Holographic Error Correction: an Introduction (for Error Correction-ers)

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Holography

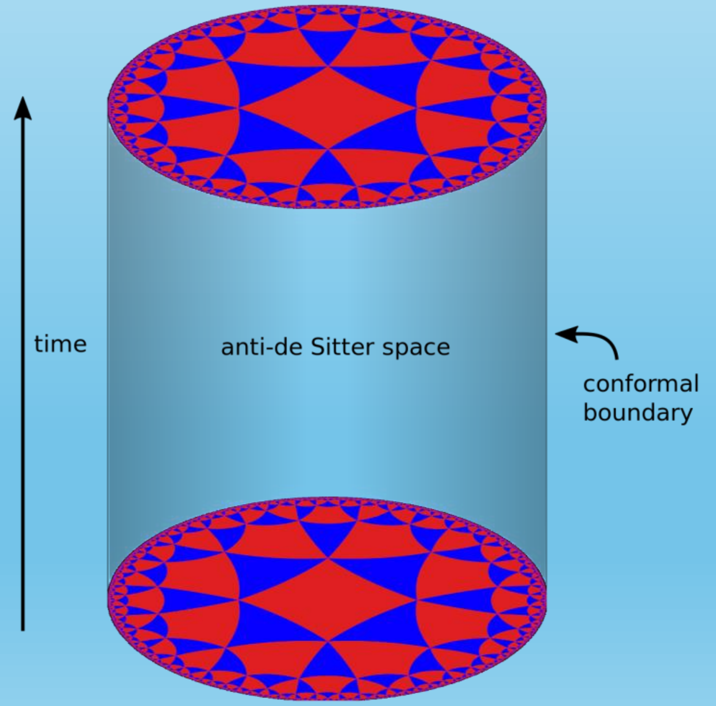
- Inspiration: **black hole thermodynamics**

$$S_{BH} = \frac{c^3 A_{\text{hor}}}{4\hbar G}$$

- Entropy scales with area
- Resolution: Planck length $\frac{4\hbar G}{c^3} = 4l_p^2$
- **Holographic principle**
- Concrete example: AdS/CFT (**A**nti **d**e-**S**itter/**C**onformal **F**ield **T**heory)

Anti de-Sitter Space

- Spacetimes with constant (scalar) curvature: Minkowski, dS, AdS
- AdS: constant **negative** curvature $R < 0$
- Boundary at spatial infinity: cannot be reached by timelike geodesics
- Horizon of AdS_{d+1} is $\mathbb{R} \times \mathbb{S}^{d-1}$
- Has $SO(d, 2)$ symmetry
- Same as a CFT



Conformal Field Theory

- In a nutshell: RQFT invariant under **conformal** group
- Transformations of Minkowski space $\mathbb{R}^d$ preserving angles but not lengths
- Simple example: free massless scalar field in 3+1-Minkowski
- Lots of nice properties
- Important for us: CFTs can be studied on cylinder $\mathbb{R} \times \mathbb{S}^{d-1}$
- Conformal group is isomorphic to $SO(d, 2)$

The Correspondence

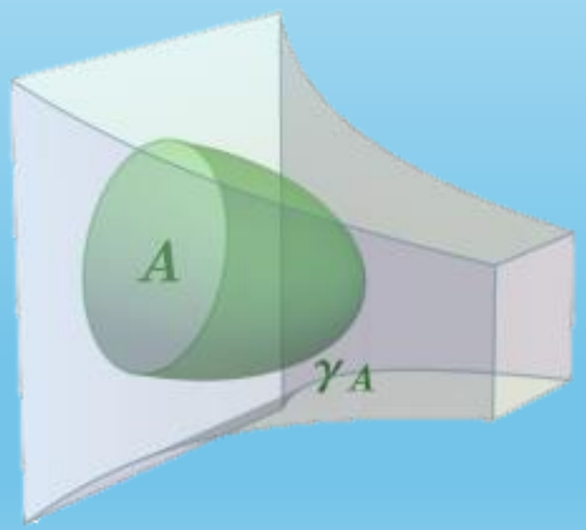
- Loosely: ‘Any relativistic CFT on $\mathbb{R} \times \mathbb{S}^{d-1}$ can be interpreted as a theory of quantum gravity on $AdS_{d+1} \times M$, where M may or may not be trivial’
- Just a definition, doesn’t say how to get the theory of quantum gravity
- Refer to the gravitational AdS as the **bulk**, and the boundary CFT as (surprise!) the **boundary**
- Map between the theories using **holographic dictionary**

