

Qudit ZX-Diagram Simplification (Draft)

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Abstract

1 Introduction

Diagrams are read bottom to top. ‘It is not hard to believe that qubits and qutrits behave differently from qupits.’ [Cui and Wang, 2015]. Can’t decide on notation for Hadamard stuff. Numbered yellow boxes are good for graph-like diagrams, but red/green split boxes are great for colour change rules:

$$\begin{array}{c} \text{Red/Green Split Boxes} \\ \vdots \\ \text{Red Circle } \tilde{\alpha} \\ \vdots \\ \text{Red/Green Split Boxes} \end{array} = \begin{array}{c} \text{Green Boxes} \\ \vdots \\ \text{Green Circle } \tilde{\alpha} \\ \vdots \\ \text{Green Boxes} \end{array} \quad \begin{array}{c} \text{Red/Green Split Boxes} \\ \vdots \\ \text{Red Circle } \tilde{\alpha} \\ \vdots \\ \text{Red/Green Split Boxes} \end{array} = \begin{array}{c} \text{Green Boxes} \\ \vdots \\ \text{Green Circle } \tilde{\alpha} \\ \vdots \\ \text{Green Boxes} \end{array} \quad (1)$$

And then the dualiser has a natural alternative notation - important because using a yellow box with a D on it now looks more like a Hadamard with decoration D :

$$\begin{array}{c} \text{Yellow Box } D \\ \vdots \end{array} = \begin{array}{c} \text{Yellow Box } D \\ \vdots \end{array} = \begin{array}{c} \text{Yellow Box } D \\ \vdots \end{array} = \begin{array}{c} \text{Yellow Box } D \\ \vdots \end{array} \quad (2)$$

But the split boxes have the drawback that often you can no longer say ‘the equation holds with the roles of green and red interchanged’ - e.g. the Euler decomposition of the Hadamard. Not a problem for dualiser though.

Qudit rules are shown in Figure 1. We’re mostly interested in stabilizer stuff so will give phase components as multiples of $\frac{2\pi}{d}$.

- $\vec{\kappa} = (1, 2, \dots, d-1)$
- $\vec{\tau}$ has components $\tau_j = \frac{j}{2}(j+d)$
- $\vec{\alpha}^n$ has components $\alpha_j^n = \alpha_{j-n} - \alpha_{-n}$

- Indices are mod d , and $\vec{\alpha}_0 = 0$ for any phase a .
- $\vec{\alpha} = (\vec{\alpha}_d - 1, \vec{\alpha}_d - 2, \dots, \vec{\alpha}_1)$

2 Stabilizer Phases

We want to know what spider phases we might encounter in the world of stabilizer qudit ZX-calculus. For odd d , components \vec{a}_j of stabilizer phases \vec{a} will be integer multiples of $\frac{2\pi}{d}$. For even d , stabilizer components \vec{a}_j can be half-integer multiples of $\frac{2\pi}{d}$ (i.e. integer multiples of $\frac{\pi}{d}$). The set of stabilizer phases is a subset of the set of all possible $(d-1)$ -tuples of such integers/half-integers. In general, this subset is strict, though we note that for qubits and qutrits it is not.

The shift gate S is uniquely defined by its action on the Pauli gates X and Z . Specifically, it satisfies:

$$SXS^\dagger = XZ \quad , \quad SZS^\dagger = Z \quad (3)$$

In ZX-parlance, these equations are:

$$\begin{array}{c} \text{S} \\ \text{---} \\ \text{---} \\ \text{S} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \quad , \quad \begin{array}{c} \text{S} \\ \text{---} \\ \text{---} \\ \text{S} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \quad (4)$$

These are satisfied by the following Z -gate whose phase \vec{s} has components \vec{s}_j :

$$\boxed{S} = \textcircled{\vec{s}} \quad , \quad \vec{s}_j = \frac{j}{2}(j+2-d) \quad (5)$$

Note that for even d , the components can be half-integers - for example, $\vec{s}_{d-1} = \frac{d-1}{2}$. For all d , the qudit Clifford group on n qubits is generated by the Hadamard gate H , the shift gate S and the generalised $CNOT$ gate [Farinholt, 2014]:

$$\text{[yellow box with 1]}, \quad \text{[green circle with } \vec{s} \text{]}, \quad \text{[green circle connected to red circle]} \quad (6)$$

