Pierre-Antoine Graham

HOMEWORK 1

Eduardo Martín-Martínez, Bindiya Arora
Quantum Information

Perimeter Institute for Theoretical Physics
March 21, 2024

Contents

1 Concurrence and negativity

(a) We are interested in the faithful measure of entanglement of two qubits provided by the concurrence. The individual states of the qubits are elements of the hilbert space $\mathcal{H}_A = \mathbb{C}^2$ and $\mathcal{H}_B = \mathbb{C}^2$ and their joint state is described with most generality through the density matrix $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ [1] representing their classical and quantum correlations. We work from a basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{H}_{A,B}$ associated with the Pauli matrices denoted σ_x, σ_y and σ_z. For states in $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the concurence can be expressed as $C[p] := \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ for the set $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ (ordered in decreasing order) of eigenvalues of $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^\star(\sigma_y \otimes \sigma_y)$. We start by computing the concurrence for a Bell state $|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. To make the calculation more transparent, we use a local operation σ_x on the second qubit to get the state $|\Psi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$[2] , their concurrences are the same. We have

$$\rho = |\Psi^+_{AB}\rangle\langle\Psi^+_{AB}| = \frac{1}{2}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since ρ is block diagonal with two vanishing blocks, the calculation of its quare root reduces to the calculation of the square root of the non-zero block. This square root con be infered through

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \implies \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

where we selected the positive branch of the square root. Next, we calculate $\tilde{\rho}$ as follows

$$\begin{aligned}
\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^\star(\sigma_y \otimes \sigma_y) &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -i(-i) \\ 0 & 0 & -i(i) & 0 \\ 0 & i(-i) & 0 & 0 \\ i(i) & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\star\begin{pmatrix} 0 & 0 & 0 & -i(-i) \\ 0 & 0 & -i(i) & 0 \\ 0 & i(-i) & 0 & 0 \\ i(i) & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

and we finally obtain the following R matrix

$$\begin{aligned}
R = (\sigma_x \otimes 1)\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}(\sigma_x \otimes 1) &= (\sigma_x \otimes 1)\left(\frac{1}{\sqrt{2}\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)^{1/2}(\sigma_x \otimes 1) \\
&= (\sigma_x \otimes 1)\left(\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2\right)^{1/2}(\sigma_x \otimes 1) = (\sigma_x \otimes 1)\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}(\sigma_x \otimes 1)
\end{aligned}$$

[1] $\mathcal{D}$ denotes the space of hermitian operators of trace one acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ with positive or zero eigenvalues. We have this restriction on eigenvalues because they provide the probabilities of each outcome of the statistical mixture represented by the diagonalized ρ

[2] Starting with $|\rho_0\rangle = |\Phi^+_{AB}\rangle\langle\Phi^+_{AB}|$ and express the associated concurrence in terms of $\rho = |\Psi^+_{AB}\rangle\langle\Psi^+_{AB}|$: we have

$$\begin{aligned}
R = \sqrt{\sqrt{\rho_0}(\sigma_y \otimes \sigma_y)\rho_0^\star(\sigma_y \otimes \sigma_y)\sqrt{\rho_0}} &= \sigma_x\sqrt{\sqrt{\rho}(\sigma_x\sigma_y\sigma_x \otimes \sigma_y)\rho^\star(\sigma_x\sigma_y\sigma_x \otimes \sigma_y)\sqrt{\rho}}\sigma_x \\
&= \sigma_x\sqrt{\sqrt{\rho}(-\sigma_y \otimes \sigma_y)\rho^\star(-\sigma_y \otimes \sigma_y)\sqrt{\rho}}\sigma_x = \sigma_x\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}\sigma_x
\end{aligned}$$

which shows what effect the real local unitary $\sigma_x \otimes 1$ has on R. The $\sigma_x \otimes 1$ operation does not change the eigenvalues of R because it preserve the caracteristic polynomial $\det(\sigma_x(R-\lambda)\sigma_x) = (-1)^2\det(R-\lambda)$ brought to 0 by the eigenvalues of R. This property is a feature of any operator U having $U^\dagger U = 1$ (unitary operators preserve eigenvalue). we note that the transformation considered preserves the concurrence because its components are all real (if they were not, there would be complications regarding the complex conjugation of ρ)

Since the determinant of the non-zero block of R is 0, one of its eigenvalues is 0 and the other is forced to equal the trace and is 1. We have the sequence $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{1, 0, 0, 0\}$ leading to the concurrence $C[\rho] = \max(0, 1/\sqrt{2}) = 1/\sqrt{2}$. For the product state $|00\rangle$ the calculation of concurrence is simplified. We have that $\rho = |00\rangle\langle 00|$ which is a projector implying $\rho = \rho^2 \implies \sqrt{\rho} = \rho$. The matrix $\tilde{\rho}$ also has a simple expression $\tilde{\rho} = (\sigma_y \otimes \sigma_y)(1^\star)|00\rangle((\sigma_y \otimes \sigma_y)|00\rangle)^\dagger = (i)^2((i)^2)^\star |11\rangle\langle 11| = |11\rangle\langle 11|$. For this preoduct state, we get $R^2 = |00\rangle\langle 00||11\rangle\langle 11||00\rangle\langle 00| = 0 \implies R = 0$ and $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{0, 0, 0, 0\}$ leading to the concurrence $C[\rho] = \max(0, 0) = 0$.

