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Homework 1

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1 Conformal invariance of the Maxwell action for $D = 4$

(a) Consider a classical abelian gauge field A_μ on $D=4$ dimensionnal Minkowski spacetime. Under an infinitesimal conformal transformation, spacetime undergoes the transformation $\tilde{x}^\mu = f(x) = x^\mu + \xi^\mu(x)$ where $\xi^\mu(x)$ is a smal deformation. We want to calculate the effect of this transformation on the gauge field A_μ. The starting point is that we expect A_μ to transform as a tensor under the Lorenz transformation subgroup of the conformal group. This implies that A_μ is a primary operator and we denote its scaling dimension Δ. The transformed field $\tilde{A}_\mu$ at $\tilde{x}$ is related to the original field A_μ at x by an internal rotation, scaling, and special conformal transformation. The rotation operation acts on the components A_μ through its spin 1 representation which is the defining representation of rotations. The scaling and special conformal transformation act together through the multiplication of A_μ by the Jacobian factor $|\partial x/\partial\tilde{x}|_x^{\Delta/D}$. Finally, translations act trivially internally. This can be summarized with the relation $\tilde{A}_\mu(\tilde{x}) = |\partial x/\partial \tilde{x}|_x^{\Delta/D} R_\mu^\nu A_\nu(x)$ where R_μ^ν is the matrix associated with the part of $\xi^\mu(x)$ that does not change the metric components (after the Weyl and diffeomorphism transformations). With this in mind, we calculate the jacobian of the infinitesimal transformation to be

$$\left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|_x = \left|\frac{\partial\tilde{x}^\mu}{\partial x^\nu}\right|_x^{-1} = |\delta^\mu_\nu + \partial_\nu\xi^\mu|_x^{-1} \approx |e^{-\partial_\nu\xi^\mu}|_x = e^{-\mathrm{Tr}\,\partial_\nu\xi^\mu(x)} = 1 - \partial_\mu\xi^\mu(x) + O(\xi^2).$$

The matrix $R_\mu^\nu(x)$ can be extracted by dividing the matrix $(\partial x/\partial\tilde{x})_x$ by a factor $\Omega(x)$ such that we extract the "metric component preserving" operation. To find this factor we consider the effect on the metric of $\Omega^{-1}(x)(\partial x/\partial\tilde{x})_x$. We can write the "metric component preserving" property as

$$\Omega^{-2}(x)\left(\frac{\partial x^\mu}{\partial\tilde{x}^\sigma}\right)_x\left(\frac{\partial x^\nu}{\partial\tilde{x}^\rho}\right)_x\eta_{\mu\nu} = \eta_{\sigma\rho}.$$

Since $\Omega(x)$ is a factor, we can extract it by taking the determinant on both sides of the previous relation to get

$$\det(\eta)\Omega(x)^{-2D}\left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|_x^2 = \det(\eta) \iff \Omega(x) = \left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|_x^{-\frac{1}{D}}.$$

This result can be intuitively understood from the fact the Jacobian measures D-volume rescaling. Since we want metric components (associated with distances) to be preserved by the rescaled transformation, we need to divide by the D-root of the jacobian. The matrix $R_\mu^\nu(x)$ provided by the rescaling is given by

$$\begin{aligned}R_\mu^\nu(x) = \frac{1}{(1-\partial_\sigma\xi^\sigma(x)+O(\xi^2))^{1/D}}\left(\frac{\partial x^\nu}{\partial\tilde{x}^\mu}\right)_x &= (1+\partial_\sigma\xi^\sigma(x)/D + O(\xi^2))(\delta^\mu_\nu + \partial_\mu\xi^\nu(x) + O(\xi^2))^{-1}\\ &= \delta^\mu_\nu(1+\partial_\sigma\xi^\sigma(x)/D) - \partial_\mu\xi^\nu(x) + O(\xi^2).\end{aligned}$$

We note that $R_\mu^\nu(x)$ will represent a rotation if $\partial_\sigma\xi^\sigma(x) = 0$ (bring the conformal Killing equation to the normal Killing equation with a rotation isometry as its solution). If $\partial_\sigma\xi^\sigma(x)\neq 0$, the rescaled transformation contains a special conformal transformation. The special conformal transformation as a Weyl transformation does not preserve distances but can be combined with a diffeomorphism to preserve the initial components of the metric. With these results, we can write the effect of the infinitesimal transformation as

$$\begin{aligned}\tilde{A}_\mu(\tilde{x}) &= (1-\partial_\rho\xi^\rho(f^{-1}(\tilde{x}))+O(\xi^2))^{\Delta/D}(A_\mu(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2))\\ &= \left(1-\frac{\Delta}{D}\partial_\rho\xi^\rho(f^{-1}(\tilde{x}))+O(\xi^2)\right)(A_\mu(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2))\\ &= A_\mu(f^{-1}(\tilde{x})) - A_\mu(f^{-1}(\tilde{x}))\frac{\Delta}{D}\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2).\end{aligned}$$

Since $\xi(f^{-1}(\tilde{x}))$ is already first order in ξ, the only term contribution to its expansion around $\xi=0$ at $O(\xi)$ is $\xi(\tilde{x})$. To go further, we expand $f^{-1}(\tilde{x})$ at first order in $\xi(\tilde{x})$ with the ansatz $f^{-1}(\tilde{x})^\nu = \tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x})$ (the first term of this ansatz is justified by noticing the transformation reduces to identity at $\xi=0$). From $f(f^{-1}(\tilde{x})) = \tilde{x}$, we find

$$\tilde{x}^\nu = \tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x}) + \xi(\tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x})) + O(\xi^2) \implies B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x}) + \xi^\nu(\tilde{x}) = 0, \quad \forall\xi(\tilde{x}) \implies B_\mu^\nu(\tilde{x}) = -\delta_\mu^\nu.$$

Using this result, we can expand $A_\mu(f^{-1}(\tilde{x}))$ as

$$A_\mu(f^{-1}(\tilde{x})) = A_\mu(\tilde{x}^\nu - \xi^\nu(\tilde{x}) + O(\xi^2)) = A_\mu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2)$$

