Pierre-Antoine Graham

HOMEWORK 1

Bindiya Arora
Quantum Mechanics

Perimeter Institute for Theoretical Physics
September 21, 2023

Contents

1 Quantum revivals

Consider a one-dimensionnal quantum harmonic oscillator with mass m, frequency ω, momentum operator p and position operator x. The Hamiltonian governing the evolution of x and p in the Heisenberg picture is

$$H = \frac{p^2(t)}{2m} + \frac{1}{2}m\omega^2 x^2(t).$$

A Operator time dependance

In the Schrodinger picture, the time dependence of x, and p is given by

$$\frac{dx}{dt} = \frac{1}{i\hbar}[x,H] = \frac{1}{i\hbar}\left([x, \frac{p^2(t)}{2m} + \frac{1}{2}m\omega^2 x^2(t)]\right) = \frac{1}{2i\hbar m}\left([x,p(t)]p + p[x,p(t)]\right) = \frac{2}{2i\hbar m}[x,p]\frac{p}{m} = \frac{p}{m}$$
$$\frac{dp}{dt} = \frac{1}{i\hbar}[p,H] = \frac{1}{i\hbar}\left([p, \frac{p^2(t)}{2m} + x^2(t)]\right) = \frac{m\omega^2}{2i\hbar}\left([p, \frac{1}{2}m\omega^2 x(t)]x + x[p,x(t)]\right) = \frac{2m\omega^2}{2i\hbar}[p,x] = -m\omega^2 x$$

because $[x,p] = -[p,x] = i\hbar\mathbf{1}$ is a multiple of the identity and commutes with x and p. To solve for the time evolution of x and p, we first differentiate the first equation to get

$$\frac{d^2x}{dt^2} = \frac{1}{m}\frac{dp}{dt} = -\omega^2 x.$$

The solution of this second-order operator differential equation can be found componentwise because all components are decoupled from each other (the initial conditions will ensure x is hermitian). For each component $\langle x'|\,x(t)\,|x"\rangle$ in the eigenbasis of $x(0)$ We get a scalar harmonic oscillator equation

$$\frac{d^2}{dt^2}\langle x'|\,x(t)\,|x"\rangle = -\omega^2\langle x'|\,x(t)\,|x"\rangle \iff \langle x'|\,x(t)\,|x"\rangle = A(x',x")\cos(\omega t) + B(x',x")\frac{\sin(\omega t)}{\omega}$$

with A, B determined by the initial conditions $x(t) = x(0)$. Evaluating the solution and its derivatives at $t = 0$ we have

$$\langle x'|\,x(0)\,|x"\rangle = A(x',x"), \quad \text{and} \quad \langle x'|\,\frac{dx}{dt}(0)\,|x"\rangle = \frac{1}{m}\langle x'|\,p(0)\,|x"\rangle = B(x',x").$$

The functions A and B are therefore components of the operators $x(0)$ and $p(0)/(m)$ (initial position and initial velocity respectively) leading to the explicit solution of the initial value problem $x(t) = x(0)\cos(\omega t) + (p(0)/m)\frac{\sin(\omega t)}{\omega}$. To obtain $p(t)$ we use the expression found for the time derivative of x to find

$$p(t) = m\frac{dx}{dt} = -m\omega x(0)\sin(\omega t) + p(0)\cos(\omega t).$$

B Correlation function

The position time-correlation function evaluated on the ground state $|0\rangle$ of the harmonic oscillator is given by

$$
\begin{aligned}
C(t) = \langle 0|\, x(0)x(t)\,|0\rangle = \langle 0| \int \mathrm{d}x'\, |x'\rangle\, \langle x'|\, x(0)(x(0)\cos(\omega t) + (p(0)/m)\frac{\sin(\omega t)}{\omega})\,|0\rangle \\
= \int \mathrm{d}x' \left(x'^2 |\psi_0(x')|^2 \cos(\omega t) + \frac{i\hbar}{m}\frac{\sin(\omega t)}{\omega}\psi_0 \frac{d}{dx'}(x'\psi_0^*) \right) \\
= \cos(\omega t) \int \mathrm{d}x' \left(x'^2 |\psi_0(x')|^2 \right) + \frac{i\hbar}{m}\frac{\sin(\omega t)}{\omega} \int \mathrm{d}x' |\psi_0(x')|^2 + \frac{i\hbar}{m}\frac{\sin(\omega t)}{\omega} \int \mathrm{d}x' \left(\psi_0 x' \frac{d}{dx'}\psi_0^* \right) \\
= \cos(\omega t) \int \mathrm{d}x' \left(x'^2 |\psi_0(x')|^2 \right) + \frac{i\hbar}{m}\frac{\sin(\omega t)}{\omega} + \frac{i\hbar}{m}\frac{\sin(\omega t)}{\omega} \int \mathrm{d}x' \left(\psi_0 x' \frac{d}{dx'}\psi_0^* \right)
\end{aligned}
$$

