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Homework 1: Equivalence principle at work: charge in a lab

Due: Monday 9pm, October 23, 2023

The goal of your homework is to calculate the electric field of a charge sitting (or hovering if you want to) in the lab on the surface of the Earth. This problem can be calculated exactly using the GR techniques and is known as Linet's solution [1]. In our approach we shall take a simplifying assumption of uniform gravitational field described by a constant gravitational acceleration g. Employing the equivalence principle 'everything' then reduces to special relativity. In order the charge remains static there is a *constant force* pushing it upwards. The electric field of such a charge is 'distorted' and is different from the familiar Coulomb's law.

The homework shall guide you through all the steps of the calculation. The basic idea is as follows: Let us imagine that we observe such a charge from a freely falling elevator. Then the charge accelerates uniformly upwards. We also know that in this frame physical laws are well described by special relativity. In particular the fields of charge are described by the Lienard–Wiechert potentials (studied in the Classical Physics course) that can easily be constructed. Final step is to look at these fields from a reference frame where the charge is static, i.e. to perform a coordinate transformation to the accelerated frame. Of course we can verify our result using the general-relativistic version of Maxwell's equations. So here we go (we take everywhere $c = 1$).

Throughout the homework you are strongly encouraged to use Mathematica/Maple, please attach the notebook.

1 Uniformly accelerated charge

To start with something familiar to you, let us consider a charged particle with charge Q and mass m_0 moving with a 4-velocity u^μ in a constant electric field E. (This coresponds to our intuition of a constant force.) The Lorentz force law reads

$$\frac{du^\mu}{d\tau} = kF^\mu{}_\nu u^\nu\,, \quad k = Q/m_0\,. \tag{1}$$

Let us take the electric field in the x-direction, i.e., $F_{01} = -F_{10} = -E$ (and all other components of $F_{\mu\nu} = 0$). The problem is axisymmetric and hence we may use the 'cylindrical coordinates' $x^\mu = (t, x, \rho, \varphi)$ with the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + d\rho^2 + \rho^2 d\varphi^2\,. \tag{2}$$

a) Show that the following trajectory $x^\mu = x^\mu(\tau)$ of the charged particle obeys (1):

$$x^\mu(\tau) = (t_Q, x_Q, 0, 0)\,, \quad t_Q = \frac{1}{g}\sinh(g\tau)\,, \quad x_Q = \frac{1}{g}\cosh(g\tau)\,. \tag{3}$$

Show that τ is the proper time, find an expression for g.

b) Show that the particle's acceleration $a^\mu(\tau) = \frac{du^\mu}{d\tau}$ obeys

$$a^\mu a_\mu = g^2 = \text{const}\,. \tag{4}$$

Of course, this is a definition of *'uniform acceleration'*.

c) Sketch/plot in Mathematica the trajectory of a particle for $\tau \in (-\infty, \infty)$ in the Minkowski (t, x)-plane for different values of g. Comment on the asymptotic behavior of the trajectory as τ approaches infinity and its physical meaning. Compare the behavior for small and large values of g.

d) Let us now imagine that we are an observer sitting on the particle. Such an observer associates with herself the *Rindler coordinates* (T, X, ρ, φ) (discussed in the lecture), related to the Minkowski coordinates (t, x, ρ, φ) as follows:

$$t = \left(\frac{1}{g} + X\right)\sinh(gT)\,, \quad x = \left(\frac{1}{g} + X\right)\cosh(gT)\,. \tag{5}$$

Show that the Rindler coordinates cover only part of the original Minkowski space, draw the lines of constant T and X. What happens on the boundary of the space that Rindler coordinates cover?

e) Show that in Rindler coordinates the metric takes the form

$$ds^2 = -(1+gX)^2 dT^2 + dX^2 + d\rho^2 + \rho^2 d\varphi^2 \,. \tag{6}$$

2 Field of a uniformly accelerated charge

In an inertial frame (Minkowski space) the field of an arbitrarily moving charge is given by the *Lienard–Wiechert potentials* (see Classical Physics course for their derivation).

