Commutative Properties of Braids

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Abstract

In this essay we explore the braid group via mapping class groups, proving a few of its key properties. We give an original proof that its centre is $\langle \Delta^2 \rangle$, based on a suggestion in [5]. We give an exposition of the work in [8] on the centralizer of a braid, and leverage it to provide an upper bound on Abelian subgroups of the braid group - a result which is completely new to the literature.

Contents

1 The Braid Group

As we shall see, the braid group can be defined in a number of ways. We will introduce the group using the original presentation given by Artin in 1925 [1]:

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & i \in \{1, \ldots, n-1\} \\ \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| > 1 \end{array} \right\rangle$$

The group's name comes from the following interpretation: we say that a braid on n strings consists of n arcs embedded in $D^2 \times [0,1]$. Each arc (called a *strand*) is transverse to each disk $D^2 \times \{t\}$, and has endpoints at n points $x_1, x_2, \ldots x_n$ in each of $D^2 \times \{0\}$ and $D^2 \times \{1\}$. These n points can be arranged in any configuration we like. The braid group is then the set of equivalence classes of braids, where two braids are equivalent if there is a level-preserving isotopy between them.

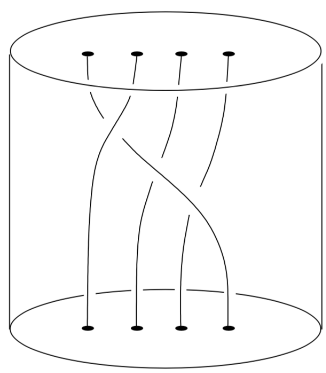

Figure 1.1: [6] A braid in the group B_4 embedded in $D^2 \times [0,1]$.

A generator σ_i of this group is the braid that crosses string i over string $i+1$, so that string i is attached to the point x_{i+1} on the upper disk, string $i+1$ is attached to the point x_i on the upper disk, and all other strings are joined to the point directly above them without any crossings. The group operation places one braid on top of another and fuses their endpoints together. Certainly this is associative. The identity is the braid whose strands are completely vertical. Every braid has an inverse under this operation; given a braid $\sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\ldots\sigma_{i_k}^{\epsilon_k}$, where each $\epsilon_i = \pm 1$, if we compose it with the braid $\sigma_{i_k}^{-\epsilon_k}\ldots\sigma_{i_2}^{-\epsilon_2}\sigma_{i_1}^{-\epsilon_1}$ this has the effect of 'untangling' the original braid, one crossing at a time, resulting in n vertical strands. Thus this is indeed a well-defined group.

Note that the group B_1 is trivial and B_2 is just the group $\mathbb{Z}$. Therefore throughout this essay we will often assume that $n \geq 3$.

We can also define braids via braid diagrams, as shown in Figure 1.3, by chaining together generating braids. These diagrams give us the following tool:

Definition 1.2: [10] In a braid diagram of some braid β, let every instance where one strand crosses over the strand directly to its right have value 1, and every instance where one strand crosses over the strand directly to its left have value -1. The *algebraic crossing number* of β is the sum of all these values. Equivalently, writing $\beta = \sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\ldots\sigma_{i_k}^{\epsilon_k}$, the algebraic crossing number is $\sum_{i=1}^{k} \epsilon_i$.

(a) (b)

Figure 1.3: [6] (a) The generator σ_i. (b) The braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

Proposition 1.4: The algebraic crossing number of a braid is well-defined.

Proof: In a group G given by a presentation $\langle X|R\rangle$, two words represent the same element exactly when they differ by a series of the following two moves:

1. Insertion or deletion of a 'spur' xx^{-1} or $x^{-1}x$, for some $x \in X$.

2. Insertion or deletion of a relation $r \in R$.

In our case, any spur $\sigma_i\sigma_i^{-1}$ or $\sigma_i^{-1}\sigma_i$ has algebraic crossing number zero. There are two possible relations, namely $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ and $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}$. Both of these again have algebraic crossing number zero. So any move of the form (1) or (2) leaves the algebraic crossing number invariant, and hence it is well-defined.

Defining B_n in this way gives the group a fairly discrete, algebraic flavour, and indeed [7] says the braid group can be treated in a purely algebraic way. But the curious thing about the braid group is how readily it can be approached from a number of different directions; for example, via configuration spaces, Garside theory, left-orderability, and mapping class groups. One of the most well-known properties of the braid group is that it is torsion-free;

that is, no braid $\beta \in B_n$ has finite order. Several proofs of this can be found in [7], each taking a different approach. In this essay, we will approach the braid group via mapping class groups. Another basic property we might want to know is: what is the centre $Z(B_n)$ of the braid group? We claim the following:

Theorem 1.5: For $n \geq 3$, $Z(B_n) = \langle \Delta^2 \rangle$, where $\Delta^2 = (\sigma_1 \sigma_2 \dots \sigma_{n-1})^n$.

We can see diagrammatically in Figure 1.6 that Δ^2 commutes with any other braid $g \in B_n$, and from there that any power of Δ^2 must do likewise. Less clear is whether $\langle \Delta^2 \rangle$ is the entire centre. We will return to this theorem once we've discussed the tools needed to prove it.

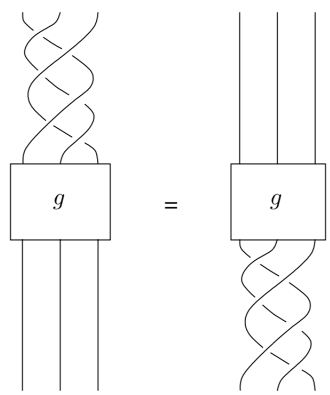

Figure 1.6: [5] The braid Δ^2 commuting with some other braid g. Consider the braid $\Delta^2 g$ on the left. We can rotate the box containing g around the vertical axis through 2π, unwinding the strands above the box and winding the strands below the box, until we have the braid $g\Delta^2$ on the right.

2 Mapping Class Groups

Given a surface S, the most general definition of its mapping class group $MCG(S)$ is the group of isotopy classes of homeomorphisms $f : S \to S$. However, there are many slight variations that we must choose between here; in this essay, we will only consider compact surfaces and orientation-preserving, boundary-fixing homeomorphisms. Further, we insist that that any isotopy between two homeomorphisms must itself fix the boundary ∂S of S throughout. Finally, if S has any marked points or punctures on it (the two terms are equivalent and can be used interchangeably), we insist that any homeomorphism must map these back to marked points/punctures. From now on, unless stated otherwise, when we say homeomorphism we in fact mean a homeomorphism that satisfies these properties.

The group operation is function composition, and it is clear this does indeed form a group. Where it does not cause confusion, we will abuse notation by speaking of a homeomorphism $f \in MCG(S)$ up to isotopy, though we actually mean the isotopy class of f. Let's first examine the mapping class group of the simple surface D^2, which we shall see is trivial. This result is known as the Alexander lemma.

Lemma 2.1 (The Alexander lemma): The group $MCG(D^2)$ is trivial.

Proof: Viewing D^2 as the unit disk in $\mathbb{R}^2$, let $f : D^2 \to D^2$ be a homeomorphism with the properties described above. We define an isotopy $H : D^2 \times [0, 1] \to D^2$ that takes f to the identity continuously while fixing the boundary throughout:

$$H(x,t) = \begin{cases} (1-t)f\left(\dfrac{x}{1-t}\right) & \text{for } |x| < 1-t \\ x & \text{for } |x| \geq 1-t \end{cases}$$

This is well-defined because f fixes the boundary throughout, so, for every $t \in (0, 1)$, $H(x, t)$ is continuous over the circle $x = |1 - t|$, and thus $H(\_, t)$ is indeed a homeomorphism with the required properties. This proof is nicely illustrated in Figure 2.2. Note that the same proof can be applied to show that the once-punctured disk also has trivial mapping class group; by isotopy, we can assume that the puncture x lies at the origin, hence $H(\_, t)$ is a homeomorphism sending x to x at each point t.

The Alexander lemma quickly leads to the Alexander method, which is used for determining when a homeomorphism on a surface is isotopic to the identity. First we require some definitions.

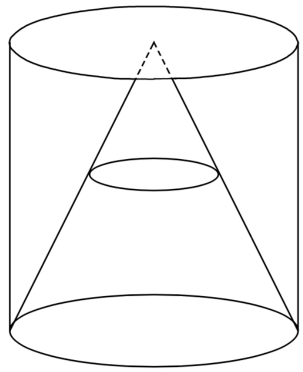

Figure 2.2: [5] The Alexander lemma.

Definition 2.3: An *essential* curve on a surface S is one that is not homotopic to a single point, a single puncture, or a boundary component.

Definition 2.4: A *proper* arc on a surface S is one that begins and ends at a puncture or boundary component, and does not pass through a puncture or boundary component at any other point.

Definition 2.5: Two curves or arcs α and β are in *minimal position* if the number of times they intersect is minimal over all γ_α and γ_β isotopic to α and β respectively.

Definition 2.6: A set Γ of curves and arcs on a surface S is said to *fill* S if cutting S along these curves and arcs would leave a disjoint union of disks and once-punctured disks.

We now outline the idea behind the Alexander method. Let Γ be a set of curves and arcs filling a compact surface S, and let $\mathcal{A}$ denote the set of disks and once-punctured disks that would result from cutting S along Γ. Suppose $f \in MCG(S)$ fixes Γ (up to isotopy) in such a way that the components $\mathcal{A}$ are not permuted, and each component $A \in \mathcal{A}$ has its orientation preserved. For each $A \in \mathcal{A}$, the restriction of f to A is a boundary-fixing homeomorphism, and so by the Alexander lemma each must be isotopic to the identity. Hence viewing f as a sum of each of these restrictions, along with the restriction to Γ itself, we see that f is isotopic to the identity on S.

