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Homework 1

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Contents

1 Conformal invariance of the Maxwell action for $D = 4$

(a) Consider a classical abelian gauge field A_μ on $D=4$ dimensionnal Minkowski spacetime. Under an infinitesimal conformal transformation, spacetime undergoes the transformation $\tilde{x}^\mu = f(x) = x^\mu + \xi^\mu(x)$ where $\xi^\mu(x)$ is a smal deformation. We want to calculate the effect of this transformation on the gauge field A_μ. The starting point is that we expect A_μ to transform as a tensor under the Lorenz transformation subgroup of the conformal group. This implies that A_μ is a primary operator and we denote its scaling dimension Δ. The transformed field $\tilde{A}_\mu$ at $\tilde{x}$ is related to the original field A_μ at x by an internal rotation, scaling, and special conformal transformation. The rotation operation acts on the components A_μ through its spin 1 representation which is the defining representation of rotations. The scaling and special conformal transformation act together through the multiplication of A_μ by the Jacobian factor $|\partial x/\partial\tilde{x}|_x^{\Delta/D}$. Finally, translations act trivially internally. This can be summarized with the relation $\tilde{A}_\mu(\tilde{x}) = |\partial x/\partial\tilde{x}|_x^{\Delta/D} R_\mu^\nu A_\nu(x)$ where R_μ^ν is the matrix associated with the part of $\xi^\mu(x)$ that does not change the metric components (after the Weyl and diffeomorphism transformations). With this in mind, we calculate the jacobian of the infinitesimal transformation to be

$$\left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|_x = \left|\frac{\partial\tilde{x}^\mu}{\partial x^\nu}\right|_x^{-1} = |\delta_\nu^\mu + \partial_\nu\xi^\mu|_x^{-1} \approx |e^{-\partial_\nu\xi^\mu}|_x = e^{-\mathrm{Tr}\partial_\nu\xi^\mu(x)} = 1 - \partial_\mu\xi^\mu(x) + O(\xi^2).$$

The matrix $R_\mu^\nu(x)$ can be extracted by dividing the matrix $(\partial x/\partial\tilde{x})_x$ by a factor $\Omega(x)$ such that we extract the "metric component preserving" operation. To find this factor we consider the effect on the metric of $\Omega^{-1}(x)(\partial x/\partial\tilde{x})_x$. We can write the "metric component preserving" property as

$$\Omega^{-2}(x)\left(\frac{\partial x^\mu}{\partial\tilde{x}^\sigma}\right)_x\left(\frac{\partial x^\nu}{\partial\tilde{x}^\rho}\right)_x\eta_{\mu\nu} = \eta_{\sigma\rho}.$$

Since $\Omega(x)$ is a factor, we can extract it by taking the determinant on both sides of the previous relation to get

$$\det(\eta)\Omega(x)^{-2D}\left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|_x^2 = \det(\eta) \iff \Omega(x) = \left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|_x^{-\frac{1}{D}}.$$

This result can be intuitively understood from the fact the Jacobian measures D-volume rescaling. Since we want metric components (associated with distances) to be preserved by the rescaled transformation, we need to divide by the D-root of the jacobian. The matrix $R_\mu^\nu(x)$ provided by the rescaling is given by

$$\begin{aligned} R_\mu^\nu(x) = \frac{1}{(1-\partial_\sigma\xi^\sigma(x)+O(\xi^2))^{1/D}}\left(\frac{\partial x^\nu}{\partial\tilde{x}^\mu}\right)_x &= (1+\partial_\sigma\xi^\sigma(x)/D + O(\xi^2))(\delta_\nu^\mu + \partial_\mu\xi^\nu(x) + O(\xi^2))^{-1} \\ &= \delta_\nu^\mu(1+\partial_\sigma\xi^\sigma(x)/D) - \partial_\mu\xi^\nu(x) + O(\xi^2). \end{aligned}$$

