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Homework 3

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Contents

1 Planar electromagnetic waves

A Maxwell equations for the four-potential

The components of the contravariant four potential are $A^\mu = (\varphi, \mathbf{A})$ ($A_\mu = (-\varphi, \mathbf{A})$ for the covariant components) where φ is the electric potential and $\mathbf{A}$ is the magnetic potential vector. The sources generating each component of A^μ can be grouped in a current four vector $j^\mu = (\rho, \mathbf{j})$ ($j_\mu = (-\rho, \mathbf{j})$ for the covariant components) where ρ is the charge density and $\mathbf{j}$ is the current observed in the reference frame where we solve for A^μ. In the lorentz gauge $0 = \nabla_\mu A^\mu$, the Maxwell equations for A^μ with sources j^μ read $\Box A^\mu = -4\pi j^\mu$ ($\Box A_\mu = -4\pi j_\mu$ for the covariant components).

B Plane wave Ansatz

We now solve the Maxwell equations in the Lorentz gauge, by introducing the plane wave ansatz $A_\mu(t, \mathbf{x}) = a_\mu \exp\left(ik_\mu x^\mu\right)$ where $k^\mu = (\mu, \mathbf{k})$ is the four wave vector and a^μ is the four amplitude. On one hand, substituting this ansatz in the Lorentz gauge condition, we get

$$0 = \nabla_\mu A^\mu = \nabla_\mu \left(a^\mu \exp\left(ik_\nu x^\nu\right)\right) = a^\mu i \delta^\nu_\mu k_\nu \exp\left(ik_\nu x^\nu\right) = (a^\mu k_\mu) \exp\left(ik_\nu x^\nu\right) \iff a^\mu k_\mu = 0.$$

On the other hand, substituting the ansatz in the vacuum Maxwell equations (j_μ) yields

$$0 = \nabla^\mu \nabla_\mu A_\nu = i\delta^\rho_\mu k_\rho \nabla^\mu (\exp\left(ik^\rho x_\rho\right)) = -k^\mu k_\mu \exp\left(ik^\rho x_\rho\right) \iff k^\mu k_\mu = 0$$

so the four wave vector is light-like in the vacuum.

C Electric and Magnetic fields

In terms of A_μ, the electric and magnetic fields $\mathbf{E}, \mathbf{B}$ can be written as

$$\begin{aligned}
\mathbf{E} &= \nabla_j A_0 - \nabla_0 \mathbf{A} = a_0 \nabla_j \exp\left(ik_\mu x^\mu\right) - \mathbf{a} \nabla_0 \exp\left(ik_\mu x^\mu\right) = (ia_0 \mathbf{k} - i\mathbf{a}k_0) \exp\left(ik_\mu x^\mu\right), \\
\mathbf{B} &= \varepsilon_i^{\ jk} \nabla_j A_k = i\varepsilon_i^{\ jk} k_j a_k \exp\left(ik_\mu x^\mu\right) = i\mathbf{k} \times \mathbf{a} \exp\left(ik_\mu x^\mu\right)
\end{aligned}$$

with $\mathbf{A}$, $\mathbf{a}$ and $\mathbf{k}$ are respectively the spatial components of A_μ, a_μ and k_μ. We consider the projection of $\mathbf{E}, \mathbf{B}$ along $\mathbf{k}$. We define $\mathbf{n} := \mathbf{k}/k$ to write the projections

$$\begin{aligned}
\mathbf{n} \cdot \mathbf{E} &= \mathbf{k}/k \cdot \mathbf{E} = (ia_0 \mathbf{k}^2 - i\mathbf{k} \cdot \mathbf{a} k_0) \exp\left(ik_\mu x^\mu\right)/k = (ia_0(k_0^2) - i(k_0 a_0) k_0) \exp\left(ik_\mu x^\mu\right)/k = 0, \\
\mathbf{n} \cdot \mathbf{B} &= \mathbf{k} \cdot \left(i\mathbf{k} \times \mathbf{a} \exp\left(ik_\mu x^\mu\right)\right)/k = 0.
\end{aligned}$$

Furthermore, we can relate $\mathbf{E}$ and $\mathbf{B}$ in the following way:

$$\begin{aligned}
\mathbf{k} \times \mathbf{B}/k_0 &= i\mathbf{k} \times (\mathbf{k} \times \mathbf{a}) \exp\left(ik_\mu x^\mu\right) \\
&= i\left((\mathbf{k} \cdot \mathbf{a})\mathbf{k} - (\mathbf{k} \cdot \mathbf{k})\mathbf{a}\right) \exp\left(ik_\mu x^\mu\right)/k_0 \\
&= i\left((k_0 a_0)\mathbf{k} - (k_0^2)\mathbf{a}\right) \exp\left(ik_\mu x^\mu\right)/k_0 \\
&= i\left(a_0 \mathbf{k} - k_0 \mathbf{a}\right) \exp\left(ik_\mu x^\mu\right) = \mathbf{E}.
\end{aligned}$$

