

Qudit ZX-Diagram Simplification (Draft)

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Abstract

1 Introduction

Diagrams are read bottom to top. ‘It is not hard to believe that qubits and qutrits behave differently from qupits.’ [Cui and Wang, 2015]. Can’t decide on notation for Hadamard stuff. Numbered yellow boxes are good for graph-like diagrams, but red/green split boxes are great for colour change rules:

$$\begin{array}{c} \text{Red/Green Split Boxes} \\ \text{Red Circle } \tilde{\alpha} \\ \text{Green Boxes} \end{array} = \begin{array}{c} \text{Crossing Lines} \\ \text{Green Circle } \tilde{\alpha} \end{array} \quad \begin{array}{c} \text{Red/Green Split Boxes} \\ \text{Red Circle } \tilde{\alpha} \\ \text{Green Boxes} \end{array} = \begin{array}{c} \text{Crossing Lines} \\ \text{Green Circle } \tilde{\alpha} \end{array} \quad (1)$$

And then the dualiser has a natural alternative notation - important because using a yellow box with a D on it now looks more like a Hadamard with decoration D :

$$\begin{array}{c} \text{Yellow Box } D \\ \text{Yellow Box } D \\ \text{Yellow Box } D \\ \text{Yellow Box } D \end{array} = \begin{array}{c} \text{Yellow Box } D \\ \text{Yellow Box } D \\ \text{Yellow Box } D \\ \text{Yellow Box } D \end{array} = \begin{array}{c} \text{Yellow Box } D \\ \text{Yellow Box } D \\ \text{Yellow Box } D \\ \text{Yellow Box } D \end{array} = \begin{array}{c} \text{Yellow Box } D \\ \text{Yellow Box } D \\ \text{Yellow Box } D \\ \text{Yellow Box } D \end{array} \quad (2)$$

But the split boxes have the drawback that often you can no longer say ‘the equation holds with the roles of green and red interchanged’ - e.g. the Euler decomposition of the Hadamard. Not a problem for dualiser though.

Qudit rules are shown in Figure 1. We’re mostly interested in stabilizer stuff so will give phase components as multiples of $\frac{2\pi}{d}$.

- $\vec{\kappa} = (1, 2, \dots, d-1)$
- $\vec{\tau}$ has components $\tau_j = \frac{j}{2}(j+d)$
- $\vec{\alpha}^n$ has components $\alpha_j^n = \alpha_{j-n} - \alpha_{-n}$

- Indices are mod d , and $\vec{\alpha}_0 = 0$ for any phase a .
- $\vec{\alpha} = (\vec{\alpha}_{d-1}, \vec{\alpha}_{d-2}, \dots, \vec{\alpha}_1)$

2 Stabilizer Phases

We want to know what spider phases we might encounter in the world of stabilizer qudit ZX-calculus. For odd d , components \vec{a}_j of stabilizer phases \vec{a} will be integer multiples of $\frac{2\pi}{d}$. For even d , stabilizer components \vec{a}_j can be half-integer multiples of $\frac{2\pi}{d}$ (i.e. integer multiples of $\frac{\pi}{d}$). The set of stabilizer phases is a subset of the set of all possible $(d-1)$ -tuples of such integers/half-integers. In general, this subset is strict, though we note that for qubits and qutrits it is not.

The shift gate S is uniquely defined by its action on the Pauli gates X and Z . Specifically, it satisfies:

$$SXS^\dagger = XZ^{-1} \quad , \quad SZS^\dagger = Z \quad (3)$$

In ZX-parlance, these equations are:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ S \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{red circle } \vec{\kappa} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{green circle } -\vec{\kappa} \\ \diagdown \\ \text{---} \end{array} \quad , \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ S \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{green circle } \vec{\kappa} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ S \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{green circle } \vec{\kappa} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (4)$$

These are satisfied by the following Z -gate whose phase \vec{s} has components \vec{s}_j :

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ S \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{green circle } \vec{s} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad , \quad \vec{s}_j = \frac{j}{2}(j+2-d) \quad (5)$$