Combining this expression with the internal transformation at first order in ξ, we get

$$\begin{aligned}\tilde{A}_\mu(\tilde{x}) &= \left(1-\frac{\Delta}{D}\partial_\sigma\xi^\sigma(\tilde{x}) + \partial_\sigma\xi^\sigma(\tilde{x}) - \partial_\mu\xi^\nu(\tilde{x})\right)(A_\mu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x})) + O(\xi^2)\\ &= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta-1}{D}\partial_\sigma\xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\partial_\mu\xi^\nu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2)\end{aligned}$$

with $\xi(f^{-1}(\tilde{x})) = \xi(\tilde{x}) + O(\xi^2)$. This result can be simplified by using the conformal killing equation $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2\eta_{\mu\nu}\partial_\sigma \xi^\sigma / D$ as follows:

$$\begin{aligned}\tilde{A}_\mu(\tilde{x}) &= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta-1}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\left(\frac{1}{2}\partial_\mu \xi^\nu(\tilde{x}) + \frac{1}{2}\partial_\mu \xi^\nu(\tilde{x})\right) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2) \\ &= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta-1}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\left(\frac{1}{2}\partial_\mu \xi^\nu(\tilde{x}) - \frac{1}{2}\partial_\nu \xi^\mu(\tilde{x}) + \delta^\nu_\mu \partial_\sigma \xi^\sigma(\tilde{x})\frac{1}{D}\right) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2) \\ &= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\underbrace{\left(\frac{1}{2}\partial_\mu \xi^\nu(\tilde{x}) - \frac{1}{2}\partial^\nu \xi_\mu(\tilde{x})\right)}_{M_\mu{}^\nu} - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2).\end{aligned}$$

From this transformed gauge field, we calculate the transformation of gauge field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ to $\tilde{F}_{\mu\nu}$. We start by writting the transformation law of the derivatives used to construct $F_{\mu\nu}$. The chain rule yields

$$\tilde{\partial}_\mu \equiv \frac{\partial}{\partial \tilde{x}^\mu} = \left(\frac{\partial f^{-1}(\tilde{x})^\nu}{\partial \tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} = \left(\frac{\partial \tilde{x}^\nu - \xi^\nu(\tilde{x})}{\partial \tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} = \left(-\frac{\partial \xi^\nu(\tilde{x})}{\partial \tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} + \left(\frac{\partial}{\partial x^\mu}\right)_{\tilde{x}} \equiv -\partial_\mu \xi^\nu(\tilde{x})\partial_\nu + \partial_\mu.$$

where the subscripts indicate that a partial derivative with respect to x^μ should be precomposed with $x = f^{-1}(x^\mu)$ to yield a function dependent on the left-hand side variable $\tilde{x}$. Now we can calculate the transformed field strength at first order in ξ to be

$$\begin{aligned}\tilde{F}_{\mu\nu} &= \tilde{\partial}_\mu \tilde{A}_\nu - (\mu \leftrightarrow \nu) \\ &= \left(-\partial_\mu \xi^\rho(\tilde{x})\partial_\rho + \partial_\mu\right)\left(A_\nu(\tilde{x}) - A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\lambda(\tilde{x})M_\nu{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x})\right) - (\mu \leftrightarrow \nu) \\ &= \partial_\mu A_\nu(\tilde{x}) - (\partial_\mu \xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x})\right) - \partial_\mu\left(A_\lambda(\tilde{x})M_\nu{}^\lambda\right) - \partial_\mu\left(\xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x})\right) - (\mu \leftrightarrow \nu) \\ &= \partial_\mu A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x})\right) - \partial_\mu A_\lambda(\tilde{x})\partial_\nu \xi^\lambda(\tilde{x}) - A_\lambda(\tilde{x})\partial_\mu\partial_\nu \xi^\lambda(\tilde{x}) - \xi^\lambda(\tilde{x})\partial_\lambda\partial_\mu A_\nu(\tilde{x}) - 2(\partial_\mu \xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - (\mu \leftrightarrow \nu) \\ &= \partial_\mu A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x})\right) - (\partial_\mu A_\lambda(\tilde{x}))M_\nu{}^\lambda - A_\lambda(\tilde{x})\partial_\mu M_\nu{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda\partial_\mu A_\nu(\tilde{x}) - 2(\partial_\mu \xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - (\mu \leftrightarrow \nu) \\ &= F_{\mu\nu}(\tilde{x}) - F_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x}) - A_{(\nu}(\tilde{x})\frac{\Delta}{D}\partial_{\mu)}\partial_\lambda \xi^\lambda(\tilde{x}) - (\partial_{(\mu}A_\lambda(\tilde{x}))M_{\nu)}{}^\lambda - A_\lambda(\tilde{x})\partial_{(\mu}M_{\nu)}{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda F_{\mu\nu}(\tilde{x}) - 2(\partial_{(\mu}\xi^\lambda(\tilde{x}))\partial_\lambda A_{\nu)}(\tilde{x}) \\ &= F_{\mu\nu}(\tilde{x}) - F_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x}) - A_{(\nu}(\tilde{x})\frac{\Delta}{D}\partial_{\mu)}\partial_\lambda \xi^\lambda(\tilde{x}) - (\partial_{(\mu}A_\lambda(\tilde{x}))M_{\nu)}{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda F_{\mu\nu}(\tilde{x}) - 2(\partial_{(\mu}\xi^\lambda(\tilde{x}))\partial_\lambda A_{\nu)}(\tilde{x})\end{aligned}$$

where we simplified further by expliciting

$$2\partial_{(\mu}M_{\nu)}{}^\lambda = \partial_\mu\partial_\nu \xi^\lambda(\tilde{x}) - \partial_\mu\partial^\lambda \xi_\nu(\tilde{x}) - \partial_\nu\partial_\mu \xi^\lambda(\tilde{x}) - \partial_\nu\partial^\lambda \xi_\mu(\tilde{x}) = 0.$$

We note that the transformation law of $F_{\mu\nu}$ involves A_μ homogeneously which is an example of mixing of CFT fields under the transformation of a descendant.

