PERIMETER SCHOLARS INTERNATIONAL

Classical Physics HW3
Fall 2023
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Electromagnetic radiation

Due: Monday October 9th @ 9PM
https://www.dropbox.com/request/jOlFzrGncD7qiHfSuE7i
(deadline and resubmission policy available in the course outline)

Late deadline: Monday October 16th @ 9PM
https://www.dropbox.com/request/rE71YXrnl47E7wcPf6C7

Pass/fail deadline: formal request needed by Monday October 30th

In this homework wee will study electromagnetic radiation and the power (energy per unit time) it carries. You can think of this as a warm-up for analogous (but different!) computation you will soon see in General Relativity.

1 Planar electromagnetic waves

a) (i) Write down the Maxwell equations in terms of a (general) vector potential A_μ. (ii) Next, write down the condition A_μ must satisfy in the Lorentz gauge as well as the ensuing Maxwell equations.

b) Consider the (complex[1]) Ansatz

$$A_\mu(t,\mathbf{x}) = a_\mu \exp\left(ik_\mu x^\mu\right) \quad \text{with} \quad \begin{cases} k^\mu = (\omega, \mathbf{k}) \\ a^\mu = (\phi, \mathbf{a}) \end{cases}$$

[1]Since the vacuum Maxwell equations are linear, we can always find real solutions by taking the imaginary and complex parts of these equations separately — but working in complex variables is algebraically much easier.

and find the conditions that the components of k^μ and a_μ must satisfy to solve the vacuum Maxwell equations in Lorentz gauge (these are the two equations you wrote at the point (ii) of the previous question, when $j^\mu = 0$).

c) Denoting $\mathbf{n}$ the unit vector in the direction $\mathbf{k}$, show that

$$\mathbf{n} \cdot \mathbf{E} = 0 = \mathbf{n} \cdot \mathbf{B} \quad \text{and} \quad \mathbf{E} = \mathbf{B} \times \mathbf{n}.$$

What does this mean for the norm of $\mathbf{E}$, $\mathbf{B}$, and their direction with respect to the direction of propagation of the way? Sketch this setup in a drawing. Are the oscillation of $\mathbf{E}$ and $\mathbf{B}$ in phase or not?

d) In the following we shall consider only the solutions with $A^0 = 0$, what does this mean for $\mathbf{A}$? Write $\mathbf{E}$ and $\mathbf{B}$ in terms of $\dot{\mathbf{A}}$ and $\mathbf{n}$. This solutions describe a (linearly polarized) electromagnetic plane wave.

e) Later in the course we will see that the momentum and energy density carried by the electromagnetic field are encoded in the Poynting vector $\mathbf{S}$ and the quantity ε respectively:

$$\mathbf{S} \doteq \mathbf{E} \times \mathbf{B} \quad \text{and} \quad \varepsilon \doteq \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2).$$

Show that Maxwell equations imply the following equation:

$$\partial_t \varepsilon + \nabla \cdot \mathbf{S} = 0.$$

Interpret its meaning: which other physical meaning can we give to the Poynting vector in view of this equation?

f) Deduce the following formula for the total power carried by energy waves traversing a "far away" sphere Ω_R at distance $R \gg 1$:

$$P = \frac{dE}{dt} = R^2 \int \vec{s} \cdot \mathbf{S} \, \sin\theta d\theta d\psi. \tag{1}$$

for longitudinal and polar angles ψ and θ.

g) Show that for the planar wave considered above,

$$\mathbf{S} = \varepsilon \mathbf{n}.$$

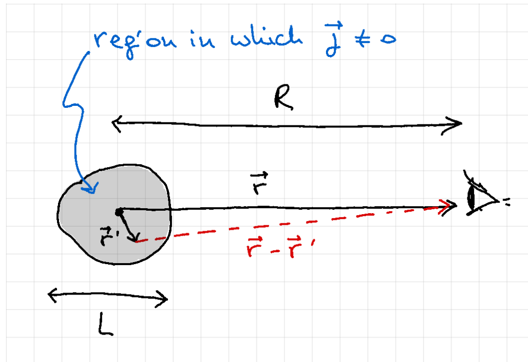

Figure 1: A schematic view of the setup for the computation of the electromagnetic field in the *wave zone*, $R \gg L$.

2 Radiation of an isolated system

a) Let us next consider the electromagnetic field produced by a motion of charges in an isolated system, see Figure 1. We shall concentrate on the wave zone of this system, i.e. on the regime where the system is small with respect to the the distance R at which we study the waves it emits. We will also take the approximation of slowly moving charges. That is, if L denotes the typical size of the system and v the typical velocity of the charges in it, we have

$$R \gg L\,, \quad v \ll 1\,.$$

Argue that the retarded potential

$$\mathbf{A}(t,\mathbf{r}) = \int d^3r' \, \frac{\mathbf{j}(t_R,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,, \quad t_R = t - |\mathbf{r}-\mathbf{r}'|\,,$$

in the radiation zone ($\mathbf{r} = O(R)$) is approximately given by

$$\mathbf{A}(t,\mathbf{r}) \approx \frac{1}{|r|}\int d^3r' \, \mathbf{j}(t_R,\mathbf{r}')\,. \tag{2}$$