Entanglement Entropy/The RT Formula

- Q: What is the dictionary entry for entanglement entropy S_A of boundary subsystem A ?
- A: The **area** of bulk minimal surface γ_A homologous to A

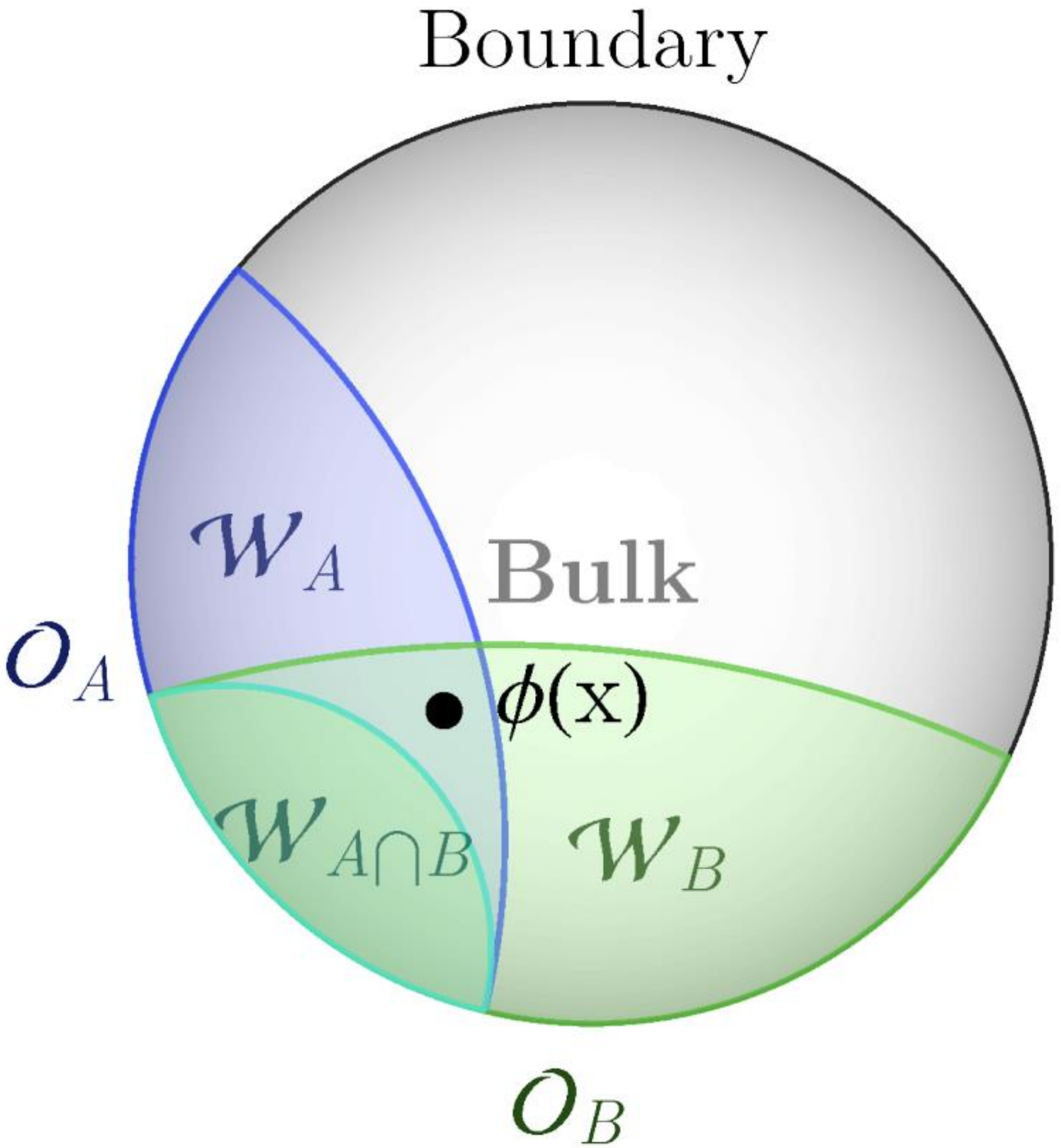
$$S_A = \frac{\mathrm{area}(\gamma_A)}{4G}$$

- Surprising link between entanglement and geometry
- Exceptionally similar to entanglement entropy bound in generic tensor networks



Operator Reconstruction

- Consider bulk field $\phi(x)$
- As x is pushed further into the bulk, requires boundary region which increases in size to reconstruct
- Given boundary subregion A, exists a wedge extending into bulk in which information can be reconstructed
- Bulk region is the **entanglement wedge**, which is bulk region enclosed by A and its RT surface γ_A
- Different boundary regions can have overlapping wedges, so bulk information can be reconstructed on different boundary regions