We note that $R_\mu^\nu(x)$ will represent a rotation if $\partial_\sigma\xi^\sigma(x) = 0$ (bring the conformal Killing equation to the normal Killing equation with a rotation isometry as its solution). If $\partial_\sigma\xi^\sigma(x) \neq 0$, the rescaled transformation contains a special conformal transformation. The special conformal transformation as a Weyl transformation does not preserve distances but can be combined with a diffeomorphism to preserve the initial components of the metric. With these results, we can write the effect of the infinitesimal transformation as

$$\begin{aligned} \tilde{A}_\mu(\tilde{x}) &= (1-\partial_\rho\xi^\rho(f^{-1}(\tilde{x}))+O(\xi^2))^{\Delta/D}(A_\mu(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2)) \\ &= \left(1-\frac{\Delta}{D}\partial_\rho\xi^\rho(f^{-1}(\tilde{x}))+O(\xi^2)\right)(A_\mu(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2)) \\ &= A_\mu(f^{-1}(\tilde{x})) - A_\mu(f^{-1}(\tilde{x}))\frac{\Delta}{D}\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2). \end{aligned}$$

Since $\xi(f^{-1}(\tilde{x}))$ is already first order in ξ, the only term contribution to its expansion around $\xi=0$ at $O(\xi)$ is $\xi(\tilde{x})$. To go further, we expand $f^{-1}(\tilde{x})$ at first order in $\xi(\tilde{x})$ with the ansatz $f^{-1}(\tilde{x})^\nu = \tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x})$ (the first term of this ansatz is justified by noticing the transformation reduces to identity at $\xi=0$). From $f(f^{-1}(\tilde{x})) = \tilde{x}$, we find

$$\tilde{x}^\nu = \tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x}) + \xi(\tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x})) + O(\xi^2) \implies B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x}) + \xi^\nu(\tilde{x}) = 0, \quad \forall\xi(\tilde{x}) \implies B_\mu^\nu(\tilde{x}) = -\delta_\mu^\nu.$$

Using this result, we can expand $A_\mu(f^{-1}(\tilde{x}))$ as

$$A_\mu(f^{-1}(\tilde{x})) = A_\mu(\tilde{x}^\nu - \xi^\nu(\tilde{x}) + O(\xi^2)) = A_\mu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2)$$

Combining this expression with the internal transformation at first order in ξ, we get

$$\begin{aligned} \tilde{A}_\mu(\tilde{x}) &= \left(1-\frac{\Delta}{D}\partial_\sigma\xi^\sigma(\tilde{x}) + \partial_\sigma\xi^\sigma(\tilde{x}) - \partial_\mu\xi^\nu(\tilde{x})\right)(A_\mu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x})) + O(\xi^2) \\ &= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta-1}{D}\partial_\sigma\xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\partial_\mu\xi^\nu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2) \end{aligned}$$

with $\xi(f^{-1}(\tilde{x})) = \xi(\tilde{x}) + O(\xi^2)$. This result can be simplified by using the conformal killing equation $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2\eta_{\mu\nu}\partial_\sigma \xi^\sigma / D$ as follows:

$$\begin{aligned}
\tilde{A}_\mu(\tilde{x}) &= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta-1}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\left(\frac{1}{2}\partial_\mu \xi^\nu(\tilde{x}) + \frac{1}{2}\partial_\mu \xi^\nu(\tilde{x})\right) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2)\\
&= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta-1}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\left(\frac{1}{2}\partial_\mu \xi^\nu(\tilde{x}) - \frac{1}{2}\partial_\nu \xi^\mu(\tilde{x}) + \delta^\nu_\mu \partial_\sigma \xi^\sigma(\tilde{x})\frac{1}{D}\right) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2)\\
&= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\underbrace{\left(\frac{1}{2}\partial_\mu \xi^\nu(\tilde{x}) - \frac{1}{2}\partial^\nu \xi_\mu(\tilde{x})\right)}_{M_\mu{}^\nu} - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2).
\end{aligned}$$

From this transformed gauge field, we calculate the transformation of gauge field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ to $\tilde{F}_{\mu\nu}$. We start by writting the transformation law of the derivatives used to construct $F_{\mu\nu}$. The chain rule yields

$$\tilde{\partial}_\mu \equiv \frac{\partial}{\partial \tilde{x}^\mu} = \left(\frac{\partial f^{-1}(\tilde{x})^\nu}{\partial \tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} = \left(\frac{\partial \tilde{x}^\nu - \xi^\nu(\tilde{x})}{\partial \tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} = \left(-\frac{\partial \xi^\nu(\tilde{x})}{\partial \tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} + \left(\frac{\partial}{\partial x^\mu}\right)_{\tilde{x}} \equiv -\partial_\mu \xi^\nu(\tilde{x})\partial_\nu + \partial_\mu.$$