The subtlety lies in defining suitable conditions on Γ and f to make this work. Below we state the full theorem, leaving a full proof to [5].

Theorem 2.7 (The Alexander method): Let S be a compact surface, and let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a set of essential simple closed curves and simple proper arcs satisfying the following:

1. Elements of Γ are in pairwise minimal position.
2. Elements of Γ are pairwise non-isotopic.
3. For any distinct γ_i, γ_j and γ_k in Γ, at least one of $\gamma_i \cap \gamma_j$, $\gamma_j \cap \gamma_k$, and $\gamma_k \cap \gamma_i$ is empty.
4. Γ fills S.

Denote by Γ^* the graph in S formed from Γ by adding vertices at the endpoints of every arc in Γ and at any intersection points of any two elements of Γ. Now suppose, up to isotopy, $f \in MCG(S)$ fixes each edge of Γ^*, including orientation. Then f is isotopic to the identity on S.

The Alexander method is one of several key ingredients that we will use in our proof of Theorem 1.5. The remaining tools we'll need all follow from the theory of Dehn twists.

3 Dehn Twists

Dehn twists are a particularly fundamental type of element of the mapping class group. Consider the homeomorphism T on the annulus $A = S^1 \times [0,1]$ given by $T(\theta, t) = (\theta + 2\pi t, t)$. Note that T is indeed orientation-preserving and fixes the two boundary components of A. This map is in fact a Dehn twist on the annulus, and it is used to define a Dehn twist on a more general surface.

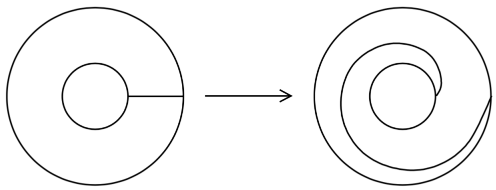

Figure 3.1: [5] The homeomorphism T, better known as the Dehn twist on the annulus.

Definition 3.2: Let α be a simple closed curve on a surface S. Let N_α be a regular neighbourhood of α, and let ψ_α be a homeomorphism from the annulus A to N_α. Then the *Dehn twist* $T_\alpha : S \to S$ about α is the homeomorphism:

$$T_\alpha(x) = \begin{cases} (\psi_\alpha \circ T \circ \psi_\alpha^{-1})(x) & \text{for } x \in N_\alpha \\ x & \text{for } x \notin N_\alpha \end{cases}$$

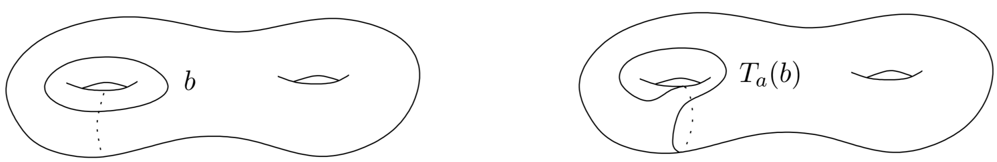

Figure 3.3: [5] The effect on a curve b of the Dehn twist about the dotted curve a.

We should quickly clarify that although this definition depends on the choice of N_α and ψ_α, the isotopy class of T_α is independent of these choices. In fact, the isotopy class of T_α is equal to that of $T_{\alpha'}$ for any α' isotopic to α. Thus when viewed as an element of $MCG(S)$ a Dehn twist is actually defined about the isotopy class a of the curve α. However, we will not usually write T_a, instead simply referring to T_α up to isotopy. Likewise we will often write α in places where we technically mean the isotopy class a of α.

Dehn twists can in turn be used to define half Dehn twists. Similarly to how we defined the full Dehn twist, we start by considering the following homeomorphism H that maps the unit disk back to itself:

$$H(\theta, t) = \begin{cases} (\theta - \pi, t) & \text{for } t \leq \dfrac{1}{2} \\ (\theta - 2\pi(1-t), t) & \text{for } t > \dfrac{1}{2} \end{cases}$$

We then use this to define the half Dehn twist on a more general surface.

Definition 3.4: Let α be a simple closed curve on a surface. Let R_α be the region enclosed by α, and let ψ_α be a homeomorphism from the unit disk D^2 to R_α. Then the *half Dehn twist* $H_\alpha : S \to S$ about α is the homeomorphism:

$$H_\alpha(x) = \begin{cases} (\psi_\alpha \circ H \circ \psi_\alpha^{-1})(x) & \text{for } x \in R_\alpha \\ x & \text{for } x \notin R_\alpha \end{cases}$$

If R_α is punctured, we must be careful to ensure that the half Dehn twist does indeed map punctures to punctures. As it happens, in this essay we will usually only be interested in the case where R_α contains two punctures and H_α permutes them, as in Figure 3.5 below.

Figure 3.5: The half Dehn twist H_α permuting the two punctures enclosed by α.

We now combine two lemmas to give a necessary and sufficient condition for an element $f \in MCG(S)$ to commute with T_α or H_α.

Lemma 3.6: For any $f \in MCG(S)$ and any simple closed curve α we have the following:

$$T_{f(\alpha)} = f \circ T_\alpha \circ f^{-1}$$
$$H_{f(\alpha)} = f \circ H_\alpha \circ f^{-1}$$

Proof: We will prove the result for the full Dehn twist, the half Dehn twist case being analogous. Recalling the definition given in Definition 3.2:

$$\begin{aligned}
T_{f(\alpha)}(x) &= \begin{cases} (\psi_{f(\alpha)} \circ T \circ \psi_{f(\alpha)}^{-1})(x) & \text{for } x \in N_{f(\alpha)} \\ x & \text{for } x \notin N_{f(\alpha)} \end{cases} \\
&= \begin{cases} ((f \circ \psi_\alpha) \circ T \circ (f \circ \psi_\alpha)^{-1})(x) & \text{for } x \in N_{f(\alpha)} \\ x & \text{for } x \notin N_{f(\alpha)} \end{cases} \\
&= \begin{cases} (f \circ (\psi_\alpha \circ T \circ \psi_\alpha^{-1}) \circ f^{-1})(x) & \text{for } x \in N_{f(\alpha)} \\ x & \text{for } x \notin N_{f(\alpha)} \end{cases} \\
&= f\left(\begin{cases} \psi_\alpha \circ T \circ \psi_\alpha^{-1})(f^{-1}(x)) & \text{for } x \in N_{f(\alpha)} \\ f^{-1}(x) & \text{for } x \notin N_{f(\alpha)} \end{cases}\right) \\
&= f\left(\begin{cases} \psi_\alpha \circ T \circ \psi_\alpha^{-1})(f^{-1}(x)) & \text{for } f^{-1}(x) \in N_\alpha \\ f^{-1}(x) & \text{for } f^{-1}(x) \notin N_\alpha \end{cases}\right) \\
&= f(T_\alpha(f^{-1}(x))) \\
&= (f \circ T_\alpha \circ f^{-1})(x)
\end{aligned}$$

Lemma 3.7: For any simple closed curves α and β, we have:

$$T_\alpha = T_\beta \iff \alpha \text{ is isotopic to } \beta$$
$$H_\alpha = H_\beta \iff \alpha \text{ is isotopic to } \beta$$

We have already justified the reverse implication of this lemma earlier in this chapter. However, a proof of the forwards implication is not as trivial as it sounds. Therefore we refer the interested reader to [5] and proceed to combine these two lemmas into the following:

Corollary 3.8: Let α be any simple closed curve on S, and let D_α denote the full or half Dehn twist about α. For any $f \in MCG(S)$ we have:

$$
\begin{aligned}
fD_\alpha = D_\alpha f \iff & fD_\alpha f^{-1} = D_\alpha \\
\iff & D_{f(\alpha)} = D_\alpha \\
\iff & f(\alpha) \text{ is isotopic to } \alpha
\end{aligned}
$$

In other words, if f is central in $MCG(S)$ then $f(\alpha) = \alpha$ up to isotopy for any simple closed curve α on S. The final result we'll need in order to prove Theorem 1.5 is the following lemma, stated without proof:

Lemma 3.9 (The capping lemma): [5] Let S be a compact subsurface of a surface S' such that S is not homeomorphic to a closed annulus and $S' - S$ is a disk with a single puncture p. Let $MCG(S', p)$ be the subgroup of $MCG(S')$ fixing the puncture p, and let ∂S denote the boundary curve of S. There exists a natural homomorphism η from $MCG(S)$ to $MCG(S', p)$ as follows: for any $f \in MCG(S)$, define $\eta(f)$ to be f on S and the identity on $S' - S$. The following two properties hold:

1. η is surjective.
2. $\mathrm{Ker}(\eta) = \langle T_{\partial S} \rangle$.

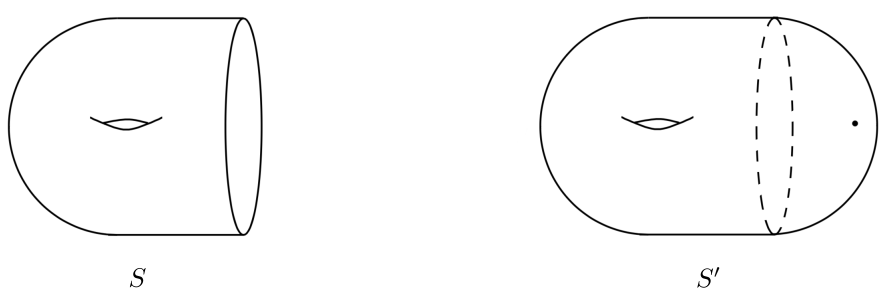

Figure 3.10: [5] A subsurface S of a surface S', such that $S' - S$ is a once-punctured disk.

