# Homework 2: How to build the spinor field from scratch

Due Oct 30th.

## 1 A strange dream

Feeling tired after juggling spinors, you fell asleep. In your dream, you traveled back to the good old scalar world and acquired a new profession.

Enter the portal.← I am a link.

## 2 Dirac field from scratch

By the end of this homework, you will discover the spinor field. Amazing property No. 1: it satisfies the renowned Dirac equation (We will introduce it again in the historical way in week 3). Amazing property No. 2: while building the field, you simultaneously (in one calculation) found out that the spinor field has spin $\frac{1}{2}$ and it leads to fermions! Spin statistics theorem any one?

## Previously

In general the spinor creation and annihilation field takes the form of

$$\psi_l^-(x) = \sum_s\!\!\!\!\!\!\int dV_{\boldsymbol{p}} u_l^s(\boldsymbol{p}) b_{\boldsymbol{p}}^s e^{-ip\cdot x}$$
$$\psi_l^+(x) = \sum_s\!\!\!\!\!\!\int dV_{\boldsymbol{p}} v_l^s(\boldsymbol{p}) (c_{\boldsymbol{p}}^s)^\dagger e^{+ip\cdot x}. \tag{1}$$

Here we assume there is a charge symmetry and $b^\dagger$ creates particle and $c^\dagger$ creates anti-particle.

And the coefficients (at 0 momentum[1]) satisfy

$$\sum_{s'} u_{l'}^{s'}(0) \boldsymbol{J}_{s's}^j(R) = \sum_l \boldsymbol{\mathcal{J}}_{l'l}(R) u_l^s(0)$$
$$\sum_{s'} v_{l'}^{s'}(0) \boldsymbol{J}_{s's}^{j*}(R) = -\sum_l \boldsymbol{\mathcal{J}}_{l'l}(R) v_l^s(0). \tag{2}$$

By using some group theory trick (It is at this step that we determined that the spinor field has spin $\frac{1}{2}$.) we come to the conclusion that

$$u_{m,\pm}^s(0) = c_\pm \delta_{ms}$$
$$v_{m,\pm}^s(0) = -id_\pm(\sigma_2)_{ms}, \tag{3}$$

where we choose our index $l$ to become index $m\pm$, where $\pm$ denotes upper or lower block and $m$ denotes within the size 2 block.

By demanding the spinor field transform properly under parity, we determined the coefficients up to some sign factors (sign factors are plus and minus signs).

$$u^{\frac{1}{2}}(0) = \begin{pmatrix} 1 \\ 0 \\ b_u \\ 0 \end{pmatrix}, u^{-\frac{1}{2}}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ b_u \end{pmatrix} \tag{4}$$

and

$$v^{\frac{1}{2}}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ b_v \end{pmatrix}, v^{-\frac{1}{2}}(0) = \begin{pmatrix} 1 \\ 0 \\ b_v \\ 0 \end{pmatrix}. \tag{5}$$

[1] This is sufficient because we can use boost to find $u_l^s(\boldsymbol{p})$ and $v_l^s(\boldsymbol{p})$.

where

$$\beta u^s(0) = b_u u^s(0)$$
$$\beta v^s(0) = b_v v^s(0). \tag{6}$$

The goal of this homework is to build the final spinor field out of the spinor creation and annihilation field using the causality condition and also determine these sign factors. Then we will show the spinor field satisfy the renowned Dirac equation.

## 3 The commutator/anti-commutator

In order to satisfy causality condition, we make linear combination of the fields (it is understood they all have four components.)

$$\psi(x) = \kappa\psi^-(x) + \lambda\psi^+(x), \tag{7}$$

note that the creation and anihilation fields are NOT adjoint of each other!

As in the scalar case, before knowing if we should compute the commutator or anti-commutator, we should compute both to be safe. Here we introduce spin sums to shorten your final result

$$N_{ll'}(\boldsymbol{p}) \equiv \sum_s u_l^s(\boldsymbol{p}) u_{l'}^{*s}(\boldsymbol{p})$$
$$M_{ll'}(\boldsymbol{p}) \equiv \sum_s v_l^s(\boldsymbol{p}) v_{l'}^{*s}(\boldsymbol{p}). \tag{8}$$

a) Show that the commutator/anti-commutator is

$$[\psi_l(x), \psi_{l'}^\dagger(y)]_\mp = \int dV_p (|\kappa|^2 N_{ll'}(\boldsymbol{p}) e^{-ip\cdot(x-y)} \mp |\lambda|^2 M_{ll'}(\boldsymbol{p}) e^{+ip\cdot(x-y)}). \tag{9}$$

b) Compute $N(0)$ using the $u$'s provided at (4). Note this is a matrix! Show that

$$N(0) = 1 + b_u\beta \tag{10}$$

where

$$\beta = \gamma^0 \equiv \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{11}$$

In the following few parts, we want to find the boosted spin sum $N(\boldsymbol{p})$ and hence the commutator.

