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HOMework 1

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1 Concurrence and negativity

- (a) We are interested in the faithful measure of entanglement of two qubits provided by the concurrence. The individual states of the qubits are elements of the Hilbert space $\mathcal{H}_A = \mathbb{C}^2$ and $\mathcal{H}_B = \mathbb{C}^2$ and their joint state is described with most generality through the density matrix $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ ¹ representing their classical and quantum correlations. We work from a basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{H}_{A,B}$ associated with the Pauli matrices denoted σ_x , σ_y and σ_z . For states in $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the concurrence can be expressed as $C[\rho] := \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ for the set $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ (ordered in decreasing order) of eigenvalues of $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. We start by computing the concurrence for a Bell state $|\Phi_{AB}^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. To make the calculation more transparent, we use a local operation σ_x on the second qubit to get the state $|\Psi_{AB}^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ ², their concurrences are the same. We have

$$\rho = |\Psi_{AB}^+\rangle\langle\Psi_{AB}^+| = \frac{1}{2}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since ρ is a block diagonal with two vanishing blocks, the calculation of its square root reduces to the calculation of the square root of the non-zero block. This square root can be inferred through

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{1/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

where we selected the positive branch of the square root. Next, we calculate $\tilde{\rho}$ as follows

$$\begin{aligned} \tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -i(-i) \\ 0 & 0 & -i(i) & 0 \\ 0 & i(-i) & 0 & 0 \\ i(i) & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & 0 & 0 & -i(-i) \\ 0 & 0 & -i(i) & 0 \\ 0 & i(-i) & 0 & 0 \\ i(i) & 0 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and we finally obtain the following R matrix

$$\begin{aligned} R = (\sigma_x \otimes 1)\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}(\sigma_x \otimes 1) &= (\sigma_x \otimes 1) \begin{pmatrix} \frac{1}{\sqrt{2}\sqrt{2}} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \frac{1}{\sqrt{2}\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix}^{1/2} (\sigma_x \otimes 1) \\ &= (\sigma_x \otimes 1) \begin{pmatrix} \frac{1}{4} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \end{pmatrix}^{1/2} (\sigma_x \otimes 1) = (\sigma_x \otimes 1) \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (\sigma_x \otimes 1) \end{aligned}$$

¹ \mathcal{D} denotes the space of hermitian operators of trace one acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ with positive or zero eigenvalues. We have this restriction on eigenvalues because they provide the probabilities of each outcome of the statistical mixture represented by the diagonalized ρ

²Starting with $|\rho_0\rangle = |\Phi_{AB}^+\rangle\langle\Phi_{AB}^+|$ and express the associated concurrence in terms of $\rho = |\Psi_{AB}^+\rangle\langle\Psi_{AB}^+|$: we have

$$\begin{aligned} R &= \sqrt{\sqrt{\rho_0}(\sigma_y \otimes \sigma_y)\rho_0^*(\sigma_y \otimes \sigma_y)\sqrt{\rho_0}} = \sigma_x \sqrt{\sqrt{\rho}(\sigma_x \sigma_y \sigma_x \otimes \sigma_y)\rho^*(\sigma_x \sigma_y \sigma_x \otimes \sigma_y)\sqrt{\rho}} \sigma_x \\ &= \sigma_x \sqrt{\sqrt{\rho}(-\sigma_y \otimes \sigma_y)\rho^*(-\sigma_y \otimes \sigma_y)\sqrt{\rho}} \sigma_x = \sigma_x \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}} \sigma_x \end{aligned}$$

which shows what effect the real local unitary $\sigma_x \otimes 1$ has on R . The $\sigma_x \otimes 1$ operation does not change the eigenvalues of R because it preserve the characteristic polynomial $\det(\sigma_x(R - \lambda)\sigma_x) = (-1)^2 \det(R - \lambda)$ brought to 0 by the eigenvalues of R . This property is a feature of any operator U having $U^\dagger U = 1$ (unitary operators preserve eigenvalue). we note that the transformation considered preserves the concurrence because its components are all real (if they were not, there would be complications regarding the complex conjugation of ρ)

