Thermodynamics of Modified Gravity : A Special State and Multi-Fluids
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Pierre-Antoine Graham
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Recently, the effective dissipative fluid formulation of scalar-tensor gravity allowed for formal identification of a temperature of gravity. As an order parameter, this temperature seems consistent with the interpretation of scalar-tensor gravity as a non-equilibrium excited state with respected to thermal equilibrium general relativity. A challenge for the validity of this temperature as a well-defined order parameter is studied here with a pathological state with undetermined temperature. This state is shown to be unstable and the origin of the indetermination is explained. Finally, challenges of the extension of the effective fluid picture to multi-fluids are discussed.

I. INTRODUCTION

Thermodynamics connects with gravity in unsuspected ways. A connection was first revealed in 1995 [1] when Jacobson used the laws of black hole thermodynamics to derive the Einstein equation as an equation of state. The connexion was then extended [2] to modified gravity with non-equilibrium considerations leading to $f(R)$ corrections. This suggests that general relativity can be seen as an equilibrium state with modified gravity theories as its excited states. Within Jacobson's theory, this idea could not be fully developed due to the lack of a clear order parameter quantifying the deviation from equilibrium [3]. The effective fluid formalism discussed here is another way to see modified gravity as a departure from the general relativistic equilibrium. It comes with a natural temperature order parameter.

A first step towards the formalism was done in 1940 [4], when Eckart proposed constitutive relations relating different components of the stress-energy tensor of a dissipative fluid with characteristics of the dissipative fluid four-velocity field. Then, in 1988 [5], Madsen wrote the stress-energy tensor of scalar-tensor gravity in the generic form of a dissipative fluid stress-energy tensor. Combining Eckart's constitutive relations with Madsen's decomposition [6] provided a way to identify an effective temperature along with other dissipative quantities such as bulk and shear viscosity [3]. The resulting first-order thermodynamics of scalar-tensor gravity was studied in the context of spatially homogeneous and isotropic cosmologies [7] where it revealed that expansion "cools" gravity towards 0 temperature general relativity while singularities "heat" it away from equilibrium.

Despite the clear success of the effective fluid picture, it was recently found [8] that temperature is not always well defined. An example of temperature indetermination was given by a family of pathological solutions from a cosmological model [9]. The model consists of a scalar field with non-minimal coupling to the Ricci scalar in a flat FLRW universe. This family of solutions is interesting because it provides an analytical example of super-acceleration of the expansion of the universe consistent with observational data [10].

An overview of the effective fluid formalism is give in sec. II. Then, the first half of the original content of the project, presented in sec.III, focuses on the stability analysis of the pathological solution solutions. The goal of this analysis was to show that the pathological states were unstable and make the nature of the temperature indetermination more precise. Section III A describes the cosmological model, then in sec. III B, homogeneous perturbation theory is applied to the pathological solutions leading to the results presented in sec. III C. The second half of the project, presented in sec. IV, is an ongoing attempt to describe multi-scalar-tensor gravity [11] with an effective multi-fluid formalism. As the formalism is at an early stage, this section mainly focuses on possible approaches to overcome its numerous challenges. The notation used in this report is that of [15]. The speed of light is set to 1, the gravitational coupling is given by $\kappa = 8\pi G$ and the metric signature is $-+++$.

II. FORMALISM

A. Scalar-Tensor Gravity

Some cosmological features of the universe can't be captured using Einstein's general theory of relativity. A clear example is the present acceleration of the expansion of the universe discovered in 1998 [12]. In the scope of general relativity, this phenomena can be modeled with an unsatisfactory *ad hoc* dark energy [6]. Another way to achieve consistency with clear cosmological constraints is to modify general relativity. The modification considered here is the theory of scalar-tensor gravity which is obtained by promoting the gravitation coupling *constant* G to a scalar *field* [13]. The action for general relativity is given by [14]

$$S_{\mathrm{GR}} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R + S^{(m)} \tag{2.1}$$

where R is the Ricci scalar, g the determinant of the metric tensor g^{ab} and $S^{(m)}$ the matter action. In parallel, the scalar-tensor gravity action can be written as [6]

$$S_{\mathrm{ST}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(m)} \tag{2.2}$$

where ϕ is the field replacing the inverse of the gravitational coupling constant, $\omega(\phi)$ is the Brans-Dicke coupling, $V(\phi)$ is the scalar field potential and ∇^c is a covariant derivative. The equations of motions obtained by variating (2.1) and (2.2) with respect to g^{ab} are respectively [6]

