Parametric Equations

Suppose a curve is given in parametric form $c(t) = (4(\theta - \sin\theta), 4(\theta - sin\ \theta)$ and $(\theta, 2\pi)$. How can we find the length of this curve?

Solution

We apply the polar form of length formula

$$l = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta$$

$$x = 4\theta - 4\ sin\ \theta\ \rightarrow \frac{dx}{d\theta} = 4 - 4\ cos\ \theta$$

$$y = 4 - 4\cos\theta \quad \rightarrow \frac{dy}{d\theta} = 4\sin\theta$$

$$l = \int_0^{2\pi} \sqrt{(4 - 4\ cos\ \theta)^2 + (4\ sin\ \theta)^2}$$

$$= \int_0^{2\pi} \sqrt{16 - 32cos\theta + 16\cos^2\theta + 16\,sin^2\theta}\ d\theta$$

$$= \sqrt{16} \int_0^{2\pi} \sqrt{1 - 2cos\theta + cos^2\theta + sin^2\theta}\ d\theta$$

$$Recall\ cos^2\theta + sin^2\theta = 1$$

$$= 4\int_0^{2\pi} \sqrt{2 - 2cos\theta}\ d\theta$$

$$= 4\sqrt{2}\int_0^{2\pi} \sqrt{1 - cos\theta}\ d\theta$$

Use the identity $\boldsymbol{cos2\theta = 1 - 2\sin^2\theta}$

$$cos2\theta = 1 - 2\,sin^2\left(\frac{\theta}{2}\right)$$

Substitute in the equation yields;

$$= 4\sqrt{2}\int_0^{2\pi}\sqrt{1-\left(1-2\,sin^2\left(\frac{\theta}{2}\right)\right)}\,d\theta$$

$$= 4\sqrt{2}\int_0^{2\pi}\sqrt{2\left(sin^2\left(\frac{\theta}{2}\right)\right)}\,d\theta$$

$$= 4\sqrt{2}\sqrt{2}\int_0^{2\pi}\sqrt{\left(sin^2\left(\frac{\theta}{2}\right)\right)}\,d\theta$$

$$= 8\int_0^{2\pi} sin\left(\frac{\theta}{2}\right)d\theta$$

Let u=$\left(\frac{\theta}{2}\right)$, $du = \left(\frac{d\theta}{2}\right)$ and $d\theta$=2du

Substituting yields:

$$= 8\int_0^{2\pi} sin\left(\frac{\theta}{2}\right)d\theta = 8\int_0^{2\pi} sin(u)\cdot 2du$$

$$= 16\int_0^{2\pi}\sin(u)d\theta = -16\cos(u)$$

Substitute u back in the equation and the limits

$$-16\,cos\left(\frac{\theta}{2}\right)\Big|_0^{2\pi} = -16cos\left(\frac{2\pi}{2}\right)-\left(-16cos\left(\frac{0}{2}\right)\right)$$

$$= -16cos(2\pi)-(-16cos(0))$$

$$= -16(-1)+16(1)$$

$$= 32\ units$$