Boundary
Bulk
$\mathcal{W}_A$
$\mathcal{O}_A$
$\phi(\mathrm{x})$
$\mathcal{W}_{A\cap B}$
$\mathcal{W}_B$
$\mathcal{O}_B$

The HaPPY Code

- Idea: inspired by the ‘QEC-like’ operator reconstruction, construct a QEC code which can reproduce entanglement properties of AdS/CFT
- In particular: RT formula, negativity of tripartite information
- HaPPY code does this
- Based on tensor network, where the entire contracted network is an encoder for a QEC code
- Identifies bulk and boundary DoFs with logical and physical DoFs respectively

Isometric Tensors

- **Isometry**: linear map $T : \mathcal{H}_A \to \mathcal{H}_B$ preserving inner product
- In basis:

$$T : |a\rangle \mapsto \sum_b |b\rangle \, T_{ba}$$

is an isometry iff

$$\sum_b T^\dagger_{a'b} T_{ba} = \delta_{a'a}$$

- Can 'push' operators through isometric tensors:

$$TO = TOT^\dagger T = (TOT^\dagger)T \equiv O'T$$

Isometric Tensors

- If input Hilbert space factorises $\mathcal{H}_A = \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_1}$, we can move an input factor to the output without messing up isometric-ness
- Explicitly, if:

$$T : |a_2 a_1\rangle \mapsto \sum_b |b\rangle\, T_{ba_2a_1}$$

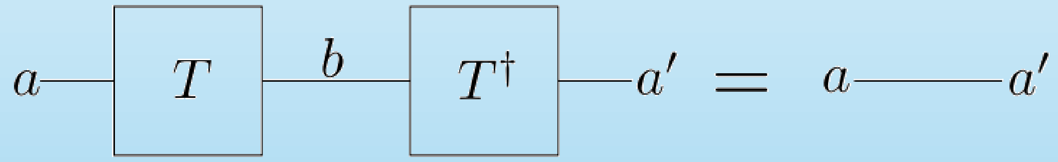
then

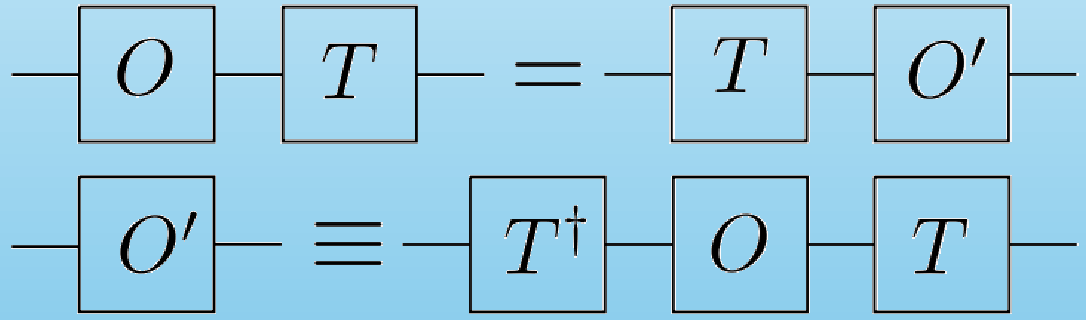
$$\tilde{T} : \mathcal{H}_{A_1} \to \mathcal{H}_B \otimes \mathcal{H}_{A_2} \qquad \tilde{T} : |a_1\rangle \mapsto \sum_{ba_2} |ba_2\rangle\, AT_{ba_2a_1}$$

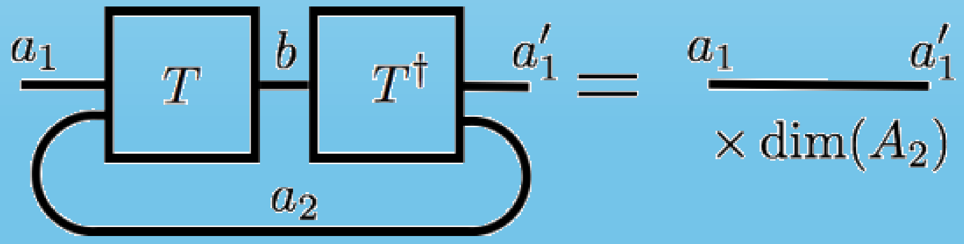
obeys

$$\tilde{T}^\dagger \tilde{T} = \dim(A_2) I_{A_1}$$

Graphical Notation







Perfect Tensors

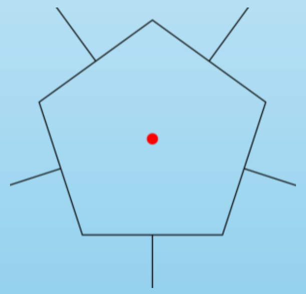
- We can divide indices of tensor with m indices $T_{a_1 \ldots a_m}$ into disjoint sets A and A^C
- T is map from A to A^C
- Each index ranges over v values (bond dimension)
- **Perfect tensors**: $2n$ index tensor T is perfect if for any index bipartition with $|A| \leq |A^C|$, T is proportional to an isometric tensor from A to A^C