(b) For a D-dimensionnal spacetime, the Maxwell action reads

$$S = \int \mathrm{d}^D x\sqrt{|g|}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \int \mathrm{d}^D x\sqrt{|g|}\; g^{\mu\sigma}g^{\nu\rho}\frac{1}{4}F_{\mu\nu}F_{\sigma\rho}.$$

where g is the metric (which we suppose conformally flat). We aim to apply the results found in (a) to determine when this action gains conformal symmetry. Under a conformal transformation given by the killing vector $\xi^\mu(x)$ and the scaling $\Omega(x) = 1 + \partial_\mu \xi^\mu(x)/D + O(\xi^2)$ of the metric components, we have

$$g_{\nu\rho}(x) = \Omega(f(x))^{-2}\tilde{g}_{\nu\rho}(f(x)) = \Omega(\tilde{x})^{-2}\tilde{g}_{\nu\rho}(\tilde{x}) \quad \text{Defining property of a conformal transformation}$$

$$|g|(x) = \Omega(f(x))^{-2D}|\tilde{g}|(f(x)), \quad g^{\nu\rho}(x) = \Omega(f(x))^{+2}\tilde{g}^{\nu\rho}(f(x)) = \Omega(\tilde{x})^2\tilde{g}^{\nu\rho}(\tilde{x}), \quad \mathrm{d}^D x\sqrt{|g|} = \mathrm{d}^D\tilde{x}\;\Omega(\tilde{x})^{-D}\sqrt{|\tilde{g}|(\tilde{x})}$$

Without loss of generality, we take the target metric $\tilde{g}$ to be the Minkowski metric. Inverting the result found in (a) for the transformation of the gauge field, we write

$$\begin{aligned}A_\mu(x) = |\partial x/\partial \tilde{x}|_{\tilde{x}}^{-\Delta/D}(R^{-1})^\nu_\mu \tilde{A}_\nu(\tilde{x}) &= \tilde{A}_\mu(\tilde{x}) + \tilde{A}_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - \tilde{A}_\mu(\tilde{x})\partial_\sigma \xi^\sigma(\tilde{x})\frac{1}{D} + \tilde{A}_\nu(\tilde{x})\partial_\mu \xi^\nu(\tilde{x}) + O(\xi^2) \\ &= \tilde{A}_\mu(\tilde{x}) + \tilde{A}_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) + \frac{1}{2}\tilde{A}_\nu(\tilde{x})\left(\partial_\mu \xi^\nu(\tilde{x}) - \partial^\nu \xi_\mu(\tilde{x})\right) + O(\xi^2).\end{aligned}$$

Then, with the derivative $(\partial_\mu)_{\tilde{x}} = \tilde{\partial}_\mu \xi^\nu(\tilde{x})\tilde{\partial}_\nu + \tilde{\partial}_\mu$, the field strength transforms as

$$\begin{aligned}F_{\mu\nu} = \partial_\mu A_\nu(x) - (\mu \leftrightarrow \nu) &= \left(\tilde{\partial}_\mu \xi^\lambda(\tilde{x})\tilde{\partial}_\lambda + \tilde{\partial}_\mu\right)\left(\tilde{A}_\nu(\tilde{x}) + \tilde{A}_\nu(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_\lambda(\tilde{x})M_\nu{}^\lambda\right) - (\mu \leftrightarrow \nu) \\ &= \tilde{\partial}_\mu\left(\tilde{A}_\nu(\tilde{x}) + \tilde{A}_\nu(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_\lambda(\tilde{x})M_\nu{}^\lambda\right) + \tilde{\partial}_\mu \xi^\lambda(\tilde{x})\tilde{\partial}_\lambda \tilde{A}_\nu(\tilde{x}) - (\mu \leftrightarrow \nu) \\ &= \tilde{F}_{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_{(\nu}(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_{\mu)}\tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{\partial}_{(\mu}(\tilde{A}_\lambda(\tilde{x}))M_{\nu)}{}^\lambda + \tilde{\partial}_{(\mu}\xi^\lambda(\tilde{x})\tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x})\end{aligned}$$

The contravariant equivalent of this result is given by

$$F^{\mu\nu} = g^{\mu\sigma} g^{\nu\rho} F_{\sigma\rho} = \Omega(\tilde{x})^4 \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} F_{\sigma\rho}$$
$$= \Omega(\tilde{x})^4 \left(\tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}^{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} (\tilde{A}_\lambda(\tilde{x})) M_{\rho)}{}^\lambda + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\rho)}(\tilde{x}) \right)$$

Next, we calculate

$$F_{\mu\nu} F^{\mu\nu} = \left(\tilde{F}_{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + O(\xi^2) \right)$$
$$\times \Omega(\tilde{x})^4 \left(\tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}^{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} (\tilde{A}_\lambda(\tilde{x})) M_{\rho)}{}^\lambda + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\rho)}(\tilde{x}) \right)$$
$$= \Omega(\tilde{x})^4 \left(\tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) \frac{2\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) \right)$$
$$= \Omega(\tilde{x})^4 \left(\tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) \frac{2\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + 4\tilde{F}^{\mu\nu} \tilde{A}_\nu(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\mu \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) \right)$$

I realized this calculation only applies a passive transformation to the action and should not not change its value without necessarly corresponding to a symmetry. I would have to redo this calculation by applying an active conformal transformation to each element of the action.

2 Axial anomaly

(a) We consider a $D = 2$-dimensionnal fermion field ψ with vector current $j^V_\mu = \overline{\psi}\gamma_\mu\psi$ where γ_μ are matrices forming a 2-dimensionnal clifford algebra. We are interested in the 2-point correlator of the vector current $\langle j^V_\mu(x_1) j^V_\nu(x_2)\rangle$. By translational symmetry, the 2-point function is forced to be a function of the relative coordinates $x = (x_1 - x_2)/2$. Translating by $-X_{12} = -(x_1 + x_2)/2$, we can bring the midpoint of the x_1, x_2 segment to the origin without changing the value of the 2-point function. Explicitly, we have $\langle j^V_\mu(x_1) j^V_\nu(x_2)\rangle = \langle j^V_\mu(x) j^V_\nu(-x)\rangle$. This property allows us to expand the 2-point function with a Fourier transform with respect to x as

$$\begin{aligned} F[\langle j^V_\mu(x_1) j^V_\nu(x_2)\rangle](q) = \frac{1}{(2\pi)^4}\int \mathrm{d}^2x e^{-iq\cdot x}\langle j^V_\mu(x) j^V_\nu(-x)\rangle &= \frac{1}{(2\pi)^4}\int \mathrm{d}^2x e^{-iq\cdot x}\langle \int \mathrm{d}^2k\ e^{+ik\cdot x} j^V_\mu(k)\int \mathrm{d}^2p\ e^{-ip\cdot x} j^V_\nu(p)\rangle \\ &= \frac{1}{(2\pi)^2}\langle \int \mathrm{d}^2k\ \mathrm{d}^2p\ \delta(-q+k-p) j^V_\mu(k)\ j^V_\nu(p)\rangle \\ &= \frac{1}{(2\pi)^2}\int \mathrm{d}^2p\ \langle\ j^V_\mu(q-p)\ j^V_\nu(p)\rangle \end{aligned}$$