where the subscripts indicate that a partial derivative with respect to x^μ should be precomposed with $x = f^{-1}(x^\mu)$ to yield a function dependent on the left-hand side variable $\tilde{x}$. Now we can calculate the transformed field strength at first order in ξ to be

$$\begin{aligned}
\tilde{F}_{\mu\nu} &= \tilde{\partial}_\mu \tilde{A}_\nu - (\mu \leftrightarrow \nu)\\
&= \left(-\partial_\mu \xi^\rho(\tilde{x})\partial_\rho + \partial_\mu\right)\left(A_\nu(\tilde{x}) - A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\lambda(\tilde{x})M_\nu{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x})\right) - (\mu \leftrightarrow \nu)\\
&= \partial_\mu A_\nu(\tilde{x}) - (\partial_\mu \xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x})\right) - \partial_\mu\left(A_\lambda(\tilde{x})M_\nu{}^\lambda\right) - \partial_\mu\left(\xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x})\right) - (\mu \leftrightarrow \nu)\\
&= \partial_\mu A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x})\right) - \partial_\mu A_\lambda(\tilde{x})\partial_\nu \xi^\lambda(\tilde{x}) - A_\lambda(\tilde{x})\partial_\mu\partial_\nu \xi^\lambda(\tilde{x}) - \xi^\lambda(\tilde{x})\partial_\lambda\partial_\mu A_\nu(\tilde{x}) - 2(\partial_\mu \xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - (\mu \leftrightarrow \nu)\\
&= \partial_\mu A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x})\right) - (\partial_\mu A_\lambda(\tilde{x}))M_\nu{}^\lambda - A_\lambda(\tilde{x})\partial_\mu M_\nu{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda\partial_\mu A_\nu(\tilde{x}) - 2(\partial_\mu \xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - (\mu \leftrightarrow \nu)\\
&= F_{\mu\nu}(\tilde{x}) - F_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x}) - A_{(\nu}(\tilde{x})\frac{\Delta}{D}\partial_{\mu)}\partial_\lambda \xi^\lambda(\tilde{x}) - (\partial_{(\mu}A_\lambda(\tilde{x}))M_{\nu)}{}^\lambda - A_\lambda(\tilde{x})\partial_{(\mu}M_{\nu)}{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda F_{\mu\nu}(\tilde{x}) - 2(\partial_{(\mu}\xi^\lambda(\tilde{x}))\partial_\lambda A_{\nu)}(\tilde{x})\\
&= F_{\mu\nu}(\tilde{x}) - F_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x}) - A_{(\nu}(\tilde{x})\frac{\Delta}{D}\partial_{\mu)}\partial_\lambda \xi^\lambda(\tilde{x}) - (\partial_{(\mu}A_\lambda(\tilde{x}))M_{\nu)}{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda F_{\mu\nu}(\tilde{x}) - 2(\partial_{(\mu}\xi^\lambda(\tilde{x}))\partial_\lambda A_{\nu)}(\tilde{x})
\end{aligned}$$

where we simplified further by expliciting

$$2\partial_{(\mu}M_{\nu)}{}^\lambda = \partial_\mu\partial_\nu \xi^\lambda(\tilde{x}) - \partial_\mu\partial^\lambda \xi_\nu(\tilde{x}) - \partial_\nu\partial_\mu \xi^\lambda(\tilde{x}) - \partial_\nu\partial^\lambda \xi_\mu(\tilde{x}) = 0.$$

We note that the transformation law of $F_{\mu\nu}$ involves A_μ homogeneously which is an example of mixing of CFT fields under the transformation of a descendant.