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa T^{(m)}_{ab} \tag{2.3}$$

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi}{\phi}T^{(m)}_{ab} + \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi\right) + \frac{1}{\phi}\left(\nabla_a\nabla_b\phi - g_{ab}\Box\phi\right) - \frac{V}{2\phi}g_{ab} \tag{2.4}$$

where $T^{(m)}_{ab}$ is the stress energy-tensor of matter, R_{ab} the Ricci tensor and $\nabla^c\nabla_c = \Box$ is the d'Alembertian. The fact that the right-hand-side of (2.4) looks like that of (2.3) with extra matter is precisely what allows to see scalar-tensor gravity as an effective dissipative fluid. This idea is developed further in the following section.

B. Imperfect Fluid

Different properties of a dissipative relativistic fluid can be identified as components of its stress-energy tensor as it is measured by a family of observers with four velocity u^a equal to that of the fluid [16] (comoving frame). The general decomposition of the stress-energy tensor in the comoving frame reads

$$T_{ab} = \rho u_a u_b + q_a u_b + u_a q_b + p h_{ab} + \pi_{ab} \tag{2.5}$$

with ρ the energy density, p the pressure, q_a the heat flux density and π_{ab} the anisotropic stress tensor.

To write the extra terms of (2.4) in the form (2.5), the four-velocity of the effective fluid is defined to be [5]

$$u^a = \frac{\nabla^a\phi}{\sqrt{-\nabla^c\phi\nabla_c\phi}}. \tag{2.6}$$

In general, the heat flux density from (2.5) is extracted from the stress-energy tensor with the projection tensor ${h_a}^b = \delta^b_a + u_a u^b$ and is given by $q_a = -T_{cb}u^c {h_a}^b$. The heat contribution extracted from the extra terms in (2.4) can be written with the four-acceleration $\dot{u}_a$ as follows

$$q_a = -\frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}\dot{u}_a. \tag{2.7}$$

Eckart's first order theory [4] of thermodynamics provides a relativistic extension of the usual fourier law describing heat transfer. The upgraded Fourier law has the following form

$$q_a = -\mathcal{K}\left({h_a}^b\nabla_b\mathcal{T} + \mathcal{T}\dot{u}_a\right). \tag{2.8}$$

where $\mathcal{K}$ is the thermal conductivity and $\mathcal{T}$ is the temperature. The term proportional to acceleration represents the effect of the inertia of energy in heat transfer. Combining (2.7) with (2.8), allows for the identifications

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}, \qquad {h_a}^b\nabla_b\mathcal{T} = 0 \tag{2.9}$$

respectively giving an expression for $\mathcal{K}\mathcal{T}$[1] and stating that the temperature gradient vanishes in the comoving frame.

III. PATHOLOGICAL STATE

The effective fluid temperature obtained in sec. II B seems to be a good order parameter quantifying deviations from the general relativity equilibrium. Testing the effective thermodynamics and its interpretation further requires looking for potentially pathological cases leading to inconsistencies. The special state studied here is pathological because it corresponds to an infinite gravitational coupling with undefined temperature.

A. Model

The simplest model [10] leading to superacceleration of the expansion of the universe is a scalar-tensor model with the action [2]

$$S = \int d^4x\sqrt{-g}\left[\left(\frac{1}{\kappa} - \xi\phi^2\right)R - \frac{1}{2}\nabla^c\phi\nabla_c\phi - V(\phi)\right], \tag{3.10}$$

where ξ is a constant describing the non minimal coupling of ϕ to R. Comparing this action to (2.1) reveals that the effective gravitational coupling of the model is given by

$$G_{\mathrm{eff}} = \frac{G}{1-\kappa\xi\phi^2}. \tag{3.11}$$

The pathological solutions studied here are associated to diverging G_{eff} achieved when the scalar field reaches the critical values $\phi = \pm\phi_c$ with

$$\phi_c = \frac{1}{\sqrt{\kappa\epsilon}}. \tag{3.12}$$

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[1] Since $\mathcal{K}$ can't be separated from $\mathcal{T}$, $\mathcal{K}\mathcal{T}$ is referred to as the temperature of gravity

