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Homework 1

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Contents

1 Cartan in a FLRW universe

(a) The Friedmann-Lemaitre-Robinson-Walker (FLRW) metric two-form describes a spacetime with spacelike foliation in homogeneous and isotropic hypersurfaces. In a coordinate chart with coordinates $x^\mu = \{t, \theta, \phi, r\}$ making the isotropy and foliation manifest, this metric reads

$$g_{\mu\nu}\underline{d}x^\mu \otimes \underline{d}x^\nu \equiv \underline{d}t \otimes \underline{d}t - a^2(t)\left(\frac{\underline{d}r \otimes \underline{d}r}{1-kr^2} + r^2\left(\underline{d}\theta \otimes \underline{d}\theta + \sin^2\theta \underline{d}\phi \otimes \underline{d}\phi\right)\right)$$

where $\{\underline{d}x^\mu\}_{\mu=0}^3 = \{\underline{d}t, \underline{d}\theta, \underline{d}\phi, \underline{d}r\}$ are the coordinate on-forms dual to the vector basis $\underline{e}_a = \{\partial_t, \partial_\theta, \partial_\phi, \partial_r\}$, $a(t) > 0$ is the scale factor and $k = 0, -1, 1$ gives the sign of the curvature of the spacelike hypersurfaces (respectively flat, Anti-de Sitter, de Sitter). In what follows, the tensor products are implicit. At every point in our chart, we define an orthonormal basis of one-forms $\underline{\omega}^a = c^a_\mu \underline{d}x^\mu$ such that $g_{\mu\nu}\underline{d}x^\mu \underline{d}x^\nu = \eta_{ab}\underline{\omega}^a\underline{\omega}^b$ where η_{ab} is the Minkowski metric components with signature $(+,-,-,-)$. We can write

$$\begin{aligned}
&g_{\mu\nu}\underline{d}x^\mu \underline{d}x^\nu \\
&= \underline{d}t\underline{d}t - \left(\frac{a(t)\underline{d}r}{\sqrt{1-kr^2}}\right)\left(\frac{a(t)\underline{d}r}{\sqrt{1-kr^2}}\right) - \left(a(t)r\underline{d}\theta\right)\left(a(t)r\underline{d}\theta\right) - (a(t)r\sin\theta\underline{d}\phi)(a(t)r\sin\theta\underline{d}\phi) \\
&= \underline{\omega}^0\underline{\omega}^0 - \underline{\omega}^1\underline{\omega}^1 - \underline{\omega}^2\underline{\omega}^2 - \underline{\omega}^3\underline{\omega}^3
\end{aligned}$$

where $\{\underline{\omega}^a\}_{a=0}^3 = \{\underline{d}t,\ a(t)r\underline{d}\theta,\ a(t)r\sin\theta\underline{d}\phi,\ \frac{a(t)}{\sqrt{1-kr^2}}\underline{d}r\}$ is shown to satisfy the orthonormality condition. We note that the resulting choice of basis is unique up to a local lorentz transformation (which preserves orthonormality).

(b) To calculate the connection one-forms $\underline{\theta}^a{}_b$, we use the orthonormal basis found in (a) and Cartan's first structure equation for vanishing torsion to get

$$\begin{aligned}
\underline{\theta}^a{}_b \wedge \underline{\omega}^b = -\underline{d}\underline{\omega}^a &= \begin{cases} -\partial_\mu(1)\ \underline{d}x^\mu \wedge \underline{d}t \\ -\partial_\mu(a(t)r)\ \underline{d}x^\mu \wedge \underline{d}\theta \\ -\partial_\mu(a(t)r\sin\theta)\ \underline{d}x^\mu \wedge \underline{d}\phi \\ -\partial_\mu\left(\frac{a(t)}{\sqrt{1-kr^2}}\right)\ \underline{d}x^\mu \wedge \underline{d}r \end{cases} \\
&= \begin{cases} 0 \\ -a'(t)r\underline{d}t \wedge \underline{d}\theta - a(t)\underline{d}r \wedge \underline{d}\theta \\ -a'(t)r\sin\theta\underline{d}t \wedge \underline{d}\phi - a(t)\sin\theta\underline{d}r \wedge \underline{d}\phi - a(t)r\cos\theta\underline{d}\theta \wedge \underline{d}\phi \\ -\frac{a'(t)}{\sqrt{1-kr^2}}\underline{d}t \wedge \underline{d}r - [\cdots]\underline{d}r \wedge \underline{d}r \end{cases} \\
&= \begin{cases} 0 \\ \frac{a'(t)}{a(t)}\underline{\omega}^1 \wedge \underline{\omega}^0 + \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1 \wedge \underline{\omega}^3 \\ \frac{a'(t)}{a(t)}\underline{\omega}^2 \wedge \underline{\omega}^0 + \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2 \wedge \underline{\omega}^3 + \frac{1}{a(t)r}\cot\theta\underline{\omega}^2 \wedge \underline{\omega}^1 \\ \frac{a'(t)}{a(t)}\underline{\omega}^0 \wedge \underline{\omega}^3 \end{cases} = \begin{cases} \underline{\theta}^0{}_b \wedge \underline{\omega}^b \\ \underline{\theta}^1{}_b \wedge \underline{\omega}^b \\ \underline{\theta}^2{}_b \wedge \underline{\omega}^b \\ \underline{\theta}^3{}_b \wedge \underline{\omega}^b \end{cases}
\end{aligned}$$

Since the $\wedge$ product with $\underline{\omega}^b$ maps $\underline{\omega}^{c\neq b}$ to linearly independant two-forms, we can read the coefficients of $\underline{\omega}^{c\neq b}$ preceeding the $\wedge$ product in the previous expressions. We have

$$\begin{cases} \underline{\theta}^1{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^1 + [\cdots]\underline{\omega}^0, \quad \underline{\theta}^1{}_3 = \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1 + [\cdots]\underline{\omega}^3 \\ \underline{\theta}^2{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^2, \quad \underline{\theta}^2{}_3 = \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2, \quad \underline{\theta}^2{}_1 = \frac{1}{a(t)r}\cot\theta\underline{\omega}^2 \\ \underline{\theta}^3{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^3 \end{cases}$$

where $[\cdots]$ terms represent the terms mapped to 0 by the $\wedge$ product from which information about $\underline{\theta}^a{}_b$ was read.

To extract the connection one-forms components from these relations, we invoke the antisymmetry relation $g_{ca}\underline{\theta}^c{}_b + g_{ca}\underline{\theta}_b{}^c = \underline{d}g_{ab}$. Recalling that in our orthonormal basis $g_{ab} = \eta_{ab}$, we have

$$\eta_{ca}\underline{\theta}^c{}_b + \eta_{ca}\underline{\theta}_b{}^c = 0 \iff \eta_{ca}\eta^{ad}\underline{\theta}^c{}_b + \eta_{ca}\eta^{ad}\underline{\theta}_b{}^c = \underline{\theta}^d{}_b + \underline{\theta}_b{}^d = 0$$

(c)

(d)

2 Acknowledgement

Thanks to Luke for help reviewing and understanding the concepts used in this assignement