Innopolis University

Control Theory (Linear Control)

Lecture 6

Linear-quadratic regulator (LQR)
Hamilton-Jacobi-Bellman Equation

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Topic of the Lecture

- Linear-quadratic regulator (LQR)
- Hamilton-Jacobi-Bellman Equation
- Algebraic Riccati equation

Hamilton-Jacobi-Bellman Equation

For cost $\int_0^\infty g(\mathbf{x},\mathbf{u})dt$, we have:

$$0 = \min_{\mathbf{u}} \left[g(\mathbf{x},\mathbf{u}) + \frac{\partial J^*}{\partial \mathbf{x}} f(\mathbf{x},\mathbf{u}) \right],$$

$$\pi^*(\mathbf{x}) = \mathrm{argmin}_{\mathbf{u}} \left[g(\mathbf{x},\mathbf{u}) + \frac{\partial J^*}{\partial \mathbf{x}} f(\mathbf{x},\mathbf{u}) \right].$$

Examples:

https://www.youtube.com/watch?v=rbEyP0xKuB4&t=2s

Linear Case

Consider a linear time-invariant system in state-space form,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$

with the infinite-horizon cost function given by

$$J = \int_0^\infty \left[\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right] dt, \quad \mathbf{Q} = \mathbf{Q}^T \geq \mathbf{0}, \mathbf{R} = \mathbf{R}^T > 0.$$

Algebraic Riccati Equation

Our goal is to find the optimal cost-to-go function $J^*(\mathbf{x})$ which satisfies the HJB:

$$\forall \mathbf{x}, \quad 0 = \min_{\mathbf{u}} \left[\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \frac{\partial J^*}{\partial \mathbf{x}}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \right].$$

There is one important step here -- it is well known that for this problem the optimal cost-to-go function is quadratic. This is easy to verify. Let us choose the form:

$$J^*(\mathbf{x}) = \mathbf{x}^T\mathbf{S}\mathbf{x}, \quad \mathbf{S} = \mathbf{S}^T \succeq 0.$$

The gradient of this function is

$$\frac{\partial J^*}{\partial \mathbf{x}} = 2\mathbf{x}^T\mathbf{S}.$$

Explanation:

https://www.youtube.com/watch?v=3heIoKDhN3E

Feedback Law

Since we have guaranteed, by construction, that the terms inside the **min** are quadratic and convex (because $\mathbf{R} \succ 0$), we can take the minimum explicitly by finding the solution where the gradient of those terms vanishes:

$$\frac{\partial}{\partial \mathbf{u}} = 2\mathbf{u}^T\mathbf{R} + 2\mathbf{x}^T\mathbf{S}\mathbf{B} = 0.$$

This yields the optimal policy

$$\mathbf{u}^* = \pi^*(\mathbf{x}) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S}\mathbf{x} = -\mathbf{K}\mathbf{x}.$$

Algebraic Riccati Equation Solution

Inserting this back into the HJB and simplifying yields

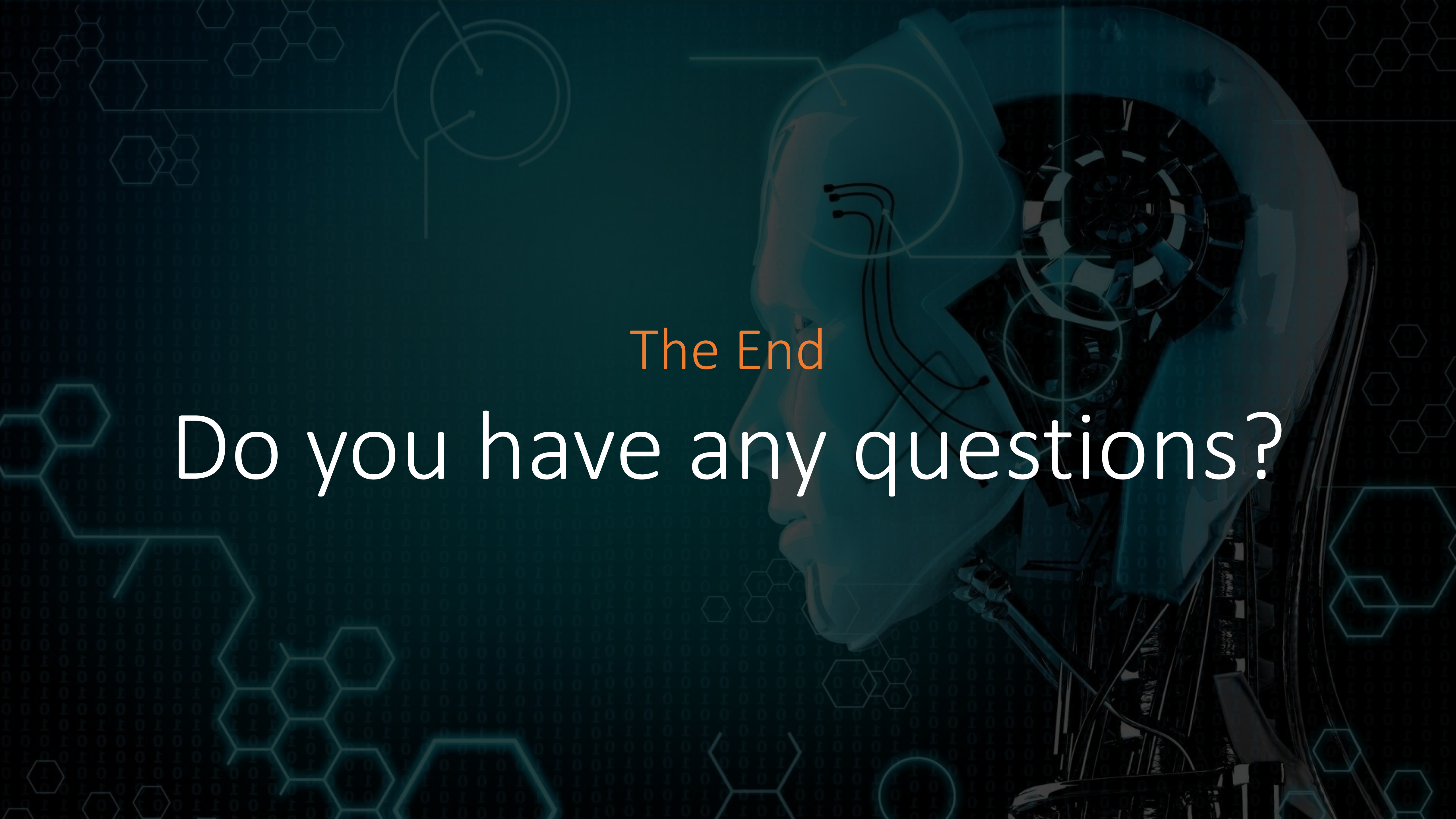
$$0 = \mathbf{x}^T \left[\mathbf{Q} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} + 2\mathbf{S}\mathbf{A}\right] \mathbf{x}.$$

All of the terms here are symmetric except for the $2\mathbf{SA}$, but since $\mathbf{x}^T\mathbf{S}\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{A}^T\mathbf{S}\mathbf{x}$, we can write

$$0 = \mathbf{x}^T \left[\mathbf{Q} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{A}^T\mathbf{S}\right] \mathbf{x}.$$

and since this condition must hold for all $\mathbf{x}$, it is sufficient to consider the matrix equation

$$0 = \mathbf{S}\mathbf{A} + \mathbf{A}^T\mathbf{S} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} + \mathbf{Q}.$$



The End

Do you have any questions?