[2] (3.10) can be written as a particular case of (2.2) under certain redefinitions

The equivalent of (2.9) for this model is given by

$$\mathcal{K}\mathcal{T} = \frac{2\xi|\phi|\sqrt{-\nabla^c\phi\nabla_c\phi}}{1-\kappa\xi\phi^2}. \tag{3.13}$$

Here is the pathology: for $\phi = \pm\phi_c$, $\mathcal{K}\mathcal{T} \sim 0\cdot\infty$ is undetermined.

To access the cosmological implications of (3.10), the form of the metric is restricted to a spatially flat FLRW ansatz given by

$$ds^2 = -dt^2 + a(t)^2(dx^2+dy^2+dz^2)$$

where $a(t)$ is the expansion factor of the universe. This metric is associated to the Ricci scalar [14]

$$R = \frac{1}{6}\left(\dot{H}+2H^2\right)$$

where $H = \dot{a}/a$ is the Hubble function. The field equations relating ϕ and H are

$$\begin{aligned} &6\left[1-\xi\left(1-6\xi\right)\kappa\phi^2\right]\left(\dot{H}+2H^2\right) - \kappa\left(6\xi-1\right)\dot{\phi}^2 - 4\kappa V \\ &+ 6\kappa\xi\phi V' = 0\,, && (3.14)\\ &\frac{\kappa}{2}\dot{\phi}^2 + 6\xi\kappa H\phi\dot{\phi} - 3H^2\left(1-\kappa\xi\phi^2\right) + \kappa V = 0\,, && (3.15)\\ &\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + V' = 0 && (3.16)\end{aligned}$$

with $V'(\phi) = \frac{dV}{d\phi}$. At the critical field values, (3.14) reduces to

$$6R = \dot{H} + 2H^2 \equiv C \tag{3.17}$$

where C is a constant. The critical solutions H_c for H at the critical field can all be obtained from (3.17). If

$$V_c \equiv V(\phi_c) = 0, \quad V_c' \equiv \left.\frac{dV}{d\phi}\right|_{\pm\phi_c} = \mp 6\xi C\phi_c, \tag{3.18}$$

they are given by [9]

$$\begin{aligned} &H_c(t) = \sqrt{\frac{C}{2}}\tanh(\sqrt{2C}t),\ C>0,\ H_c \neq \pm\sqrt{\frac{C}{2}} && (3.19)\\ &H_c(t) = \pm\sqrt{\frac{C}{2}},\ C>0 && (3.20)\\ &H_c(t) = \frac{1}{2t},\ C=0,\ H_c\neq 0 && (3.21)\\ &H_c = 0,\ C=0 && (3.22)\\ &H_c(t) = -\sqrt{\frac{|C|}{2}}\tan(\sqrt{2|C|}t),\ C<0. && (3.23)\end{aligned}$$

In the following section, homogeneous perturbations of the critical solutions are computed to study stability and explain the temperature indetermination.

B. Perturbation theory

Homogeneous perturbations to the solutions (3.19)-(3.23) are computed with the ansatz

$$H(t) = H_c(t) + \delta H(t),\ \phi(t) = \pm\phi_c + \delta\phi(t) \tag{3.24}$$

where $\delta H(t)$ and $\delta\phi(t)$ are the spatially uniform shifts from the critical solutions. These shifts are taken to be *initially* small compared to H_c and $\pm\phi_c$. This property is exploited when (3.24) are substituted in (3.14)-(3.16) by neglecting all second order shift combinations. Applying this procedure yields the following *linearized* equations

$$\begin{aligned} &\left[1-\xi(1-6\xi)\kappa\phi_c^2\right]\delta R \mp 2\xi(1-6\xi)\kappa\phi_c\delta R\phi \\ &\pm 6\kappa\xi\phi_c\delta\phi\left(V''-R\xi\right), && (3.25)\\ &\pm 6\kappa\xi\phi_c H_c(\delta\dot{\phi} + H_c\delta\phi) + H_c(6\kappa\xi\phi_c^2-1)\delta H \\ &+ \kappa V'\delta\phi = 0\,, && (3.26)\\ &\delta\ddot{\phi} + 3H_c\delta\dot{\phi} + \xi R\delta\phi \pm \xi\phi_c\delta R + V''\delta\phi = 0 && (3.27)\end{aligned}$$

where

$$\delta R = 6(\delta\dot{H} + 4H_c\delta H) \tag{3.28}$$

is the perturbed Ricci scalar computed by substituting (3.24) in (3.17).