using the wavefunction $\psi_0(x') = \langle x'|0\rangle$, $\langle x'|\, x(0) = \langle x'|\, x'$ and $\langle x'|\, p(0)\,|0\rangle = \frac{d}{dx'}\psi_0$. To evaluate the first integral, we use the explicit expression

$$
\psi_0(x') = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x'^2\right) \implies |\psi_0(x')|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(-\frac{m\omega}{\hbar}x'^2\right)
$$

to get

$$
\begin{aligned}
\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \cos(\omega t) \int \mathrm{d}x' x'^2 \exp\left(-\frac{m\omega}{\hbar}x'^2\right) &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \cos(\omega t) \frac{-\hbar}{\omega}\frac{d}{dm} \int \mathrm{d}x' \exp\left(-\frac{m\omega}{\hbar}x'^2\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \cos(\omega t) \frac{-\hbar}{\omega}\frac{d}{dm} \left(\frac{\pi\hbar}{m\omega}\right)^{\frac{1}{2}} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \cos(\omega t) \frac{-\hbar}{2m\omega} \left(\frac{\pi\hbar}{m\omega}\right)^{\frac{1}{2}} = -\frac{\hbar}{2m\omega}\cos(\omega t).
\end{aligned}
$$

The last integral reads

$$
\begin{aligned}
\frac{i\hbar}{m}\frac{\sin(\omega t)}{\omega} \int \mathrm{d}x' \left(\psi_0 x' \frac{d}{dx'}\psi_0^* \right) &= \frac{-m\omega}{\hbar}\frac{i\hbar}{m}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{\sin(\omega t)}{\omega} \int \mathrm{d}x' \left(x'^2 \exp\left(-\frac{m\omega}{\hbar}x'^2\right)\right) \\
&= \frac{-m\omega}{\hbar}\frac{i\hbar}{m}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{\sin(\omega t)}{\omega}\frac{-\hbar}{2m\omega}\left(\frac{\pi\hbar}{m\omega}\right)^{\frac{1}{2}} = i\sin(\omega t)\frac{-\hbar}{2m\omega}.
\end{aligned}
$$

Combining all terms, we get

$$
C(t) = -\frac{\hbar}{2m\omega}\cos(\omega t) + \frac{i\hbar}{m}\frac{\sin(\omega t)}{\omega} + i\frac{-i\hbar}{2m\omega}\sin(\omega t) = -\frac{\hbar}{2m\omega}e^{-i\omega t}.
$$

2 Composite Spin

The Hilbert space $\mathcal{H}$ of two particles of spin 1/2 with hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ is given by the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. We are interested in the matrix representation of the total spin component operators. In the tensor product basis $\{|11\rangle, |01\rangle, |10\rangle, |00\rangle\}$, they are expressed as

$$
\begin{aligned}
\sigma_x &:= \sigma_x^{(1)} \otimes 1^{(2)} + 1^{(1)} \otimes \sigma_x^{(2)} \\
\sigma_y &:= \sigma_y^{(1)} \otimes 1^{(2)} + 1^{(1)} \otimes \sigma_y^{(2)} \\
\sigma_z &:= \sigma_z^{(1)} \otimes 1^{(2)} + 1^{(1)} \otimes \sigma_z^{(2)}
\end{aligned}
$$

where $1^{(i)}$ and $\sigma^{(i)}_{x,y,z}$ are respectively the identity matrix and the Pauli matrices in the $|1\rangle, |0\rangle$ basis of $\mathcal{H}_i$. The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The tensor product operation leads to the following $\sigma_{x,y,z}$ matrices :