We close the chapter by noting that Corollary 3.8 and the Alexander method of Theorem 2.7 carry over from $MCG(S)$ to the subgroup $MCG(S, p)$ without any trouble.

4 The Centre of the Braid Group

Let $S_{g,n,b}$ denote the genus-g surface with n marked points and b boundary components, with the shorthand $S_{g,n} = S_{g,n,0}$. Our first aim in this section is to prove the following theorem:

Theorem 4.1: For $n \geq 3$, the centre of $MCG(S_{0,n+1}, p_{n+1})$ is trivial, where p_{n+1} denotes the $(n+1)$th puncture.

We will prove the theorem by defining a set Γ that satisfies the properties in the Alexander method for the $(n+1)$-punctured sphere $S_{0,n+1}$. We'll then use Corollary 3.8 to show that any $f \in Z(MCG(S_{0,n+1}, p_{n+1}))$ must fix each edge of the induced graph Γ^*, including orientation. By the Alexander method, this f must then be isotopic to the identity, and hence it must be that $Z(MCG(S_{0,n+1}, p_{n+1}))$ is indeed trivial. Without loss of generality, assume that the sphere's marked points $\{p_1, \ldots, p_n, p_{n+1}\}$ are evenly spaced along the equator. Let γ_i denote an arc joining p_i to p_{i+1}. Assume all arcs are in minimal position, so that no arcs γ_i and γ_j intersect each other.

Proposition 4.2: $\Gamma = \{\gamma_1, \ldots, \gamma_{n-1}\}$ satisfies the conditions in the Alexander method.

Proof: By construction, the members of Γ are in pairwise minimal position. They are pairwise non-isotopic; every γ_i has different marked points as endpoints. For any distinct i, j, k, at least one of $\gamma_i \cap \gamma_j, \gamma_j \cap \gamma_k$, and $\gamma_k \cap \gamma_i$ is empty. Finally, cutting along every arc leaves us with a once-punctured disk, so Γ fills $S_{0,n+1}$.

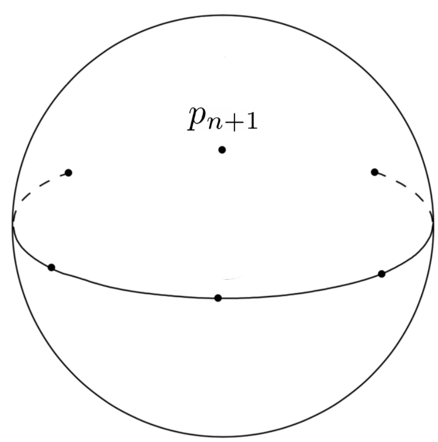

Figure 4.3: $S_{0,6}$ filled by Γ.

Now we we wish to show that $f \in Z(MCG(S_{0,n+1}, p_{n+1}))$ fixes each edge of Γ^*, including orientation.

Lemma 4.4: If $f \in Z(MCG(S_{0,n+1}, p_{n+1}))$ then f fixes the $(n+1)$ punctures.

Proof: Certainly f fixes the $(n+1)$th puncture. Furthermore, the action of f on the punctures induces a surjective homomorphism from $MCG(S_{0,n+1}, p_{n+1})$ to $\mathrm{Sym}(n)$. Any surjective homomorphism takes central elements to central elements, so, recalling that $Z(\mathrm{Sym}(n))$ is trivial for all $n \geq 3$, we see that any central element in $MCG(S_{0,n+1}, p_{n+1})$ must fix each of the $n+1$ punctures.

Lemma 4.5: For any $f \in Z(MCG(S_{0,n+1}, p_{n+1}))$ and any of the arcs $\gamma_i \in \Gamma$, we have $f(\gamma_i) = \gamma_i$.

Proof: Let $i \in \{1, \ldots, n-1\}$ and consider the simple closed curve α_i enclosing punctures p_i and p_{i+1}. Since f is central, Corollary 3.8 tells us that $f(\alpha_i) = \alpha_i$ up to isotopy. From Lemma 4.4 we know too that the punctures p_i and p_{i+1} are fixed by f. These two facts force $f(\mathrm{Int}(\alpha_i)) = \mathrm{Int}(\alpha_i)$. We then note that up to isotopy there is exactly one arc between two marked points on a twice punctured disk such as $\mathrm{Int}(\alpha_i)$. Thus f must map the arc γ_i back to itself.

Corollary 4.6: f fixes each edge of Γ^* including orientation.

Proof: Lemma 4.5 says f maps the arc γ_i back to itself, and Lemma 4.4 says it fixes the endpoints of γ_i. Thus f must fix each edge of Γ^* including orientation.

Proof of Theorem 4.1: We've shown that $f \in Z(MCG(S_{0,n+1}, p_{n+1}))$ and Γ satisfy the conditions set out in the Alexander method. So f is isotopic to the identity, and thus $Z(MCG(S_{0,n+1}, p_{n+1}))$ is trivial.

It remains to relate what we've established about $Z(MCG(S_{0,n+1}, p_{n+1}))$ to the centre of the braid group. Firstly, we claim the following:

Proposition 4.7: Let D_n denote the n-times punctured disk. The braid group B_n is isomorphic to $MCG(D_n)$.

Proof: We only sketch a proof here, referring the reader to [3] for a full discussion. Identify the endpoints $x_1, \ldots, x_n$ of the strands of the braid with the punctures in the disk D_n. We can freely assume these points lie along a diameter of the disk. Let α_i be a simple closed curve containing exactly the points x_i and x_{i+1}. Then the set of half Dehn twists $\{H_{\alpha_1}, \ldots, H_{\alpha_{n-1}}\}$ generates $MCG(B_n)$, and the homomorphism given by $H_{\alpha_i} \mapsto \sigma_i$ is in fact an isomorphism between $MCG(D_n)$ and B_n.

It can be shown that under this isomorphism the braid $\Delta^2 = (\sigma_1\sigma_2\ldots\sigma_{n-1})^n$ corresponds to the full Dehn twist about the boundary of the disk. We are now finally ready to prove Theorem 1.5.

Proof of Theorem 1.5: View D_n as a subsurface of the $(n+1)$-times punctured sphere $S_{0,n+1}$, such that that $S_{0,n+1} - D_n$ is a disk with a single puncture p_{n+1}. Hence the capping lemma of Lemma 3.9 applies; there is a surjective homomorphism $\eta : MCG(D_n) \to MCG(S_{0,n+1}, p_{n+1})$ with kernel $\langle T_{\partial D_n}\rangle = \langle \Delta^2\rangle$. Since we've shown $Z(MCG(S_{0,n+1}, p_{n+1}))$ is trivial for $n \geq 3$, and any surjective homomorphism between groups takes central elements to central elements, it follows that $Z(B_n) = Z(MCG(D_n)) = \mathrm{Ker}(\eta) = \langle \Delta^2\rangle$, proving Theorem 1.5.

The structure of this original proof is based on a suggestion in [5]. However, we believe this suggestion to be flawed, as it relies upon the incorrect claim that the homomorphism η from the capping lemma is a surjective homomorphism between B_n and $MCG(S_{0,n+1})$. We believe the proof given above constitutes a correction.

5 The Nielsen-Thurston Classification

Having found the centre of the braid group on n strands by applying ideas from mapping class group theory, we might next wonder whether the same approach can tell us anything about the centralizer of a braid β in B_n. Indeed, this does turn out to be the case, and in this section we explore some of the results that will help us reason about the structure of a braid's centralizer. The main such result is the Nielsen-Thurston classification, which says that all mapping classes fall into at least one of three categories; periodic, reducible and pseudo-Anosov.

However, we must first provide a few definitions. Where appropriate, we have restricted these definitions to the domain we're interested in.

Definition 5.1: A *reduction system* for an element $f \in MCG(S)$ is a non-empty set Γ of isotopy classes of essential simple closed curves such that:

1. $f(\Gamma) = \Gamma$.
2. For every $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \cap \gamma_2 = \emptyset$ when γ_1, γ_2 in minimal position.

A reduction system is *maximal* if it cannot be extended to include any further isotopy classes without violating (1) or (2). Note that it's possible for f to have more than one maximal reduction system. Thus we can define the *canonical reduction system* for f to be the intersection of all maximal reduction systems. Unlike a reduction system, a canonical reduction system can be empty.

We now define the first two categories into which all elements of $MCG(S)$ fall. In what follows, remember that we need only consider the homeomorphism f up to isotopy.

Definition 5.2: An element $f \in MCG(S)$ is *reducible* if it has a reduction system. The name *reducible* comes from the fact that in order to further understand f we can cut S along (representatives of) the elements of Γ and consider the effect of f on each of the components of the new surface.

Definition 5.3: An element $f \in MCG(S)$ is *periodic* if it has finite order.

As we will see, a mapping class f can be both periodic and reducible, but often those that are non-periodic reducible have interesting properties, such as the following, which we give without proof:

Theorem 5.4: [9] If $f \in MCG(S)$ non-periodic reducible, then its canonical reduction system is non-empty.

Next, we introduce notions that will be used to define the third and final category of mapping classes.

Definition 5.5: [11] Let S be a surface. A *chart* on S is a pair (U, ϕ), where U is an open subset of S and $\phi : U \to \tilde{U}$ is a homeomorphism from U to an open subset $\tilde{U} = \phi(U) \subseteq \mathbb{R}^n$.