(b) For a D-dimensionnal spacetime, the Maxwell action reads

$$S = \int \mathrm{d}^D x \sqrt{|g|}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \int \mathrm{d}^D x \sqrt{|g|}\; g^{\mu\sigma}g^{\nu\rho}\frac{1}{4}F_{\mu\nu}F_{\sigma\rho}.$$

where g is the metric (which we suppose conformally flat). We aim to apply the results found in (a) to determine when this action gains conformal symmetry. Under a conformal transformation given by the killing vector $\xi^\mu(x)$ and the scaling $\Omega(x) = 1 + \partial_\mu \xi^\mu(x)/D + O(\xi^2)$ of the metric components, we have

$$\begin{aligned}
&g_{\nu\rho}(x) = \Omega(f(x))^{-2}\tilde{g}_{\nu\rho}(f(x)) = \Omega(\tilde{x})^{-2}\tilde{g}_{\nu\rho}(\tilde{x}) \quad \text{Defining property of a conformal transformation}\\
&|g|(x) = \Omega(f(x))^{-2D}|\tilde{g}|(f(x)), \quad g^{\nu\rho}(x) = \Omega(f(x))^{+2}\tilde{g}^{\nu\rho}(f(x)) = \Omega(\tilde{x})^2\tilde{g}^{\nu\rho}(\tilde{x}), \quad \mathrm{d}^D x\sqrt{|g|} = \mathrm{d}^D\tilde{x}\;\Omega(\tilde{x})^{-D}\sqrt{|\tilde{g}|(\tilde{x})}
\end{aligned}$$

Without loss of generality, we take the target metric $\tilde{g}$ to be the Minkowski metric. Inverting the result found in (a) for the transformation of the gauge field, we write

$$\begin{aligned}
A_\mu(x) = |\partial x/\partial \tilde{x}|_{\tilde{x}}^{-\Delta/D}(R^{-1})_\mu^\nu \tilde{A}_\nu(\tilde{x}) &= \tilde{A}_\mu(\tilde{x}) + \tilde{A}_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - \tilde{A}_\mu(\tilde{x})\partial_\sigma \xi^\sigma(\tilde{x})\frac{1}{D} + \tilde{A}_\nu(\tilde{x})\partial_\mu \xi^\nu(\tilde{x}) + O(\xi^2)\\
&= \tilde{A}_\mu(\tilde{x}) + \tilde{A}_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) + \frac{1}{2}\tilde{A}_\nu(\tilde{x})\left(\partial_\mu \xi^\nu(\tilde{x}) - \partial^\nu \xi_\mu(\tilde{x})\right) + O(\xi^2).
\end{aligned}$$

Then, with the derivative $(\partial_\mu)_{\tilde{x}} = \tilde{\partial}_\mu \xi^\nu(\tilde{x})\tilde{\partial}_\nu + \tilde{\partial}_\mu$, the field strength transforms as

$$\begin{aligned}
F_{\mu\nu} = \partial_\mu A_\nu(x) - (\mu \leftrightarrow \nu) &= \left(\tilde{\partial}_\mu \xi^\lambda(\tilde{x})\tilde{\partial}_\lambda + \tilde{\partial}_\mu\right)\left(\tilde{A}_\nu(\tilde{x}) + \tilde{A}_\nu(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_\lambda(\tilde{x})M_\nu{}^\lambda\right) - (\mu \leftrightarrow \nu)\\
&= \tilde{\partial}_\mu\left(\tilde{A}_\nu(\tilde{x}) + \tilde{A}_\nu(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_\lambda(\tilde{x})M_\nu{}^\lambda\right) + \tilde{\partial}_\mu \xi^\lambda(\tilde{x})\tilde{\partial}_\lambda \tilde{A}_\nu(\tilde{x}) - (\mu \leftrightarrow \nu)\\
&= \tilde{F}_{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_{(\nu}(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_{\mu)}\tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{\partial}_{(\mu}(\tilde{A}_\lambda(\tilde{x}))M_{\nu)}{}^\lambda + \tilde{\partial}_{(\mu}\xi^\lambda(\tilde{x})\tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x})
\end{aligned}$$

The contravariant equivalent of this result is given by

$$
\begin{aligned}
F^{\mu\nu} &= g^{\mu\sigma} g^{\nu\rho} F_{\sigma\rho} = \Omega(\tilde{x})^4 \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} F_{\sigma\rho} \\
&= \Omega(\tilde{x})^4 \left(\tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}^{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} (\tilde{A}_\lambda(\tilde{x})) M_{\rho)}{}^\lambda + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\rho)}(\tilde{x}) \right)
\end{aligned}
$$

