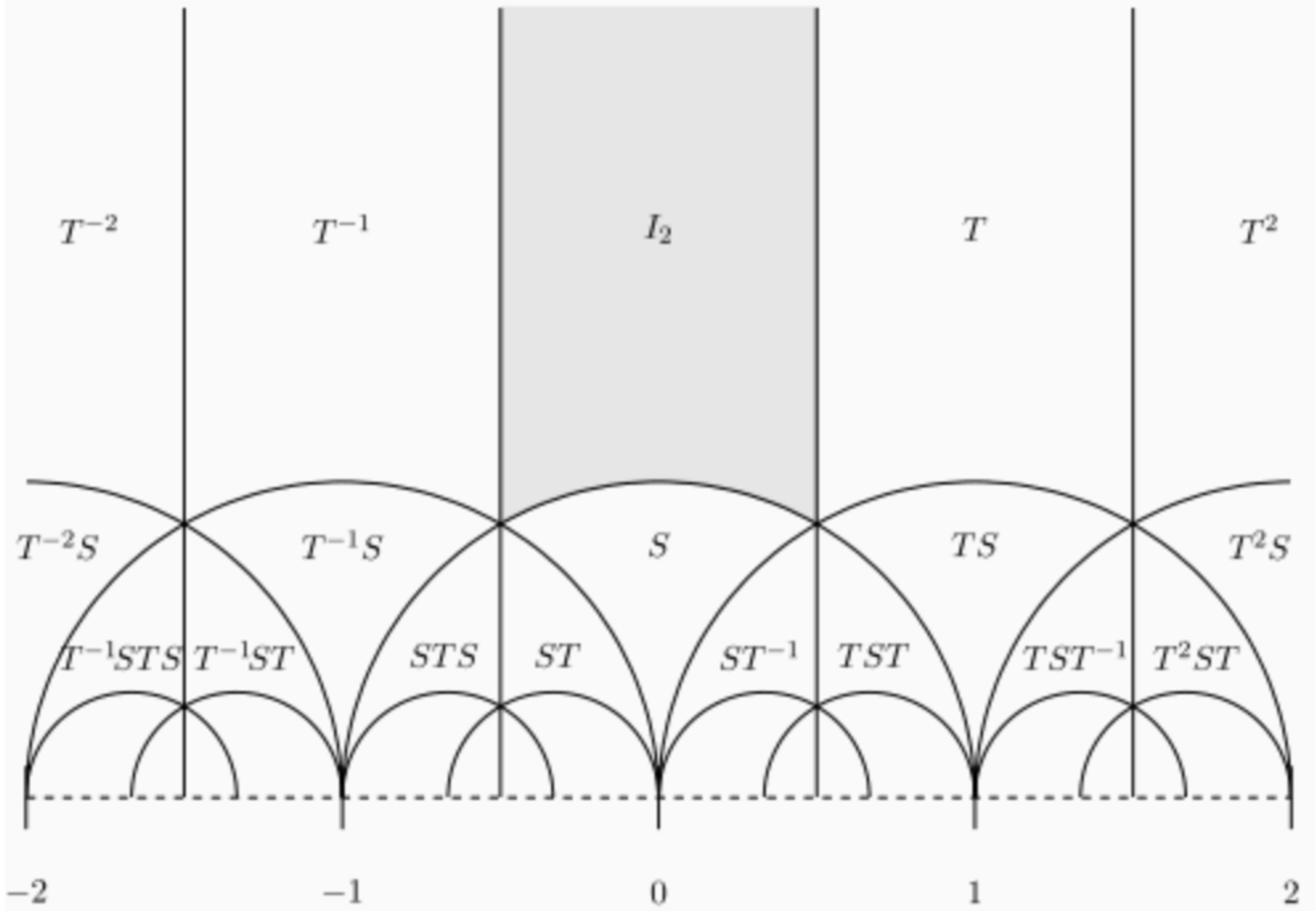
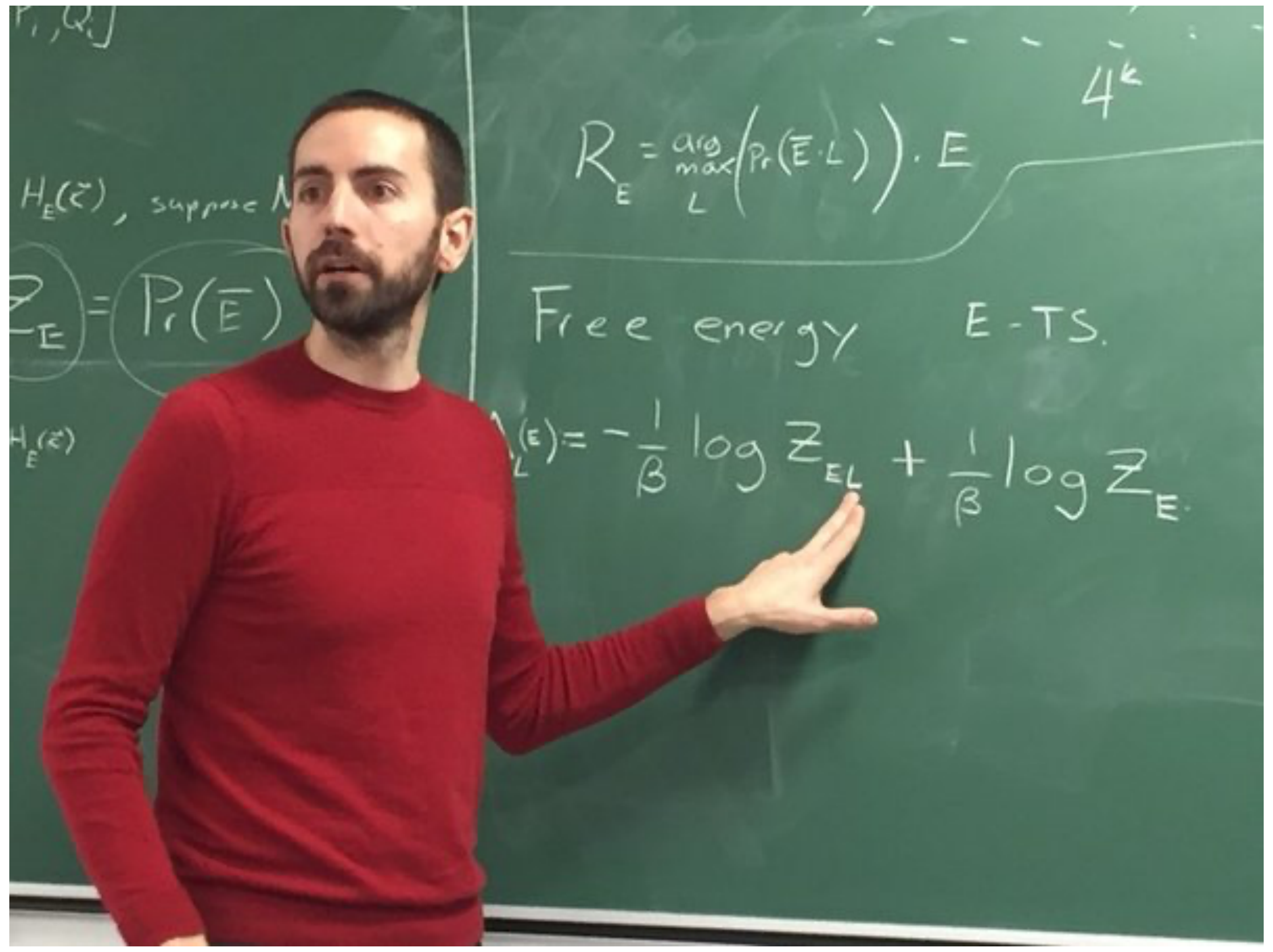
Logical GKP Shadows

