

Simplification strategies for the qutrit ZX-calculus

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Abstract. The ZX-calculus is a graphical language for suitably represented tensor networks - namely ZX-diagrams. Calculations are done by transforming ZX-diagrams with rewrite rules, which preserve semantics, and two ZX-diagrams representing the same tensor can be transformed into each other *only* by a sequence of rewrites. The ZX-calculus has found applications in reasoning about low-level quantum circuits, high-level proofs of quantum algorithms, and recently in recasting complexity results about counting problems in an intuitive graphical form. A key notion of such results is the stabiliser fragment of the ZX-calculus, a subfamily of ZX-diagrams for which rewriting can be done efficiently in terms of derived simplifying rewrites. In the context of quantum computation with qubits or of Boolean counting problems, one employs the qubit ZX-calculus. However, higher-dimensional cases are of interest as well. Recently, qudits have become more prominent in quantum computing, and there are counting problem families whose complexity depends on the dimension. The main contribution of this work is the derivation of *efficient* rewrite strategies for the stabiliser fragment of the qutrit ZX-calculus. This constitutes a first non-trivial step towards simplification of qutrit quantum circuits. Finally, we reinterpret known complexity-theoretic results about evaluating the Jones polynomial, an important link-invariant, and counting graph colourings. ***Eligible for the best student paper award.***

1 Introduction

The ZX-calculus originates in quantum foundations [6]. It is a graphical language that allows for reasoning about ZX-diagrams, which are composed of primitive generators [23] called ‘spiders’. A ZX-diagram is interpreted as a tensor network over a (semi)ring. The calculus is defined by a set of rewrite rules involving spiders. Performing a rewrite amounts to a subdiagram substitution, thus transforming the diagram. Note that the set of rewrite rules *depends* on the semantics, i.e. the (semi)ring over which the spiders are interpreted as tensors. Importantly, ZX rewrites are sound and complete for linear maps over the chosen (semi)ring.

In the context of quantum computing, the Gottesman-Knill theorem states that stabilizer quantum circuits can be simulated *efficiently* on classical computers [1], where by simulation here we mean the *exact* computation of quantum probability amplitudes. An alternative - and in fact more general - proof of the Gottesman-Knill theorem for qubits has been obtained graphically in terms of the qubit ZX-calculus [ToDo - ref?]. Stabilizer circuits can be expressed by the *stabilizer* fragment of the ZX-calculus, and stabilizer ZX-diagrams can be rewritten efficiently in terms of derived rewrite strategies [9]. Moreover, these simplifying rewrites can be viewed as theorems that are useful for simplifying or simulating universal quantum circuits, not just stabilizer circuits. The key rewrite rules that enable the efficient rewriting of qubit stabilizer ZX-diagrams are called *local complementation* and *pivot*, the latter being a composition of three of the former [9]. In this work, we recall the *qutrit* version of local complementation [21] and derive the corresponding pivot rule. We then derive simplification strategies which allow for the *efficient* rewriting of qutrit stabilizer ZX-diagrams, the core result of this work.

Interestingly, though the motivation for studying ZX diagrams stems from quantum computation, they have broader applicability as they can express arbitrary linear maps; every quantum circuit is a ZX-diagram but not vice versa. A plethora of hard problems in physics and computer science regard interacting many-body multi-state systems - both classical and quantum - and reduce to *exactly* computing a single scalar. These range from quantum amplitudes, partition functions in classical statistical mechanics, counting problems, probabilistic inference, and many more. Problem instances can be encoded as closed tensor networks over an appropriate (semi)ring. The scalar can be evaluated by *full tensor contraction*, which in general is #P-hard [7], and finding the optimal contraction path for a general tensor network is NP-hard.

If a scalar of interest is expressed as a closed ZX-diagram, one can rewrite the diagram by applying rewrite rules; one's goal is then to perform *full diagram simplification* in order to compute the desired scalar. Again, simplifying arbitrary closed ZX-diagrams is #P-hard and finding the optimal rewrite strategy is NP-hard. Note that given an arbitrary tensor network, one can attempt to invent rewrite rules by inspecting the contents of the tensors and performing linear-algebra operations [12].

ZX-diagrams have been leveraged to study the complexity of boolean satisfiability (SAT) and model counting (#SAT) [5]. Model counting entails computing the number of satisfying assignments to a boolean formula, and this number can be expressed as a closed diagram. Specifically, one finds that in the case of XORSAT and #XORSAT, known to be tractable, the corresponding ZX-diagrams representing the problems exactly correspond to qubit stabilizer diagrams. Going further, the ZH calculus [3], an extension of the ZX-calculus, can be imported to make graphically obvious why 2SAT is in P but #2SAT is #P-complete, whereas 3SAT and #3SAT are both hard. The key realisation is that the decision problem is in fact a counting problem but where the diagram is interpreted over the Boolean semiring. Then the *rewrite rules change accordingly* and quickly imply the above result by allowing efficient simplification strategies.

Continuing in the spirit of [5], we treat the ZX-calculus as a library of rewrite rules, from which we import the appropriate tools for the job depending on the occasion. Here we present two case studies: evaluating the Jones polynomial at lattice roots of unity, and graph colouring. Both of these problem families reduce to evaluating closed tensor networks and show a transition in complexity at a particular dimension, below which the tensor network corresponds to a stabilizer ZX-diagram. We underline that throughout this work, all rewrites are valid *up to a scalar* since we are only concerned about making statements about complexity. Keeping track of multiplicative scalar factors that arise under diagram rewriting can be done efficiently.