Note that for even d , the components can be half-integers - for example, $\vec{s}_{d-1} = \frac{d-1}{2}$. For all d , the qudit Clifford group on n qubits is generated by the Hadamard gate H , the shift gate S and the generalised $CNOT$ gate [Farinholt, 2014]:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{yellow square } 1 \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad , \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{green circle } \vec{s} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad , \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{green circle } \vec{s} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{red circle } \vec{\kappa} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (6)$$

The Clifford group on one qudit (the *local Clifford group*) is generated by H and S [Farinholt, 2014]. We seek a normal form for such gates, à la [Wang, 2018, Theorem 6.2.12]. A method for constructing any such gate is from H and S is given in Farinholt [2014]! In

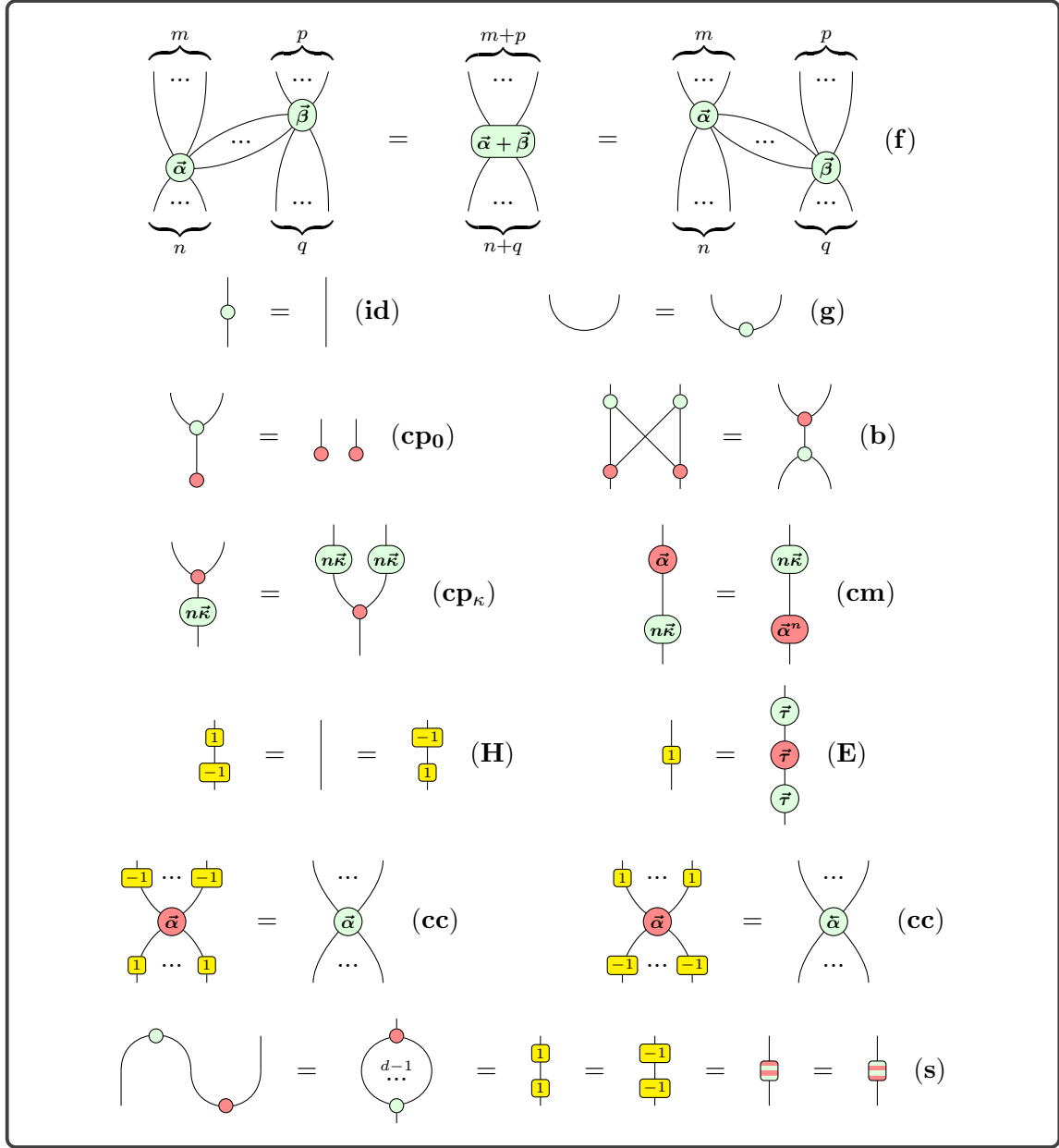


Figure 1: Qudit ZX-calculus rules

particular, for prime d at least, this has a nice form. A local Clifford operator C is uniquely defined by its action on X and Z under unitary conjugation. If it sends X to $X^p Z^r$ and Z to $X^q Z^s$, we represent C by the matrix:

$$C = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \quad (7)$$

where entries are in \mathbb{Z}_d . C is a local Clifford operator if and only if it is symplectic, i.e. has determinant 1. Let $x_y = x^{-1}(1+y)$. If q is invertible, C has decomposition:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = S^{-q_s} H^{-1} S^{-q} H^{-1} S^{-q_p} \quad (8)$$

