5月15日作业

1

1. Use a linear combination of $f(t)$, $f(t+h)$, $f(t+2h)$ to approximate $f'(t)$ (as accurately as you can). Estimate the truncation error.

解：$f(t+h) = f(t) + f'(t)\cdot h + \frac{1}{2}f''(t)h^2 + \frac{1}{6}f'''(\hat{t})h^3$

$f(t+2h) = f(t) + 2f'(t)\cdot h + 2f''(t)h^2 + \frac{4}{3}f'''(\tilde{t})h^3$

其中，$\hat{t} \in (t, t+h)$，$\tilde{t} \in (t, t+2h)$

做如下线性组合：

$$2(f(t+h)-f(t)) - \frac{1}{2}(f(t+2h)-f(t))$$

$$= 2f'(t)\cdot h + f''(t)h^2 + \frac{1}{3}f'''(\hat{t})h^3 - f'(t)h - f''(t)h^2 - \frac{2}{3}f'''(\tilde{t})\cdot h^3$$

$$= f'(t)\cdot h + O(h^3)$$

那么：

$$f'(t) \approx \frac{2f(t+h) - \frac{1}{2}f(t+2h) - \frac{3}{2}f(t)}{h}$$

$$E_{trunc} = O(h^2)$$

□

2. Use a linear combination of nine function values $f(x+ih, y+jh)$ (for $i,\ j \in \{-1,0,1\}$) to approximate

$$\frac{\partial^2}{\partial x^2} f(x,y) + \frac{\partial^2}{\partial y^2} f(x,y)$$

(as accurately as you can). Estimate the truncation error.

2

解: $f(x\pm h, y) = f(x,y) \pm \frac{\partial}{\partial x} f(x,y)\cdot h + \frac{1}{2}\cdot \frac{\partial^2}{\partial x^2} f(x,y) h^2 \pm \frac{1}{6}\frac{\partial^3}{\partial x^3} f(x,h)\cdot h^3 + O(h^4)$

$$\Rightarrow \frac{f(x+h,y)+f(x-h,y)-2f(x,y)}{h^2}$$

$$= \frac{\partial^2}{\partial x^2} f(x,y) + O(h^2)$$

同理: $\dfrac{f(x,y+h)+f(x,y-h)-2f(x,y)}{h^2} = \dfrac{\partial^2}{\partial y^2} f(x,y) + O(h^2)$

$$\Rightarrow \frac{\partial^2}{\partial x^2} f(x,y) + \frac{\partial^2}{\partial y^2} f(x,y)$$

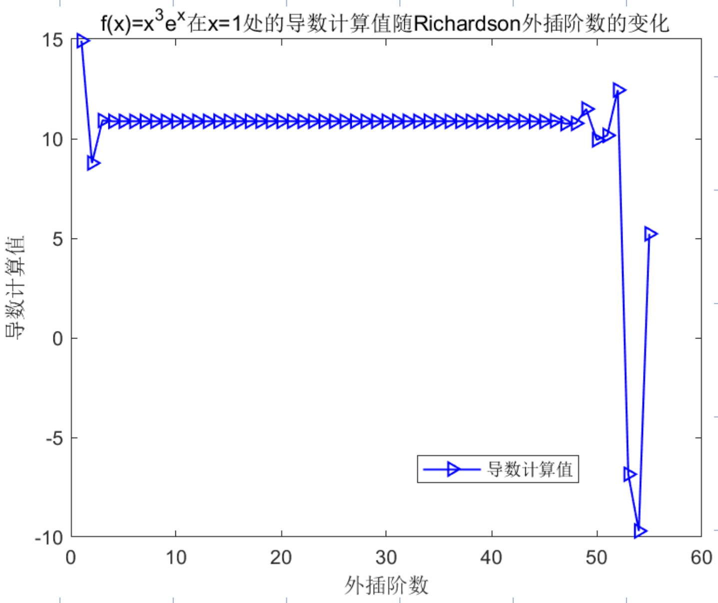
$$\approx \frac{f(x+h,y)+f(x-h,y)+f(x,y+h)+f(x,y-h)-4f(x,y)}{h^2}$$

