Mathematical physics (core course)

Giuseppe Sellaroli

PSI 2023/2024

Lecture 8

8.1 Poisson bracket

Definition 8.1 (Poisson bracket and Poisson manifold)**.** A Poisson bracket on a manifold M is a bilinear[1] map

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(M,\mathbb{R}) \times \mathcal{C}^\infty(M,\mathbb{R}) \to \mathcal{C}^\infty(M,\mathbb{R}) \tag{8.1.1}$$

satisfying

- $\{f,g\} = -\{g,f\}$ (anti-symmetry)
- $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (Jacobi identity)
- $\{f,gh\} = g\{f,h\} + \{f,g\}h$ (Leibniz's rule)

The pair $(M,\{\cdot,\cdot\})$ is called a Poisson manifold.

The first two properties of the Poisson bracket make it into a Lie bracket for the vector space of smooth functions, while Leibniz's rule allows us to interpret

$$\{\cdot,f\} : \mathcal{C}^\infty(M,\mathbb{R}) \to \mathcal{C}^\infty(M,\mathbb{R}) \tag{8.1.2}$$

as a derivation and therefore as a vector field.

A symplectic manifold (M,ω) is automatically also a Poisson manifold with Poison bracket defined as

$$\{f,g\} = \omega(X_f,X_g) \equiv \mathrm{d}f(X_g) \equiv X_g f. \tag{8.1.3}$$

In local Darboux coordinates we have

$$\{f,g\} = \frac{\partial \hat{f}}{\partial q^i}\frac{\partial \hat{g}}{\partial p_i} - \frac{\partial \hat{g}}{\partial q^i}\frac{\partial \hat{f}}{\partial p_i} \tag{8.1.4}$$

which is the classical expression of the Poisson bracket from classical mechanics, and it can be checked (with a bit of effort) that it satisfies the required properties.

Conversely, a Poisson manifold is enough to define Hamiltonian vector fields (and hence dynamics) by defining

$$X_f = \{\cdot,f\} \tag{8.1.5}$$

seen as a derivation. In the case of the Poisson bracket induced by the symplectic structure we obtain exactly the same Hamiltonian vector field, but Poisson brackets are more general since they also exist on odd-dimensional manifolds.

We can use Poisson brackets to check how a function $f : M \to \mathbb{R}$ changes along a physical path by noticing that if γ is an integral curve of X_H then

$$(f\circ\gamma)'(t) = \dot{\gamma}(t)f = X_H|_{\gamma(t)} f = (X_H f)(\gamma(t)) = \{f,H\}(\gamma(t)), \tag{8.1.6}$$

which in particular tells us that if $\{f,H\} = 0$ then f is conserved along physical paths.

[1] Linearity is with respect to the vector space structure on $\mathcal{C}^\infty(M,\mathbb{R})$ given by point-wise addition and scalar multiplication.

8.2 Symplectic and Poisson diffeomorphisms

If we want to check if two symplectic/Poisson manifolds are the same, we need something stronger that diffeomorphisms, since those only care about the differential structure.

Definition 8.2 (Symplectic diffeomorphism)**.** Let (M, ω) and (N, ρ) be symplectic manifolds. A diffeomorphism $F : M \to N$ is symplectic (also called a symplectomorphism) if

$$F^*\rho = \omega, \tag{8.2.1}$$

that is the symplectic form that we obtain by pulling the structure of N to M coincides with the symplectic form already there.

Definition 8.3 (Poisson diffeomorphism)**.** Let $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ be Poisson manifolds. A diffeomorphism $F : M \to N$ is Poisson if

$$F^*\{f, g\}_N = \{F^*f, F^*g\}_M, \quad \forall f, g \in \mathcal{C}^\infty(N, \mathbb{R}), \tag{8.2.2}$$

that is pulling back the Poisson bracket (in N) of two functions is the same as taking the Poisson bracket (in M) of the pullbacks of the functions.

Additionally, in the case of a symplectic manifold the following holds:

Theorem. *Let (M, ω) and (N, ρ) be symplectic manifolds. A function $F : M \to N$ is a symplectic diffeomorphism if and only if it is a Poisson diffeomorphism with respect to the induced Poisson brackets.*

8.3 Group actions

Given a Lie group G, we define a (smooth) action of G on a manifold M as a smooth map

$$\Phi : G \times M \to M \tag{8.3.1}$$

such that, if we define for each $g \in G$

$$\Phi_g : a \in M \mapsto \Phi(g, a) \in M, \tag{8.3.2}$$

we have[2]

$$\Phi_e = \mathrm{id}_M, \quad \Phi_{gh} = \Phi_g \circ \Phi_h. \tag{8.3.3}$$

The idea here is that the action of each element of the group is a diffeomorphism of M into itself (as we want the action to be invertible), with the identity acting trivially and the composition of two group actions being the same as the action of the product of the group elements.

Since we are planning on interpreting these actions as symmetries, we will also need to require that they preserve the symplectic or Poisson structure. Specifically, we say that a group action Φ is

- symplectic if each Φ_g is a symplectic diffeomorphism
- Poisson if each Φ_g is a Poisson diffeomorphism.

[2] Here $e \in G$ is the identity element of the group.

8.4 Infinitesimal group action

For the purposes of Noether's theorem we are interested in infinitesimal group actions. If Φ is the action of G on M and $\xi \in \mathfrak{g}$ is an element of the Lie algebra of G, we can define the smooth curve

$$\Phi_a^\xi : t \in \mathbb{R} \mapsto \Phi(e^{t\xi}, a) \in M \tag{8.4.1}$$

and use it to define the vector field on $\xi_M : M \to TM$ given by

$$\xi_M\big|_a = \dot{\Phi}_a^\xi(0), \tag{8.4.2}$$

which acts on functions $f : M \to \mathbb{R}$ as

$$(\xi_M f)(a) = \frac{\mathrm{d}}{\mathrm{d}t} f(\Phi(e^{t\xi}, a))\bigg|_{t=0}. \tag{8.4.3}$$

The vector field ξ_M is called the *infinitesimal generator* of the group action associate to ξ.

8.5 Noether's theorem

Noether's theorem takes the following form in the Hamiltonian picture:

Theorem. *Let $\Phi : G \times M \to M$ be a Poisson group action preserving the Hamiltonian, i.e.,*

$$H(\Phi_g(a)) = H(a), \quad \forall g \in G, \forall a \in M. \tag{8.5.1}$$

If $\xi \in \mathfrak{g}$ and there is a function $\mu^\xi : M \to \mathbb{R}$ such that ξ_M is the Hamiltonian vector field of μ^ξ, then μ^ξ is conserved along integral curves of X_H.

Proof. For each $a \in M$ we have

$$\{H, \mu^\xi\}(a) = (\xi_M H)(a) = \frac{\mathrm{d}}{\mathrm{d}t} H(\Phi(e^{t\xi}, a))\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} H(a)\bigg|_{t=0} = 0 \tag{8.5.2}$$

so μ^ξ is conserved. □