Homework 2

Due on Wednesday, February 7

*Submit a **single file** (pdf or zip) online using the dropbox submission link for the appropriate deadline. Acknowledge any references you use as well as any other students with whom you collaborate.*

1 Dynamics on the tangent bundle

In this question you will explore how, in the case of regular Lagrangians, the tangent bundle to a manifold can be made into a symplectic manifold, allowing us to define dynamics in the same way as the cotangent bundle.

Let Q be a smooth n-manifold describing the possible positions of a system of particles. In this question we will work on the tangent bundle TQ and interpret its points $(\mathbf{q}, \mathbf{v})$ as configurations of the system, where $\mathbf{q} \in Q$ represents positions and $\mathbf{v} \in T_{\mathbf{q}}Q$ represents velocities.

We will use coordinates charts on TQ induced from the ones on Q, i.e., if $(q^1, \dots, q^n)$ are coordinates on Q then $(q^1, \dots, q^n, v^1, \dots, v^n)$ are coordinates on TQ with

$$v^i(\mathbf{q}, \mathbf{v}) = \mathrm{d}q^i_{\mathbf{q}}(\mathbf{v}). \tag{1}$$

Let $L : TQ \to \mathbb{R}$ be a smooth function that we will call the Lagrangian of the system. We will use it to construct the *Legendre transform*

$$\mathbf{F}L : (\mathbf{q}, \mathbf{v}) \in TQ \mapsto (\mathbf{q}, DL_{\mathbf{q}}(\mathbf{v})) \in T^*Q \tag{2}$$

where in coordinates we define

$$DL_{\mathbf{q}} : \mathbf{v} \in T_{\mathbf{q}}Q \mapsto \frac{\partial \hat{L}}{\partial v^i}(\hat{\mathbf{q}}, \hat{\mathbf{v}})\, \mathrm{d}q^i_{\mathbf{q}} \in T^*_{\mathbf{q}}Q. \tag{3}$$

We can use the Legendre transform to pull back the canonical forms from the cotangent bundle to the tangent bundle.

(a) Find the coordinate expression for the 1-form $\theta_L = \mathbf{F}L^*\theta$.

(b) Find the coordinate expression for *Lagrange 2-form* $\omega_L = \mathbf{F}L^*\omega$.

(c) Show that the Lagrange 2-form is symplectic if and only if the Langrangian is *regular*, which in coordinates means that at each point the determinant of

$$\left(\frac{\partial^2 \hat{L}}{\partial v^i\, \partial v^j} \right) \tag{4}$$

is non-zero.

In the following assume that the Lagrangian is regular.

(d) We can use the Legendre transform to define the *energy* function

$$E : (\mathbf{q}, \mathbf{v}) \in TQ \mapsto (DL_{\mathbf{q}}(\mathbf{v}))(\mathbf{v}) - L(\mathbf{q}, \mathbf{v}) \in \mathbb{R}. \tag{5}$$

Note that this is not the same as the Hamiltonian, since it lives on TQ. Find the coordinate expression of the energy function.

(e) We define the *Lagrangian vector field* X_E on TQ by

$$\omega_L(X_E, \cdot) = \mathrm{d}E \tag{6}$$

in analogy with Hamiltonian vector fields on T^*Q. Find the coordinate expression for X_E.

(f) Show that the integral curves of X_E satisfy in coordinates the Euler–Lagrange equations

$$\begin{cases} \dot{q}^i(t) = v^i(t) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \hat{L}}{\partial v^i}(q(t), v(t)) = \dfrac{\partial \hat{L}}{\partial q^i}(q(t), v(t)). \end{cases} \tag{7}$$