Conformal Field Theory

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Contents

Introduction

Jaume Gomis, the lecturer is a faculty member here at PI.

The basic textbook for the course, which will not be followed exactly, is *Conformal Field Theory* by Di Francesco, Mathieu and Seńećhal. Interested readers can also take a look at *Applied Conformal Field theory* by Ginsparg at ArXiv hep-th/9108028. These two references focus more on the 2 dimensional conformal invariant theories.

The assumed background material for this course is contained in Peskin and Schroeder, specifically chapter 8 and chapter 12.1

Part I

The Many Uses of Conformal Field Theories

1 A Broader Context

We will start by looking at conformal field theory (CFT) taking a wider perspective. We want to show that CFT lies at the heart of quantum field theory (QFT) and statistical mechanics. We will delve into greater details later in the course.

QFT's are usually taken to be invariant under the Poincaré group, CFT's have a larger symmetry group called the *conformal group* which contains the Poincaré group as a subgroup. In this sense a CFT[1] has many more symmetries than regular QFT's and is therefore more tractable.

The conformal group can be defined as the set of transformations of spacetime that preserves angles. In general a conformal transformation won't preserve distances, as we saw in our QFT course the set of transformations that preserve distances form the Poincaré group which is generated by transformations of the following form:

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{1}$$

which are a combination of a Lorentz transformation $\Lambda^\mu{}_\nu$ and a translation a^μ. A Poincaré transformation preserves distances and leaves the metric invariant $\eta_{\mu\nu} \to \eta_{\mu\nu}$. In general such transformations are called *isometries* of the metric.

[1] Theories with conformal anomalies still satisfy conformal ward identity in flat space, they are still considered to be theories that have conformal symmetry.

1.1 Comments on Galilean symmetry

The non-relativistic QFTs enjoy Galilean symmetry instead of Poincaré symmetry instead. The generators of Galilean symmetry include H (time translation), P_μ (space translation), M_{ij} the space rotation, and Galilean Boost B_i. In addition, there is a central charge Z for an additional $U(1)$ symmetry, Z shows up in the commutator of the other generators,

$$[P_i, B_j] = iZ\delta_{ij}$$

Z commutes with all the generators. This $U(1)$ symmetry means particle number is conserved, which makes particle production impossible. While relativistic QFT features particle production. Galilean symmetry can be extended to non-relativistic symmetry: Schrödinger symmetry, which is the maximal symmetry of the free Schrödinger equation.

1.2 The Scale Transformation

The scale transformation is the simplest conformal transformation, it acts by rescaling the spacetime coordinates (zooming in/out). For $x^\mu = (t, x^i)$ a scale transformation takes the general form:

$$t \rightarrow \lambda^z t \tag{2}$$

$$x^i \rightarrow \lambda x^i \tag{3}$$

where z is a dynamical critical exponent.

In a relativistic framework, space and time must be treated the same or in other words, the theory needs to be covariant and commute with Lorentz transformation. Thus $z = 1$ and we have $x^\mu \rightarrow \lambda x^\mu$. Then the distance will transform in the following way, which is obviously not preserved.

$$\eta_{\rho\sigma} d\tilde{x}^\rho d\tilde{x}^\sigma = \lambda^2 \eta_{\mu\nu} dx^\mu dx^\nu \tag{4}$$

However, angles are preserved since

$$\cos\theta_{x,y} = \frac{\eta_{\mu\nu} x^\mu y^\nu}{\sqrt{\eta_{\mu\nu} x^\mu x^\nu}\sqrt{\eta_{\mu\nu} y^\mu y^\nu}} \tag{5}$$

which is clearly invariant under scaling.

For non-relativistic free particle

$$\mathcal{L} = \psi^\dagger (i\partial_t - \nabla^2)\psi \tag{6}$$

with equation of motion being the Schrödinger equation,

$$i\frac{\partial}{\partial t}\psi = -\nabla^2\psi$$

to preserve the equations of motion of this system, we have dynamical exponent $z = 2$. For other systems, the dynamical exponent might be different. Another example, with the following action,

$$S = \int dx dt \frac{1}{2}(\partial_t\phi)^2 - (\nabla^2\phi)^3$$

we have dynamical exponent $z = 3$. It can be inferred by dimensional analysis that on momenta, the scale transformation acts in the opposite way - to zoom-in and probe smaller regions in spacetime requires modes with higher momentum.

$$p_\mu \to \frac{1}{\lambda}p_\mu. \tag{7}$$

There are more conformal transformations than scaling and Poincaré transformations.

1.3 The Conformal Transformation

A conformal transformation is a coordinate transformation that preserves angles but not distance necessarily

$$x^\mu \to \tilde{x}^\mu(x) \tag{8}$$

such that the distance transforms in the following way

$$dx^\mu dx_\mu \to e^{2\sigma(x)} dx_\mu dx^\mu \tag{9}$$

For example, for the scaling transformation, we have

$$ds^2 \to \lambda^2 ds^2$$

In four dimensions there are a finite number of conformal transformations: 10 Poincaré transformations (with $\sigma = 0$ which preserves both angles and distances.) plus the other

5 we will elaborate on soon, but we'll see that in two dimensions we have an infinite set of conformal transformations.

Some of the most beautiful equations/interactions in physics are *conformally invariant*. For example,

1. Maxwell's equations: $\partial^\mu F_{\mu\nu} = 0$
2. Massless Dirac Equation: $\Gamma^\mu \partial_\mu \Psi = 0$
3. *Classical* $\lambda\phi^4$ in $D = 4$: $\Box\phi = \frac{\lambda}{3!}\phi^3$
4. *Classical* Yang-Mills in $D = 4$: $D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}] = 0$
5. *Classical* Yukawa coupling in $D = 4$: $\lambda\phi\bar{\psi}\psi$.

These equations have no dimensionful parameters like masses or coupling constants, if they did they couldn't be conformally invariant. Recall that quantum effects in general give an energy dependence for the coupling constants, this quantum effect will spoil the classical conformal invariance and this symmetry is anomalous at the quantum level.

The energy dependence of the coupling constant is encoded in the beta function

$$\beta_i(\vec{\lambda}) = \frac{d\lambda}{d\ln\mu} = \mu\frac{d\lambda_i}{d\mu} \tag{10}$$

In general this function won't vanish for non-trivial theories, one of the most studied theories with a vanishing beta function is $\mathcal{N} = 4$ SYM, which is also quantum mechanically conformal invariant.

The conformal group can also be characterized as the group that preserves the causal structure of spacetime, meaning null geodesics are always mapped to null geodesics. In Euclidean space conformal transformations map spheres to spheres.

1.4 Why Study CFT?

1. CFTs play a *central* role in our understanding of QFT, being fixed points of the RG flow its structure underlies our understanding of QFT's. QFT's can be seen as a perturbation of CFT's by a relevant operator.

2. CFTs help us understand how physics changes under a change of scale. It is important for understanding the domain of validity of our theories. Whenever we ask a physical question, we have to ask which energy scale we are trying to probe and decide which effective description is more appropriate to describe the relevant degrees of freedom.

3. When we study QFT or any other physical theory, we need to be aware that they come with a breakdown length in which our theory is not suited anymore to describe the physics. QFT comes with an ultraviolet cutoff Λ beyond which it breaks down. We believe that this energy scale is of the order of the Planck scale where gravity effects should come into play. We can think of Λ as parameterizing the energy scale above which we know nothing about the fundamental degrees of freedom or alternatively the distance scale $\frac{1}{\Lambda}$ as the distance below which we know nothing.

4. The Renormalization Group (RG) of Wilson is a way of parameterizing our ignorance about physics at $E \geq \Lambda$ in terms of interactions between the low energy degrees of freedom.

2 Flow Equations

2.1 Constructing Renormalization Group Flows

How do we use the Renormalization Group to study low energy physics?

In order to study some physical system we need to know the low energy degrees of freedom and the symmetries of the theory. In a spin chain we have the magnetization which can be calculated taking a scalar field as the low energy degrees of freedom $\varphi(x)$, the symmetry would be $\mathbb{Z}_2$.

Now we write down the most general action possible compatible with the symmetries

$$S = S_0 + \int d^D x \sum_i \lambda_i \theta_i(\varphi) \tag{11}$$

where S_0 is the free field theory or a *Gaussian Fixed Point*, $\theta_i(\varphi)$ are a set of interactions or operators and λ_i are coupling constants telling us how strong the interaction is.

This seems like a useless way to proceed since we have a infinite number of parameters in this Lagrangian. But there is a way to describe the low-energy physics of this system. Wilson told us that at low energy only a finite number of interactions are actually important for the physics. We can focus our attention on these parameters and forget about the irrelevant ones.

Once we have a *low energy effective description* the first thing we do is to fix the coupling constants $\lambda_i(\mu)$ by measuring them at some energy scale. Then using the RG equations we know how they flow with the energy and now we can do predictions with our effective theory.

We can now write

$$Z = \int [D\varphi]_\Lambda \, e^{-S} \tag{12}$$

where we are integrating over all modes of all fields $0 \leq |\vec{p}| \leq \Lambda$. This is equivalent to integrating inside a sphere of radius Λ in momentum space, but not outside as we know nothing about the region outside.

When taking into account quantum effects the coupling constants in general become functions of the energy scale $\lambda_i(\mu)$, we say that they are *running coupling constants.* We compute them using the renormalization group flow. We want to know how the coupling constants in the regime $0 \leq |\vec{p}| \leq \Lambda$ are related to $0 \leq |\vec{p}| \leq b\Lambda$. Where with $b < 1$ we have reduced the radius of our sphere of integration. It is valid to eliminate these high energy modes or degrees of freedom if we are studying low energy physics. This allows use to determine the RG flow equations, which tell us how $\lambda_i(\mu)$ evolve with energy scales. These flow equations are usually called Beta-Functions;

$$b\frac{d\lambda_i}{db} = \beta_i(\vec{\lambda}). \tag{13}$$

These are very complicated (relating all of the λ_is) and are in general very difficult to solve. Some of the most important physical theories have the λ_is scaling logarithmically with energy.

2.2 Fixed Points

Consider the space of coupling constants for some theory. The general form of the beta function is

$$\beta_i(\vec{\lambda}) = \frac{d\lambda_i}{d\ln\mu} = \mu\frac{d\lambda_i}{d\mu} \tag{14}$$

These are flow equations in the space of coupling constants. The topology of the flow lines is controlled by its fixed points,

$$\beta_i(\vec{\lambda}^*) = 0. \tag{15}$$

At these fixed points the theory is scale invariant, as clearly the coupling constants do not change with a change of scale. Close to these fixed points the coupling constants might either flow towards or away from the fixed point. This behavior is useful for classifying the interactions.

In general a theory that is scale invariant is not necessarily conformal invariant (there are other *special conformal transformations* that we will study later) in any dimension greater than 2, in two dimensions it is a well know result that scale invariance and unitarity implies conformal invariance. However there are no counter examples so far and in this course we will assume that scale invariance implies conformal invariance. This remains an important open problem in CFTs. Jaume would bet that if a theory is scale invariant, unitary and relativistic then it is conformal invariant!

Now we want to understand the flow (from high to low energies in the space of coupling constants) of the couplings away from the fixed point λ_i^*. Any particular direction can be 'unstable' (going away from the fixed point) or 'stable' (going towards the fixed point).

2.3 Interactions

Quantum mechanically loop diagrams will induce scale dependence of the coupling when $\beta(\lambda) \neq 0$. The operator will trigger a RG flow that depends on the mass/scaling dimension of the operator. For operator $\int d^Dx\lambda\mathcal{O}$, with dimension of $\mathcal{O}$ being Δ, the scaling dimension of λ is $D - \Delta$. We can construct the dimensionless coupling

$$\bar{\lambda} \equiv \lambda\mu^{\Delta - D}$$

then we have

$$\beta(\bar{\lambda}) = \mu \frac{d\bar{\lambda}}{d\mu} = (\Delta - D)\bar{\lambda}$$

Following the above recipe we have an infinite number of coupling constants, depending on their scaling behavior under the flow (given below by the *exact* quantum dimension Δ[2]) we can classify the interactions they control in the following way:

- Relevant: Unstable interactions with $\Delta < D$, they flow away from the fixed points. These operators dominate the infrared (IR) low energy regime.

- Irrelevant: Stable or attractive with $\Delta > D$, they flow towards the fixed point. They are not important in the IR. Most of the operators are irrelevant and that's what allows us to do physics.

- Marginal: (invariant under scale transformations), there is a line of CFT's (fixed points) which differ by marginal deformations. $\Delta = D$

Given a set of degrees of freedom (fields) usually the number of relevant or marginal operators is *finite* and *small* in order to extract useful information from the theory. We have parametrized our ignorance about long distance physics with a small number of finite operators.

By thinking in this language, we have quantitatively understood how we can make progress in physics and extend physics to new regimes.

QFTs are perturbations of CFTs by relevant operators.

RG also explains why two systems that are microscopically very different can have the same behavior near a fixed point. These two theories have the same low energy degrees of freedom and the same symmetries so they flow to the same fixed point. This is the concept of *universality* which pervades modern physics.

[2]Note the classically marginal operators could become either irrelevant, relevant or exact with quantum corrections. For example, the coupling constant in $\lambda\phi^4$ is marginal irrelevant, and the one in Yang-Mills is marginal relevant. We will emphasize this point again later as well.

2.4 Example: Scalar Field

Let us consider a system with degrees of freedom which at low energies can be described by a scalar field $\phi(x)$. From before, we have

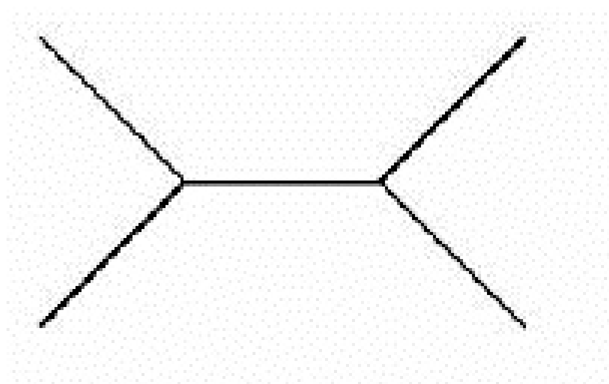
$$S = S_0 + \sum \int d^D x \lambda_i \theta_i(\phi) \tag{16}$$

where the fixed point is given by the free field action

$$S_0 = \int d^D x \frac{1}{2} \left(\partial\phi\right)^2 \tag{17}$$

From this we have the classical dimension of the field $[\phi] = \frac{D-2}{2}$. We can have θ_is that do not have any derivatives such as ϕ^2, ϕ^3, ϕ^4... which are called the potential interactions. We can also have θ_i that do have derivatives such as $\left(\Box\phi\right)^2, \phi\left(\partial\phi\right)^2, \phi^2\left(\partial\phi\right)^2, ...$ which are called higher derivative interactions. We could list all of them but so could the reader so we won't.