2 Simplifying Qubit ZX-Diagrams

In this section we briefly review the qubit ZX-calculus, recalling the definitions of graph-like diagrams and stabilizer diagrams. Crucially, we also recall the simplifying rewrites that enable the efficient simplification of stabilizer diagrams.

2.1 The Qubit ZX-Calculus

The qubit ZX-calculus is a diagrammatic language for quantum processes generated by *spiders*. Spider legs, or *wires*, represent vector spaces of dimension $d = 2$. Diagrams are read bottom-to-top; bottom open wires (not connected to anything) are *input wires* and top ones are *outputs*. Diagrams with only outputs are called *states* and those with only inputs are called *effects*. A *closed* diagram is one with no input nor output wires and it represents a scalar.

Spiders can be *composed*; placing diagrams side by side represents parallel composition (\otimes), whose concrete interpretation is the tensor product. Vertically stacking diagrams corresponds to

sequential composition (\circ) and concretely it means matrix multiplication. Specifically, it means tensor contraction of two spider tensors along the wires connecting them, i.e. the common tensor indices represented by these wires are summed over. The concrete representation of these operations of course depends on the (semi)ring over which the spiders are interpreted as tensors. For spiders S and S' , we denote these as

$$\llbracket S \otimes S' \rrbracket = \llbracket S \rrbracket \otimes \llbracket S' \rrbracket \quad , \quad \llbracket S \circ S' \rrbracket = \llbracket S \rrbracket \cdot \llbracket S' \rrbracket \quad (1)$$

Spiders come in two species: green Z -spiders and red X -spiders, decorated by a *phase* $\alpha \in [0, 2\pi)$. When $\alpha = 0$, we will omit it. The *standard representation* of the spider generators as tensors over \mathbb{C} is:

$$\left[\begin{array}{c} \overbrace{\quad \quad \quad}^m \\ \vdots \\ \text{green spider with } \alpha \\ \vdots \\ \underbrace{\quad \quad \quad}_n \end{array} \right] = |0\rangle^m \langle 0|^n + e^{i\alpha} |1\rangle^m \langle 1|^n, \quad \left[\begin{array}{c} \overbrace{\quad \quad \quad}^m \\ \vdots \\ \text{red spider with } \alpha \\ \vdots \\ \underbrace{\quad \quad \quad}_n \end{array} \right] = |+\rangle^m \langle +|^n + e^{i\alpha} |-\rangle^m \langle -|^n \quad (2)$$

where $\{|0\rangle, |1\rangle\}$ is the Z -basis and $|\pm\rangle = |0\rangle \pm |1\rangle$ the X -basis in \mathbb{C}^2 , in Dirac notation. The Hadamard gate H , whose function is to switch between the Z and X bases, is denoted as a yellow box. Often we will instead draw a dashed blue line to represent a *Hadamard edge*:

$$\begin{array}{c} | \\ \text{yellow box} \\ | \end{array} = \text{dashed blue line} = \begin{array}{c} \text{green circle } \pi/2 \\ | \\ \text{red circle } \pi/2 \\ | \\ \text{green circle } \pi/2 \end{array}, \quad \left[\begin{array}{c} | \\ \text{yellow box} \\ | \end{array} \right] \simeq |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| - |1\rangle \langle 1| \quad (3)$$

The ZX-calculus is universal for multilinear maps; any tensor over a (semi)ring has a corresponding ZX-diagram [22]. The rewrite rules of the ZX-calculus (see Fig.?? in Appendix ??) allow manipulation of the diagrams by *local tensor-rewrites*. Importantly, up to a scalar, the rewrites are sound; they *preserve the semantics*, i.e. the concrete tensor representation over \mathbb{C} . The ZX-calculus is also *complete* in the sense that any true equation between tensors can be proven *only* in terms of rewrites; the diagram on the left hand side of the equation can be transformed to that on the right hand side by applying rewrite rules. This gives ZX the status of a ‘calculus’. What is important about qubit ZX-diagrams is that *only topology matters*; the concrete tensor semantics of the diagram are invariant under deformations of the network as long as the inter-spider connectivity is respected.

2.2 Stabilizer Qubit ZX Diagrams

The *stabilizer fragment* of the calculus consists of all diagrams in which all phases are $\alpha = \frac{\pi n}{2}$, $n \in \mathbb{Z}$. In (Theorem 5.4, [9]) the authors give an efficient algorithm for simplifying any qubit ZX-diagram to an equivalent diagram with fewer spiders. The algorithm consists of consecutive applications of spider-eliminating rewrites.

First it is shown that every diagram is equivalent to a *graph-like* diagram: every spider is green, every edge is a Hadamard edge, there are no parallel edges or self-loops, every input and output wire is connected to a spider and every spider has at most one input or output wire. The

following derivable rules are key to this:

$$\begin{array}{c} \text{...} \\ \curvearrowright \\ \alpha \\ \curvearrowleft \\ \text{...} \end{array} = \begin{array}{c} \text{...} \\ \curvearrowright \\ \alpha \\ \curvearrowleft \\ \text{...} \end{array}, \quad \begin{array}{c} \text{...} \\ \curvearrowright \\ \alpha \\ \curvearrowleft \\ \text{...} \end{array} = \begin{array}{c} \text{...} \\ \curvearrowright \\ \alpha \\ \curvearrowleft \\ \text{...} \end{array}, \quad \begin{array}{c} \text{...} \\ \curvearrowright \\ \alpha \\ \curvearrowleft \\ \text{...} \end{array} = \begin{array}{c} \text{...} \\ \curvearrowright \\ \alpha + \pi \\ \curvearrowleft \\ \text{...} \end{array} \quad (4)$$