YIT Workshop
Mar 22th 2023





Jonathan Conrad
FU Berlin/Helmholtz-Center Berlin

Steve Flammia
AWS CQC
Caltech

Overview

1. GKP and GKP Cliffords
2. Twirling and Shadows
3. Logical GKP Shadows

GKP codes

[GKP (2001)]
[Harrington & Preskill (2001)]
[Harrington Thesis (2004)]
[JC, Eisert, Arzani (2022)]

Stabilizer Group:

$$\mathcal{S} = \langle D\left(\xi_1\right), D\left(\xi_2\right), \ldots, D\left(\xi_{2n}\right)\rangle = \left\{e^{i\phi(\xi)} D\left(\xi\right), \xi \in \mathcal{L}\right\}$$

Lattice:

$$\mathcal{L} = \mathrm{span}_{\mathbb{Z}}\left\{\xi_1, \xi_2, \ldots, \xi_{2n}\right\} \subset \mathbb{R}^{2n}$$

Centralizer:

$$\mathcal{C}\left(\mathcal{S}\right)$$

Symplectic Dual Lattice:

$$\mathcal{L}^{\perp} = \left\{\xi^{\perp} \in \mathbb{R}^{2n} :\ (\xi^{\perp})^T J \xi \in \mathbb{Z}\ \forall \xi \in \mathcal{L}\right\} \subset \mathbb{R}^{2n}$$

Lattice basis

$$M = \begin{pmatrix} \boldsymbol{\xi}_1^T \\ \vdots \\ \boldsymbol{\xi}_{2n}^T \end{pmatrix}$$

Symplectic Gram matrix

$$A = MJM^T$$

$$M = AM^{\perp}$$

$$\mathcal{L} \subseteq \mathcal{L}^{\perp}$$

Quantifies symp. dual quotient

$\#\{$logical Pauli operators$\} = d^2 = |\det A|$

Commutation:

$$D\left(\xi\right) D\left(\eta\right) = e^{-i2\pi\xi^T J\eta} D\left(\eta\right) D\left(\xi\right)$$

Closure:

$$D\left(\xi\right) D\left(\eta\right) = e^{-i\pi\xi^T J\eta} D\left(\xi + \eta\right)$$

Image: [GKP (2001)]