Since $k_0^2 - \mathbf{k}^2 = 0$ and $\mathbf{n} = \mathbf{k}/\sqrt{\mathbf{k}^2}$, $\mathbf{k} \times \mathbf{B}/k_0 = \mathbf{n} \times \mathbf{B} = \mathbf{E}$. The conclusion of these calculations is that $\mathbf{E}, \mathbf{B}$ are orthogonal to each oother and to the direction of propagation of the wave given by $\mathbf{k}$. To analyse the phase difference between $\mathbf{E}$ and $\mathbf{B}$, we notice that the global phase in $\mathbf{E}$ is the phase of the complex quantity $a_0 \mathbf{k} - k_0 \mathbf{a}$ and that the global phase in $\mathbf{B}$ is the phase in $\mathbf{a}$.

D Linearly polarized waves

In what follows, we set $A^0 = -\varphi = 0$, $a^0 = 0$ which corresponds to having a 0 electric potential everywhere. The time derivative of the spatial components of four potential is

$$\dot{\mathbf{A}} = ik_0 \mathbf{a} \exp\left(ik_\mu x^\mu\right).$$

It can be used to express $\mathbf{E}, \mathbf{B}$ when the $a^0 = 0$. Indeed

$$\mathbf{E} = -\dot{\mathbf{A}} = (i(0)\mathbf{k} - i\mathbf{a}k_0) \exp\left(ik_\mu x^\mu\right), \mathbf{B} = \mathbf{n} \times \dot{\mathbf{A}}$$

E Poynting vector

The energy-momentum transport associated to the electromagnetic field is described by the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}$. Here, we want to relate $\mathbf{S}$ to the electromagnetic energy density $\varepsilon = (\mathbf{E}^2 + \mathbf{B}^2)/2$. To do so, we differenciate ε with respect to time to get

$$\frac{\partial \varepsilon}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot (\mathbf{E} \times \mathbf{B}) \iff 0 = \frac{\partial \varepsilon}{\partial t} + \nabla \cdot \mathbf{S}$$

where we have used the Faraday and Vacuum Ampere laws to express the partial derivatives. A continuity equation is found and we interpret $\mathbf{S}$ as the energy current density.

F Asymptotic Power

Following the analogy with the charge continuity equation, we can write an integral form of the energy continuity equation. We choose a spherical volume V surrounded by a sphere surface ∂V at radius R with outward normal $\mathbf{n}$. Integrating he continuity equation for ε and $\mathbf{S}$, we get

$$0 = \int_V \mathrm{d}^3 r \left(\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \mathbf{S} \right) = \frac{\partial}{\partial t} \left(\int_V \mathrm{d}^3 r \varepsilon \right) + \int_V \mathrm{d}^3 r \nabla \cdot \mathbf{S} = \frac{dE}{dt} + R^2 \int_{\partial V} \sin(\theta) \mathrm{d}\phi \mathrm{d}\theta \mathbf{n} \cdot \mathbf{S}$$

Where E represents the total electromagnetic energy in V. If R is big enough compared to the caracteristic siuze of the emitting system, only radiation directed to infinity goes trough it and $\frac{dE}{dt}$ represents the total radiation power of the system.

G Poynting vector for planar waves

For planar waves, we have the following poynting vector:

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} = -\mathbf{B} \times (\mathbf{n} \times \mathbf{B}) = -(\mathbf{B} \cdot \mathbf{n})\mathbf{B} + (\mathbf{B} \cdot \mathbf{B})\mathbf{n} = \frac{\mathbf{B}^2 + \mathbf{E}^2}{2} \mathbf{n} = \varepsilon \mathbf{n}$$

where we used $\mathbf{E} = \mathbf{n} \times \mathbf{B}$, $0 = \mathbf{n} \cdot \mathbf{B}$ and $\mathbf{B}^2 = (\mathbf{n} \times \mathbf{B})^2 = \mathbf{E}^2$.

2 Radiation of an isolated system

A Lienard–Wiechert potential with isolated sources

The Lienard–Wiechert potential provides an expression for the four-potential generated by a charge moving on a world line.