Further simplifications of (3.26) can be done using the defining property (3.12) and (3.18) leading to

$$6\xi\kappa H_c\phi_c\left(\delta\dot{\phi} + H_c\delta\phi\right) - \kappa R\xi\phi_c\delta\phi = 0. \tag{3.29}$$

When $H_c \neq 0$, (3.29) becomes

$$\delta\dot{\phi} + \left[H_c - \frac{C}{H_c}\right]\delta\phi = 0 \tag{3.30}$$

which has the following solution

$$\delta\phi = \exp\left\{-\int\left[H_c - \frac{C}{H_c}\right]dt\right\} = \delta\phi_0 a_c H_c \tag{3.31}$$

where $\delta\phi_0$ is an integration constant giving the initial state of the perturbation.

When $H_c = C = 0$, (3.29) can't be used to solve for $\delta\phi$. However, in this special case, (3.27) reduces to

$$\delta\ddot{\phi} + V''\delta\phi = 0 \tag{3.32}$$

with solutions

$$\delta\phi(t) = \delta\phi_0\cos\left(\sqrt{V_c''}t\right) + \delta\phi_1\sin\left(\sqrt{V_c''}t\right) \tag{3.33}$$

where the additional integration constant $\delta\phi_1$ is necessary because (3.27) is second order. The solution for δH is given by (3.25) which simplifies to $\delta\dot{H} = 0$ for this special case. It follows that

$$\delta H = \delta H_0 \tag{3.34}$$

where δH_0 is yet another integration constant.

Combining (3.27) with (3.30) provides a first order differential equation for δH sources by $\delta\phi$. One has

$$\delta\dot{H} + 4H_c\delta H = \mp\frac{1}{6\xi\phi_c}\left(6\xi C + 2C + V_c''\right)\delta\phi \equiv -D_\pm\delta\phi.$$

This forced linear ODE can be solved using an integrating factor. To compute it generally, (3.17) and (3.30) are combined to produce

$$\delta\dot{\phi} = \left(-H_c + 2H_c + \frac{\dot{H}_c}{H_c}\right)\delta\phi = \left(H_c + \frac{\dot{H}_c}{H_c}\right)\delta\phi$$
$$\iff H_c = \frac{\delta\dot{\phi}}{\delta\phi} - \frac{\dot{H}_c}{H_c}.$$

The integrating factor is

$$\exp\left\{4\int dt H_c\right\} = \exp\left\{4\int dt\left(\frac{\delta\dot{\phi}}{\delta\phi} - \frac{\dot{H}_c}{H_c}\right)\right\} \sim \left(\frac{\delta\phi}{H_c}\right)^4.$$

Finally, using (3.31) the solution for δH when $H \neq 0$ is

$$\begin{aligned}\delta H &= -\frac{D_\pm}{5}a_c^{-4}\int H_c a_c^5 dt = -\frac{D_\pm}{5}\delta\phi_0 a_c^{-4}\int \dot{a}_c a_c^4 dt\\ &= \delta\phi_0\left(-\frac{D_\pm}{5}a_c\right) + \delta H_0\left(\frac{1}{a_c^4}\right).\end{aligned} \tag{3.35}$$

The first term of this result is the particular solution and the second one is the homogeneous solution. They are respectively identified (P) and (H) in the figures of sec. III C.