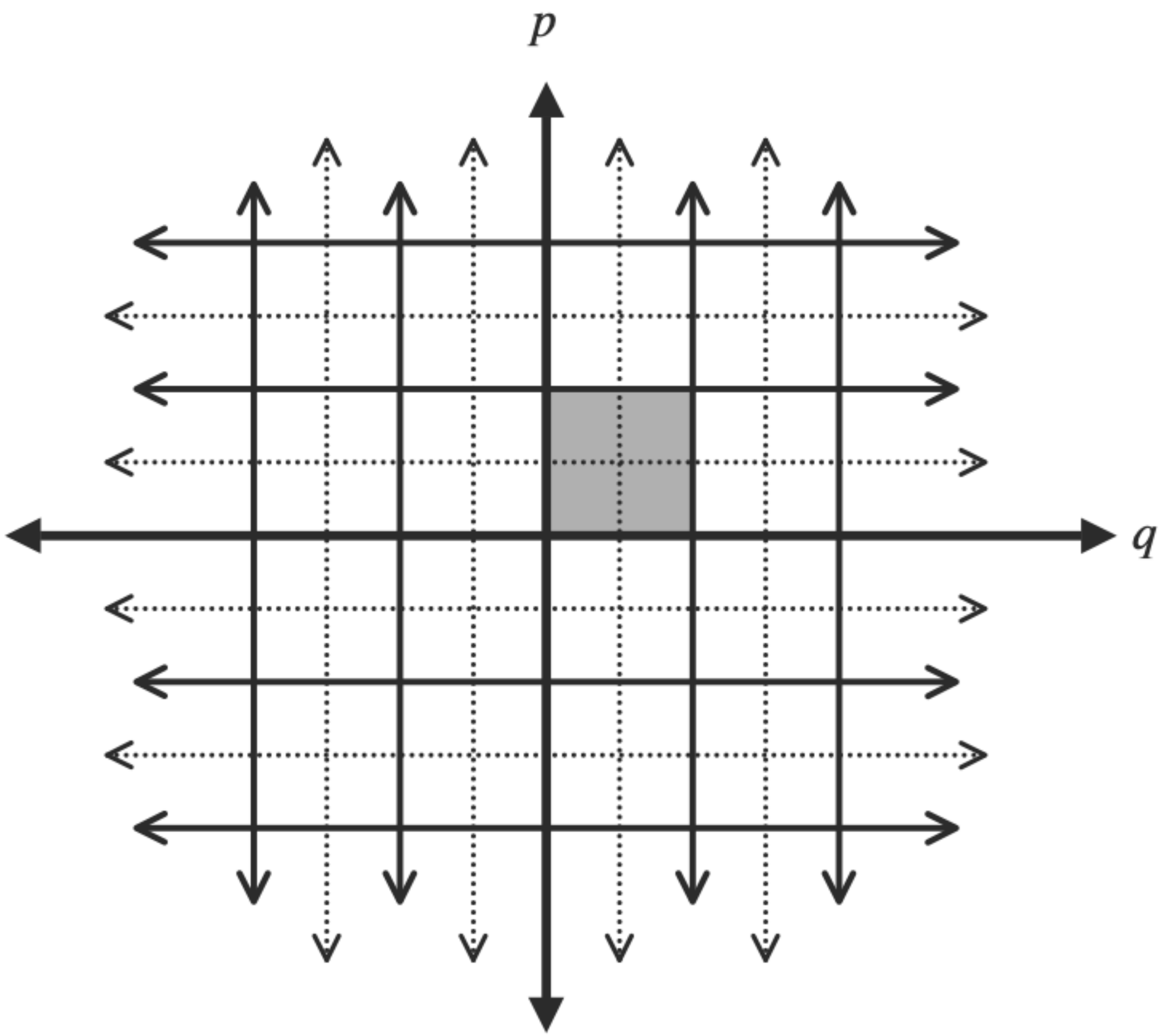
Scaled GKP Codes

- Scaled GKP codes: rescale symplectic lattice

$$M_0 \in \mathbb{R}^{2n\times 2n}: \; M_0 J M_0^T = J \;\Rightarrow\; M = \sqrt{\lambda} M_0, \, \lambda \in \mathbb{N}$$

$$D = \lambda^n;$$

n	$\dim(\mathcal{L}_0)$ $(\mathcal{L})$	$\mathcal{L}_0$	$(\lambda_1(\mathcal{L}))^2$	Symp. self-dual	Eucl. self-dual
1	2	$\mathbb{Z}^2$	1	✓	✓
1	2	A_2	$\frac{2}{\sqrt{3}}$	✓	✓
2	4	D_4	$\sqrt{2}$	✓[10]	✓
4	8	E_8	2	✓	✓
6	12	K_{12}	$\frac{4}{\sqrt{3}}$ [10]	✓[10]	✓
12	24	Λ_{24}	4 [32]	✓[28]	✓
n	$2n$	$\sqrt{\lambda/q}L_{\mathrm{cs}}$	$\Delta \sim O\left(\sqrt{n/q\lambda}\right)$	✓	✓
N	$2N$	$\Lambda_{\square}(\mathcal{Q})$	$\Delta \geq \sqrt{d/2}$	x	x
N	$2N$	$\Lambda_{\hexagon}(\mathcal{Q})$	$\Delta = \sqrt{d/\sqrt{3}}$	x	x



[GKP (2001)]

GKP Cliffords = $\mathrm{Aut}(\mathscr{L}^\perp)$

- Logical Paulis = Trivial Automorphisms $\mathrm{Aut}_0(\mathscr{L}^\perp)$ = translations by vectors in $\mathscr{L}^\perp$

- Logical non-trivial Cliffords = non-trivial symp. Automorphisms

  $\mathrm{Aut}^S(\mathscr{L}^\perp) = \mathrm{Aut}(\mathscr{L}^\perp) \cap \mathrm{Sp}_{2n}(\mathbb{R})$

- Scaled GKP codes: $\mathscr{L}^\perp \propto \mathscr{L} \Rightarrow \mathrm{Aut}^S(\mathscr{L}^\perp) = \mathrm{Aut}^S(\mathscr{L})$

GKP Cliffords

- $S \in \mathrm{Aut}(\mathscr{L})$:

$$MS^T = UM$$

GKP Cliffords

- $S \in \mathrm{Aut}(\mathscr{L})$:

$$MS^T = UM$$

GKP Cliffords

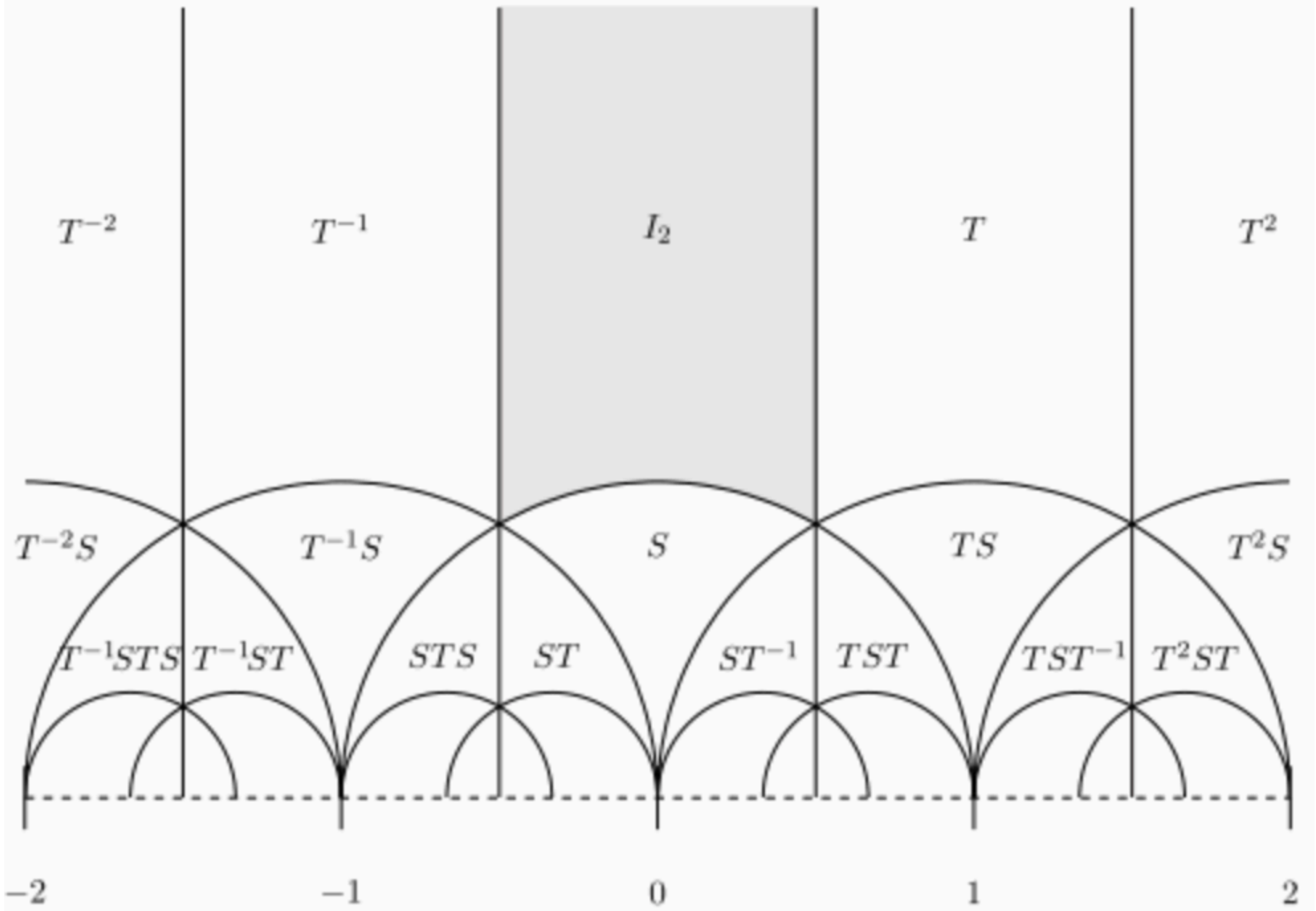
$$
\begin{tikzcd}
\mathrm{Aut}^S\left(\mathcal{L}^\perp\right) \arrow[d, "\bmod \mathcal{L}"] & \mathrm{SL}_{2n}\left(\mathbb{Z}\right) \supseteq \mathrm{Sp}_{2n}\left(\mathbb{Z}\right) \arrow[l] \arrow[d, "\bmod d"] \\
\mathrm{Aut}^S\left(\mathcal{L}^\perp/\mathcal{L}\right) \arrow[r, leftrightarrow] & \mathrm{Sp}_{2n}\left(\mathbb{Z}/d\mathbb{Z}\right)
\end{tikzcd}
$$

GKP Cliffords in a single mode

- $\Gamma = \mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) = \langle S, T \,|\, S^4 = 1,\, (ST)^3 = 1 \rangle$
- Mapping Class group of Torus $\quad T = \mathbb{R}^2/\mathscr{L}^\perp$
- Action can be represented in $\quad \mathfrak{h} = \mathbb{R} \oplus i\mathbb{R}_{>0}$ as $\quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}): \quad \gamma z = \dfrac{az+b}{cz+d}$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