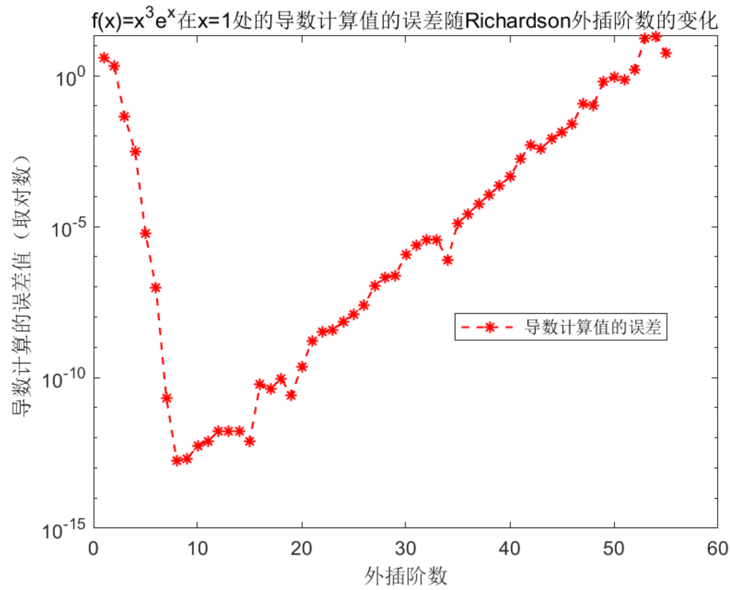
$$E_{trunc} = O(h^2)$$

$\square$

3

3. Use Richardson extrapolation to estimate the derivative of $f(x) = x^3e^x$ at $x = 1$. Keep a record for intermediate results. What happens if you iterate for many steps?