Now that the solution for the linearized equations are known, it is possible to compute their temperature. The calculations are restricted to positive gravitational coupling [3] following evidence against the possibility of effective anti-gravity [17]. This restriction is acheived by taking $\delta\phi$ to have a sign opposite to that of the pertubred scalar field so that (3.11) stays positive under the perturbation. With this in mind, the following general formula for perturbed temperature is obtained

$$\mathcal{KT} = \frac{2\xi|\phi|\sqrt{-\nabla^c\phi\nabla_c\phi}}{1-\kappa\xi\phi^2} = \frac{1}{\kappa}\left|\frac{\delta\dot{\phi}}{\delta\phi}\right|.$$

For $H_c \neq 0$, (3.30) can be used to write

$$\mathcal{KT} = \frac{1}{\kappa}\left|\frac{C}{H_c} - H_c\right|. \tag{3.36}$$

[3] Note that allowing negative G_{eff} would lead to a negative temperature which has no clear interpretation in this context

C. Stability analysis

As a sufficient criterion for the instability of a critical solution (3.19)-(3.23), the divergences of the ratios $\delta H/H_c$ and $\pm\delta\phi/\phi_c$ are studied in this section. If any of these ratios diverges for a given critical solution, the perturbation fails to remain small; it grows without bounds away from the critical solution. Using the results of sec. III B, the perturbations δH and $\delta\phi$ associated to (3.19)-(3.23) are respectively

$$\delta H = \delta\phi_0\left(-\frac{D_\pm}{5}a_0\cosh^{1/2}[\sqrt{2C}t]\right) + \delta H_0\left(\frac{1}{a_0^4}\operatorname{sech}^2[\sqrt{2C}t]\right), \quad \delta\phi = \sqrt{\frac{C}{2}}\delta\phi_0 a_0\frac{\sinh[\sqrt{2C}t]}{\cosh^{1/2}[\sqrt{2C}t]}, \quad C>0,\ H_c \neq \pm\sqrt{\frac{C}{2}},$$

$$\delta H = \delta\phi_0\left[-\frac{D_\pm a_0}{5}\exp\left(\pm\sqrt{\frac{C}{2}}t\right)\right] + \delta H_0\left[\frac{1}{a_0^4}\exp\left(\mp 4\sqrt{\frac{C}{2}}t\right)\right], \quad \delta\phi = \pm\sqrt{\frac{C}{2}}\delta\phi_0 a_0\exp\left(\pm\sqrt{\frac{C}{2}}t\right), \quad H_c = \pm\sqrt{\frac{C}{2}},\ C>0,$$

$$\delta H = \delta\phi_0\left(-\frac{D_\pm}{5}a_0\sqrt{t}\right) + \delta H_0\left(\frac{1}{a_0^4 t^2}\right), \quad \delta\phi = \delta\phi_0\frac{a_0}{2\sqrt{t}}, \quad C=0,\ H_c \neq 0,$$

$$\delta H = \delta H_0, \quad \delta\phi(t) = \delta\phi_0\cos\left(\sqrt{V_c''}t\right) + \delta\phi_1\sin\left(\sqrt{V_c''}t\right), \quad H_c = 0,\ C=0,$$

$$\delta H = \delta\phi_0\left(-\frac{D_1 a_0}{5}\cos^{1/2}[\sqrt{2C}t]\right) + \delta H_0\left(\frac{1}{a_0^4}\sec^2[\sqrt{2C}t]\right), \quad \delta\phi = -\sqrt{\frac{C}{2}}\delta\phi_0\alpha_0\frac{\sin[\sqrt{2C}t]}{\cos^{1/2}[\sqrt{2C}t]}, \quad C<0.$$

where a_0 is a constant appearing in the expression of a_c.

The power law critical solution (3.21) is used here to exemplify the instability analysis performed for each case. The relevant ratios are plotted against comoving time in fig. III C. The homogeneous part of $\delta H/H_c$ and of the ratio $\delta\phi/\phi_c$ is bounded at times after the big bang ($t=0$) and perturbations with $\delta H_0 \neq 0$ and $\delta\phi_0 \neq 0$ seem to be stable from the point of view of those curves. However, if