$$\sigma_x = \begin{pmatrix} 1\cdot\sigma_x^{(1)} & 0\cdot\sigma_x^{(1)} \\ 0\cdot\sigma_x^{(1)} & 1\cdot\sigma_x^{(1)} \end{pmatrix} + \begin{pmatrix} (\sigma_x)_{11}\cdot 1^{(1)} & (\sigma_x)_{10}\cdot 1^{(1)} \\ (\sigma_x)_{01}\cdot 1^{(1)} & (\sigma_x)_{00}\cdot 1^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{array}{l} |11\rangle \\ |01\rangle \\ |10\rangle \\ |00\rangle \end{array}$$

$$\sigma_y = \begin{pmatrix} 1\cdot\sigma_y^{(1)} & 0\cdot\sigma_y^{(1)} \\ 0\cdot\sigma_y^{(1)} & 1\cdot\sigma_y^{(1)} \end{pmatrix} + \begin{pmatrix} (\sigma_y)_{11}\cdot 1^{(1)} & (\sigma_y)_{10}\cdot 1^{(1)} \\ (\sigma_y)_{01}\cdot 1^{(1)} & (\sigma_y)_{00}\cdot 1^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1\cdot\sigma_z^{(1)} & 0\cdot\sigma_z^{(1)} \\ 0\cdot\sigma_y^{(1)} & 1\cdot\sigma_z^{(1)} \end{pmatrix} + \begin{pmatrix} (\sigma_z)_{11}\cdot 1^{(1)} & (\sigma_z)_{10}\cdot 1^{(1)} \\ (\sigma_z)_{01}\cdot 1^{(1)} & (\sigma_z)_{00}\cdot 1^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

3 Free Path Integral

The lagrangian of a free one-dimensional particle with mass m described by a generalised coordinate q, is $L = \frac{1}{2}m\dot{q}$. To use the path integral formalism, we need to discretize the trajectory $q(t)$ in N steps. Each step is associated with an independent variable q_n corresponding to the coordinate of the particle at time nT/N where T is the final time at which we wish to observe the particle. The time interval for a step is $\Delta t = T/N$ and we have $\dot{q} = \frac{q_{n+1}-q_n}{\Delta t}$. Going further, the action integral is replaced by a discrete sum expressed as

$$S = \sum_{n=0}^{N-1} \frac{1}{2}m\left(\frac{q_{n+1}-q_n}{\Delta t}\right)^2 \Delta t.$$

The path integral representation of the amplitude A for the particle to scatter from q_0 to q_N is given in the discretized picture by

$$A = \left(\frac{im}{2\hbar\pi\Delta t}\right)^{N/2}\left(\prod_{n=1}^{N-1}\int_{-\infty}^{\infty}\mathrm{d}q_n\right)\exp\left(\frac{i}{\hbar}\sum_{n=0}^{N-1}\frac{1}{2}m\left(\frac{q_{n+1}-q_n}{\Delta t}\right)^2\Delta t\right)$$

To compute it, we consider the sequence

$$\begin{aligned}
S(r) &= \left(\frac{im}{2\hbar\pi\Delta t}\right)^{N/2}\left(\prod_{n=1}^{N-r}\int_{-\infty}^{\infty}\mathrm{d}q_n\right)\exp\left(\frac{i}{\hbar}\sum_{n=0}^{N-r-1}\frac{1}{2}m\left(\frac{q_{n+1}-q_n}{\Delta t}\right)^2\Delta t + \frac{i}{\hbar}\frac{1}{2}m\left(\frac{q_N-q_{N-r}}{\Delta tr}\right)^2\Delta tr\right)\\
&= \left(\frac{im}{2\hbar\pi\Delta t}\right)^{N/2}\left(\prod_{n=1}^{N-r-1}\int_{-\infty}^{\infty}\mathrm{d}q_n\right)\exp\left(\frac{i}{\hbar}\sum_{n=0}^{N-r-2}\frac{1}{2}m\left(\frac{q_{n+1}-q_n}{\Delta t}\right)^2\Delta t\right)\int_{-\infty}^{\infty}\mathrm{d}q_{N-r}\exp\left(\frac{1}{2}m\left(\frac{q_N-q_{N-r}}{\Delta tr}\right)^2\Delta tr + \frac{1}{2}m\left(\frac{q_{N-r}-q_{N-r-1}}{\Delta t}\right)^2\Delta t\right)\\
&= \left(\frac{im}{2\hbar\pi\Delta t}\right)^{N/2}\left(\prod_{n=1}^{N-(r+1)}\int_{-\infty}^{\infty}\mathrm{d}q_n\right)\exp\left(\frac{i}{\hbar}\sum_{n=0}^{N-(r+1)-1}\frac{1}{2}m\left(\frac{q_{n+1}-q_n}{\Delta t}\right)^2\Delta t + \frac{mi}{2\hbar}\left(\frac{q_N-q_{N-(r+1)}}{(r+1)\Delta t}\right)^2(r+1)\Delta t\right)\left(\frac{2\hbar\pi\Delta tr}{mi(r+1)}\right)^{1/2}\\
&= \left(\frac{im}{2\hbar\pi\Delta t}\right)^{N/2}\left(\frac{2\hbar\pi\Delta tr}{mi(r+1)}\right)^{1/2}S(r+1)
\end{aligned}$$