随着迭代次数(外插的阶数)增加，导数值的误差先下降后上升。上升的原因是，外插最终会归结于计算

$$\frac{f(x+h/2^n)-f(x-h/2^n)}{h/2^{n-1}}$$

而 n 太大，也即 $h/2^n$ 太小时，舍入误差就变大了。

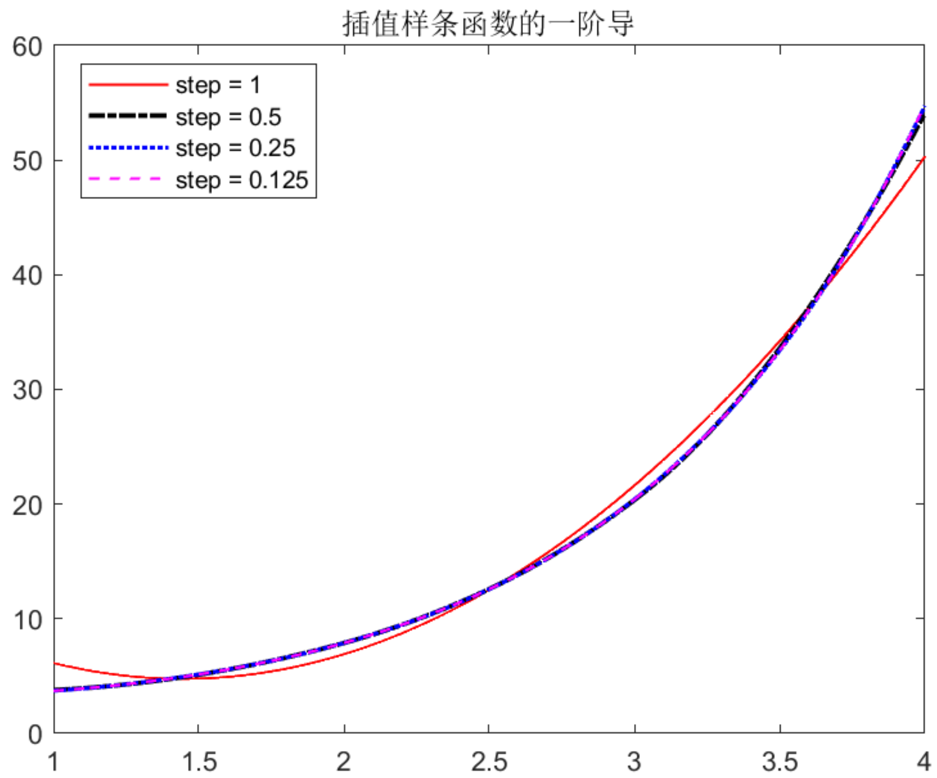
另一方面，当 n 变大时 计算递推式：

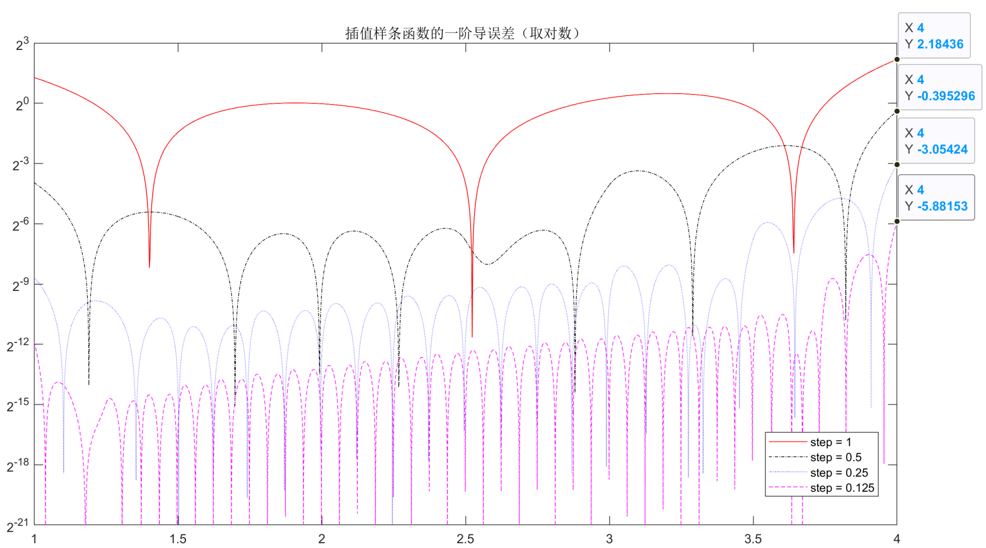
$$F_n(h) = \frac{2^{n-1}F_{n-1}(h/2) - F_{n-1}(h)}{2^{n-1}-1}$$