Since the determinant of the non-zero block of R is 0, one of its eigenvalues is 0 and the other is forced to equal the trace and is 1. We have the sequence $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{1, 0, 0, 0\}$ leading to the concurrence $C[\rho] = \max(0, 1/\sqrt{2}) = 1/\sqrt{2}$. For the product state $|00\rangle$ the calculation of concurrence is simplified. We have that $\rho = |00\rangle\langle 00|$ which is a projector implying $\rho = \rho^2 \implies \sqrt{\rho} = \rho$. The matrix $\tilde{\rho}$ also has a simple expression $\tilde{\rho} = (\sigma_y \otimes \sigma_y)(1^*)|00\rangle\langle 00|(\sigma_y \otimes \sigma_y)^\dagger = (i)^2((i)^2)^*|11\rangle\langle 11| = |11\rangle\langle 11|$. For this preduct state, we get $R^2 = |00\rangle\langle 00||11\rangle\langle 11||00\rangle\langle 00| = 0 \implies R = 0$ and $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{0, 0, 0, 0\}$ leading to the concurrence $C[\rho] = \max(0, 0) = 0$.

- (b) For states in $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we can measure entanglement with negativity defined as $\mathcal{N}[\rho] = (\text{tr}(\sqrt{(\rho^\Gamma)^\dagger \rho^\Gamma}) - 1)/2$ where ρ^Γ is the partial transpose of ρ taken with respect to its subsystem A indices. Here we want to calculate the negativity of $\rho = |\Phi_{AB}^+\rangle\langle \Phi_{AB}^+|$. We start by calculating the partial transpose

$$\rho^\Gamma = \frac{1}{2} \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}^\Gamma = \frac{1}{2} \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

We then calculate the matrix norm

$$\text{tr}(\sqrt{(\rho^\Gamma)^\dagger \rho^\Gamma}) = \text{tr} \left(\left(\begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \right)^{1/2} \right) = \frac{1}{2} \text{tr} \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)^{1/2} = 2$$

leading to a negativity $\mathcal{N}[\rho] = (2 - 1)/2 = 1/2$ which is half of the concurrence calculated for the same state in (a) as expected of a two-qubit system. Repeating the calculation for the state $\rho = |00\rangle\langle 00|$, we find $\rho^\Gamma = \rho$ and $\text{tr}(\sqrt{|00\rangle\langle 00|(|00\rangle\langle 00|)^\dagger}) = 1 + 0 \times 3$ leading to the negativity $\mathcal{N}[\rho] = (1 - 1)/2 = 0$ as expected for a separable state.

- (c) Here we are interested in the negativity of the bipartite state $\rho = 1_A \times 1_B/4$. We have $\rho^\Gamma = 1_A \times 1_B^T/4 = \rho^\Gamma$ and $\text{tr}(1_A \times 1_B/4) = 1$ implying $\mathcal{N}[\rho] = (1 - 1)/2 = 0$ which is again expected for a separable state. Vanishing of negativity for separable states is a property differentiating it from Von Neumann entropy which fails to distinguish separable states from states that are not because it also grows with classical correlations.

2 POVM accounts for errors

- (a) We consider a source producing a statistical mix of qubit pure states described in the basis $\{|0\rangle, |1\rangle\}$ by the fully general density matrix $\rho = \frac{1}{2}(1 + \mathbf{r} \cdot \boldsymbol{\sigma})$ where $\mathbf{r} = (r_x, r_y, r_z)$ and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ (vector of Pauli matrices associated to the chosen basis). Since the trace of all Pauli matrices vanishes, the only term contributing to the trace of ρ is $\frac{1}{2}$ which ensures $\text{tr}(\rho) = 1$. Through a basis transformation, we can always bring the density matrix to the form $\rho = \frac{1}{2} \text{diag}(1 + |\mathbf{r}|, 1 - |\mathbf{r}|)$ showing that ρ has positive or zero probability eigenvalues if and only iff $0 \leq |\mathbf{r}| \leq 1$. The $|\mathbf{r}| = 1$ saturation produces states with purity 1 and we can identify the surface of a ball of radius 1 in the \mathbf{r} parameter space with the Bloch sphere of pure qubit states. The $|\mathbf{r}| = 1$ state produces a unique maximally mixed state living at the origin of the \mathbf{r} parameter space. We see that in the ball of allowed parameters, we interpolate between maximally mixed at the origin and pure on the surface. Using the fact $\sigma_i \sigma_j = i \varepsilon_{ijk} \sigma_k$ and $\sigma_i^2 = 1$ we have

$$\text{tr}(\sigma_i \sigma_j) = \begin{cases} \text{tr}(\sigma_i^2) = 2, & i = j \\ \text{tr}(i \varepsilon_{ijk} \sigma_k) = i \varepsilon_{ijk} \text{tr}(\sigma_k) = 0, & i \neq j \end{cases} = 2 \delta_{ij}.$$