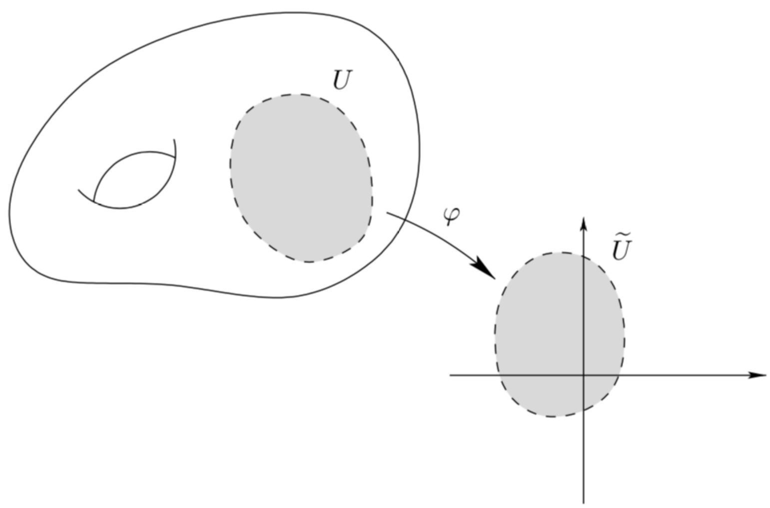

Figure 5.6: [11] A chart (U, ϕ) on the torus.

Definition 5.7: [11] Let S be a surface. If $(U, \phi), (V, \psi)$ are two charts such that $U \cap V \neq \emptyset$, then the *transition map* is defined to be the homeomorphism $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$.

Definition 5.8: [5] A *singular foliation* $\mathcal{F}$ on a closed surface S is a decomposition of S into a disjoint union of subsets of S, called the *leaves* of $\mathcal{F}$, and a finite set of points of S, called *singular points* of $\mathcal{F}$, such that the following two conditions hold:

1. For each non-singular point $p \in S$, there is a smooth chart from a neighbourhood of p to $\mathbb{R}^2$ that takes leaves to horizontal line segments, and the transition map between any two of these charts are smooth maps taking horizontal lines to horizontal lines.

2. For each singular point $p \in S$, there is a smooth chart from a neighbourhood of p to $\mathbb{R}^2$ that takes leaves to the level sets of a k-pronged saddle for some $k \geq 3$, as in Figure 5.9. If p is in fact a puncture or marked point of S, we allow $k = 1$ as well (but a puncture of S need not be a singularity of $\mathcal{F}$ at all).

(a) (b)

Figure 5.9: [5] (a) A three-pronged saddle. (b) A one-pronged saddle at a puncture.

Definition 5.10: [5] Let $\mathcal{F}$ be a singular foliation on a surface S. A smooth arc α in S is *transverse* to $\mathcal{F}$ if α misses the singular points of $\mathcal{F}$ and is transverse to each leaf of $\mathcal{F}$ at each point in its interior.

Definition 5.11: [5] Let α and β be smooth arcs transverse to a foliation $\mathcal{F}$. An isotopy $H : I \times I \to S$ from α to β is *leaf preserving* if:

1. For all $t \in [0, 1]$, the arc $H(I \times \{t\})$ is transverse to $\mathcal{F}$.
2. $H(\{0\} \times I)$ and $H(\{1\} \times I)$ are each contained in a single leaf.

Definition 5.12: [5] Let A denote the set of all smooth arcs transverse to a foliation $\mathcal{F}$. A function $\mu : A \to \mathbb{R}_{>0}$ is a *transverse measure* on $\mathcal{F}$ if:

1. For any smooth arcs α and β isotopic under a leaf-preserving isotopy, $\mu(\alpha) = \mu(\beta)$.
2. μ is absolutely continuous with respect to Lebesgue measure.

Definition 5.13: [5] A *measured foliation* $(\mathcal{F}, \mu)$ on a surface S is a foliation $\mathcal{F}$ of S equipped with a transverse measure μ.

Definition 5.14: Two measured foliations are *transverse* if their leaves are transverse away from the singularities. Note that this means transverse measured foliations must have the same set of singularities, and have the same number of prongs at each singularity.

We are now at a point where we can say what it means for a mapping class to be pseudo-Anosov.

Definition 5.15: [5] An element $g \in MCG(S)$ is *pseudo-Anosov* if there exists $\lambda > 1$ and a pair of transverse measured foliations $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$ on S such that:

$$g((\mathcal{F}_u, \mu_u)) \coloneqq (g(\mathcal{F}_u), g \circ \mu_u) = (\mathcal{F}_u, \lambda\mu_u)$$
$$g((\mathcal{F}_s, \mu_s)) \coloneqq (g(\mathcal{F}_s), g \circ \mu_s) = (\mathcal{F}_s, \frac{1}{\lambda}\mu_s)$$

The measured foliations $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$ are called the *unstable foliation* and the *stable foliation*, respectively. We call λ the *stretch factor* of g.

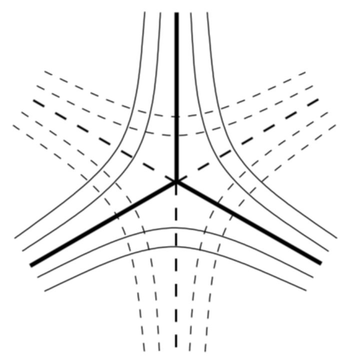

Figure 5.16: [6] A singularity of the stable (solid line) and unstable (dotted line) foliation.

In other words, a pseudo-Anosov homeomorphism on S has the effect of 'shrinking' the stable leaves, while 'stretching' the unstable leaves. For example, suppose we have a pseudo-Anosov g that fixes a singularity p of $\mathcal{F}_s$. If we take a point x on a stable leaf that emanates from p then $g^n(x)$ approaches p as n goes to infinity. Likewise if y is on an unstable leaf emanating from p then $g^n(y)$ becomes further from p as n tends to infinity.

We develop here a few further results pertaining specifically to pseudo-Anosov maps.

Definition 5.17: [6] Two measured foliations are *Whitehead equivalent* if one can be transformed to the other via a sequence of *Whitehead equivalences*, as shown in Figure 5.18.

The Whitehead equivalence classes of measured foliations on a surface S form a topological space which we denote by $\mathcal{MF}(S)$. However, we will rarely acknowledge the fact that we

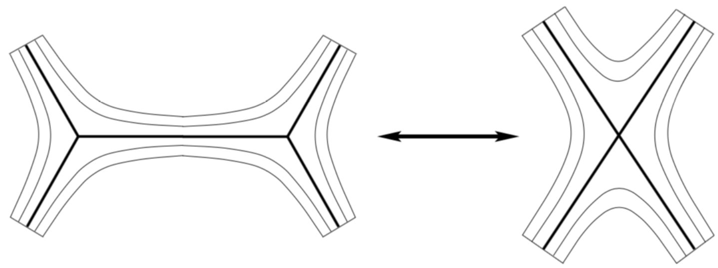

Figure 5.18: [6] A Whitehead equivalence.

are working over equivalence classes, instead just calling $\mathcal{MF}(S)$ the space of measured foliations on S.

Definition 5.19: The space $\mathbb{R}_{>0}$ acts naturally on the group $\mathcal{MF}(S)$ by multiplying the transverse measures by a constant. Therefore we can define the quotient space $\mathcal{PMF}(S) = \mathcal{MF}(S)/\mathbb{R}_{>0}$ of *projective measured foliations* on S, in which we consider measured foliations only up to constant scalings of the measure.

Below are two fairly straightforward but useful results about this space.

Proposition 5.20: All powers of a pseudo-Anosov mapping class have the same stable and unstable projective foliations.

Proof: This follows from the definitions; let g be a pseudo-Anosov element of $MCG(S)$, and note that $g^n((\mathcal{F}_u, \mu_u)) \coloneqq (g^n(\mathcal{F}_u), g^n \circ \mu_u) = (\mathcal{F}_u, \lambda^n \mu_u)$. The foliations $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_u, \lambda^n \mu_u)$ differ only in a scaling of the measure, and so as projective foliations they are equal. The case of the stable foliation is analogous.

Proposition 5.21: [5] The group $MCG(S)$ acts naturally on the space $\mathcal{PMF}(S)$.

Proof: In fact, there is a natural action of $MCG(S)$ on the space $\mathcal{MF}(S)$, which carries over to the quotient space $\mathcal{PMF}(S)$. Namely, for $g \in MCG(S)$ and $(\mathcal{F}, \mu) \in \mathcal{MF}(S)$ we define the action $g.(\mathcal{F}, \mu) = (g(\mathcal{F}), g^*(\mu))$, where $g^*(\mu)$ is defined by $g^*(\mu)(\gamma) = \mu(g^{-1}(\gamma))$ for any smooth arc γ transverse to $g(\mathcal{F})$. Certainly $g(\mathcal{F})$ is again a foliation of S, and one can check too that $g^*(\mu)$ satisfies all the requirements of being a transverse measure on $g(\mathcal{F})$.

Finally, we close this chapter by stating the Nielsen-Thurston classification.

Theorem 5.22 (The Nielsen-Thurston Classification): Every element of $MCG(S_{g,n,b})$ is periodic, reducible, or pseudo-Anosov. Mapping classes can be both periodic and reducible, but pseudo-Anosov mapping classes are neither periodic nor reducible.

6 The Centralizer of a Braid

Equipped with Thurston's classification theorem and a little pseudo-Anosov theory, we may now characterise the centralizer of any braid β in B_n, following the work in [8]. The characterisation is given in terms of mixed braid groups, defined as follows:

Definition 6.1: [8] Let $X = \{x_1, \ldots, x_n\}$ denote the endpoints of the braids in B_n. Given a partition $\mathcal{P}$ of X, the *mixed braid group* $B_{\mathcal{P}}$ consists of those braids whose associated permutation preserves each subset of $\mathcal{P}$.

