Low-overhead non-Clifford topological fault-tolerant circuits for all non-chiral abelian topological phases

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03/28/24

Two theories of topological quantum computation

Passive/**physical** fault tolerance	Active/**engineered** fault tolerance

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Arbitrary topological **phases** are studied	Almost exclusively the **toric code** is studied
Universality achieved purely topologically	Universality achieved through magic state distillation

Active fault tolerance beyond the toric-code phase: **Literature**

- Dauphinais et al, *Quantum Error Correction with the Semion Code* (2018)
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- Schotte et al, *Quantum error correction thresholds for the universal Fibonacci Turaev-Viro code* (2020)
  Syndrome-extraction circuit for the Fibonacci string-net model

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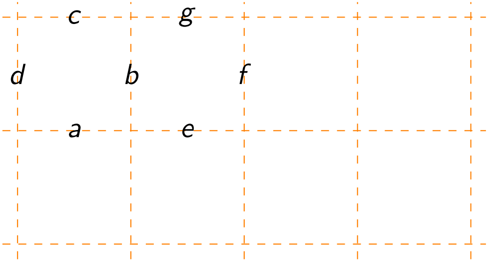
- Error correction beyond Pauli measurements + Clifford operations

Discrete path integrals

- Sum over variables a, b, c, ... on regular lattice



$$\Rightarrow \quad Z = \sum_{a,b,c,d,e,f,g,\ldots}$$

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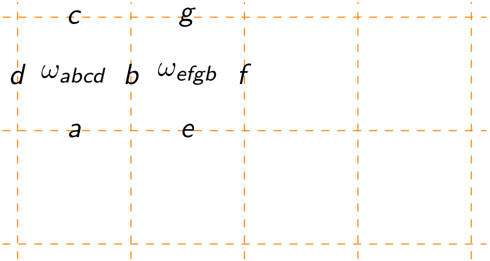

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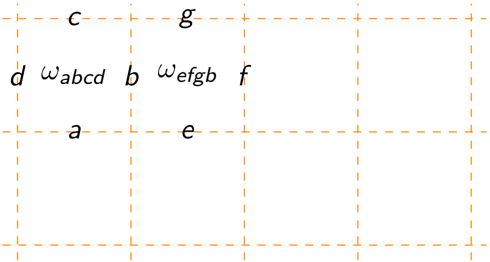

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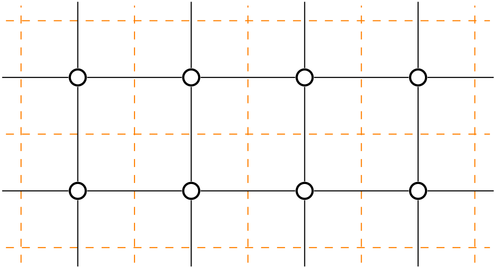
- Alternative language: Tensor networks in spacetime



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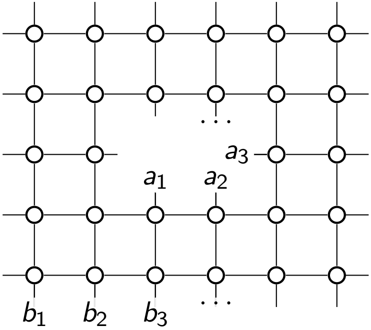
Fixed-point path integrals

- Zero correlation length → Annulus operator = rank 1:



$$= T_{\vec{a},\vec{b}} = v_{\vec{a}} \cdot w_{\vec{b}} \ .$$

Example: Toric code

Ground state: Closed-loop superposition $\in$ 2D

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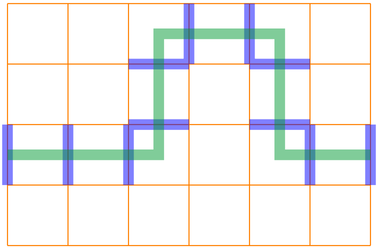

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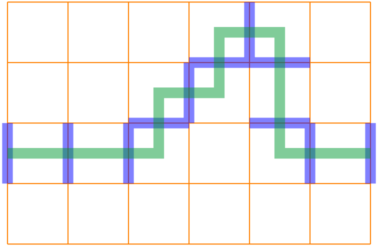

