=1$

下面证明对于任何连续函数 $\lim_{\eta\to0^+}\int_{-1}^{+1}\frac{\eta f(x)}{x^2+\eta^2}\,dx=\pi f(0)$

$$\int_{-1}^{+1}\frac{\eta f(x)}{x^2+\eta^2}\,dx$$

$$=\int_{-1}^{+1}\frac{\eta^2 f(\eta\cdot\frac{x}{\eta})}{\eta^2((\frac{x}{\eta})^2+1)}\,d\frac{x}{\eta}$$

$$=\int_{-\frac{1}{\eta}}^{\frac{1}{\eta}}\frac{f(\eta\cdot s)}{s^2+1}\,ds$$

$$= \int_{-1/\eta}^{1/\eta} f(\eta \cdot s) \cdot d \arctan s$$

$$= f(\eta \cdot s) \arctan s \Big|_{-1/\eta}^{1/\eta} - \int_{-1/\eta}^{1/\eta} \arctan s \cdot \eta \cdot \underbrace{f'(\eta \cdot s)}_{?} \cdot ds$$

Q：这一步就卡住了，题目没说 $f(x)$ 可导……