This property allow to extract the components of \mathbf{r} from ρ with

$$r_i = \text{tr}(\rho \sigma_i) = \text{tr} \left(\frac{1}{2} (1 + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z) \sigma_i \right) = \frac{1}{2} (\text{tr}(\sigma_i) + r_x \text{tr}(\sigma_x \sigma_i) + r_y \text{tr}(\sigma_y \sigma_i) + r_z \text{tr}(\sigma_z \sigma_i)) = r_x \delta_{ix} + r_y \delta_{iy} + r_z \delta_{iz}$$

which shows r_i can be interpreted as the average outcome of a σ_i measurement on ρ . Repeating many experiments on qubits prepared in the state ρ for each direction allows to statistically determine ρ .

- (b) The method for determining ρ described in (a) can be brought closer to the experiment by replacing the notion of projective measurements of σ_i with a POVM. More precisely, we use a POVM to model a measurement which has probability $1/2$ to be performed in the desired basis $\{|0\rangle, |1\rangle\}$ and probability $1/2$ to be performed in a closely accidental basis $\{|0'\rangle = \sqrt{1-\varepsilon}|0\rangle + \sqrt{\varepsilon}|1\rangle, |1'\rangle = -\sqrt{1-\varepsilon}|1\rangle + \sqrt{\varepsilon}|0\rangle\}$ where $0 < \varepsilon \ll 1$ is a small parameter representing how far the accidental basis is from the desired basis. The POVM element associated with a measurement of 0 is formed by summing the 0 projector in each basis weighted by the probability of measuring in that basis and actually applying this projector. We have $E_0 = \frac{1}{2}(|0\rangle\langle 0| + |0'\rangle\langle 0'|)$. The same construction can be done for a measurement of 1 to write the POVM element $E_1 = \frac{1}{2}(|1\rangle\langle 1| + |1'\rangle\langle 1'|)$. Next, we verify that the proposed E_0, E_1 representing the imperfect measurement satisfies the required properties for POVM elements. We have $E_1 + E_0 = \frac{1}{2}(|0\rangle\langle 0| + |0'\rangle\langle 0'| + |1\rangle\langle 1| + |1'\rangle\langle 1'|) = 1$ and from the matrix representation,

$$E_0 = \frac{1}{2} \begin{pmatrix} 2-\varepsilon & \sqrt{\varepsilon(1-\varepsilon)} \\ \sqrt{\varepsilon(1-\varepsilon)} & \varepsilon \end{pmatrix}, \quad E_1 = \frac{1}{2} \begin{pmatrix} \varepsilon & -\sqrt{\varepsilon(1-\varepsilon)} \\ -\sqrt{\varepsilon(1-\varepsilon)} & 2-\varepsilon \end{pmatrix}$$

we see that we have positive trace $1/2$ and determinant $\varepsilon/4 > 0$ implying that their two eigenvalues have positive sum and positive product. The positive product forces the eigenvalues to share their sign and the positive trace can never be realized if they are both negative. We conclude that for $\varepsilon > 0$, E_0, E_1 are positive definite (they would be positive semidefinite at $\varepsilon = 0$ as expected for projectors) and are valid POVM elements. If we apply this POVM to the state $|0\rangle\langle 0|$, the probability of measuring 1 is given by $\text{tr}(E_1 |0\rangle\langle 0|) = \frac{1}{2}\text{tr}(|1'\rangle\langle 1'| |0\rangle\langle 0|) = \frac{1}{2}\langle 0| |1'\rangle\langle 1'| |0\rangle = \frac{1}{2}\langle 0| |1'\rangle\langle 1'| |0\rangle = \frac{1}{2}\langle 0| |1'\rangle\langle 1'| |0\rangle = \varepsilon/2$.