These terms are generated by quantum corrections, for example in ϕ^3 quantum effects generates a ϕ^4 interactions as the following picture:



where the internal particle at low energies can be integrated out (if it is sufficiently massive) and we have an effective ϕ^4 vertex.

Now let us list all the relevant operators, keeping in mind that $[\phi] = \frac{D-2}{2}$

All the higher derivative interactions will be irrelevant since by Lorentz invariance we always need two derivatives acting on the fields and $(\partial\phi)^2$ already has $\Delta = D$, with that in mind:

- $D = 6$, $[\phi] = 2$: we have ϕ^2; ϕ^3 is classically marginal (ϕ^3 is marginally irrelevant when quantum effects are taken into account, it gets smaller very slowly in the IR)

- $D = 4$, ϕ^2,ϕ^3,ϕ^4(ϕ^4 is marginally relevant via quantum process)

- $D = 3$, $[\phi] = \frac{1}{2}$: $\phi^2, \phi^3, \phi^4, \phi^5$ (ϕ^6 is marginally irrelevant due to quantum processes).

- $D = 2$, ϕ^2, ϕ^3... all of them!

Note that we cannot have fractional powers of ϕ as such operators would not be analytic.

It is a general feature that in lower dimensions we get more relevant operators, in 2 dimensions there are a rich landscape of CFTs which is not seen in higher dimensions. Why is that?

Yang-Mills is an example where an operator that is classically marginal becomes marginally relevant.

2.5 Multiple or Non-Gaussian Fixed Points

Relevant operators can generate flows between fixed points. By starting at one fixed point say a Gaussian fixed point and deforming by a relevant operator we can flow to another fixed point and hence get a non-trivial CFT. Since in two dimensions we have an infinite number of relevant operators we can flow in several directions to other CFTs.

This is important in Condensed Matter physics as the critical behaviors in 2nd order phase transitions are described by one of these non-Gaussian fixed points. For example, the Wilson-Fischer fixed point plays an important role in critical physics in 3 and 4 dimensional spacetime.

In $D = 2$, there are an infinite number of non-Gaussian fixed points, and a huge set of 2D CFTs. There are some cases where spacetime symmetries allow us to completely solves these CFTs.

How to get non-trivial CFTs by Wilson-Fischer: Start with a theory in some dimension D which has a classically marginal coupling λ, quantum mechanically the beta function tells us how this coupling will run with energy.

Analytic continuation to dimension $D - \epsilon$ turns a marginal operator into a relevant operator, its beta function in this dimension gets a contribution

$$\frac{d\lambda}{d\ln\mu} = -\epsilon\lambda + \beta(\lambda) \tag{18}$$

from this we can find other zeros of the beta function and thus other CFTs! For example for a $\lambda\phi^4$ theory $\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)$, so a new fixed point will be at $\lambda^* = \frac{16\pi^2}{3}\epsilon$.

2.6 CFT flows and an example of 3d Ising model

There is a RG flow from CFT_{UV} to CFT_{IR}. In principle even if the theory in UV is weakly coupled, in the IR, the theory could be arbitrarily strongly coupled. It is hard to know what theory we can get in the IR. Certain quantities appear during the study of anomalies will play a crucial role in establishing certain theorems we are going to prove using abstract arguments. There is a certain quantity that is monotonically decreasing $c_{UV} > c_{IR}$. c_{UV} and c_{IR} are central charges related to the anomaly. As the renormalization flows, this quantity c decreases. This gives rises to sets of selection rules. If we are given two CFTs, we can compute the central charges and discern if these two theories are related by RG flow. This also tells us the RG flow is irreversible. So we can have a theory A that flows to theory B but there is no way theory B can flow back to theory A. Note the inequality $c_{UV} > c_{IR}$ is strict. We will see a big difference between anomalies in even and odd dimensions. We will see a simple reason why. Nevertheless, the spherical partition function allows us to postulate certain quantity even in odd dimensions that is monotonic under RG flow. This again gives us nonperturbative selection rules for what kind of flows can exist just using basic principles of field theory such as unitarity, Lorentz invariance, diffeomorphism invariance and so on.

In general it is possible the RG flow will bring the theory to the regime of strong coupling and thus it is possible we are not able to study it perturbatively. There are clever ways to choose $\beta\lambda^* = 0$ while $\lambda^* \ll 1$ still holds, thus we will still have control over the IR CFT we flow to. These are the “perturbative” RG flows, two of the more famous examples are

1. Wilson-Fischer fixed-point CFTs (scalars)

2. Banks-Zachs CFT (Yang-Mills+quarks)

We will use Wilson-Fischer as an example. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

In four dimension, the interaction term is marginal and the β function receives 1-loop correction,

$$\beta(\lambda) = |A|\lambda^2$$

where $|A|$ is some positive constant that is computed to be $\frac{3}{16\pi^2}$. To find the Wilson-Fischer CFT, we study this theory in $4-\epsilon$ dimensions, then the interaction is no longer marginal, and as $\dim(\lambda) \propto \epsilon$, the β function now have a tree-level contribution as well,

$$\beta(\lambda) = -\epsilon\lambda + |A|\lambda^2$$

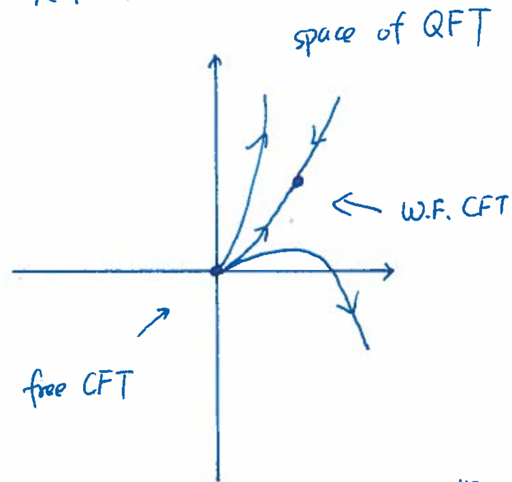
Thus we can solve for $\beta(\lambda^*) = 0$, which occurs at $\lambda^* = \frac{\epsilon}{|A|}$. See the following RG flow diagram for more comments. Now we will take a short detour to talk about $D = 3$ Ising model, which is given by the following nearest neighbor interaction term

$$H = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j. \tag{19}$$

with $\sigma_i = \{1, -1\}$. See the following phase diagram The theory is in general not scale invariant. By tuning the strength of the interaction J, at the critical point J_c, the lattice Ising model becomes the Ising CFT. When $T > T_c$, the theory has a Z_2 symmetry, and the symmetry is spontaneously broken at $T < T_c$. We can draw analogy between this theory with the scalar theory we just studied. In the scalar theory, when $m^2 > 0$, because $\langle\phi\rangle = 0$, there is a Z_2 symmetry when we take $\phi \to -\phi$. When $m^2 < 0$, $\langle\phi\rangle \neg 0$, the Z_2 symmetry is spontaneously broken. Thus at $T = T_c$, the theory corresponds to the massless scalar theory. The Ising CFT has two relevant primaries σ and ε, which corresponds to spin and energy respectively. The scaling dimension of σ is about $\Delta_\sigma \simeq 0.51$,

$$\langle\sigma(x)\sigma(0) = \frac{1}{x^{2\Delta_\sigma}}$$

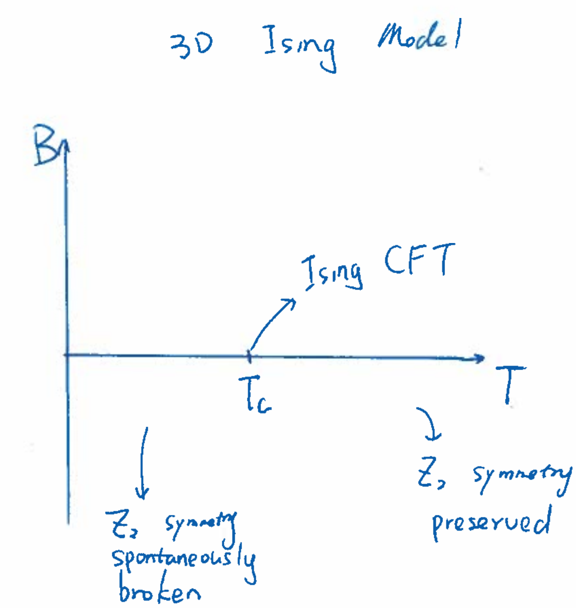
RG flow diagram



arrows means to go to IR

relevant operator that is responsible of the flow near UV CFT becomes irrelevant at the end of flow near IR CFT.

and the scaling dimension of ε is about $\Delta_\varepsilon \simeq 1.5$. These are related to the critical exponents in the Ising model. To compare with the scalar model in $D = 3$, we have $\Delta_{0,\phi} = 0.5$(σ is analogous to ϕ), while $\Delta_{0,\phi^2} = 1$(ε is analogous to ϕ^2), which shows that



the Ising CFT is strongly coupled, and very far away from the free Gaussian theory. At the Wilson-Fischer fixed point, the operator ϕ^4 will be irrelevant, thus have dimension bigger than 3.

It is surprising, but starting in $D = 4 - \epsilon$ and study the perturbative theory, and then taking $\epsilon \to 1$ to reduce to $D = 3$(which is not perturbative), we are able to predict critical exponents of $D = 3$ Ising model.

3 Critical Phenomena

3.1 Recap and Summary

1. CFTs are the scaffolding in the space of QFTs. Renormalization Group (RG) flows in the space of couplings has an action given by

$$S = S_0 \sum \int d^D x \lambda_i \theta_i(x) \tag{20}$$

λ_is obey RG flow,

$$\mu \frac{d\lambda_i}{d\mu} = \beta_i(\vec{\lambda}) \tag{21}$$

which have fixed points

$$\beta_i(\vec{\lambda}^*) = 0 \tag{22}$$

and points where this happens defines our conformal field theory.

2. The more traditional answer of why conformal field theories are important is that they are directly relevant for nature.

   a It's useful for studying Critical phenomena in statistical mechanics at non-zero temperature, at continuous phase transitions (second order phase transitions). Near these critical points the correlation length goes to infinity and the system can be described by a CFT. The thermodynamic quantities exhibit scaling behavior near these points which are encoded in the dimensions of the operators in the CFT.

   b Quantum Critical Phenomena. These occur in quantum systems at zero temperature when quantum fluctuations drive the system through a phase transition to a gap-less system. This is described by a CFT.

We'll come back to these applications later when we have the tools to solve some simple examples.

3.2 Criticality and Correlation Lengths

An indication of whether the system is approaching criticality can be found from the *correlation length* ξ. You could find the criticality of the system by, for example, calculating the two point correlation function.

$$\langle\sigma(x)\sigma(0)\rangle - \langle\sigma(x)\rangle\langle\sigma(0)\rangle \sim \exp\left(-\frac{|x|}{\xi}\right) \tag{23}$$

where we use σ as the basic degree of freedom (ϕ from previous sections).

If we had a massive scalar field of mass m, then we tend to find that $\xi = \frac{1}{m}$.

In order to find a phase transition, we need to tune an external parameter (such as pressure, a magnetic field, temperature etc). We tune in such a way that as we approach the critical point, $\xi \to \infty$ and we observe critical behavior.

$\xi \to \infty$ means that we have no length scale, and thus scale invariance.

3.3 Phase Diagrams and Universality

[3] We can look at phase diagrams to learn more about critical points. For example, in the case of Ferromagnets, we want to study how the magnetic field changes with temperature. The critical transition is called the Curie transition in this case, and it is at T_c where CFTs become relevant.

***Insert Diagram: Phase B-T Diagram for Ferromagnet, $B \sim 0$ for $T \leq T_c$

In the region below T_c, when $B < 0$, $M < 0$ and at $B > 0$, $M > 0$. At $T = T_c$ a second order phase transition occurs, which has a singularity for $\frac{\partial^2 F}{\partial B^2}$ (hence the name *second* order). This is related to *magnetic susceptibility*. For a Ferromagnet, $T_c = 6000K$,

A similar thing can be observed in water, which has a second order phase transition at around $600K$ where the line between gas and liquid ends. This is called critical opalescence.

While these two systems appear very different, as long as they have the same basic degrees of freedom and symmetries, they will have the same critical exponent. This is called *Universality* and is one of the big successes of the Renormalization Group ideas.

[3] Not edited beyond spellcheck

In order to characterize a critical point, we need to determine several things.

3.4 Example: Ferromagnetization

Phase transitions are characterized by non-analyticities of macroscopic thermodynamic quantities (ie. temperature, density etc). Recall that such thermodynamic quantities are found by writing the partition function, summing over all possible configurations of the Hamiltonian,

$$Z = \sum_{states} e^{-\frac{H(S,B,P,..)}{T}} = e^{-\frac{F}{T}}. \tag{24}$$

This gives us the free energy F of the system.

For example, the magnetization M and magnetic susceptibility χ are defined as derivatives of the free energy,

$$M = -\frac{\partial F}{\partial B}, \quad \chi = \frac{\partial M}{\partial B} = -\frac{\partial^2 F}{\partial B^2} \tag{25}$$

Jumps in M characterize the first order phase transitions while χ is continuous.

Near the critical point the thermodynamic quantities and correlation length scale with specific exponents

$$\xi \sim (T - T_c)^{-v} \tag{26}$$

$$M \sim (T_c - T)^{\beta} \tag{27}$$

$$M \sim B^{\delta}. \tag{28}$$

The heat capacity c_v is given by

$$c_v \sim \frac{\partial E}{\partial T} \sim T\frac{\partial^2 F}{\partial T^2} \sim (T - T_c)^{-\alpha} \tag{29}$$

and diverges at the critical point.

At a the fixed point the scaling behavior of the two point functions gets an anomalous dimension η

$$\langle \sigma(x)\sigma(0)\rangle \sim \frac{1}{|x|^{D-2+\eta}} \tag{30}$$

Due to the power of CFT, all the critical exponents are determined by two of them ν and η. These have been experimentally observed.

3.5 Example: 3D Ising Model

The 3D Ising model Hamiltonian, with $\sigma_i = \{1, -1\}$ is given by the following nearest neighbor interaction term

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j. \tag{31}$$

This system has a critical point at $K = J/T$ and has a second order phase transition when $\sinh(2K_c) = 1$.

This is one of the few exactly solvable models in the statistical physics, we can also use symmetries and the RG flow to determine the physics. This model at the critical point is described by a two dimensional CFT which we'll study later. Different CFTs characterize different classes of Universality. Systems which has the same critical exponents are said to be in the same *Universality Class*.