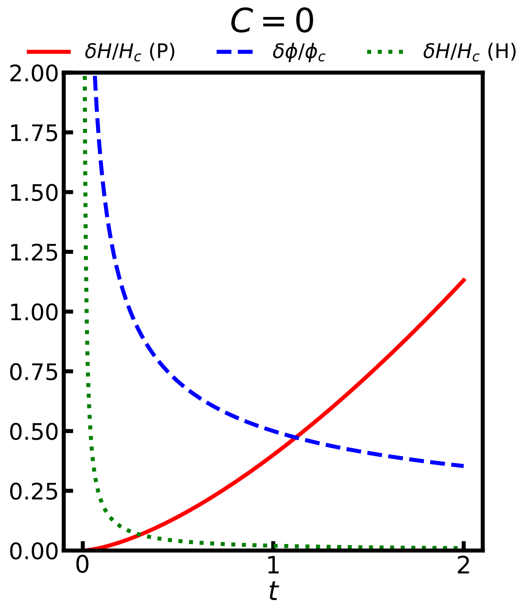

FIG. 1. Particular (P) and homogeneous (H) contributions to the ratio $\delta H/H_c$ (see (3.35)) and ratio $\delta\phi/\phi_c$ plotted against comoving time t. For simplicity of representation, the constants are set to $a_0 = 1, \delta\phi_0 = 1, D_\pm = -1$ and $\delta H_0 = 1/100$. Also, $\phi_c = 1$ was taken even if $\delta\phi > 0$ as it doesn't change the discussion.

$\delta\phi_0 \neq 0$, the particular part of the perturbation gives an instable behavior to the solution. As $t \to \infty$, $\delta H/H_c \sim t^{3/2}$ diverges and the hypothesis used for linearization is lost. The existence a family of unstable perturbations suffices to deem the critical solution unstable.

The other cases are treated in detail in [8] and all of them, except (3.22), end up being unstable for at least one perturbation. The oscillatory behavior of the perturbation (3.33) of the critical solution (3.22) leaves all perturbations stable. Even thought it is stable, this solution allows for dynamical crossing of $\phi = \pm\phi_c$ which would introduce oscillations from gravity to anti-gravity going against considerations of sec. III B. Instead of being instable this critical solution is looked at as being *unphysical*. In the end, all solutions do not harm the validity of temperature by letting it be undetermined because they are not realised in nature.

At this point, a question remains: what made temperature undetermined in the first place? The answer is provided by (3.36). This equation shows that temperature is independent of the perturbation parameters $\delta\phi_0$ and δH_0. Normally the temperature of a solution could be computed from nearby (perturbed) solutions by letting the perturbation size go to 0. Here nearby $\mathcal{KT}$ always depend on H_c and C while $\mathcal{KT}$ computed exactly at the critical field doesn't. The only way to make this consistent is to have $\mathcal{KT}$ undetermined at the critical field.

IV. MULTI-FLUIDS

As the natural extension of *scalar-tensor* gravity is the theory of *scalar-multi-tensor* gravity, the natural extension of the *single fluid* picture is a *multi-fluid* picture. Scalar-multi-tensor gravity describes a collection of N scalar fields ϕ^A non-minimally coupled to the Ricci scalar by the coupling factor $f(\phi^A)$. The vacuum action for this theory [11] reads

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(fR - Z_{AB}g^{ab}\partial_a\phi^A\partial_b\phi^B - 2\kappa U\right) \tag{4.37}$$

where Z_{AB} is a quadratic form with implicit summation for repeated *capital* Latin indices and U the potential. The equation of motion obtained from variations of (4.37) with respect to g^{ab} is

$$\begin{aligned} R_{ab} - \frac{1}{2}g_{ab}R &= \frac{1}{f}\frac{\partial f}{\partial\phi^A}\left(\nabla_a\nabla_b\phi^A - g_{ab}\Box\phi^A\right) \\ &+ D_{AB}\nabla_a\phi^A\nabla_b\varphi^B + g_{ab}\left(\frac{Z_{AB}}{2f} - D_{AB}\right)\nabla_c\phi^A\nabla^c\phi^B - g_{ab}\frac{U}{f} \end{aligned} \tag{4.38}$$

where $fD_{ab} = Z_{ab} + \frac{\partial^2 f}{\partial\varphi^a\partial\varphi^b}$.