where the fourier decomposition $j^V_\rho(x_i) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2p\ e^{ip\cdot x_i} j^V_\rho(p)$ (this definition does not have the proper normalization and leads to divergences. In (c), we use an alternate definition of the Fourier modes of the currents through correlators) of the vector current was used. In what follows, we focus on the Fourier space 2-point functions $\langle\ j^V_\mu(-p)\ j^V_\nu(p)\rangle$ contribution to the $q = 0$ Fourier component of the spacetime 2-point function. Lorentz invariance requires that $\langle\ j^V_\mu(q-p)\ j^V_\nu(p)\rangle$ is a sum of tensors (it can be extracted from a Fourier transform linearly combining tensor so it is a tensor). Furthermore, it only depends on components p_μ of p. The only tensors with two indices built can be constructed by combining the Minkowski metric $\eta_{\mu\nu}$, the components p_μ, the norm p^2 and the matrices γ^μ (we only need to include a term $\gamma_\mu\gamma_n u$ since the anticommutator $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ relates it to $\gamma_\nu\gamma_\mu$). The most general form for the Fourier space 2-point function consistent with Lorentz invariance reads

$$\langle\ j^V_\mu(-p)\ j^V_\nu(p)\rangle = F_1(p^2)\varepsilon_{\mu\nu} + F_2(p^2)\eta_{\mu\nu} + F_3(p^2)p_\mu p_\nu + F_4(p^2)\gamma_\mu\gamma_\nu + F_5(p^2)\gamma_\mu p_\nu + F_6(p^2)\gamma_\nu p_\mu$$

where the functions $F_i : \mathbb{R} \to \mathbb{C}$ provide full generality and $\varepsilon_{\mu\nu}$ is the 2-dimensionnal Levi-Civita tensor. Since the current operator follows a Bose statistic (they each contain an even number of fermion operators), we can exchange them without changing the value of the 2-point function. This property can be expressed as

$$\langle\ j^V_\mu(-p)\ j^V_\nu(p)\rangle = \langle\ j^V_\nu(p)\ j^V_\mu(-p)\rangle = -F_1(p^2)\varepsilon_{\mu\nu} + F_2(p^2)\eta_{\mu\nu} + F_3(p^2)(-p_\nu)(-p_\mu) + F_4(p^2)\gamma_\nu\gamma_\mu - F_5(p^2)\gamma_\nu p_\mu - F_6(p^2)\gamma_\mu p_\nu.$$

Subtracting this exchanged expression from the initial expression, we get the constraint

$$\begin{aligned} &0 = 2F_1(p^2)\varepsilon_{\mu\nu} + F_4(p^2)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) + (F_6(p^2) + F_5(p^2))\gamma_\nu p_\mu + (F_6(p^2) + F_5(p^2))\gamma_\mu p_\nu, \forall p \\ &\implies F_4(p^2) = F_1(p^2) = 0, \quad F_6(p^2) = -F_5(p^2) \end{aligned}$$

The current we are interested in is conserved as a result of the global symmetry $\psi \to e^{i\theta}\psi$, $\theta \in \mathbb{R}$. We note that applying the infinitesimal version of this symmetry transformation to the current leads to a vanishing variation $\delta j^\nu(x) = 0$. The ward identity corresponding to this symmetry reads

$$\langle \partial^\mu j_\mu(x_1) j_\nu(x_2)\rangle = \delta(x_1 - x_2)\langle \delta j_\nu(x_2)\rangle = 0.$$

We then compute the Fourier transformation with respect to x_1, x_2 to get

$$\begin{aligned} 0 = F[\langle \partial^\mu j^V_\mu(x_1) j^V_\nu(x_2)\rangle](q,p) &= \int \mathrm{d}^2x_1 e^{-iq\cdot x_1}\int \mathrm{d}^2x_2 e^{-ip\cdot x_2}\langle \partial^\mu j^V_\mu(x_1) j^V_\nu(x_2)\rangle \\ &= \langle (+iq^\mu)\int \mathrm{d}^2x_1 e^{-iq\cdot x_1} j^V_\mu(x_1)\int \mathrm{d}^2x_2 e^{-ip\cdot x_2} j^V_\nu(x_2)\rangle \quad \text{with integration by parts} \\ &= iq^\mu\langle j^V_\mu(q) j^V_\nu(p)\rangle \end{aligned}$$

At $q = -p$, we find $p^\mu\langle j^V_\mu(-p)\ j^V_\nu(p)\rangle = 0$ wich implies

$$\begin{aligned} 0 = p^\mu\langle\ j^V_\mu(-p)\ j^V_\nu(p)\rangle &= p^\mu\left(F_2(p^2)\eta_{\mu\nu} + F_3(p^2)p_\mu p_\nu\right) + F_5(p^2)p^\mu\left(\gamma_\mu p_\nu - \gamma_\nu p_\mu\right) \\ &= \left(F_2(p^2) + F_3(p^2)p^2\right)p_\nu + F_5(p^2)\left(p^\mu\gamma_\mu p_\nu - \gamma_\nu p^2\right), \forall p \\ &\implies F_2(p^2) = -F_3(p^2)p^2, \quad F_5(p^2) = 0. \end{aligned}$$

More explicitly, starting with the constraint $F_5(p^2)(p^\mu\gamma_\mu p_\nu - \gamma_\nu p^2) = 0$, we can have either $F_5(p^2) = 0$ or $p^\mu\gamma_\mu p_\nu - \gamma_\nu p^2$. The latter case is associated with $0 = \gamma_0 + \gamma_1$ for $p^\mu = (1,1)$ showing it can't hold for all p and forcing $F_5(p^2)$. The updated expression for the 2-momentum current correlator is

$$\langle j^V_\mu(-p)\ j^V_\nu(p)\rangle = F_3(p^2)\left(-p^2\eta_{\mu\nu} + p_\mu p_\nu\right).$$