where we used

$$\begin{aligned}
&\int_{-\infty}^{\infty}\mathrm{d}q_{N-r}\exp\left(\frac{im}{2\hbar}\left(\frac{q_N-q_{N-r}}{r\Delta t}\right)^2 r\Delta t + \frac{im}{2\hbar}\left(\frac{q_{N-r}-q_{N-r-1}}{\Delta t}\right)^2\Delta t\right)\\
&= \int_{-\infty}^{\infty}\mathrm{d}q_{N-r}\exp\left(\frac{im}{2\hbar\Delta t}\left(\left(\frac{r+1}{r}\right)q_{N-r}^2 - 2\left(\frac{q_M}{r}+q_{N-r-1}\right)q_{N-r}\right)\right)\exp\left(\frac{im}{2\hbar\Delta t}\left(q_{N-r-1}^2+\frac{q_N^2}{r}\right)\right)\\
&= \int_{-\infty}^{\infty}\mathrm{d}q_{N-r}\exp\left(\frac{im}{2\hbar\Delta t}\left(\frac{r+1}{r}\right)\left(q_{N-r}^2 - \left(\frac{2}{r+1}\right)(q_N+rq_{N-r-1})q_{N-r} + \left(\frac{q_N+rq_{N-r-1}}{r+1}\right)^2\right)\right)\exp\left(\frac{im}{2\hbar\Delta t}\left(q_{N-r-1}^2+\frac{q_N^2}{r}-\left(\frac{r+1}{r}\right)\left(\frac{q_N+rq_{N-r-1}}{r+1}\right)^2\right)\right)\\
&= \left(\frac{2\hbar\pi\Delta tr}{mi(r+1)}\right)^{1/2}\exp\left(\frac{im}{2\hbar\Delta t(r+1)}\left((r+1)q_{N-r-1}^2+\frac{q_N^2(r+1)}{r}-\left(\frac{q_N^2}{r}+rq_{N-r-1}^2+2q_Nq_{N-r-1}\right)\right)\right)\\
&= \left(\frac{2\hbar\pi\Delta tr}{mi(r+1)}\right)^{1/2}\exp\left(\frac{im}{2\hbar\Delta t(r+1)}\left((r+1)q_{N-r-1}^2+q_N^2-rq_{N-r-1}^2-2q_Nq_{N-r-1}\right)\right)\\
&= \left(\frac{2\hbar\pi\Delta tr}{mi(r+1)}\right)^{1/2}\exp\left(\frac{mi}{2\hbar}\left(\frac{q_N-q_{N-r-1}}{(r+1)\Delta t}\right)^2(r+1)\Delta t\right)
\end{aligned}$$

Comparing S with A we see $A = S(1)$ and we also note that the maximal value for r is provided by $N-r=1 \iff N-1=r$ which corresponds to

$$\begin{aligned}
S(N-1) &= \left(\frac{im}{2\hbar\pi\Delta t}\right)^{N/2}\left(\prod_{n=1}^{1}\int_{-\infty}^{\infty}\mathrm{d}q_n\right)\exp\left(\frac{i}{\hbar}\frac{1}{2}m\left(\frac{q_{0+1}-q_0}{\Delta t}\right)^2\Delta t + \frac{i}{\hbar}\frac{1}{2}m\left(\frac{q_N-q_1}{\Delta t(N-1)}\right)^2\Delta t(N-1)\right)\\
&= \left(\frac{im}{2\hbar\pi\Delta t}\right)^{N/2}\left(\frac{\hbar\pi\Delta t(N-1)}{mi(N)}\right)^{1/2}\exp\left(\frac{i}{\hbar}\frac{1}{2}m\left(\frac{q_N-q_0}{N\Delta t}\right)^2 N\Delta t\right).
\end{aligned}$$

