Innopolis University

Control Theory (Linear Control)

Lecture 3

Transfer Functions

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Topic of the Lecture

- Transfer Functions
- From State Space Model to Transfer Function
- Stability and Asymptotic Stability
- Solution for an ODE

Transfer Functions

The relationship between the input $X(t)$ and the output $Y(t)$ for an analog signal system is called a transfer function $Y(t) = F[X(t)]$.

Transfer Function Representation (1/2)

Consider the following ODE:

$$3\ddot{x} + 56\dot{x} - 7x = u \quad \Leftrightarrow \quad 3\frac{d^2}{dt^2}x + 56\frac{d}{dt}x - 7x = u$$

We can introduce differentiation operator $p = \frac{d}{dt}$:

$$3p^2x + 56px - 7x = u$$

Transfer Function Representation (2/2)

Next we have:

$$(3p^2 + 56p - 7)x = u \quad \Rightarrow \quad x = \frac{1}{3p^2 + 56p - 7}u$$

If we denote $W(p) = \frac{1}{3p^2+56p-7}$, then we have:

$$\boxed{x = W(p) \cdot u \quad \text{(transfer function representation)}}$$

Another Example

$$\ddot{x} - 2\dot{x} + 3x = -2\dot{u} + 3u$$

$$p^2x - 2px + 3x = -2pu + 3u$$

$$(p^2 - 2p + 3)x = (-2p + 3)u$$

$$x = \frac{-2p + 3}{p^2 - 2p + 3}u$$

Transfer function:

$$W(p) = \frac{-2p + 3}{p^2 - 2p + 3}$$

From State Space Model to Transfer Function

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$\begin{cases} px = Ax + Bu \\ y = Cx \end{cases}$$

$$\begin{cases} (Ip - A)x = Bu \\ y = Cx \end{cases}$$

$$\begin{cases} x = (Ip - A)^{-1}Bu \\ y = Cx \end{cases}$$

$$y = C(Ip - A)^{-1}Bu$$

Transfer function:

$$W(p) = C(Ip - A)^{-1}B$$

Why Do We Need to Know Transfer Functions?

- All **textbooks** have them.
- A lot of **old methods** use them.
- Many scientists prefer to “talk in” transfer functions rather than ODEs or State-Space.
- Easy to make **visual representations** of Transfer Function.
- If you don’t have computer but know Laplace transformations, you can solve linear ODEs, and that is done in the way closely related to Transfer Function.
- If you don’t have computer and $\mathbf{eig()}$, you can still access stability of the system knowing its TF.
- Frequency response of a SISO is easier to study with Transfer Functions.
- Your lecturer and all your TAs studied them in their college years, now they want to share the experience. 😈