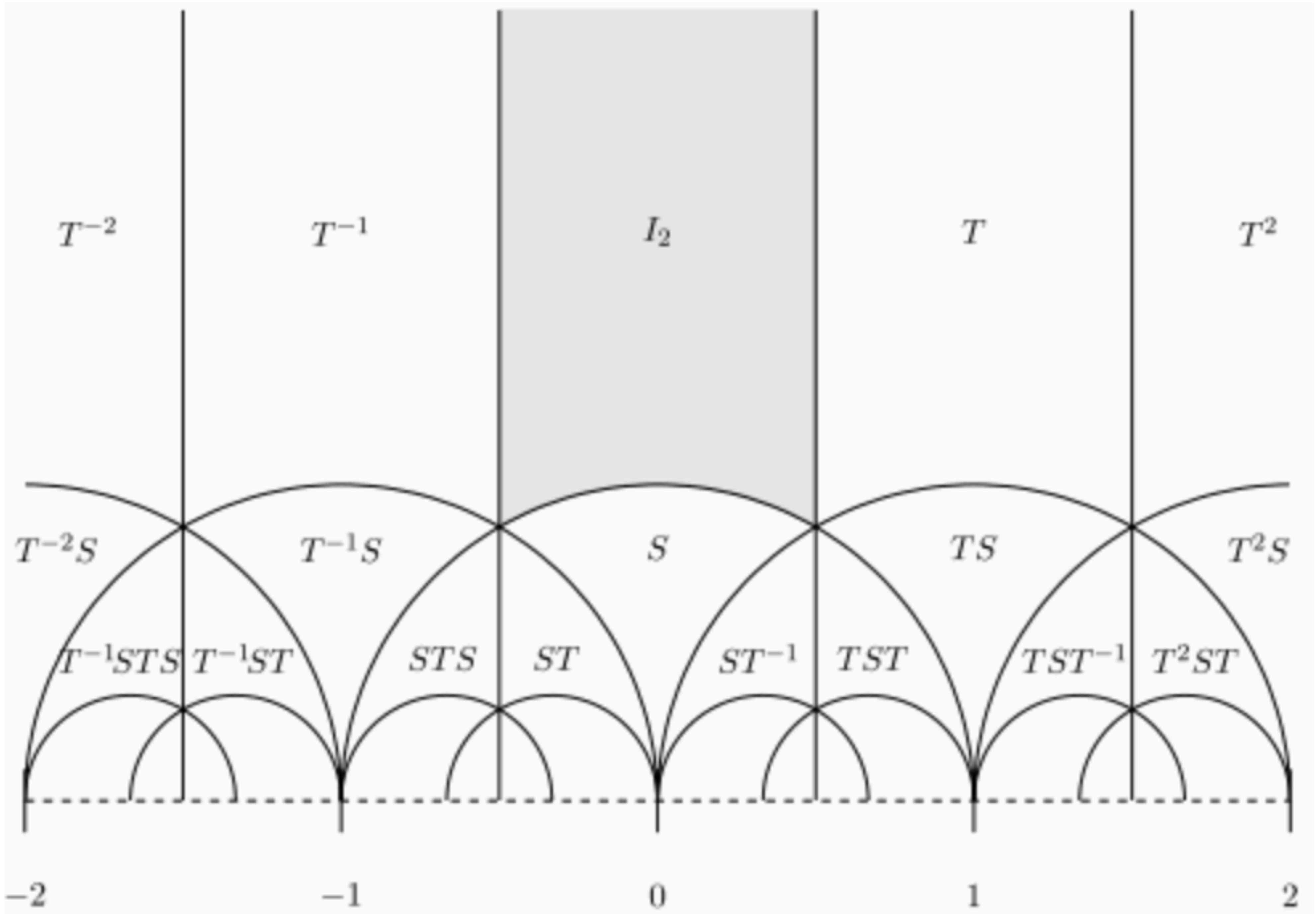
$$Sz = \frac{-1}{z}, \;\; Tz = z + 1$$



GKP Cliffords in a single mode

- $\Gamma := \mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) = \langle S, T \,|\, S^4 = 1,\, (ST)^3 = 1 \rangle$
- Principal Congruence subgroup $\Gamma(2) = \{\gamma \in \Gamma : \ \gamma \mod 2 = I\}$

$$\begin{tikzcd}
\mathrm{Aut}^S\left(\mathcal{L}^\perp\right) \arrow[d, "\bmod \mathcal{L}"] & & \mathrm{SL}_2\left(\mathbb{Z}\right) \arrow[ll] \arrow[d, "\bmod \Gamma(2)"] \\
\mathrm{Aut}^S\left(\mathcal{L}^\perp/\mathcal{L}\right) \arrow[rr, leftrightarrow] & & \mathrm{SL}_2\left(\mathbb{Z}\right)/\Gamma(2) = \mathrm{SL}_2\left(\mathbb{Z}_2\right)
\end{tikzcd}$$



Twirling Theory

Flavours of Twirling

State twirl [Bennet et al (1996)]

$$\rho \mapsto \Pi(\rho)$$

- $\mathscr{G} = \{R \otimes R, R \in \mathrm{SU}(2)\}$
- $\mathscr{G} = \mathrm{SU}(2)$ $\leftarrow$ depolarizing channel

Channel Twirl

$$\mathscr{C} = \sum_i K_i \otimes \overline{K}_i \mapsto \Pi_{\mathscr{G}}(\mathscr{C})$$

- standard twirl : $\mathscr{G} = \{U \otimes \overline{U}, U \in \mathscr{G}_0\}$
- $\mathscr{G}_0 = \mathscr{P}$ $\leftarrow$ Pauli twirl, $\Pi(\mathscr{C})$ is diagonal in Paulis

Group Projector

$$\Pi_{\mathscr{G}}(\,\cdot\,) = \sum_{g \in \mathscr{G}} g \cdot g^{-1} \in \mathscr{C}\left(\mathscr{G}\right)$$

Hamiltonian Twirl = Dynamical Decoupling

$$H \mapsto \Pi_{\mathscr{G}}(H) = \langle H \rangle_T$$

- $\mathscr{G} = \langle X \rangle \leftarrow$ Spin Echo

POVM Twirl = Shadow Tomography

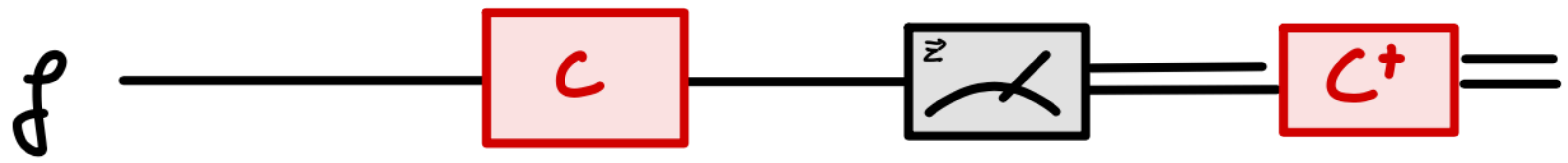
$$\mathscr{M}_0 = \mapsto \Pi_{\mathscr{G}}\left(\mathscr{M}_0\right)$$

- Standard setup [Huang et al (2022)]:

$$\mathscr{M}_0 = \sum_{\vec{z}} |\vec{z}\rangle\!\rangle\langle\!\langle\vec{z}|$$

- $\mathscr{G} = \{C \otimes \overline{C}, C \in \mathrm{Cl}_n\}$

Shadow Tomography



$$|S\rangle\rangle = \underbrace{\mathcal{U}^\dagger |\vec{z}\rangle\rangle\langle\langle\vec{z}| \mathcal{U}}_{\mathcal{M}} |\rho\rangle\rangle \qquad \mathscr{U} = U \otimes \overline{U},\ U \in \mathrm{Cl}_n$$

- $\mathscr{M} = |I\rangle\rangle\langle\langle I| + f(I \otimes \bar{I} - |I\rangle\rangle\langle\langle I|)$ is the depolarizing channel, invertible as matrix.
- Shadow is set of states $\mathscr{U}^\dagger |\vec{z}\rangle\rangle$. Clifford + Z-measurements → Stabilizer States
- Channel Twirl **projects** onto depol. channel. Robust to imperfections in POVM [Chen et al (2021)].