Then the following two rewrite rules, derived via *local complementation* and *pivoting*, can be used to eliminate spiders:

$$\begin{array}{c} \alpha_1 \quad \pm \frac{\pi}{2} \quad \alpha_n \\ \vdots \quad \vdots \quad \vdots \\ \alpha_2 \quad \vdots \quad \alpha_{n-1} \\ \vdots \quad \vdots \quad \vdots \end{array} = \begin{array}{c} \alpha_1 \mp \frac{\pi}{2} \quad \alpha_n \mp \frac{\pi}{2} \\ \vdots \quad \vdots \\ \alpha_2 \mp \frac{\pi}{2} \quad \alpha_{n-1} \mp \frac{\pi}{2} \\ \vdots \quad \vdots \end{array} \quad (5)$$

$$\begin{array}{c} \alpha_1 \quad j\pi \quad \alpha_n \\ \vdots \quad \vdots \quad \vdots \\ \beta_n \\ \vdots \\ \beta_1 \end{array} = \begin{array}{c} \alpha_1 + k\pi \quad \gamma_1 + j\pi \\ \vdots \quad \vdots \\ \alpha_n + k\pi \quad \gamma_n + j\pi \\ \vdots \quad \vdots \\ \beta_n + (j+k+1)\pi \\ \vdots \\ \beta_1 + (j+k+1)\pi \end{array} \quad (6)$$

For more details, see (Section 4, [9]). Eq. (5) says that we can remove any spider with phase $\pm \frac{\pi}{2}$ at the cost of performing a local complementation at said spider. Furthermore, Eq. (6) says we can remove any pair of spiders with phases in $\{0, \pi\}$ connected by a Hadamard edge at the cost of performing a pivot along said edge. After each application of (5) or (6), we can use (4) to remove any parallel Hadamard edges and ensure the diagram remains graph-like.

In particular, and importantly to this work, this algorithm will *efficiently* simplify any closed stabilizer diagram until it contains at most one spider, at which point the scalar it represents can be easily read off. By ‘efficiently’ we mean via a sequence of spider elimination rewrites whose cost is polynomial in the initial number of spiders. Note each such rewrite updates only a polynomial number of edges in the diagram, which prevents an overwhelming memory cost of the simplification procedure.

3 Simplifying Qutrit ZX-Diagrams

We now turn to the qutrit ZX-calculus and examine the analogous story to that of the previous subsection, but now for the case where the dimension of the vector space carried by the wires is $d = 3$.

3.1 The Qutrit ZX-Calculus

As in the qubit case, the qutrit ZX-calculus concerns spiders connected by wires, but there are key differences, some subtler than others. Again, spiders come in two species, Z (green) and X (red), with the three-dimensional Z -basis denoted as $\{|0\rangle, |1\rangle, |2\rangle\}$. Let $\omega = e^{i\frac{2\pi}{3}}$ denote the

third root of unity with $\bar{\omega} = \omega^2$ its complex conjugate. The qutrit X -basis consists of the three vectors:

$$|+\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) \quad , \quad |\omega\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \bar{\omega}|2\rangle) \quad , \quad |\bar{\omega}\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \bar{\omega}|1\rangle + \omega|2\rangle) \quad (7)$$

In the qutrit ZX-calculus, spiders carry *two* phases α and β , and have the following *standard representation* as linear maps:

$$\left\| \begin{array}{c} \overbrace{\quad\quad\quad}^m \\ \vdots \\ \text{spider with } \alpha, \beta \\ \vdots \\ \underbrace{\quad\quad\quad}_n \end{array} \right\| = |0\rangle^{\otimes m} \langle 0|^{\otimes n} + e^{i\alpha} |1\rangle^{\otimes m} \langle 1|^{\otimes n} + e^{i\beta} |2\rangle^{\otimes m} \langle 2|^{\otimes n} \quad (8)$$

$$\left\| \begin{array}{c} \overbrace{\quad\quad\quad}^m \\ \vdots \\ \text{spider with } \alpha, \beta \\ \vdots \\ \underbrace{\quad\quad\quad}_n \end{array} \right\| = |+\rangle^{\otimes m} \langle +|^{\otimes n} + e^{i\alpha} |\omega\rangle^{\otimes m} \langle \omega|^{\otimes n} + e^{i\beta} |\bar{\omega}\rangle^{\otimes m} \langle \bar{\omega}|^{\otimes n} \quad (9)$$

When $\alpha = \beta = 0$ we will again omit the angles entirely, and just draw a small green or red dot. Throughout our work, whenever we use an integer n as a spider decoration, this is a shorthand for $\frac{2\pi}{3}n$. Since spider phases hold mod 2π , these integer decorations hold mod 3. Unless otherwise stated, we will use Greek letters to denote general angles, and Roman letters for these integer shorthands.

