Stochastic Optimal Control and Application in Finance

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Abstract

This document is to draft a basic understanding and theorems important for stochastic optimal control. This document is mainly based on the book "*Estimation and Control of Dynamical Systems*" by Dr. Alain Bensoussan.

1 Introduction

Stochastic control problems in finance are a branch of mathematical optimization that deals with decision-making in uncertain financial environments. They provide a framework for determining optimal strategies or policies for individuals or institutions to maximize their expected utility or minimize risk while considering the random nature of financial markets.

In finance, uncertainty is inherent due to various factors such as unpredictable market conditions, fluctuating asset prices, and uncertain future events. Stochastic problems allow to model and analyze these uncertainties mathematically, making them an essential tool for decision-making in the financial world. Examples of those problems are:

- Consumer-Investor problem
- Entrepreneur Decision-Making problem etc

Before discussing specific problems and their respective solutions, it is important to familiarize ourselves with the tools necessary for constructing these solutions.

2 Preliminary

Definition 2.1. A stochastic process in continuous time (Note: that we could also define this on the discrete case but we are dealing with the continuous case here.) is a function X(t;ω) that maps each point in the probability space (Ω, $\mathcal{A}$, P) to a continuous-time function of t. In other words, for each fixed value of ω, X(t; ω) is a function of time t that is measurable with respect to the Borel σ-algebra on R^+ and the σ-algebra $\mathcal{A}$ on the probability space (Ω, $\mathcal{A}$, P).

A trajectory is a function of time t, denoted by X(t; ω), that describes the behaviour of the stochastic process for the fixed sample point ω. Each trajectory represents one possible outcome of the stochastic process for the given sample point ω. $t \mapsto X(t;\omega), \quad \forall, \omega; \text{fixed}$

A Filtration is family of σ-algebra indexed with time such that, $\mathcal{F}^t = \mathcal{F}^s : s \leq t, \quad \mathcal{F}^s \subseteq \mathcal{F}^t; \text{for}; s \leq t$.

Consider a process $X(t)$ that is adapted to $\mathcal{F}^t$, we say that its a $P, \mathcal{F}^t$ martingale if

1. $E|X(t)| < +\infty \quad \forall t$
2. $E(X(t)|\mathcal{F}^s) = X(s), \quad \forall t \geq s$

- Submartingale if $E(X(t)|\mathcal{F}^s) \geq X(s), \quad \forall t \geq s$
- Supermartingle if $-X(t)$ is a submartingle.

Martingale's theory can be used to establish the existence and uniqueness of solutions to stochastic control problems

2.1 Wiener Process

A standard wiener process $(1D)$ is a stochastic process with the following properties:

1. w_t is continuous and $w_0 = 0$
2. The value of $w_t \sim \mathcal{N}(0, t)$
3. $\forall$ times $0 < t_1 < t_2 ... < t_n$, the increments are iid

$E(w(t)w(s)) = min(t, s) \quad \forall t, s$ A common definition of the wiener process, $w(t)$ have the property $E(w(t) - w(s))^2|\mathcal{F}^s) = t - s, \quad \forall t \geq s$, introducing filtration and extending to $n-$dimensions, $P, \mathcal{F}^t$ standard wiener process
$E(w(t)w(s)^*) = Imin(t, s)$
$E((w(t) - w(s))(w(t) - w(s))^*|\mathcal{F}^s) = I(t - s)$, where I: identity matrix.
Where * mean transpose.

2.2 Stochastic Integrals

We define a process $\phi(t)$ adapted to $\mathcal{F}^t$ and satisfies

$$E \int_0^T |\phi(t)|^2 dt < +\infty \tag{1}$$

The space of processes satisfying 2 are called $\mathcal{L}^2_{\mathcal{F}}(0, T)$. We then define Stochastic Integral $I\phi(t) = \int_0^T \phi(t) dw(t)$
and if $\phi(t)$ is a piece-wise constant, we have:

$$I(\phi(t)) = \sum_{n=1}^{N-1} \phi_n(w(t_{n+1} - w(t_n))$$

And to consider it a stochastic process, we have the property :

$$E(I(t))^2 = E \int_0^T |\phi(s)|^2 ds$$

With Martingale Inequality:

$$E \sup_{0 \leq s \leq t} E|I(s)^2| \leq CE(I(t))^2$$

we can obtain more.