The form of $F_3(p^2)$ can be made more precise by imposing scale invariance. To use scale invariance, we first relate scale invariance in position space to scale invariance in momentum space. Since the conserved current operators have no anomalous scaling dimension (leaving the scaling dimension $\Delta = D = 2$), we have

$$\begin{aligned}\langle j_\mu(x_1) j_\nu(x_2)\rangle = \lambda^{2+2}\langle j_\mu(x_1\lambda) j_\nu(x_2\lambda)\rangle \implies \langle j_\mu^V(-p)\, j_\nu^V(p)\rangle &= \lambda^{2+2} F[\langle \partial^\mu j_\mu^V(\lambda x_1) j_\nu^V(\lambda x_2)\rangle](-p,p)\\ &= \lambda^4 \langle \int \mathrm{d}^2 x_1 e^{-iq\cdot x_1} j_\mu^V(\lambda x_1) \int \mathrm{d}^2 x_2 e^{-ip\cdot x_2} j_\nu^V(\lambda x_2)\rangle\\ &= \lambda^4 \langle \frac{1}{\lambda^2} \int \mathrm{d}^2 x_1 e^{-iq\cdot x_1/\lambda} j_\mu^V(x_1) \frac{1}{\lambda^2} \int \mathrm{d}^2 x_2 e^{-ip\cdot x_2/\lambda} j_\nu^V(x_2)\rangle\\ &= \langle j_\mu^V(-p/\lambda)\, j_\nu^V(p/\lambda)\rangle.\end{aligned}$$

Applying this constraint to the 2-momentum curent correlator yields

$$F_3(p^2)\left(-p^2\eta_{\mu\nu} + p_\mu p_\nu\right) = \frac{1}{\lambda^4} F_3(p^2/\lambda^2)\left(-p^2\eta_{\mu\nu} + p_\mu p_\nu\right) \implies F_3(p^2) = \frac{1}{\lambda^2} F_3(p^2/\lambda^2).$$

The previous implication suggests the power-law ansatz $F_3 = ap^b$ depending on constants a, b. Substituting this ansatz, we get $ap^b = ap^b\lambda^{-b-2} \implies b = -2$. The final form for the correlator is

$$\langle j_\mu^V(-p)\, j_\nu^V(p)\rangle = \frac{a}{p^2}\left(-p^2\eta_{\mu\nu} + p_\mu p_\nu\right).$$

(b) We consider the axial current defined by $j_\mu^A = \varepsilon_{\mu\nu} j^{V,\nu}$. In $D = 2$, the non zero components of the levi-civita tensor are $\varepsilon_{01} = -\varepsilon_{10} = 1$. To determine if the axial current is classically conserved, we calculate its divergence as follows

$$\begin{aligned}\partial^\mu j_\mu^A = \partial^\mu \varepsilon_{\mu\nu} j^{V,\nu} &= \partial^\mu(\bar\psi \varepsilon_{\mu\nu}\gamma^\nu \psi) = \partial^\mu(\bar\psi \eta_{\mu\sigma}\gamma^1\gamma^0\gamma^\sigma\psi) = \partial_\mu(\bar\psi\gamma^1\gamma^0\gamma^\mu\psi)\\ &= (\partial_\mu\bar\psi)\gamma^1\gamma^0\gamma^\mu\psi + \bar\psi\gamma^1\gamma^0\gamma^\mu(\partial_\mu\psi) = 0\end{aligned}$$

where we used the property $\varepsilon_{\mu\nu}\gamma^\nu = \eta_{\mu\sigma}\gamma^1\gamma^0\gamma^\sigma$, the classical equation of motion for a free massless fermion field $0 = \gamma^1\gamma^0\gamma^\mu\partial_\mu\psi$ and its conjugate $0 = \partial_\mu\psi^\dagger\gamma^0\gamma^1\gamma^0\gamma^\mu\gamma^0 = \partial_\mu\bar\psi\gamma^1\gamma^0\gamma^\mu\gamma^0$. We explicitly check

$$\begin{aligned}\gamma^1 = \varepsilon_{01}\gamma^1 &= \eta_{00}\gamma^1\gamma^0\gamma^0 = (\eta_{00}\eta^{00})\gamma^1 = \gamma^1,\\ -\gamma^0 = \varepsilon_{10}\gamma^0 &= \eta_{11}\gamma^1\gamma^0\gamma^1 = -(\eta_{11}\eta^{11})\gamma^0 = -\gamma^0.\end{aligned}$$

We conclude that the classical equations of motion imply both the classical conservation of the vector current and the classical conservation of the axial current.

(c) To test if the classical conservation of the axial current survives quantum effects, we calculate

$$\begin{aligned}p^\mu\langle j_\mu^A(-p) j_\nu^V(p)\rangle = \eta^{\sigma\rho}\varepsilon_{\mu\sigma}p^\mu\langle j_\rho^V(-p) j_\nu^V(p)\rangle &= \eta^{\sigma\rho}\varepsilon_{\mu\sigma}p^\mu\frac{a}{p^2}\left(-p^2\eta_{\rho\nu} + p_\rho p_\nu\right)\\ &= \varepsilon_{\mu\sigma}\frac{a}{p^2}\left(-p^2 p^\mu\delta_\nu^\sigma + p^\mu p^\sigma p_\nu\right)\\ &= \frac{a}{p^2}(-p^2\varepsilon_{\mu\nu}p^\mu + \underbrace{\varepsilon_{\mu\sigma}p^\mu p^\sigma}_{p^0p^1 - p^1p^0 = 0} p_\nu) = -a\varepsilon_{\mu\nu}p^\mu\end{aligned}$$

and Fourier transform the result to get (An alternate definition of the current Fourier modes is used here)