Encoding Maps

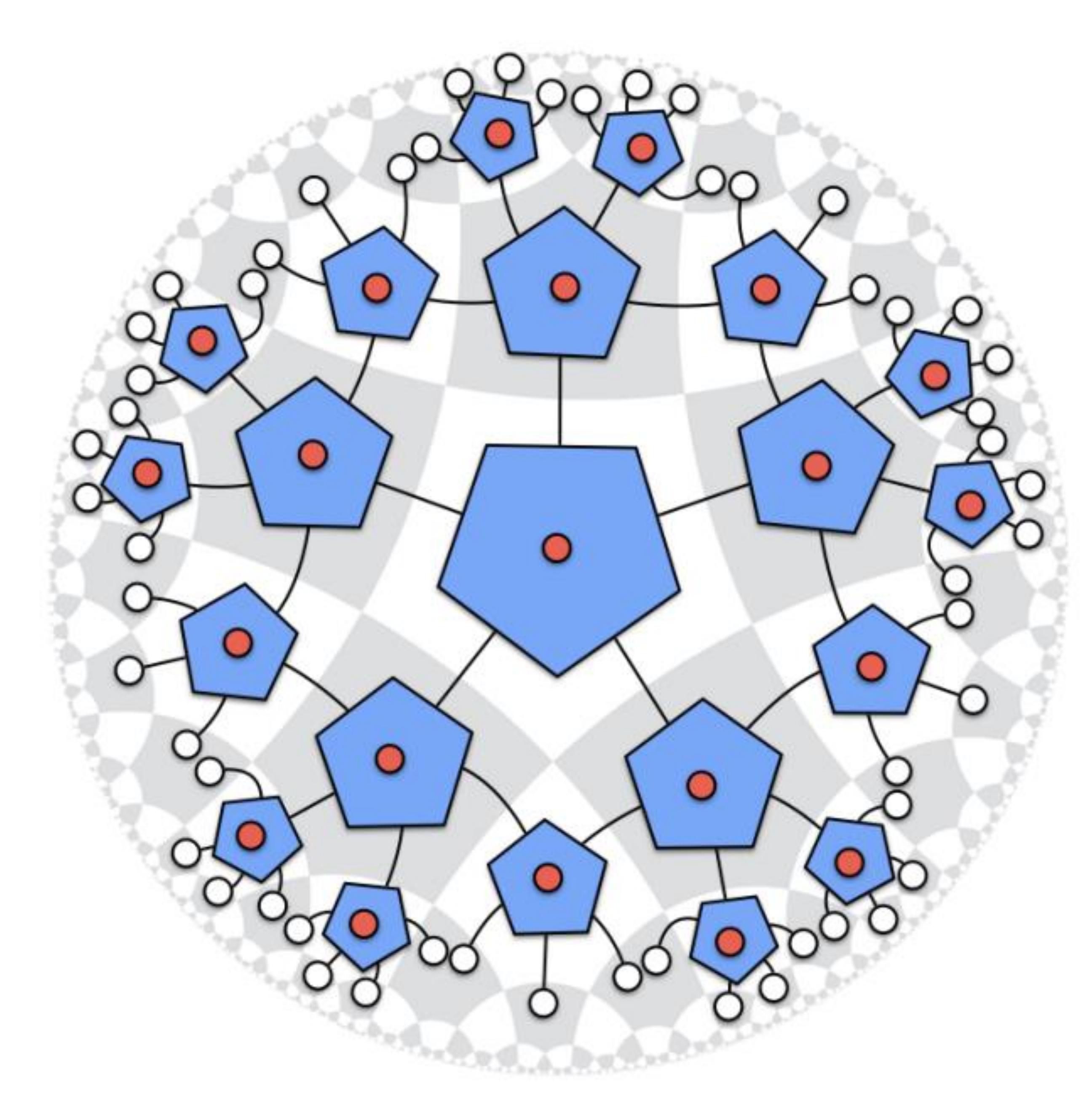
- As map from one spin to $2n-1$ spins, perfect tensor is the **isometric encoding map of a QEC code**
- One logical spin encoded in $2n-1$ physical spins
- Logical spin protected against erasure of $n-1$ physical spins, so have a $[[2n-1,1,n]]_v$ code
- All this is very general
- Specialise to $[[5,1,3]]_2$ code, associated with six-index perfect tensor for qubits