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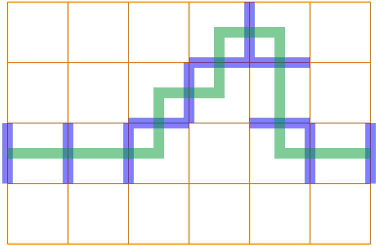
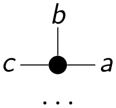


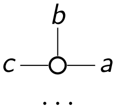
Path integral: Closed-membrane superposition $\in$ 2+1D,

$$Z \propto \sum_{\text{1-cocycle } A} 1 = \sum_{\text{1-cochain } A} \prod_{\text{plaquette } p} \delta_{p_0+p_1+p_2+\ldots=0}$$

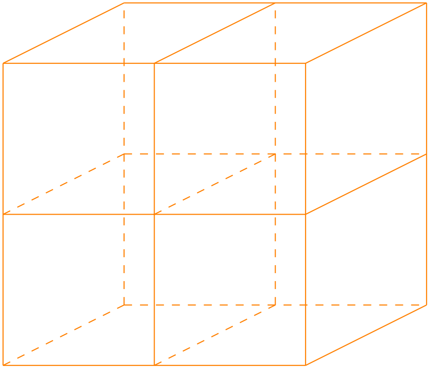
Example: Toric code



$$= \begin{cases} 1 & \text{if } a = b = c = \dots \\ 0 & \text{otherwise} \end{cases}$$



$$= \begin{cases} 1 & \text{if } a + b + c + \dots \\ & = 0 \mod 2 \\ 0 & \text{otherwise} \end{cases}$$



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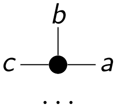

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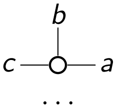

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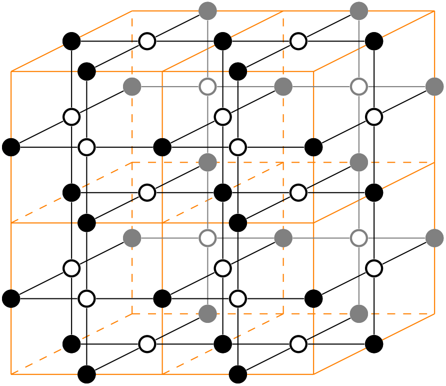

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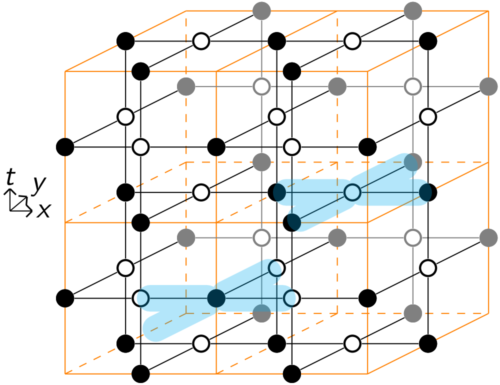
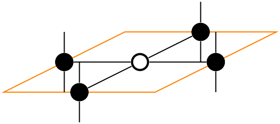


$$T_1 :=$$



$$, \quad = \frac{1}{2}(1 + ZZZZ)$$

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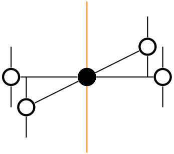

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$$Z = \sum_{\text{1-cocycle } A} \prod_{\text{volume } v} i^{(\overline{A} \cup d\overline{A})(v)} .$$

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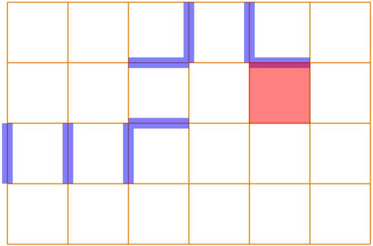
$$A \quad \Rightarrow \quad dA$$



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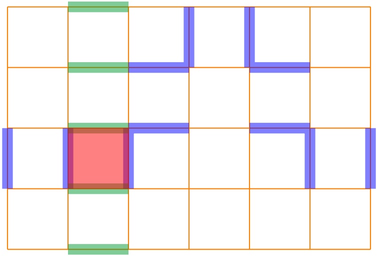

