Qudit ZX-Diagram Simplification (Draft)

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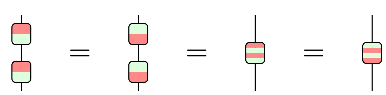
Abstract

1 Introduction

Diagrams are read bottom to top. 'It is not hard to believe that qubits and qutrits behave differently from qupits.' [Cui and Wang, 2015]. Can't decide on notation for Hadamard stuff. Numbered yellow boxes are good for graph-like diagrams, but red/green split boxes are great for colour change rules:

(1)

And then the dualiser has a natural alternative notation - important because using a yellow box with a D on it now looks more like a Hadamard with decoration D:



(2)

But the split boxes have the drawback that often you can no longer say 'the equation holds with the roles of green and red interchanged' - e.g. the Euler decomposition of the Hadamard. Not a problem for dualiser though.

Qudit rules are shown in Figure 1. We're mostly interested in stabilizer stuff so will give phase components as multiples of $\frac{2\pi}{d}$.

- $\vec{\kappa} = (1, 2, ..., d-1)$
- $\vec{\tau}$ has components $\vec{\tau}_j = \frac{j}{2}(j+d)$
- $\vec{\alpha}^n$ has components $\vec{\alpha}^n_j = \vec{\alpha}_{j-n} - \vec{\alpha}_{-n}$

- Indices are mod d, and $\vec{\alpha}_0 = 0$ for any phase a.
- $\overleftarrow{\alpha} = (\vec{\alpha}_{d-1}, \vec{\alpha}_{d-2}, ..., \vec{\alpha}_1)$

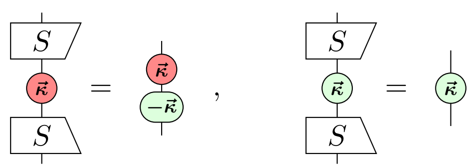
2 Stabilizer Phases

We want to know what spider phases we might encounter in the world of stabilizer qudit ZX-calculus. For odd d, components $\vec{a}_j$ of stabilizer phases $\vec{a}$ will be integer multiples of $\frac{2\pi}{d}$. For even d, stabilizer components $\vec{a}_j$ can be half-integer multiples of $\frac{2\pi}{d}$ (i.e. integer multiples of $\frac{\pi}{d}$). The set of stabilizer phases is a subset of the set of all possible $(d-1)$-tuples of such integers/half-integers. In general, this subset is strict, though we note that for qubits and qutrits it is not.

The shift gate S is uniquely defined by its action on the Pauli gates X and Z. Specifically, it satisfies:

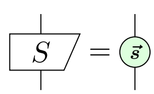
$$SXS^\dagger = XZ^{-1} \quad , \quad SZS^\dagger = Z \tag{3}$$

In ZX-parlance, these equations are:



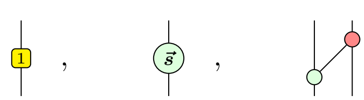
(4)

These are satisfied by the following Z-gate whose phase $\vec{s}$ has components $\vec{s}_j$:



$$, \quad \vec{s}_j = \frac{j}{2}(j + 2 - d) \tag{5}$$

Note that for even d, the components can be half-integers - for example, $\vec{s}_{d-1} = \frac{d-1}{2}$. For all d, the qudit Clifford group on n qubits is generated by the Hadamard gate H, the shift gate S and the generalised $CNOT$ gate [Farinholt, 2014]:



(6)

The Clifford group on one qudit (the *local Clifford group*) is generated by H and S [Farinholt, 2014]. We seek a normal form for such gates, à la [Wang, 2018, Theorem 6.2.12]. A method for constructing any such gate is from H and S is given in Farinholt [2014]! In