Supposing the charges are moving slowly compared to the speed of light, the three potential $\mathbf{A}$ contribution at time t and position $\mathbf{r}$ of a point charge charge q with three-velocity $\mathbf{v}$ and three-position $\mathbf{r}'$ at time $t_R = t - |\mathbf{r} - \mathbf{r}'|$ reads:

$$\mathbf{A} = \frac{q\mathbf{v}(t_R)}{|\mathbf{r} - \mathbf{r}'| - \mathbf{v}(t_R) \cdot (\mathbf{r} - \mathbf{r}')} \approx \frac{q\mathbf{v}(t_R)}{|\mathbf{r} - \mathbf{r}'|} + O(|\mathbf{v}|^2)$$

Here we are interested in the integrated potential generated by a continum of charges described by charge density $\rho(t, \mathbf{r})$ and a three-curent $\mathbf{j}(t, \mathbf{r})$ at time t and cartesian three-position $\mathbf{r}$. In the limit of small velocities, the previous expression can be formulated in the charge continuum by replacing $q\mathbf{v}(t_R)$ by the integral expression of the magnetic potential is given by $\mathbf{j}(t_R, \mathbf{r}')$ and integrating over a space-slice to combine the contribution of all sources. We have

$$\mathbf{A}(t, \mathbf{r}) = \int \mathrm{d}^3 r' \frac{\mathbf{j}(t_R, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.$$

If the observation point $\mathbf{r}$ of the three-potential is far from the sources, we can write

$$\begin{aligned} \mathbf{A}(t, \mathbf{r}) &= \int_{\mathbf{j}(t_R, \mathbf{r}') \sim 0,\ |\mathbf{r}| \sim |\mathbf{r}'|} \mathrm{d}^3 r' \frac{\mathbf{j}(t_R, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \int_{\mathbf{j}(t_R, \mathbf{r}') \not\sim 0,\ |\mathbf{r}| \gg |\mathbf{r}'|} \mathrm{d}^3 r' \frac{\mathbf{j}(t_R, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &\approx \frac{1}{|\mathbf{r}|} \int_{\mathbf{j}(t_R, \mathbf{r}') \sim 0,\ |\mathbf{r}| \sim |\mathbf{r}'|} \mathrm{d}^3 r' \underbrace{\mathbf{j}(t_R, \mathbf{r}')}_{\sim 0} + \frac{1}{|\mathbf{r}|} \int_{\mathbf{j}(t_R, \mathbf{r}') \not\sim 0,\ |\mathbf{r}| \gg |\mathbf{r}'|} \mathrm{d}^3 r' \mathbf{j}(t_R, \mathbf{r}') = \frac{1}{|\mathbf{r}|} \int \mathrm{d}^3 r' \mathbf{j}(t_R, \mathbf{r}') \end{aligned}$$

where we have used the expansion

$$
\begin{aligned}
|\mathbf{r}-\mathbf{r}'| &= |\mathbf{r}-\mathbf{r}'|\Big|_{\mathbf{r}'=0} + \mathbf{r}' \cdot \frac{\partial}{\partial \mathbf{r}'}|\mathbf{r}-\mathbf{r}'|\Bigg|_{\mathbf{r}'=0} + O(|\mathbf{r}'|^2) = |\mathbf{r}| - \mathbf{r}' \cdot \frac{\mathbf{r}}{|\mathbf{r}|} + O(|\mathbf{r}'|^2, 1/|\mathbf{r}|^2) \\
\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{|\mathbf{r}| - \mathbf{r}' \cdot \frac{\mathbf{r}}{|\mathbf{r}|} + O(|\mathbf{r}'|^2)} = \frac{1}{|\mathbf{r}|}\frac{1}{1 - \mathbf{r}' \cdot \frac{\mathbf{r}}{|\mathbf{r}|^2} + O(|\mathbf{r}'|^2)} = \frac{1}{|\mathbf{r}|}\left(1 + \mathbf{r}' \cdot \frac{\mathbf{r}}{|\mathbf{r}|^2}\right) + O(|\mathbf{r}'|^2) = \frac{1}{|\mathbf{r}|} + O(|\mathbf{r}'|^2, 1/|\mathbf{r}|^2) \\
\mathbf{j}(t_R, \mathbf{r}') &= \mathbf{j}\left(t - |\mathbf{r}| + \mathbf{r}' \cdot \frac{\mathbf{r}}{|\mathbf{r}|}\right) =
\end{aligned}
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B

C

D

E

3 Beyond radiation

A

B

C

4 Acknowledgement

References

[1] Aldo Riello. *Fourteen Lectures in CLASSICAL PHYSICS*. 2023.