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- Topological path integral $\Rightarrow$ anyon worldlines

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b: fluxes or m-anyons
c: charges or e-anyons

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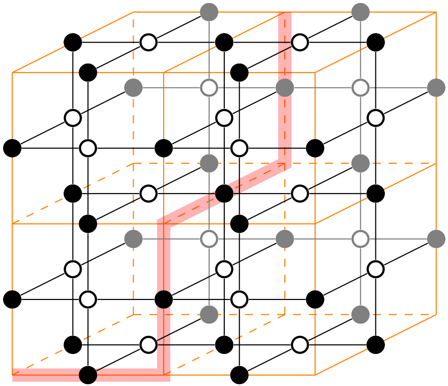
c defects along 1-cycles:



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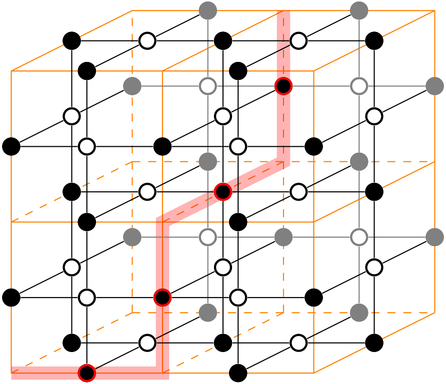

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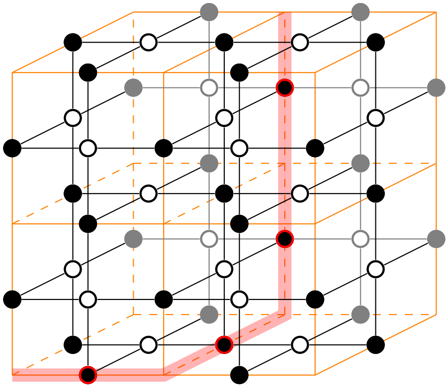

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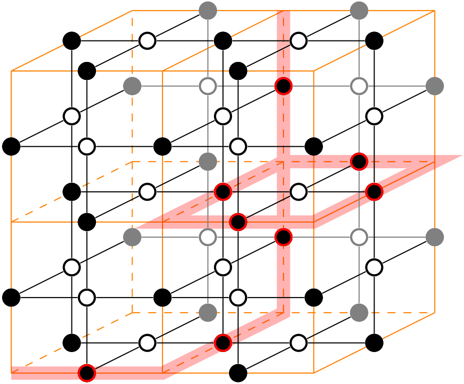

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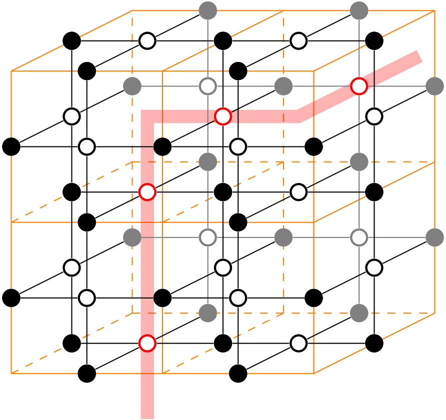
b defects along 2-cocycles:



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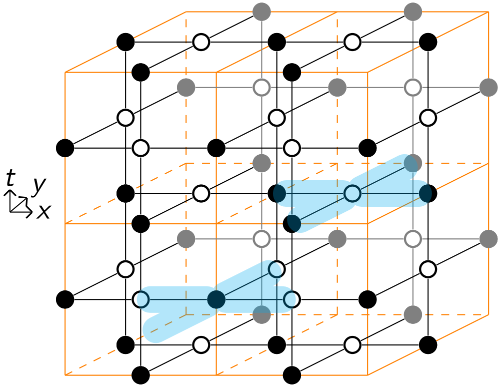
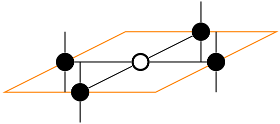
$\cup_1$: *First order cup product*

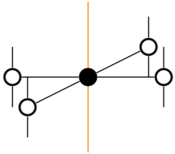
$$d(A \cup_1 B) = dA \cup_1 B + (-1)^a A \cup_1 dB$$
$$+ (-1)^{a+b+1} A \cup B + (-1)^{a+b+ab} B \cup A$$

Example: Stabilizer toric code

Path integral $\rightarrow$ non-unitary circuit



$T_1 :=$  ,

$V_1 :=$  .