Hadamard gates are no longer self-adjoint, so we change our notation: we let a yellow box decorated with a 1 (mod 3) denote a Hadamard gate, while decorating with a 2 (mod 3) denotes its adjoint. We will shortly explain this choice. We also use a dashed blue line for the *Hadamard edge* (H -edge) and a purple dashed line for its adjoint (H^\dagger -edge). Those familiar with the ZH-calculus should note that this notation is *not* analogous to the H-boxes therein.

$$\begin{array}{c} \text{yellow box } 1 \\ \hline \end{array} = \text{dashed blue line} = \begin{array}{c} \text{green circle } \frac{2}{2} \\ \hline \text{red circle } \frac{2}{2} \\ \hline \text{green circle } \frac{2}{2} \end{array} , \quad \begin{array}{c} \text{yellow box } 2 \\ \hline \end{array} = \text{dashed purple line} = \begin{array}{c} \text{green circle } \frac{1}{1} \\ \hline \text{red circle } \frac{1}{1} \\ \hline \text{green circle } \frac{1}{1} \end{array} , \quad \begin{array}{c} \text{yellow box } 0 \\ \hline \end{array} = \begin{array}{c} \text{green circle } \frac{1}{1} \\ \hline \text{green circle } \frac{1}{1} \end{array} \quad (10)$$

The last equation above says that in a graph-like diagram (which we will define shortly), a Hadamard edge decorated by a 0 is in fact not an edge at all, thanks to spider fusion. A very important difference from the qubit case is that in qutrit ZX-calculus there is no *plain* cap or cup:

$$\begin{array}{c} \text{cup with green dot} \end{array} \neq \begin{array}{c} \text{cup with red dot} \end{array} , \quad \begin{array}{c} \text{cap with green dot} \end{array} \neq \begin{array}{c} \text{cap with red dot} \end{array} \quad (11)$$

This has several consequences. Firstly, the maxim that ‘*only topology matters*’ *no longer applies*. That is, it is now important to make clear the distinction between a spider’s input and output wires, unlike in the qubit case where we could freely interchange the two. Diagram components

can still be isotoped around the plane but only so long as this input/output distinction is respected. This gives the qutrit calculus a slightly more rigid flavour than its qubit counterpart. That said, this rigidity is loosened in certain cases; in particular, this distinction is irrelevant for H - and H^\dagger -edges [11]:

$$(12)$$

The full set of rules governing the qutrit ZX-calculus is shown in Figure ?? in Appendix ??.

3.2 Graph-Like Qutrit ZX Diagrams

A *graph-like* qutrit ZX-diagram is one where every spider is green, spiders are only connected by blue Hadamard edges (H -edges) or their purple adjoints (H^\dagger -edges), every pair of spiders is connected by at most one H -edge or H^\dagger -edge, every input and output wire is connected to a spider, and every spider is connected to at most one input or output wire. A graph-like qutrit ZX-diagram is a *graph state* when every spider has zero phase (top and bottom) and is connected to an output.

Note the difference compared to the qubit case: we need not worry about self-loops because the qutrit ZX-calculus doesn't define a 'plain' cap or cup. But this comes at a cost: spiders in the qutrit case fuse more fussily. Specifically, when two spiders of the same colour are connected by at least one plain edge and at least one H - or H^\dagger -edge, fusion is not possible. Instead, should we want to ensure we have a graph-like diagram, we can replace the plain wire:

$$(13)$$

Indeed, we can show that every qutrit ZX-diagram is equivalent to a graph-like one. The following equations, derived in Appendix ??, are vital to this:

$$(14)$$

This justifies our notation for Hadamard gates: we can think of Hadamard edges as 1-weighted edges and their adjoints as 2-weighted edges, then work modulo 3, since every triple of parallel edges disappears. Where the previous equations relate single H - and H^\dagger -boxes across multiple edges, the next three relate multiple H - and H^\dagger -boxes on single edges. They hold for $h \in \{1, 2\}$, and are proved via simple applications of rules (id), (H) and (s).

$$(15)$$

Proposition 1. *Every qutrit ZX-diagram is equivalent to one that is graph-like.*

Proof. First use the colour change rule to turn all X -spiders into Z -spiders. Then use (15) to remove excess H - and H^\dagger -boxes, inserting a spider between any remaining consecutive pair of such boxes, so that all spiders are connected only by plain edges, H -edges or H^\dagger -edges. Fuse together as many as possible, and apply (13) where fusion is not possible, so that no plain edge connects two spiders. Apply (14) to all connected pairs of spiders until at most one H - or H^\dagger -edge remains between them. Finally, to ensure every input and output is connected to a spider and every spider is connected to at most one input or output, we can use **(H)** and **(id)** to add a few spiders, H - and H^\dagger -edges as needed:

$$\text{---} = \text{---} \circ \text{---} \circ \text{---}, \quad \text{---} \circ \text{---} = \text{---} \circ \text{---}, \quad \text{---} \circ \text{---} = \text{---} \circ \text{---} \quad (16)$$

□

A graph state is described fully by its underlying multigraph, or equivalently by an adjacency matrix, where edges take weights in \mathbb{Z}_3 (Lemma 4.2, [21]). Nodes correspond to phaseless green spiders, edges of weight 1 correspond to Hadamard edges, and edges of weight 2 correspond to H^\dagger -edges. As in the qubit case, graph states admit a *local complementation* operation (Definition 2.6, [21]), though the effect is now slightly more complicated. We'll give the intuition after the formal definition:

Definition 1. *Given $a \in \mathbb{Z}_3$ and a graph state G with adjacency matrix $W = (w_{i,j})$, the a -local complementation at node x is the new graph state $G *_a x$, whose adjacency matrix $W' = (w'_{i,j})$ is given by $w'_{i,j} = w_{i,j} + aw_{i,x}w_{j,x}$.*

So only those edges between neighbours of node x are affected. Specifically, for two nodes i and j both connected to x by the *same* colour edge, a -local complementation at x *increases* weight $w_{i,j}$ by a . If instead i and j are connected to x by edges of *different* colours, a -local complementation at x *decreases* $w_{i,j}$ by a . This is shown graphically below, and holds with the roles of blue and purple interchanged:

$$\text{---} \circ \text{---} \circ \text{---} \Rightarrow \text{---} \circ \text{---} \circ \text{---}, \quad \text{---} \circ \text{---} \circ \text{---} \Rightarrow \text{---} \circ \text{---} \circ \text{---} \quad (17)$$