4 Quantum Critical Points

4.1 Requirements for QCPs

Recall that Quantum Critical Points (QCPs) are zero temperature transitions with $\xi \to \infty$, driven by quantum fluctuations. The spectrum of excitation of the Hamilton, contains a length scale determined by the the energy gap Δ between E_0 and the first excited state. In general $\xi \sim \frac{1}{\Delta}$ and thus to observe quantum critical points, we requires that this energy gap $\Delta \to 0$.

If we want a QCP described by a 2 dimension CFT, then the quantum system must be in 1 dimension.

4.2 Example: Heisenberg Spin Chain

For example there are a QCP in the Anti-ferromagnetic Heisenberg spin chain in $D = 1$, which is a chain or loop of quantum spins whose Hamiltonian is given by

$$H = J \sum_i \vec{S}_i \vec{S}_{i+1} \tag{32}$$

where $S_i^a = \sigma_i^a$ are the Pauli spin matrices and $i = 1, ...N$, $a = 1, 2, 3$. This system has an SU(2) symmetry. This model was solved by Bethe exactly.

The critical behavior can be described in two dual ways

a) $D = 2$ free boson which takes values in a circle

$$\phi(x) \simeq \phi(x) + 2\pi R \tag{33}$$

This model exhibits a hidden symmetry when the radius $R = \frac{1}{\sqrt{2}}$ this symmetry is the SU(2) symmetry of the Heisenberg Spin Chain.

b) $D = 2$ SU(2) Wess-Zumino-Witten (WZW) model at level $k = 1$. (We may not get to this in the course - however it makes the SU(2) symmetry manifest)

4.3 Example: 1D Quantum Ising Model

The Quantum Ising model in $D = 1$ in a perpendicular magnetic field has a Hamiltonian given by

$$H = -J \sum_i \left(\sigma_i^z \sigma_{i+1}^z - g\sigma_i^x \right). \tag{34}$$

This system has a Quantum Critical Point at $g = 1$, when the system becomes gap-less, which will be described by a $D = 2$ CFT of free fermions.

5 More Reasons to Study CFTs

We now state even more reasons to study CFT

1. There is an important theorem that states that the maximal spacetime symmetry of a QFT is actually the conformal group (Coleman-Mandula Theorem). One has to be careful applying this theorem to massless fields. But in general, if the scattering S matrix is non-trivial, the maximal spacetime symmetry is given by the conformal group. We tend to learn most about physics by studying the most symmetric cases, thus to learn about QFT, we should study CFT.

2. There is a loophole in the previous theorem, if anti-commuting charges are allowed then the most symmetric theory is a *supersymmetric* CFT. We can't get beyond the superconformal group as symmetries of spacetime (Haag-Lopuszanski-Sohnius Theorem).

3. CFTs can be defined without referring to a Lagrangian. Knowing the spectrum of the theory, we can calculate the 3 point correlation functions $\langle\theta(x_1)\theta(x_2)\theta(x_3)\rangle$ which gives us a 'conformal bootstrap' to calculate everything. We can reduce the computation of a 4 point function to the product of 3 point functions. It is becoming clear that there are a large number of theories that do not have a Lagrangian description, and a current area of research resolves around how to deal with such theories. [4]

4. CFTs provide our best understanding of quantum gravity. This is in the framework of the AdS/CFT correspondence, the simplest and most well understood AdS/CFT correspondence is found in the Super-Yang Mills Theory in $D = 4$ and $N = 4$. Briefly this is studying gravity in a Anti-de-Sitter like background with periodic boundary conditions. It seems to be true that all the quantum gravity $D + 1$ phenomena inside the spacetime can be described exactly by a CFT in D dimensions on the *boundary* of the space.

5. CFT is also connected to perturbative string theory. The classical solutions of string theory involve the worldsheet of a string which lives a 2D QFT, and the solutions are described by $D = 2$ CFTs. By demanding that the theory on the world-sheet is a CFT we recover precisely (a generalization) of Einstein's Equations of GR.

6. $D = 2$ CFT describes the dynamics of relativistic string theories. The CFT lives in the embedding of the string in spacetime which is a surface (called world-sheet). To have a consistent theory we must have a CFT instead of a QFT. The spectrum of the string contains a spin two particle, the graviton. Demanding that the beta

[4] Pedro Vieira is an expert in this field.

function of the world-sheet QFT vanishes implies Einstein's equations plus higher order corrections.

7. Our best understanding of quantum gravity so far comes from the AdS_{D+1}/CFT_D correspondence. AdS is the maximally symmetric space with negative cosmological constant, this space has a boundary, light bounces at the boundary and comes back in a finite time. The conjecture is that the quantum theory of gravity in the bulk is described by a CFT in the boundary.

Part II

Basic Mechanics of Conformal Field Theories

6 Computing Observables

6.1 Overview

Our goal: to extract physics from CFT. For that we will

1. study the constraints that conformal symmetry imposes on observables
2. determine exactly the observables

We can compute correlation functions of local operators

$$\langle \theta_1(x_1)...\theta_n(x_n)\rangle \tag{35}$$

as we will see n-point correlators in CFT are completely determined in terms of 2 and 3 pt function. local operators are denoted by $\theta(x)$ where x is a point in spacetime. We can also study non-local operators denoted $O(\varepsilon)$, where ε is a sub-manifold, the simplest example is the Wilson loop. We will also study the effect of boundaries in the observables of a CFT's.

First we need to determine the set of all conformal transformations in flat spacetime, Minkowski or Euclidean, given by $\eta = diag(-,+,+,+)$ or $\eta = diag(+,+,+,+)$ respectively.

If a QFT has a global continuous symmetry, i.e. a Poincaré symmetry, that is associated with a particular transformation then Noether's theorem implies that there is a conserved current j^μ

$$\partial_\mu j^\mu = 0 \tag{36}$$

which in turn gives a conserved charge

$$Q = \int j^0 d^{D-1}x \tag{37}$$

We will eventually find the conserved currents associated with conformal symmetry which can be written in terms of the stress-energy tensor for the fields we are studying.

6.2 Transformations and Isometries

As a warm-up we'll recall the definition of isometries. Let D be the dimension of spacetime. Under a general coordinate transformation

$$x^\mu \to \tilde{x}^\mu(x) \tag{38}$$

the metric tensor $g_{\mu\nu}$ transforms as

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x(\tilde{x})) \tag{39}$$

For a general spacetime its isometries are the transformations that leave the metric invariant, which means

$$g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x) \tag{40}$$

or equivalently

$$g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu \tag{41}$$

for flat Minkowski metric the isometry group is called the Poincaé group. To figure out if a coordinate transformation is an isometry it's possible to use local methods which are encoded in differential equations obeyed by the Killing vector fields.

6.3 The Killing Equation

Consider an infinitesimal coordinate transformation

$$\tilde{x}^\mu = x^\mu + v^\mu(x). \tag{42}$$

which preserves the flat spacetime metric

$$\eta_{\mu\nu} = \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} \eta_{\rho\sigma} \tag{43}$$

We want to ascertain all possible isometries of flat spacetime.

Note that to leading order we can invert the x transformation to compute the derivatives

$$\tilde{x}^\mu = x^\mu + v^\rho(x), \quad \Longrightarrow \quad x^\rho = \tilde{x}^\rho - v^\rho(\tilde{x}) \tag{44}$$

which gives

$$\eta_{\mu\nu} = \left(\delta^\rho_\mu - \partial_\mu v^\rho\right)\left(\delta^\sigma_\nu - \partial_\nu v^\sigma\right)\eta_{\rho\sigma} \tag{45}$$

$$= \eta_{\mu\nu} - \partial_\mu v_\nu - \partial_\nu v_\mu \tag{46}$$

$$0 = \partial_\mu v_\nu + \partial_\nu v_\mu \tag{47}$$

These are the Killing equations for flat spacetime. In case of a general metric the partial derivative of a vector field is not a tensor so in general we have the *Killing Equation* for curved manifolds

$$\nabla_\mu v_n + \nabla_\nu v_\mu = 0 \tag{48}$$

where we call v_μ the Killing vector, and

$$\nabla_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\lambda{}_{\mu\nu} v_\lambda \tag{49}$$

is the covariant derivative.

Going back to the flat spacetime, it is easy to be convinced that the solutions of the equation $\partial_{(\mu} v_{\nu)} = 0$ are of the form

$$v^\mu = a^\mu + \omega^{\mu\nu} x_\nu \tag{50}$$

where a^μ is a constant and $\omega^{\mu\nu} = -\omega^{\nu\mu}$, taken together these generate the Poincaré transformations

$$a^\mu : x^\mu \to \quad x^\mu + a^\mu \quad \text{(Translation)} \tag{51}$$

$$\omega^{\mu\nu} : x^\mu \to \quad x^\mu + \omega^\mu{}_\nu x^\nu \quad \text{(Lorentz)} \tag{52}$$

The Lorentz transformations are elements of the group $O(1,3)$, in QFT we usually take that the symmetry group is the connected part of this group $SO_+(1,3)$ called the proper orthochronous Lorentz group. The full group can be retrieved by adding space parities and time reversal transformations. On top of that we also require QFTs to be invariant under the CPT transformation.

We have an associated Lie algebra called the Poincaré algebra obeyed by the generators of the group. The Lie bracket is given by the commutator of vector fields. The vector fields which correspond to the generators are

$$P^\mu = -i\partial^\mu \tag{53}$$

$$M^{\mu\nu} = -i(x^\mu \partial^\nu - x^\nu \partial^\mu) \tag{54}$$

with commutator relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\sigma} M^{\mu\rho}) \tag{55}$$

$$[M^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu). \tag{56}$$

To get the full elements of the group (at least the ones close to the identity) we exponentiate the generators

$$g = e^{-ia_\mu P^\mu + \frac{i}{2} w_{\mu\nu} M^{\mu\nu}} \tag{57}$$

this is a finite element of the Poincaré group.

6.4 The Conformal Killing Equation

Consider flat spacetime and again do an infinitesimal coordinate transformation that is connected to the identity,

$$x^\mu \to \tilde{x}^\mu = x^\mu + v^\mu(x) \tag{58}$$

but now we require that

$$\eta_{\rho\sigma}d\tilde{x}^\rho d\tilde{x}^\sigma = \Omega^2(x)\eta_{\mu\nu}dx^\mu dx^\nu \tag{59}$$

where $\Omega(x)$ is called the conformal factor. When $\Omega(x) = 1$ we are back to the previous case of isometries.

Conformal transformation is local Lorentz with a dilatation, as we can define

$$\Lambda_\mu^{\ \nu}(x) = \frac{\partial x^\nu}{\partial \tilde{x}^\mu}\Omega(x)$$

It is easy to show,

$$\Lambda^T \eta \Lambda = \eta$$

Expanding $\Omega^2(x) \equiv e^{2\sigma(x)} = 1 + 2\sigma(x)$ we find that the Killing Equation from before now has a non-zero RHS

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\sigma(x)\eta_{\mu\nu} \tag{60}$$

By taking the trace, and denoting $\partial \cdot v = \partial_\lambda v^\lambda$, we find that

$$2\partial \cdot v = 2D\sigma(x) \ \rightarrow \qquad \sigma(x) = \frac{1}{D}\partial \cdot v \tag{61}$$

then the Conformal Killing Equation is

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{D}\eta_{\mu\nu}\partial \cdot v \tag{62}$$

we call v^μ a *conformal Killing vector*.

In curved spacetime, this generalizes to

$$\nabla_\mu v_\nu + \nabla_\nu v_\mu = \frac{2}{D}g_{\mu\nu}(\nabla \cdot v). \tag{63}$$

However we will focus on flat spacetime, as the Mikowski spacetime is one of the spacetimes with the most solutions to the Killing equation.

7 Solutions to the Killing Equation

7.1 More equations from the Killing Equation

We can derive other differential equations from the Killing equation

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{D}\eta_{\mu\nu}\partial \cdot v \tag{64}$$

The two equations below will be useful in solving the Killing equation

1. Act with ∂^ν to find

$$\begin{aligned} \partial_\mu(\partial\cdot v)+\Box v_\mu &= \frac{2}{D}\partial_\mu\left(\partial\cdot v\right) &(65)\\ \Box v_\mu+\left(\frac{D-2}{D}\right)\partial_\mu\left(\partial\cdot v\right) &= 0 &(66)\end{aligned}$$

2. And with ∂^μ again to find

$$(D-1)\Box\cdot(\partial\cdot v)=0 \tag{67}$$

7.2 2 Dimensional Properties

We notice that $D=2$ is special because we can write the conformal Killing equations as simply

$$\Box v_\mu=0 \tag{68}$$

whose solutions, for Euclidean metric, are given by harmonic functions which satisfy the Cauchy-Riemann equations

$$\begin{aligned} \partial_1 v_1 &= \partial_2 v_2 &(69)\\ \partial_1 v_2 &= -\partial_2 v_1. &(70)\end{aligned}$$

We can see this by going over to complex coordinates

$$z=x^1+ix^2, \qquad v=v^1+iv^2 \tag{71}$$

and the above condition becomes identical to

$$\bar{\partial}_{\bar{z}}v=0 \tag{72}$$

which expresses the equivalence of local conformal transformations in $D=2$ and holomorphic mappings. Since any holomorphic function defines a conformal transformation we have in a sense an infinite dimensional symmetry. We will later see that the Lie-algebra generated by such functions is infinite dimensional and it's called the Witt algebra.

7.3 General Solution In $D \neq 2$

One of the equations derived from the Killing equation was $\Box(\partial \cdot v) = 0$, which means that $\partial \cdot v$ is at most linear

$$\partial \cdot v = A + B_\mu x^\mu \tag{73}$$

throwing this back in the Killing equation we have

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \tfrac{2}{D}\eta_{\mu\nu}(A + B_\mu x^\mu) \tag{74}$$

$$\implies \partial_\rho \partial_\lambda(\partial_\mu v_\nu + \partial_\nu v_\mu) = 0. \tag{75}$$

Relabeling the dummy indices

$$\partial_\rho \partial_\lambda \partial_\mu v_\nu + \partial_\rho \partial_\lambda \partial_\nu v_\mu = 0 \tag{76}$$

$$\partial_\mu \partial_\lambda \partial_\rho v_\nu + \partial_\mu \partial_\lambda \partial_\nu v_\rho = 0 \tag{77}$$

$$\partial_\nu \partial_\lambda \partial_\rho v_\mu + \partial_\nu \partial_\lambda \partial_\mu v_\rho = 0 \tag{78}$$

$$\tag{79}$$

summing the first two equations and subtracting the last we have

$$\partial_\rho \partial_\lambda \partial_\mu v_\nu = 0 \tag{80}$$

which implies that the vector v^ν is at most quadratic in x.