2.3 Representation of Martingales theorem

Suppose $\mathcal{F}^t = \sigma(w(s), s \leq t)$, i.e filtration generated by σ-algebra. The every $P, \mathcal{F}^t$ sqaure integrable martingale $M(t)$ is given by:

$$M(t) = M(0) + \int_0^t \phi(s) dw(s), \quad \phi(.) \in \mathcal{L}^2_{\mathcal{F}} < +\infty \quad \forall T$$

A process is called semimartingale if defined as follows:

$$\xi(t) = \xi(0) + \int_0^t a(s) ds + \int_0^t b(s) dw \quad a, b \in \mathcal{L}^2_{\mathcal{F}}(0, T)$$

with the assumption that $\xi(0)$ is $\mathcal{F}^0$- measurable. A process of the form above equation is said to have Itô differential. Take a bounded function $\phi(x,t)$, C^2 in x and C^1 in t, the the process $\phi(\xi(t),t)$ is also a semimartingale. It's Itô differential :

$$d\phi(\xi(t),t) = (\frac{\partial\phi(\xi(t),t)}{\partial t} + \frac{1}{2}\frac{\partial^2\phi(\xi(t),t)}{\partial x^2}b(t)^2)dt + \frac{\partial\phi(\xi(t),t)}{\partial x}d\xi(t)$$

Generalizing into an n-dimensional semimartingale:

$$d\phi(\xi(t),t) = [\frac{\partial\phi(\xi(t),t)}{\partial t} + \frac{1}{2}b^*(t)\frac{\partial^2\phi(\xi(t),t)}{\partial x^2}b(t)]dt + \frac{\partial\phi(\xi(t),t)}{\partial x}d\xi(t)$$

By applying martingale representation theorems, one can prove the existence of optimal control strategies that achieve the maximum or minimum of an objective function. Martingales also help ensure the uniqueness of solutions, providing a solid mathematical basis for studying and analyzing stochastic control systems.

2.4 Stochastic Differential Equation

Take a and b depending on w only through X_t i.e $a_t = a(X_t,t)$, $b_t = b(X_t,t)$. Then, we define :

$$dX_t = a(X_t,t)dt + b(X_t,t)dw_t$$

Consider m- dimensional $P, \mathcal{F}^t$ weiner process $w(t)$ and functions $g(x,t) \in \mathbb{R}^n$, $\sigma(x,t) \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$ We find the process $y(t)$ whose Itô differential satisfies the equation:

$$dy(t) = g(y(t),t)dt + \sigma(y(t),t)dw(t) \tag{2}$$

$$y(0) = y_0$$

$y(0)$ is a given $\mathcal{F}^0$- measurable r.v y_0

Theorem 2.1. *Assume that:*

$$|g(x,t) - g(x^{'},t)| + |\sigma(x,t) - \sigma(x^{'},t)| \leq k|x - x^{'}|,$$

$$|g(x,t)|^2 + |\sigma(x,t)|^2 \leq K_0^2(1 + |x|^2)$$

$$E|y_0|^2 < \infty$$

Then $\exists$ a unique semimartingale $y(t)$ solution of 2 s.t $y(0) = y_0$ and

$$E \sup_{0\leq t\leq T} |y(t)|^2 \leq C_T(1 + E|y_0|^2)$$

The intuition around the above theorem: Take two SDE solutions say $Q(t)$ and $R(t)$, then the solution are equal almost surely, i.e

$$P(\omega \in \Omega | Q(t) - R(t)| = 0) = 1$$

Theorem 2.2. *Girsanov's Theorem: Consider $P, \mathcal{F}^t$ wiener process $w(.)$ Let $b(.)$ be adapted and bounded. We define*

$$\eta(t) = \exp\,[\int_o^t b(s)dw(s) - \frac{1}{2}\int_0^t |b(s)|^2 ds]$$

and

$$\hat{w}(t) = w(t) - \int_0^t b(s)ds.$$

Then, $\eta(t)$ satisfies

$$d\eta(t) = \eta b dw$$

and $E(\eta(t) = 1$ We define a probability P^t, on each measurable space $\Omega, \mathcal{F}^t$ by setting

$$P^t(A) = \mathbb{E}[1_A \eta(t)], \quad A \in \mathcal{F}^t$$

if the σ-algebra $\mathcal{A}$ on Ω is

$$\mathcal{A} = \cup_{t \geq 0} \mathcal{F}^t,$$

then by Kolmogorov's theorem, the probability measure $\hat{P}$ on ω gives the family P^t and with the compactibility property

$$P^t(A) = \hat{P}(A) \quad \forall A \in \mathcal{F}^t$$

We then define

$$\frac{d\hat{P}}{dP}|_{\mathcal{F}^t} = \eta(t),$$

and where $\eta(t)$ is the Randon-Nikodym of $\hat{P}$ w.r.t P Now, stating the theorem formally;
$\hat{w}$ is a $\hat{P}, \mathcal{F}^t$ standard wiener process, $\hat{w}$ is not $P, \mathcal{F}^t$ standard wiener process and $w(t)$ is not $\hat{P}, \mathcal{F}^t$ standard wiener process.