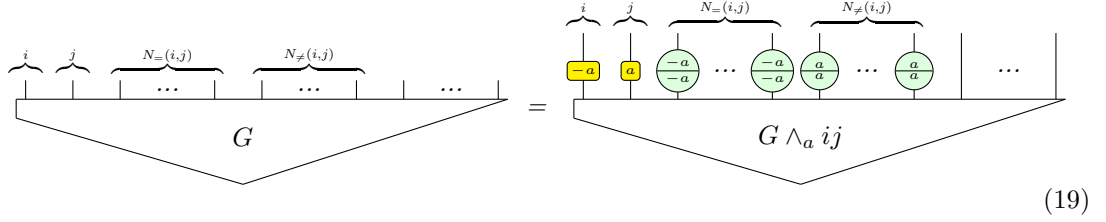
Theorem 1. (Theorem 4.4, Corollary 4.5, [21]) *Given $a \in \mathbb{Z}_3$ and a graph state (G, W) containing a node x , let $N(x)$ denote the neighbours of x ; that is, nodes i with weight $w_{i,x} \in \{1, 2\}$. Then the following equality holds:*

$$\text{---} \circ \text{---} \circ \text{---} = \text{---} \circ \text{---} \circ \text{---} \quad (18)$$

Definition 2. *Given $a, b, c \in \mathbb{Z}_3$ and a graph state G containing nodes i and j , the (a, b, c) -pivot along ij is the new graph state $G \wedge_{(a,b,c)} ij := ((G *_a i) *_b j) *_c i$.*

This pivot operation again leads to an equality, up to introducing some extra gates on outputs, whose proof is found in Appendix ???. Here we shall only consider an $(a, -a, a)$ -pivot along an edge ij of non-zero weight, for $a \in \{1, 2\}$. We will call this a *proper a -pivot* along ij , and denote it $G \wedge_a ij$.

Theorem 2. *Given $a \in \mathbb{Z}_3$ and a graph state (G, W) containing connected nodes i and j , define $N_=(i, j) := \{x \in N(i) \cap N(j) \mid w_{x,i} = w_{x,j}\}$ and $N_\neq(i, j) := \{x \in N(i) \cap N(j) \mid w_{x,i} \neq w_{x,j}\}$. Then the following equation relates G and its proper a -pivot along ij :*



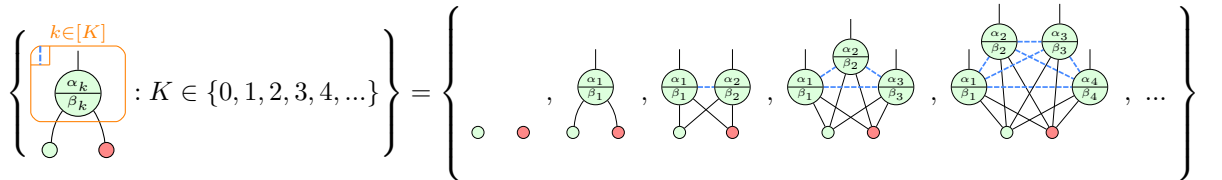
3.3 Qutrit Elimination Theorems

We classify spiders into three families exactly as in (Theorem 3.1, [21]):

$$\mathcal{M} = \left\{ \begin{pmatrix} \cdots \\ 0 \\ \cdots \end{pmatrix}, \begin{pmatrix} \cdots \\ 1 \\ \cdots \end{pmatrix}, \begin{pmatrix} \cdots \\ 2 \\ \cdots \end{pmatrix} \right\}, \quad \mathcal{N} = \left\{ \begin{pmatrix} \cdots \\ 0 \\ \cdots \end{pmatrix}, \begin{pmatrix} \cdots \\ 1 \\ \cdots \end{pmatrix}, \begin{pmatrix} \cdots \\ 0 \\ \cdots \end{pmatrix}, \begin{pmatrix} \cdots \\ 2 \\ \cdots \end{pmatrix} \right\}, \quad \mathcal{P} = \left\{ \begin{pmatrix} \cdots \\ 1 \\ \cdots \end{pmatrix}, \begin{pmatrix} \cdots \\ 2 \\ \cdots \end{pmatrix} \right\} \quad (20)$$

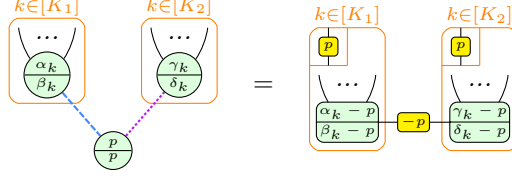
We call a spider in a graph-like ZX-diagram *interior* if it isn't connected to an input or output. Given any graph-like ZX-diagram, we will show that we can eliminate standalone interior \mathcal{P} - and \mathcal{N} -spiders by local complementation, and pairs of connected interior \mathcal{M} -spiders by pivoting.

