Lecture 5- MIMO, LTI, LTV

Objectives:

- Multi-Input Multi-Output (MIMO) systems.
- Linear Time Invariant (LTI) systems and their properties.
- Linear Time Varying (LTV) systems and their properties.
- Equilibrium points

Recap:

Single-Input Single-Output (SISO) system simply put is a system that takes in one input and produces one output. If we have an ODE

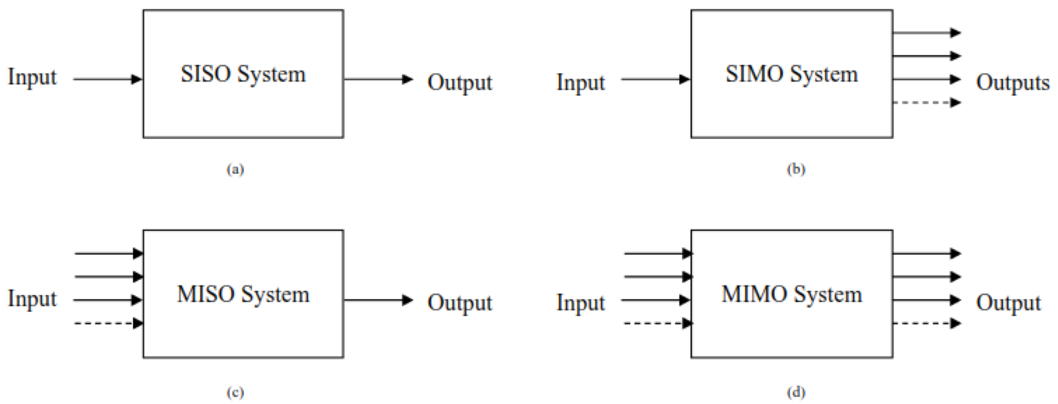
$$\ddot{x} + a_1\dot{x} + a_2 x + a_3 = u \tag{1}$$

For the input ***u*** and for a single chosen output, the system is a SISO system. The output can be ***x*** or we can introduce output variable ***y*** where

$$y = a_4\dot{x} + a_5 x + a_6 \tag{2}$$

Introduction

We can generally classify systems based on their input and output. So aside from SISO, we could have Single-Input Mulit-Output (SIMO), Mulit-Input Single-Output (MISO) and Multi-Input Multi-Output (MIMO).



Today we will focus on MIMO systems.

MIMO systems

Multi-Input Multi-Output (MIMO) is a system that takes in multiple inputs and produces multiple outputs. For example, given:

$$\begin{aligned} \dot{x_1} &= a_1x_1 + a_2x_2 + a_3u_1 \\ \dot{x_2} &= b_1x_1 + b_2x_2 + b_3u_2 - b_4u_1 \end{aligned} \tag{3}$$

And

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_1 + cx_2 \end{aligned} \tag{4}$$

The system is a MIMO system . Below is a brief comparison.

SISO	SIMO	MISO	MIMO
$y \in \mathbb{R}, u \in \mathbb{R}, x \in \mathbb{R}^n$	$y \in \mathbb{R}^k, u \in \mathbb{R}, x \in \mathbb{R}^n$	$y \in \mathbb{R}, u \in \mathbb{R}^m, x \in \mathbb{R}^n$	$y \in \mathbb{R}^k, u \in \mathbb{R}^m, x \in \mathbb{R}^n$

Linear systems

A control system can be classified according a lot of categories, some of which are

- Linear and Non-linear Systems
- Time Variant and Time Invariant Systems
- linear Time variant and linear Time invariant systems
- Static and Dynamic Systems
- Causal and Non-causal Systems
- Invertible and Non-Invertible Systems
- Stable and Unstable Systems

The focus in this lecture is on **Linear systems** which are systems satisfy superposition and homogenate principles. A **Time Invariant** system means that the parameters of the system do not change with time and a **Time Variant** system means that the parameters of the system vary with time.

LTI system

If a system is both linear and time Invariant then that system is called Linear Time Invariant (LTI) system.

Example

Consider a piano, where the loudness of a played note is (linearly) proportional with the force you use on the keyboard.

Soft touch -> low volume
Hard touch -> high volume
Press A -> listen A
Press B -> listen B
Press A+B -> listen to A+B

Press now A -> listen to A after t sec.
Press A in 5 min -> listen to A in 5 min + t sec

This is a Linear Time Invariant system.

Stability (using eigenvalues)

1. A LTI system is ***asymptotically stable*** if and only if all the eigenvalues have strictly negative real part
2. A LTI system is ***marginally stable*** if and only if all the eigenvalues have non positive real part and those which have zero real part have scalar Jordan blocks. [1] **
3. A LTI system is ***unstable*** if and only if there exists at least one eigenvalue with positive real part or a Jordan block corresponding to an eigenvalue with zero real part of dimension greater than 1

Phase portrait

A phase portrait is a graphical tool to visualize how the solutions of a given system of differential equations would behave in the long run. Below will be illustrated phase portraits of linear systems.

LTV system

If a system is both linear and time variant, then it is called Linear Time Variant (LTV) system.

Example

The human vocal tract is a time variant system, with its transfer function at any given time dependent on the shape of the vocal organs.