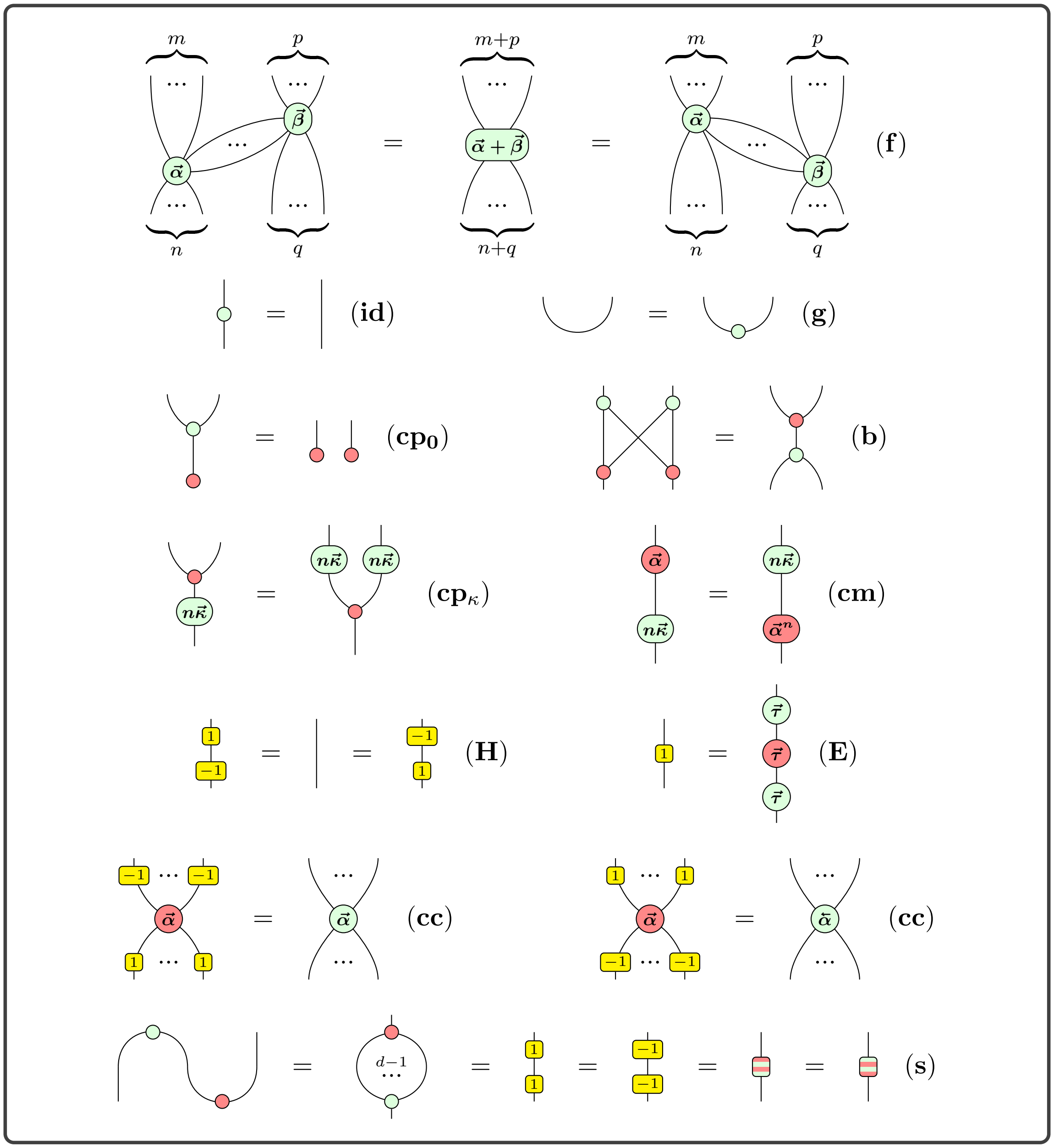


Figure 1: Qudit ZX-calculus rules

particular, for prime d at least, this has a nice form. A local Clifford operator C is uniquely defined by its action on X and Z under unitary conjugation. If it sends X to $X^p Z^r$ and Z to $X^q Z^s$, we represent C by the matrix:

$$C = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \tag{7}$$

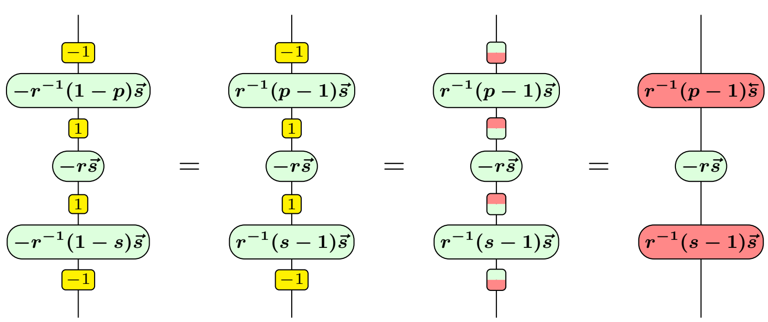
where entries are in $\mathbb{Z}_d$. C is a local Clifford operator if and only if it is symplectic, i.e. has determinant 1. Let $x_y = x^{-1}(1+y)$. If q is invertible, C has decomposition:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = S^{-q_s} H^{-1} S^{-q} H^{-1} S^{-q_p} \tag{8}$$