Next, we calculate

$$
\begin{aligned}
F_{\mu\nu} F^{\mu\nu} &= \left(\tilde{F}_{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + O(\xi^2) \right) \\
&\quad \times \Omega(\tilde{x})^4 \left(\tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}^{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} (\tilde{A}_\lambda(\tilde{x})) M_{\rho)}{}^\lambda + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\rho)}(\tilde{x}) \right) \\
&= \Omega(\tilde{x})^4 \left(\tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) \frac{2\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) \right) \\
&= \Omega(\tilde{x})^4 \left(\tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) \frac{2\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + 4\tilde{F}^{\mu\nu} \tilde{A}_\nu(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\mu \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) \right)
\end{aligned}
$$

I realized this calculation only applies a passive transformation to the action and should not not change its value without necessarly corresponding to a symmetry. I would have to redo this calculation by applying an active conformal transformation to each element of the action.

2 Axial anomaly

(a) We consider a $D = 2$-dimensionnal fermion field ψ with vector current $j^V_\mu = \overline{\psi}\gamma_\mu\psi$ where γ_μ are matrices forming a 2-dimensionnal clifford algebra. We are interested in the 2-point correlator of the vector current $\langle j^V_\mu(x_1) j^V_\nu(x_2)\rangle$. By translational symmetry, the 2-point function is forced to be a function of the relative coordinates $x = (x_1 - x_2)/2$. Translating by $-X_{12} = -(x_1 + x_2)/2$, we can bring the midpoint of the x_1, x_2 segment to the origin without changing the value of the 2-point function. Explicitly, we have $\langle j^V_\mu(x_1) j^V_\nu(x_2)\rangle = \langle j^V_\mu(x) j^V_\nu(-x)\rangle$. This property allows us to expand the 2-point function with a Fourier transform with respect to x as

$$\begin{aligned} F[\langle j^V_\mu(x_1) j^V_\nu(x_2)\rangle](q) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2x e^{-iq\cdot x}\langle j^V_\mu(x) j^V_\nu(-x)\rangle &= \frac{1}{(2\pi)^2}\int \mathrm{d}^2x e^{-iq\cdot x}\langle \int \mathrm{d}^2k\, e^{+ik\cdot x} j^V_\mu(k) \int \mathrm{d}^2p\, e^{-ip\cdot x} j^V_\nu(p)\rangle \\ &= \frac{1}{2\pi}\langle \int \mathrm{d}^2k\, \mathrm{d}^2p\, \delta(-q+k-p) j^V_\mu(k)\, j^V_\nu(p)\rangle \\ &= \frac{1}{2\pi}\int \mathrm{d}^2p\, \langle\, j^V_\mu(q-p)\, j^V_\nu(p)\rangle \end{aligned}$$

where the fourier decomposition $j^V_\rho(x_i) = \frac{1}{2\pi}\int \mathrm{d}^2p\, e^{ip\cdot x_i} j^V_\rho(p)$ of the vector current was used. In what follows, we focus on the Fourier space 2-point functions $\langle\, j^V_\mu(-p)\, j^V_\nu(p)\rangle$ contribution to the $q = 0$ Fourier component of the spacetime 2-point function. Lorentz invariance requires that $\langle\, j^V_\mu(q-p)\, j^V_\nu(p)\rangle$ is a sum of tensors (it can be extracted from a Fourier transform linearly combining tensor so it is a tensor). Furthermore, it only depends on components p_μ of p. The only tensors with two indices built can be constructed by combining the Minkowski metric $\eta_{\mu\nu}$, the components p_μ, the norm p^2 and the matrices γ^μ (we only need to include a term $\gamma_\mu\gamma_n u$ since the anticommutator $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ relates it to $\gamma_\nu\gamma_\mu$). The most general form for the Fourier space 2-point function consistent with Lorentz invariance reads

$$\langle\, j^V_\mu(-p)\, j^V_\nu(p)\rangle = F_1(p^2)\varepsilon_{\mu\nu} + F_2(p^2)\eta_{\mu\nu} + F_3(p^2)p_\mu p_\nu + F_4(p^2)\gamma_\mu\gamma_\nu + F_5(p^2)\gamma_\mu p_\nu + F_6(p^2)\gamma_\nu p_\mu$$

where the functions $F_i : \mathbb{R} \to \mathbb{C}$ provide full generality and $\varepsilon_{\mu\nu}$ is the 2-dimensionnal Levi-Civita tensor. Since the current operator follows a Bose statistic (they each contain an even number of fermion operators), we can exchange them without changing the value of the 2-point function. This property can be expressed as