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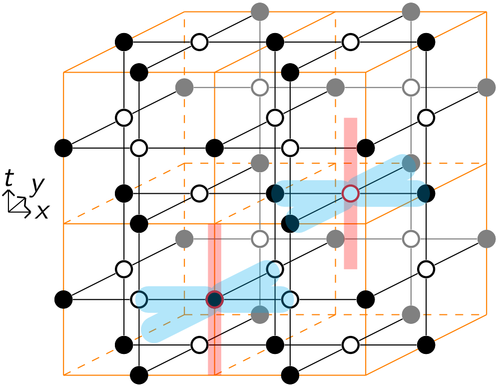
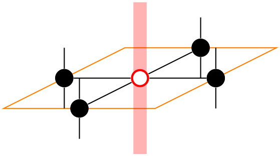
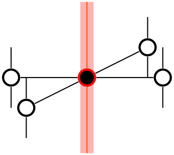


$T_m :=$ 

$V_e :=$  .

Double-semion by measuring 1-form symmetries

- Resolve stabilizer measurements using ancillas and CX gates

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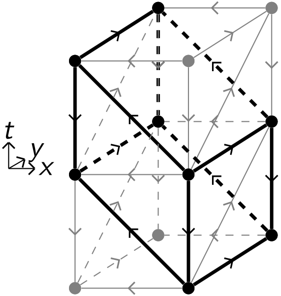
- Slanted cubic lattice, 

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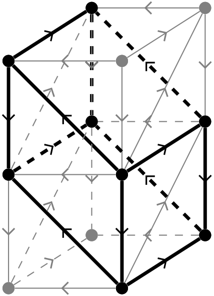

- Slanted cubic lattice, 

- Weight

$$i^{(\overline{A}\cup d\overline{A}+\overline{b}\cup_1 d\overline{A})(v)}$$

implemented by phase gates

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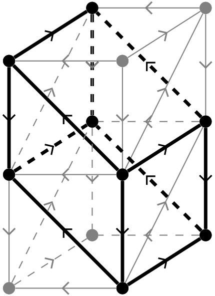

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- $CS\,|x,y\rangle = i^{\overline{xy}}\,|x,y\rangle$

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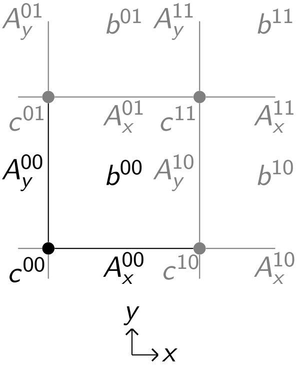
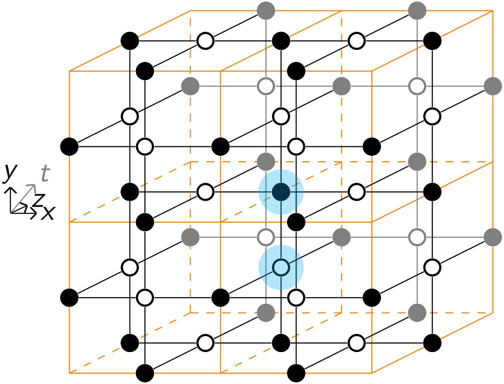