The Girsanov theorem is used to define "weak solutions" in stochastic differential equations. And the main advantage of the weak solution concept for control theory is that there is no requirement that the dependence of g on y in 2 be smooth (e.g., Lipshitz as the standard Ito conditions require). If we have the conditions as in Theorem 2.1, then we have a strong solution.

3 Stochastic Optimal Control

3.1 Setting Of the Problem

Consider the state of the system whose equation is the solution to the SDE

$$\begin{aligned} dx &= g(x(t), v(t))dt + \sigma(x(t), v(t))dw \\ x(0) &= \xi \end{aligned} \tag{3}$$

Assume that ξ is $\mathcal{F}^0$-measurable and independent of the Wiener Process. We also have the following estimate

$$E[\sup_{0 \leq t \leq T} \left|x(t)\right|^2] \leq C(1 + E|\xi|^2 + E\int_0^T \left|v(t)\right|^2 \ dt) \tag{4}$$

$$\begin{gathered} g(x,v) : \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n \\ \sigma(x,v) : \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{L}(\mathcal{R}^k; \mathcal{R}^n) \\ g, \sigma \text{ are continuously differentiable and have bounded derivatives} \\ \left|g(x,v)\right| \leq \overline{g}(1 + |x| + |v|) \\ \left|\sigma(x,v)\right| \leq \overline{\sigma}(1 + |x| + |v|) \end{gathered} \tag{5}$$

Let $(\Omega, \mathcal{A}, P)$ be a probability space equipped with a filtration $\mathcal{F}^t$ and a standard $P, \mathcal{F}^t$ Wiener Process with values in $\mathcal{R}^k$. Set

$$\mathcal{U}_{ad} = \text{nonempty closed subset of } \mathcal{R}^m \tag{6}$$

An admissible control is a process $v(.) \in L^2_{\mathcal{F}}(0, T; \mathcal{R}^m)$ s.t

$$v(t) \in \mathcal{U}_{ad}, \ a.e, \ a.s$$

Now, to define the performance measure, we make some assumptions:

$$\begin{aligned}
&l(x,v) : \mathcal{R} \times \mathcal{R}^m \to \mathcal{R}, \text{ is continuously differentiable,} \\
&h(x) : \mathcal{R}^n \to \mathcal{R}, \text{ is continuously differentiable,} \\
&\left|l(x,v)\right| \le \bar{l}(1+|x|^2+|v|^2),\ \left|h(x)\right| \le \bar{h}(1+|x|^2) \\
&|l_x|,|l_v| \le \bar{l}(1+|x|+|v|),\ |h_x| \le \bar{h}(1+|x|)
\end{aligned} \tag{7}$$

$$\text{The payoff: } J(v(.)) = E[\int_0^T l(x(t),v(t))\,dt + h(x(T))], \tag{8}$$

This performance measure is well-defined because of the assumptions imposed above.

3.2 Gâteaux Differential

Let $u(.)$ be an optimal control and $y(.)$ the corresponding optimal state. We introduce the process $z(.) \in L^2_{\mathcal{F}}(0,T;\mathcal{R}^m)$, the solution of the linear equation

$$\begin{aligned}
dz &= (g_x(y,u)z + g_v(y,u)v)dt + \sum_{j=1}^{k} (\sigma_x^j(y,u)z + \sigma_v^j(y,u)v)dw_j \\
z(0) &= 0
\end{aligned} \tag{9}$$

Assume $(5),(7)$. Then the Gateaux differential is

$$\begin{aligned}
&\frac{d}{d\theta}J(u(.)+\theta v(.))|_{\theta=0} = \\
&\quad E\int_0^T [l_x(y,u)z + l_v(y,u)v]dt + E[h_x(y(T))z(T)]
\end{aligned} \tag{10}$$

The drawback of (10) is that z remains on the right-hand side. However, it can be shown that there exist $p,r \in L^2_{\mathcal{F}}(0,T;\mathcal{R}^n)$

$$\begin{aligned}
&\frac{d}{d\theta}J(u(.)+\theta v(.))|_{\theta=0} = \\
&\quad E\int_0^T [l_v(y,u)v + p.g_v(y,u)v + \sum_{j=1}^{k} r^j.\sigma_v^j(y,u)v]dt
\end{aligned} \tag{11}$$

The Gâteaux derivative-based maximum principle is a fundamental result in stochastic optimal control theory that provides the necessary conditions for a control policy to be optimal. It establishes relationships between the cost functional, the system dynamics, and the adjoint variables associated with the optimization problem.