To make life easier for ourselves, we'll instead suppose r invertible, and consider:

$$\begin{aligned} H^{-1} C^{-1} H^{-1} &= \begin{pmatrix} -p & -r \\ -q & -s \end{pmatrix} \\ &= S^{-(-r)-s} H^{-1} S^{-(-r)} H^{-1} S^{-(-r)-p} \\ &= S^{r-s} H^{-1} S^r H^{-1} S^{r-p} \\ \therefore C^{-1} &= H S^{r-s} H^{-1} S^r H^{-1} S^{r-p} H \\ \therefore C &= H^{-1} S^{-r-p} H S^{-r} H S^{-r-s} H^{-1} \end{aligned}$$

As a ZX-diagram, this is:

What if r not invertible? Since d prime and $r \in \mathbb{Z}_d$, this only happens if $r = 0$. If so, since C has determinant 1, this forces $p \neq 0$, i.e. p invertible. So then consider:

$$\begin{aligned}
H^{-1}C^{-1} &= \begin{pmatrix} r & -p \\ s & -q \end{pmatrix} \\
&= S^{-(p)-q} H^{-1} S^{(-p)} H^{-1} S^{(-p)r} \\
&= S^{p-q} H^{-1} S^p H^{-1} S^{pr} \\
\therefore C^{-1} &= H S^{p-q} H^{-1} S^p H^{-1} S^{pr} \\
\therefore C &= S^{-pr} H S^{-p} H S^{-p-q} H^{-1}
\end{aligned}$$

Note since $r = 0$ we have $p_r = p^{-1}(1 + r) = p^{-1}$. As a ZX-diagram, this is:

$$(10)$$

So we have a normal form. Negating all of p, q, r and s changes nothing, so put more palatably, our theorem is:

Theorem 2.1. For d prime, every local Clifford operator is of exactly one of the following two forms, and conversely every such diagram is a local Clifford operator:

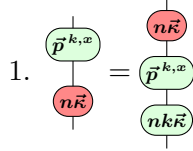
$$(11)$$

for $a, b, c \in \mathbb{Z}_d$ with $a > 0$.

I've been playing around a bit with gates with the phase $\vec{p}^{k,x}$, whose components are:

$$\vec{p}_j^{k,x} = -j \left(\frac{j+1}{2} k + x \right) \quad (12)$$

These are analogous to those in Wang [2018] Lemma 6.2.8 and have the following properties:

1. 

