Innopolis University

Control Theory (Linear Control)

Lecture 2

Linear Systems Stability

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Topic of the Lecture

- Linearity
- Examples of Linear Systems
- Stability and Asymptotic Stability
- Solution for an ODE

Linearity

RECAP

A system G is linear with respect to its inputs and output

$$\mathrm{u}(t) \rightarrow \boxed{G(s)} \rightarrow \mathrm{y}(t)$$

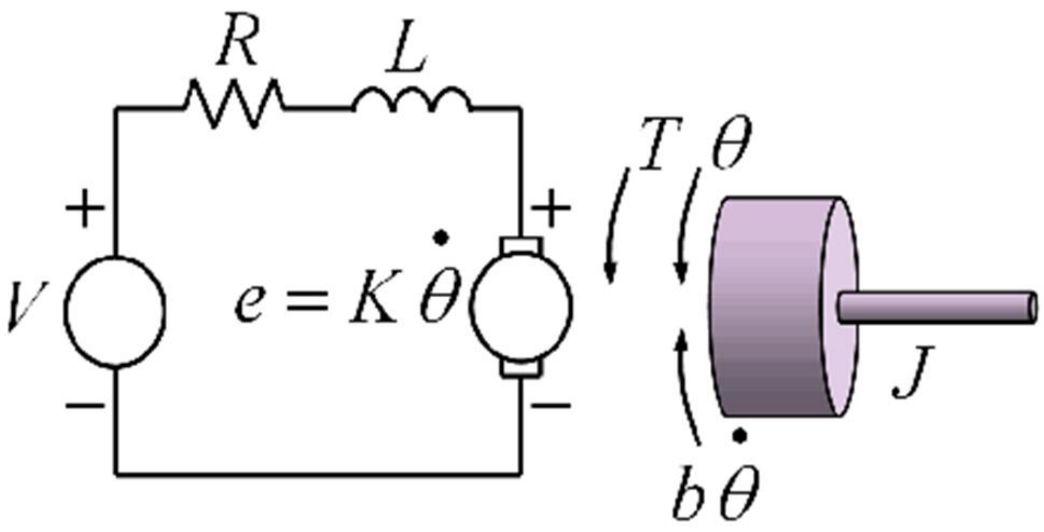
if and only if superposition holds:

$$G(\alpha_1 \mathrm{u}_1 + \alpha_2 \mathrm{u}_2) = \alpha_1 G \mathrm{u}_1 + \alpha_2 G \mathrm{u}_2$$

So, if y_1 is the response of G to u_1 ($\mathrm{y}_1 = G\mathrm{u}_1$), and y_2 is the response of G to u_2 ($y_2 = Gu_2$), then the response to $\alpha_1 \mathrm{u}_1 + \alpha_2 \mathrm{u}_2$ is $\alpha_1 \mathrm{y}_1 + \alpha_2 \mathrm{y}_2$

Examples of Linear Systems (1/2)

DC motor



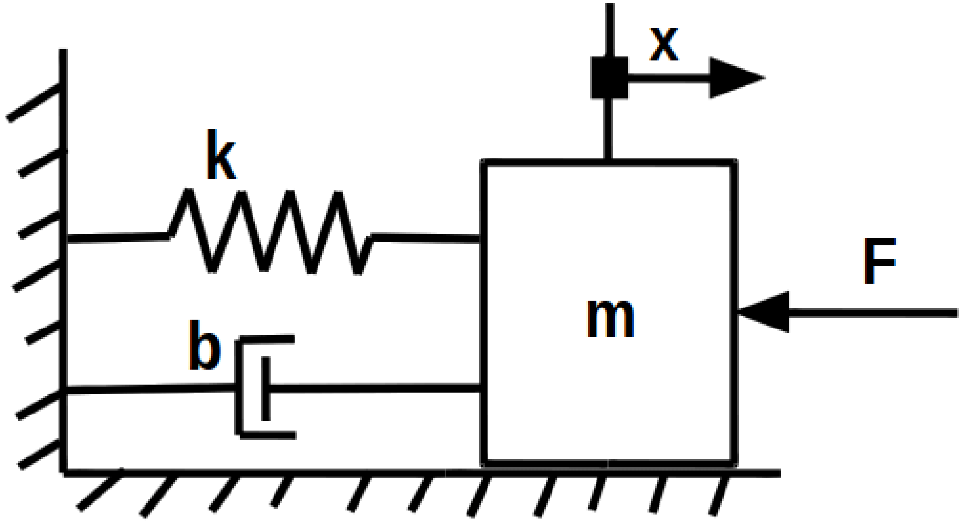
$$L\frac{d}{dt}I + RI + \eta C_e \omega = u$$

$$\tau = \eta C_\tau I$$

Linear model on black board!

Examples of Linear Systems (2/2)

Spring-damper



Linear model on whiteboard!

The Concept of Stability of a Control System

Stability and Asymptotic Stability (1/2)

Suppose that c is a critical point of the system $y'(t) = f\big(y(t)\big)$.

- We say that c is stable if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that every solution of the system satisfies

$$\left|\left|y(0) - c\right|\right| < \delta \quad \Rightarrow \quad \left|\left|y(t) - c\right|\right| < \varepsilon \quad \text{for all } t \geq 0$$

- We say that c is asymptotically table, if it is stable and there exists $\delta > 0$ such that every solution of the system satisfies

$$\left|\left|y(0) - c\right|\right| < \delta \quad \Rightarrow \quad \lim_{t\to\infty} y(t) = c$$

Stability and Asymptotic Stability (2/2)

Loosely speaking, stability means that every solution which is initially close to the critical point c must remain close to c at all times.

Solution for an ODE

$$\dot{x} + ax = 0$$

Let’s write the solution on the black board!

When does it converge to zero?

Solution for an ODE
(when A has an eigendecomposition)

$$\dot{x} = Ax$$

$$A = VDV^{-1}$$

$$V^{-1}\dot{x} = V^{-1}VDV^{-1}x$$

$$V^{-1}\dot{x} = DV^{-1}x$$

$$\Rightarrow$$

$$y \equiv V^{-1}x$$

$$\dot{y} = Dy$$

$$y = e^{Dt}y_0$$

Solution for an ODE

$$y = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_1 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_1 t} \end{bmatrix} y_0$$

The End

Do you have any questions?