$$\langle\, j^V_\mu(-p)\, j^V_\nu(p)\rangle = \langle\, j^V_\nu(p)\, j^V_\mu(-p)\rangle = -F_1(p^2)\varepsilon_{\mu\nu} + F_2(p^2)\eta_{\mu\nu} + F_3(p^2)(-p_\nu)(-p_\mu) + F_4(p^2)\gamma_\nu\gamma_\mu - F_5(p^2)\gamma_\nu p_\mu - F_6(p^2)\gamma_\mu p_\nu.$$

Subtracting this exchanged expression from the initial expression, we get the constraint

$$\begin{aligned} &0 = 2F_1(p^2)\varepsilon_{\mu\nu} + F_4(p^2)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) + (F_6(p^2) + F_5(p^2))\gamma_\nu p_\mu + (F_6(p^2) + F_5(p^2))\gamma_\mu p_\nu, \forall p \\ &\implies F_4(p^2) = F_1(p^2) = 0, \quad F_6(p^2) = -F_5(p^2) \end{aligned}$$

The current we are interested in is conserved as a result of the global symmetry $\psi \to e^{i\theta}\psi$, $\theta \in \mathbb{R}$. We note that applying the infinitesimal version of this symmetry transformation to the current leads to a vanishing variation $\delta j^\nu(x) = 0$. The ward identity corresponding to this symmetry reads

$$\langle \partial^\mu j_\mu(x_1) j_\nu(x_2)\rangle = \delta(x_1 - x_2)\langle \delta j_\nu(x_2)\rangle = 0.$$

We then compute the Fourier transformation with respect to x_1, x_2 to get

$$\begin{aligned} 0 = F[\langle \partial^\mu j^V_\mu(x_1) j^V_\nu(x_2)\rangle](q,p) &= \int \mathrm{d}^2x_1 e^{-iq\cdot x_1}\int \mathrm{d}^2x_2 e^{-ip\cdot x_2}\langle \partial^\mu j^V_\mu(x_1) j^V_\nu(x_2)\rangle \\ &= \langle (+iq^\mu)\int \mathrm{d}^2x_1 e^{-iq\cdot x_1} j^V_\mu(x_1)\int \mathrm{d}^2x_2 e^{-ip\cdot x_2} j^V_\nu(x_2)\rangle \quad \text{with integration by parts} \\ &= iq^\mu\langle j^V_\mu(q) j^V_\nu(p)\rangle \end{aligned}$$

At $q = -p$, we find $p^\mu\langle j^V_\mu(-p)\, j^V_\nu(p)\rangle = 0$ wich implies

$$\begin{aligned} 0 = p^\mu\langle\, j^V_\mu(-p)\, j^V_\nu(p)\rangle &= p^\mu\left(F_2(p^2)\eta_{\mu\nu} + F_3(p^2)p_\mu p_\nu\right) + F_5(p^2)p^\mu\left(\gamma_\mu p_\nu - \gamma_\nu p_\mu\right) \\ &= \left(F_2(p^2) + F_3(p^2)p^2\right)p_\nu + F_5(p^2)\left(p^\mu\gamma_\mu p_\nu - \gamma_\nu p^2\right), \forall p \\ &\implies F_2(p^2) = -F_3(p^2)p^2, \quad F_5(p^2) = 0. \end{aligned}$$

More explicitly, starting with the constraint $F_5(p^2)(p^\mu\gamma_\mu p_\nu - \gamma_\nu p^2) = 0$, we can have either $F_5(p^2) = 0$ or $p^\mu\gamma_\mu p_\nu - \gamma_\nu p^2$. The latter case is associated with $0 = \gamma_0 + \gamma_1$ for $p^\mu = (1,1)$ showing it can't hold for all p and forcing $F_5(p^2)$. The updated expression for the 2-momentum curent correlator is

$$\langle j^V_\mu(-p)\, j^V_\nu(p)\rangle = F_3(p^2)\left(-p^2\eta_{\mu\nu} + p_\mu p_\nu\right).$$