Let us parameterize the most general solution as

$$v^\mu \;=\; a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + x^2 b^\mu - 2x^\mu x_\lambda b^\lambda. \tag{81}$$

We can identify our friends translations, a^μ, and Lorentz transformations, $\omega^\mu{}_\nu$, which generates the Poincaré group. The other parameters generate the rest of the conformal group, λ is a scale transformation (dilatation) and b^μ is a special conformal transformation.

This is in striking difference from the two dimensional case where the space of solutions was infinite dimensional, here it is parameterized by $\{a^\mu, \omega^{\mu\nu}, \lambda, b^\mu\}$, in four dimensions there are 15 components.

For D dimensions the conformal group is $\frac{1}{2}(D+2)(D+1)$ dimensional.

8 Conformal Algebra

Just a reminder, the conformal transformations are

1. Translations: $a^\mu \to P^\mu$, translations transform our space as $x^\mu \to x^\mu + a^\mu$
2. Lorentz: $\omega^{\mu\nu} \to M^{\mu\nu}$; coordinates transform as $x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$
3. Scale: $\lambda \to \hat{D}$; coordinates transform as $x^\mu \to (1+\lambda)x^\mu$
4. Special conformal: $b^\mu \to K^\mu$; coordinates transform as $x^\mu \to x^\mu + b^\mu x^2 - 2x^\mu x \cdot b$

They are easy to visualize except for the special conformal transformation since they act non-linearly on the coordinates.

8.1 Generators of the Algebra

From the conformal Killing vector field

$$v^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + x^2 b^\mu - 2x^\mu b \cdot x. \tag{82}$$

we can deduce the form of the generators of the Poincaré group acting as derivations on the space functions.

$$\begin{aligned} \delta f(x) &= \tilde{f}(x) - f(x) = -v^\mu \partial_\mu f & (83) \\ &= \left(-ia^\mu P_\mu - ib^\mu K_\mu + \frac{i}{2}\omega^{\mu\nu} M_{\mu\nu} + i\lambda \hat{D}\right) f & (84) \end{aligned}$$

Thus we write down the generators

$$\begin{aligned} P^\mu &= -i\partial^\mu & (85) \\ M^{\mu\nu} &= -i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) & (86) \\ \hat{D} &= ix^\mu \partial_\mu & (87) \\ K^\mu &= -i\left(x^2 \partial^\mu - 2x^\mu x^\nu \partial_\nu\right) & (88) \end{aligned}$$

.

8.2 Commutators of the Algebra

Using these, we can now work out the commutators for these quantities by computing the commutator of the correponding vector fields. Then we find

$$\begin{aligned} [M^{\mu\nu}, P^{\rho}] &= i\left(\eta^{\mu\rho}P^{\nu} - \eta^{\nu\rho}P^{\mu}\right), & (89) \\ [M^{\mu\nu}, M^{\rho\sigma}] &= i\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho}\right). & (90) \end{aligned}$$

for the Poincaré algebra. We also have

$$\begin{aligned} [M^{\mu\nu}, \hat{D}] &= 0 & (91) \\ \left[\hat{D}, P^{\mu}\right] &= -iP^{\mu} & (92) \\ \left[\hat{D}, K^{\mu}\right] &= iK^{\mu} & (93) \end{aligned}$$

We can calculate

$$\begin{aligned} \left(\hat{D}P^{\mu}\right)f - \left(P^{\mu}\hat{D}\right)f = & \quad x^{\nu}\partial_{\nu}\partial^{\mu}f - \partial^{\mu}\left(x^{\nu}\partial_{\nu}f\right) & (94) \\ = & \quad x^{\nu}\partial_{\nu}\partial^{\mu}f - \partial^{\mu}f - x^{\nu}\partial^{\mu}\partial_{\nu}f = -\partial^{\mu}f = -iP^{\mu}f & (95) \end{aligned}$$

We also find the final commutators

$$\begin{aligned} [M^{\mu\nu}, K^{\rho}] &= i\left(\eta^{\mu\rho}K^{\nu} - \eta^{\nu\rho}K^{\mu}\right) & (96) \\ [P^{\mu}, K^{\nu}] &= -2i\left(\eta^{\mu\nu}\hat{D} + M^{\mu\nu}\right) & (97) \end{aligned}$$

The commutators between $\hat{D}, P^{\mu}, K^{\mu}$ reminds us of the harmonic oscillator of quantum mechanics operators which behave similarly $[H, a] = -a$ and $[H, a^{\dagger}] = a^{\dagger}$.

The conformal algebra is generated by the Poincaré generators plus the new ones

$$\hat{D} = ix \cdot \partial \tag{98}$$

$$K^{\mu} = -i(x^2\partial^{\mu} - 2x^{\mu}x \cdot \partial) \tag{99}$$

$$P^{\mu} = -i\partial^{\mu} \tag{100}$$

$$M^{\mu\nu} = -i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) \tag{101}$$

with the commutator relations

$$[M^{\mu\nu}, K^\rho] = i(\eta^{\mu\rho}K^\nu - \eta^{\nu\rho}K^\mu) \tag{102}$$

$$[M^{\mu\nu}, \hat{D}] = 0 \tag{103}$$

$$[\hat{D}, P^\mu] = -iP^\mu \tag{104}$$

$$[\hat{D}, K^\mu] = iK^\mu \tag{105}$$

$$[P^\mu, K^\nu] = -2i(\eta^{\mu\nu}\hat{D} + M^{\mu\nu}) \tag{106}$$

which tells us that K^μ transforms as a vector and $\hat{D}$ as a scalar.

The commutator $[\hat{D}, P^\mu]$ tells us that P^μ is like a creation operator, it raises the scaling dimension while K^μ is like a lowering operator for the "Hamiltonian" $\hat{D}$. This can be seen by simply notice that under scaling transformation $x \to \lambda x$, translation $P^\mu \sim \partial_x \to \lambda^{-1}\partial_x$ and special conformal $K^\mu \sim x^2\partial_x \to \lambda x^2\partial_x$.

The conformal algebra can be used to to

1. Bound the spectrum of $\hat{D}$, using also unitarity $\Delta \geq \frac{D-2}{2}$.

2. Constrain the form of the correlation functions.

8.3 Commutators of the vector field leads to conformal algebra

We can construct the following vector field from the Killing vector,

$$v = v^\mu \partial_\mu$$

Then we can deduce the conformal algebra, by calculating

$$[v_1, v_2] = v_3$$

or

$$v_3^\mu = v_1^\nu \partial_\nu v_2^\mu - v_2^\nu \partial_\nu v_1^\mu$$

For example, for Poincaré transformation, we have translation vector field,

$$v_P = a^\mu \partial_\mu$$

and rotation vector field,

$$v_M = w^\lambda_\nu x^\nu \partial_\lambda$$

It is clear that

$$[v_{P1}, v_{P2}] = 0$$

which we can deduce the generators for translations must commute

$$[P^\mu, P^\nu] = 0$$

It is easy to show that

$$[v_M, v_P] = a^\mu {w_\mu}^\lambda \partial_\lambda = v_{P3}$$

where $a_3^\lambda = a^\mu w_\mu^\lambda$, taking into account the anti-symmetry of w, we can deduce the commutator of the Lorentz generator and translation generator,

$$[M^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu$$

The i is just a convention how generators are defined. Another example with dilatation

$$v_D = \lambda x^\mu \partial_\mu$$

so

$$[v_D, v_P] = -a^\nu \lambda \partial_\nu = -v_{P'}$$

where $a'^\nu = a^\nu \lambda$, and the commutator is

$$\hat{D}, P^\mu = -iP^\mu$$

8.4 AdS$_{D+1}$ Correspondence

From the commutator relations we can find an isomorphism between the conformal algebra in D spacetime dimensions and orthogonal groups in D+2 dimensions, $SO(2, D)$ for Minkowski signature or $SO(1, D+1)$ for Euclidean.

Just like the Poincaré are isometries of Minkowski space, $SO(2, D)$ are isometries of AdS$_{D+1}$.

For $SO(2,D)$ or $SO(1,D+1)$ we can define L_{MN} where $M,N=0,...,D-1,D,D+1$ where the first D entries correspond to Greek letter μ such that $M=(\mu,D,D+1)$. We have

$$[L_{MN},L_{PQ}]=i\left(\eta_{MP}L_{NQ}+...\right) \tag{107}$$

We find now

$$\begin{aligned} L_{\mu\nu} &= M_{\mu\nu} && (108)\\ L_{D+1\,D} &= \hat{D} && (109)\\ L_{\mu D} &= \frac{1}{2}\left(P_\mu+K_\mu\right) && (110)\\ L_{\mu D+1} &= \frac{1}{2}\left(P_\mu-K_\mu\right) && (111)\end{aligned}$$

which we identify with the orthogonal transformations $y^M \to \Lambda^M_N y^N$ and $\Lambda^T\eta\Lambda=\eta$. These are also the isometries of AdS space. This is at the heart of the AdS$_{D+1}$ correspondence.

The number of generators of SO(2,D) is

$$\underbrace{\frac{1}{2}(D+2)(D+1)}_{L_{MN}} \tag{112}$$

The number of generators of conformal algebra is

$$\underbrace{\frac{1}{2}D(D-1)}_{M_{\mu\nu}}+\underbrace{D}_{P_\mu}+\underbrace{1}_{\hat{D}}+\underbrace{D}_{K_\mu}=\frac{1}{2}(D+2)(D+1) \tag{113}$$

as before.

Consider now a $\mathbb{R}^D$ plane

$$\begin{aligned} ds^2 &= dr^2+r^2d\Omega^2_{D-1} && (114)\\ &= r^2\left[\frac{dr^2}{r^2}+d\Omega^2_{D-1}\right] && (115)\end{aligned}$$

where Ω_{D-1} is the metric of $D-1$ sphere $S^{(D-1)}$.

We see that CFT in a $\mathbb{R}^D$ plane is equivalent to CFT in $R\times S^{D-1}$, a cylinder since they are conformally equivalent. The operator

$$\hat{D}\sim r\frac{\partial}{\partial r}\sim\frac{\partial}{\partial t}. \tag{116}$$

can be interpreted as a “Hamiltonian” that gives the time evolution.

8.5 Discrete Transformations

We have classified all infinitesimal transformations in flat space

$$x^\mu \to \tilde{x}^\mu = x^\mu + v^\mu \tag{117}$$

such that

$$\eta_{\rho\sigma}\frac{\partial\tilde{x}^\rho}{\partial x^\mu}\frac{\partial\tilde{x}^\sigma}{\partial x^\nu} = \Omega^2(x)\eta_{\mu\nu} \tag{118}$$

where $\Omega(x) = 1 + \sigma(x)$. which are called conformal transformations. The vector field that generates these transformations obeys the conformal Kiling vector equation (CKV)

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{D}\eta_{\mu\nu}\left(\partial \cdot v\right) \tag{119}$$

and $\sigma(x) = \frac{1}{D}(\partial \cdot v)$. and the most general solution was

$$v^\mu \quad = \quad a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + x^2 b^\mu - 2x^\mu b \cdot x \tag{120}$$

with the respective arbitrary real constants. From this we can identify

$$\omega(x) = \lambda - 2b \cdot x. \tag{121}$$

There are other transformations which satisfy the condition for being a conformal transformation but are not continuous so they are not encoded by the conformal Killing vectors. These includes the well know parity and time-reversal transformations.

To them we add another discrete transformation the *inversion*

$$x^\mu \to \tilde{x}^\mu = \frac{x^\mu}{x^2} \tag{122}$$

This transformation reverses the orientation of the space so it cannot be connected to the identity.

Note that we often need to add points at infinity in order to have a globally well defined transformation, in Euclidean signature it is enough to add a point at infinity to which the origin maps, in Minkowski signature we need to add a whole lightcone at infinity which is exchanged with the lightcone at the origin.

This transformation has the property that performing it twice returns to the identity. The conformal transformations can be generated by combining Poincaré transformations with the inversion.

Note that we always need an even number of inversions to get a transformation connected to the Identity. For example conjugating the momentum we get the generator $IP^\mu I$ which induces the transformation

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + a^2x^2 + 2a \cdot x} \tag{123}$$

which for small a^μ this transformation reduces to $x^\mu + a^\mu x^2 - 2x^\mu a \cdot x$ which is the infinitesimal form of the special conformal transformation.

What is the conserved currents that corresponds to conformal generators? It should be,

$$j_\mu(\xi) = \xi^\nu T_{\mu\nu}$$

To prove it is conserved, we have

$$\partial^\mu j_\mu = \xi^\nu \partial^\mu T_{\mu\nu} + \frac{1}{2} T_{\mu\nu}(\partial^\mu \xi^\nu + \partial^\nu \xi^\mu) \propto T_\mu \nu \eta^\mu \nu (\partial \cdot \xi) = 0$$

For example, we have the scaling current

$$j_\mu^D = T_{\mu\nu} x^\nu$$

We can define charge

$$Q = \int d^{D-1}x j^0$$

$$j_0^D = T_{0\nu} x^\nu$$

Thus dilatation charge is defined to be

$$D \equiv \int d^{D-1}x T_{0\nu} x^\nu$$

and the translation charge, i.e., momentum is

$$P_\mu \equiv \int d^{D-1}x T_{0\mu}$$

Poison bracket generates scale transformation of field.

8.6 Inversion

Inversion transformation I:

$$\tilde{x}^\mu = \frac{x^\mu}{x^2}$$

exchange 0 with ∞. We have

$$I^2 = \mathbf{1}$$

So inversion is a Z_2 transformation. Conformal group is generated by Poincaré transformations and inversions. An inversion followed by translation and then followed by inversion gives special conformal transformation,

$$IP_\mu I : x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2} \sim x^\mu + b^\mu x^2 - 2b \cdot x x^\mu$$

which gives us the familiar infinitesimal symmetry. We can get scaling transformation through the commutator of the translation and the special conformal transformation.

Conformal group has four disconnected components which are related by discrete transformation like parity and time reversal. All CFTs have the $SO(2,4)_+$ which is connected to identity as their symmetry group, denote the transformation by Λ. Some other theories are also invariant under $I\Lambda$ which is parity P. $T\Lambda$, which is time reversal. and $IT\Lambda$. Provide embedding Minkowski to higher dimensional, the non-linear conformal transformations follow from the linear transformations from the higher dimensions.

8.7 More about symmetry

The conformal group is the biggest spacetime symmetry group in four dimensions allowed for non-trivial (S-matrix $\neq 1$) QFTs.

This is the Coleman-Mandula theorem. One of the assumptions of this theorem is that the generators of the symmetry are bosonic. By allowing fermionic generators we can get a bigger spacetime symmetry group which is the superconformal group, there can be no more extensions of the spacetime symmetry group (Haag-Lopuszanski-Sohnius theorem). For spin $= 1$ the maximally supersymmetric theory in four dimensions is $\mathcal{N} = 4$ super Yang-Mills.