a) Convince our TA that in the relativistic notation they can write

$$A_\mu(x^\nu) = -\frac{Q u_\mu}{R^\nu u_\nu}\bigg|_{t=t(\tau)+|\vec{x}-\vec{x}(\tau)|} \,, \tag{7}$$

where $A_\mu = A_\mu(x^\nu)$ is the 4-potential at a point $x^\nu = (t, \vec{x})$, $u^\mu(\tau)$ is the 4-velocity of the charge with trajectory $x^\nu(\tau)$ and $R^\nu \equiv x^\nu - x^\nu(\tau)$. The r.h.s. is evaluated at τ such that $t = t(\tau) + |\vec{x} - \vec{x}(\tau)|$, which reflects the fact that fields propagate at the speed of light.

b) Let us now consider the uniformly accelerated charge Q described by Eq. (3). Note that

$$x_Q = \sqrt{L^2 + t_Q^2}\,, \quad L = 1/g\,. \tag{8}$$

Show that the 4-velocity of the charge can be written as

$$u_\mu(\tau) = g(-x_Q, t_Q, 0, 0)\,. \tag{9}$$

Hence show that the Lienard–Wiechert potential of the uniformly accelerated charge reads

$$A_\mu = \frac{Q}{\xi}(-x_Q, t_Q, 0, 0)\,, \quad \xi \equiv t x_Q - x t_Q\,. \tag{10}$$

Of course, here x_Q and t_Q have to be expressed in terms of x and t.

c) Show that using (8) to evaluate t_Q and x_Q and the fact that $t - t_Q = \sqrt{\rho^2 + (x - x_Q)^2}$, one can write

$$t_Q = \frac{t\delta - 2x\xi}{2(x^2 - t^2)}, \quad x_Q = \frac{x\delta - 2t\xi}{2(x^2 - t^2)}, \tag{11}$$

where

$$\xi = \left((L^2+t^2-\rho^2-x^2)^2/4 + L^2\rho^2\right)^{\frac{1}{2}}\,, \qquad \delta \equiv \rho^2+x^2+L^2-t^2\,. \tag{12}$$

d) Which components of $F_{\mu\nu}$ are nontrivial? What fields do they describe?

3 Sitting on the charge

Let us now transform back to the frame where the charge is static.

a) Using transformation (5), transform A_μ (10) to show that in the rest frame of the charge we have

$$A_\mu{}' = \left(-\Phi, \frac{-Q}{X+L}, 0, 0\right), \quad \Phi = \frac{Q}{r}\frac{1+gX+g^2r^2/2}{\sqrt{1+gX+g^2r^2/4}}\,, \tag{13}$$

where we have defined $r = \sqrt{X^2+\rho^2}\,$.

b) Find an appropriate gauge function f such that the 4-potential can be written as $A_\mu{}' = -\Phi\delta_\mu^T$, i.e., it has only the electrostatic part Φ. This means that in an orthonormal frame the observer would see the electrostatic potential

$$\varphi = \frac{\Phi}{1+gX}\,. \tag{14}$$

This is the *modified Coulomb potential* of the charge under the influence of constant acceleration g.

c) Derive the linear and quadratic in g correction to the Coulomb potential law.

d) Sketch the equipotentials of full φ in (ρ, X) plane for several g's. Comment on your findings and try to interpret them.

e) Derive the nontrivial components of $F'_{\mu\nu}$ and verify that Maxwell's equations in 'curved' spacetime

$$\frac{1}{\sqrt{-\det(g_{\alpha\beta})}}\partial_\mu\left(\sqrt{-\det(g_{\alpha\beta})}F^{\mu\nu\prime}\right) = 0 \tag{15}$$

are satisfied in the Rindler space (6).

Remark: The field of uniformly accelerated charge was studied for the first time in 1909 by Born [2]. Since then many papers have appeared. The attention focuses especially on the following two questions: whether the uniformly accelerated charge emits radiation, and whether the principle of equivalence holds. Although satisfactory explanation can be found e.g. in [3], [4], contradictory statements appear until now. If you are interested, nice relatively recent reference is [5].

References

[1] Linet B. (1976) *J. Phys. A: Math. Gen.* 9, 1081.

[2] Born M. (1909): *Ann. Physik* 30, 1.

[3] Rohrlich F. (1965): *Classical charged particles: foundation of their theory*, Addison-Wesley, Reading.

[4] Boulware D. G. (1980): *Radiation from a uniformly accelerated charge*, Ann. Phys. 124, 169.

[5] Eriksen E., Grøn Ø. (2000): *Electrodynamics of hyperbolically accelerated charges*, Ann. Phys. 286, 320-399.