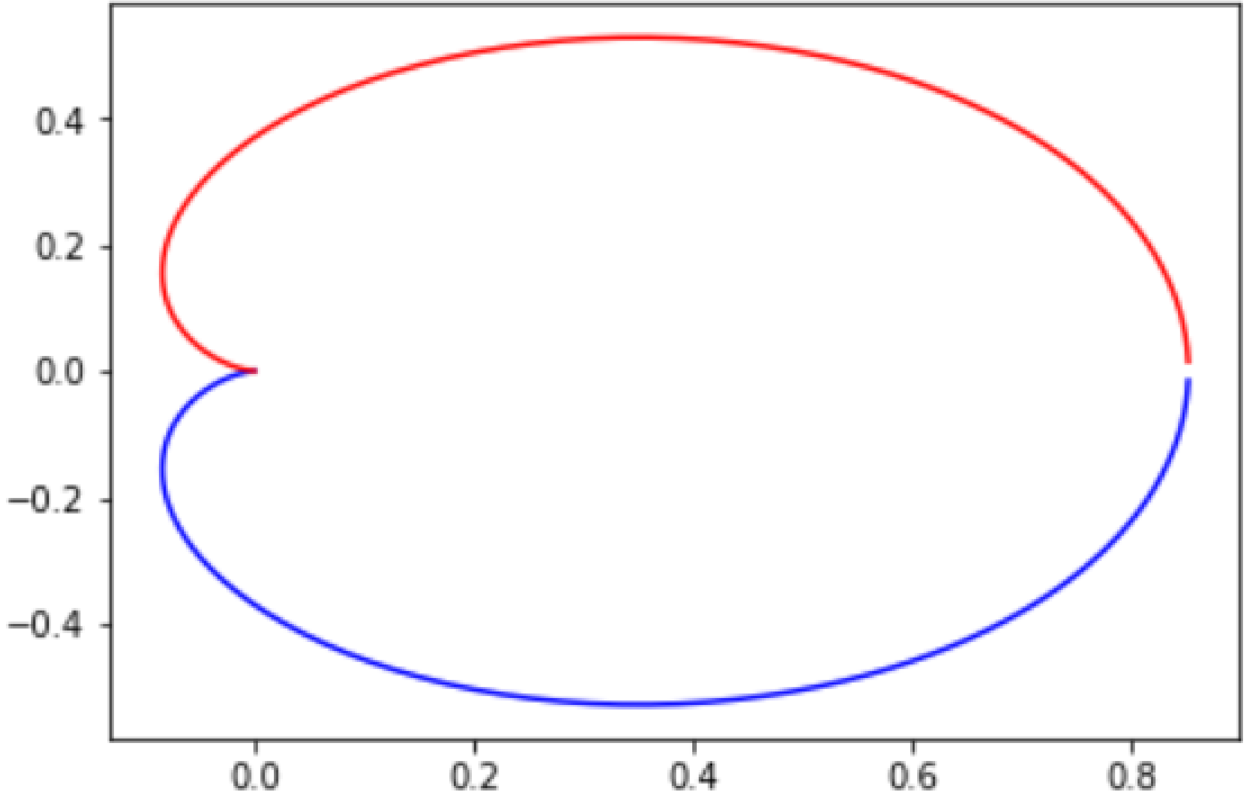
Reasons to Avoid Transfer Functions

- Transfer Functions only work for very simple systems, State-Space works for everything.
- Old methods based on Transfer Functions are replaced with numeric tools: $\mathbf{eig()}$, $\mathbf{ode45()}$, $\mathbf{lqr()}$. All of them are better, faster, and easier to use.
- A lot of important details and restrictions that you need to know before using them, otherwise you will make a mistake sooner or later.

Nyquist Diagram

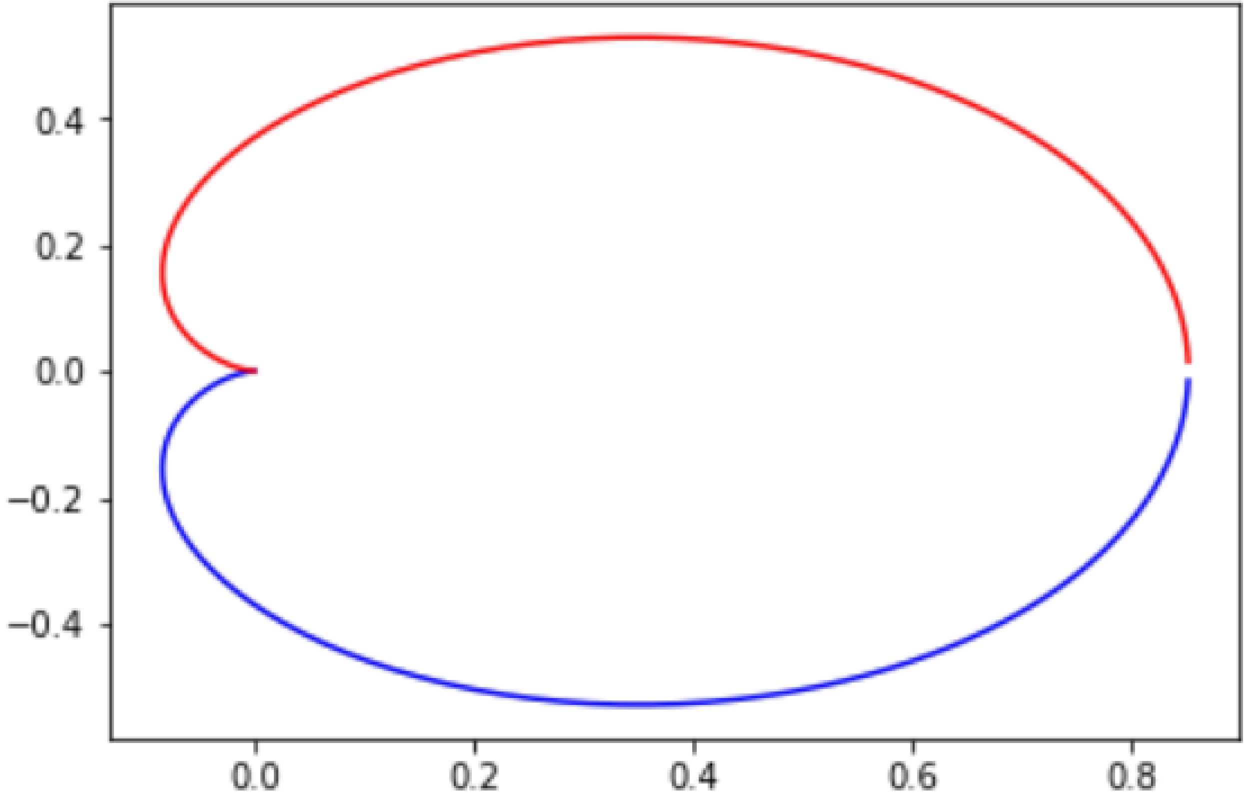
Nyquist diagram is:

1. take TF $W(p)$
2. substitute $j\omega$ for p: $W(j\omega)$
3. plot the curve $W(j\omega)$ on complex plane, varying ω



Nyquist Diagram and Bode Plot (1/3)

Nyquist diagram is obtained by substituting ωj instead of s into the transfer function of the system, and then plotting the real and imaginary parts of the resulting complex number as a parametric curve given by ω.



Nyquist Diagram and Bode Plot (2/3)

There are other plots, besides Nyquist. Bode is one of them. Unlike Nyquist, it is two real-valued plots, where as Nyquist is complex-valued. To understand their connection, you can think of it this way. Let $Z(\omega)$ be a Nyquist plot. Then phase plot is given as:

$$\varphi(\omega) = \arg(Z(\omega))$$

and the amplitude plot is:

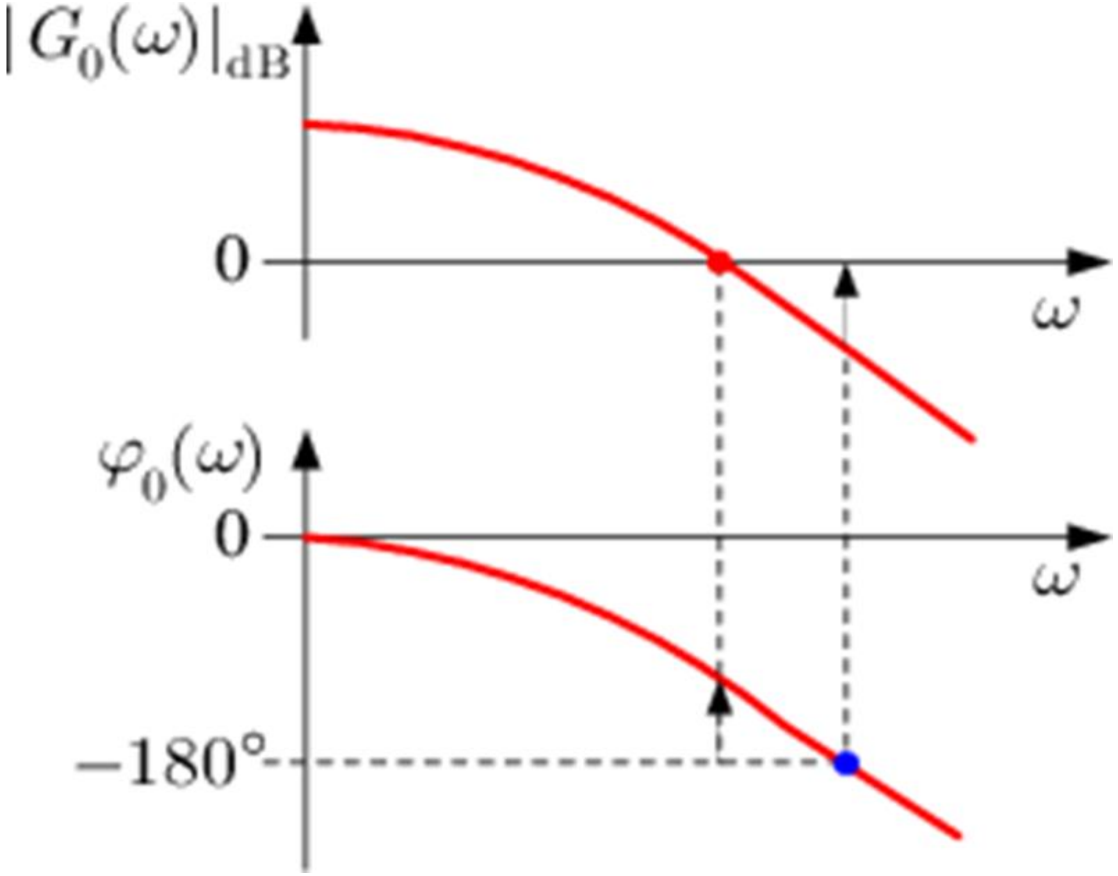
$$A(\omega) = |Z(\omega)|$$

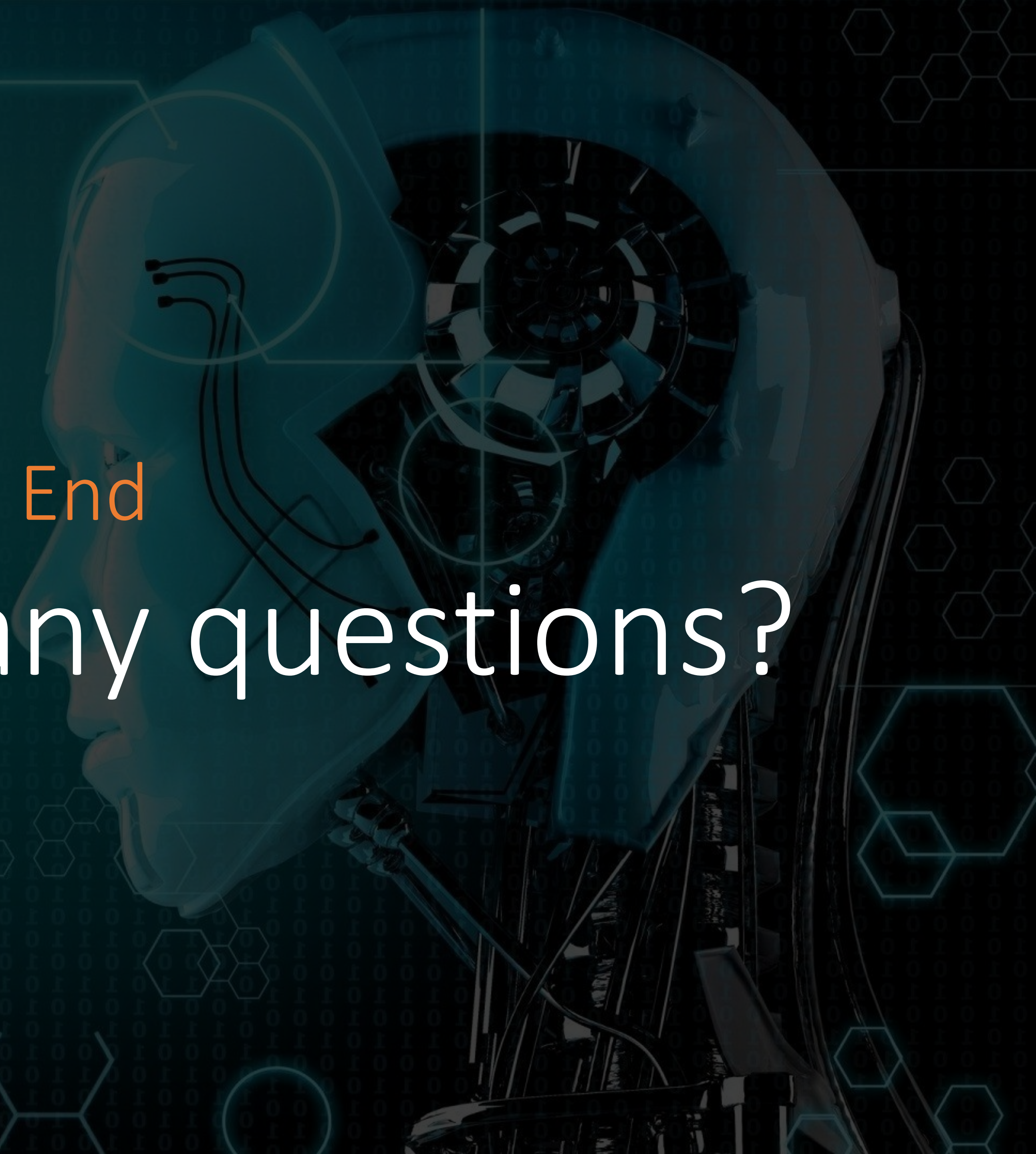
In case of Bode plot, we have phase plot $H(\omega)$, and magnitude plot $B(\omega)$ in decibels. Phase plot $H(\omega) = \varphi(\omega) = \arg(Z(\omega))$, and magnitude plot:

$$B(\omega) = 20 \log_{10}|Z(\omega)|$$

$$B(\omega) = 20 \log_{10} A(\omega)$$

Nyquist Diagram and Bode Plot (3/3)





The End

Do you have any questions?