The form of $F_3(p^2)$ can be made more precise by imposing scale invariance. To use scale invariance, we first relate scale invariance in position space to scale invariance in momentum space. Since the conserved current operators have no anomalous scaling dimension (leaving the scaling dimension $\Delta = D = 2$), we have

$$\begin{aligned}\langle j_\mu(x_1) j_\nu(x_2)\rangle = \lambda^{2+2}\langle j_\mu(x_1\lambda) j_\nu(x_2\lambda)\rangle \implies \langle j^V_\mu(-p)\ j^V_\nu(p)\rangle &= \lambda^{2+2} F[\langle \partial^\mu j^V_\mu(\lambda x_1) j^V_\nu(\lambda x_2)\rangle](-p,p) \\ &= \lambda^4 \langle \int \mathrm{d}^2 x_1 e^{-iq\cdot x_1} j^V_\mu(\lambda x_1) \int \mathrm{d}^2 x_2 e^{-ip\cdot x_2} j^V_\nu(\lambda x_2)\rangle \\ &= \lambda^4 \langle \frac{1}{\lambda^2}\int \mathrm{d}^2 x_1 e^{-iq\cdot x_1/\lambda} j^V_\mu(x_1) \frac{1}{\lambda^2}\int \mathrm{d}^2 x_2 e^{-ip\cdot x_2/\lambda} j^V_\nu(x_2)\rangle \\ &= \langle j^V_\mu(-p/\lambda)\ j^V_\nu(p/\lambda)\rangle.\end{aligned}$$

Applying this constraint to the 2-momentum curent correlator yields

$$F_3(p^2)\big(-p^2\eta_{\mu\nu} + p_\mu p_\nu\big) = \frac{1}{\lambda^4} F_3(p^2/\lambda^2)\big(-p^2\eta_{\mu\nu} + p_\mu p_\nu\big) \implies F_3(p^2) = \frac{1}{\lambda^2} F_3(p^2/\lambda^2).$$

The previous implication suggests the powerlaw ansatz $F_3 = ap^b$ depending on constants a, b. Substituting this ansatz, we get $ap^b = ap^b\lambda^{-b-2} \implies b = -2$. The final form for the correlator is

$$\langle j^V_\mu(-p)\ j^V_\nu(p)\rangle = \frac{a}{p^2}\big(-p^2\eta_{\mu\nu} + p_\mu p_\nu\big).$$

(b) We consider the axial current defined by $j^A_\mu = \varepsilon_{\mu\nu} j^{V,\nu}$. In $D = 2$, the non zero components of the levi-civita tensor are $\varepsilon_{01} = -\varepsilon_{10} = 1$. To determine if the axial current is classicaly conserved, we calculate its divergence as follows

$$\begin{aligned}\partial^\mu j^A_\mu &= \partial^\mu \varepsilon_{\mu\nu} j^{V,\nu} = \partial^\mu(\overline{\psi}\varepsilon_{\mu\nu}\gamma^\nu\psi) = \partial^\mu(\overline{\psi}\eta_{\mu\sigma}\gamma^1\gamma^0\gamma^\sigma\psi) = \partial_\mu(\overline{\psi}\gamma^1\gamma^0\gamma^\mu\psi) \\ &= (\partial_\mu\overline{\psi})\gamma^1\gamma^0\gamma^\mu\psi + \overline{\psi}\gamma^1\gamma^0\gamma^\mu(\partial_\mu\psi) = 0\end{aligned}$$

where we used the property $\varepsilon_{\mu\nu}\gamma^\nu = \eta_{\mu\sigma}\gamma^1\gamma^0\gamma^\sigma$, the classical equation of motion for a free massless fermion field $0 = \gamma^1\gamma^0\gamma^\mu\partial_\mu\psi$ and its conjugate $0 = \partial_\mu\psi^\dagger\gamma^0\gamma^1\gamma^0\gamma^\mu\gamma^0 = \partial_\mu\overline{\psi}\gamma^1\gamma^0\gamma^\mu\gamma^0$. We explicitly check

$$\begin{aligned}\gamma^1 = \varepsilon_{01}\gamma^1 &= \eta_{00}\gamma^1\gamma^0\gamma^0 = (\eta_{00}\eta^{00})\gamma^1 = \gamma^1, \\ -\gamma^0 = \varepsilon_{10}\gamma^0 &= \eta_{11}\gamma^1\gamma^0\gamma^1 = -(\eta_{11}\eta^{11})\gamma^0 = -\gamma^0.\end{aligned}$$

We conclude that the classical equations of motion imply both the classical conservation of the vector current and the classical conservation of the axial current.