- (c) We now consider the effect of the faulty measurement described in (b) on the coefficients of \mathbf{r} effectively extracted from an adaptation of the method described in (a). We consider that only the measurement in the z direction is faulty (the x and y directions are still associated with projective measurements). As before we will suppose r_z (now denoted r'_z for the faulty measurement) is the expectation value of the z direction measurement outcome. In terms of the POVM elements defined in (b), this expectation value is

$$\begin{aligned} r'_z &= 1 \times \text{tr}(E_0 \rho) - 1 \times \text{tr}(E_1 \rho) \\ &= \text{tr} \left[\frac{1}{2} \begin{pmatrix} 2-\varepsilon & \sqrt{\varepsilon(1-\varepsilon)} \\ \sqrt{\varepsilon(1-\varepsilon)} & \varepsilon \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix} \right] - \text{tr} \left[\frac{1}{2} \begin{pmatrix} \varepsilon & -\sqrt{\varepsilon(1-\varepsilon)} \\ -\sqrt{\varepsilon(1-\varepsilon)} & 2-\varepsilon \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix} \right] \\ &= \text{tr} \left[\frac{1}{4} \begin{pmatrix} (2-\varepsilon)(1+r_z) + \sqrt{\varepsilon(1-\varepsilon)}(r_x + ir_y) & \dots \\ \dots & \varepsilon(1-r_z) + \sqrt{\varepsilon(1-\varepsilon)}(r_x - ir_y) \end{pmatrix} \right] \\ &\quad - \text{tr} \left[\frac{1}{4} \begin{pmatrix} \varepsilon(1+r_z) - \sqrt{\varepsilon(1-\varepsilon)}(r_x + ir_y) & \dots \\ \dots & (2-\varepsilon)(1-r_z) - \sqrt{\varepsilon(1-\varepsilon)}(r_x - ir_y) \end{pmatrix} \right] \\ &= \frac{1}{4} \left[(2-\varepsilon)(1+r_z) + \varepsilon(1-r_z) + 2\sqrt{\varepsilon(1-\varepsilon)}r_x - ((2-\varepsilon)(1-r_z) + \varepsilon(1+r_z) - \sqrt{\varepsilon(1-\varepsilon)}r_x) \right] \\ &= \frac{1}{4} \left[2(2-\varepsilon)r_z - 2\varepsilon r_z + 4\sqrt{\varepsilon(1-\varepsilon)}r_x \right] = (1-\varepsilon)r_z + \sqrt{\varepsilon(1-\varepsilon)}r_x. \end{aligned}$$

3 Entanglement-breaking channel

- (a) Provided two Hilbert spaces \mathcal{H} and \mathcal{H}' and their respective density matrix spaces $\mathcal{D}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H}')$, a quantum channel $\Phi : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H}')$ is a linear completely-positive trace-preserving map representing quantum state manipulation in a general way. To study the entanglement breaking of a quantum channel, we consider a system with Hilbert space \mathcal{H}_A and its environment with Hilbert space \mathcal{H}_B represented by a density matrix $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Given a quantum channel $\Phi : \mathcal{D}(\mathcal{H}_A) \rightarrow \mathcal{D}(\mathcal{H}'_A)$ with target Hilbert space \mathcal{H}'_A we construct an extended channel $\Phi_E : \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathcal{D}(\mathcal{H}'_A \otimes \mathcal{H}_B)$ applying Φ to the system part of a density matrix and applying the identity channel to the environment part. The map Φ is said to be entanglement breaking if an extension Φ_E by any environment is such that $\Phi_E(\rho_{AB}) = \rho_{A'B}$ is a separable state. In other words, this channel produces a density matrix featuring no entanglement between the environment state and the target state in \mathcal{H}'_A of the system.

