Homework Assignment 2: Partial Differential Equations

Main deadline: Feb 11, 2024

The main goal of this Homework is to solve the scalar wave equation

$$\partial_t^2 \phi = \Delta \phi \tag{1}$$

on the surface of a sphere. To discretize $\phi(t, \theta, \phi)$ in space, we recommend using spherical harmonic functions Y_{lm}. The Laplace operator for spherical harmonics is a diagonal operator given by the factor $-l(l+1)$ for each c^{lm} (i.e., it is independent of the mode number m). This will lead to a system of ODEs for the coefficient vector c^{lm}.

To discretize ϕ in time, we want to use standard ODE methods. These standard methods assume that there is only a first derivative in time. We thus introduce a new function $\psi = \partial_t \phi$, arriving at the system

$$\partial_t \phi = \psi \tag{2}$$
$$\partial_t \psi = \Delta \phi \tag{3}$$

Both functions ϕ and ψ need to be expanded in spherical harmonics, i.e., the coefficient vector is now twice as long.

Questions

a) Numerically implement the discretization of ϕ in terms of spherical harmonics.

b) Use an initial condition that is peaked around the North Pole, i.e., that looks similar to a Gaussian with a width equal to 0.2. (The exact initial condition does not matter).

c) Evolve the system in time to see from $t = 0$ to $t = 10$ using your favorite ODE integrator. The resulting evolution should look similar to water waves moving on the surface of a pond, except that the pond is the surface of a sphere.

d) Create a series of figures or a movie that shows how the solution ϕ evolves in time. Perform the simulation three times with different choices of $l_{\max}$, and at least one of these with a small $l_{\max}$ (e.g., $l_{\max} = 4$) to study the influence of the cut-off $l_{\max}$.

Present your work, both the results and source code, in a Github repository. This should have the form of a brief report, i.e., there should be a (brief) description of the methods used and the results you obtained as text, not just the source code and figures.