Recall that we can boost the coefficients using the representation of the standard boost.

$$\begin{aligned} u^s_{l'}(\boldsymbol{p}) &= \sum_l D_{l'l}(L(p))u^s_l(0) \\ u^{*s}_{l'}(\boldsymbol{p}) &= \sum_l u^{*s}_l(0)D^\dagger_{ll'}(L(p)), \end{aligned} \tag{12}$$

where the second line is a complex conjugate of the first line.

c) Rewrite the boosts (12) as matrix equations.

d) What does $N(\boldsymbol{p})$ become now?

e) Now use

$$\beta(\gamma^\mu)^\dagger\beta = \gamma^\mu \tag{13}$$

to show

$$D(\Lambda)^\dagger = \beta D(\Lambda)^{-1}\beta. \tag{14}$$

*Hint: Expand* $D(\Lambda), D^\dagger(\Lambda)$ *and* $D^{-1}(\Lambda)$ *and compare. Note* $J^{\mu\nu} \equiv -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ *is not hermitian!*

f) Use the previous part to show

$$N(\boldsymbol{p}) = (D(L(p))\beta D^{-1}(L(p)) + b_u)\beta. \tag{15}$$

Now we are in business: use

$$\begin{aligned} D^{-1}(\Lambda)\gamma^\mu D(\Lambda) &= \Lambda^\mu{}_\rho\gamma^\rho \\ D(\Lambda)\gamma^\mu D^{-1}(\Lambda) &= \Lambda_\rho{}^\mu\gamma^\rho, \end{aligned} \tag{16}$$

we find

$$D(L(p))\beta D^{-1}(L(p)) = L_\rho{}^0(p)\gamma^\rho = \frac{p_\mu}{m}\gamma^\mu. \tag{17}$$

It is left as optional to show this is true, we make use the specific form of standard boost[2].

g) Now, finally show that the spin sums become

$$
\begin{aligned}
N(\boldsymbol{p}) &= \frac{1}{m}(p_\mu \gamma^\mu + b_u m)\beta \\
M(\boldsymbol{p}) &= \frac{1}{m}(p_\mu \gamma^\mu + b_v m)\beta.
\end{aligned} \tag{19}
$$

h) Now show the final form of the commutator/anti-commutator is

$$
[\psi_l(x), \psi^\dagger_{l'}(y)]_\mp = |\kappa|^2 (i\gamma^\mu \partial_\mu + b_u m)\beta D(x-y) \mp |\lambda|^2 (i\gamma^\mu \partial_\mu + b_v m)\beta D(y-x). \tag{20}
$$

where

$$
D(x-y) \equiv \int dV_p e^{-ip\cdot(x-y)}. \tag{21}
$$

Note that here the derivative is respect to the entire argument of $D(x-y)$. And recall from the scalar QFT course that $D(x-y)$ is even in the entire argument at spacelike separations.

# 4 Causality Condition

Now we have the commutator and anti-commutator, we can use causality condition to proceed. Recall that the causality condition requires either the commutator or the anti-commutator to be zero at space separation. Consider the terms involving a derivative $\partial_\mu$ and the terms that do not involve derivative separately.

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$$
\begin{aligned}
L^i{}_k(p) &= \delta^i{}_k + \left(\frac{p^0}{m} - 1\right)\frac{p^i p^k}{|\boldsymbol{p}|^2} \\
L^i{}_0(p) &= L^0{}_i(p) = \frac{p^i}{m} \\
L^0{}_0(p) &= \frac{p^0}{m}
\end{aligned} \tag{18}
$$

.

a) First consider the derivative terms. What kind of symmetry does the derivative of $D(x-y)$ have knowing that $D(x-y)$ itself is even at spacelike separations? What condition do we have on $|\kappa|^2$ and $|\lambda|^2$? Make a simple choice of them. Consider both the commutator and anti-commutator. Which should we pick? Is a spinor field quantized to bosons or fermions?

b) What is the condition indicated by the terms that do not involve a derivative? How is $b_u$ related to $b_v$?

c) Choose $b_u = 1$, write the final spinor field. Keep $u(\boldsymbol{p})$ and $v(\boldsymbol{p})$ in their form.

d) Use

$$D(L(p))\beta D^{-1}(L(p)) = \frac{p_\mu}{m}\gamma^\mu. \tag{22}$$

to show that

$$\begin{aligned} (p^\mu\gamma_\mu - m)u^s(\boldsymbol{p}) &= 0 \\ (p^\mu\gamma_\mu + m)v^s(\boldsymbol{p}) &= 0. \end{aligned} \tag{23}$$

e) Show that $u^s(\boldsymbol{p})e^{-ip\cdot x}$ and $v^s(\boldsymbol{p})e^{ip\cdot x}$ satisfy the renowned Dirac equation.

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0. \tag{24}$$

In other words, the spinor field $\psi(x)$ is the mode expansion of the solution of the Dirac equation.

# 5 Acknowledgement

Who did you collaborate with on this homework assignment?