The Clifford group on one qudit (the *local Clifford group*) is generated by H and S [Farinholt, 2014]. We seek a normal form for such gates, à la [Wang, 2018] Theorem 6.2.12. **So**

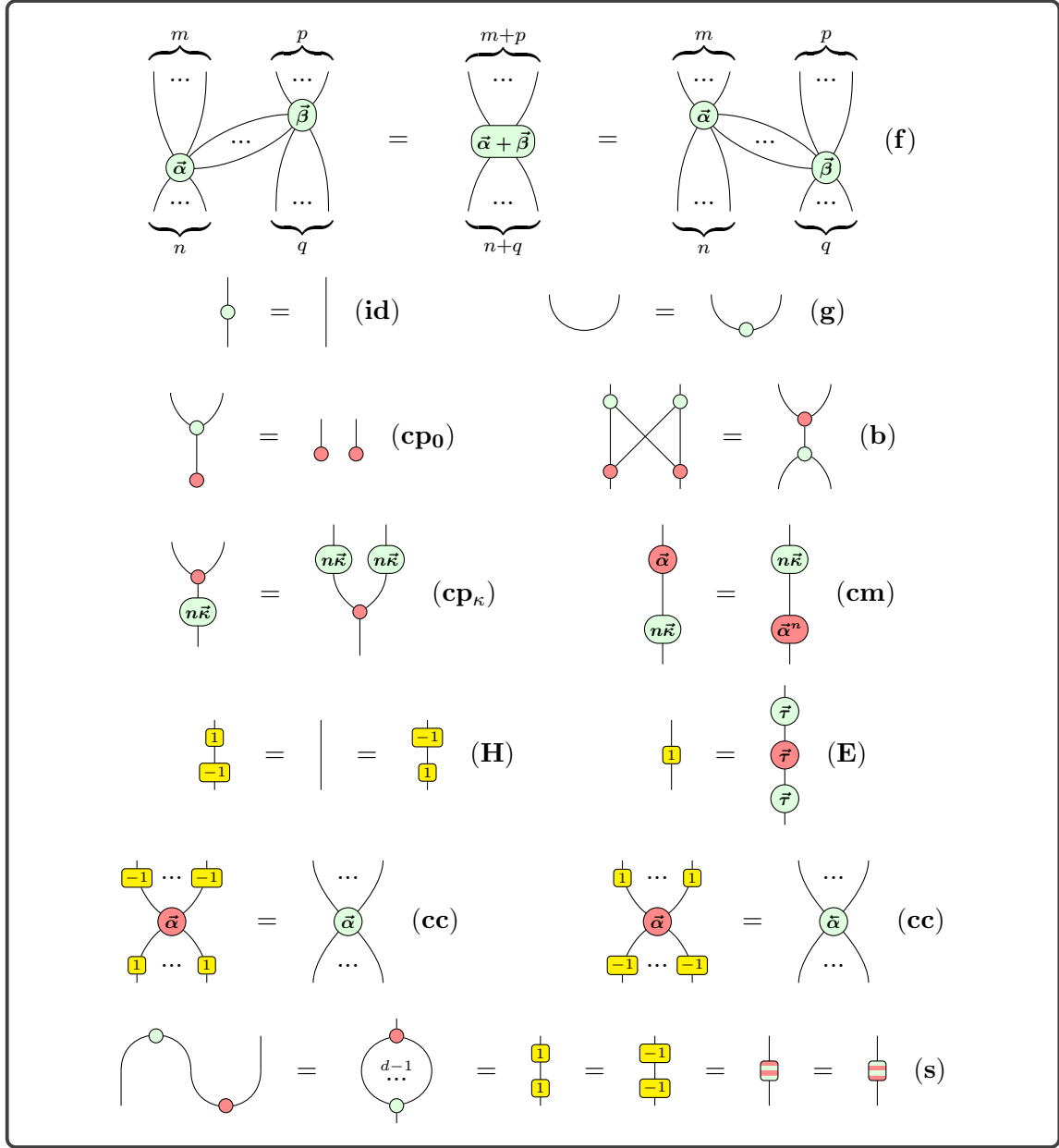
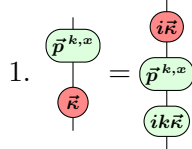