3.3 Hamiltonian

Introducing the Hamiltonian function as a function of the adjoint states:

$$H(x,v,q,r) = l_v(y,u)v + q.g_v(y,u)v + \sum_{j=1}^{k} r^j.\sigma_v^j(y,u)v \tag{12}$$

Then we have

$$\frac{d}{d\theta}J(u(.)+\theta v(.))|_{\theta=0} = E\int_0^T H_v(y(t),u(t),p(t),r(t))dt \tag{13}$$

A necessary condition for Optimal control $u(.)$:

Proposition 1. *We assume* (5), (6), (7)*. If* $u(.)$ *is an optimal control for* (3), (8)*, then*

$$\begin{aligned} H_v(y(t), u(t), p(t), r(t)).(v - u(t)) \geq 0, a, s., a.e. \\ \forall v \in \mathcal{U}_{ad} \end{aligned} \tag{14}$$

If $u(.)$ optimal exist, then $y(t)$ is the optimal trajectory defined in following the SDE:

$$\begin{aligned} dx &= g(y(t), u(t))dt + \sigma(y(t), u(t))dw \\ y(0) &= \xi \end{aligned} \tag{15}$$

$$\begin{aligned} -dp =& (g_x^*(y, u)p + l_x(y, u) + \sum_{j=1}^{k} \sigma_x^{j*}(y, u)r^j)dt - \sum_{j=1}^{k} r^j dw_j \\ p(T) =& h_x(y(t)) \end{aligned} \tag{16}$$

$$H_v(y(t), u(t), p(t), r(t)).(v - u(t)) \geq 0, a, s., a.e. \forall v \in \mathcal{U}_{ad} \tag{17}$$

3.4 Stochastic Dynamic Programming

Similar to that of the deterministic case, we consider a family of stochastic controls problems

$$\begin{aligned} dx &= g(x(s), v(s))ds + \sigma(x(s), v(s))dw, \ t \leq s \leq T, \\ x(t) &= x \end{aligned} \tag{18}$$

$$J_{x,t}(v(.)) = E[\int_t^T l(x(s), v(s)) \, ds + h(x(T))] \tag{19}$$

3.4.1 Optimality Principle

Let's introduce $\mathcal{F}_t^s = \sigma(w(\tau) - w(t)), t \leq \tau \leq s$, and $v(.) \in L^2_{\mathcal{F}}()t, T; \mathcal{R}^m, \ v(s) \in \mathcal{U}_{ad}, \ a.e., a.s.$

The value function: $$\Phi(x, t) = \inf_{v(.)} J_{x,t}(v(.)) \tag{20}$$

Using the optimality principle: every point of an optimal path is optimal, we have

$$\Phi(x, t) = \inf_{v(.)} E[\int_t^{t+\epsilon} l(x(s), v(s))ds + \Phi(x(t + \epsilon), t + \epsilon)], \forall t \leq T - \epsilon \tag{21}$$

4 HJB Equation

Using the SDE solution and Itô differential rule, we can derive the HJB equation for a stochastic control system. In general, the HJB equation can be written as

$$\begin{aligned} \frac{\partial \Phi}{\partial t} + \inf_{v \in \mathcal{U}_{ad}} [l(x, v) + D\Phi.g(x, v) + \frac{1}{2} tr D^2 \Phi \sigma \sigma^*(x, v)] = 0, \\ \Phi(x, T) =& h(T) \end{aligned} \tag{22}$$

In practice, uncertainties are external meaning that we don't have control over them, thus we need to do some modifications. So we assume that

$$\sigma(x, v) = \sigma(x) \tag{23}$$

and set

$$a(x) = \frac{1}{2} \sigma(x) \sigma^*(x) \tag{24}$$

To avoid degeneracy, we also assume:

$$\alpha I \leq a(x) \leq MI,\ \alpha > 0 \tag{25}$$

$$A\phi(x) = -\sum a_{ij}(x)\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \tag{26}$$

The HJB Equation (22) can be rewritten

$$\begin{gathered} -\frac{\partial \Phi}{\partial t} + A\Phi = H(x, D\Phi) \\ \Phi(x,T) = h(x) \end{gathered} \tag{27}$$

