

# Observer Design, Kalman Filter

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- Observer Design
  - Observer as a controllable LTI
  - Estimation error dynamics
  - State estimation error stability
  - Observer Design vs Controller design
  - Special case
  - General case: design via Riccati eq.
- Kalman Filter
  - System with noise and disturbances
  - Estimates definitions
  - Estimate updates
  - Knowns and unknowns
  - Covariance update and Kalman gain

# Observer Design

## Observer as a controllable LTI

- From the previous lecture we remember that we wanted the estimate of the state to have the same dynamics as the actual state.
- We also remember that we want to take into account the error between the estimated and measured output  $y$

Assume the measurement is perfect:  $\hat{y} = y$ . Then we can propose observer as the following dynamical system:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

# Observer Design

## Estimation error dynamics

Remember that state estimation error is  $\epsilon = \hat{x} - x$  and the actual dynamics of the system is:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Then, we can write the state estimation error dynamics:

$$\dot{\hat{x}} - \dot{x} = A\hat{x} - Ax + Bu - Bu + L(y - \hat{y})$$

or:

$$\dot{\epsilon} = A\epsilon + L(Cx - C\hat{x})$$

or:

$$\dot{\epsilon} = A\epsilon - LC\epsilon$$

# Observer Design

## State estimation error stability

Thus, with the proposed observer, state estimation error is:

$$\dot{\epsilon} = (A - LC)\epsilon$$

From which immediately follows that the observer is *stable* (i.e., the state estimation error tends to zero), as long as the following matrix is negative-definite:

$$A - LC < 0$$

We need to find L.

# Observer Design

## Observer Design vs Controller design

Let us observe the key difference between observer design and controller design.

Controller design: find such  $K$  that:

$$A - BK < 0$$

Observer design: find such  $L$  that:

$$A - LC < 0$$

We have instruments for finding  $K$ , what about  $L$ ?

# Observer Design

## Special case

Assume that  $C$  is identity  $I$ . Then the observer design problem becomes:

- find such  $L$  that  $A - LI < 0$ .

Which is equivalent to the problem:

- find such  $L$  that  $A - BL < 0$ , where  $B = I$ .

And in that formulation, it is equivalent to the controller design problem, which we know how to solve.

# Observer Design

General case: design via Riccati eq.

In general, we can observe that if  $A - LC$  is negative-definite, then  $(A - LC)^\top$  is negative-definite too (by definition of the negative-definiteness).

Therefore, we can solve the following *dual problem*:

- find such  $L$  that  $A^\top - C^\top L^\top < 0$ .

The dual problem is *equivalent* to the control design problem. We can solve it by producing and solving algebraic Riccati equation, as in the LQR formulation. In pseudo-code it can be represented the following way:

$$L^\top = \text{lqr}(A^\top, C^\top, Q, R).$$

where  $Q$  and  $R$  are weight matrices, determining the "sensitivity" or "aggressiveness" of the observer.



# Kalman Filter

## System with noise and disturbances

Assume we have a discrete linear system with disturbance and noise:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i + \mathbf{w} \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i + \mathbf{v} \end{cases}$$

where  $\mathbf{w}$  and  $\mathbf{v}$  are disturbance and noise, both are random processes. Covariance matrix of  $\mathbf{w}$  is  $\mathbf{W}$ , covariance matrix of  $\mathbf{v}$  is  $\mathbf{V}$ .

Our estimation  $\hat{\mathbf{x}}$  is now done in two steps: after dynamics update, and after sensor update. We will use the following notation for it:

- $\hat{\mathbf{x}}_{i|i-1}$  is the  $i$ -th step estimate after dynamics update; also called an *a priori* estimate.
- $\hat{\mathbf{x}}_{i|i}$  is the estimate after sensor update; also called an *a posteriori* estimate.

# Kalman Filter

## Estimates covariances

Both estimates  $\hat{x}_{i|i-1}$  and  $\hat{x}_{i|i}$  have associated state estimation errors:

- $\epsilon_{i|i-1} = x_i - \hat{x}_{i|i-1}$
- $\epsilon_{i|i} = x_i - \hat{x}_{i|i}$

Those state estimation errors have their covariance matrices:

- $P_{i|i-1} = \text{cov}(\epsilon_{i|i-1})$ ,
- $P_{i|i} = \text{cov}(\epsilon_{i|i})$

We can consider the output estimation error  $\varepsilon_i = y_i - C\hat{x}_{i|i-1}$  and its covariance matrix  $Y_i$ :

$$Y_i = \text{cov}(y_i - C\hat{x}_{i|i-1})$$

# Kalman Filter

## Estimate updates

We know that  $\hat{x}_{i|i-1}$  can be found as follows:

$$\hat{x}_{i|i-1} = A\hat{x}_{i-1|i-1} + Bu_i$$

where  $\hat{x}_{i-1|i-1}$  is the estimate on the previous time step.

We will search for the update law for the a posteriori estimate  $\hat{x}_{i|i}$  in the following form:

$$\hat{x}_{i|i} = \hat{x}_{i|i-1} + K_i(y_i - C\hat{x}_{i|i-1})$$

We need to find  $K_i$ , which is the Kalman filter gain.

# Kalman Filter

## Knows and unknowns

We know or measure:

- $A$ ,  $B$  and  $C$
- $W$  and  $V$
- $y_i$

We need to find:

- $\hat{x}_{i|i-1}$  and  $\hat{x}_{i|i}$
- $K_i$
- $Y_i$ ,  $P_{i|i-1}$  and  $P_{i|i}$

# Kalman Filter

## Covariance update and Kalman gain

There is a formula for the a priori estimation covariance update:

$$P_{i|i-1} = A P_{i-1|i-1} A^T + W$$

And a formula for the measurement covariance:

$$Y_i = C P_{i|i-1} C^T + V$$

With those two, we can find the Kalman filter gain:

$$K_i = P_{i|i-1} C^T Y_i^{-1}$$

Now we can find the a posteriori covariance:

$$P_{i|i} = (I - K_i C) P_{i|i-1}$$

Lecture slides are available via Moodle.

You can help improve these slides at:

<https://github.com/SergeiSa/Linear-Control-Slides-Spring-2020>

Check Moodle for additional links, videos, textbook suggestions.