的次数会变多，也会累积舍入误差 □

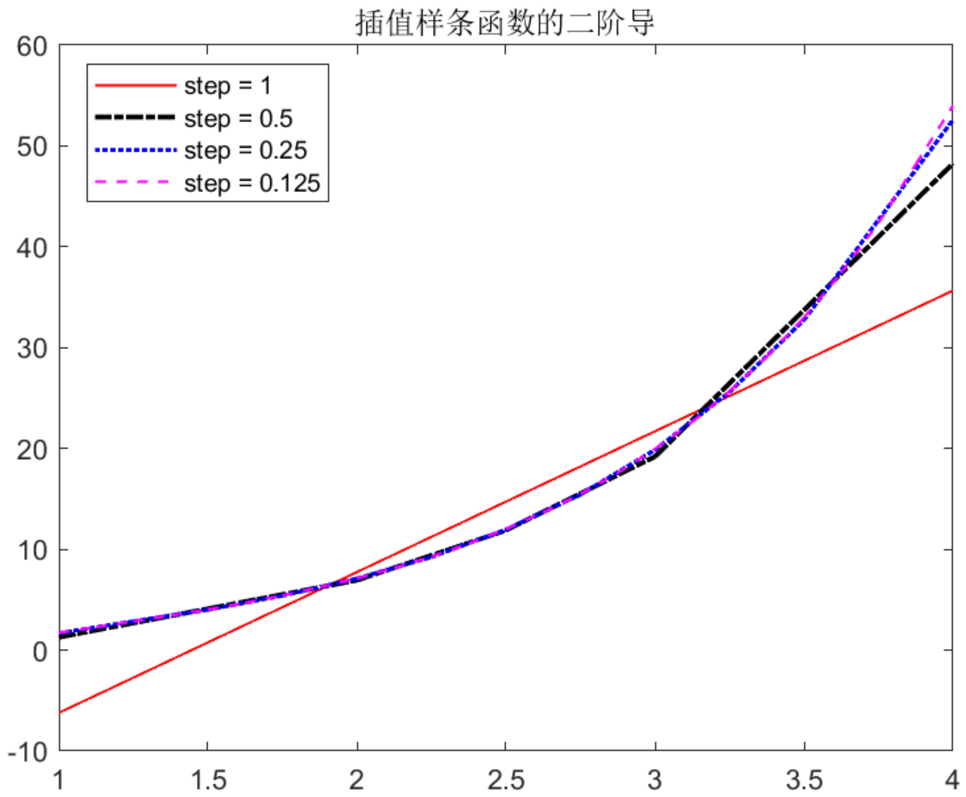
4

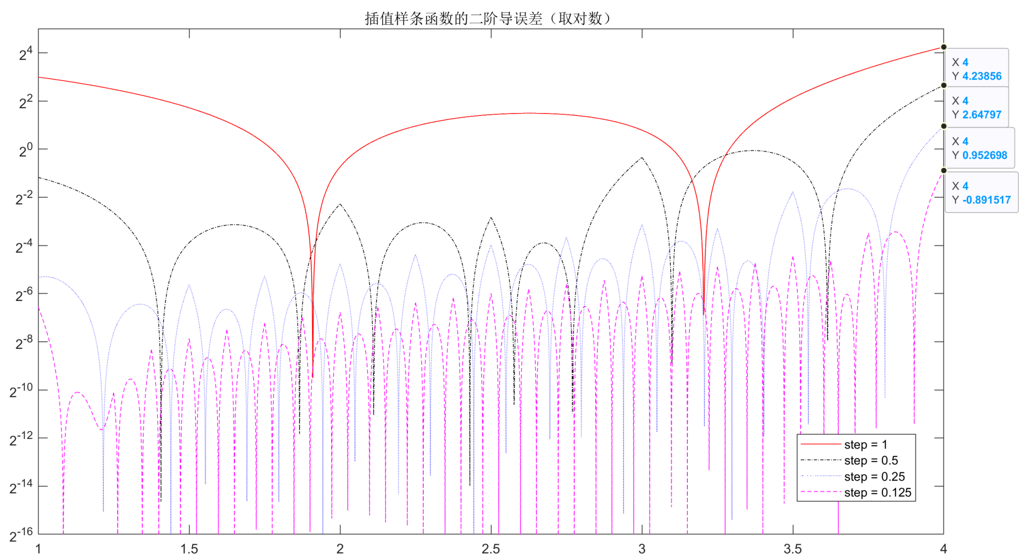
4. Use a cubic spline function $s(x)$ to approximate $f(x) = e^x + \ln x$ over $[1, 4]$. Plot the first and second derivatives, as well as the errors. You are encouraged to try different step sizes and observe the behavior of the error with respect to the step size.





h减半，最大误差大约变为 2^{-3}





h减半，最大误差大约变为2^{-2}

随步长减小，插值函数的一、二阶导数的误差也在下降
但下降的倍数并不一样。其规律基本符合之前上课讲的下述

定理：

定理5 设被插函数 $f(x) \in C^4[a,b]$, $s(x)$ 是它的 D_1, D_2 型三次样条插值, 则在插值区间 $[a,b]$ 上成立

$$|f^{(j)}(x) - s^{(j)}(x)| \leqslant c_j h^{4-j} \|f^{(4)}\|_\infty, \qquad j = 0, 1, 2, \tag{3.19}$$

且 $c_0 = 1/16, c_1 = c_2 = 1/2$. □

5

5. Try to find constants c_1, c_2, and c_3 such that the degree of exactness of the following quadrature rule is maximized:

$$\int_{-2a}^{2a} f(x)\,\mathrm{d}x \approx c_1 f(-a) + c_2 f(0) + c_3 f(a).$$

解: $c_1 = c_3 = \frac{8}{3}a$, $c_2 = -\frac{4}{3}a$，代数精度为 3

以下是证明:

对于 $f(x) = 1$ 而言

$$\int_{-2a}^{2a} f(x)\,dx = \int_{-2a}^{2a} 1 \cdot dx = 4a$$

$$c_1 f(-a) + c_2 \cdot f(0) + c_3 f(a)$$
$$= \frac{8}{3}a - \frac{4}{3}a + \frac{8}{3}a$$
$$= 2a$$