The conformal transformations are non-linearly realized in Minkowski space and we are used to linear transformations. We can do a trick, by embedding Minkowski space in $\mathbb{R}^{2,D}$ the conformal transformations will be linearly realized in this space and by restricting to the hypersurface we get back to Minkowski space.

8.8 Form of the Energy momentum tensor from symmetry argument

CFT is a theory that has Poincaré symmetry, which tell us that the energy momentum tensor is conserved and can be made symmetric

$$\partial^\mu T_{\mu\nu} = 0,\ T_{\mu\nu} = T_{\nu\mu}$$

What properties must $T_{\mu\nu}$ have so that the theory is dilatation invariant and special conformal invariant? We can find the energy momentum tensor by looking at the variation of action respect to the metric,

$$\delta S = \frac{1}{2}\int T_{\mu\nu}\delta h^{\mu\nu}$$

where

$$\delta h^{\mu\nu} = -\partial^\mu v^\nu - \partial^\nu v^\mu$$

is the induced metric change by a conformal Killing vector.

Consider diffeomorphism, we have conformal Killing equation

$$\partial^\mu v^\nu + \partial^\nu v^\mu = \frac{1}{2}(\partial \cdot v)\eta^{\mu\nu}$$

Thus we have

$$\delta S = \frac{1}{4}\int T_{\mu\nu}(\partial \cdot v)\eta^{\mu\nu} = \frac{1}{4}\int T^\mu_\mu(\partial \cdot v)$$

If the theory is scale invariant, then we have $(\partial \cdot v) =$ constant, Thus we need $\int T^\mu_\mu = 0$, in other words, we need the trace to be a surface term $T^\mu_\mu = \partial_\mu L^\mu$. Now consider the theory is special conformal invariant, then we have $(\partial \cdot v) \propto x$, thus we need $\int T^\mu_\mu x = 0$, using integrate by parts, we need the trace to be $T^\mu_\mu = \partial_{\mu\nu} L^{\mu\nu}$. In CFT, we can always add improvement terms, such that $\tilde{T}^\mu_\mu = 0$. Note this will not spoil the conserve property, as current

$$\hat{j}_\mu \equiv j_\mu + \partial^\rho O_{[\rho\mu]}$$

For $T_{\mu\nu}$, we can have improvement term of

$$\hat{T}_{\mu\nu} = T_{\mu\nu} + \partial^\rho \partial^\sigma Y_{\mu\rho\nu\sigma}$$

where $Y_{\mu\rho\nu\sigma}$ has symmetry of Riemann tensor. We can get terms like this by including non-minimal coupling to the background metric.

$$\mathcal{L}_{\text{flat}} \to \mathcal{L}_{\text{covariant}} + \int R_{\mu\rho\nu\sigma} Y_{\mu\rho\nu\sigma}$$

8.9 Comments on anomalous dimensions

Anomalous dimensions show up as wave function renormalization of operators. (This is even true in quantum mechanics.) Consider n-point correlation function $G_n p, \lambda, \mu$, we have

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n\gamma\right) G_n p, \lambda, \mu = 0$$

To compensate the change due to change of scale, the coupling constant needs to change, and we need to renormalize the wave function:

$$\phi_0 = Z\phi$$

and the anomalous dimension is

$$\gamma \propto \frac{1}{Z} \mu \frac{dZ}{d\mu}$$

At a fixed point of RG

$$\beta(\lambda^*) = 0$$

Thus the equation simplifies, taking 2-pt function as an example,

$$\left(\mu \frac{\partial}{\partial \mu} + 2\gamma(\lambda^*)\right) G_2 p, \lambda, \mu = 0$$

Solve this differential equation, we have

$$G_2 = \frac{1}{p^2} \left(\frac{p^2}{\mu^2}\right)^{\gamma(\lambda^*)}$$

change momentum dependence by γ_0 Take the Fourier transform, we have

$$\langle\phi(x)\phi(0)\rangle = G_2(x) \sim \int d^D p e^{ip\cdot x}\frac{1}{p^{2-2\gamma(\lambda^*)}} \sim \frac{1}{x^{D-2-2\gamma(\lambda^*)}}$$

In Gaussian free theory the correlation function is given by $\frac{1}{x^{D-2}}$, now with interactions, it is modified to be $\frac{1}{x^{D-2-2\gamma(\lambda^*)}}$, that is why γ is known as the anomalous dimension.

In this case with scaling transformation $x^\mu \to \lambda x^\mu$, the scalar field changes in the following way,

$$\hat{\phi}(x) = \lambda^{-\frac{D-2}{2}-\gamma(\lambda^*)}\phi(\lambda^{-1}x)$$

conformal invariance point

9 Conserved Currents and Charges

Noether has told us that if our theory has a continuous symmetry then associated to it there is a conserved current. How does that works for conformal symmetry?

9.1 Wee bit of Classical Mechanics

Our physical systems are usually specified by an action[5]

$$\mathcal{S} = \int \mathcal{L}(\phi(x), \partial_\mu\phi(x))d^D x \tag{124}$$

We are going to omit the field dependence of x from now on. The equations of motion are given by varying the Lagrangian with respect to the field

$$0 = \delta S = \int d^D x \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \tag{125}$$

where we kept explicitly the total derivative. And the equation of motion is given by

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0 \tag{126}$$

To construct the Noether current we vary the action with respect to the symmetry, which takes the infinitesimal form of

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\delta\phi(x) \tag{127}$$

[5] In principle we can have more than one fields, but we will simplify our discussion here by assuming there is only one field.

where α is an infinitesimal parameter. The change of the Lagrangian must be invariant up to a 4-divergence,

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu \mathcal{K}^\mu \tag{128}$$

We find then the conserved current is given by

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - \mathcal{K}^\mu & (129) \\ \partial_\mu j^\mu &= -\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right) \delta\phi & (130) \end{aligned}$$

This current is conserved on-shell

$$\partial_\mu j^\mu = 0 \Rightarrow \text{there is a conserved charge Q} \tag{131}$$

with

$$Q \equiv \int j^0 d^{D-1}x \tag{132}$$

9.2 Conserved Currents

For each generator in the conformal Killing vector v^μ, there is an associated operator.

$$\begin{aligned} \text{Translation} \quad a^\mu : &\quad P_\mu & (133) \\ \text{Lorentz} \quad \omega^\mu{}_\nu : &\quad M_{\mu\nu} & (134) \\ \text{Scale Transformation} \quad \lambda : &\quad \hat{D} & (135) \\ \text{Special Conformal Transformation} \quad b^\mu : &\quad K_\mu & (136) \end{aligned}$$

we can associate a conserved charge.

Since those are all continuous transformations they have a conserved current associated to them. We should be familiar with the stress-energy tensor $T_{\mu\nu}$, it measures the variation of our system in response to a small change in the metric. Which means that it should encode the conserved currents for the symmetries.

For example take a translation $x^\mu \to x^\mu + a^\mu \Rightarrow j^\mu = T^{\mu\nu}_{(c)} a_\nu$. This is the canonical stress energy tensor

$$T^{(c)}_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L} \tag{137}$$

it is usually not symmetric or gauge invariant.

From GR we have another notion of stress-energy tensor which is the variation of the action with respect to the metric

$$\delta S = \frac{1}{2}\int d^D x\sqrt{g}T_{\mu\nu}\delta g^{\mu\nu} \tag{138}$$

which we will take to be the definition of the (Belifante-Rosenfeld) stress-energy tensor[6] To require the total change in action vanish, we have

$$\delta S = 0 = -\frac{1}{2}\int d^D x\delta g_{\mu\nu}\left(T^{\mu\nu} + 2\frac{\delta S}{\delta g_{\mu\nu}}\right) \tag{139}$$

where the change of the metric is due to the coordinate transformation,

$$\delta g_{\mu\nu} = -(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \tag{140}$$

So we have the energy momentum tensor as (evaluated in flat space)

$$T_{\mu\nu} = -2\frac{\delta S}{\delta g^{\mu\nu}}. \tag{141}$$

We can thus write the conserved currents as

$$j^\mu = T^{\mu\nu}v_\nu. \tag{142}$$

For example, for translation $v_\nu = a_\nu$, we have

$$P^\mu = T^{\mu\nu}a_\nu \tag{143}$$

Notice that, contrary to the canonical stress-energy tensor, the Belinfante-Rosenfeld stress-energy tensor is always symmetric by definition.

For Lorentz transformations some barbarians take the parameter $\omega^{\mu\nu}$ out of the current and define the three indice tensor

$$\mathcal{M}^{\mu\nu\lambda} = T^{\mu\nu}x^\lambda - T^{\mu\lambda}x^\nu. \tag{144}$$

[6]which agrees with Hilbert stress-energy tensor.

9.3 The Stress-Energy Tensor

We can take the divergence of the current

$$\partial_\mu j^\mu_{(v)} = \partial_\mu T^{\mu\nu} v_\nu + T^{\mu\nu}\partial_\mu v_\nu \tag{145}$$

which we expect to vanish.

1. If v^μ is an isometry (Poincaré transformation). We can write (using the symmetric property of $T^{\mu\nu}$ and the Killing equation)

   $$T^{\mu\nu}\partial_\mu v_\nu = T^{\mu\nu}\frac{1}{2}\left(\partial_\mu v_\nu + \partial_\nu v_\mu\right) = 0 \tag{146}$$

   The current is conserved implies that

   $$\partial_\mu T^{\mu\nu} = 0. \tag{147}$$

   Thus diffeomorphism requires that $T^{\mu\nu}$ is divergence free.

2. if v^μ is a conformal transformation

   $$\partial_\mu j^\mu_{(v)} = T^{\mu\nu}\frac{1}{2}\left(\partial_\mu v_\nu + \partial_\nu v_\mu\right) \tag{148}$$

   The fact that the current is conserved implies

   $$T^{\mu\nu}\eta_{\mu\nu} = T^\mu_{\ \mu} = 0. \tag{149}$$

   Thus Conformal symmetry requires that $T_{\mu\nu}$ is traceless.

1. Invariance under special conformal transformations $\Leftrightarrow$ $T^\mu_{\ \mu} = \partial_\nu\partial_\lambda X^{\nu\lambda}$ for some $X^{\nu\lambda}$. In this case we can cook-up an improvement term to make it traceless.

2. Invariance under scale transformationa $\Leftrightarrow$ $T^\mu_{\ \mu} = \partial_\mu K^\mu$ for some K^μ. In this case we can also cook-up an improvement term to make it traceless.

(1)$\Rightarrow$(2) but the inverse is not necessary true in dimensions other than 2 (though we like to believe in it).

$T_{\mu\nu}$ will be playing a crucial role in our work, as it is the object that tells us about spacetime.

9.4 Relations to QFTs

CFTs are QFTs that have conformal symmetry, which has a conserved current given by

$$j^\mu = T^{\mu\nu} v_\nu, \quad \partial_\mu j^\mu = 0 \tag{150}$$

Recall we had

$$T^{\mu\nu} = -2\frac{\delta S}{\delta g^{\mu\nu}} \tag{151}$$

which is called the Belinfante-Rosenfeld (BR) stress-energy tensor. Let us consider how this expression compares to the traditional expression for $T_{\mu\nu}$ found by Noether's procedure in QFTs, that is

$$T^{(\mathrm{Can})}_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial^\mu\varphi)}\partial_\nu\varphi - \eta_{\mu\nu}\mathcal{L} \tag{152}$$

One important difference is that the CFT expression is symmetric and gauge invariant where as the old expression is not!

Let us introduce the *improved* current

$$\hat{j}^\mu = j^\mu + \partial_\nu V^{\nu\mu} \tag{153}$$

with the antisymmetric $V^{\nu\mu} = -V^{\mu\nu}$. We note that if $\partial_\mu j^\mu = 0$ then $\partial_\mu \hat{j}^\mu = 0$, and the corresponding conserved charge is

$$\hat{Q} = \int d^{D-1}x \hat{j}^0 = \int d^{D-1}x j^0 = Q \tag{154}$$

By a similar procedure, we can construct the BR tensor from the canonical as

$$T_{\mu\nu} = T^{(\mathrm{c})}_{\mu\nu} + \partial^\rho V_{\rho\mu\nu} \tag{155}$$

such that $V_{\rho\mu\nu} = -V_{\mu\rho\nu}$.

An explicit form for the improvement term $V_{\rho\mu\nu}$ is

$$V^{\rho\mu\nu} = \frac{1}{2}(B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu}) \tag{156}$$

$$B^{\mu\nu\rho} = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}(M^{\nu\rho})_A{}^B\phi_B(x) \tag{157}$$

where $(M^{\nu\rho})_A^{\ B}$ is the matrix for the representation of the Lorentz group under which the field ϕ transforms.[7]

The BR stress tensor has some advantages over the canonical stress tensor

- Easy to Calculate,
- Symmetric $T_{\mu\nu} = T_{\nu\mu}$
- Couples to gravity

We can now calculate

$$\delta S = \frac{1}{2} \int d^D x \sqrt{g} T_{\mu\nu} \delta g^{\mu\nu} \tag{158}$$

with the scale transformation $\delta g^{\mu\nu} = \lambda g^{\mu\nu}$

$$\delta S \propto \int d^D x \sqrt{g}\, T_\mu^{\ \mu} \tag{159}$$

The stress tensor is traceless. Thus from here on, we shall use only traceless Ts. So the two stress-tensors are equivalent so we'll stick to the BR stress-energy tensor.

We can see the relation between these tensors as coming from additional terms in the action when written for curved spacetime

$$\mathcal{L}_f \quad \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 \rightarrow \text{ flat space Lagrangian} \tag{160}$$

$$\mathcal{L}_c \quad \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 + R\phi^2 \rightarrow \text{ curved space Lagrangian} \tag{161}$$

the difference between them is a term proportional to the Ricci scalar which is called the conformal coupling. If we restrict the second action to flat metrics we get back to the original Lagrangian. [8].

For ϕ^4 in four dimension the stress-energy tensor is

$$T^{\mu\nu}_{(c)} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \left(\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4!}\phi^4 \right) \tag{162}$$

we can calculate the trace

$$T^{\mu}_{\mu(c)} = -(\partial\phi)^2 + \frac{\lambda}{3!}\phi^4 \tag{163}$$

[7] For further discussion on BR tensor, see Weinberg, "The quantum theory of fields", ch 7.4.

[8] There is an ambiguity in the generalization to curved spacetimes

Our equation of motion is

$$\Box\phi = -\frac{\lambda}{3!}\phi^3 \tag{164}$$

where $\Box \equiv \partial_\mu \partial^\mu$. We can use it to rewrite the trace of the stress-energy tensor as

$$T^{\mu}_{\mu(c)} = \partial_\mu(-\phi\partial^\mu\phi) = -\frac{1}{2}\Box\phi^2 \tag{165}$$

which means the theory is invariant under special conformal transformations.