Following the ideas presented in sec. II, the next step would be to define a four-velocity using the fields and decompose the right-hand side of (4.38) as an imperfect fluid in the frame comoving with the chosen four-velocity. However, each field is consistent with its own four-velocity given by

$$u^a_A = \frac{\nabla^a\phi^A}{\sqrt{-\nabla^c\phi^A\nabla_c\phi^A}}$$

and choosing one of them for the decomposition is arbitrary. This problem disappears in the limiting case where all the fields are equal. In this case, a common velocity exists and decomposition of the right-hand side of (4.38) leads to a temperature of

$$\mathcal{KT} = \frac{1}{\kappa f}\frac{\partial f}{\partial\phi^A}\sqrt{-\nabla^c\phi^A\nabla_c\phi^A}. \tag{4.39}$$

To solve the problem in general, another view of the collection of fields was proposed. In this approach following [11], the Nth fluid is replaced by f which becomes an independent field renamed ψ while the $N-1$ first fluids are renamed φ^A. Also, the quadratic form $Z_{AB}(\phi^C)$ is replaced by $W_{AB}(\varphi^C, \psi)$ and the potential $U(\phi^A)$ is replaced by $V(\varphi^C, \psi)$. Elements W_{NA} with A different from N are taken to be 0 without loss of generality. Finally, one takes $W_{NN} \equiv \omega/\psi$ to make the action take the following familiar form

$$\begin{aligned} S = \frac{1}{2k}\int d^4x\sqrt{g}\Big(&\psi R - W_{AB}\nabla^c\varphi^A\nabla_c\varphi^B \\ &- \frac{\omega}{\psi}\nabla^c\psi\nabla_c\psi - V\Big). \end{aligned}$$

The right-hand side of the equivalent of (4.38) is the decomposed in the comoving frame of the ψ-fluid with velocity

$$u^a = \frac{\nabla^a \psi}{\sqrt{-\nabla^c \psi \nabla_c \psi}}.$$

It turns out that the ψ-fluid is the only dissipative fluid involved. All other fields correspond to effective perfect fluids. The only non-zero temperature that can be formally identified with (2.8) is that of the ψ-fluid and it is given by

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{2X}}{\kappa\psi} = \frac{1}{\kappa f}\sqrt{-\frac{\partial f}{\partial \varphi^A}\frac{\partial f}{\partial \phi^B}\nabla_c \phi^A \nabla^c \phi^B}. \quad (4.40)$$

Since all the ϕ^A fluids are coupled to each other, it has been proposed that they always are at thermal equilibrium thus sharing a global temperature possibly given by (4.40).

Comparison of (4.39) with (4.40) reveals that the ψ-fluid picture does not agree with the only limiting case where a clear choice of velocity exists. This makes the global temperature status of (4.40) questionnable and other approaches must be considered. What is clear is that understanding how the fluids interact when they are all flowing with the same four-velocity is crucial to understand the most general case.

V. CONCLUSION

The stability analysis of the critical field has shown that critical solutions (3.19)-(3.21) and (3.23) were unstable under at least one type of homogeneous perturbations. While critical solution (3.22) was shown to be stable for all types of homogeneous perturbations it was deemed unphysical because its perturbation crossed dynamically from gravity to anti-gravity. In the model studied here, there is no stable *and* physical state of equilibrium with undetermined temperature for gravity. Furthermore, the origin of the indetermination of temperature was found to be related to a dependency of $\mathcal{K}\mathcal{T}$ on H_c arbitrarily close to the critical field where the temperature is independent of H_c.

After showing that the pathology is harmless for the formalism, the extension of the fluid picture has been undertaken. The multi-fluid picture was introduced and a clear problem with the choice of frame to use to decompose the effective stress-energy tensor was given. This problem was tackled by going to a limiting case and by using a different formulation of scalar-multi-tensor gravity. Both ideas gave different temperatures of gravity. Questions were raised about thermal equilibrium of the fluids and equivalent of quantities like the comoving temperature in a more general frame. The next part of the research will focus on the exchanges between the fluids when they are all aligned.

The future of the thermodynamic analysis of modified gravity is full of challenges and interesting ideas. On of the challenges revolves around problems in Eckart's thermodynamics. The theory has many flaws like acausality of the heat density flux and instability [18]. To get away from these problems, the formalism would have to be rebuilt using a more modern formulation of relativistic thermodynamics like [19]. Finally, it was proposed that the formalism could lead to a phase diagram of modified theories of gravity.

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