Doubled Color Codes

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- Big picture
- Notation
- Transversal Logical Gates of the Color Code
- Logical Clifford+T circuits with the 15-qubit code
- Conclusion

15:45

- Discussion
- Feedback

Big picture - (Handwavy)

Eastin & Knill: TLGs are not universal for any error detecting code

Bravyi & Koenig, Pastawski & Yoshida: 2D stabilizer codes have only Clifford TLGs

You can get around this using the gauge-fixing method from:

Universal Fault-Tolerant Quantum Computation with Only Transversal Gates and Error Correction

Adam Paetznick[1] and Ben W. Reichardt[2]

Big picture - (Handwavy)

TLG:

$$V^{\otimes n} = \begin{array}{c} \begin{array}{cc} \text{codespace} & \end{array} \\ \left[\begin{array}{c|c} U & 0 \\ \hline 0 & * \end{array}\right] \end{array}$$

Gauge fixing method

$$V^{\otimes n} = \begin{array}{c} \begin{array}{cc} \text{codespace} & \end{array} \\ \left[\begin{array}{c|c} U & * \\ \hline * & * \end{array}\right] \end{array} \xrightarrow{\text{error correction}} \begin{array}{c} \begin{array}{cc} \text{codespace} & \end{array} \\ \left[\begin{array}{c|c} U & 0 \\ \hline 0 & * \end{array}\right] \end{array}$$

Big picture - (Handwavy)

This paper: transversal implementation of the Clifford + T gate set by the gauge fixing method in a 2D architecture

Notation

- Given a linear subspace $S \subseteq \mathbb{F}_2^n$ let $S^\perp$ be the orthogonal subspace

$$S^\perp = \{x \in \mathbb{F}_2^n : x^T y = 0 \text{ for all } y \in S\}$$

- define $\dot{S} = S^\perp \cap \text{Even}$
- define $S = \text{CSS}(A, B) = \langle X(A), Z(B) \rangle$

$$A, B \subseteq \mathbb{F}_2^n, \quad A \subseteq B^\perp, \quad A \subseteq \text{Even}, \quad B \subseteq \text{Even}$$

$$P(f) = \prod_{j \in \text{supp}(f)} P_j$$

pauli

Here $[n, k, d]$

odd 1

Notation

- Examples of logical operators

$$X_L = X(\bar{1}),\quad Y_L = Y(\bar{1}),\quad Z_L = Z(\bar{1})$$

- A logical state encoding a single-qubit state $\eta = I + c_1 X + c_2 Y + c_3 Z$ is defined as

$$\rho_L(\eta) = \delta\left(I + c_1 X_L + c_2 Y_L + c_3 Z_L\right)\frac{1}{|S|}\sum_{G\in S} G$$

Notation

- A subspace $\mathcal{A} \leq \mathbb{F}_2^n$ is doubly-even iff there exist disjoint subsets $M^{\pm} \subseteq [n]$ such that

$$|f \cap M^+| - |f \cap M^-| = 0 \pmod 4) \text{ for all } f \in \mathcal{A}$$

$[1, \ldots, n], \emptyset$ M^+ or M^- $\emptyset$

$M^+ + M^- = [n]$

Transversal Logical Gates

$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

Lemma 1: $\mathcal{H} = CSS(\mathcal{A}, \mathcal{B})$ has a tranversal gate S^m, with $m = |M^+| - |M^-| = 1 \bmod 2$

$$S_{all}\, \rho_L(\eta)\, S_{all} = \rho_L(S^m \eta S^{-m})$$

where $S_{all} = \prod_{j \in M^+} S_j \cdot \prod_{j \in M^-} S_j^{-1}$

whenever ① $\mathcal{A} \subseteq \mathcal{B}$

② $\mathcal{A}$ is doubly even wrt $M^{\pm}$

Transversal Logical Gates

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$S_{all} = \prod_{j \in M^+} S_i \cdot \prod_{j \in M^-} S_j^{-1}$$

① $\mathcal{A} \subseteq \mathcal{B}$ ② $\mathcal{A}$ is doubly even wrt $M^{\pm}$

First: $S_{all} \mathcal{H} S_{all}^{-1} = \mathcal{H}$ (S commutes with Z-stabilizers)

$$S_{all} X(f) S_{all}^{-1} = \prod_{j \in M^+} S_j \prod_{j \in M^-} S_j^{-1} X(f) \prod_{j \in M^+} S_j^{-1} \prod_{j \in M^-} S_j$$

Now what?