To make life easier for ourselves, we'll instead suppose r invertible, and consider:

$$\begin{aligned}
H^{-1}C^{-1}H^{-1} &= \begin{pmatrix} -p & -r \\ -q & -s \end{pmatrix} \\
&= S^{-(-r)_{-s}} H^{-1} S^{-(-r)} H^{-1} S^{-(-r)_{-p}} \\
&= S^{r_{-s}} H^{-1} S^{r} H^{-1} S^{r_{-p}} \\
\therefore C^{-1} &= H S^{r_{-s}} H^{-1} S^{r} H^{-1} S^{r_{-p}} H \\
\therefore C &= H^{-1} S^{-r_{-p}} H S^{-r} H S^{-r_{-s}} H^{-1}
\end{aligned}$$

As a ZX-diagram, this is:

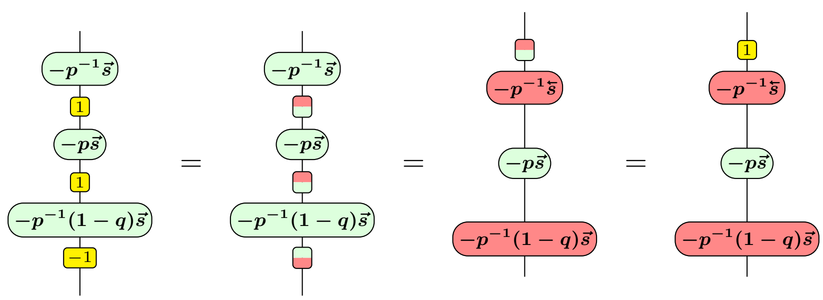


(9)

What if r not invertible? Since d prime and $r \in \mathbb{Z}_d$, this only happens if $r = 0$. If so, since C has determinant 1, this forces $p \neq 0$, i.e. p invertible. So then consider:

$$
\begin{aligned}
H^{-1}C^{-1} &= \begin{pmatrix} r & -p \\ s & -q \end{pmatrix} \\
&= S^{-(-p)_{-q}} H^{-1} S^{-(-p)} H^{-1} S^{-(-p)_r} \\
&= S^{p_{-q}} H^{-1} S^{p} H^{-1} S^{p_r} \\
\therefore C^{-1} &= H S^{p_{-q}} H^{-1} S^{p} H^{-1} S^{p_r} \\
\therefore C &= S^{-p_r} H S^{-p} H S^{-p_{-q}} H^{-1}
\end{aligned}
$$

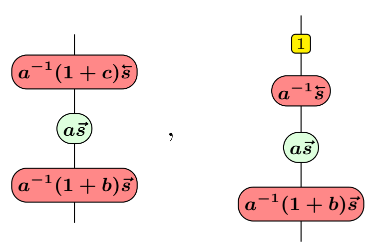
Note since $r = 0$ we have $p_r = p^{-1}(1 + r) = p^{-1}$. As a ZX-diagram, this is:



(10)

So we have a normal form. Negating all of p, q, r and s changes nothing, so put more palatably, our theorem is:

Theorem 2.1. For d prime, every local Clifford operator is of exactly one of the following two forms, and conversely every such diagram is a local Clifford operator:



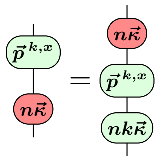
(11)

for $a, b, c \in \mathbb{Z}_d$ with $a > 0$.

I've been playing around a bit with gates with the phase $\vec{p}^{k,x}$, whose components are:

$$\vec{p}_j^{k,x} = -j\left(\frac{j+1}{2}k + x\right) \tag{12}$$

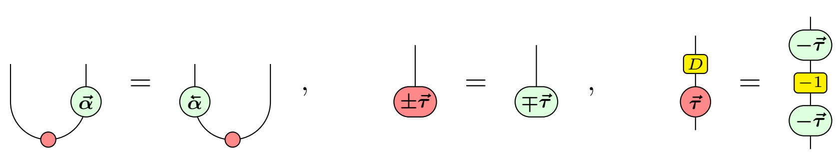
These are analogous to those in Wang [2018]Lemma 6.2.8 and have the following properties:

1. 
2. They form an Abelian subgroup:
   (a) $\vec{p}^{k,x} + \vec{p}^{l,y} = \vec{p}^{k+l,x+y}$
   (b) $-\vec{p}^{k,x} = \vec{p}^{-k,-x}$
   (c) identity is $\vec{p}^{0,0} = (0, ..., 0)$
3. They include a lot of our favourite phases:
   (a) $\vec{\kappa} = \vec{p}^{0,-1}$
   (b) $\vec{\tau} = \vec{p}^{-1,\frac{1-d}{2}}$
   (c) $\vec{s} = \vec{p}^{-1,\frac{d-1}{2}}$
   (d) $\overleftarrow{s} = \vec{p}^{-1,\frac{d+3}{2}}$
4. The last component $\vec{p}_{d-1}^{k,x}$ is exactly x.

