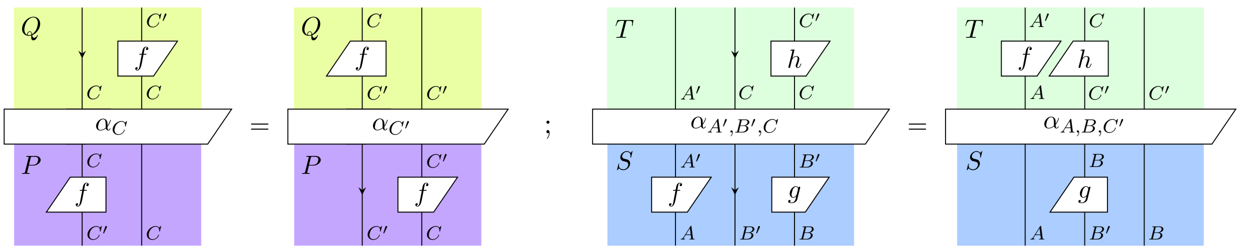
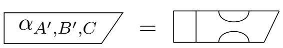
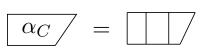
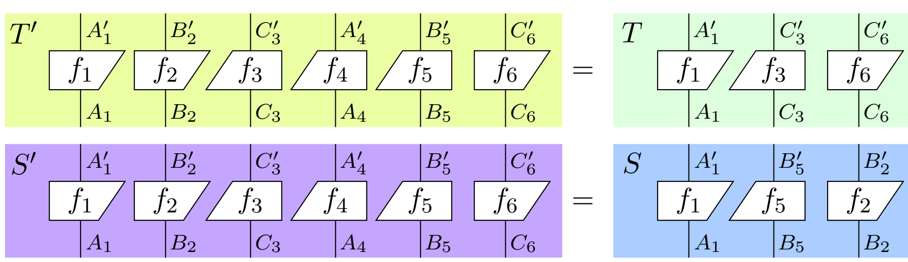
Graphical Aspects of (Co)ends

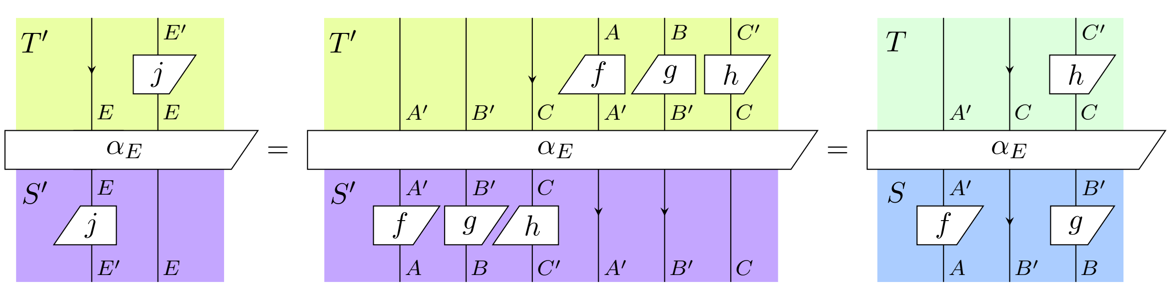
We humans have a knack for geometric reasoning, but have to work harder at our algebraic intuition. Hence it makes sense to leverage the former wherever possible. Here we present two distinct graphical aspects of (co)ends, starting with a closer look at the naturality definitions underlying them. In [1] a *dinatural transformation* between functors $P, Q : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is a family of morphisms $\alpha_C : P(C,C) \to Q(C,C)$ for each $C \in \mathcal{C}$ satsisfying the graphical equation on the left below. Similarly an *extranatural transformation* between functors $S : \mathcal{A} \times \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{D}$ and $T : \mathcal{A} \times \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is a family of morphisms $\alpha_{A,B,C}$ for all $(A,B,C) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C}$ satisfying the equation on the right:



The graphical calculus used is based on the sound and complete calculus for monoidal categories, as in (for example) [2]. Sadly we do not have space to justify our use of it. Functors are depicted as in [3]. Objects and morphisms in opposite categories are indicated by a downwards arrow and rotated box respectively. The less detailed graphical presentation mentioned in [1] set  and allowed morphisms to slide along wires; if we do the same we see the extranaturality equation becomes trivial. Similarly, our presentation is suggestive of the fact that dinaturality is strictly more general than extranaturality, as in [1, Exercise 1.5]. That is, we can satisfy dinaturality by setting , which is not in any way extranatural, then sliding morphisms up and down. Conversely, we can give a much clearer proof that extranaturality is dinaturality [1, Proposition 1.1.12]. Given functors S and T as above, set $\mathcal{E} = \mathcal{A}^{op} \times \mathcal{B}^{op} \times \mathcal{C}$ and define functors S', $T' : \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{D}$ by:



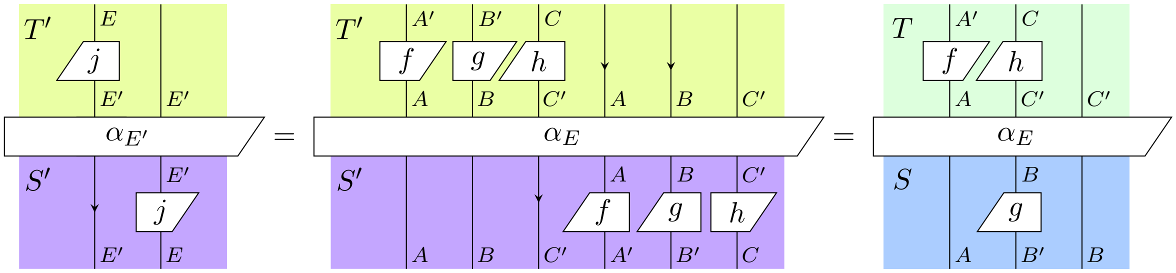
Noting that any morphism $j : E \to E'$ in $\mathcal{A}$ is a triple of morphisms $f : A \to A'$ in $\mathcal{A}^{op}$, $g : B \to B'$ in $\mathcal{B}^{op}$ and $h : C \to C'$ in $\mathcal{C}$, we get:



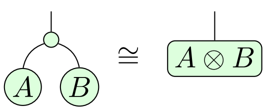
[1] Coend Calculus, Fosco Loregian, arXiv:1501.02503

[2] A survey of graphical languages for monoidal categories, Peter Selinger, arXiv:0908.3347

[3] Functorial boxes in string diagrams, Paul-André Melliès

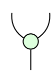
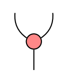


So extranaturality of S and T is exactly dinaturality of S' and T'. There are many other ways we could have chosen to demonstrate the benefit of using this calculus - for example, it gives a very illustrative proof that (co)ends in $\mathcal{C}$ are precisely (co)limits in $TW(\mathcal{C})$. A graphical use of (co)ends with a different flavour can be found in [4], where string diagrams represent functors into **Set**. Sequential composition of diagrams of functors $P : \mathcal{A}^{op} \times \mathcal{B} \to \mathbf{Set}$ and $Q : \mathcal{B}^{op} \times \mathcal{C} \to \mathbf{Set}$ gives the functor $(P \diamond Q) : \mathcal{A}^{op} \times \mathcal{C} \to \mathbf{Set}$ defined by the coend $(P \diamond Q)(A, C) = \int^{B \in \mathcal{B}} P(A, B) \times Q(B, C)$. Parallel composition of functors $P : \mathcal{A}_1^{op} \times \mathcal{B}_1 \to \mathbf{Set}$ and $Q : \mathcal{A}_2^{op} \times \mathcal{B}_2 \to \mathbf{Set}$ yields $(P \otimes Q) : (\mathcal{A}_1 \times \mathcal{A}_2)^{op} \times (\mathcal{B}_1 \times \mathcal{B}_2) \to \mathbf{Set}$ defined by $(P \otimes Q)(A_1, A_2, B_1, B_2) = P(A_1, B_1) \otimes Q(A_2, B_2)$. So for objects A, B in a monoidal category $\mathcal{C}$, if we define  $= \mathcal{C}(A, -)$ and  $= \mathcal{C}(- \otimes -, -)$ then we have the following equation [4, Proposition 2.3]:

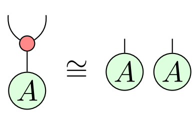


$$\left(\begin{array}{lr} (\mathcal{C}(A,-) \otimes \mathcal{C}(B,-)) \diamond \mathcal{C}(- \otimes -, -) & \\ \cong \displaystyle\int^{(X,Y)} (\mathcal{C}(A \otimes B, X \otimes Y) \times \mathcal{C}(X \otimes Y, -) & \text{(definition)} \\ \cong \mathcal{C}(A \otimes B, -) & \text{(Yoneda)} \end{array}\right)$$

This equation is much more digestible in its graphical form on the left. Let's further define:

 $= \mathcal{C}(-, A)$  $= \mathcal{C}(-, - \otimes -)$  $= \mathcal{C}(-^0, -^1) \otimes \mathcal{C}(-^0, -^2)$

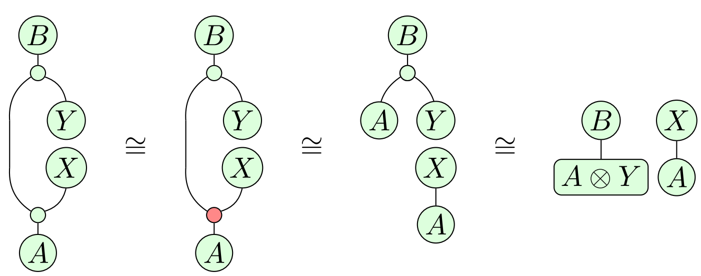
We can show that this 'red spider' on the far right satisfies a 'copy law':



$$\left(\begin{array}{lr} \mathcal{C}(A,-) \diamond (\mathcal{C}(-^0, -^1) \otimes \mathcal{C}(-^0, -^2)) & \\ \cong \displaystyle\int^{X} \mathcal{C}(A, X) \times (\mathcal{C}(X,-) \otimes \mathcal{C}(X,-)) & \text{(definition)} \\ \cong \mathcal{C}(A,-) \otimes \mathcal{C}(A,-) & \text{(Yoneda)} \end{array}\right)$$

Now, given objects A, B, X, Y in a monoidal category $\mathcal{C}$, a *lens* or *optic* from (A, B) to (X, Y) is an element of the set $\int^M \mathcal{C}(A, M \otimes X) \times \mathcal{C}(M \otimes Y, B)$. It is shown in [4, Proposition 2.4] that $\mathcal{C}$ is Cartesian if and only if  $\cong$ . So for Cartesian $\mathcal{C}$ we can easily show graphically that such a lens is (isomorphic to) just a pair of morphisms $A \otimes Y \to B$ and $A \to X$ as in [4, Proposition 2.7]:

$$\int^M \mathcal{C}(A, M \otimes X) \times \mathcal{C}(M \otimes Y, B) =$$



[4] Open Diagrams via Coend Calculus, Mario Román, arXiv:2004.04526