To build an example of an entanglement breaking channel, we study the combination of a qubit system S (with Hilbert space \mathcal{H}_S) with a toy environment E (with Hilbert space \mathcal{H}_E) also modeled as a qubit. Initially, S and E are respectively in states ρ_S and ρ_E . We then act on them with the unitary $U = e^{i\theta\sigma_z \otimes \sigma_z}$ parametrised by θ with the pauli matrices σ_z on S and E associated to the respective basis $\{|0\rangle, |1\rangle\}_S$ and $\{|0\rangle, |1\rangle\}_E$. A quantum channel $\Phi : \mathcal{D}(\mathcal{H}_S) \rightarrow \mathcal{D}(\mathcal{H}_E)$ acting on ρ_S is constructed by tracing the system S out the result of the action of U on $\rho_S \otimes \rho_E$. Explicitly, we have

$$\begin{aligned} \rho'_E &:= \Phi(\rho_S) = \text{tr}_S(U \rho_S \otimes \rho_E U^\dagger) = \langle 0| e^{i\theta\sigma_z \otimes \sigma_z} \rho_S \otimes \rho_E e^{i\theta\sigma_z \otimes \sigma_z} |0\rangle + \langle 1| e^{i\theta\sigma_z \otimes \sigma_z} \rho_S \otimes \rho_E e^{i\theta\sigma_z \otimes \sigma_z} |1\rangle \\ &= \langle 0| \rho_S |0\rangle \otimes e^{+i\theta\sigma_z} \rho_E e^{-i\theta\sigma_z} + \langle 1| \rho_S |1\rangle \otimes e^{-i\theta\sigma_z} \rho_E e^{+i\theta\sigma_z}. \end{aligned}$$

Precision: Here, the spectral decomposition $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ was used in the same way it is used in part (c). We write $U = e^{i\theta(|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes \sigma_z} = |0\rangle\langle 0| \otimes e^{i\theta\sigma_z} + |1\rangle\langle 1| \otimes e^{-i\theta\sigma_z}$ to simplify the calculation of the trace to

$$\begin{aligned}\rho'_E &= \langle 0| e^{i\theta\sigma_z \otimes \sigma_z} \rho_S \otimes \rho_E e^{i\theta\sigma_z \otimes \sigma_z} |0\rangle + \langle 1| e^{i\theta\sigma_z \otimes \sigma_z} \rho_S \otimes \rho_E e^{i\theta\sigma_z \otimes \sigma_z} |1\rangle \\ &= \langle 0| |0\rangle\langle 0| \otimes e^{i\theta\sigma_z} \rho_S \otimes \rho_E |0\rangle\langle 0| \otimes e^{i\theta\sigma_z} |0\rangle + \langle 1| |1\rangle\langle 1| \otimes e^{-i\theta\sigma_z} \rho_S \otimes \rho_E |1\rangle\langle 1| \otimes e^{-i\theta\sigma_z} |1\rangle \\ &= \langle 0| \rho_S |0\rangle \otimes e^{+i\theta\sigma_z} \rho_E e^{-i\theta\sigma_z} + \langle 1| \rho_S |1\rangle \otimes e^{-i\theta\sigma_z} \rho_E e^{+i\theta\sigma_z}\end{aligned}$$

- (b) To simplify further calculation, we set $\rho_E = |0\rangle\langle 0|$. Combining system S with second system S' with qubit Hilbert space $\mathcal{H}_{S'}$, we can form the joint state $\rho_{SS'} \in \mathcal{D}(\mathcal{H}_S \otimes \mathcal{H}_{S'})$ which we take to be the maximally entangled state $\rho_{SS'} = |\Phi^+\rangle\langle \Phi^+|$ where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ ($\{|0\rangle, |1\rangle\}_{S'}$ is a basis of $\mathcal{H}_{S'}$). Extending the action of Φ on $\mathcal{D}(\mathcal{H}_S)$ to the action of $\Phi_{S'}$ on $\mathcal{D}(\mathcal{H}_S \otimes \mathcal{H}_{S'})$ as described above, we get

$$\begin{aligned}\rho'_{ES'} &:= \Phi_{S'}(\rho_{SS'}) = \text{tr}_S(U \rho_{SS'} \otimes \rho_E U^\dagger) = \langle 0| e^{i\theta\sigma_z \otimes \sigma_z} \rho_{SS'} \otimes \rho_E e^{i\theta\sigma_z \otimes \sigma_z} |0\rangle + \langle 1| e^{i\theta\sigma_z \otimes \sigma_z} \rho_{SS'} \otimes \rho_E e^{i\theta\sigma_z \otimes \sigma_z} |1\rangle \\ &= \langle 0| \rho_{SS'} |0\rangle \otimes e^{+i\theta\sigma_z} \rho_E e^{-i\theta\sigma_z} + \langle 1| \rho_{SS'} |1\rangle \otimes e^{-i\theta\sigma_z} \rho_E e^{+i\theta\sigma_z} \\ &= \langle 0| |\Phi^+\rangle\langle \Phi^+| |0\rangle \otimes e^{+i\theta\sigma_z} |0\rangle\langle 0| e^{-i\theta\sigma_z} + \langle 1| |\Phi^+\rangle\langle \Phi^+| |1\rangle \otimes e^{-i\theta\sigma_z} |0\rangle\langle 0| e^{+i\theta\sigma_z} \\ &= \left(\frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| \right) \otimes |0\rangle\langle 0|\end{aligned}$$