b) Next, show that in the same approximation

$$|\mathbf{r}-\mathbf{r}'| \approx |\mathbf{r}| - \mathbf{r}'\cdot\mathbf{n}\,, \quad \mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}\,, \quad \text{and} \quad t' = t - |\mathbf{r}|,$$

and show that (2) yields

$$\mathbf{A}(t,\mathbf{r}) \approx \frac{1}{|\mathbf{r}|}\int dr' \; \mathbf{j}(t',\mathbf{r}') + \frac{1}{|\mathbf{r}|}\frac{\partial}{\partial t'}\int d^3r' \; (\mathbf{r}'\cdot\mathbf{n}) \; \mathbf{j}(t',\mathbf{r}') \,. \tag{3}$$

c) Focusing on the first term in the previous equation, show that under the approximations we made, one has

$$\mathbf{A}(t,\mathbf{r}) = \frac{\dot{\mathbf{d}}(t')}{|\mathbf{r}|}\,, \quad t' = t - |\mathbf{r}|\,,$$

where

$$\mathbf{d}(t') = \int d^3r' \; \rho(r',t')\mathbf{r}' = \sum_{\alpha=1}^{N} e_\alpha \mathbf{r}'_\alpha\,,$$

is the *(electric) dipole moment of the system* (the rightmost expression holds for a system constituted of N point particles, explain).

d) Using the results from the first part of this homework, show that the dipole radiation power is

$$P = \frac{2}{3}|\ddot{\mathbf{d}}|^2\,.$$

e) (i) Comment on the fact that this formula does not depend on the scale $|\mathbf{r}| \sim R$ (provided it is much larger than L). (ii) Argue that charges can only radiate if they move with acceleration. (iii) Show that for a closed system of charged particles whose ratio of charge to mass is the same, there is no dipole radiation. (iv) Do we expect such a system as those discussed at point (iii) to be physically relevant? Justify.

3 Beyond dipole radiation

a) Let us now return to the full expression (3). Show that in this case we recover the following expression:

$$\mathbf{A} = \frac{1}{|\mathbf{r}|}\left(\dot{\mathbf{d}} + \frac{\ddot{\mathbf{Q}}\cdot n}{6} + \dot{\mathbf{m}}\times\mathbf{n}\right)\,, \tag{4}$$

where all quantities on the right hand side are computed at the retarded time and

$$\mathbf{d}^i = \sum_\alpha e_\alpha \mathbf{r}^i_\alpha \,, \quad \mathbf{Q}^{ij} = \sum_\alpha e_\alpha (3\mathbf{r}^i_\alpha \mathbf{r}^j_\alpha - \delta^{ij} r_\alpha |\mathbf{r}|^2) \,, \quad \mathbf{m}^i = \frac{1}{2}\sum_\alpha e_\alpha (\mathbf{r}_\alpha \times \mathbf{v}_\alpha)^i \,, \tag{5}$$

are, respectively, the dipole, quadrupole, and magnetic moments of a N point-like charges (you can write down their continuous analogue in terms of integral of charge densities and currents).

b) Denote by $\langle \bullet \rangle = \int_{S^2} (\bullet) \sin\theta d\theta d\psi$ the angular average. By using the following averaging trick for $\mathbf{n} = \mathbf{r}/|\mathbf{r}|$:

$$\langle \mathbf{n}^i \rangle = 0 \,, \quad \langle \mathbf{n}^i \mathbf{n}^j \rangle \sim \delta^{ij} \,, \quad \langle \mathbf{n}^i \mathbf{n}^j \mathbf{n}^k \rangle = 0 \,, \quad \langle \mathbf{n}^i \mathbf{n}^j \mathbf{n}^k \mathbf{n}^l \rangle \sim \delta\delta + \delta\delta + \delta\delta \,, \tag{6}$$

show that we have the following power:[2]

$$P = \frac{2}{3c^3} \ddot{\mathbf{d}} \cdot \ddot{\mathbf{d}} + \frac{1}{180 c^5} \dddot{\mathbf{Q}} : \dddot{\mathbf{Q}} + \frac{2}{3c^3} \ddot{\mathbf{m}} \cdot \ddot{\mathbf{m}}, \tag{7}$$

where we restored the speed of light c. Which of these is typically the strongest contribution?

c) Argue that for a system in which the charge-to-mass ratio is the same for all charges, only the quadrupole radiation survives. As you will see in the Relativity course, this is precisely the case of the gravitating systems and the "strongest" gravitational waves correspond to quadrupole radiation. Why are such systems physical in the gravitational case but not in the electromagnetic one?

4 One more question...

Who did you collaborate with?

When handing in your assignment, you *must* acknowledge the contributions and support received from fellow PSIons on specific questions, as well as the resources (online or otherwise) you consulted to complete the homework. If you worked alone, say so, but do not leave this question unanswered—even if *what* you answer will not

[2] Here: $\ddot{\mathbf{Q}} : \ddot{\mathbf{Q}} \equiv \sum_{i,j} \ddot{Q}^{ij} \ddot{Q}_{ij}$.

impact our evaluation of your assignment. Science is a collaborative endeavour and as scientists we have the ethical duty to give credit where credit is due.