Figure 1: Qudit ZX-calculus rules

far this has proved difficult! I've been playing around a bit with gates with the phase $\vec{p}^{k,x}$, whose components are:

$$\vec{p}_j^{k,x} = -j \left(\frac{j+1}{2} k + x \right) \quad (7)$$

These are analogous to those in Wang [2018] Lemma 6.2.8 and have the following properties:



2. They form an Abelian subgroup:
 - (a) $\vec{p}^{k,x} + \vec{p}^{l,y} = \vec{p}^{k+l,x+y}$
 - (b) $-\vec{p}^{k,x} = \vec{p}^{-k,-x}$
 - (c) identity is $\vec{p}^{0,0} = (0, \dots, 0)$
3. They include a lot of our favourite phases:
 - (a) $\vec{\kappa} = \vec{p}^{0,-1}$
 - (b) $\vec{\tau} = \vec{p}^{-1, \frac{1-d}{2}}$
 - (c) $\vec{s} = \vec{p}^{-1, \frac{d-1}{2}}$
 - (d) $\vec{\bar{s}} = \vec{p}^{-1, \frac{d+3}{2}}$
4. The last component $\vec{p}_{d-1}^{k,x}$ is exactly x .

3 Local Complementation and Pivot

Local complementation carries over to qudit case without much fuss. Some lemmas we'll need:

Lemma 3.1.

[illegible]

Proof. For the first equation:

Diagrammatic equation (9) showing transformations of a loop with a green circle labeled $\bar{\alpha}$ and a red dot. The sequence of diagrams is as follows:

- A vertical line with a red dot at the bottom, a green circle labeled $\bar{\alpha}$ at the top, and a curved line connecting them.
- An equals sign.
- A vertical line with a green circle at the bottom, a red dot at the top, and a green circle labeled $\bar{\alpha}$ at the top.
- An equals sign.
- A vertical line with a green circle at the bottom, a red dot at the top, and a yellow box labeled D at the top.
- An equals sign.
- A vertical line with a green circle at the bottom, a red dot at the top, and a green circle labeled $\bar{\alpha}$ at the top.
- An equals sign.
- A vertical line with a green circle at the bottom, a red dot at the top, and a yellow box labeled D at the top.

(9)

For the second:

Diagrammatic equation (10) showing transformations of a vertical line with a red circle labeled $\pm\bar{\tau}$. The sequence of diagrams is as follows:

- A vertical line with a red circle labeled $\pm\bar{\tau}$.
- An equals sign.
- A vertical line with a yellow box labeled ∓ 1 at the top and a green circle labeled $\pm\bar{\tau}$ at the bottom.
- An equals sign.
- A vertical line with a green circle labeled $\mp\bar{\tau}$ at the top, a red circle labeled $\mp\bar{\tau}$ in the middle, and a green circle labeled $\pm\bar{\tau}$ at the bottom.
- An equals sign.
- A vertical line with a green circle labeled $\mp\bar{\tau}$ at the top, a red circle labeled $\mp\bar{\tau}$ in the middle, and a green circle at the bottom.
- An equals sign.
- A vertical line with a green circle labeled $\mp\bar{\tau}$ at the bottom.

(10)

And the third:

Diagrammatic equation (11) showing transformations of a vertical line with a red circle labeled $\bar{\tau}$ and a yellow box labeled D . The sequence of diagrams is as follows:

- A vertical line with a yellow box labeled D at the top and a red circle labeled $\bar{\tau}$ at the bottom.
- An equals sign.
- A vertical line with a yellow box labeled -1 at the top, a yellow box labeled -1 in the middle, and a red circle labeled $\bar{\tau}$ at the bottom.
- An equals sign.
- A vertical line with a yellow box labeled -1 at the top, a green circle labeled $\bar{\tau}$ in the middle, and a yellow box labeled -1 at the bottom.
- An equals sign.
- A vertical line with a green circle labeled $-\bar{\tau}$ at the top, a red circle labeled $-\bar{\tau}$ in the middle, a green circle labeled $-\bar{\tau}$ in the middle, a green circle labeled $\bar{\tau}$ in the middle, and a yellow box labeled -1 at the bottom.
- An equals sign.
- A vertical line with a green circle labeled $-\bar{\tau}$ at the top, a red circle labeled $-\bar{\tau}$ in the middle, and a yellow box labeled -1 at the bottom.
- An equals sign.
- A vertical line with a green circle labeled $-\bar{\tau}$ at the top, a yellow box labeled -1 in the middle, and a green circle labeled $-\bar{\tau}$ at the bottom.