3 Local Complementation and Pivot

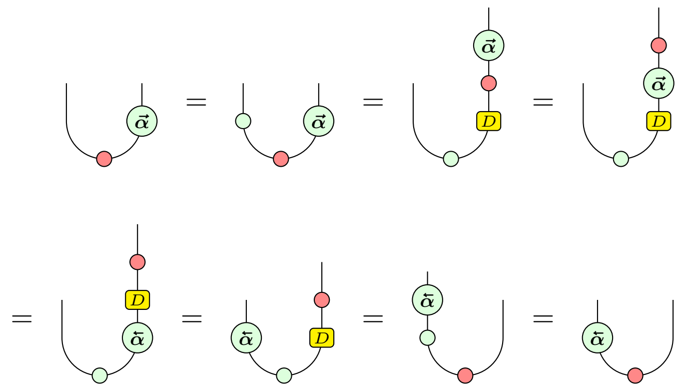
Local complementation carries over to qudit case without much fuss. Some lemmas we'll need:

Lemma 3.1.



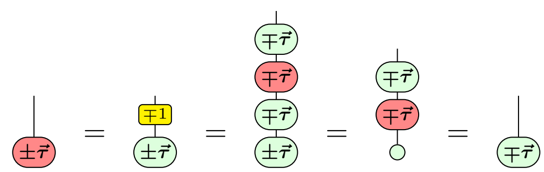
(13)

Proof. For the first equation:



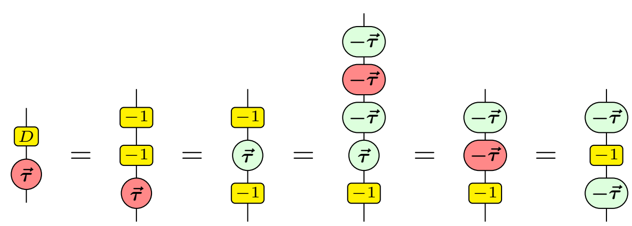
(14)

For the second:



(15)

And the third:

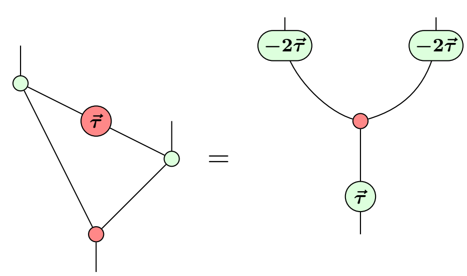


(16)