(c) To test if the classical conservation of the axial current survives quantum effects, we calculate

$$\begin{aligned}p^\mu\langle j^A_\mu(-p) j^V_\nu(p)\rangle = \eta^{\sigma\rho}\varepsilon_{\mu\sigma}p^\mu\langle j^V_\rho(-p) j^V_\nu(p)\rangle &= \eta^{\sigma\rho}\varepsilon_{\mu\sigma}p^\mu\frac{a}{p^2}\big(-p^2\eta_{\rho\nu} + p_\rho p_\nu\big) \\ &= \varepsilon_{\mu\sigma}\frac{a}{p^2}\big(-p^2 p^\mu\delta^\sigma_\nu + p^\mu p^\sigma p_\nu\big) \\ &= \frac{a}{p^2}(-p^2\varepsilon_{\mu\nu}p^\mu + \underbrace{\varepsilon_{\mu\sigma}p^\mu p^\sigma}_{p^0p^1 - p^1p^0 = 0} p_\nu) = -a\varepsilon_{\mu\nu}p^\mu\end{aligned}$$

and Fourier transfrom the result to get

$$\begin{aligned}\frac{1}{2\pi}\int \mathrm{d}p\ e^{ip\cdot x}p^\mu\langle j^A_\mu(-p) j^V_\nu(p)\rangle &= \frac{-i}{2\pi}\partial^\mu\langle\int \mathrm{d}p\ e^{ip\cdot x}\int \mathrm{d}x_1\ e^{+ip\cdot x_1} j^A_\mu(x_1)\int \mathrm{d}x_2\ e^{-ip\cdot x_2} j^V_\nu(x_2)\rangle \\ &= -i\partial^\mu\langle\int \mathrm{d}x_1\mathrm{d}x_2\ j^A_\mu(x_1) j^V_\nu(x_2)\delta(x + x_1 - x_2)\rangle \\ &= -i\langle\int \mathrm{d}x_1\ \partial^\mu j^A_\mu(x_1) j^V_\nu(x + x_1)\rangle = -i\langle\ \partial^\mu j^A_\mu(0) j^V_\nu(x)\rangle\int \mathrm{d}x_1\end{aligned}$$

and

$$-\frac{1}{2\pi}\int \mathrm{d}p\ e^{ip\cdot x}a\varepsilon_{\mu\nu}p^\mu = a\ i\ \varepsilon_{\mu\nu}\partial^\mu\delta(x)$$

which shows that the divergence of the axial current has a non-vanishing correlator (can't be explained by contact terms since they would vanish through $\delta j_\nu(x) = 0$) with the vector current contradicting the Ward identity for the axial current.

(d) When a background gauge field A^μ is added, the axial anomaly affects the expectation value of the axial current divergence dirrectly. This expectation value is evaluated at first non-trivial order

$$\begin{aligned}\big\langle\partial_\mu j^{A,\mu}(0)\big\rangle_{A^\mu} \equiv \Big\langle\partial_\mu j^{A,\mu}(0)e^{\int dx A^\nu j^V_\nu}\Big\rangle &\approx \big\langle\partial_\mu j^{A,\mu}(0)1\big\rangle + \Big\langle\partial_\mu j^{A,\mu}(0)\int dx A^\nu j^V_\nu\Big\rangle \\ &\approx \Big\langle\partial_\mu j^{A,\mu}(0)\int dx A^\nu j^V_\nu\Big\rangle\end{aligned}$$

3 OPE coefficients from three-point functions

(a) Given scalar CFT operators O_{Δ_1}, O_{Δ_2} et O_{Δ_3} with respective scaling dimensions Δ_1, Δ_2 and Δ_3, there is only one two-point function consistent with conformal symmetry given by

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{x_{12}^{2\Delta_1}}$$

where $x_{ij} = |x_i - x_j|$. The three-point function has residual freedom in the coefficients C_{123} depending on the operators involved. The most general expression reads

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

(b)

(c)

(d)

4 Acknowledgement

Thanks to Thiago for a discussion about question 1 (b)