Notation: The space $L^p_{loc}(\mathcal{R}^n)$ is the space of functions z s.t $z\phi \in L^p(\mathcal{R}^n), \forall \phi \in C_0^\infty(\mathcal{R}^n)$ Additional Assumption:
For some l_0 and c_0, the following inequalities hold

$$\begin{aligned} l(x,v) &\geq l_0|v|^2 - c_0 \\ h(x) &\geq -c_0 \end{aligned} \tag{28}$$

4.0.1 Existence of a Unique Solution

Theorem 4.1. *We assume* (3), (6), (7), (25), (27)*, and $h_{xx}(x)$ is bounded. Then there is a unique soultion of* (27) *s.t*

$$\Phi, \frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial x_i}, \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \in L^p(0,T; L^p_{loc}(\mathcal{R}^n))\ \forall p, 2 \leq p < +\infty \tag{29}$$

$$\begin{aligned} \left|\Phi(x,t)\right| &\leq C(1+|x|^2) \\ \left|D\Phi(x,t)\right| &\leq C(1+|x|) \end{aligned} \tag{30}$$

4.1 Feedback

Let's introduce the Lagrangian $L(x,v,q) = l(x,v) + q.g(x,v)$. We define the feedback $\overline{v}(x,q)$ that minimizes $L(x,v,q)$ and

$$\left|\overline{v}(x,q)\right| \leq C(1+|x|+|q|) \tag{31}$$

as

$$\begin{aligned} \hat{v}(x,t) &= \overline{v}(x, D\Phi(x,t)) \text{ with} \\ \left|\hat{v}(x,t)\right| &\leq C(1+|x|) \end{aligned} \tag{32}$$

Finally, If we assume $\Phi(x,t)$ is $C^{2,1}$ and $\hat{v}(x,t)$ Lipschitz continuous in x , then we have a solution to

$$\begin{aligned} dy &= g(y(s), \hat{v}(y(s),s))ds + \sigma(y(s))dw,\ t \leq s \leq T, \\ y(t) &= x \end{aligned} \tag{33}$$

And we get the optimal control

$$u(s) = \hat{v}(y(s), s)$$

5 Stochastic Control Problems in Finance

Considering a financial market with n risky assets, and the price of asset i at time t is represented by $Y^i(t)$. This is a random process defined on a probability space denoted by (Ω, A, P). Additionally, we have a filtration $\mathcal{F}^t$ and a standard $P, \mathcal{F}^t$ wiener process with values in R^n, where the components of the Wiener process are represented by $w_i(t)$
The uncertainty in the market is caused by the wiener process.
The price $Y^i(t)$ evolves as a geometric Brownian motion (GBM)

$$\begin{aligned} dY_i = Y_i(\alpha_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)w_j), \quad i = 1, 2, ..., n, \\ Y^i(0) = Y_i^0 \end{aligned} \tag{34}$$

where α_i is the drift coefficient, the volatility matrix $\sigma = (\sigma_{ij})$ is invertible, and $w_j(t)$, $j = 1, 2, ..., n$, is the standard Brownian motion. Also, we assume that $\alpha, \sigma(t), \sigma^-1$ bounded. The assumption for σ^-1 existence is such that an individual can manage risky assets due to the uncertainty in the market. In addition, there is a risk-free asset(cash) $Y^0(t)$ defined by

$$\begin{aligned} dY^0(t) = rY^0(t)dt, \\ Y^0(0) = Y_0^0, \end{aligned} \tag{35}$$

where r is the return per unit time. Introducing the Sharpe ratio helps us to do a smooth transformation of the model.

$$\theta = \sigma^{-1}(\alpha(t) - r1) \tag{36}$$

such that we can define a stochastic process $Z(t)$ $w.r.t$ the Sharpe ratio ;

$$\begin{aligned} dZ = -Z\theta(t).dw(t) \\ Z(0) = 1. \end{aligned} \tag{37}$$

5.1 Optimal Consumption and Investment Problem

The consumer-investor problem is a fundamental issue in mathematical finance, involving the allocation of an individual's wealth between consumption and investment in the financial market. The problem is typically solved by considering the individual's utility functions, which are used to inform their decision-making process.
Let $X(t)$ denote the wealth of an individual at time t. Assuming that there is no extra source of income(like salaries). The only income comes from the profit on the investments. At time s, the individual porftfolio

$$X(s) = \pi_0(t)Y^0(t) + \sum_{j=1}^{n} \pi_i(t)Y^i(t) \tag{38}$$

where $\pi_0(s)$ and $\pi_i(s), i = 1, 2, ..., n$, represent the amount invested in the risk-free asset $Y_0(t)$ and risky assets $Y^i, i = 1, 2, ..., n,$. The processes π_0 and $\pi(s)$ are the control variables. Mathematically, $X(s)$ has an Itô differential given by

$$\begin{aligned} dX(s) = \pi_0(t)Y^0(t) + \sum_{j=1}^{n} \pi_i(t)Y^i(t) - C(s)ds \\ X(0) = x \end{aligned} \tag{39}$$