$f \in \mathcal{A}$

$X(f) = \prod_{i \in supp(f)}$

Transversal Logical Gates

$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

$S_{all} = \prod_{j \in M^+} S_i \cdot \prod_{j \in M^-} S_j^{-1}$

① $\mathcal{A} \subseteq \mathcal{B}$

$\mathcal{A} = \mathcal{B}$

② $\mathcal{A}$ is doubly even wrt $M^\pm$

$\mathcal{A} \subset \mathcal{B}$

First: $S_{all} \mathcal{H} S_{all}^{-1} = \mathcal{H}$ (S commutes with Z-stabilizers)

$$S_{all} X(f) S_{all}^{-1} = \prod_{j \in M^+} S_j \prod_{j \in M^-} S_j^{-1} X(f) \prod_{j \in M^+} S_j^{-1} \prod_{j \in M^-} S_j$$

$[n] = M^+ \cup M^-$

$$= \prod_{j \in M^+ \cap f} S_j X_j S_j^{-1} \prod_{l \in M^- \cap f} S_l^{-1} X_l S_l \qquad SXS = iXZ$$

$M^+ = [n]$

$$= \prod_{j \in M^+ \cap f} i X_j Z_j \prod_{l \in M^- \cap f} i^{(-1)} X Z_l$$

$M^- = \emptyset$

$$= i^{|f \cap M^+| - |f \cap M^-|} X(f) Z(f) = X(f) Z(f) \in \mathcal{H} \quad \ddot\smile$$

Transversal Logical Gates

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$S_{all} = \prod_{j \in M^+} S_j \cdot \prod_{j \in M^-} S_j^{-1}$$

① $\mathcal{A} \subseteq \mathcal{B}$ ② $\mathcal{A}$ is doubly even wrt $M^\pm$

where have I used this?

First: $S_{all} \mathcal{H} S_{all}^{-1} = \mathcal{H}$ (S commutes with Z-stabilizers)

$$S_{all} X(f) S_{all}^{-1} = \prod_{j \in M^+} S_j \prod_{j \in M^-} S_j^{-1} X(f) \prod_{j \in M^+} S_j^{-1} \prod_{j \in M^-} S_j$$

$$= \prod_{j \in M^+ \cap f} S_j X_j S_j^{-1} \prod_{l \in M^- \cap f} S_l^{-1} X_l S_l$$

$$= \prod_{j \in M^+ \cap f} i X_j Z_j \prod_{l \in M^- \cap f} i^{(-1)} X_l Z_l$$

$$= i^{|f \cap M^+| - |f \cap M^-|} X(f) Z(f) = X(f) Z(f)$$

Transversal Logical Gates

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$S_{all} = \prod_{j \in M^+} S_j \cdot \prod_{j \in M^-} S_j^{-1}$$

$$S_{all} \underbrace{\rho_L}(\eta) S_{all} = \rho_L(S^m \eta S^{-m})$$

$$\rho_L(\eta) = \delta(I + c_1 X_L + c_2 Y_L + c_3 Z_L) \frac{1}{|S|} \sum_{G \in S} G = \Pi$$

To show that it implements S_L prove:

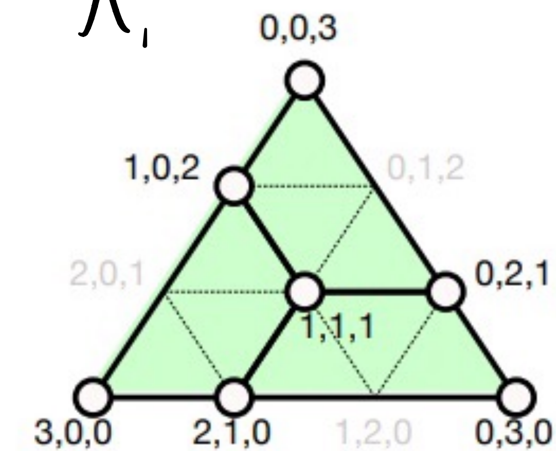
$$S^m X S^m = i^m X Z \leftarrow S_{all} X_L \Pi S_{all}^{-1} = S_{all} X_L S_{all}^{-1} \Pi = i^m X_L Z_L \Pi$$

$$S^m Z S^m Z S^{-m} = Z \leftarrow S_{all} Z_L \Pi S_{all}^{-1} = Z_L \Pi$$

TLG's of the colour code

- Consider a lattice Δ_t with lattice sites $\vec{j} = (j_1, j_2, j_3)$ satisfying $j_1 + j_2 + j_3 = 3t$, $j_i \geq 0$

Λ_1



- Consider sublattices

$$\Delta_t^b = \{ \vec{j} \in \Delta_t : j_2 - j_1 = b \pmod 3 \}$$

- Define the color code lattice $\Lambda_t = \Delta_t^0 \cup \Delta_t^2$

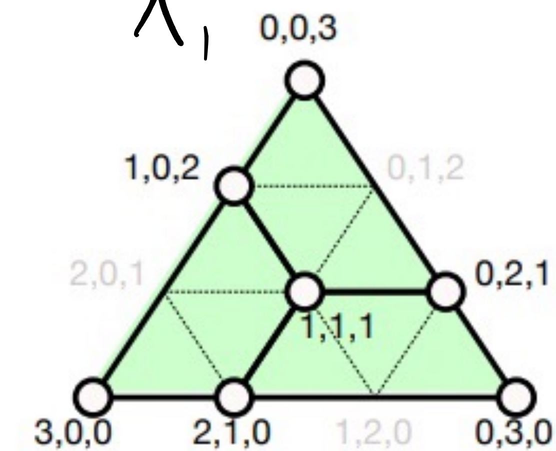
Δ_t^1

TLG's of the colour code

Remember for transversal S we need:

① $\mathcal{A} \subseteq B$ ② $\mathcal{A}$ is doubly even wrt $M^{\pm}$

Λ_1



For color codes $\mathcal{A} = B$.

One can show $\mathcal{A}$ is doubly even wrt Δ_t^0 and Δ_t^2. (just believe me)

$M^+ = \Delta_t^0$, $M^- = \Delta_t^2$

$$S_L^m = \prod_{j \in \Delta_t^0} S_j \cdot \prod_{j \in \Delta_t^2} S_j^{-1}$$ with $m = |M^+| - |M^-| = 1$

$S_{all} =$

$M^+ = \Delta_t^0 \cup \Delta_t^2$ $M^- = \emptyset$

TLG's of the colour code

- A subspace $A \subseteq \mathbb{F}_2^n$ is <u>triply-even</u> iff there exist disjoint subsets $M^\pm \subseteq [n]$ such that $|f \cap M^+| - |f \cap M^-| = 0 \mod 8$ for all $f \in A$

- A code $CSS(A, B)$ has a transversal gate T^m,

$$T_{all}\, \rho_L(\eta)\, T_{all}^{-1} = \rho_L(T^m \eta T^{-m}), \quad T_{all} = \prod_{j \in M^+} T_j \cdot \prod_{j \in M^-} T^{-1}$$

whenever $B = \mathring{A}$ and A is triply even wrt $M^\pm$.

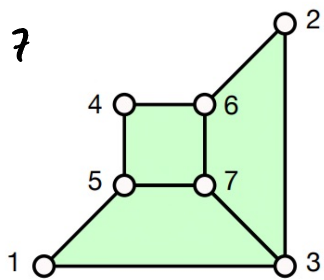
$\begin{bmatrix} 1 & \\ & e^{i\pi/4} \end{bmatrix}$ $\qquad$ $(A^\perp \cap \text{even})$