which is separable making Φ an entanglement breaking channel.

- (c) A classical-quantum channel (QC) acting on the density matrices $\rho \in \mathcal{D}(\mathcal{H})$ associated to the Hilbert space \mathcal{H} is defined by $\Phi_{qc}(\rho) := \sum_j \text{tr}(\hat{E}_j \rho) \sigma_j$ where σ_j are density operators and \hat{E}_j are POVM elements. This channel takes a quantum system and returns the mixed state resulting from a POVM where the measurement output is not recorded. It can be shown that all QC channels are entanglement breaking. We are interested in the channel $\Phi(\rho_S) = \text{tr}_S(U \rho_S \rho_E U^\dagger)$ which is a generalization of the qubit channel described in (b) with \mathcal{H}_S and \mathcal{H}_E general Hilbert spaces and $U = e^{i\theta \hat{A} \otimes \hat{B}}$ for A, B arbitrary Hermitian operators respectively acting on S and E . To show that Φ as an entanglement breaking map is also a QC map, we write the spectral decomposition $A = \sum_i A_i |A_i\rangle\langle A_i|$ with A_i the eigenvalues of A and $|A_i\rangle$ their associated eigenvectors. With this decomposition, we have

$$U = e^{i\theta \hat{A} \otimes \hat{B}} = e^{i\theta \sum_j A_j |A_j\rangle\langle A_j| \otimes \hat{B}} = \sum_j |A_j\rangle\langle A_j| e^{i\theta A_j \times 1 \otimes \hat{B}}$$

and

$$\begin{aligned}\Phi(\rho_S) &= \text{tr}_S(U \rho_S \otimes \rho_E U^\dagger) = \sum_k \langle A_k| \sum_j |A_j\rangle\langle A_j| e^{i\theta A_j \times 1 \otimes \hat{B}} \rho_S \otimes \rho_E \sum_l |A_l\rangle\langle A_l| e^{-i\theta A_l \times 1 \otimes \hat{B}} |A_k\rangle \\ &= \sum_k \langle A_k| \rho_S |A_k\rangle \otimes e^{i\theta A_k \hat{B}} \rho_E e^{-i\theta A_k \hat{B}}\end{aligned}$$

which is a separable state and shows the considered entanglement breaking map is a QC map consistently with the fact all QC channels are entanglement breaking.

Precision: $\Phi(\rho_S)$ is a density matrix because it has trace 1 ($\text{tr}_E(\Phi(\rho_S)) = \text{tr}_E \text{tr}_S(U \rho_S \otimes \rho_E U^\dagger) = \text{tr}(\rho_S \otimes \rho_E) = 1$ using the cyclic property of the total trace), is positive semidefinite ($\langle A_k| \rho_S |A_k\rangle$ is positive because ρ_S is a density matrix and each term in the k sum is the product M_k of a positive number by a basis transformation on a positive semidefinite matrix ρ_E and constitute a positive semidefinite matrix. For any vector $|\psi\rangle$, we have the inner product $\langle \psi| M_k |\psi\rangle \geq 0$ leading to $\langle \psi| \Phi(\rho_S) |\psi\rangle = \langle \psi| \sum_k M_k |\psi\rangle \geq 0$ which shows $\Phi(\rho_S)$ is positive semidefinite) and is because it is the sum of Hermitian operator $(e^{i\theta A_k \hat{B}} \rho_E e^{-i\theta A_k \hat{B}})^\dagger = e^{-(-i)\theta A_k \hat{B}} \rho_E e^{+(-i)\theta A_k \hat{B}}$ weighted by positive real numbers $\langle A_k| \rho_S |A_k\rangle$.

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