Unpacking the telescopic expression for $S(0)$ we have

$$\begin{aligned}
S(1) &= \left(\frac{2\hbar\pi\Delta t(1)}{mi(1+1)}\right)^{1/2}S(1) = \left(\frac{2\hbar\pi\Delta t2}{mi(2+1)}\right)^{1/2}S(2) = \left(\frac{2\hbar\pi\Delta t(1)}{mi(1+1)}\right)^{1/2}\left(\frac{2\hbar\pi\Delta t(2)}{mi(2+1)}\right)^{1/2}S(3) = S(N-1)\prod_{r=1}^{N-2}\left(\frac{2\hbar\pi\Delta t(r)}{mi(r+1)}\right)^{1/2}\\
&= \left(\frac{im}{2\pi\Delta t}\right)^{N/2}\left(\frac{2\hbar\pi\Delta t(N-1)}{mi(N)}\right)^{1/2}\exp\left(\frac{i}{\hbar}\frac{1}{2}m\left(\frac{q_N-q_0}{N\Delta t}\right)^2 N\Delta t\right)\left(\frac{2\hbar\pi\Delta t}{mi}\right)^{(N-2)/2}\left(\frac{(1)}{(N-1)}\right)^{1/2}\\
&= \left(\frac{im}{2\hbar\pi\Delta t}\right)^{N/2}\exp\left(\frac{i}{\hbar}\frac{1}{2}m\left(\frac{q_N-q_0}{N\Delta t}\right)^2 N\Delta t\right)\left(\frac{2\hbar\pi\Delta t}{mi}\right)^{(N-1)/2}\left(\frac{1}{N}\right)^{1/2} = \exp\left(\frac{mi}{2\hbar T}\frac{(q_N-q_0)^2}{2}\right)\left(\frac{mi}{2\hbar\pi T}\right)^{1/2}
\end{aligned}$$

This result coincides with the quantum mechanical free particle propagator $K(q_0, q_1, 0, T) = \langle q_1| e^{-iH_{\text{free}}T/\hbar} |q_0\rangle$ (evolving with H_{free}) which is the imaginary time heat kernel (the free Schrodinger equation is a wick rotated heat equation so it makes sense that its propagator can be deduced by a wick rotation of the heat kernel). Furthermore, for fixed q_0 this propagator is the diffusing wave function of a free wave packet initially localized at q_0.

4 Mach-Zehnder Interferometer

The Mach-Zehnder interferometer is a sequence consisting of a beam splitter followed by a phase element on one of the split paths finally ending with a second beam splitter. In a quantum computer, we can represent the two paths with the orthogonal states $|0\rangle$ and $|1\rangle$ of a qubit. The first beam splitter is represented by the action of a Hadamard gate on $|0\rangle$ and the phase is represented by a phase gate leaving $|0\rangle$ unchanged while multiplying $|1\rangle$ by a phase. The final beam splitter is again represented with a Hadamard gate. The probabilities to find the qubit in state $|0\rangle$ or $|1\rangle$ will oscillate as the phase shift applied increases (we get an interference pattern). To go further, we can implement the effect of a probe on the qubit. The probe is represented by a second qubit. After the first Hadamard gate, the state of the first qubit represents the path: it should activate the probe (flip its state from $|0\rangle$ to $|1\rangle$ state) only if the path is $|1\rangle$. The quantum computation gate that does this is the CNOT gate. It entangles the path with the probe state and no interference pattern is observed. Quantum circuit for Mach-Zehnder

5 Acknowledgement

Thanks to Luke for pointing out mistakes in my calculation for the first question and for a discussion about the time reversal effect of conjugating a correlation function.

References

[1] *Wave packet, Wikipedia*. 2023. URL: https://en.wikipedia.org/wiki/Wave_packet.