(11)

□

So, first we have a triangle lemma:

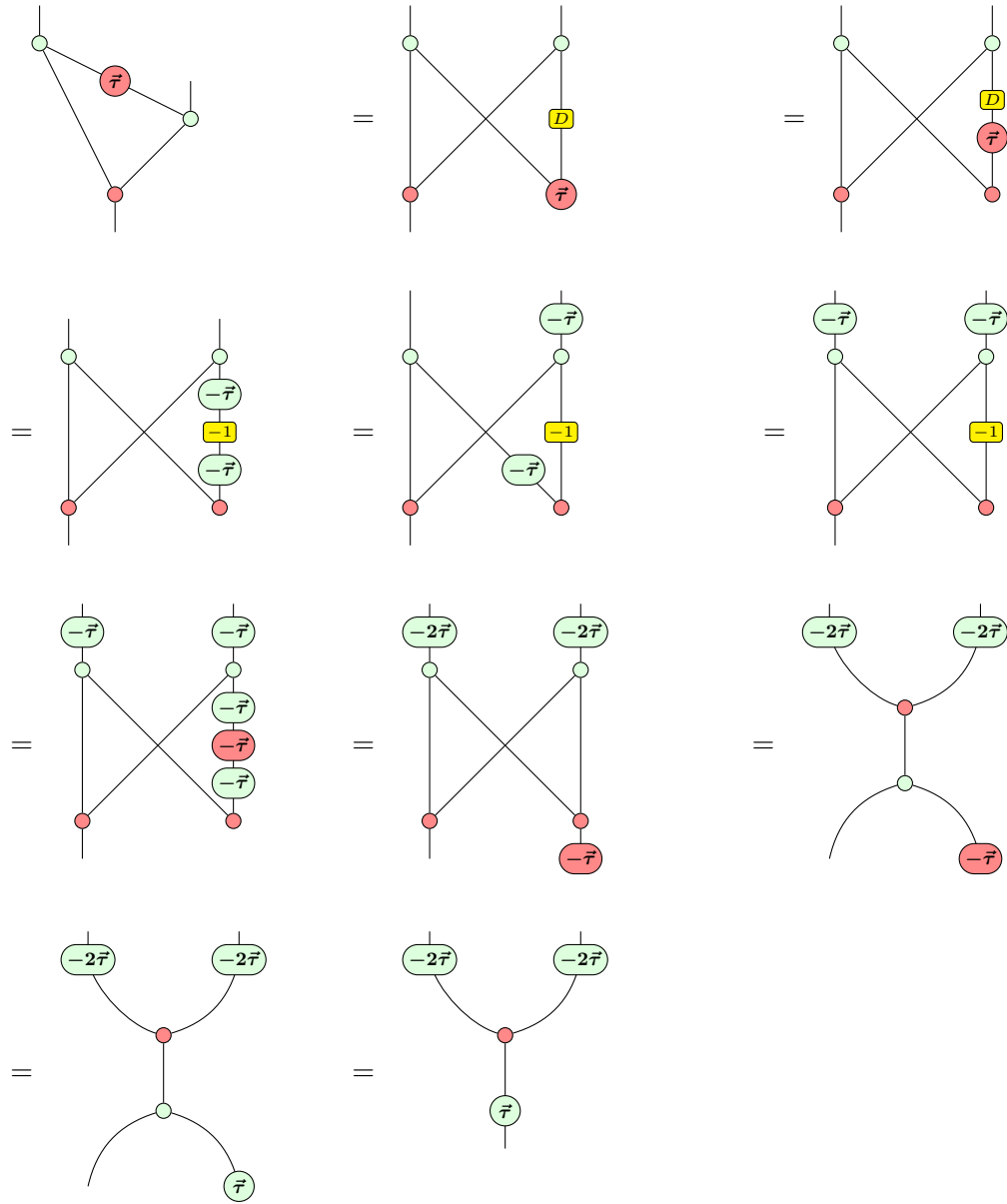
Lemma 3.2.