Displacement twirling

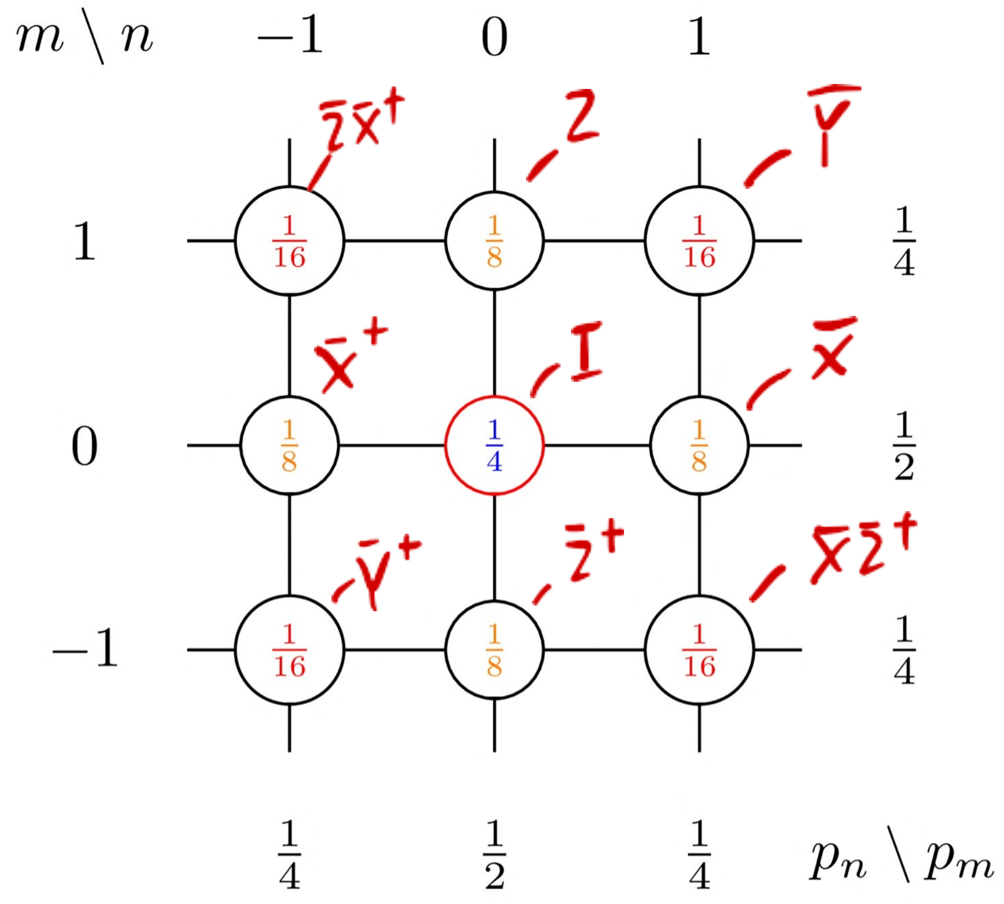
$$N(\,\cdot\,) = \int d^2\alpha d^2\beta\, c(\alpha,\beta)\, D(\alpha)\cdot D^\dagger(\beta)$$

[JC (2020)]:

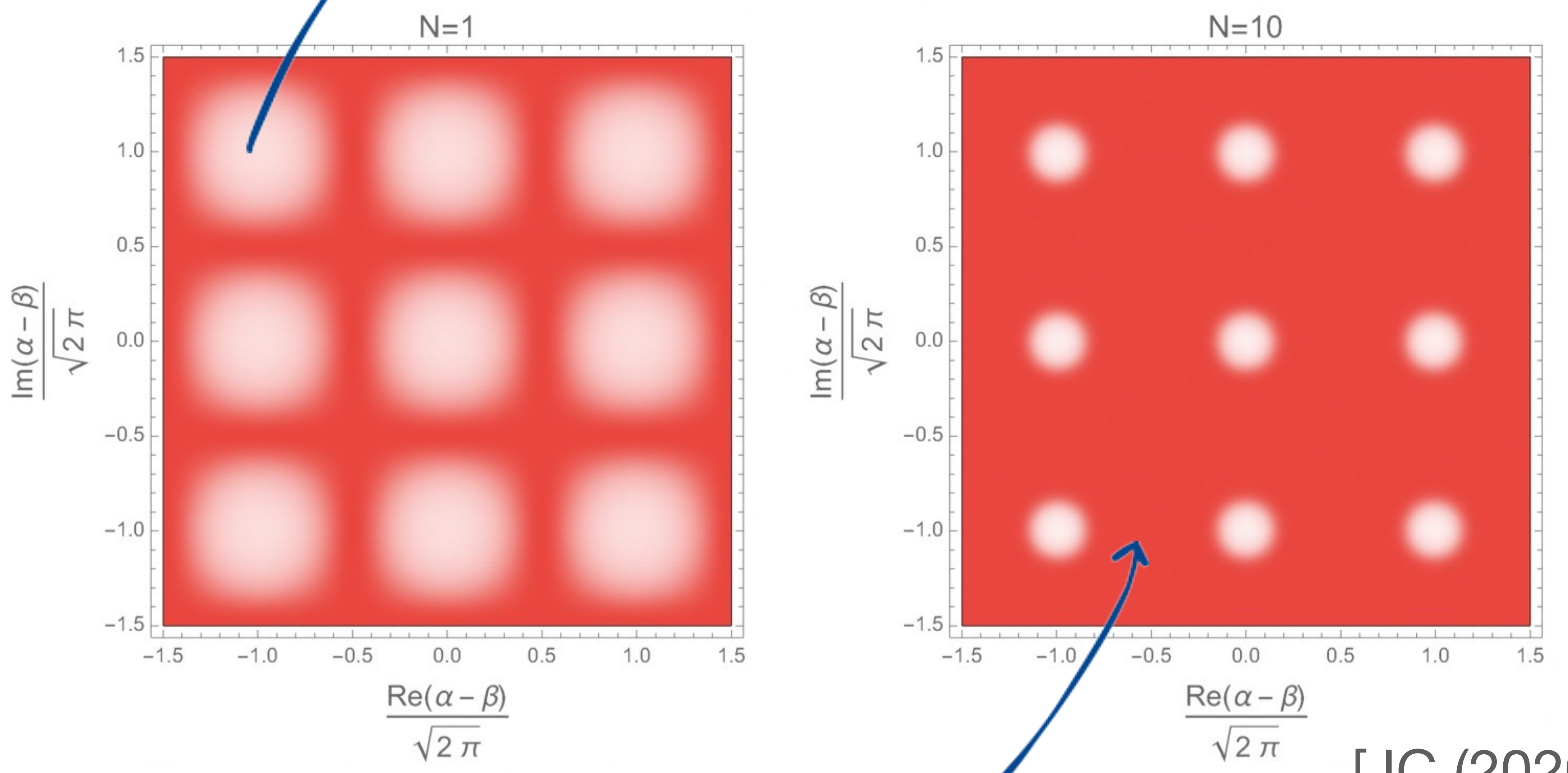
- Approximate displacement twirl by random walk over $\mathscr{L}^\perp$.
- Suppresses non-stabilizer coherences.