We proceed to describe the centralizer of β by considering in turn the cases where β is pseudo-Anosov, periodic, or non-periodic reducible. However, we must first discuss a small caveat. We said in chapter one that the braid group is torsion free, so no non-trivial element has finite order, and yet we also said that an element $f \in MCG(S)$ is periodic if and only if it has finite order. Since B_n is isomorphic to $MCG(D_n)$, aren't these two ideas at odds with each other? How can a braid be periodic if no braid has finite order?

The answer is that we actually define a braid to be pseudo-Anosov, periodic or reducible according to whether its image in the quotient group $B_n/Z(B_n)$ is pseudo-Anosov, periodic or reducible respectively. Recall from our earlier chapters that $Z(B_n) = \langle \Delta^2 \rangle = \text{Ker}(\eta)$, where η is the surjective homomorphism from B_n to $MCG(S_{0,n+1}, p)$. Thus via the first isomorphism theorem we see that the natural projection $B_n \to B_n/\langle \Delta^2 \rangle$ is exactly the function η from the capping lemma.

With that out of the way, we can begin the characterisation. We proceed recursively; there are two base cases, the first of which we will prove in full.

Theorem 6.2: If $\beta \in B_n$ is pseudo-Anosov, $Z(\beta)$ is isomorphic to $\mathbb{Z}^2$.

In fact, as the paragraphs above hint at, we will initially be working in the quotient group $B_n/\langle \Delta^2 \rangle$, wherein we will prove a slightly more specific theorem. It will at times help to visualise elements β of $B_n/\langle \Delta^2 \rangle$ as elements of $MCG(S_{0,n+1}, p)$. The theorem we will prove is the following:

Theorem 6.3: [8] If $\beta \in B_n/\langle \Delta^2 \rangle$ is pseudo-Anosov, then the centralizer of β is Abelian. It is generated by some pseudo-Anosov element which has the same stable and unstable projective measured foliation as β, and possibly one periodic element ρ.

We begin with two lemmas. Throughout the following, let β denote some pseudo-Anosov element of $B_n/\langle \Delta^2 \rangle$.

Lemma 6.4: If $\alpha \in B_n/\langle\Delta^2\rangle$ is non-periodic reducible then α and β don't commute. In other words, all elements of $Z(\beta)$ are either pseudo-Anosov or periodic.

Proof: By Theorem 5.4, any such α must have non-empty canonical reduction system Γ. Note that the canonical reduction system of $\beta^{-1}\alpha\beta$ is $\beta(\Gamma)$. Now suppose α and β did commute; then we'd have $\alpha = \beta^{-1}\alpha\beta$. Thus the reduction systems for α and for $\beta^{-1}\alpha\beta$ would be identical; that is, we'd have $\beta(\Gamma) = \Gamma$. But this would imply that β has a reduction system, and hence is reducible, contradicting the fact that pseudo-Anosov mapping classes are neither periodic nor reducible.

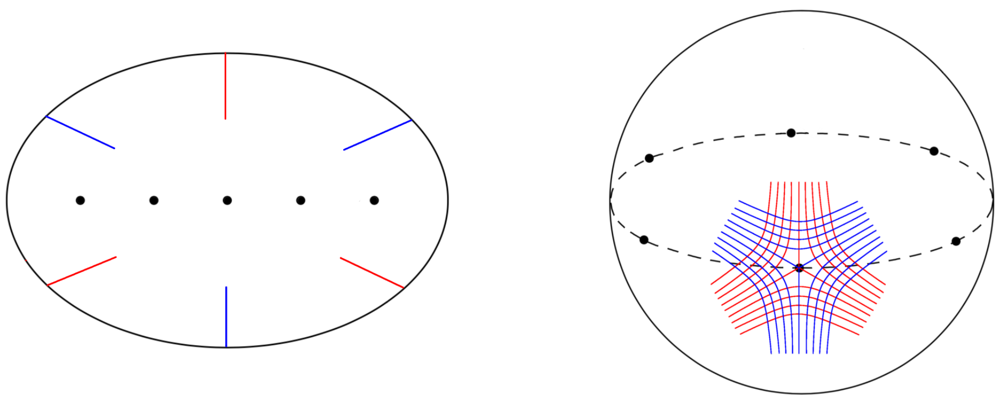

Figure 6.5: The natural projection η taking D_5 to $S_{0,6}$. Shown in red and blue respectively are the stable and unstable foliations of β in a neighbourhood of the point p that comes from collapsing the boundary of D_5. The preimages of the singular leaves in this neighbourhood are drawn on D_5. We use a 3-pronged saddle for illustrative purposes only; p may have a different number of prongs, or may not be a singularity at all.

At this point we view our $\beta \in B_n/\langle\Delta^2\rangle$ as an element of $MCG(S_{0,n+1}, p)$, and consider what happens at the distinguished puncture p fixed by β. By the definition of transverse foliations we know that singular leaves of the stable and unstable foliation of β emanate alternately from p. We also have the following lemma:

Lemma 6.6: If $\alpha \in Z(\beta)$ then the action of α must preserve the stable and unstable projective foliations of β.

Proof: Recall from Proposition 5.21 that $B_n/\langle\Delta^2\rangle$ acts naturally on $\mathcal{PMF}(B_n/\langle\Delta^2\rangle)$; indeed, this is what we mean by the action of α. Let $\mathrm{Fix}(\beta) = \{\mathcal{F} \in \mathcal{PMF}(B_n/\langle\Delta^2\rangle) \mid \beta\mathcal{F} = \mathcal{F}\}$. We use the well-known group theoretic result that if α and β commute then

$\alpha(\text{Fix}(\beta)) = \text{Fix}(\beta)$, and we claim that $\text{Fix}(\beta)$ is the set of size two consisting of the stable and unstable foliations $\mathcal{F}_s$ and $\mathcal{F}_u$ of β. Intuitively we can see why this might be the case; given any foliation $\mathcal{F} \in \mathcal{PMF}(B_n/\langle\Delta^2\rangle)$, the effect of the action of β is to shrink $\mathcal{F}$ along the leaves of $\mathcal{F}_s$ and stretch $\mathcal{F}$ along the leaves of $\mathcal{F}_u$. It would not be surprising, then, if the only fixed points under this action were the stable and unstable foliations themselves. However, a rigorous proof would likely require another chapter altogether, so instead we take this lemma on faith from [8].

Corollary 6.7: If $\alpha \in Z(\beta)$ then the action of α can only induce a cyclic permutation of the singular leaves at p, taking stable leaves to stable leaves and unstable leaves to unstable leaves throughout.

We now prove the following key property of elements in the centralizer $Z(\beta)$:

Lemma 6.8: Any $\alpha \in Z(\beta)$ is uniquely determined by two data λ_α and π_α.

Proof: We denote by λ_α the stretch factor by which the action of α multiplies the measure of the unstable measured foliation of β. Note that if α is periodic then $\lambda_\alpha = 1$, else if α is pseudo-Anosov then $\lambda_\alpha \in \mathbb{R}_{>0}\backslash\{1\}$. We denote by π_α the permutation of the singular leaves of p induced by the action of α. Suppose now that $\lambda_\alpha = \lambda_\gamma$ and $\pi_\alpha = \pi_\gamma$ for two elements $\alpha, \gamma \in Z(\beta)$. Note this means the inverse γ^{-1} of γ has $\lambda_{\gamma^{-1}} = \lambda_\gamma^{-1}$ and $\pi_{\gamma^{-1}} = \pi_\gamma^{-1}$. Therefore, $\alpha\gamma^{-1}$ has $\lambda_{\alpha\gamma^{-1}} = 1$ and $\pi_{\alpha\gamma^{-1}} = id$, the trivial permutation. It can be shown that this suffices to ensure $\alpha\gamma^{-1}$ is the identity in $B_n/\langle\Delta^2\rangle$, so $\alpha = \gamma$.

Corollary 6.9: The periodic elements of $Z(\beta)$ form a subgroup that is trivial or isomorphic to $\mathbb{Z}/k\mathbb{Z}$ for some k dividing the number of singular leaves of p.

Proof: Let K be the number of singular leaves at p in either the stable or unstable foliation. For any cyclic permutation π, let $|\pi|$ denote the number of 'clicks' through which it permutes a particular leaf. The lemma above implies that there is an injective homomorphism between $Z(\beta)$ and $\mathbb{Z}/K\mathbb{Z} \times \mathbb{R}_{>0}$, given by $\alpha \mapsto (|\pi_\alpha|, \lambda_\alpha)$. Now, the periodic elements in $Z(\beta)$ have stretch factor 1, so their image in $\mathbb{Z}/K\mathbb{Z} \times \mathbb{R}_{>0}$ is in fact a subgroup of $\mathbb{Z}/K\mathbb{Z} \times \{1\} \cong \mathbb{Z}/K\mathbb{Z}$. The only such subgroups are of the form $\mathbb{Z}/k\mathbb{Z}$, where k divides K. The trivial case occurs when $k = 1$.

From this point onwards we let ρ denote the generator of this subgroup with minimal $|\pi_\rho|$. We aim to show that ρ is one of two generators of $Z(\beta)$, along with some pseudo-Anosov element τ. We begin by proving the following:

Lemma 6.10: ρ commutes with every $\alpha \in Z(\beta)$.

Proof: Consider $[\rho, \alpha] = \rho^{-1}\alpha^{-1}\rho\alpha$. Note that $\lambda_{[\rho,\alpha]} = \lambda_\rho^{-1}\lambda_\alpha^{-1}\lambda_\rho\lambda_\alpha = 1$, and by the same method we can show $\pi_{[\rho,\alpha]} = id$. Thus the commutator is trivial.

Lemma 6.11: There is a pseudo-Anosov element $\tau \in Z(\beta)$ with stretch factor λ_τ such that the stretch factor λ_α of every $\alpha \in Z(\beta)$ is some power of λ_τ.