We can define a new tensor by

$$T^{\mu\nu} = T^{\mu\nu}_{(c)} + A(\eta^{\mu\nu}\Box\phi^2 - \partial^\mu\partial^\nu\phi^2) \tag{166}$$

which is again conserved $\partial_\mu T^{\mu\nu} = 0$ and is traceless

$$T^{\mu}_{\ \mu} = T^{\mu}_{(c)\mu} + 3A\Box\phi^2 = 0 \text{ for } A = \frac{1}{6}. \tag{167}$$

The added term is not so arbitrary, if we compute the BR stress-tensor from the action with the improvement term that couples it to gravity we get precisely the tensor we just defined.

The tracelessness of the stress-energy tensor for ϕ^4 means that it is classically conformal invariant

$$D^\mu \quad = T^{\mu\nu}x_\nu \tag{168}$$

$$\partial_\mu D^\mu \quad = T^{\mu}_{\ \mu} = 0. \tag{169}$$

This symmetry is broken by anomalies quantum mechanically, the trace will get an anomalous contribution.

9.5 Conformal Anomaly

We know that in general quantum effect gives an energy dependence to the coupling constants of our theories and thus breaks scale invariance. We call classical symmetries which are broken by quantum effects anomalous. From our discussion above we infer that if the scale symmetry is anomalous then its associated current cannot be conserved anymore and that is indeed the case.

The current for a scale transformation is

$$D^\mu = T^{\mu\nu} x_\nu \tag{170}$$

so the conservation law $\partial_\mu D^\mu = T^\mu_{\ \mu} = 0$ has to be corrected quantum mechanically.

The statement of running coupling constant is contained in the beta function of the theory. Under a dilation the coupling transforms as $\lambda \to \lambda + \beta(\lambda)$ and the change in the Lagrangian is

$$\delta\mathcal{L} = \beta(\lambda)\frac{\partial\mathcal{L}}{\partial\lambda} \tag{171}$$

On the other hand the change in the Lagrangian is related to the current that corresponds to the dilatation.

So quantum mechanically we have

$$\langle \partial_\mu D^\mu \rangle = \beta(\lambda)\frac{\partial\mathcal{L}}{\partial\lambda} = \langle T^\mu_{\ \mu} \rangle \tag{172}$$

this anomaly is called the *trace anomaly*.

For example for a free massive scalar field $\beta(m^2) = -m^2$ and the trace of the stress-energy tensor is

$$T^\mu_{\ \mu} = (-m^2)\left(-\frac{1}{2}\phi^2\right) = \frac{1}{2}m^2\phi^2 \tag{173}$$

already at the classical level the conformal symmetry is broken by a mass term.

For QED this is a one-loop effect

$$\mathcal{L} = -\frac{1}{4e^2}(F^{\mu\nu})^2 + \bar{\Psi} i \not{D} \Psi \tag{174}$$

$$\langle T^\mu_{\ \mu} \rangle = \beta(e)\frac{1}{2e^3}(F^{\mu\nu})^2 = \frac{1}{24\pi^2}(F^{\mu\nu})^2 \tag{175}$$

where $\beta(e) = \frac{e^3}{12\pi^2}$.

For a CFT in flat spacetime $T^\mu_{\ \mu} = 0$ but coupling it to a curved background will in general break conformal symmetry but in a very controlled way by the geometry.

$$\langle T^\mu_{\ \mu} \rangle = \frac{c}{16\pi^2}(W_{ijkl})^2 - \frac{a}{16\pi^2}(\tilde{R}_{ijkl})^2 \tag{176}$$

$$(W_{ijkl})^2 = (R_{ijkl})^2 - (R_{ij})^2 + \frac{1}{3}R^2 \quad \text{(Weyl Tensor)} \tag{177}$$

$$(\tilde{R}_{ijkl})^2 = (R_{ijkl})^2 - 4(R_{ij})^2 + R^2 \quad \text{(Euler characteristic)} \tag{178}$$

The breaking of conformal symmetry in four dimensions is parametrized by two parameters c, a.

In two dimensions things are simpler

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{c}{12} R \tag{179}$$

c is a function that measures the effective number of degrees of freedom of a QFT, it has been proved that

$$\frac{\partial c}{\partial \ln \mu} < 0 \tag{180}$$

this is made precise by the Zamolodchikov c-theorem.

In four dimensions it has been conjectured (and is widely believed) that the role of counting degrees of freedom is encoded in the a-anomaly.

10 Conformal Anomaly: New development in 2014

10.1 Motivation

There are many reasons we shall study the Weyl anomalies:

1. Anomalies are at the heart of QFTs and CFTs. They are the foundations of the QFTs. We have anomalies for all kinds of symmetries, both global and gauge. Understanding how anomalies arise is what QFT is all about. We will see using very simple physical principles, we will be able to discern what kind of Weyl anomalies CFT can have in various dimensions.

2. There is a RG flow from CFT_{UV} to CFT_{IR}. In principle even if the theory in UV is weakly coupled, in the IR, the theory could be arbitrarily strongly coupled. It is hard to know what theory we can get in the IR. Certain quantities appear during the study of anomalies will play a crucial role in establishing certain theorems we are going to prove using abstract arguments. There is a certain quantity that is monotonically decreasing $c_{UV} > c_{IR}$. c_{UV} and c_{IR} are central charges related to the anomaly. As the renormalization flows, this quantity c decreases. This gives rises to sets of selection rules. If we are given two CFTs, we can compute the

central charges and discern if these two theories are related by RG flow. This also tells us the RG flow is irreversible. So we can have a theory A that flows to theory B but there is no way theory B can flow back to theory A. Note the inequality $c_{UV} > c_{IR}$ is strict. We will see a big difference between anomalies in even and odd dimensions. We will see a simple reason why. Nevertheless, the spherical partition function allows us to postulate certain quantity even in odd dimensions that is monotonic under RG flow. This again gives us nonperturbative selection rules for what kind of flows can exist just using basic principles of field theory such as unitarity, Lorentz invariance, diffeomorphism invariance and so on.

10.2 How do anomalies arise?

Let us couple an arbitrary CFT to a curved background with a fiducial metric $g_{\mu\nu}$. Now we will consider the generating function of the correlation functions of the stress tensor $T_{\mu\nu}$.

$$Z[g_{\mu\nu}] = e^{W[g_{\mu\nu}]} \tag{181}$$

where $W[g_{\mu\nu}]$ captures the connected correlators of the stress tensor $T_{\mu\nu}$. For simplicity, we set all sources j^A to zero. Weyl invariance implies

$$W[g_{\mu\nu}] = W[e^{2\sigma(x)} g_{\mu\nu}] \Leftrightarrow T^{\mu}_{\mu} = 0 \tag{182}$$

Let us now discuss what happens when we include quantum effects. Here are a few comments about QFT:

1. In QFTs, generating functions like $W[g_{\mu\nu}]$ have UV divergences that come from very short distance fluctuations and need to be renormalized. The renormalization procedure parametrized our ignorance about what happens at extremely short distances(quantum gravity) in terms of the coupling constants and the fields in the theory. This procedure gives rise some predictions to some low-energy physics for that particular field theory.

2. We need to regulate these divergences. The physical way to do so is to add some massive degrees of freedom in the UV and consider the "massive deformation"

of the CFT. Essentially at a UV scale, we declare we dont know what happens there.

After this procedure, we need to ask if our classical symmetry is still there. That is the subject of anomalies. In other words, even the classical action can be symmetric under certain symmetry, the measure of the infinite dimensional integration cannot have that symmetry. We can think of the anomalies arise from the measure of the integration.

If we have a classical symmetry, by Noether theorem, we have a conserved current of $\partial_\mu j^\mu = 0$. Now we ask the question if this symmetry is preserved by renormalization. We call a symmetry anomalous if it is not preserved with quantum fluctuations and we have

$$\partial_\mu j^\mu = O(\hbar) \tag{183}$$

We can study for all symmetries: flavor symmetry, discrete symmetry(no current). But we will focus on the conformal symmetry, specifically we want to know if $T^\mu_\mu = 0$(an operator equation) in the quantum theory. Even in flat space, we can have violation of $T^\mu_\mu = 0$ coming from the running coupling constant. $T^\mu_\mu = \beta\lambda\frac{\delta\mathcal{L}}{\delta\lambda}$. $T^\mu_\mu = 0$ is also the dilatation current conservation.

10.3 Some simple physical arguments

The anomaly will be captured in the variation of $W[g_{\mu\nu}]$ in an infinitesimal transformation: $\delta_\sigma W[g_{\mu\nu}]$, which is by definition the anomaly. Under an infinitesimal transformation, we have $\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}$. We will use general principles to constrain what we can have for $\delta_\sigma W[g_{\mu\nu}]$. Anomalies arise from very short distance physics, because we integrate out the very massive excitations. What we get is a local function of the fields. This is not true if we integrated out massless degrees of freedom, then we can have non-local behavior. So what we have is a local function of the metric and it is diffeomorphism invariant. So the general form will be

$$\delta_\sigma W[g_{\mu\nu}] = \int d^D x\sqrt{g} A_\sigma[g_{\mu\nu}] \tag{184}$$

This needs to obey several rules

1. It is linear in σ. (Infinitesimal transformation)

2. Most general diff invariant operator of dimension that equals to D. (No scale in system)

3. Weyl transformation is abelian. The anomaly should represent the symmetry properly. If we perform two consecutive transformation, the order should not matter.

$$[\delta_{\sigma 2}, \delta_{\sigma 1}]W[g_{\mu\nu}] = \delta_{\sigma 2}\int d^D x\sqrt{g}A_{\sigma 1}[g_{\mu\nu}] - \delta_{\sigma 1}\int d^D x\sqrt{g}A_{\sigma 2}[g_{\mu\nu}] = 0 \tag{185}$$

If the symmetry is non-Abelian,we have $[\delta_{\sigma 2}, \delta_{\sigma 1}]W[g_{\mu\nu}] = \delta_{\sigma 3}W[g_{\mu\nu}]$. This is known as the Wess-Zumino consistency condition. This will constrain the terms that will appear in the anomaly.

10.3.1 $D = 2$

We have $\sigma(x)$ and $g_{\mu\nu}(x)$, and we need dimensional 2 scalars. Dimension of the Ricci scalar is 2. We can have the following terms: $\sqrt{g}\sigma R$ and $\sqrt{g}\Box\sigma$, but the latter one is a total derivative that will integrate to 0. It turns out we have

$$\delta_\sigma W[g_{\mu\nu}] = \int d^D x\sqrt{g}\left(\sigma\frac{c}{64}R\right) \tag{186}$$

where the numerical factor is just a way of normalization. Now we need to check this satisfies Wess-Zumino condition: we need to do a Weyl transformation of this anomaly. c is the anomaly coefficient of the theory which we need to compute. This coefficient is position independent after imposing the Wess-Zumino condition. We can compute the trace of the stress tensor and we have

$$T^\mu_\mu = -\frac{c}{24\pi}R \tag{187}$$

c is the central charge of the Virasoro algebra of 2d CFT. The central charge commutes with the sets of the generators of conformal symmetry. The normalization is that $c = 1$ for free scalar field. In flat space, define $T(z) = T_{zz}$, where $z = x_1 + ix_2$. We can prove that

$$\langle T(z)T(0)\rangle = \frac{c}{2z^4} \tag{188}$$

Even the flat space knows the anomaly.

c determines important physical quantities: it measures the Casmir energy of CFT, if we put CFT on a cylinder with finite radius, $E = -\frac{\pi c}{6L}$. It also controls the density of states in the high energy in CFT. The entropy at high energy is given $S(E) \sim \sqrt{cE}$. c gives us a measure of number of degrees of freedom in CFT and it is the quantity that monotonically decreases in the RG flow $c_{UV} > c_{IR}$. We are integrating out the massive degrees of freedom, and losing number of degrees of freedom. As in 2D, the entropy scales with c, c is the quantity we are looking for. We have thus restricted the RG flow. In the following, we will try to prove this statement, known as Zamolodchikov theorem (c-theorem).

In two dimension we have $T_{\mu\nu}\eta^{\mu\nu} = T_{z\bar{z}}$. The two point function of the stress energy tensor is constrained by Ward identities. We have,

$$\langle T_{zz}(z)T_{zz}(0)\rangle = \frac{F(r^2)}{z^4} \tag{189}$$

$$\langle T_{zz}(z)T_{z\bar{z}}(0)\rangle = \frac{G(r^2)}{4z^3\bar{z}} \tag{190}$$

$$\langle T_{z\bar{z}}(z)T_{z\bar{z}}(0)\rangle = \frac{H(r^2)}{16z^2\bar{z}^2} \tag{191}$$

where $r^2 = z\bar{z}$. The stress tensor is conserved

$$\bar{\partial}T_{zz} + \partial T_{z\bar{z}} = 0 \tag{192}$$

This imposes equation for F, G, H. Take a particular linear combination of this function $c = 2F - G - \frac{3}{8}H$. and we have $\mu^2 \frac{dc}{d\mu^2} = -\frac{3}{4}H < 0$ as $H > 0$ in a unitary theory. This shows c decrease towards IR. This reduces to the center charge c_{UV} and c_{IR} asymptotically.

10.3.2 $D = 3$

As the dimension of the anomaly is 3, and $R^{3/2}$ is non local. As we cannot write down a term with dimension 3, there are actually no conformal anomalies in odd dimensions. The only dimension 3 term we can write down in the Chern-Simons invariant with spin-connection

$$w_{cs} = d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega \tag{193}$$

but $\int \sigma(x) w_{cs}$ is not diffeomorphism invariant. We can define this term in any dimension $2k+1$: $dw_{cs} = TrR \wedge \cdots \wedge R$, with k Ricci scalars. There is a quantity c but it is not related to conformal anomaly.