First, we define a modification of the *!-box* notation, as introduced in [8] for general string diagrams. A *!-box* in a ZX-diagram is a compressed notation for a family of diagrams; the contents of the *!-box*, along with any wires into or out of it, are ‘unfolded’ (copied) $n \geq 0$ times and placed side-by-side. Following the style of [3], we also allow a parameter denoting the maximum number of copies that a *!-box* unfolds over. We extend this notation as follows: we decorate the *!-box* with an edge-type, which denotes that the spiders unfolded by the *!-box* are all-to-all connected by an edge of this type. In qutrit ZX-diagrams this notation will only be well-defined in certain scenarios: in particular, it is well-defined when the corner diagram is a H - or H^\dagger -edge (since then the distinction between a spider's input and output wires disappears) and the main contents of the *!-box* is equivalent to a single spider (since then there is no ambiguity about which spiders are connected). For example:

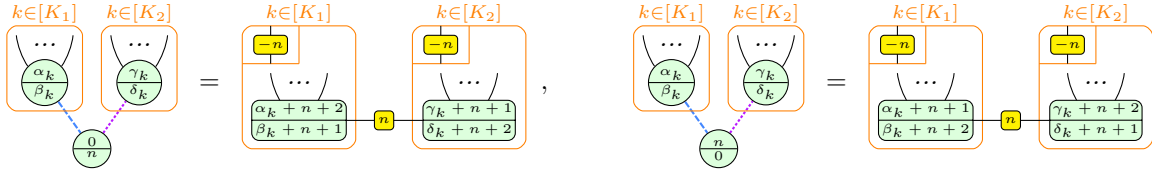


Theorem 3. *Given any graph-like ZX-diagram containing an interior \mathcal{P} -spider x with phase $\frac{p}{p}$ for $p \in \{1, 2\}$, suppose we perform a p -local complementation at x . Then the new ZX-diagram is*

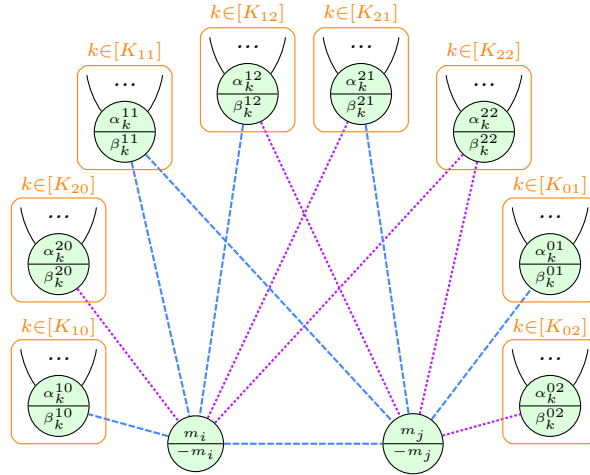
related to the old one by the following equality:

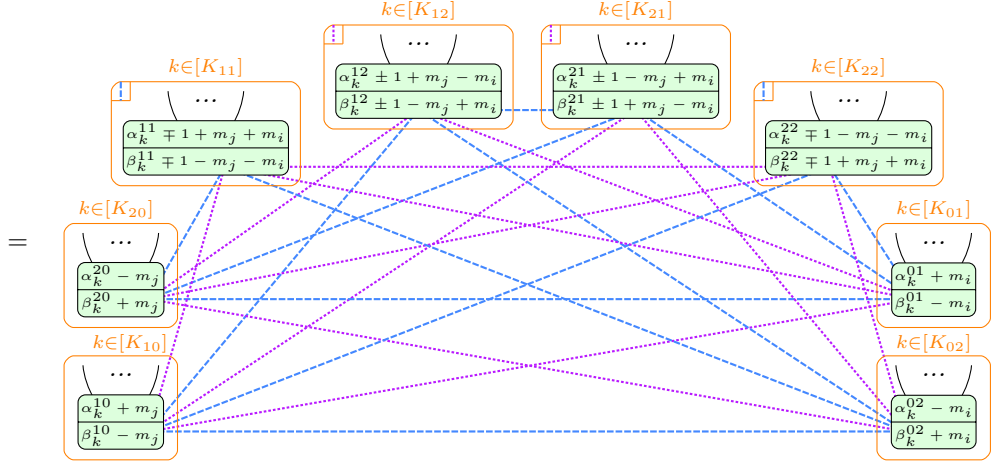


Theorem 4. Given any graph-like ZX-diagram containing an interior \mathcal{N} -spider x with phase $\frac{0}{n}$ or $\frac{n}{0}$ for $n \in \{1, 2\}$, suppose we perform a $(-n)$ -local complementation at x . Then, treating the two cases separately, the new ZX-diagrams are related to the old ones by the equalities:



Theorem 5. Given any graph-like ZX-diagram containing two interior \mathcal{M} -spiders i and j connected by edge ij of weight $w_{i,j} =: w \in \{1, 2\}$, suppose we perform a proper $\pm w$ -pivot along ij (both choices give the same result). For $w = 1$, the new ZX-diagram is related to the old one by the following equality:





The case $w = 2$ differs only as follows: in the first diagram (the left hand side of the equation), the edge ij is purple (by definition), while in the lower diagram (the right hand side of the equation), $a \pm 2 = \mp 1$ will replace all occurrences of ± 1 , and the roles of purple and blue will be swapped throughout.

Proofs of these theorems are found in Appendix ??, ?? and ?? respectively. We can now combine them into an algorithm for efficiently simplifying a *closed* graph-like ZX-diagram. First note that after applying any one of the three elimination theorems to such a diagram, and perhaps removing parallel H - or H^\dagger -edges via (14), we again have a graph-like diagram.

Theorem 6. *Given any closed graph-like ZX-diagram, the following algorithm will always terminate after a finite number of steps, returning an equivalent graph-like ZX-diagram with no \mathcal{N} -spiders, \mathcal{P} -spiders, or adjacent pairs of \mathcal{M} -spiders. Repeat the steps below until no rule matches. After each step, apply (14) as needed until the resulting diagram is graph-like:*

1. Eliminate a \mathcal{P} -spider via Theorem 3.
2. Eliminate an \mathcal{N} -spider via Theorem 4.
3. Eliminate two adjacent \mathcal{M} -spiders via Theorem 5.