□

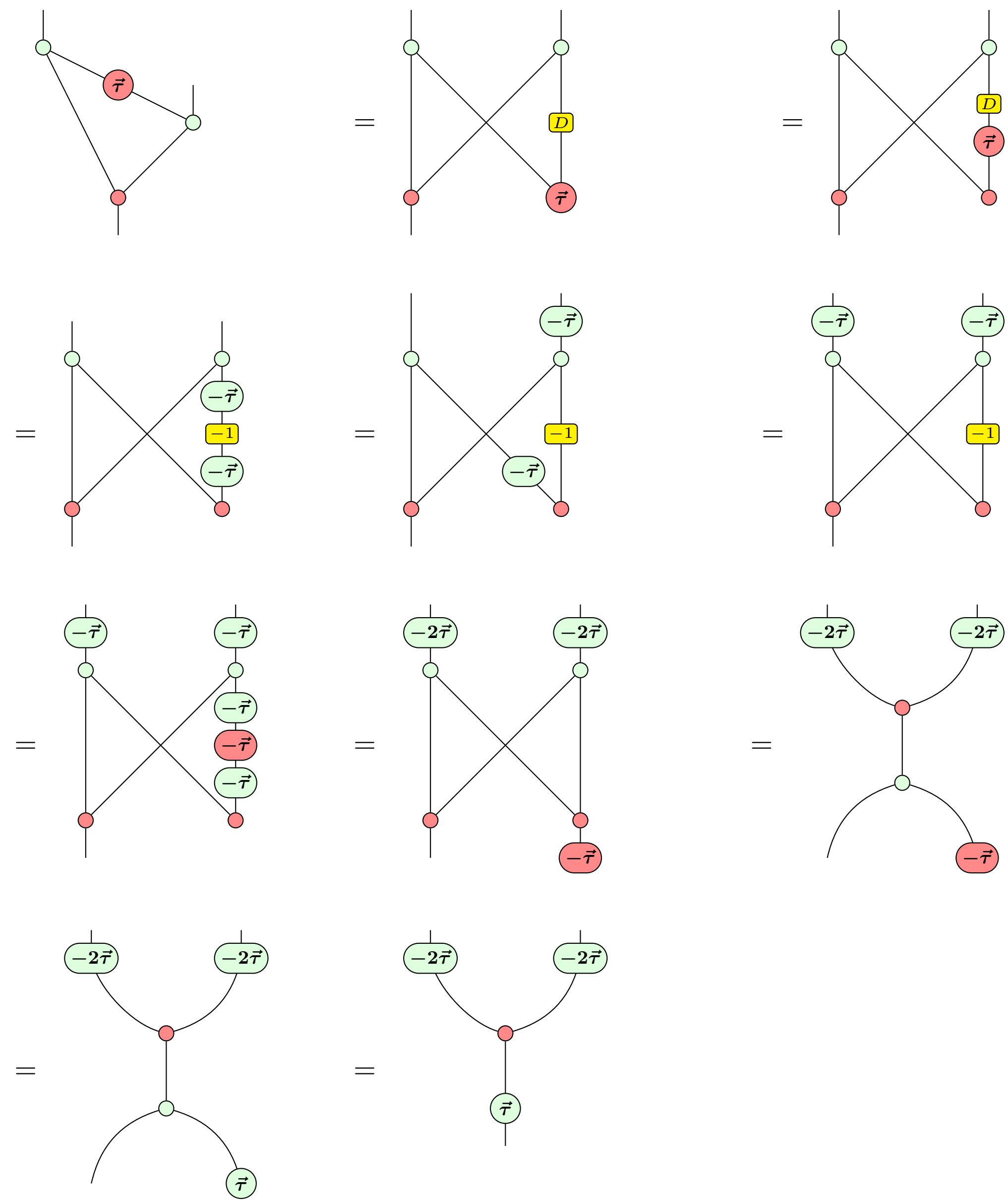
So, first we have a triangle lemma:

Lemma 3.2.



(17)

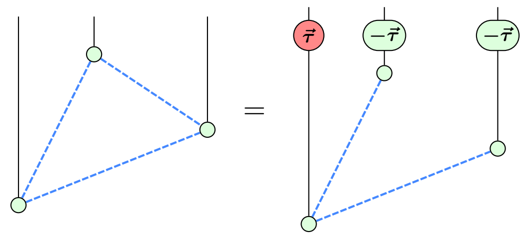
Proof.