$$N(\,\cdot\,) \mapsto \int d\mu(\gamma) D^\dagger(\gamma)\, N\left(D(\gamma)\cdot D^\dagger(\gamma)\right)\, D(\gamma)$$

$$c(\alpha,\beta) \mapsto c(\alpha,\beta)\left[\int d\mu(\gamma)\, e^{i2\pi\gamma^T J(\alpha-\beta)}\right]^N$$



$\alpha - \beta = (k + i l)\sqrt{2\pi}$: $f(\alpha,\beta) = 1$



[JC (2020)]

Gaussian twirling

$$N(\,\cdot\,) = \int d^2\alpha d^2\beta\, c(\alpha,\beta)\, D(\alpha)\cdot D^\dagger(\beta)$$

- Approximate Gaussian Unitary twirling as random walk over $\mathrm{Aut}^S\left(\mathscr{L}^\perp\right)$.
- If $\mu_{SL_2(\mathbb{Z})}(S) = \mu^{\mathrm{Haar}}_{SL_2(\mathbb{Z}_2)}(S)$ this realizes logical Clifford twirl.

$$N(\,\cdot\,) \mapsto \int d\mu_{SL_2(\mathbb{Z})}(S) U_S^\dagger N\left(U_S \cdot U_S^\dagger\right) U_S$$

$$c(\alpha,\beta) \mapsto \int d\mu_{SL_2(\mathbb{Z})}(S)\; c(S\alpha, S\beta)$$

Representation of POVMs:

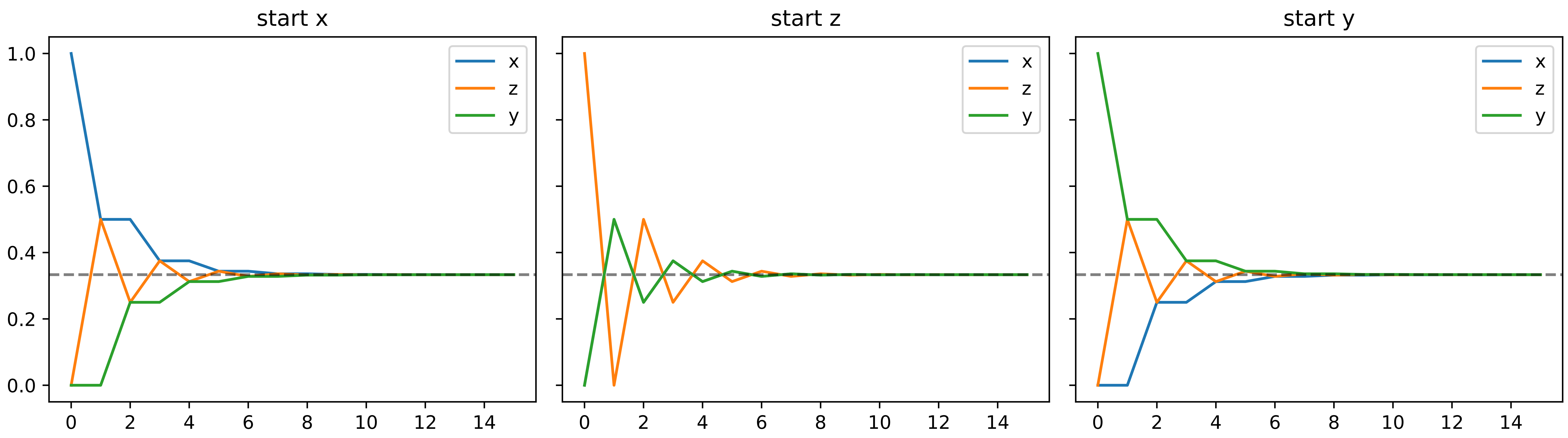
- Homodyne: $c(\alpha,\beta) \propto \delta(\Re(\alpha))\delta(\Re(\beta))\delta(\Im(\alpha-\beta))e^{i\Re(\alpha-\beta)\Im(\alpha)}$
- Heterodyne: $c(\alpha,\beta) \propto e^{-|\alpha|^2}\delta(\alpha-\beta)$
- Photon Counting: $c(\alpha,\beta) \propto e^{-|\alpha|^2/2-|\beta|^2/2}\sum_n L_n(|\alpha|^2)L_n(|\beta|^2)$

[JC (2020)]

