Discrete Systems

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Consider linear time-invariant autonomous system:

$$\dot{x} = Ax$$

The time derivative $\dot{x}$ can be replaces with a finite difference:

$$\dot{x} \approx \frac{1}{\Delta t}(x(t + \Delta t) - x(t))$$

Note that we could have also used other definitions of a finite difference:

$$\dot{x} \approx \frac{1}{\Delta t}(x(t + 0.5\Delta t) - x(t - 0.5\Delta t))$$

or

$$\dot{x} \approx \frac{1}{\Delta t}(x(t) - x(t - \Delta t))$$

We can introduce notation:

$$\begin{cases} \mathsf{x}[0] = \mathsf{x}(0) \\ \mathsf{x}[1] = \mathsf{x}(\Delta t) \\ \mathsf{x}[2] = \mathsf{x}(2\Delta t) \\ \dots \\ \mathsf{x}[n] = \mathsf{x}(n\Delta t) \end{cases}$$

We say that $x[i]$ is the value of x at the time step i. Then the finite difference can be written, for example, as follows:

$$\dot{\mathsf{x}} \approx \frac{1}{\Delta t}(\mathsf{x}[i+1] - \mathsf{x}[i])$$

We can rewrite our original autonomous LTI as follows:

$$\frac{1}{\Delta t}(\mathsf{x}[i+1] - \mathsf{x}[i]) = \mathsf{A}\mathsf{x}[i]$$

Isolating $\mathsf{x}[i+1]$ on the left hand side, we get:

$$\mathsf{x}[i+1] = (\mathsf{A}\Delta t + \mathsf{I})\mathsf{x}[i]$$

Or alternatively:

$$\frac{1}{\Delta t}(\mathsf{x}[i+1] - \mathsf{x}[i]) = \mathsf{A}\mathsf{x}[i+1]$$

Isolating $\mathsf{x}[i+1]$ on the left hand side, we get:

$$\mathsf{x}[i+1] = (\mathsf{I} - \mathsf{A}\Delta t)^{-1}\mathsf{x}[i]$$

If we have a non-autonomous dynamics in the form:

$$\dot{\mathsf{x}} = \mathsf{A}\mathsf{x} + \mathsf{B}\mathsf{u}$$

we can do the same transformation as before:

$$\frac{1}{\Delta t}(\mathsf{x}[i+1] - \mathsf{x}[i]) = \mathsf{A}\mathsf{x}[i] + \mathsf{B}\mathsf{u}[i]$$

$$\mathsf{x}[i+1] = (\mathsf{A}\Delta t + \mathsf{I})\mathsf{x}[i] + \mathsf{B}\Delta t\mathsf{u}[i]$$

Defining *discrete state space matrix* $\bar{\mathsf{A}}$ and *discrete control matrix* $\bar{\mathsf{B}}$ as follows:

$$\bar{\mathsf{A}} = (\mathsf{A}\Delta t + \mathsf{I}), \quad \bar{\mathsf{B}} = \mathsf{B}\Delta t$$

We get discrete dynamics:

$$\mathsf{x}[i+1] = \bar{\mathsf{A}}\mathsf{x}[i] + \bar{\mathsf{B}}\mathsf{u}[i]$$

This way of defining discrete dynamics is called *Zero order hold.*

ZOH and other types of discretization

Zero order hold vs First order hold

Graphically, we can understand what zero order hold is, by comparing it to the first order hold:

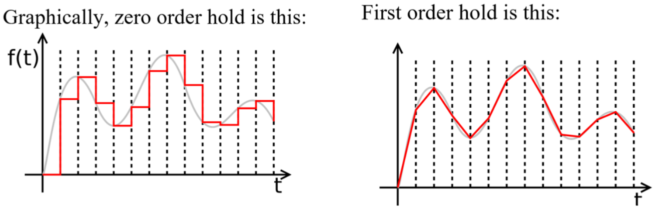

Figure: Different types of discretization

ZOH and other types of discretization

Precise discretization

Let the discrete state $\mathsf{x}[i]$ correspond to the moment of time continuous state x at the moment of time t_i, and $\mathsf{x}[i+1]$ - to the point of time t_{i+1}. Then, we can say that the discretization is *exact* if there is the following implication relation

$$\mathsf{x}[i] = \mathsf{x}(t_i) \to \mathsf{x}[i+1] = \mathsf{x}(t_{i+1})$$

or in other words, if state of both systems are identical for the discrete time points where the discrete states are defined.
We can compute exact discretization as follows:

$$\bar{\mathsf{A}} = e^{\mathsf{A}\Delta t}$$

$$\bar{\mathsf{B}} = \mathsf{B} \int_{t_0}^{t_0+\Delta t} e^{\mathsf{A}s} ds$$

Stability criterion

For continuous-time systems $\dot{\mathsf{x}} = \mathsf{A}\mathsf{x}$, the criterion is that the real parts of the eigenvalues of the matrix A are all negative.

For a discrete time system $\mathsf{x}[i+1] = \bar{\mathsf{A}}\mathsf{x}[i]$, the criterion for the stability in the sense of Lyapunov is that all eigenvalues should be less than 1 by magnitude:

$$|\lambda_i(\bar{\mathsf{A}})| \leq 1, \ \forall i$$

if $|\lambda_i(\bar{\mathsf{A}})| < 1, \ \forall i$, the system is asymptotically stable (or just stable).

Lecture slides are available via Moodle.

Check Moodle for additional links, videos, textbook suggestions.