Proof: Let $\psi : Z(\beta) \to \mathbb{R}_{>0}$ be the homomorphism given by $\psi(\alpha) = \lambda_\alpha$. We showed in Proposition 5.20 that all pseudo-Anosov elements in $Z(\beta)$ have the same stable and unstable projective foliations. Combining this with a theorem in [9] which states that the stretch factor λ_α of a given foliation can have only discretely many values, we see that $\mathrm{Im}(\psi)$ must be a cyclic subgroup of $\mathbb{R}_{>0}$. Let τ be an element of $Z(\beta)$ whose image $\psi(\tau) = \lambda_\tau$ generates this subgroup, so that the stretch factor of every $\alpha \in Z(\beta)$ is a power of λ_τ. Note that λ_τ cannot equal 1 because $\beta \in Z(\beta)$, so τ must be pseudo-Anosov.

Corollary 6.12: ρ and τ generate $Z(\beta)$.

Proof: Take any $\alpha \in Z(\beta)$. We know that $\lambda_\alpha = \lambda_\tau^n$ for some integer m, and therefore $\alpha\tau^{-n}$ has stretch factor 1, i.e. it's a power ρ^m of ρ. So $\alpha = \rho^m\tau^n$ as required.

This final corollary proves Theorem 6.3, but we now need to use this theorem about the quotient group $B_n/\langle\Delta^2\rangle$ to prove Theorem 6.2 concerning B_n itself. To avoid confusion, we will change our notation, writing $\bar{\beta}, \bar{\rho}, \bar{\tau}$ for elements $\beta\langle\Delta^2\rangle, \rho\langle\Delta^2\rangle, \tau\langle\Delta^2\rangle$ of the quotient group $B_n/\langle\Delta^2\rangle$, where we previously wrote β, ρ and τ.

In [8], the authors prove Theorem 6.2 in part by using the following claim: since $\langle\Delta^2\rangle$ is the centre of B_n, it is contained in the centralizer of any element $\alpha \in B_n$, hence $Z(\alpha) \leq B_n$ is just the preimage of the centralizer $Z(\bar{\alpha}) \leq B_n/Z(B_n) = B_n/\langle\Delta^2\rangle$. However, generally we don't believe this claim to be true; for example, the group U_3 of unitriangular 3×3 matrices is non-Abelian, so there is a matrix $M \in U_3$ with $Z(M) \neq U_3$. But the quotient $U_3/Z(U_3)$ is Abelian, so $Z(\bar{M}) = U_3/Z(U_3)$, and the preimage of the whole quotient group is the whole group U_3. This contradicts the claim used in [8]. Therefore, we will take a different approach.

Lemma 6.13: Δ^2 is a power of ρ.

Proof: In the quotient group, $\bar{\rho}$ cyclically permutes the singular leaves at p. The only elements of B_n that can project to such a $\bar{\rho}$ are the rotations of the punctured disk D_n by $\frac{2\pi r}{K}$, where K is the number of singular leaves in the stable (equivalently, unstable) foliation, and r is some integer. We claim that there's an integer m such that ρ^m is the full twist Δ^2.

Indeed, recall that in Corollary 6.9 we defined $\bar{\rho}$ to be the 'smallest' generator of a subgroup $\mathbb{Z}/k\mathbb{Z} \leq \mathbb{Z}/K\mathbb{Z}$, in the sense that the number of 'clicks' about which it rotated any given leaf was minimal over all generators. So suppose for a contradiction that the rotation $\frac{2\pi rm}{K}$ corresponding to ρ^m doesn't equal 2π for any m. Let M be the least integer such that $\frac{2\pi rM}{K} > 2\pi$. Modulo 2π we certainly have $\frac{2\pi rM}{K} \leq \frac{2\pi r}{K}$. However, if $\frac{2\pi rM}{K} < \frac{2\pi r}{K}$ then we contradict the minimality of $\bar{\rho}$, and if $\frac{2\pi rM}{K} = \frac{2\pi r}{K}$ then no power of $\bar{\rho}$ is trivial, contradicting the fact that $\bar{\rho}$ is periodic. So our assumption that the rotation corresponding to ρ^m doesn't equal 2π for any m must have been wrong.

Proposition 6.14: ρ and τ generate $Z(\beta)$.

Proof: Any $\alpha \in Z(\beta)$ projects to $\alpha\langle\Delta^2\rangle \in Z(\bar{\beta})$, which can be written $\tau^i\rho^j\langle\Delta^2\rangle$ because $\bar{\rho}$ and $\bar{\tau}$ commute and generate $Z(\bar{\beta})$. So $\alpha = \tau^i\rho^j(\Delta^2)^k$ for some integer k. By Lemma 6.13, Δ^2 is a power of ρ, so $\alpha = \tau^i\rho^l$ for some l.

Lemma 6.15: ρ and τ commute.

Proof: We showed in Lemma 6.10 that the commutator $[\bar{\rho}, \bar{\tau}]$ is trivial, thus in the braid group itself we have $[\rho, \tau] \in \langle\Delta^2\rangle$. So $[\rho, \tau] = (\Delta^2)^k$ for some k. Recall that $\Delta^2 = (\sigma_1\sigma_2 \ldots \sigma_{n-1})^n$, and thus has algebraic crossing number $n(n-1)$. Therefore $(\Delta^2)^k$ has crossing number $kn(n-1)$. If two braids are equal then their crossing numbers must be equal. However, the algebraic crossing number of $[\rho, \tau] = \rho\tau\rho^{-1}\tau^{-1}$ is zero. Thus we have $kn(n-1) = 0$, so $k = 0$, so ρ and τ commute.

Since the braid group is torsion-free, no further relations can hold. Thus $\mathbb{Z}^2 \cong \langle\rho, \tau \mid \rho\tau = \tau\rho\rangle = Z(\beta)$. This concludes our proof of the first base case in the characterisation of the centralizer of braids. The second base case concerns periodic braids, and we state it without proof.

Theorem 6.16: [8] If $\beta \in B_n$ is periodic, $Z(\beta)$ is equal to B_n, or is isomorphic to a mixed braid group.

Stating the characterisation for the case where $\beta \in B_n$ is non-periodic reducible requires us to define the notion of a tubular braid and its corresponding interior braids. We briefly do so here, leaving a full exposition to [8]. We can freely assume the endpoints of β (which we shall interchangeably call punctures) lie along a diameter of the disk D^2, and we need only work with β up to conjugacy, because for $\alpha, \beta \in B_n$ we have $Z(\alpha^{-1}\beta\alpha) = \alpha^{-1}Z(\beta)\alpha$.

Now, since β is non-periodic reducible, its canonical reduction system is non-empty. For some suitable conjugate of β, the set of outermost curves in the canonical reduction system is a family of circles. Some punctures may not lie within any such curve. Define $R(\beta)$ to be the union of this set of outermost curves and the set consisting of a small circle around each unenclosed puncture. It can be shown that $R(\beta)$ has the following properties:

1. The interiors of the curves in $R(\beta)$ are pairwise disjoint.
2. Every puncture belongs to the interior of exactly one curve in $R(\beta)$.
3. β fixes $R(\beta)$, though possibly permutes the curves' interiors.

We note that $1 < |R(\beta)| < n$. The lower bound comes from the fact that a reduction system consists of *essential* curves; that is, no curve in $R(\beta)$ can be isotopic to the boundary of the disk. The upper bound comes from the fact if $|R(\beta)| = n$ then it implies each of the n curves in $R(\beta)$ is a small circle around one puncture. But given how we constructed $R(\beta)$ this would mean the canonical reduction system of β was empty, which cannot be the case.

Definition 6.17: By collapsing each curve in $R(\beta)$ to a point, β and $R(\beta)$ together induce a *tubular braid* $\hat{\beta} \in B_{|R(\beta)|}$.

However, collapsing these curves to a point is a poor way of visualising $\hat{\beta}$; we're better off visualising each strand of $\hat{\beta}$ as a hollow tube, because that is consistent with the following:

Definition 6.18: Each strand i of a tubular braid $\hat{\beta}$ contains within it an *interior braid* $\beta_i \in B_{m_i}$, where m_i is the number of punctures enclosed in the curve at the endpoints of strand i.

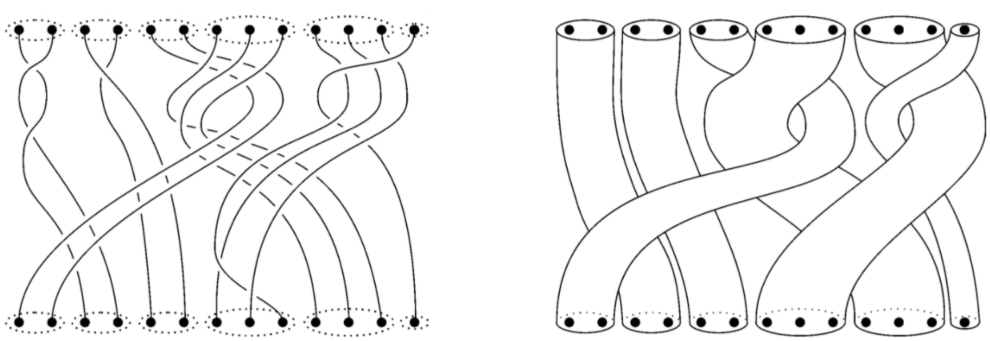

Figure 6.19: [8] A reducible braid in B_{13} and its corresponding tubular braid.