Constructing the full code

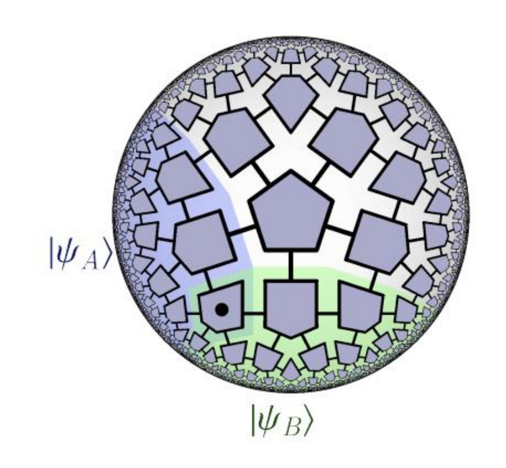
- Take aforementioned $[[5,1,3]]_2$, and draw as a pentagon



- Associate red dot as logical input qubit, and each leg/edge as a physical qubit
- Adjoin tensors to initial ones by contracting along edges, following adjacency as specified by discrete geometry



Bulk Reconstruction



- Follows naturally from code properties
- Start from subset of the uncontracted physical ‘boundary’ edges
- Can reconstruct state of adjacent logical ‘bulk’ states from three physical sites
- Use recovered physical state on remaining edges as input for next layer of tensors, and repeat
- This builds up a discrete wedge until we can no longer find three physical sites around the same pentagon
- Called the **greedy algorithm**

Stabiliser Codes

- Usual definition: $[[n,k]]$ code is a stabiliser code if the code space is the simultaneous eigenspace of $n-k$ commuting Pauli operators
- Call an n-index tensor a **stabiliser tensor** if the state it corresponds to is a stabiliser state
- ***Theorem**: a holographic code defined by a contracted network of perfect stabiliser tensors where the greedy algorithm starting at the boundary reaches the entire network is a stabiliser code*
- Sketch proof for pentagon code

Proof Sketch

- Six-index perfect tensor defines $[[6,0]]$ stabiliser code, but can also be regarded as $[[6-k,k]]$ code
- When greedy algorithm condition is satisfied, encoding algorithm for the holographic code is just composition of encoding isometry associated to each perfect tensor
- Each tensor has 0,1,2, or 3 incoming legs, including dangling bulk leg and all contracted legs
- Proof therefore reduces to showing that composing encoding isometries of two stabiliser codes is the encoding isometry of a stabiliser code (which is true)

Detail for Last Step of Proof

- Suppose $\mathcal{S}_1, M_1$ and $\mathcal{S}_2, M_2$ are stabiliser group and encoding isometry for $[[n_1, k_1]]$ and $[[n_2, k_2]]$ codes respectively
- Can apply M_2 to m of n_1 output qubits from M_1 along with $k_2 - m$ additional input qubits, obtaining a $[[n_1 - m + n_2, k_1 + k_2 - m]]$ code
- This new code is indeed a stabiliser code, with stabiliser group generated by $\mathcal{S}_2$ and $M_2(\mathcal{S}_1)$
- Elements of $M_2(\mathcal{S}_1)$ are Paulis since M_2 is a Clifford isometry by definition

Some QEC Fun Facts

- The QEC code corresponding to a pentagon is also known as the five-qubit perfect code
- It is the smallest qubit stabiliser code correcting a single qubit error
- It has generators

$$
\begin{aligned}
S_1 &= X \otimes Z \otimes Z \otimes X \otimes I \\
S_2 &= I \otimes X \otimes Z \otimes Z \otimes X \\
S_3 &= X \otimes I \otimes X \otimes Z \otimes Z \\
S_4 &= Z \otimes X \otimes I \otimes X \otimes Z.
\end{aligned}
$$

- Has actually been experimentally realised on a superconducting qubit device

Future Research

- HaPPY codes are on a static timeslice of the full theory – is there a way of introducing **dynamics**?
- Would also be nice to find a way to realise a universal set of logical operators acting on the code space
- Our line of attack for the above relies on an underlying fermionic structure of the HaPPY code – makes use of **Majorana dimers**

Thank you for listening!

Any questions?