2. They form an Abelian subgroup:

- (a) $\vec{p}^{k,x} + \vec{p}^{l,y} = \vec{p}^{k+l,x+y}$
- (b) $-\vec{p}^{k,x} = \vec{p}^{-k,-x}$
- (c) identity is $\vec{p}^{0,0} = (0, \dots, 0)$

3. They include a lot of our favourite phases:

- (a) $\vec{\kappa} = \vec{p}^{0,-1}$
- (b) $\vec{\tau} = \vec{p}^{-1, \frac{1-d}{2}}$
- (c) $\vec{s} = \vec{p}^{-1, \frac{d-1}{2}}$
- (d) $\vec{\tilde{s}} = \vec{p}^{-1, \frac{d+3}{2}}$

4. The last component $\vec{p}_{d-1}^{k,x}$ is exactly x .

3 Local Complementation and Pivot

Local complementation carries over to qudit case without much fuss. Some lemmas we'll need:

Lemma 3.1.

$$\begin{array}{c} \text{Diagram 1: A line with a green circle } \vec{\alpha} \text{ and a red dot below it, connected by a curved line.} \\ \text{Diagram 2: A line with a green circle } \vec{\alpha} \text{ and a red dot below it, connected by a curved line.} \end{array} = \begin{array}{c} \text{Diagram 3: A line with a green circle } \vec{\alpha} \text{ and a red dot below it, connected by a curved line.} \\ \text{Diagram 4: A line with a green circle } \vec{\alpha} \text{ and a red dot below it, connected by a curved line.} \end{array}, \quad \begin{array}{c} \text{Diagram 5: A red circle labeled } \pm\vec{\tau} \\ \text{Diagram 6: A green circle labeled } \mp\vec{\tau} \end{array}, \quad \begin{array}{c} \text{Diagram 7: A red circle labeled } \vec{\tau} \text{ with a yellow box labeled } D \text{ above it.} \\ \text{Diagram 8: A green circle labeled } -\vec{\tau} \text{ with a yellow box labeled } -1 \text{ above it.} \\ \text{Diagram 9: A green circle labeled } -\vec{\tau} \end{array} \quad (13)$$

Proof. For the first equation:

Diagrammatic equation (14) showing transformations of a loop with a green circle labeled $\bar{\alpha}$ and a red dot. The sequence of diagrams is as follows:

- A vertical line with a green circle labeled $\bar{\alpha}$ and a red dot on the left side, connected by a curved line.
- An equals sign.
- A vertical line with a green circle labeled $\bar{\alpha}$ and a red dot on the right side, connected by a curved line.
- An equals sign.
- A vertical line with a green circle labeled $\bar{\alpha}$ and a red dot on the left side, connected by a curved line, with a yellow box labeled D on the right side.
- An equals sign.
- A vertical line with a green circle labeled $\bar{\alpha}$ and a red dot on the right side, connected by a curved line, with a yellow box labeled D on the left side.

(14)

For the second:

Diagrammatic equation (15) showing transformations of a vertical line with a red circle labeled $\pm\bar{\tau}$. The sequence of diagrams is as follows:

- A vertical line with a red circle labeled $\pm\bar{\tau}$.
- An equals sign.
- A vertical line with a green circle labeled $\pm\bar{\tau}$ and a yellow box labeled ∓ 1 on top.
- An equals sign.
- A vertical line with a green circle labeled $\pm\bar{\tau}$ and a red circle labeled $\mp\bar{\tau}$ on top.
- An equals sign.
- A vertical line with a green circle labeled $\mp\bar{\tau}$ and a red circle labeled $\mp\bar{\tau}$ on top.
- An equals sign.
- A vertical line with a green circle labeled $\mp\bar{\tau}$.

(15)

And the third:

Diagrammatic equation (16) showing transformations of a vertical line with a red circle labeled $\bar{\tau}$ and a yellow box labeled D . The sequence of diagrams is as follows:

- A vertical line with a red circle labeled $\bar{\tau}$ and a yellow box labeled D on top.
- An equals sign.
- A vertical line with a red circle labeled $\bar{\tau}$ and a yellow box labeled -1 on top.
- An equals sign.
- A vertical line with a green circle labeled $\bar{\tau}$ and a yellow box labeled -1 on top.
- An equals sign.
- A vertical line with a green circle labeled $\bar{\tau}$ and a red circle labeled $-\bar{\tau}$ on top.
- An equals sign.
- A vertical line with a green circle labeled $-\bar{\tau}$ and a red circle labeled $-\bar{\tau}$ on top.
- An equals sign.
- A vertical line with a green circle labeled $-\bar{\tau}$ and a yellow box labeled -1 on top.
- An equals sign.
- A vertical line with a green circle labeled $-\bar{\tau}$ and a yellow box labeled -1 on top.