Layer	Gates
0^+	$CX[A_y^{10}, b^{00}],\ CX[c^{00}, A_x^{00}]$
1	$\overline{CS}[A_y^{10}, A_x^{01}],\ \overline{CS}[b^{00}, c^{10}]$
$1'$	$\overline{CS}[A_x^{00}, A_y^{00}],\ \overline{CS}[c^{10}, b^{00}]$
$1''$	$cS[\hat{b}^{00(-1)}, A_y^{00}],\ CS[A_x^{00}, c^{01}]$
1^+	$CX[A_x^{00}, b^{00}],\ CX[c^{01}, A_y^{00}]$
2^+	$CX[A_x^{01}, b^{00}],\ CX[c^{00}, A_y^{00}]$
3	$\overline{cS}[\hat{b}^{00(-1)}, A_y^{00}],\ CS[c^{10}, A_x^{00}]$
3^+	$CX[A_y^{00}, b^{00}],\ \overline{CS}[A_x^{00}, c^{00}]$
4	$CS[A_y^{10}, A_x^{01}],\ \overline{cS}[\hat{b}^{00(-1)}, c^{01}]$
$4'$	$\overline{cS}[\hat{b}^{00(-1)}, A_x^{00}],\ CS[c^{10}, A_y^{00}]$
4^+	$P_{\vert 0\rangle} \circ M_Z[b^{00}, \hat{b}^{000}],\ CS[c^{10}, A_x^{01}]$
5	$cS[\hat{b}^{000}, A_x^{00}],\ CS[A_y^{10}, c^{11}]$
$5'$	$cS[\hat{b}^{000}, c^{10}],\ CS[A_x^{00}, A_y^{00}]$
5^+	$CX[c^{10}, A_x^{00}]$
6^+	$P_{\vert +\rangle} \circ M_X[c^{00}, \hat{c}^{000}]$

Other fault-tolerant protocols

Diagonal time direction ⇒ CSS honeycomb Floquet code [1] [2]



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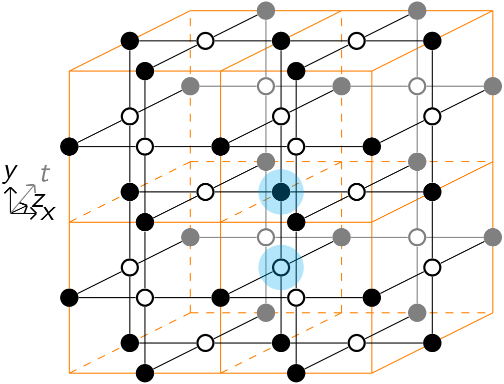

Turn $2+1$D protocol into $3+0$D protocol ⇒ Measurement-based quantum computation

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General abelian twisted quantum doubles

Abelian gauge group G. Path integral:

$$\sum_{G\text{-valued 1-cocycle } A} \prod_{\text{volume } v} e^{2\pi i (\overline{A}^T F \cup d\overline{A})(v)}$$

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1-form symmetries:

$$\sum_{A:dA=b} \prod_{v} e^{2\pi i(\overline{A}^T F \cup d\overline{A} - \overline{A}^T (F+F^T) \cup \overline{b} + \overline{b}^T F \cup_1 d\overline{A})(v)} \prod_{e} e^{2\pi i \overline{A}^T (f^{-1})^T \overline{c}}$$

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$$\sum_{G\text{-valued 1-cocycle } A} \prod_{\text{volume } v} e^{2\pi i(\overline{A}^T F \cup d\overline{A})(v)}$$

1-form symmetries:

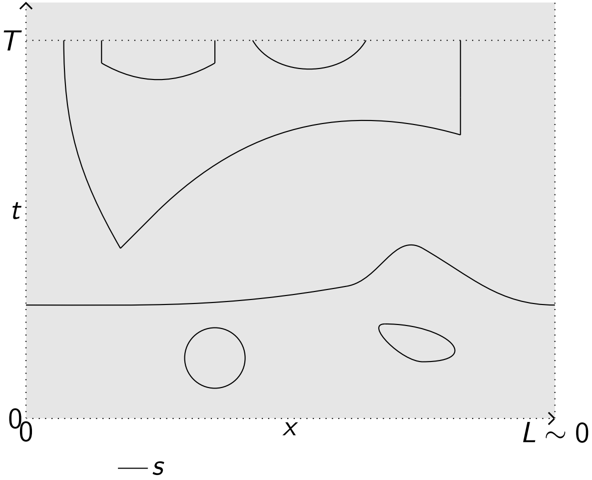
$$\sum_{A:dA=b} \prod_{v} e^{2\pi i(\overline{A}^T F\cup d\overline{A} - \overline{A}^T(F+F^T)\cup\overline{b} + \overline{b}^T F\cup_1 d\overline{A})(v)} \prod_{e} e^{2\pi i\overline{A}^T (f^{-1})^T\overline{c}}$$