$\square$

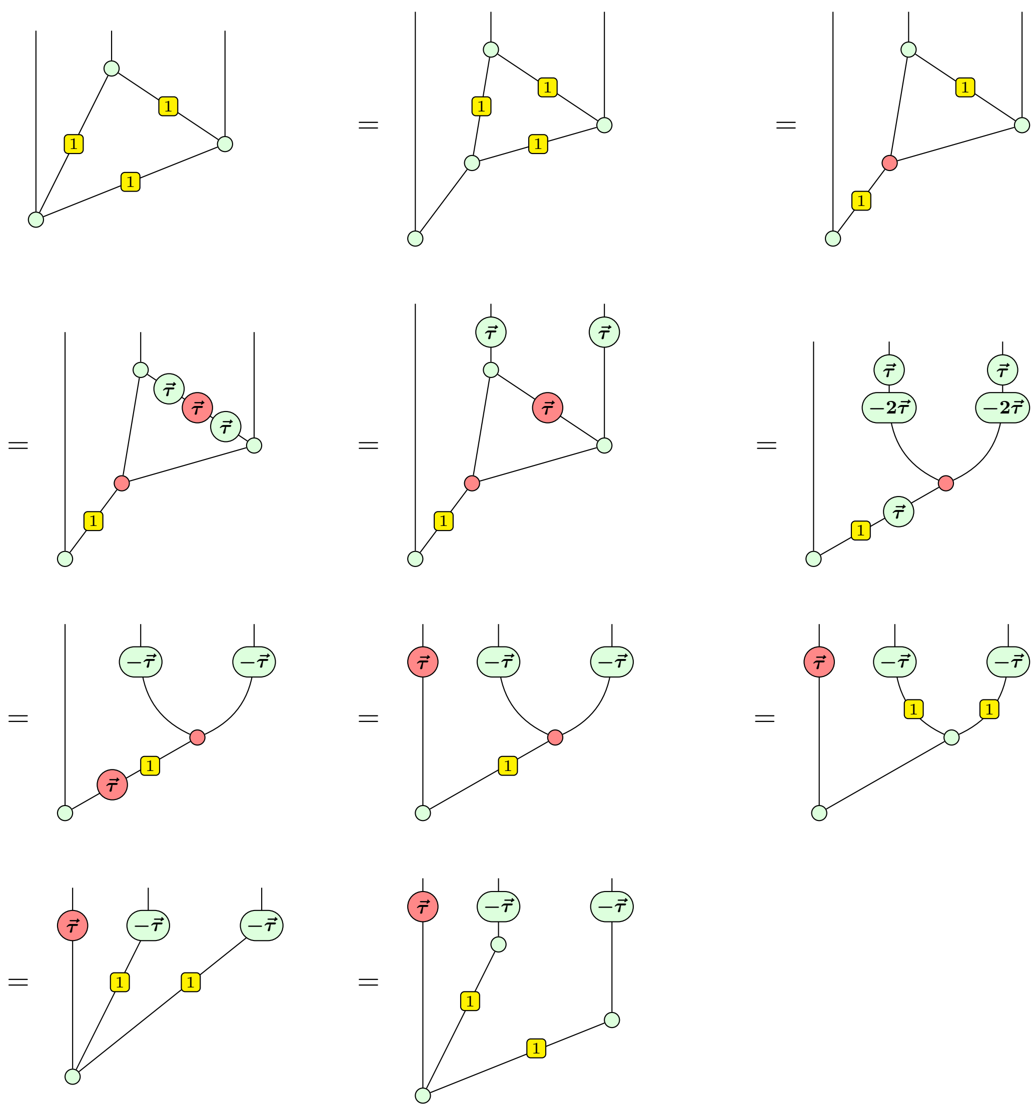
Then a lemma that tells us how to remove one edge in the K_3 graph state:

Lemma 3.3.



(18)

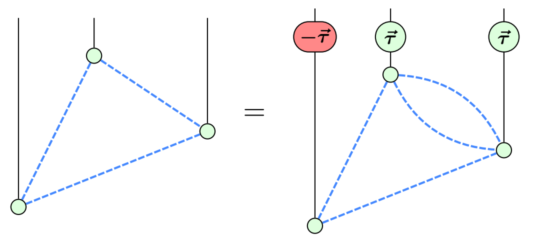
Proof.



□

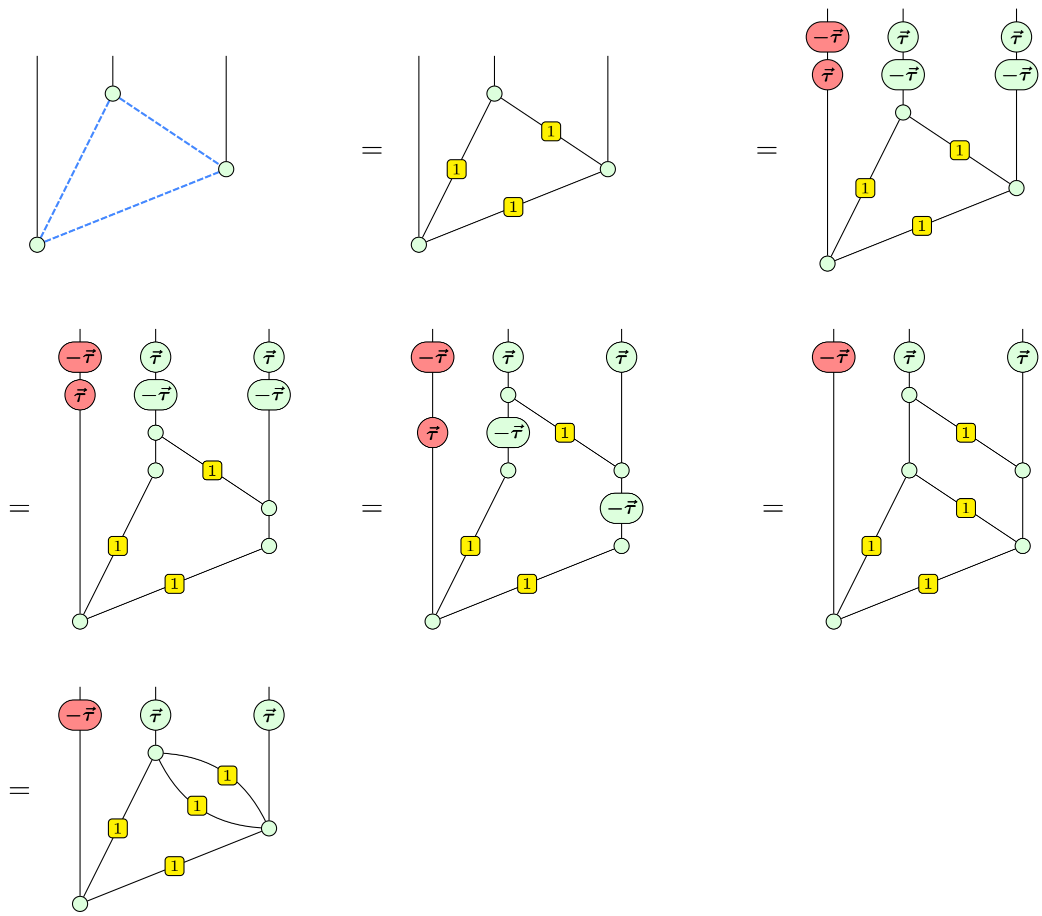
And finally the local complementation result for the complete graph K_3:

Lemma 3.4.



(19)

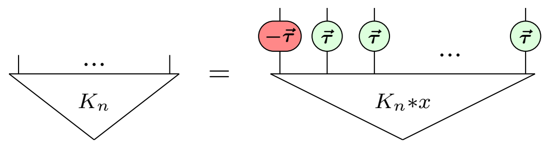
Proof.



□

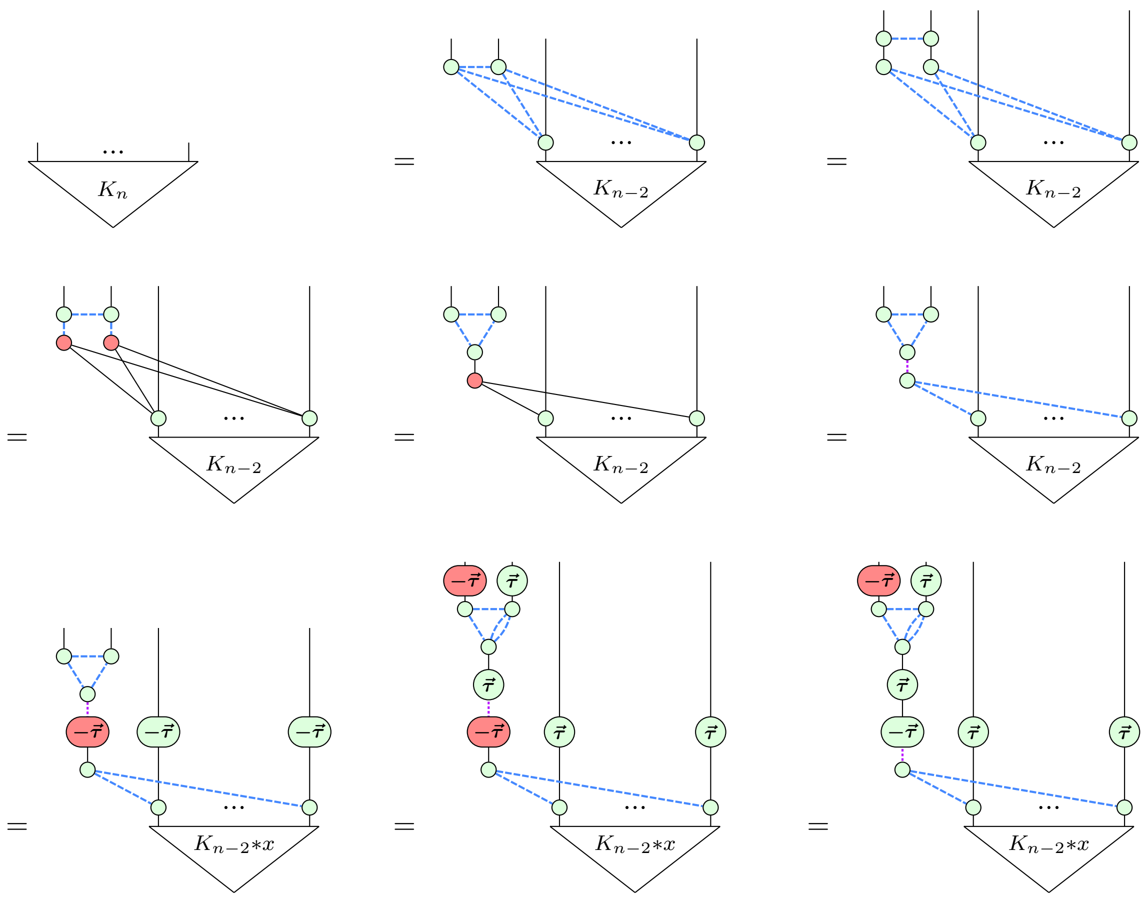
Other base cases K_0, K_1 and K_2 are easy enough to prove. For larger complete graphs K_n we proceed inductively:

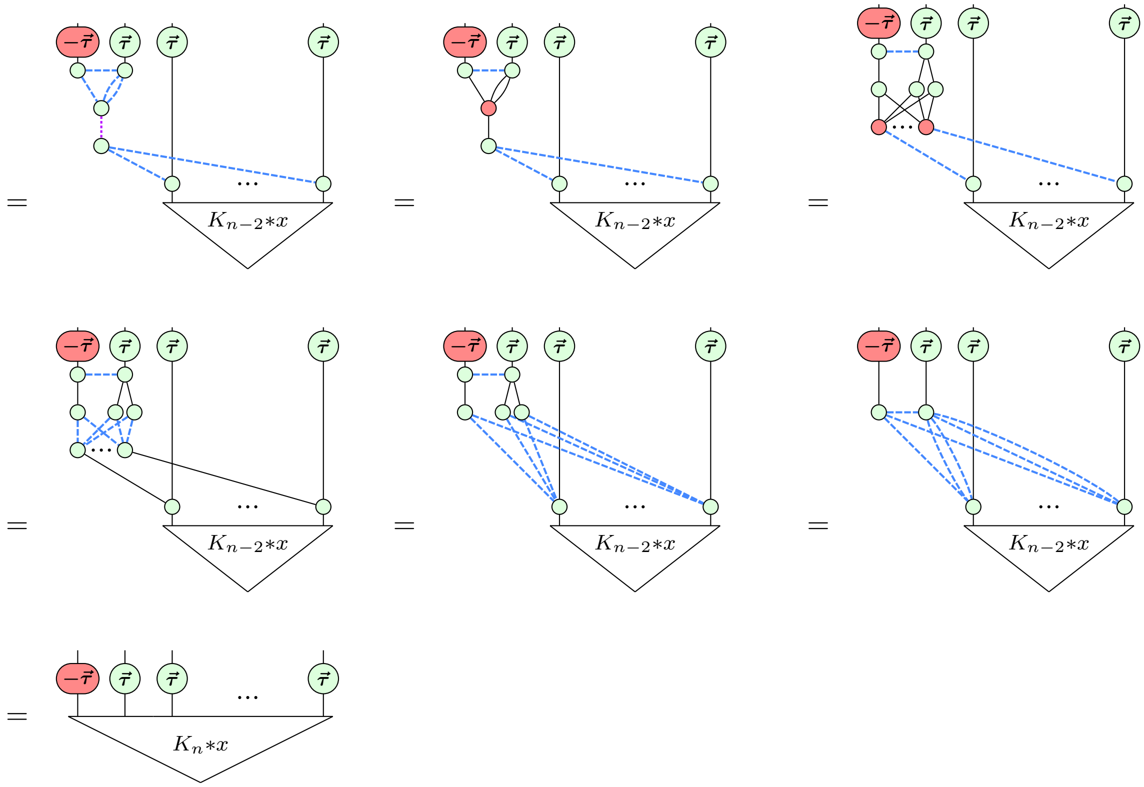
Lemma 3.5.



(20)

Proof.





□

For general (non-complete) graphs, can derive the result from the K_n case. Then need to derive pivot equation.

4 Elimination Theorems

Find as many eliminatable spiders as possible!

References

S. X. Cui and Z. Wang. Universal quantum computation with metaplectic anyons. *Journal of Mathematical Physics*, 56(3):032202, Mar 2015. ISSN 1089-7658. doi: 10.1063/1.4914941. URL `http://dx.doi.org/10.1063/1.4914941`.

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