Diagrammatic equation (12) showing a triangle lemma transformation. The sequence of diagrams is as follows:

- A triangle with a green circle at the top-left, a red circle labeled $\bar{\tau}$ at the top-right, and a red circle at the bottom. Each vertex has a vertical line extending from it.
- An equals sign.
- A diagram with two green circles labeled $-2\bar{\tau}$ at the top, connected by a curved line. A red circle is at the bottom of this curve, and a green circle labeled $\bar{\tau}$ is at the bottom of the entire diagram.

(12)

Proof.



□

Then a lemma that tells us how to remove one edge in the K_3 graph state:

Lemma 3.3.

$$(13)$$

Proof.

□

And finally the local complementation result for the complete graph K_3 :

Lemma 3.4.

$$(14)$$

Proof.

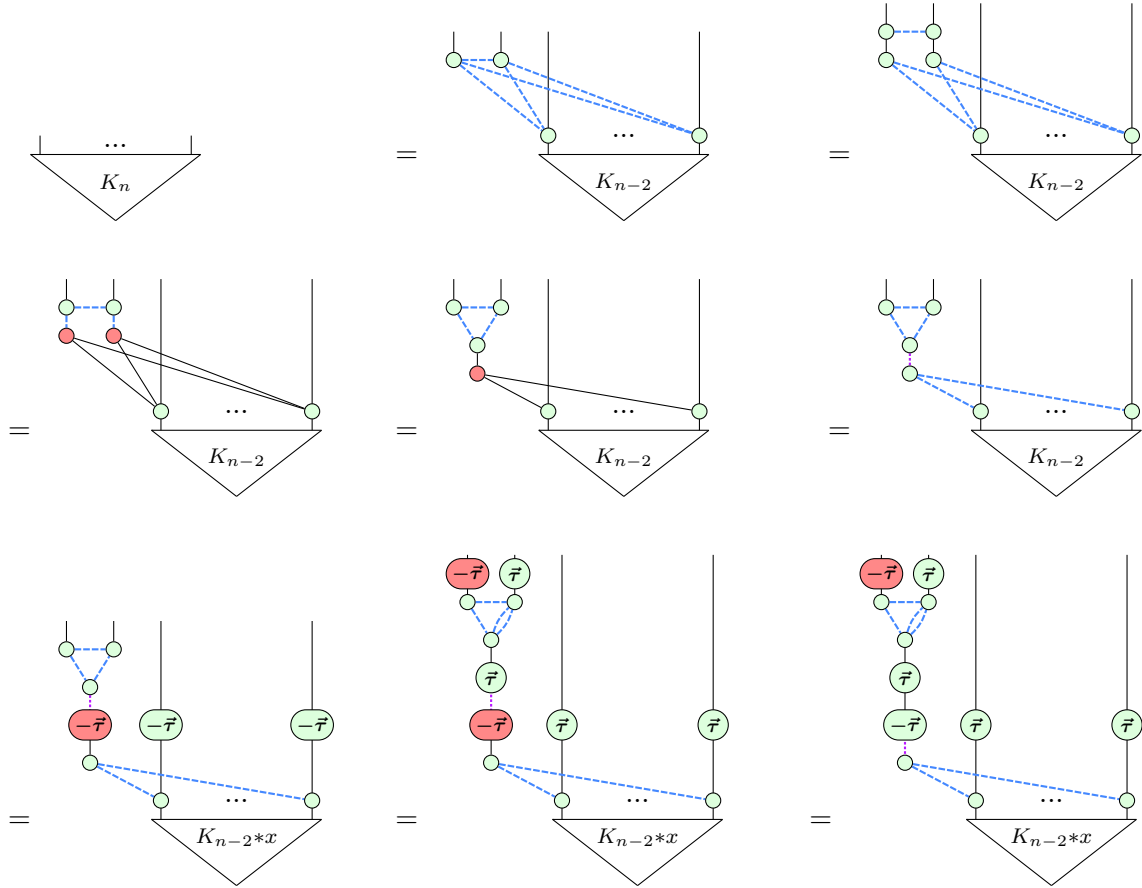
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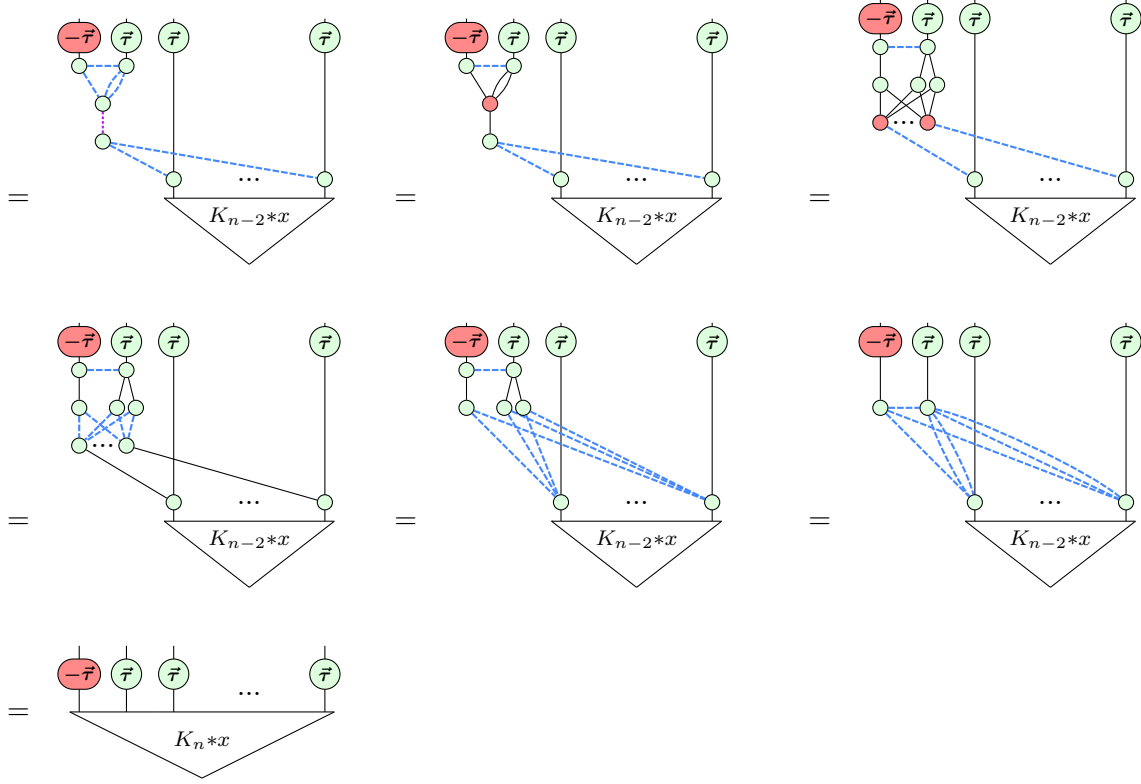
Other base cases K_0, K_1 and K_2 are easy enough to prove. For larger complete graphs K_n we proceed inductively:

Lemma 3.5.

$$\begin{array}{c} \dots \\ \triangle \\ K_n \end{array} = \begin{array}{c} \begin{array}{c} \textcolor{red}{-\tau} \quad \textcolor{green}{\tau} \quad \textcolor{green}{\tau} \quad \dots \quad \textcolor{green}{\tau} \end{array} \\ \triangle \\ K_{n*x} \end{array} \quad (15)$$

Proof.





□

For general (non-complete) graphs, can derive the result from the K_n case. Then need to derive pivot equation.

4 Elimination Theorems

Find as many eliminatable spiders as possible!

References

- S. X. Cui and Z. Wang. Universal quantum computation with metaplectic anyons. *Journal of Mathematical Physics*, 56(3):032202, Mar 2015. ISSN 1089-7658. doi: 10.1063/1.4914941. URL <http://dx.doi.org/10.1063/1.4914941>.
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