Logical GKP Clifford twirl from Random Walks

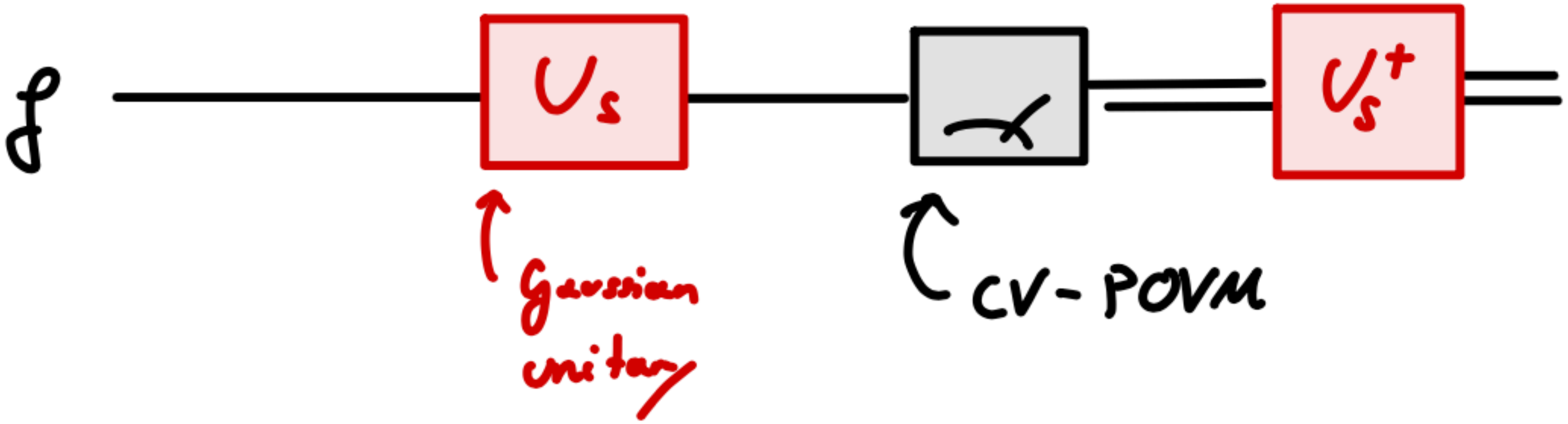
Compute effective marginal distribution of Logical Paulis in random walk:

Independent of P, random walk yields $\mathrm{prob}(P \mapsto P' = X, Y, Z) \to \dfrac{1}{3}$



Together with displacement twirling, this yields full twirl over $\mathrm{Aut}(\mathscr{L}^{\perp})$ and projects any POVM onto logical depolarizing channel with $p = 2/3$.

Logical GKP Clifford twirl from Random Walks



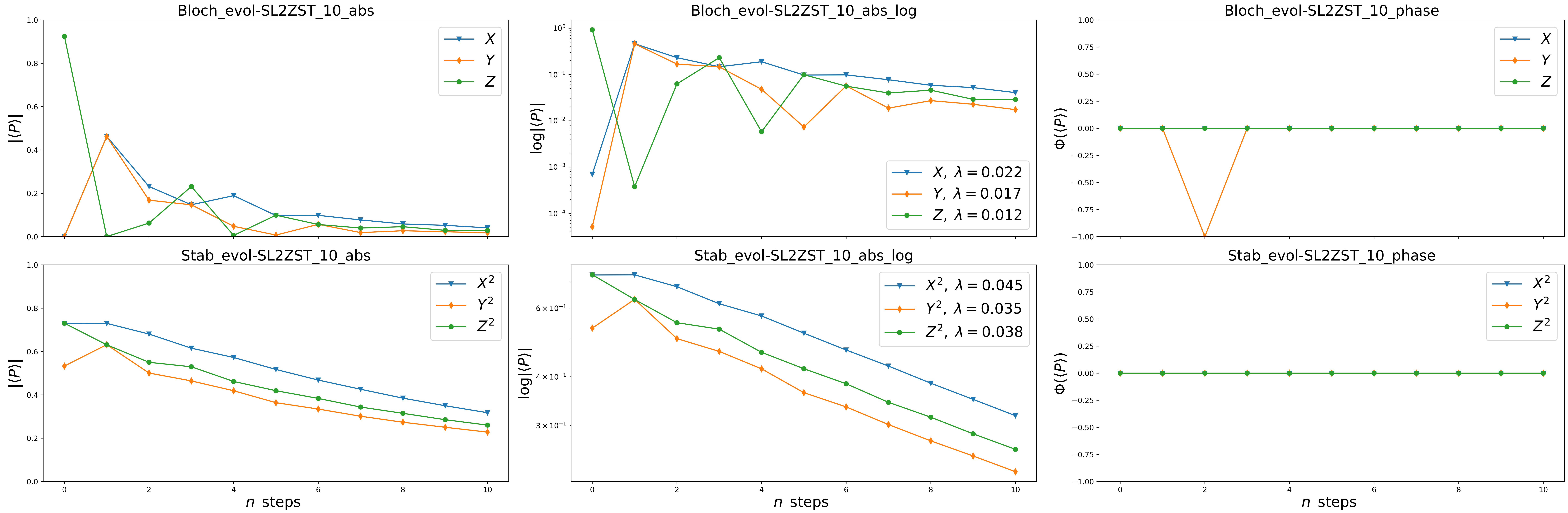
- Projects arbitrary CV-POVM onto logical depolarizing channel, can be clasically inverted.

- Heterodyne: Shadow = generalized Gaussian decomposition of input states that reproduces logical expectation values

Appendix

Finite Squeezing?

- For Simplicity, look at Clifford twirling of **States**



Finite Squeezing errors ?

- For Simplicity, look at Clifford twirling of **States**

TODO

- Use Logically decoded metric, sufficient to combat finite squeezing dynamics?

$$P_0 = \langle \mathrm{rect}_{1,\sqrt{\pi},2\sqrt{\pi}}\left(\hat{x}\right)\rangle,$$
$$P_1 = \langle \mathrm{rect}_{1,\sqrt{\pi},2\sqrt{\pi}}\left(\hat{x} - \sqrt{\pi}\right)\rangle,$$

$$\mathrm{rect}_{A,\tau,T}\left(x\right) = \sum_n c_n\left(\tau\right) e^{i\frac{2\pi}{T}x}$$

$$P_0 = \sum_n c_n \langle e^{i\sqrt{\pi}n\hat{x}}\rangle,$$
$$P_1 = \sum_n c_n (-1)^n \langle e^{i\sqrt{\pi}n\hat{x}}\rangle,$$

Can obtain logically-decoded statistics from Displacement- expectations:

Improve phase estimation results in controlled-Displacement implementations