The fact that $1 < |R(\beta)| < n$ implies that each m_i is strictly less than n. Note also that $m_1 + \ldots + m_{|R(\beta)|} = n$. We can now characterise the centralizer of $\beta \in B_n$ in the final case:

Theorem 6.20: If $\beta \in B_n$ is non-periodic reducible, $Z(\beta)$ is isomorphic to

$$(Z(\beta_1) \times \cdots \times Z(\beta_{|R(\beta)|})) \rtimes Z_0(\hat{\beta})$$

where $Z_0(\hat{\beta})$ is a subgroup of $Z(\hat{\beta})$, isomorphic either to $\mathbb{Z}^2$ or to a mixed braid group on $m \leq |R(\beta)| < n$ strands.

Of course, each of $\beta_{i_1}, \ldots, \beta_{i_t}$ will be either pseudo-Anosov, periodic or non-periodic reducible, hence the recursive structure of the characterisation.

We end this chapter with a small detour; we have stated a few times that the braid group is torsion-free, but we can now provide a short proof. We include it at this point specifically because it relies on the same theorem that the proof of Theorem 6.16 (the periodic case) uses; namely, all periodic braids are classified up to conjugacy. Below, assume $n \geq 3$.

Definition 6.21: Suppose the endpoints of braids in B_n are arranged in a ring on the top and bottom disks. Define δ_n to be the braid that rotates the top disk by $\frac{2\pi}{n}$. Similarly, now suppose $n-1$ of the endpoints of braids in B_n are arranged in a ring, with the nth endpoint in the centre. Define γ_n to be the braid that rotates the top disk by $\frac{2\pi}{n-1}$.

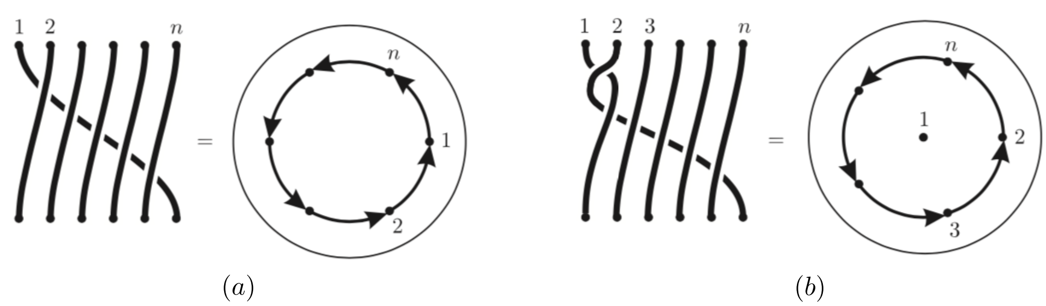

Figure 6.22: [7] (a) The braid $\delta_n = \sigma_1 \ldots \sigma_{n-1}$. (b) The braid $\gamma_n = \sigma_1(\sigma_1 \ldots \sigma_{n-1})$.

Theorem 6.23: [4] Any periodic $\beta \in B_n$ is conjugate to either δ_n^k or γ_n^k for some k.

Now, if a braid is to have finite order in B_n then it must also have finite order in $B_n/\langle \Delta^2 \rangle$ - i.e. it must be a periodic braid. But we've just characterised all such braids up to conjugacy, and we can prove that no such braid has zero algebraic crossing number, thus no power of it can be trivial. First note that δ_n and γ_n have crossing numbers $n-1$ and n respectively. Conjugation leaves the crossing number invariant; if a braid β has algebraic crossing number b then β^{-1} has crossing number $-b$, so that $\beta\delta_n^k\beta^{-1}$ and $\beta\gamma_n^k\beta^{-1}$ have crossing numbers $k(n-1)$ and kn respectively. These are both non-zero for all $k \neq 0$.

7 Abelian Subgroups of The Braid Group

In this final chapter, we seek to use the characterization of centralizers of braids to give an upper bound on the size of any Abelian subgroup of the braid group. To the best of our knowledge, this is material that does not currently appear anywhere in the literature. In [2] Birman, Lubotzky and McCarthy prove an upper bound for Abelian subgroups of the mapping class group. However, this bound is on the torsion free rank of the Abelian subgroups - that is, it disregards Abelian subgroups containing periodic elements. We aim here to do better, giving an upper bound that is linear in n for any Abelian subgroup of B_n. As usual, we assume $n \geq 3$.

Since B_n is torsion free, any *finitely generated* Abelian $A \leq B_n$ will be of the form $\mathbb{Z}^p$ for some p. The proof for our upper bound will be inductive, and the following will be our inductive hypothesis:

Inductive hypothesis: any Abelian subgroup A of B_n is (isomorphic to) a subgroup of $\mathbb{Z}^{43(n-1)}$.

Of course, if we can prove this upper bound, then it will follow that any such Abelian A must be finitely generated. Throughout this chapter, let A be an Abelian subgroup of B_n. The following simple result, which is not specific to B_n, motivates the structure of our proof:

Proposition 7.1: Any Abelian subgroup $A \leq B_n$ satisfies $A \leq \bigcap_{\beta \in A} Z(\beta)$.

Proof: For any $\alpha \in A$, we must have $\alpha\beta = \beta\alpha$ for all $\beta \in A$. In other words, $\alpha \in Z(\beta)$ for all $\beta \in A$.

We want to examine three cases. The first is where A contains a pseudo-Anosov braid, and the second is where A consists *entirely* of periodic braids. These will be our two base cases. The third case, where A contains a non-periodic reducible braid, is recursive. It is worth stating that this does indeed cover all possible scenarios.

7.1 The Pseudo-Anosov Case

In fact, there is very little to prove here.

Proposition 7.2: If some pseudo-Anosov $\beta \in A$ then $A \leq \mathbb{Z}^2$.

Proof: Follows from Proposition 7.1 and the fact that $Z(\beta) \cong \mathbb{Z}^2$.

Certainly it's true that $\mathbb{Z}^2 \leq \mathbb{Z}^{43(n-1)}$, so we have proved our inductive hypothesis for this base case.

7.2 The Periodic Case

One might expect that we would follow the same strategy again; that is, to consider the case when A contains a periodic braid β and look at the centralizer of β, using Proposition 7.1 to bound A. However, this approach runs into trouble because the centralizer $Z(\beta)$ may well be isomorphic to B_n, in which case we have gained no new information. Instead, we suppose A consists *entirely* of periodic braids, and bound A using a different idea.

Definition 7.3: For a surface $S = S_{g,n,b}$, the *Euler characteristic* is defined as $\chi(S) = 2 - 2g - (n + b)$.

The following theorem is a generalisation of a result in [5] attributed to Hurwitz:

Theorem 7.4 (Hurwitz' Theorem): For a surface $S = S_{g,n,b}$ with $\chi(S) < 0$, the order of any finite subgroup of $MCG(S)$ is at most $42\,|\chi(S)|$.

The proof of this seemingly strange theorem has its roots in hyperbolic geometry, a topic which we have not touched on in this essay. We instead use it without proof, in a rather sledgehammer-like fashion, to put a bound on A:

Proposition 7.5: If $A \leq B_n$ consists entirely of periodic elements then $A \leq \mathbb{Z}^{43(n-1)}$.

Proof: Once again we shall employ our trick of passing into the quotient group $B_n/\langle\Delta^2\rangle \cong MCG(S_{0,n+1}, p) \leq MCG(S_{0,n+1})$. By definition, any periodic braid β has finite order in this quotient group, and so the image $A\langle\Delta^2\rangle$ is a finite subgroup. Since $\chi(S_{0,n+1}) = 2 - 0 - (n + 1 + 0) = 1 - n < 0$, we can apply Theorem 7.4, which tells us that $\left|A\langle\Delta^2\rangle\right| \leq 42(n-1)$. In particular, this means $A\langle\Delta^2\rangle$ has at most $42(n-1)$ generators. Now, the effect of passing back into the original group B_n is to add at most one extra generator Δ^2 to this group. Thus our Abelian A certainly has at most $43(n-1)$ generators, and so is a subgroup of $\mathbb{Z}^{43(n-1)}$. We have therefore proved our inductive hypothesis for our two base cases.

7.3 The Non-Periodic Reducible Case

We now look at the recursive case, where A contains a non-periodic reducible braid β. Letting $r = |R(\beta)|$, recall $Z(\beta) = (Z(\beta_1) \times \cdots \times Z(\beta_r)) \rtimes Z_0(\hat{\beta})$, where:

1. $\beta_i \in B_{m_i}$ are interior braids induced by β.
2. $\sum_{i=1}^{r} m_i = n$, with each m_i strictly less than n.
3. $\hat{\beta} \in B_r$ is the tubular braid induced by β, and is pseudo-Anosov or periodic.
4. If $\hat{\beta}$ is pseudo-Anosov then $Z_0(\hat{\beta})$ is isomorphic to $\mathbb{Z}^2$, else if $\hat{\beta}$ is periodic then $Z_0(\hat{\beta})$ is isomorphic to a mixed braid group on $m \leq r$ strands.

We begin by clarifying that the recursion terminates, as we would hope; an interior braid β_i may again be non-periodic reducible, but it will have strictly fewer than n strands. Eventually the number of strands will reach two or fewer, and any braid on two or fewer strands is periodic. Next we prove results about semi-direct products, by changing our viewpoint to that of split exact sequences.