The initial condition which is the initial wealth follows the invariant embedding principle, and $C(s) > 0$ is a new control variable representing the rate of consumption per unit of wealth. Combining $38, 39, 35$ and then using 36 we eliminate π_0 and also defining the proportion of wealth invested in the risky assets $Y^i, i = 1, ..., n,$ as

$$\bar{\omega}_i(s) = \frac{\pi(s)Y^i(s)}{X(s)}$$

provided $X(s) \neq 0$. Preferably, $X(s) > 0$. With the new control, we obtain the evolution of the individual's wealth as follows

$$
\begin{aligned}
dX = rXds + X\sigma^*\bar{\omega}.(\theta ds + dw) - Cds \\
X(t) = x
\end{aligned}
\tag{40}
$$

The control variables are processes; $\bar{\omega}(s)$ and $C(s)$. These processes are adapted to the filtration $\mathcal{F}_t^s = \sigma(w(\tau) - w(t), t \leq \tau \leq s)$ The explicit solution of 40 with $C(s) = 0$ would be given by

$$
X(t) = x + \int_s^t rXd\tau + \int_s^t X\sigma^*\bar{\omega}.\theta d\tau + \int_s^t X\sigma^*\bar{\omega}.dw(\tau) - \int_s^t Cd\tau
$$

Our controls are square-integrable such that

$$
E\int_0^T X(s)^2|\bar{\omega}(s)|^2 ds, \quad E\int_0^T C(s)^2 ds < +\infty \tag{41}
$$

This defines the class of admissible controls. Now, we need to define the payoff for the individual to maximize his/her utility by

$$
J_{x,t}(\bar{\omega}(.), C(.)) = E[\int_t^T U_1(C(s)) \exp(-r(s-t))ds + U_2(X(T)) \exp(-r(T-t))], \tag{42}
$$

The value function is then given by

$$
\Phi(x,t) = \sup_{\bar{\omega}(.),C(.)} J_{x,t}(\bar{\omega}(.), C(.)) \tag{43}
$$

By observation, it's obvious that

$$
\Phi(x,T) = U_2(x) \tag{44}
$$

$U_1(c)$ is the utility function for the individual consumption and $U_2(x)$ for his or her final wealth. With the assumption:

$$
\begin{aligned}
U_1(c), U_2(x) : R^+ \mapsto R^+ \\
\textit{concave} \quad C^2, \\
U_i^{'}(0) = +\infty \quad U_i^i(+\infty) = 0, i = 1,2
\end{aligned}
$$

5.2 Solution

5.2.1 Dynamic Equation Approach using HJB

By standard invariant embedding arguments of dynamic programming, we can now write the HJB equation as follows:

$$
\begin{aligned}
\frac{\partial \Phi}{\partial t} + rx\frac{\partial \Phi}{\partial x} + \sup_c(U_1(c) - c\frac{\partial \Phi}{\partial x}) + \sup_{\bar{\omega}}(x\sigma^*\bar{\omega}.\theta\frac{\partial \Phi}{\partial x} + \frac{1}{2}x^2|\sigma^*\bar{\omega}|^2\frac{\partial^2 \Phi}{\partial x^2}) - r\Phi = 0 \\
\Phi(x,T) = U_2(x)
\end{aligned}
\tag{45}
$$

Introducing a dual problem

$$
\sup_c(U_1(c) - c\lambda),
$$

where λ is a positive paramter. The solution to the above optimization problem is given by

$$
U_1^{'}(\hat{c}) = \lambda \tag{46}
$$

using the assumption that $U_1^{'}(c)$ is a continuous decreasing function. $\exists$ a unique solution to the equation 46 same is true for $U_2(x)$. Let β_i be this unique solution of

$$
U_i^{'}(\beta_i) = x, \tag{47}
$$

Note that the sup in 45 was removed by solving an optimization problem to obtain the feedback.β_i id the inverse $U_i^{'}(x)$ Thus, the sup can be attained at the following optimal controls

$$\hat{c}(x,t) = \beta_1(\frac{\partial \Phi}{\partial x}(x,t)), \quad \hat{\omega}(x,t) = -\frac{\frac{\partial \Phi}{\partial x}}{\frac{\partial^2 \Phi}{\partial x^2}} \frac{(\sigma^*)^{-1}\theta(t)}{x}, \tag{48}$$