$$\begin{aligned}&p_1^\mu\langle j_\mu^A(p_1) j_\nu^V(p_2)\rangle\delta(p_1+p_2) = i\partial_1^\mu\langle \int \mathrm{d}x_1\; e^{-ip_1\cdot x_1} j_\mu^A(x_1) \int \mathrm{d}x_2\; e^{-ip_2\cdot x_2} j_\nu^V(x_2)\rangle\\ &\implies \frac{1}{(2\pi)^2}\int \mathrm{d}p_2\; e^{ip_2\cdot x} p_1^\mu\langle j_\mu^A(p_1) j_\nu^V(p_2)\rangle\delta(p_1+p_2) = \frac{i}{(2\pi)^2}\partial_1^\mu\langle \int \mathrm{d}x_1\; e^{-ip_1\cdot x_1} j_\mu^A(x_1)\int \mathrm{d}x_2\left(\int \mathrm{d}p_2\; e^{ip_2\cdot x}e^{-ip_2\cdot x_2}\right) j_\nu^V(x_2)\rangle\\ &\implies e^{-ip_1\cdot x} p_1^\mu\langle j_\mu^A(p_1) j_\nu^V(-p_1)\rangle = i\partial_1^\mu\langle \int \mathrm{d}x_1\; e^{-ip_1\cdot x_1} j_\mu^A(x_1) j_\nu^V(x)\rangle, \quad \longrightarrow \text{change of variables } p_1 = -p\\ &\implies -\frac{1}{(2\pi)^2}\int \mathrm{d}p\; e^{ip\cdot x} p^\mu\langle j_\mu^A(-p) j_\nu^V(p)\rangle = \frac{i}{(2\pi)^2}\partial_1^\mu\langle \int \mathrm{d}p\int \mathrm{d}x_1\; e^{+ip\cdot x_1} j_\mu^A(x_1) j_\nu^V(x)\rangle = i\partial_1^\mu\langle j_\mu^A(0) j_\nu^V(x)\rangle\end{aligned}$$

and

$$-\frac{1}{(2\pi)^2}\int \mathrm{d}p\; e^{ip\cdot x} a\varepsilon_{\mu\nu}p^\mu = a\; i\; \varepsilon_{\mu\nu}\partial_1^\mu\delta(x) \implies \langle \partial_1^\mu j_\mu^A(0) j_\nu^V(x)\rangle = a\;\varepsilon_{\mu\nu}\partial_1^\mu\delta(x)$$

which shows that the divergence of the axial current has a non-vanishing correlator (can't be explained by contact terms since they would vanish through $\delta j_\nu(x) = 0$) with the vector current contradicting the Ward identity for the axial current.

(d) When a background gauge field A^μ is added, the axial anomaly affects the expectation value of the axial current divergence directly. This expectation value is evaluated at first non-trivial order

$$\begin{aligned}\left\langle \partial_\mu j^{A,\mu}(0)\right\rangle_{A^\mu} \equiv \left\langle \partial_\mu j^{A,\mu}(0) e^{\int dx\, A^\nu(x) j_\nu^V(x)}\right\rangle &\approx \left\langle \partial_\mu j^{A,\mu}(0) 1\right\rangle + \left\langle \partial_\mu j^{A,\mu}(0)\int dx\, A^\nu(x) j_\nu^V(x)\right\rangle\\ &\approx \int dx\, A^\nu(x)\left\langle \partial_\mu j^{A,\mu}(0) j_\nu^V(x)\right\rangle = a\int dx\, A^\nu(x)\;\varepsilon_{\mu\nu}\partial^\mu\delta(x)\\ &\approx -a\int dx\;\partial^\mu\left(\varepsilon_{\mu\nu}A^\nu(x)\delta(x)\right) + a\int dx\;\left(\delta(x)\varepsilon_{\mu\nu}\partial^\mu A^\nu(x)\right) = a\varepsilon_{\mu\nu}\partial^\mu A^\nu(0)\end{aligned}$$

where we supposed $\delta(x \neq 0) = 0$ to remove the boundary term in the integration by parts.

3 OPE coefficients from three-point functions

(a) Given scalar CFT operators O_{Δ_1} and O_{Δ_2} with respective scaling dimensions Δ_1 and Δ_2, there is only one two-point function consistent with conformal symmetry given by

$$\left\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\right\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{|x_{12}|^{2\Delta_1}}$$

where $x_{ij} = x_i - x_j$. Adding a third operator O_{Δ_3} with scaling dimension Δ_3, the three-point function has residual freedom in the coefficients C_{123} depending on the operators involved. The most general expression reads

$$\left\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\right\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}.$$

Assumming[1] that the operator algebra of the CFT is closed (multiplying operators produces an algebra operator which is in the same vector space) and has the primary and their descendants as a vector basis, we have that any product of CFT operators can be decomposed as a sum of primary and descendants operators. Explicitly we have the general operator product expansion

$$\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) = \sum_{\Delta'}\sum_{l} \alpha^{\Delta',l}_{\Delta_1,\Delta_2}(x_1,x_2)\partial_l O_{\Delta'}(x_2)$$

where l is a string representing the sequence of components of x_2[2] with respect to which we differentiate to reach all descendants, α represents linear combinations of CFT operators with full generality. We sum over Δ' to include all conformal towers and we sum over l to reach all the members of a given tower. We can make the dependence on x_1 and x_2 more precise by imposing the transformation properties of each operator. We first use the translation symmetry. Having fixed the expansion point x_2 (it will not be affected by a translation of the operators involved in the product), the uniqueness of the expansion imposes that we find the same coefficients if we translate x_1 and x_2 by the same amount a (this gives us back the same operator because it consists in a *representation* of the translation operation on the operator product). We conclude that translationnal symmetry forces $\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(x_1,x_2) = \alpha^{\Delta',l}_{\Delta_1,\Delta_2}(x_1+a,x_2+a) = \alpha^{\Delta',l}_{\Delta_1,\Delta_2}(x_{12})$. Then, for a scale transformation with scale factor λ, we have

$$\begin{aligned}
\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) = \lambda^{\Delta_1+\Delta_2}\mathcal{O}_{\Delta_1}(\lambda x_1)\mathcal{O}_{\Delta_2}(\lambda x_2) &\implies \sum_{\Delta'}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(x_{12})\partial_l O_{\Delta'}(x_2) = \sum_{\Delta'}\sum_{l}\lambda^{\Delta_1+\Delta_2}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(\lambda x_{12})\partial_l O_{\Delta'}(\lambda x_2)\\
&\implies \sum_{\Delta'}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(x_{12})\partial_l O_{\Delta'}(x_2) = \sum_{\Delta'}\sum_{l}\lambda^{\Delta_1+\Delta_2-\Delta'}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(\lambda x_{12})\lambda^{-|l|}\partial_l O_{\Delta'}(x_2)\\
&\implies \alpha^{\Delta',l}_{\Delta_1,\Delta_2}(x_{12}) = \lambda^{\Delta_1+\Delta_2-\Delta'-|l|}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(\lambda x_{12})
\end{aligned}$$

where we used the uniqueness of the expansion to provide a point x_2 and denoted $|l|$ the number of elements in l. To apply the scale transformation to the descendant operators we used

$$\frac{\partial}{\partial x^l}O_{\Delta'}(x)\bigg|_{x=\lambda x_2} = \frac{\partial x/\lambda}{\partial x^l}\frac{\partial}{\partial y^l}O_{\Delta'}(\lambda y)\bigg|_{y=x/\lambda, y=x_2} = \lambda^{|l|}\lambda^{-\Delta'}\frac{\partial}{\partial y^l}O_{\Delta'}(y)\bigg|_{y=x_2} = \lambda^{-|l|}\lambda^{-\Delta'}\frac{\partial}{\partial y^l}\partial_l O_{\Delta'}(x_2).$$