Lemma 7.6: Let A be any subgroup of $G = N \rtimes K$. The following diagram is well-defined; that is, $\psi^{-1}(A)$ and $\phi(A)$ are subgroups of N and K respectively, and the bottom sequence is exact and splits.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \xrightarrow{\psi} & G & \underset{\sigma}{\overset{\phi}{\rightleftarrows}} & K & \longrightarrow & 1 \\
 & & \vee\!| & & \vee\!| & & \vee\!| & & \\
1 & \longrightarrow & \psi^{-1}(A) & \xrightarrow{\psi|_{\psi^{-1}(A)}} & A & \underset{\sigma|_{\phi(A)}}{\overset{\phi|_A}{\rightleftarrows}} & \phi(A) & \longrightarrow & 1
\end{array}$$

Proof: It is clear that $\phi(A) \leq K$, because ϕ is a homomorphism. We can show $\psi^{-1}(A)$ satisfies the subgroup lemma; if $x, y \in \psi^{-1}(A)$ then $\psi(xy^{-1}) = \psi(x)\psi(y)^{-1} \in A$. Now we check that the bottom sequence is also exact and splits. Indeed, since ψ is injective, any restriction of it is injective too, and any restriction $\phi|_A$ is surjective onto its image $\phi(A)$. Exactness in the top sequence says that $\mathrm{Im}(\psi) = \mathrm{Ker}(\phi)$, and so we see that $\mathrm{Im}(\psi|_{\psi^{-1}(A)}) = \mathrm{Im}(\psi) \cap A = \mathrm{Ker}(\phi) \cap A = \mathrm{Ker}(\phi|_A)$. Finally, as the top sequence splits, $\phi \circ \sigma = id_K$, so certainly $\phi|_A \circ \sigma|_{\phi(A)} = id_{\phi(A)}$.

We will need to extend this result for the case where A is Abelian.

Lemma 7.7: Consider the group $G = N \rtimes K$, as in the diagram above. Let $A \leq G$ be an Abelian subgroup. Then we can write $A = \psi^{-1}(A) \rtimes \phi(A)$, where $\psi^{-1}(A)$ and $\phi(A)$ are Abelian subgroups of N and K respectively.

Proof: We have shown that $A = \psi^{-1}(A) \rtimes \phi(A)$, and we can quickly note that $\phi(A)$ is Abelian because ϕ is a homomorphism. So it only remains to show that $\psi^{-1}(A)$ is Abelian. Consider $x, y \in \psi^{-1}(A)$. Since ψ is injective, if $\psi(xy) = \psi(yx)$ then $xy = yx$. But $\psi(xy) = \psi(x)\psi(y) = \psi(y)\psi(x) = \psi(yx)$ as required, where the middle equality follows from the fact that A is Abelian.

In our case, this means that our Abelian subgroup $A \leq (Z(\beta_1) \times \ldots \times Z(\beta_r)) \rtimes Z_0(\hat{\beta})$ can be written $A = N \rtimes K$, where $N \leq Z(\beta_1) \times \ldots \times Z(\beta_r)$ and $K \leq Z_0(\hat{\beta})$ are Abelian subgroups.

Proposition 7.8: If $K \leq Z_0(\hat{\beta})$ is Abelian then $K \leq \mathbb{Z}^{43(r-1)}$.

Proof: We know that $Z_0(\hat{\beta})$ is isomorphic to either $\mathbb{Z}^2$ or to a mixed braid group on $m \leq r$ strands. Certainly in the former case this proves $K \leq \mathbb{Z}^{43(r-1)}$, but also in the latter case; a mixed braid group on $m \leq r$ strands is simply a subgroup of B_m, and so our inductive hypothesis applies. Thus the Abelian subgroup $K \leq B_m$ is a subgroup of $\mathbb{Z}^{43(m-1)} \leq \mathbb{Z}^{43(r-1)}$.

Before proving an analogous result for our Abelian $N \leq Z(\beta_1) \times \ldots \times Z(\beta_r)$, we need to introduce the following:

Lemma 7.9: Given a group $G = G_1 \times G_2$, any Abelian subgroup $A \leq G$ can be written as $A = A_1 \times A_2$, where A_1 and A_2 are (isomorphic to) Abelian subgroups of G_1 and G_2 respectively.

Proof: We've actually already done most the work here. Let the map C_n denote conjugation by n, and recall that direct products $N \times K$ are special cases of semi-direct products $N \rtimes K$, where the 'twist' map $\tau : K \to Aut(N)$ given by $k \mapsto C_{\psi^{-1}(\sigma(k))}$ is in fact just the map $k \mapsto id_N$. Thus Lemma 7.7 applies; an Abelian subgroup A of $G = G_1 \times G_2$ can certainly be written as $A = A_1 \rtimes A_2$, where A_1 and A_2 are (isomorphic to) Abelian subgroups of G_1 and G_2 respectively. We need to show that in fact $A = A_1 \times A_2$. But this follows from the fact that if conjugation by $\psi^{-1}(\sigma(k))$ is the identity on N then conjugation by $\psi|^{-1}_{\psi^{-1}(A)}(\sigma|_{\phi(G)}(k))$ is the identity on the subgroup $\psi^{-1}(A) \leq N$.

Proposition 7.10: If $N \leq Z(\beta_1) \times \ldots \times Z(\beta_r)$ is Abelian then $N \leq \mathbb{Z}^{43(m_1-1)+\ldots+43(m_r-1)}$.

Proof: Inductively applying the lemma above we see that $N = N_1 \times \cdots \times N_r$, where each N_i is (isomorphic to) an Abelian subgroup of $Z(\beta_i)$. But every $Z(\beta_i)$ is a subgroup of the braid group B_{m_i}, for some m_i strictly less than n. So our inductive hypothesis applies;

each N_i is a subgroup of $\mathbb{Z}^{43(m_i-1)}$. In other words:

$$\begin{aligned} N &= N_1 \times \ldots \times N_r \\ &\leq \mathbb{Z}^{43(m_1-1)} \times \ldots \times \mathbb{Z}^{43(m_r-1)} \\ &= \mathbb{Z}^{43(m_1-1)+\ldots+43(m_r-1)} \end{aligned}$$

Having found a bound for N and K, we now just need one final lemma:

Lemma 7.11: If $G = N \rtimes K$, where N and K are finitely generated by a and b generators respectively, then G can be written in terms of $a+b$ generators.

Proof: Let $\{n_1, \ldots, n_a\}$ and $\{k_1, \ldots, k_b\}$ generate N and K respectively. Since ϕ is surjective, there are elements $g_1, \ldots, g_b$ in G such that $\phi(g_i) = k_i$ for each i. Now consider any element $g \in G$. Since $\phi(g)$ is in K, we can write:

$$\begin{aligned} \phi(g) &= k_{i_1}^{\epsilon_1} k_{i_2}^{\epsilon_2} \ldots k_{i_m}^{\epsilon_m} \\ &= \phi(g_{i_1})^{\epsilon_1} \phi(g_{i_2})^{\epsilon_2} \ldots \phi(g_{i_m})^{\epsilon_m} \\ &= \phi(g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \ldots g_{i_m}^{\epsilon_m}) \end{aligned}$$

where each $\epsilon_i = \pm 1$. This implies that the image under ϕ of $g^{-1} g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \ldots g_{i_m}^{\epsilon_m}$ is trivial in K, so in particular it's in $\mathrm{Ker}(\phi)$. But since $\mathrm{Im}(\psi) = \mathrm{Ker}(\phi)$, this means we can write $g^{-1} g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \ldots g_{i_m}^{\epsilon_m}$ as a product of terms $\psi(n_{j_k})^{\epsilon_k}$. In other words, we've shown that the set $\{\psi(n_1), \ldots, \psi(n_a), g_1, \ldots, g_b\}$ generates G. The size of this set is $a+b$.

Proposition 7.12: For Abelian $A \leq Z(\beta)$ we have $A \leq \mathbb{Z}^{43(n-1)}$.

Proof: Combining Proposition 7.10 and Proposition 7.8 we see that $A = N \rtimes K$ where $N \leq \mathbb{Z}^{43(m_1-1)+\ldots+43(m_r-1)}$ and $K \leq \mathbb{Z}^{43(r-1)}$. Thus N has at most $43(m_1-1)+\ldots+43(m_r-1)$ generators, and K has at most $43(r-1)$ generators, so by the lemma above A is an Abelian group with at most $(\sum_{i=1}^{r} 43(m_i-1)) + 43(r-1)$ generators. Simplifying, we get:

$$\begin{aligned} &\left(\sum_{i=1}^{r} 43(m_i - 1) \right) + 43(r-1) \\ &= 43 \left(\left(\sum_{i=1}^{r} (m_i - 1) \right) + (r-1) \right) \\ &= 43 \left(\left(\sum_{i=1}^{r} m_i \right) - r + r - 1 \right) \\ &= 43 \left(\left(\sum_{i=1}^{r} m_i \right) - 1 \right) \\ &= 43\,(n-1) \end{aligned}$$

In other words, A is a subgroup of $\mathbb{Z}^{43(n-1)}$.

Corollary 7.13: If some non-periodic reducible $\beta \in A$ then $A \leq \mathbb{Z}^{43(n-1)}$.

Proof: Follows from Proposition 7.1 and the proposition above.

7.4 Discussion

Is this bound tight? Almost certainly not. This is due to our generous use of Hurwitz' theorem in 7.5; we add a single generator to a set of at most $42(n-1)$ generators, but then give ourselves lots of breathing space by saying that the resulting group has at most $43(n-1)$ generators. This could be remedied, but there is a more fundamental issue; although the bound in Hurwitz' theorem itself is known to be tight for infinitely many surfaces S, it seems unlikely to be tight in the more specific case of *Abelian* subgroups of $MCG(S_{0,n+1})$.

However, this bound is at least tight up to a constant factor. That is, there are certainly Abelian subgroups of the braid group generated by a set whose size is linear in n. One such example is the subgroup $\langle \sigma_1, \sigma_3, \ldots, \sigma_{2n-1} \rangle \leq B_{2n}$, which is isomorphic to $\mathbb{Z}^n$.

Word Count

According to TexCount, this essay contains 7211 words in text, 43 in headers and 238 outside text (captions etc.), for a total of 7492 words.

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