10.3.3 $D = 4$

The only terms we can write down for $D = 4$ are

$$\sqrt{g}\sigma(x)[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R^2, R_{\mu\nu}R^{\mu\nu}, \Box R, \text{signature invariant} S = \epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\alpha\beta}R^{\alpha\beta}_{\rho\sigma}] \tag{194}$$

We define Euler density,

$$E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{195}$$

$$W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \tag{196}$$

where the Weyl tensor $W_{\mu\nu\rho\sigma}$ measures the conformal flatness of a manifold. Now we can rewrite the anomaly as

$$\delta_\sigma W = \int d^4x \sqrt{g}\sigma(x)\left(-\frac{c}{64\pi^2}W^2 + \frac{a}{64\pi^2}E_4 + \beta R^2 + \alpha\Box R - \frac{e}{64\pi^2}S_4\right) \tag{197}$$

Now we need to impose Wess-Zumino condition, and we find βR^2 violates the condition. $\alpha\Box R$ is regularization scheme(where two schemes differ by local counter term of $D = 4$) dependent, which is not physical. If we take the counter term to be $S_{\text{counter term}} = \int d^4x\sqrt{g}R^2$, we have $\delta_\sigma S_{\text{counter term}} = \int d^4x\sqrt{g}\sigma\Box R$. For scalar field, we have $a_s = \frac{1}{360}$ and $c_s = \frac{1}{120}$. The anomaly is described by a, c, e. e is parity odd and is missing from literature. We have not found $e \neq 0$ but does not mean $e = 0$. It could receive correction non-perturbatively. For a free Weyl Fermion, e is imaginary. And the trace anomaly is

$$T^\mu_\mu = \frac{c}{16\pi^2}W^2 - \frac{a}{16\pi^2}E_4 + \frac{e}{16\pi^2}S_4 \tag{198}$$

When we take cutoff to infinity, the e term does not survive in the continuum limit. c enters two point function of stress tensor and a and c enters three point function of stress tensor in flat space. It is proved a is the quantity that decreases along the RG flow to IR. [9]. We can think about this by considering a sphere, where W^2 and S_4

[9]This is proved in hep-th/1107.3987

vanishes, the sphere is conformal flat and does not have two cycles. What could be this object in the odd dimensions? How can we unify c in 2D and a in 4D?

10.4 The quantity that unifies in different dimensions

Take a CFT_D we can conformally map it to a sphere with radius r. The partition function does not depend on the radius if the theory is really conformally invariant. The conformal anomaly indicates the partition function depends on the radius in a particular way. We ask how the partition function changes as $logr$. Let us use the convenient coordinate

$$ds^2 = \frac{1}{1+\frac{x^2}{4r^2}} dx^\mu dx_\mu \tag{199}$$

The scaling of the radius is just the rescale of the metric. Thus we have

$$\frac{\delta Z_{S^D}}{\delta \log r} \propto \int d^D x \sqrt{g} T^\mu_\mu \tag{200}$$

where $T_{\mu\nu} = 2\frac{\delta S}{\sqrt{g}\delta g^{\mu\nu}}$ and $\delta g^{\mu\nu} \propto g^{\mu\nu} \log r$, and thus

$$\delta Z = \delta \log r \int_{S^D} \sqrt{g} T^\mu_\mu =' c' \delta \log r \tag{201}$$

On the sphere, only E_4 survives. Thus we have

$$Z = r^{'c'} \tag{202}$$

And we have $'c' = c$ in 2d and $'c' = a$ in 4d.

10.5 Relation with entanglement entropy

[10] What if we try this in odd dimension? Is there a quantity that monotonically in RG flow decreases in odd dimension CFT? The answer again lies in the sphere partition function $Z_{S^{2D+1}} \equiv e^F$, where F is the free energy. The finite piece of F is the monotonic quantity we are looking for. (conjecture.) $F_{UV} > F_{IR}$. It does not depend on the radius as there is no conformal anomaly. Let us study CFT in Minkowski space and look at

[10] Robert Myers is the expert to talk to.

the fixed time slice. In this fixed time slice, we can look at a $D-2$ sphere, and call the sphere region A and the outside of the sphere region B. We can define entanglement entropy between region A and region B that depends on the density matrix $S(\rho)$. After conformal transformation, we can map the entropy with the partition function Z_{S^D}. First due to locality, the Hilbert space separates $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. For the vacuum state of the whole space $|0\rangle$, with density matrix $\rho = |0\rangle\langle 0|$. We can define reduced density matrix,

$$\rho_A = Tr_B \rho, \rho_B = Tr_A \rho \tag{203}$$

The reduced state will not be pure any more. We can define Von Neumann entropy

$$S(\rho) = -Tr(\rho \log \rho) \tag{204}$$

For pure state, the entropy is 0. and we also have $S(\rho_A) = S(\rho_B)$. We can define a formal Hamiltonian $\rho_A = e^{-\beta H}$, where $\beta = 2\pi r$ and H is a charge symmetry called modular Hamiltonian. Let us choose some suitable coordinates,

$$ds^2 = -dt^2 + r^2 dr^2 + r^2 d\Omega_{D-2} \tag{205}$$

we can choose some coordinate transformation (unspecified) to map it to dS space.

$$ds^2 = \Omega^2(-\cos^2\theta d\tau^2 + R^2(d\theta^2 + \sin^2\theta d\Omega_{D-2}) \tag{206}$$

now we can identify the density matrix with $\rho = e^{-2\pi r H_\tau}$. $Z_{dS} \sim e^S$, is the thermal partition function in dS_D with $\frac{1}{T} = 2\pi r$. The way to compute is to go to Euclidean time $\tau = i\tau_E$, and compactify such that the Euclidean time is periodic $\tau_E = \tau_E + 2\pi r$. We map $e^{S_{EE}}$ to $Z_{S^D}^{CFT}$. If we add symmetries(SUSY) to CFT to make it SCFT, and study the partition function on the sphere, we have less ambiguity as the theory is more constrained. This quantity can be computed exactly(with a trick called "SUSY localization") that encodes classes of correlation functions of operators.

11 Conformal Fields and Operators

We label field operators according to how they transform under the conformal algebra, just as we do in regular QFT using the Poincaré group. Up till now, this has all be kinematics, but now we need to understand how symmetries act on operators.

11.1 Reminder: Poincaré Fields

Consider the Poincaré transformation with the new notation

$$x^\mu \to \tilde{x}^\mu = a^\mu + \Lambda^\mu{}_\nu x^\nu \equiv (g \cdot x)^\mu \,. \tag{207}$$

We recall how fields transform under the rotation

$$\Phi^A(x) \to \tilde{\Phi}^A(\tilde{x}) = L^A{}_B(g)\Phi^B(x) \tag{208}$$

where the index A characterizes the Lorentz properties of the field.

We have to differentiate between two aspects of the transformation

1. Geometrical transformations or "orbital" transformation which acts on spacetime only.

2. Internal or "spin" transformations that depends on the choice of Φ^A.

The index A labels a representation of the Lorentz group under which Φ^A transforms.

We write the transformation $\Phi^A \to \tilde{\Phi}^A(gx)$ as

$$\begin{aligned} \tilde{\Phi}^A(gx) &= L^A{}_B(g)\Phi^B(x) && (209) \\ \tilde{\Phi}^A(x) &= L^A{}_B(g)\Phi^B(g^{-1}x) && (210) \end{aligned}$$

notice that in the second line the transformation on the coordinates is the inverse to the one used in the representation matrices $L^A{}_B$.

There's a consistency condition on the matrices L given by the group properties of the transformations. We perform two transformations $x \to (g_1 g_2)x$ and calculate the answer two different ways:

1. $$\tilde{\Phi}^A(x) = L^A{}_C(g_1 g_2)\Phi^C\left((g_1 g_2)^{-1}x\right) \tag{211}$$

2. $$\tilde{\Phi}^A(x) = L^A{}_B(g_1)L^B{}_C(g_2)\Phi^C\left(g_2^{-1}g_1^{-1}x\right) \tag{212}$$

where in the first case we do the transformation in one step with the elemente $g_1 g_2$ and in the second step we did two transformations, first with g_1 then with g_2.

Comparing the two, we find that

$$L^A{}_C(g_1 g_2) = L^A{}_B(g_1) L^B{}_C(g_2) \tag{213}$$

so $L^A{}_C(g)$ must be representations of Lorentz group.

To obtain how the field transform under any Poincaré transformation we can always calculate how the field transform under Lorentz transformations at the origin $\phi(0)$ and then translate the field to any other point. The translation for the operator is given by

$$\Phi(x) = e^{-iP_\mu x^\mu} \Phi(0) e^{iP_\mu x^\mu} \tag{214}$$

Now perform an infinitesimal Poincaré transformation

$$x^\mu \to \tilde{x}^\mu = (g \cdot x)^\mu = x^\mu + v^\mu \tag{215}$$

where $v^\mu = a^\mu + \omega^{\mu\nu} x_\nu$ as before. We also have that $(g^{-1}x) = x^\mu - v^\mu$. We plug it back into the constraint equation for the Ls we get

$$L^A{}_B(g) = \exp\left(\frac{i}{2}\omega_{\mu\nu} M_R^{\mu\nu}\right)^A_B \tag{216}$$

Where $M_R^{\mu\nu}$ are representation matrices for the Lorentz algebra.

$$(M_R^{\mu\nu})^A{}_B = \begin{cases} \text{scalar} & 0 \\ \text{vector} & \frac{1}{i}\left(\eta^{\mu A}\delta^\nu_B - \eta^{\nu A}\delta^\mu_B\right) \\ \text{spinors} & \frac{i}{2}\left[\Gamma^\mu, \Gamma^\nu\right]^A{}_B \end{cases} \tag{217}$$

where Γ^μ are the Dirac matrices which obey $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$.

We can now calculate the infinitesimal transformation of the field

$$\begin{aligned}
\tilde{\Phi}^A(x) &= \left(\delta^A_B + \frac{i}{2}\omega_{\mu\nu}\left(M_R^{\mu\nu}\right)^A_B\right)\Phi^B(x-v) && (218)\\
\Phi^B(x-v) &= \Phi^B(x) - a^\mu\partial_\mu\Phi^B(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \Phi^B(x) && (219)\\
\delta\Phi^A(x) &= \tilde{\Phi}^A(x) - \Phi^A(x) && (220)\\
&= -a^\mu\partial_\mu\Phi^A(x) + \frac{i}{2}\omega_{\mu\nu}\left(M_R^{\mu\nu}\right)^A_B \Phi^B(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\Phi^A(x) && (221)\\
&= -a^\mu\partial_\mu\Phi^A(x) - \omega_{\mu\nu}\left(\delta^A{}_B x^\nu\partial^\mu - \frac{i}{2}(M_R^{\mu\nu})^A{}_B\right)\Phi^B(x) && (222)
\end{aligned}$$

we can clearly see now what is the effect of a translation and a Lorentz transformation on the fields.

Writing in terms of the killing vector field v^μ we identify what is the geometrical effect of the transformation and which is the internal rotation

$$\delta\Phi^A(x) = \underbrace{-v^\mu\partial_\mu\Phi^A(x)}_{\text{'orbital' transformation}} + \underbrace{\frac{i}{2}\omega_{\mu\nu}\left(M^{\mu\nu}\right)^A_B}_{\text{'spin' transformation}} \Phi^B(x). \tag{223}$$

For the Poincaré group we have only to specify its properties under a Lorentz transformation. We'll see that for the conformal group we need another number to classify the irreducible representations which turns out to be the eigenvalue of the dilatation operator, the conformal dimension of the field.

We still need to check the closing of the algebra, which means that two infinitesimal transformations should give another transformation

$$[\delta_v, \delta_u]\Phi^A(x) = \delta_w\Phi^A(x) \tag{224}$$

where $w = [v, u]$.

11.2 Detour: Symmetries

Symmetries in quantum mechanics are realized by unitary transformations $U(g)$ acting on the Hilbert space of states

$$|\Psi\rangle \to U(g)|\Psi\rangle \tag{225}$$

and field operators transform as

$$\tilde{\Phi}^A(x) = U^{-1}(g)\Phi^A(x)U(g) = L^A{}_B(g)\Phi^B(g^{-1}x) \tag{226}$$

where L is a representation matrix for the group. The compatibility condition for this matrix can be deduced by doing two consecutive transformations

$$\begin{aligned} U^{-1}(g_1g_2)\Phi^A(x)U(g_1g_2) &= U^{-1}(g_2)U^{-1}(g_1)\Phi^A(x)U(g_1)U(g_2) && (227)\\ &= L^A{}_B(g_1)U^{-1}(g_2)\Phi^B(g_1^{-1}x)U(g_2) && (228)\\ &= L^A{}_B(g_1)L^B{}_C(g_2)\Phi^C(g_2^{-1}g_1^{-1}x) && (229)\\ &= L^A{}_C(g_1g_2)\Phi^C((g_1g_2)^{-1}x) && (230) \end{aligned}$$

For a Poincaré transformation we have

$$U(g) = \exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} - ia_\mu P^\mu\right) \tag{231}$$

we only need to know how the field transform at the origin since

$$[M^{\mu\nu}, \Phi^A(x)] = e^{-ix_\mu P^\mu}[e^{ix_\mu P^\mu}M^{\mu\nu}e^{-ix_\mu P^\mu}, \Phi^A(0)]e^{ix_\mu P^\mu} \tag{232}$$

$$e^{ix_\mu P^\mu}M_{\mu\nu}e^{-ix_\mu P^\mu} = M_{\mu\nu} + x_\mu P_\nu - x_\nu P_\mu \tag{233}$$

then the commutator is simply

$$[M^{\mu\nu}, \Phi^A(x)] = (M_R^{\mu\nu})^A{}_B\Phi^B(x) - i(x^\mu\partial^\nu - x^\nu\partial^\mu)\Phi^A(x) \tag{234}$$

11.3 Conformal Fields and Algebras

In a CFT, there is a net of *fundamental operators* or primary operators θ_a. All other operators in the CFT are derived from these ($\partial_{\mu_1,\ldots\mu_L}\theta_a$) and called *descendants*. How do the operators transform under conformal transformations?

1. *Primary*: $\theta^A(x)$ transform as a tensor under conformal transformations

2. *Descendant*: obtained from the primary operators by taking derivatives $\partial_\mu\theta^A(x), \ldots$

Knowing the properties of the primaries we can deduce the properties of its family of descendants.

To study the action of the conformal transformations we need more information about the field: this means the field will carry more quantum numbers. We need two pieces of information,

1. R: The representation of the Lorentz group

2. Δ: The eigenvalues of θ under $\hat{D}$, with

$$\hat{D}\cdot\theta_a = \Delta_a\theta_a \text{ no sum} \tag{235}$$

where Δ_a's are the conformal dimension of the operators.

We want to compute the set of all dimensions for the primary operators $\{\Delta_1, \dots\}$.

The primary operator θ will have a dimension Δ, and the descendants will have higher dimensions since the momentum operator is like a raising operator for the dilation. For example, $\partial_\mu \theta$ has dimension $\Delta + 1$ and $\partial_{\mu_1 \dots \mu_L} \theta$ has dimension $\Delta + L$.

The spectrum of Δ_a is contained in the 2 point functions of the primary operators

$$\langle \theta_a(x) \theta_b(y) \rangle = \frac{c\delta_{ab}}{|x-y|^{2\Delta_a}} \tag{236}$$

We note that the stress-energy tensor always has $\Delta = D$, it can be easily seen from

$$P^\mu = \int d^{D-1} x T^{\mu 0} \tag{237}$$

P^μ has dimension 1 (from the commutation relations with $\hat{D}$) so the above implies $1 = -(D-1) + [T]$.