[1]This assumption can be seen to hold with the state-operator correspondence of the radial quantization of the CFT. This state operator correspondence associates each CFT operator (primary and descendants) at a point x to an eigenstate of the dilation operator. Using these states, we can write a resolution of identity allowing us to decompose any other state as a sum of states corresponding to primary and descendant CFT operators. This decomposition can then be brought back to operators using the state operator correspondence in the reverse direction.

[2]This point is the point at which we take our asymptotic radial quantized states for the usage of the state operator correspondance

We can also extract constraints using the rotational invariance of the product. Supposing the operators in the product are scalar primaries, we can write $\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) = \mathcal{O}_{\Delta_1}(R(x_1))\mathcal{O}_{\Delta_2}(R(x_2))$ where R is a rotation in $\mathbb{R}^D$. Substituting this relation in the expansion yields

$$\begin{aligned}
\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) = \mathcal{O}_{\Delta_1}(R(x_1))\mathcal{O}_{\Delta_2}(R(x_2)) &\implies \sum_{\Delta'}\sum_{l}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(x_{12})\partial_l O_{\Delta'}(x_2) = \sum_{\Delta'}\sum_{l}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(R(x_{12}))\partial_l O'_{\Delta}(R(x_2)) \\
&\implies \sum_{\Delta'}\sum_{l}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(x_{12})\partial_l O'_{\Delta}(x_2) = \sum_{\Delta'}\sum_{l}\sum_{l'}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(R(x_{12}))L(R^{-1})^{l'}_{l}\,\partial_{l'} O_{\Delta'}(x_2) \\
&\implies \sum_{\Delta'}\sum_{l'}\alpha^{\Delta',l'}_{\Delta_1,\Delta_2}(x_{12})\partial_{l'} O_{\Delta'}(x_2) = \sum_{\Delta'}\sum_{l'}\left(\sum_{l}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(R(x_{12}))L(R^{-1})^{l'}_{l}\right)\partial_{l'} O_{\Delta'}(x_2) \\
&\implies \alpha^{\Delta',l'}_{\Delta_1,\Delta_2}(x_{12}) = \sum_{l}\alpha^{\Delta',l}_{\Delta_1,\Delta_2}(R(x_{12}))L(R^{-1})^{l'}_{l}
\end{aligned}$$

which shows that $\alpha^{\Delta',l}_{\Delta_1,\Delta_2}$ transforms like a $(0,l)$ tensor under rotation. Since the only translationnaly invariant tensor accessible is x_{12} we have to build $\alpha^{\Delta',l}_{\Delta_1,\Delta_2}$ as a product of the components $x_{12}^{l_i}$ for $l_i \in l$ with some power of the scalar $|x_{12}|$ to account for scaling. In other words, we build a tensor *density* representation from the tensor product of the spin 1 vector representations (the only tensor building block available). As a result of the rotation constraint, we have $\alpha^{\Delta',l}_{\Delta_1,\Delta_2} = \gamma^{\Delta',l}_{\Delta_1,\Delta_2}|x_{12}|^{\Delta}(x_{12})^l$ (where $\gamma^{\Delta',l}_{\Delta_1,\Delta_2}$ is now a set of x_{12} independant coefficients). We could include (but we do not for simplification) contractions of the derivatives with each other and contractions of the x_{12}^{μ} with each other which both consists in singlet contributions. Substituting this ansatz in the scaling constraint found above, we find

$$\beta^{\Delta',l}_{\Delta_1,\Delta_2}|x_{12}|^{\Delta}(x_{12})^l = \lambda^{\Delta_1+\Delta_2-\Delta'-|l|}\gamma^{\Delta',l}_{\Delta_1,\Delta_2}\lambda^{\Delta}|x_{12}|^{\Delta}\lambda^{|l|}(x_{12})^l \implies \Delta = -\Delta_1-\Delta_2+\Delta'$$

leading to the expansion

$$\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) = \sum_{\Delta'}\sum_{l}\gamma^{\Delta',l}_{\Delta_1,\Delta_2}|x_{12}|^{-\Delta_1-\Delta_2+\Delta'}(x_{12})^l\partial_l O_{\Delta'}(x_2)$$

with $l_i \geq 0,\ \forall l_i \in l$.

(b) We note that the multi-index l is associated with a contraction of indices and it is therefore running over all multiplets. If $\gamma^{\Delta',l}_{\Delta_1,\Delta_2}$ distinguished between the ordering of the multiplet up to identical elements, it would spoil the Lorentz invariance of the contraction $(x_{12})^l\partial_l O_{\Delta'}(x_2)$. Without loss of generality, we rewrite the expansion by fixing $\gamma^{\Delta',l}_{\Delta_1,\Delta_2} = \beta^{\Delta',|l|}_{\Delta_1,\Delta_2}$ where $\beta^{\Delta',|l|}_{\Delta_1,\Delta_2}$ is a set of numbers depending only on the number of indices $|l|$ of the multi-index l.