(16)

□

So, first we have a triangle lemma:

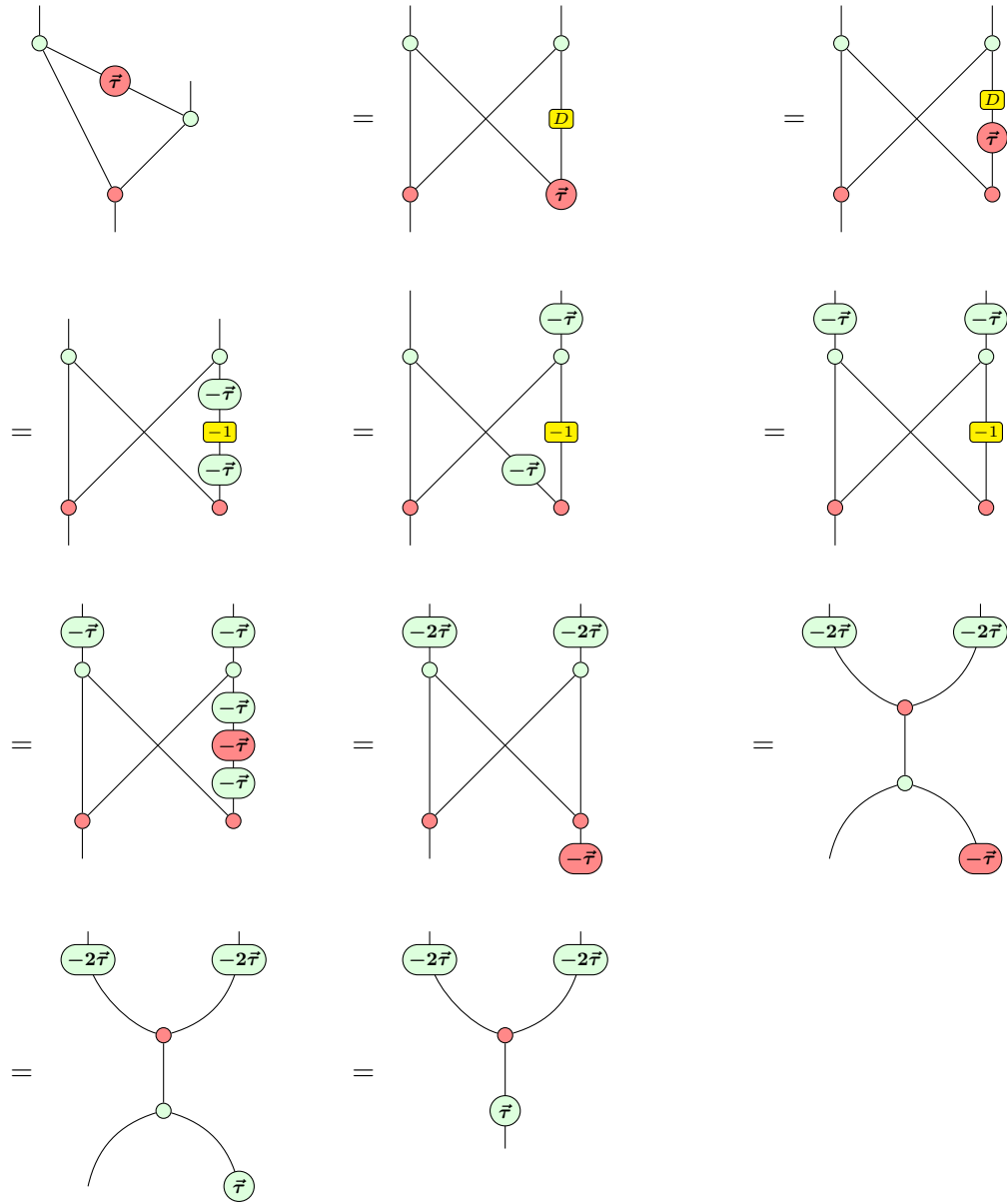
Lemma 3.2.