With these feedbacks. the HJB equation becomes

$$\begin{aligned} \frac{\partial \Phi}{\partial t} + rx\frac{\partial \Phi}{\partial x} + U_1(\beta_1(\frac{\partial \Phi}{\partial x})) - \beta_1(\frac{\partial \Phi}{\partial x})\frac{\partial \Phi}{\partial x} - \frac{1}{2}|\theta|^2 \frac{\frac{\partial \Phi}{\partial x}}{\frac{\partial^2 \Phi}{\partial x^2}} - r\Phi = 0, \\ \Phi(x,T) = U_2(x). \end{aligned} \tag{49}$$

Since,

$$\lambda(x,t) = \frac{\partial \Phi(x,t)}{\partial x}$$

the we can re-write HJB as

$$\frac{\partial \Phi}{\partial t} + rx\lambda + U_1(\beta_1(\lambda)) - \lambda\beta_1(\lambda) - \frac{1}{2}|\theta|^2 \frac{\lambda^2}{\frac{\partial \lambda}{\partial x}} - r\Phi = 0. \tag{50}$$

$\frac{\partial \Phi}{\partial x} = \lambda$, helps to obtain easily that Φ satifsies the HJB equation. If we differentiate 50 in x and use the substitution $x = G(\lambda,t)$. We then obtain a linear equation

$$\begin{aligned} -\frac{\partial G}{\partial t} - |\theta|^2\lambda(\frac{\partial G}{\partial \lambda} + \frac{1}{2}\lambda\frac{\partial^2 G}{\partial \lambda^2}) + rG = \beta_1(\lambda), \\ G(\lambda,T) = \beta_2(\lambda) \end{aligned} \tag{51}$$

Since $\beta_1(\lambda)$ and $\beta_2(\lambda)$ are unbounded, we need to make some assumptions for the existence and uniqueness of the solution in 51 Assumptions:

$$0 < \delta \le \lambda^m \beta_i(\lambda) \le C, m > 0 \tag{52}$$

Considering

$$G_m(\lambda,t) = \lambda^m G(\lambda,t), \tag{53}$$

which solves

$$\begin{aligned} -\frac{\partial G_m}{\partial t} - |\theta|^2\lambda((1-m)\frac{\partial G_m}{\partial \lambda} + \frac{1}{2}\lambda\frac{\partial^2 G_m}{\partial \lambda^2}) + (r + \frac{|\theta|^2}{2}m(1-m))G_m = \lambda^m\beta_1(\lambda), \\ G_m(\lambda,T) = \lambda^m\beta_2(\lambda). \end{aligned} \tag{54}$$

Growth condition:

$$\frac{\delta_0}{\lambda^m} \le G(\lambda,t) \le \frac{C_0}{\lambda^m} \tag{55}$$

Since $\beta_1(\lambda)$ and $\beta_2(\lambda)$ are decreasing, the function $G(\lambda,t)$ is also decreasing in λ.
Also, we could define in a unique way that the derivative $\frac{\partial \Phi}{\partial x}$, with the growth condition

$$\frac{\delta_1}{x^{\frac{1}{m}}} \le \frac{\partial \Phi}{\partial x} \le \frac{C}{x^{\frac{1}{m}}} \tag{56}$$

We define

$$\Phi(x,t) = \Psi(\lambda(x,t),t) \tag{57}$$

where $\Psi(\lambda(x,t),t)$ solves the a linear PDE, which is

$$\begin{aligned} -\frac{\partial \Psi}{\partial t} - \frac{1}{2}|\theta|^2\lambda^2\frac{\partial \Psi}{\partial \lambda^2} + r\Psi = U_1(\beta_1(\lambda)), \\ \Psi(\lambda,T) = U_2(\beta_2(\lambda)). \end{aligned} \tag{58}$$

To solve the above problem, we need to assume:

$$\lambda^p|U_i(\beta_i(\lambda))| \le C_0 + C_1\lambda^q, p,q \ge 0. \tag{59}$$

Theorem 5.1. *Assume that α_i, $\sigma(t)$, σ^{-1} bounded, and also the assumption on $U_1(c)$, $U_2(x)$, (52), (59). The HJB equation (49) has a unique solution, which is C^2 in , with the growth condition (56) and $\frac{\partial^2\Phi}{\partial x^2} < 0$. It is given explicitly by the formula (57).*

