House Numbers

To understand the problem, we first need to define a couple of terms. The first is a **triangular number**. A triangular number is one that can be arranged as an equilateral triangle like so:

```
O   O     O       O
1  O O   O O     O O
    3   O O O   O O O
          6    O O O O
                 10
```

This gives the formula $\frac{n(n+1)}{2}$

A number is said to be **square-triangular** if it is both triangular and a perfect square (i.e. its square root is a whole integer). We will use **36** as a case study. 36 is both a triangular number ($\frac{8(8+1)}{2}$) and a perfect square ($\sqrt{36} = 6$) making it a square-triangular number. We can write it out like so:

```
1         O
2        O O
3       O O O
4      O O O O
5     O O O O O
6    O O O O O O
7   O O O O O O O
8  O O O O O O O O
          36
```

In our analogy, the house numbers are the ‘rows’ of the triangular number as illustrated above. Now, if we take the house at the square root of the number, 6, we can see that the totals of all the house numbers greater than 6 is 15, and the totals of all the house numbers before 6 is also 15. We can see this more clearly if we arrange the number in its square form:

```
X O O O O O
O X O O O O
O O X O O O
O O O X O O
O O O O X O
O O O O O X
```

The line formed by the ‘X’s represent the dividing line highlighted in red on the triangular number (i.e. the square root). We can see that the ‘O’s on either side of this dividing line are both the triangular number 15. This leads us to the conclusion that if a number is square triangular, we can divide the square number diagonally into two equal triangular numbers. These triangular numbers are therefore equal to the sum of the original triangular number’s rows from 1 to k - 1; and k + 1 to n where k is the square root of the number and n is the triangular root of the number.

Now we have determined that k and n are the square root and the triangular root of the square-triangular number respectively. Conveniently there are two simple recurrences we can use to determine the square and triangle roots of square-triangular numbers:

$$k_x = 6 \times k_{x-1} - k_{x-2} \quad with\ k_0 = 0\ and\ k_1 = 1$$
$$n_x = 6 \times n_{x-1} - n_{x-2} + 2 \quad with\ n_0 = 0\ and\ n_1 = 1$$

References:

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Wikipedia - Square-Triangular Numbers
Wikipedia - Triangular Numbers

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