(b) For states in $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we can measure entanglement with negativity defined as $\mathcal{N}[\rho] = (\mathrm{tr}(\sqrt{(\rho^\Gamma)^\dagger \rho^\Gamma}) - 1)/2$ where ρ^Γ is the partial transpose of ρ taken with respect to its subsystem A indices. Here we want to calculate the negativity of $\rho = |\Phi^+_{AB}\rangle\langle\Phi^+_{AB}|$. We start by calculating the partial transpose

$$\rho^\Gamma = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}^\Gamma = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

We then calculate the matrix norm

$$\mathrm{tr}(\sqrt{(\rho^\Gamma)^\dagger \rho^\Gamma}) = \mathrm{tr}\left(\left(\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)^{1/2}\right) = \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{1/2}\right) = 2$$

leading to a negativity $\mathcal{N}[\rho] = (2-1)/2 = 1/2$ which is half of the concurrence calculated for the same state in (a) as expected of a two-qubit system. Repeating the calculation for the state $\rho = |00\rangle\langle 00|$, we find $\rho^\Gamma = \rho$ and $\mathrm{tr}(\sqrt{|00\rangle\langle 00|(|00\rangle\langle 00|)^\dagger}) = 1 + 0 \times 3$ leading to the negativity $\mathcal{N}[\rho] = (1-1)/2 = 0$ as expected for a separable state.

(c) Here we are interested in the negativity of the bipartite state $\rho = 1_A \times 1_B/4$. We have $\rho^\Gamma = 1_A \times 1_B^T/4 = \rho^\Gamma$ and $\mathrm{tr}(1_A \times 1_B/4) = 1$ implying $\mathcal{N}[\rho] = (1-1)/2 = 0$ which is again expected for a separable state. Vanishing of negativity for separable states is a property differenciating it from Von Neumann entropy which fails to distinguish separable states from states that are not because it also grows with classical correlations.

2 POVM accounts for errors

(a) We consider a source producing a statistical mix of qubit pure sates described in the basis $\{|0\rangle, |1\rangle\}$ by the fully general density matrix $\rho = \frac{1}{2}(1 + \mathbf{r}\cdot\sigma)$ where $\mathbf{r} = (r_x, r_y, r_z)$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ (vector of Pauli matrices associated to the chosen basis). Since the trace of all Pauli matrices vanishes, the only term contributing to the trace of ρ is $\frac{1}{2}$ which ensures $\mathrm{tr}(\rho) = 1$. Through as basis transformation, we can alsway bring the density matrix to the form $\rho = \frac{1}{2}\mathrm{diag}(1+|\mathbf{r}|, 1-|\mathbf{r}|)$ showing that ρ has positive or zero probability eigenvalues if and only iff $0 \leq |\mathbf{r}| \leq 1$. The $|\mathbf{r}| = 1$ saturation produces states with purity 1 and we can identify the surface of a ball of radius 1 in the $\mathbf{r}$ parameter space with the bloch sphere of pure qubit states. The $|\mathbf{r}| = 1$ state produces a unique maximally mixed state living at the origin of the $\mathbf{r}$ parameter space. We see that in the ball of allowed parametres, we interpolate between maximally mixed at the origin and pure on the surface. Using the fact $\sigma_i \sigma j = i\varepsilon_{ijk}\sigma_k$ and $\sigma_i^2 = 1$ we have

$$\mathrm{tr}(\sigma_i\sigma_j) = \begin{cases} \mathrm{tr}(\sigma_i^2) = 2, & i = j \\ \mathrm{tr}(i\varepsilon_{ijk}\sigma_k) = i\varepsilon_{ijk}\mathrm{tr}(\sigma_k) = 0, & i \neq j \end{cases} = 2\delta_{ij}.$$

This property allow to extract the components of $\mathbf{r}$ from ρ with

$$\begin{aligned} r_i = \mathrm{tr}(\rho\sigma_i) &= \mathrm{tr}\left(\frac{1}{2}(1 + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)\sigma_i\right) \\ &= \frac{1}{2}\mathrm{tr}(\sigma_i) + \frac{1}{2}r_x\mathrm{tr}(\sigma_x\sigma_i) + \frac{1}{2}r_y\mathrm{tr}\left(\sigma_y\sigma_i\right) + \frac{1}{2}r_z\mathrm{tr}(\sigma_z\sigma_i) = r_x\delta_{ix} + r_y\delta_{iy} + r_z\delta_{iz} \end{aligned}$$

(b)

(c)

3 Entanglement-breaking channel

(a)
(b)
(c)

4 Acknowledgement

References

[1] *Frobenius inner product*. https://en.wikipedia.org/wiki/Frobenius_inner_product. Accessed: 2024-03-14.

[2] *Schmidt decomposition*. https://en.wikipedia.org/wiki/Schmidt_decomposition. Accessed: 2024-03-14.