5.3 Solution of the Consumer-Investor Problem

Considering the feedbacks (48), and a corresponding state denoted by $\hat{X}(s)$, the solution of (40) with initial conditions $\hat{X}(0) = x$, given by:

$$\begin{aligned} d\hat{X} = (r\hat{X} - |\theta|^2 \frac{\frac{\partial\Phi}{\partial x}}{\frac{\partial^2\Phi}{\partial x^2}}(\hat{X}) - \beta_1(\frac{\partial\Phi}{\partial x}(\hat{X})))ds - \frac{\frac{\partial\Phi}{\partial x}}{\frac{\partial^2\Phi}{\partial x^2}}(\hat{X})\theta.dw \\ \hat{X}(0) = x. \end{aligned} \tag{60}$$

Obtaining an explicit solution of (60) is quite challenging, however we know the solution exists. Now, we introduce martingale

$$\begin{aligned} d\zeta &= -\zeta.dw, \\ \zeta(0) &= \lambda(x, 0). \end{aligned} \tag{61}$$

We define,

$$\hat{X}(s) = G(\zeta(s), s). \tag{62}$$

60 satisfies Itô differential, and consequently we have

$$\zeta(s) = \frac{\partial\Phi}{\partial x}(\hat{X}(s), s).$$

Also,

$$\frac{\partial G}{\partial\lambda}(\zeta(s), s) = \frac{1}{\frac{\partial^2\Phi}{\partial x^2}(\hat{X}(s), s)}.$$

Using Itô formula on (62), we obtain:

$$d\hat{X} = (\frac{\partial G}{\partial s}(\zeta(s), s) + \frac{1}{2}|\theta|^2\zeta^2(s)\frac{\partial^2 G}{\partial\lambda^2}(\zeta(s), s))ds - (\frac{\partial G}{\lambda}(\zeta(s), s)\zeta(s)\theta.dw \tag{63}$$

Putting together (52) and above equation, we obtain (60). Our controls then become:

$$\begin{aligned} \hat{C}(s), \beta_1(\zeta(s)), \\ \hat{\bar{\omega}}\hat{X}(s) = -\frac{\partial\Psi}{\partial\lambda}(\zeta, s)(\sigma^*)^{-1}\theta(s). \end{aligned} \tag{64}$$

Next, is to show that these controls are admissible i.e they satisfy the set of constraints imposed (41). The assumption (52) helps to check that (the second part of the condition)

$$E\int_0^T \frac{ds}{(\zeta(s))^{2m}} < +\infty. \tag{65}$$

But then, $\eta(s) = \frac{1}{\zeta(s)}$ satisfies the SDE

$$d\eta = \eta|\theta|^2 ds + \eta\theta dw(s),$$

$$\eta(0) = \frac{1}{\lambda(x, 0)},$$

and (65) follows easily. And to check the first part of the condition (41), we need to show that

$$E\int_0^T (\frac{\partial\Psi}{\partial\lambda}(\zeta(s)))^2 ds < +\infty, \tag{66}$$

Taking an estimate $\chi(\lambda) = \frac{\partial \Psi}{\partial \lambda}(\lambda)$. $\chi(\lambda)$ is the solution to this PDE

$$
\begin{aligned}
-\frac{\partial \chi}{\partial t} - \frac{1}{2}|\theta|^2 \lambda^2 \frac{\partial^2 \chi}{\partial \lambda^2} - |\theta^2| \lambda \frac{\partial \chi}{\partial \lambda} + r\chi = \lambda \beta_1^{'}(\lambda), \\
\chi(\lambda, T) = \lambda \beta_2^{'}(\lambda).
\end{aligned}
\tag{67}
$$

Just like how we tackle $G(\lambda, t)$, we assume some condition for the existence of the PDE's solution. We assume that

$$
\lambda^{m+1} |\beta_i^{'}| \leq C, \tag{68}
$$

and above equation $\implies$ the growth condition $\lambda^m |\chi(\lambda)| \leq C$. Then, the property (66) follows.

Theorem 5.2. *Assuming Theorem* (5.1) *and* (68). *The processes* $\hat{\hat{w}}(.)$, $\hat{C}(.)$ *obtained from the feedbacks* $\hat{\hat{w}}(x, t)$, $\hat{C}(x, t)$ *provide a solution of the consumption-investment problem.*