Diagrammatic equation (17) showing a triangle lemma transformation. The sequence of diagrams is as follows:

- A triangle with three vertices. The top-left vertex is a green circle, the top-right vertex is a green circle, and the bottom vertex is a red circle. The edges are labeled with $\bar{\tau}$.
- An equals sign.
- A vertical line with a green circle labeled $\bar{\tau}$ at the bottom, and two green circles labeled $-2\bar{\tau}$ at the top, connected by a curved line.

(17)

Proof.



□

Then a lemma that tells us how to remove one edge in the K_3 graph state:

Lemma 3.3.

$$(18)$$

Proof.

□

And finally the local complementation result for the complete graph K_3 :

Lemma 3.4.

$$(19)$$

Proof.

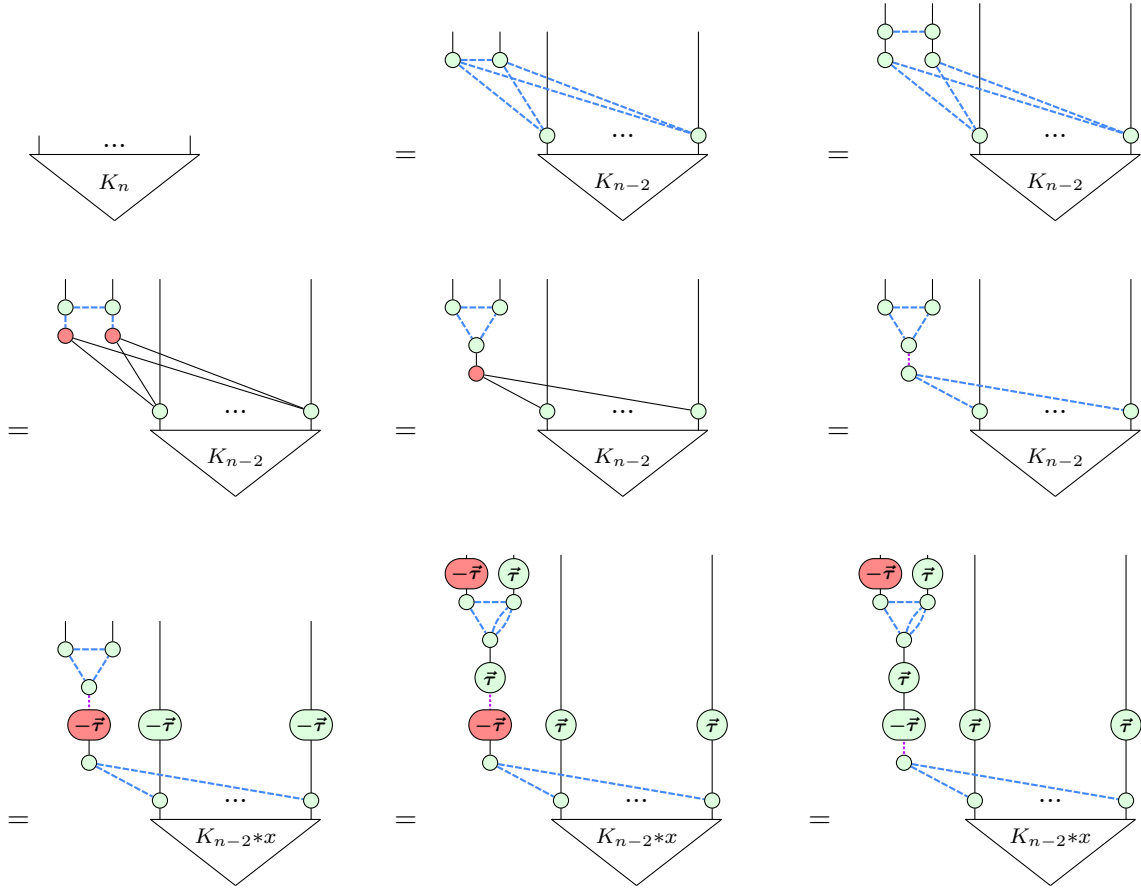
□

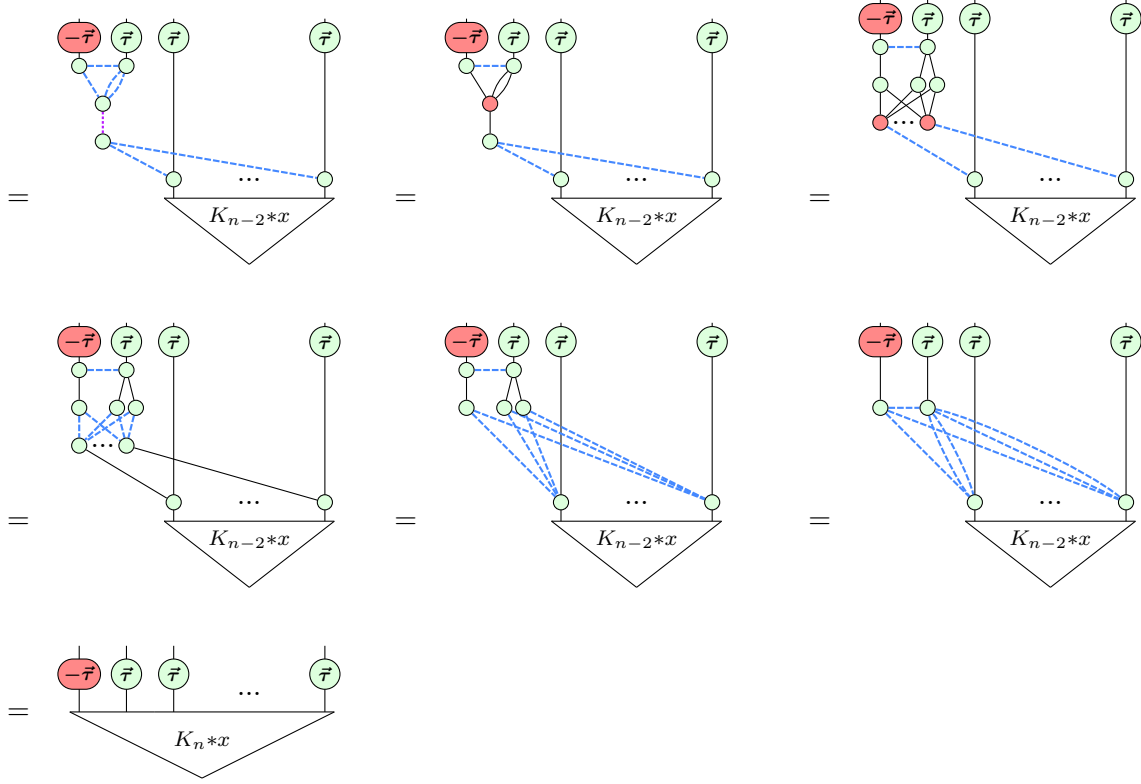
Other base cases K_0, K_1 and K_2 are easy enough to prove. For larger complete graphs K_n we proceed inductively:

Lemma 3.5.

$$\begin{array}{c} \dots \\ \triangle \\ K_n \end{array} = \begin{array}{c} \begin{array}{c} \textcolor{red}{-\tau} \quad \textcolor{green}{\tau} \quad \textcolor{green}{\tau} \quad \dots \quad \textcolor{green}{\tau} \end{array} \\ \triangle \\ K_{n*x} \end{array} \quad (20)$$

Proof.





□

For general (non-complete) graphs, can derive the result from the K_n case. Then need to derive pivot equation.

4 Elimination Theorems

Find as many eliminatable spiders as possible!

References

- S. X. Cui and Z. Wang. Universal quantum computation with metaplectic anyons. *Journal of Mathematical Physics*, 56(3):032202, Mar 2015. ISSN 1089-7658. doi: 10.1063/1.4914941. URL <http://dx.doi.org/10.1063/1.4914941>.
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Q. Wang. *Completeness of the ZX-calculus*. PhD thesis, University of Oxford, 2018.