Proof. At every step the total number of spiders decreases by at least one, so since we start with a finite diagram the algorithm terminates after a finite number of steps. By construction, when it does so we are left with an equivalent graph-like ZX-diagram with no \mathcal{N} -spiders, \mathcal{P} -spiders, or adjacent pairs of \mathcal{M} -spiders. \square

In particular, if we start with a stabilizer diagram, we can eliminate all but perhaps one spider, depending on whether the initial number of \mathcal{M} -spiders was odd or even. This is because no step introduces any non-stabilizer phases. The algorithm above could be extended to a *non-closed* graph-like diagrams as in (Theorem 5.4, [9]) - for example, as part of a qutrit circuit optimisation algorithm. We leave this for future work.

4 Case studies

In this section we present two problems which can naturally be cast in tensor network form. These problem families are interesting in that they show a transition in complexity from easy to hard when the dimension d carried by the wires is greater than a specific value. We recast known results about evaluating the *Jones polynomial*, and finally we briefly look at graph colouring.

4.1 Jones Polynomial at Lattice Roots of Unity

A *knot* K is a circle embedded in \mathbb{R}^3 . A set of knots tangled together make a *link* L . A link L can be represented by a *link diagram* by projecting it on to the plane but retaining the information of over or under crossings. We say that $L \simeq L'$ iff the diagram of link L can be deformed to that of link L' without cutting or gluing strands, or passing strands through each other. The Jones polynomial $V_L(t)$ is a Laurent polynomial in a variable $t \in \mathbb{C}$ and is a *link invariant*. This means that $V_L(t) \neq V_{L'}(t) \Rightarrow L \not\simeq L'$.

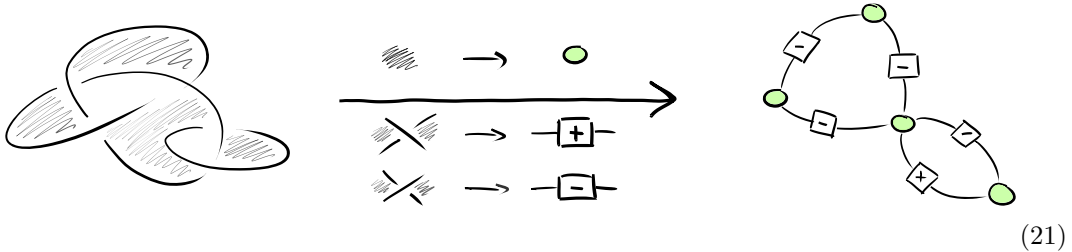
In general, computing $V_L(t)$ is exponentially costly in the number of crossings c , something made explicit when one uses the Kauffman bracket method [14]. Exactly evaluating the Jones polynomial at points $t \in \mathbb{C}$ is #P-hard, *except* at the *lattice roots of unity* $\Lambda = \{\pm 1, \pm i, \pm e^{i2\pi/3}, \pm e^{i4\pi/3}\}$, where it can be evaluated at cost $O(\text{poly}(c))$ [13].

In the context of quantum computation, additively approximating the Jones polynomial at *non-lattice roots of unity* is a paradigmatic BQP-complete problem [2,15]. Topological quantum computation [10,18] is the most natural model for such knot theoretic questions. In this model, quantum states are defined in the fusion space of *anyons*, emergent quasiparticles with nontrivial exchange statistics arising in two-dimensional exotic phases of matter. Quantum computation is performed by creating anyons from the vacuum, then braiding them, and finally fusing them back to the vacuum, where braids play the role of unitary gates. The world-lines of the anyons define a closed braid, i.e. a link. Such a link encodes a quantum amplitude corresponding to its Jones polynomial evaluated at a root of unity depending on the anyon theory at hand [24]. Specifically, in the case of $SU(2)_k$ anyons, the Jones polynomial is evaluated at $t(k) = e^{i2\pi/(2+k)}$ [19].

Also, the evaluation of the Jones polynomial at certain points can be expressed, up to an efficiently computable scalar that depends on the link diagram, as the partition function $Z_{G_L}(d)$ of a d -state Potts model with suitable spin-spin interactions [25]. This Potts model is defined on a signed graph G_L , obtained as follows. The link diagram is bicoloured checkerboard-style, then every coloured area is mapped to a vertex and every crossing is mapped to a signed edge according to its orientation relative to the surrounding colours, as in (21) below. The relation between the point $t(d)$ at which the Jones polynomial is evaluated and the dimension d of the spins is $d = t + t^{-1} + 2$, which can be solved for $t(d)$ [17].

Note the correspondence between the dimension d in the Potts approach and the level k in the anyon-braiding approach to the Jones polynomial: $\{t(d) \mid d \in \{1, 2, 3, 4\}\} = \{t(k) \mid k \in \{1, 2, 4, \infty\}\} \subseteq \Lambda$. This is consistent with the fact that braiding $SU(2)_2$ anyons (Ising) or $SU(2)_4$ anyons is not universal (unless the $SU(2)_4$ anyons are augmented by fusion and measurements [16]).