Stability

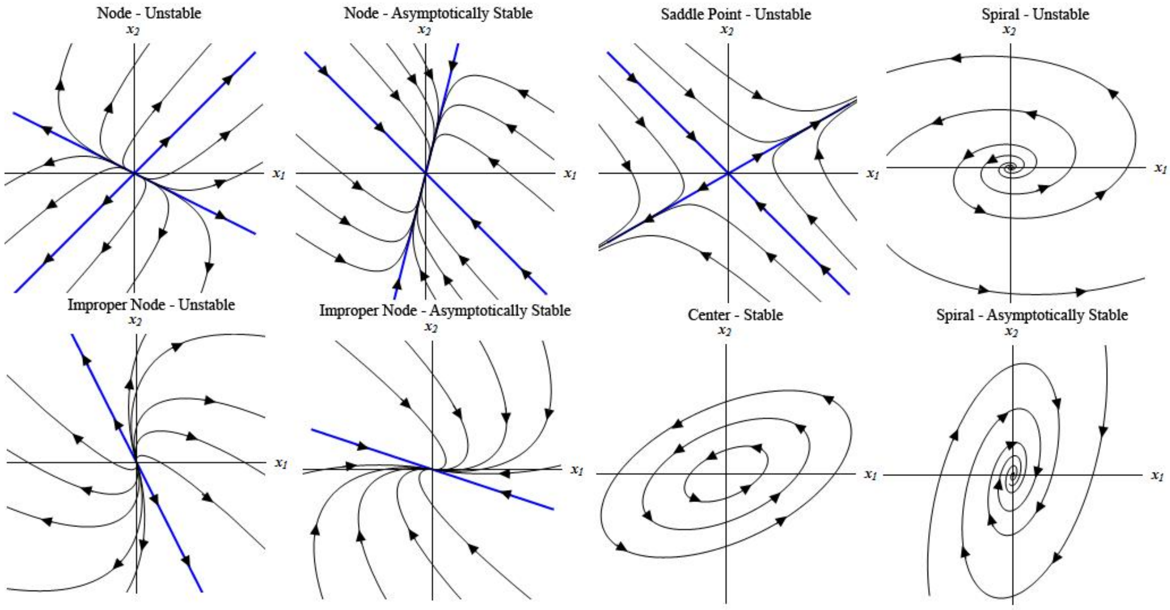
The stability analysis of time varying systems is more complicated than the time invariant systems. This is because the time varying parameters of the system would add more complexity when using Lyapunov stability theory in the analysis.

However, determining the stability of LTI helps us understand better the stability of LTV.

LTI	LTV
A = const, B = const, C = const $x = x(t)$ $u = u(t)$	$A = A(t)$ and/or $B = B(t)$ and/or $C = C(t)$ $x = x(t)$ $u = u(t)$

Phase portraits of Linear Systems

Consider a systems of linear differential equations x' = Ax. Its phase portrait is a representative set of its solutions, plotted as parametric curves (with t as the parameter) on the Cartesian plane tracing the path of each particular solution $(x, y) = (x1(t), x2(t)), -\infty < t < \infty$.



<u>Note</u>: It is quite labor-intensive, but it is possible to sketch the phase portrait by hand

We will classify the type and stability the equilibrium point of a given linear system by the shape and behavior of its phase portrait.

	Equilibrium Point Type	λ_1, λ_2	Stability
1	Node	Two distinct real numbers, both of the ***same sign***	It is ***unstable*** if $\lambda_1 > 0; \lambda_2 > 0;$ ***asymptotically stable*** if $\lambda_1 < 0; \lambda_2 < 0.$
2	Saddle	Two distinct real numbers, but ***opposite signs***	It is always ***unstable***
3	Proper node	***Repeated*** real numbers, both linearly independent	It is ***unstable*** if $\lambda_1 = \lambda_2 > 0$; ***asymptotically stable*** if $\lambda_1 = \lambda_2 < 0$
4	Improper node	***Repeated*** real numbers, one linearly independent	It is ***unstable*** if $\lambda_1 = \lambda_2 > 0$; ***asymptotically stable*** if $\lambda_1 = \lambda_2 < 0$
5	Center	Complex conjugate numbers, with real part zero (purely imaginary numbers)	***Stable*** (but not asymptotically stable); sometimes it is referred to as neutrally stable.
6	Spiral	Complex eigenvalues, with nonzero real part	It is ***unstable*** if the eigenvalues have positive real part; ***asymptotically stable*** if the eigenvalues have negative real part.
7	Focus	Complex numbers, real parts equal and non-zero	It is ***unstable*** if the eigenvalues have positive real part; ***asymptotically stable*** if the eigenvalues have negative real part.

References

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4. Systems Classification
   https://www.tutorialspoint.com/signals_and_systems/systems_classification.htm
5. Example of LTI
   https://www.quora.com/What-are-some-real-life-examples-that-helps-to-understand-the-LTI-Linearly-Time-Invariant-system

1. (A Jordan block is a square matrix which has zero entries everywhere except on the diagonal, where the entries are a fixed scalar, and except on the superdiagonal, where the entries are either all 0s or all 1s) ↩