(c) From the OPE expansion derived in (a), we can evaluate three-point functions. We start by reducing the product $O_{\Delta_1}(x_1)O_{\Delta_2}(x_2)$ to a sum of operators. Then we use the linearity of the expectation value to break the correlator of this sum with $O_{\Delta_3}(x_3)$ into a sum of two-point functions which we can evaluate. Explicitly, we have

$$\begin{aligned}
\left\langle\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\right\rangle &= \left\langle\sum_{\Delta'}\sum_{l}\beta^{\Delta',|l|}_{\Delta_1,\Delta_2}|x_{12}|^{-\Delta_1-\Delta_2+\Delta'}(x_{12})^l\partial_l O_{\Delta'}(x_2)\mathcal{O}_{\Delta_3}(x_3)\right\rangle \\
&= \sum_{\Delta'}\sum_{l}\beta^{\Delta',|l|}_{\Delta_1,\Delta_2}|x_{12}|^{-\Delta_1-\Delta_2+\Delta'}(x_{12})^l\partial_l\left\langle O_{\Delta'}(x_2)\mathcal{O}_{\Delta_3}(x_3)\right\rangle \\
&= \sum_{\Delta'}\delta_{\Delta',\Delta_3}\sum_{l}\beta^{\Delta',|l|}_{\Delta_1,\Delta_2}|x_{12}|^{-\Delta_1-\Delta_2+\Delta'}(x_{12})^l\partial_l\frac{1}{|x_{23}|^{2\Delta_3}} \\
&= \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}}\sum_{l}\beta^{\Delta_3,|l|}_{\Delta_1,\Delta_2}(x_{12})^l\frac{\partial}{\partial x_2^l}\frac{1}{|x_{23}|^{2\Delta_3}}, \quad \text{supposing only one primary operator for each } \Delta_3.
\end{aligned}$$

Comparing this result with the first expression given for the three-point function, we get

$$\begin{aligned}
&\frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}}\sum_{l}\beta^{\Delta_3,|l|}_{\Delta_1,\Delta_2}(x_{12})^l\partial_l\frac{1}{|x_{23}|^{2\Delta_3}}. \\
\iff &\frac{C_{123}}{|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} = \sum_{l}\beta^{\Delta_3,|l|}_{\Delta_1,\Delta_2}(x_{12})^l\partial_l\frac{1}{|x_{23}|^{2\Delta_3}}.
\end{aligned}$$

To bring the two sides of this relation to a form explicitly constraining the $\beta^{\Delta_3,|l|}_{\Delta_1,\Delta_2}$ for $|l| = 0,1,2$, we write $|x_{31}| = |x_{12} - x_{23}|$ and perform an expansion of the left-hand side in components of x_{12} to get

$$\frac{C_{123}}{|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} = \frac{C_{123}}{|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \sum_l \frac{1}{N_l M_l}(x_{12})^l \frac{\partial}{\partial x_{12}^l} \frac{1}{|x_{23}+x_{12}|^{\Delta_3+\Delta_1-\Delta_2}}\Bigg|_{x_{12}=0}$$

where N_l is the number of distinguishable permutations of the multi-index l (this factor accounts for the overcounting associated with the commutation of derivatives summed over all orderings of l) and $M_l = m_{l_1}!\cdots$ (m_{l_1} is the multiplicity of the index l_1 in l) is the Taylor series factor. For $l = \mu\nu$, $N_l = 2$ if $\mu \neq \nu$ ($N_l = 1$ if $\mu = \nu$) and $M_l = 1$ if $\mu \neq \nu$ ($M_l = 2$ if $\mu = \nu$) and the global factor is $1/(N_l M_l) = 1/2$ in all cases. Up to second order in x_{12}, we have

$$\begin{aligned}\frac{C_{123}}{|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} &\approx \frac{C_{123}}{|x_{23}|^{2\Delta_3}} + (x_{12})^\mu \frac{-(2/2)(\Delta_3+\Delta_1-\Delta_2)C_{123}(x_{23})_\mu}{|x_{23}|^{2\Delta_3+2}} \\ &- \frac{1}{2}(x_{12})^\mu(x_{12})^\nu(2/2)(\Delta_3+\Delta_1-\Delta_2)C_{123}\left(\frac{-(2/2)(\Delta_3+\Delta_1-\Delta_2+1)(x_{23})_\mu(x_{23})_\nu}{|x_{23}|^{2\Delta_3+4}} + \frac{\eta_{\mu\nu}}{|x_{23}|^{2\Delta_3+2}}\right) + O(x_{12}^l, |l|=3)\end{aligned}$$

On the OPE side, we expand to second order in x_{12} to get

$$\begin{aligned}\sum_l \beta^{\Delta_3,|l|}_{\Delta_1,\Delta_2}(x_{12})^l \partial_l \frac{1}{|x_{23}|^{2\Delta_3}} &\approx \beta^{\Delta_3,0}_{\Delta_1,\Delta_2}\frac{1}{|x_{23}|^{2\Delta_3}} - \beta^{\Delta_3,1}_{\Delta_1,\Delta_2}(x_{12})^\mu \frac{(4/2)\Delta_3(x_{23})_\mu}{|x_{23}|^{2\Delta_3+2}} \\ &- \beta^{\Delta_3,2}_{\Delta_1,\Delta_2}(x_{12})^\mu(x_{12})^\nu(4/2)\Delta_3\left(\frac{-(2/2)(2\Delta_3+2)(x_{23})_\nu(x_{23})_\mu}{|x_{23}|^{2\Delta_3+4}} + \frac{\eta_{\mu\nu}}{|x_{23}|^{2\Delta_3+2}}\right) \\ &- \beta^{\Delta_3,\star}_{\Delta_1,\Delta_2}(x_{12})^\mu(x_{12})^\nu\eta_{\mu\nu}(4/2)\Delta_3\left(\frac{-(2/2)(2\Delta_3+2)|x_{23}|^2}{|x_{23}|^{2\Delta_3+4}} + \frac{\eta^{\mu\nu}\eta_{\mu\nu}}{|x_{23}|^{2\Delta_3+2}}\right) + O(x_{12}^l, |l|=3)\end{aligned}$$

where we included the additional allowed singlet terms with free coefficient $\beta^{\Delta_3,\star}_{\Delta_1,\Delta_2}$ mentioned above to avoid a contradiction regarding the expression of $\beta^{\Delta_3,2}_{\Delta_1,\Delta_2}$. Comparing the coefficients, we have the relations

$$\begin{aligned}\beta^{\Delta_3,0}_{\Delta_1,\Delta_2} &= C_{123}, \\ \beta^{\Delta_3,1}_{\Delta_1,\Delta_2} &= C_{123}\frac{2(\Delta_3+\Delta_1-\Delta_2)}{2\Delta_3}, \\ \beta^{\Delta_3,2}_{\Delta_1,\Delta_2} &= C_{123}\frac{(\Delta_3+\Delta_1-\Delta_2)(\Delta_3+\Delta_1-\Delta_2+1)}{4\Delta_3(2\Delta_3+2)}.\end{aligned}$$

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