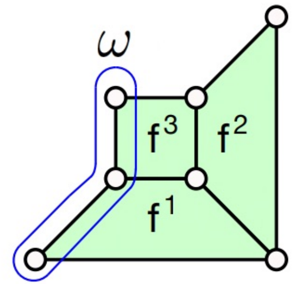
15 qubit code

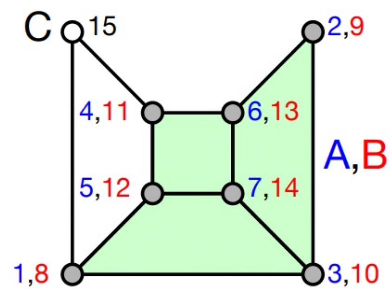
$$|f \cap M^+| - |f \cap M^-| = 0 \mod 8$$

can't be satisfied for the color code"

$$H_L = H^{\otimes 7}$$
$$S_L = S^{\otimes 7}$$







T-code: CSS $T \subseteq \mathbb{F}_2^{15}$

basis vectors of T:

1		1		1		1		1		1		1		1
	1	1			1	1			1	1			1	1
			1	1	1	1					1	1	1	1
							1	1	1	1	1	1	1	1

$|f| = 0 \mod 8$

$M^+ = [n]$

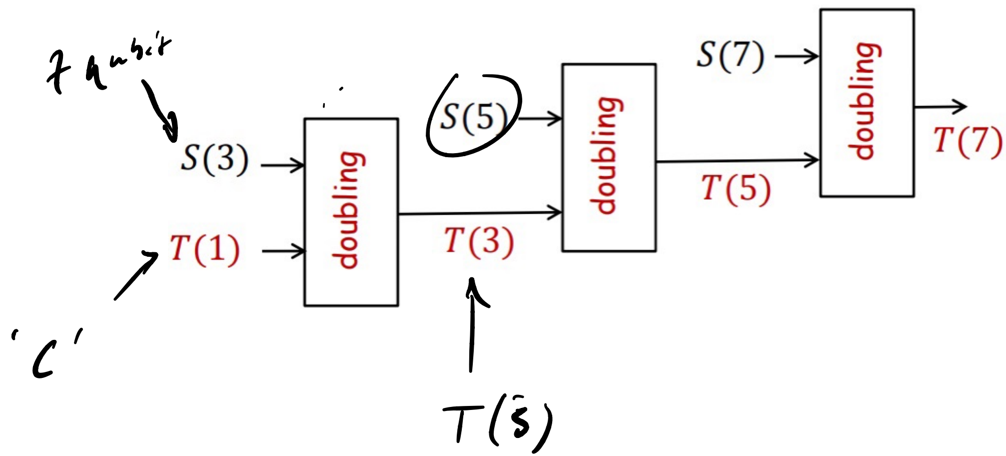
$M^- = \emptyset$

	Transversal gates	X-stabilizers	Z-stabilizers	stabilizer group	gauge group
C-code	Clifford group	single faces	single faces	$\mathrm{CSS}(\mathcal{C}, \mathcal{C})$	$\mathrm{CSS}(\mathcal{C}, \mathcal{C})$
T-code	T gate	double faces BC	single faces double edges	$\mathrm{CSS}(\mathcal{T}, \dot{\mathcal{T}})$	$\mathrm{CSS}(\mathcal{T}, \dot{\mathcal{T}})$
Base code		double faces BC	single faces	$\mathrm{CSS}(\mathcal{T}, \mathcal{C})$	$\mathrm{CSS}(\mathcal{C}, \dot{\mathcal{T}})$

Table 1: The family of codes used in the protocol.

$$\mathcal{C} \subseteq \dot{\mathcal{T}} \text{ and } \dot{\mathcal{T}} = \mathcal{C} + G$$

$\boxed{\text{Both } \mathcal{C} \text{ and } \mathcal{T} \text{ can be obtained by gauge fixing from the base code.}}$


7 qubit
'C'
S(3)
T(1)
doubling
S(5)
T(3)
doubling
S(7)
T(5)
doubling
T(7)
T(5)

Conclusion

- Infinite family with diverging code distance
- Transversal Clifford + T gates by gauge fixing
- Related work is given by Bravyi in his 2016 QIP talk. (Available on youtube.)