故 $f(x) = 1$ 时 有严格等号成立

$f(x) = x$ 时.

$$\int_{-2a}^{2a} f(x)\,dx = \int_{-2a}^{2a} x\,dx = 0$$

$$c_1 f(-a) + c_2 \cdot f(0) + c_3 f(a)$$
$$= \frac{8}{3} \cdot (-a) - \frac{4}{3}a \cdot 0 + \frac{8}{3}a$$
$$= 0$$

故 $f(x) = x$ 时等号严格成立

$f(x) = x^2$ 时.

$$\int_{-2a}^{2a} f(x)\,dx = \int_{-2a}^{2a} x^2 dx = \frac{1}{3}x^3 \Big|_{-2a}^{2a} = \frac{16}{3}a^3$$

$$C_1 f(-a) + C_2 \cdot f(0) + C_3 f(a)$$
$$= \frac{8}{3}a \cdot a^2 + \frac{4}{3}a \cdot 0 + \frac{8}{3}a \cdot a^2$$
$$= \frac{16}{3}a^3$$

故 $f(x) = x^2$ 时也满足

$f(x) = x^3$ 时.

$$\int_{-2a}^{2a} f(x)\,dx = \int_{-2a}^{2a} x^3 dx = 0$$

$$C_1 f(-a) + C_2 \cdot f(0) + C_3 f(a)$$
$$= \frac{8}{3}a \cdot (-a^3) + \frac{4}{3} \cdot a \cdot 0 + \frac{8}{3}a \cdot a^3$$
$$= 0$$

故 $f(x) = x^3$ 也成立

而当 $f(x) = x^4$ 时

$$\int_{-2a}^{2a} x^4 dx = \frac{1}{5}x^5 \Big|_{-2a}^{2a} = \frac{64}{5}a^5$$

$$C_1 f(-a) + C_2 \cdot f(0) + C_3 f(a) = \frac{16}{3}a^5$$

并不满足!

因此 $C_1 = C_3 = \frac{8}{3}a$, $C_2 = -\frac{4}{3}a$ 时, 代数精度为 3. □

6

6. (optional) An ancient way of computing π can be interpreted in modern terms as $\pi \approx n\sin(\pi/n)$, where n is typically chosen as $n = 3\cdot 2^k$ for $k \in \mathbb{N}$. Unfortunately, the convergence of such an approximation is very slow, and the calculation is expensive—it involves a lot of square roots (because ancient mathematicians had to use geometry instead of Taylor series to compute $\sin(\pi/n)$). It is believed that Chinese mathematicians, Liu Hui and Zu Chongzhi, discovered a way to largely accelerate the calculation using some sort of extrapolation.

Use Richardson extrapolation to calculate π to seven correct digits after the decimal point, based on the asymptotic expansion

$$n\sin\frac{\pi}{n} = \pi - \frac{\pi^3}{3!\,n^2} + \frac{\pi^5}{5!\,n^4} - \cdots.$$

What is the largest value of n in your calculation?

For simplicity, you may perform calculations such as `sin(pi/n)` directly in your program.

解：记 $F_2(n) = \pi - \frac{\pi^3}{3!n^2} + \frac{\pi^5}{5!n^4} \cdots$

$= \pi + k_2\cdot\frac{1}{n^2} + k_4\cdot\frac{1}{n^4} + \cdots$

那么，可构造：

$$F_4(n) = \frac{2^2 F_2(2n) - F_2(n)}{2^2-1}$$

$$= \pi + \tilde{k}_4\cdot\frac{1}{n^4} + \cdots$$

一般的：

$$F_{2k}(n) = \frac{2^{2(k-1)} F_{2(k-1)}(2n) - F_{2(k-1)}(n)}{2^{2(k-1)}-1}$$

从 $n_0 = 3$ 开始迭代，到达 F_8 时，也即 n 最大为 24 时已经计算得到 7 位正确数字：

```
3     6     12     24

inter_results =

2.598076211353316    3.133974596215561    3.141580063335317    3.141592650573243
```

□