The partition function $Z_{G_L}(d \in \mathbb{N})$ can be expressed as a closed tensor network in terms of phaseless (green) d -dimensional Z -spiders connected via wires that go through \pm -boxes [17]:



The \pm -boxes have the following concrete interpretation as matrices:

$$\left[\begin{array}{c} | \\ \boxed{\pm} \\ | \end{array} \right] = \sum_{i,j=0}^{d-1} (1 - (1 + t(d)^{\mp 1})\delta_{ij}) |i\rangle \langle j| = \begin{pmatrix} -t(d)^{\mp 1} & 1 & \dots & 1 \\ 1 & -t(d)^{\mp 1} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -t(d)^{\mp 1} \end{pmatrix} \quad (22)$$

We can now express the \pm -boxes in the ZX-calculus. The \pm -matrices above for $d \in \{2, 4\}$ are equal up to a scalar to concrete interpretations of *qubit stabilizer* ZX-diagrams, and the \pm -matrices for $d = 3$ of *qutrit stabilizer* ZX-diagrams (see Appendix ??):

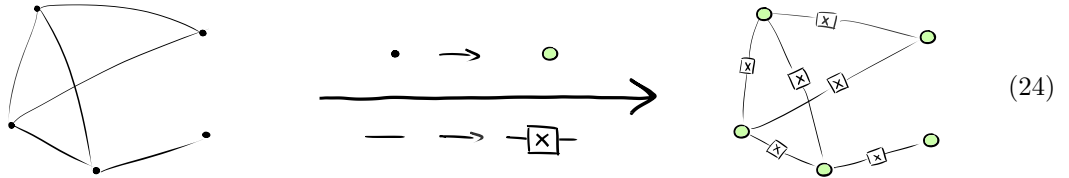
$$\left[\begin{array}{c} | \\ \boxed{\pm} \\ | \end{array} \right]_{d=2} \simeq \left[\begin{array}{c} | \\ \textcircled{\pm \frac{\pi}{2}} \\ | \end{array} \right], \quad \left[\begin{array}{c} | \\ \boxed{\pm} \\ | \end{array} \right]_{d=3} \simeq \left[\begin{array}{c} | \\ \textcircled{\pm 1 \atop \pm 1} \\ | \end{array} \right], \quad \left[\begin{array}{c} | \\ \boxed{\pm} \\ | \end{array} \right]_{d=4} \simeq \left[\begin{array}{c} | \\ \pi \text{---} \text{---} \pi \\ | \end{array} \right] \quad (23)$$

Since these generators decompose as stabilizer diagrams, $Z_{G_L}(d \in \{2, 3, 4\})$ can be evaluated *efficiently* via stabilizer ZX-diagram simplification. Thus, we recover the known result that evaluating the Jones polynomial at $t \in \Lambda$ is in P. On the other hand, computing $Z_{G_L}(d \geq 5)$ is #P-hard.

4.2 Graph Colouring

Finally, let us briefly look at the *graph colouring problem*, or evaluating the chromatic polynomial. A d -colouring of a graph G is an assignment of colours $\{1, \dots, d\}$ to the vertices of G so that no neighbouring vertices have the same colour. Given a graph G and an integer d , we wish to *count* the number of such d -colourings. Again, this problem can be interpreted as the zero-temperature partition function of an antiferromagnetic d -state Potts model [20]. Through the lens of our graphical exposition we see that counting problems and computing partition functions are essentially the same problem, since both can be straightforwardly encoded as closed tensor networks.

Given a graph G , the graph colouring problem can be encoded as a ZX-diagram as follows. Every vertex of the graph is mapped to a d -dimensional phaseless (green) Z -spider which copies spin states so that they can enter into the X -boxes, each of which in turn enforces that spin states entering it are not the same.



The X -boxes have the following concrete interpretation (as special case of the \pm -boxes, where $t = 0$), and for $d = 2$ they are just the Pauli X :

$$\left[\begin{array}{c} | \\ \boxed{X} \\ | \end{array} \right] = \sum_{i,j=0}^{d-1} (1 - \delta_{ij}) |i\rangle \langle j| = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{pmatrix}, \quad \left[\begin{array}{c} | \\ \boxed{X} \\ | \end{array} \right]_{d=2} \simeq \left[\begin{array}{c} | \\ \textcircled{\pm \pi} \\ | \end{array} \right] \quad (25)$$

Counting k -colourings for $k \geq 3$ is a canonical $\#P$ -complete problem. For $k = 0, 1, 2$ the problem is in P [13]. For $k = 2$ this is witnessed by the fact that the problem reduces to simplifying a qubit stabilizer diagram, which can be done efficiently. However, for $d = 3$ the X -matrix cannot be expressed as a stabilizer qutrit diagram; we prove this using our qutrit elimination theorems in Appendix ???. This is consistent with the fact that counting 3-colourings is $\#P$ -complete.

5 Outlook

We have advocated for the use of the ZX-calculus as a convenient graphical framework for treating a broad range of many-body problems on equal footing, and for diagnosing the complexity of interesting families of problems. When the solution to a problem is encoded as a closed ZX-diagram, this solution can be obtained via full diagram simplification by applying the rules of the calculus. The problem can be solved *efficiently* when the diagram encoding its answer is in the stabilizer fragment of the calculus. Specifically, we fleshed out the simplifications strategies that make this apparent for the case of qutrits.

We also looked at two case studies. We reviewed complexity results on evaluating the Jones polynomial and counting graph colourings, two problems which can be cast in tensor network form. We then recovered known complexity results using only the language of the ZX-calculus.

A main path for future work entails the generalisation of our stabilizer simplification rules to all prime dimensions. This in turn also motivates work in circuit extraction [4] for qudit circuits; one could hope to find graph-theoretic simplification [9] strategies analogous to those for the case of qubits. Another direction regards defining scalar-exact versions of the rewrite rules, which would be necessary for concrete applications to problems where d -state systems are the native degrees of freedom.

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