b and c no longer independent:

$$\delta c = f^T \backslash \big(\cup f^T (F + F^T) d\overline{b} \big)$$

Classical decoding

No noise: measured 1-form symmetry defects form 2-cocycle s



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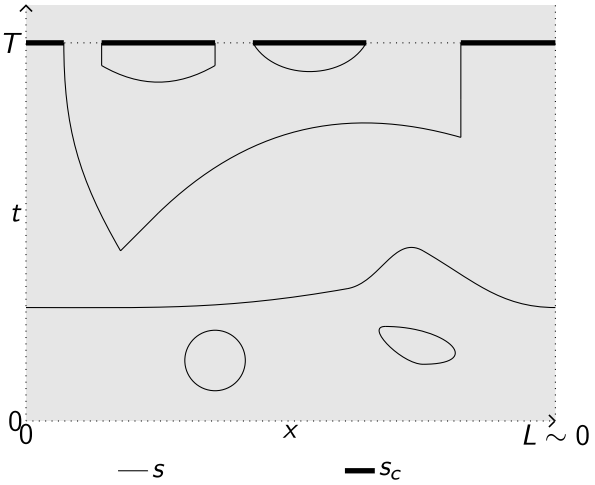

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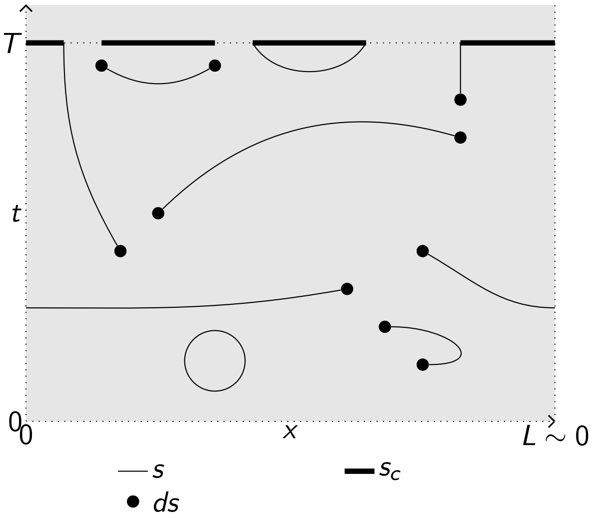
With noise: Fix 1-form symmetry defects



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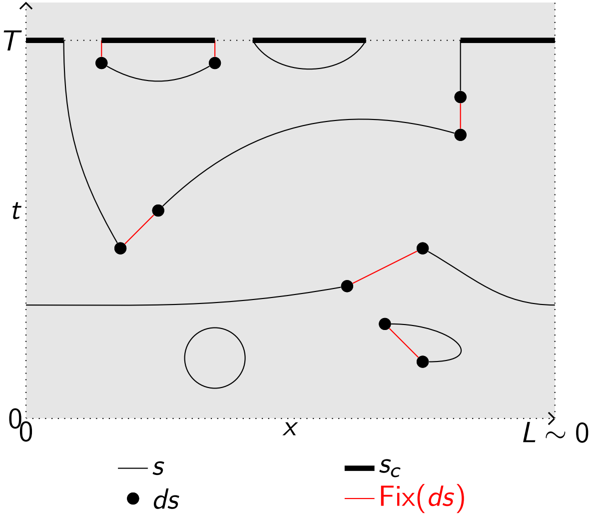

Non-abelian example

- Gauge group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, with non-abelian "twist":

$$Z = \sum_{A_i} \prod_{v} (-1)^{(A_0 \cup A_1 \cup A_2)(v)}$$

[3] Bombin, *2D quantum computation with 3D topological codes* (2018)
[4] Brown, *A fault-tolerant non-Clifford gate for the surface code in two dimensions* (2019)

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- Requires just-in-time decoding [3] [4]

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