12 Transformation of Conformal Fields

Again we first study the transformations that fix the origin of spacetime, these are the Lorentz transformations $M^{\mu\nu}$, the Dilatation $\hat{D}$ and the special conformal transformations K^μ.

$$[M^{\mu\nu}, \Phi^A(0)] = (M^{\mu\nu})^A_B \Phi^B(0) \tag{238}$$

$$[\hat{D}, \Phi^A(0)] = -i\Delta_\Phi \Phi^A(0) \tag{239}$$

$$[K^\mu, \Phi^A(0)] = 0 \tag{240}$$

In principle we need a quantum number for the special conformal transformations but for primary fields $[K^\mu, \Phi^A(0)] = 0$ by definition, they are the highest weight states for the Dilatation operator.

From the conformal algebra we have

$$[(M_R^{\mu\nu}), \hat{D}] = 0 \tag{241}$$

$$[\hat{D}, K_\mu] = iK_\mu \tag{242}$$

by Schur's Lemma $\hat{D}$ is proportional to the identity and then $K^\mu = 0$ on the space of primaries. In the end as was pointed out before the fields will carry a Lorentz transformation and a Dilatation label.

12.1 Constructing the Jacobian

Above we have seen how the fields transform under the generators of the conformal group, we can compute from them the finite transformation by exponentiating this action.

Primary fields transform as follows

$$\tilde{\Phi}^A(x) = \left|\frac{\partial x}{\partial \tilde{x}}\right|^{\Delta/D} L^A{}_B(R)\Phi^B(g^{-1}x) \tag{243}$$

where L are the Lorentz group representation matrices and from GR we recognize this as the transformation rule for a tensor density.

Tensor densities are quantities which transform similar to tensors but have a Jacobian factor in front. More specifically, after the following coordinate transformation $x^\mu \to \tilde{x}^\mu = (gx)^\mu$ tensors transform in the usual way with the matrices $\frac{\partial x^\mu}{\partial \tilde{x}^\nu}$

$$V^\mu(x) \to \tilde{V}^\mu(x) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} V^\nu(g^{-1}x) \tag{244}$$

while tensor densities get a factor of the determinant of the transformation to some power

$$\tilde{V}^{\overbrace{\cdots}^{p}}{}_{\underbrace{\cdots}_{q}} = \left|\frac{\partial x}{\partial \tilde{x}}\right|^{\omega} \overbrace{\left(\frac{\partial \tilde{x}}{\partial x}\cdots\frac{\partial \tilde{x}}{\partial x}\right)}^{p}\overbrace{\left(\frac{\partial x}{\partial \tilde{x}}\cdots\frac{\partial x}{\partial \tilde{x}}\right)}^{q} V^{\cdots}{}_{\cdots}(g^{-1}x) \tag{245}$$

Where we call $\left|\frac{\partial x}{\partial \tilde{x}}\right|^{\omega}$ the density weight.

Now we define R as a Jacobian that is 'enriched' by one power of the conformal factor

$$R^\mu_\nu(x) = \Omega^{-1}(x)\frac{\partial \tilde{x}^\mu}{\partial x^\nu}; \quad R^T\eta R = \eta \tag{246}$$

where $R \in O(D)$ is an orthogonal transformation.

Another way we can derive the Jacobian factor by taking the determinant of the following equation

$$\eta_{\rho\sigma}\frac{\partial \tilde{x}^\rho}{\partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\nu} = \Omega^2(x)\eta_{\mu\nu} \tag{247}$$

and get

$$\left|\frac{\partial \tilde{x}}{\partial x}\right|^2 = \Omega^{2D} \tag{248}$$

for a conformal transformation. The transformation of primary fields, Eq. (243) can be written

$$\tilde{\theta}^A(x) = \Omega^{-\Delta(x)} L^A_B(R)\theta^B\left(g^{-1}x\right) \tag{249}$$

12.2 Examples

We can see how these primary fields transform under some simple operations

- Poincaré transformations recover the usual transformations since $\Omega(x) = 1$
- Under a scale transformation

  $$x^\mu \to \tilde{x}^\mu = \lambda^{-1}x^\mu \tag{250}$$

  and $\Omega(x) = \lambda^{-1}$ because

  $$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \lambda^{-1}\delta^\mu_\nu, \qquad \left|\frac{\partial x}{\partial \tilde{x}}\right| = \lambda^D \tag{251}$$

  For a scale operator, the primary fields transform homogeneously

  $$\tilde{\theta}(x) = \lambda^\Delta \theta(\lambda x) \tag{252}$$

From these we can constrain the form of the two point function for scalars in a CFT in the same way as for a regualr QFT

$$\langle\theta_1(x)\theta_2(y)\rangle = \langle\tilde{\theta}_1(x)\tilde{\theta}_2(y)\rangle = \lambda^{\Delta_1+\Delta_2}\langle\theta_1(\lambda x)\theta_2(\lambda y)\rangle \tag{253}$$

Translational invariance implies the two point function only depends on $x^\mu - y^\mu$ and rotational invariance implies the two pint function only depends on $|x-y|$, which means that

$$\langle\theta_1(x)\theta_2(y)\rangle = f(|x-y|) \tag{254}$$

where f has the property

$$f(|x-y|) = \lambda^{\Delta_1+\Delta_2} f(\lambda|x-y|) \tag{255}$$

this property fixes f to be of the form

$$f(|x-y|) = \frac{G_{12}}{|x-y|^{\Delta_1+\Delta_2}} \tag{256}$$

where G_{12} is just a constant. We will see in later sections that the special conformal transformation will end up requiring that the 2 point function vanishes $\langle\theta_1(x)\theta_2(y)\rangle = 0$ if $\Delta_1 \neq \Delta_2$, so this is a very powerful constraint.

12.3 Infinitesimal Transformations

We start by taking Eq. (249) and expanding in a small parameter to give the variation of a primary field

$$\delta\theta^A(x) = \underbrace{-v^\mu\partial_\mu\theta^A(x)}_{\text{`orbital'}} + \underbrace{\frac{i}{2}\Omega_{\mu\nu}(x)\,(M_R^{\mu\nu})^A_{\ B}\,\theta^B(x) - \Delta\sigma(x)\theta^A(x)}_{\text{`spin'}} \tag{257}$$

where v^ν is the conformal killing vector for the infinitesimal transformation, R is a representation of the Lorentz group and Δ is the dimension of the operators. More explicitly

$$\Omega_{\mu\nu}(x) = -\frac{1}{2}(\partial_\mu v_\nu - \partial_\nu v_\mu) = \omega_{\mu\nu} - 2\,(x_\mu b_\nu - x_\nu b_\mu) \tag{258}$$

$$\sigma(x) = \frac{1}{D}\partial\cdot v = \lambda - 2x\cdot b \tag{259}$$

the first term is a position dependent Lorentz transformation and the second a position dependent scale transformation. Notice that there is a special conformal parameter b^μ inside the Lorentz transformation and the scale transformation. This shows that special conformal transformation does weird things like inducing a rotation of the Lorentz frame and a scale transformation.

For a scale transformation, $x^\mu \to x^\mu + \lambda x^\mu$, of a scalar primary field we have,

$$\delta\Phi(x) = -\lambda(x^\mu\partial_\mu + \Delta)\Phi(x) \tag{260}$$

To show that this is a consistent representation of the conformal algebra in the space of fields we need to show that two consecutive infinitesimal transformations are equivalent to another transformation, that is

$$[\delta_{v_1}, \delta_{v_2}]\theta^A(x) = \delta_{v_3}\theta^A(x) \tag{261}$$

where $v_3 = [v_1, v_2]$.

Descendant operators do not transform like densities under a general conformal transformation. Consider $\theta(x)$ a scalar operator of dimension Δ, one of its descendants is given by the derivative $\partial_\mu \theta(x)$. Under conformal transformations such descendant transforms as

$$\delta\left(\partial_\mu \theta\right) = -v^\nu \partial_\nu \partial_\mu \theta(x) - (\Delta+1)\sigma(x)\partial_\mu \theta + \Omega_{\mu\nu}(x)\partial^\nu \theta(x) + 4b_\mu \theta(x). \tag{262}$$

notice that the last term is inhomogeneous, it is proportional to the primary. This means that conformal transformations of descendants mix the elements of their conformal family in a non-trivial way, while primaries are mapped into primaries.

12.4 Primary Operators are Special

Recall the analogy between the harmonic oscillator operators $H, a^\dagger, a$ and the generators D, P_μ, K_ν, P^μ was a raising operator while K^μ was a lowering operator for D.

$$[D, P^\mu] = -iP^\mu \tag{263}$$

$$[D, K^\mu] = iK^\mu. \tag{264}$$

Consider a field $\Phi(x)$ of dimension $[D, \Phi] = -i\Delta\Phi$, the dimension of its descendants are specified completely by the dimension of its primary. for example the dimension of the descendant $[P_\mu, \Phi] = \partial_\mu \Phi$ can be easily calculated

$$\begin{aligned} [D, [P_\mu, \Phi]] &= -[P_\mu, [\Phi, D]] - [\Phi, [D, P_\mu]] && (265) \\ &= -i\Delta[P_\mu, \Phi] + i[\Phi, P_\mu] && (266) \\ &= -i(\Delta+1)[P_\mu, \Phi]. && (267) \end{aligned}$$

We see that the dimension of the descendant is increased by P^μ. In order to have a bound on the dimensions we assume, analogously to the harmonic oscillator, that there exists a state with lowest "energy", in this case a state of lowest dimension, these states are the primaries, $[K_\mu, \theta] = 0$.

Part III

Working in Conformal Field Theories

13 Brief Review

13.1 Operators in a CFT: Review

Recall from the previous sections that we have studied two kinds of operators:

- Primary Operators: $\theta^A(x)$, transform nicely under conformal transformations
- Descendant Operators: $\partial_{\mu_1\ldots\mu_2}\theta^A(x)$, transform inhomogeneously under conformal transformations

In any CFT there always exists

1. The identity operator $\mathbb{I}$ (Conformal/Mass Dimension $\Delta = 0$)
2. The Stress tensor $T_{\mu\nu}$ ($\Delta = D$), there is no anomalous dimension.

These form the minimum requirement for a CFT. On top of that CFTs can also have global continuous symmetry $\phi^a \to \phi^a \to R^a{}_b\phi^b$, therefore Noether's theorem implies a conserved current j_μ ($[J_\mu] = D-1$) for such symmetry. These currents are going to be give us more operators for the CFT.

The variation of the action can always be written

$$\delta S = \int d^D x \, (\partial^\mu \varepsilon) \, j_\mu \tag{268}$$

where ε is the transformation parameter. If ε is constant, we can act the derivative on j and recover our conservation law.

What constraints does conformal invariance give on the correlation function of operators in CFT? We can classify operators into representations of conformal group. CFT operators are labeled by the Lorentz quantum number and the dilatation eigenvalue (scaling dimension).

P_μ is a raising operator, if $\mathcal{O}_\Delta$ has dimension Δ, $P_\mu \mathcal{O}_\Delta$ has dimension $\Delta + 1$. K_μ is a lowering operator, if $\mathcal{O}_\Delta$ has dimension Δ, $K_\mu \mathcal{O}_\Delta$ has dimension $\Delta - 1$.

primary operator O with scaling dimension Δ satisfies,

$$K_\mu O_\Delta(0) = 0$$

$P_{\mu,1} \dots P_{\mu,n} O$ are descendants with dimension $\Delta + n$.

13.2 Transformation of Operators: Review

For conformal transformations we have

$$[\xi_1, \xi_2] = \xi_3$$

then for primaries, we should have

$$[\delta_{\xi_1}, \delta_{\xi_2}] O_A = \delta_{\xi_3} O_A$$

they form a rep of the algebra. The descendants, $\partial_\mu O_A$ for example, does not transform like this, but it is completely determined. Primaries transform as follows (conformal transformation) for $x \to \tilde{x}$

$$\tilde{\theta}^A(x) = \left| \frac{\partial x}{\partial \tilde{x}} \right|^{\Delta/d} L^A{}_B(R)\ \theta^B(g^{-1}x) \tag{269}$$

where R is the position dependent Lorentz transform, and

$$R^\mu_\nu(x) = \Omega^{-1}(x) \frac{\partial \tilde{x}^\mu}{\partial x^\nu}, \quad R^T \eta R = \eta, \quad \Omega(x)^{-\Delta} = \left| \frac{\partial x}{\partial \tilde{x}} \right|^{\Delta/d} \tag{270}$$

The variation of the operator can be written

$$\delta\theta^A(x) = -v^\mu \partial_\mu \theta^A(x) + \frac{i}{2} \Omega_{\mu\nu}(x) \left(M_R^{\mu\nu}\right)^A{}_B\, \theta^B(x) - \Delta\omega(x)\theta^A(x) \tag{271}$$

where

$$\Omega_{\mu\nu}(x) = \omega_{\mu\nu} - 2(x_\mu b_\nu - x_\nu b_\mu)$$

is a local rotation and

$$omega(x) = \lambda - 2x \cdot b$$

is a local dilatation. The spinor $M_{\mu\nu} \propto [\Gamma_\mu, \Gamma_\nu]$. The second part is local Lorentz and last part is scaling transformation. R denotes the representation of the Lorentz group. L is a representation means $L(g_1)L(g_2) = L(g_1 g_2)$.

We are going to use the conformal symmetry to determine the form of correlation functions

$$\langle \theta_1^{A_1}(x_1) \dots \theta_n^{A_n}(x_n) \rangle \tag{272}$$

Conformal symmetry constraints the correlation function of the primary operators.

$$\langle O_1(x_1) O_2(x_2) \rangle = F(x_1, x_2)$$

We need

$$\delta_\xi \langle O_1(x_1) O_2(x_2) \rangle = 0$$

Thus

$$\langle (\delta_\xi O_1(x_1)) O_2(x_2) \rangle + \langle O_1(x_1) \delta_\xi O_2(x_2) \rangle = 0$$

This gives differential equation for $F(x_1, x_2)$. Poincaré means $F(x_1, x_2) = f(|x_1 - x_2|)$.

14 Correlation Functions in CFT

The conformal symmetry gives strong restriction on the form of the correlation functions, we now study what are these constraints.

Given an n-point correlation function of primaries, the statement of invariance of the correlator under conformal transformations is that

$$\left\langle \theta_1^{A_1}(x_1) \theta_2^{A_2}(x_2) \dots \theta_n^{A_n}(x_n) \right\rangle = \left\langle \tilde{\theta}_1^{A_1}(x_1) \tilde{\theta}_2^{A_2}(x_2) \dots \tilde{\theta}_n^{A_n}(x_n) \right\rangle \tag{273}$$

or infinitesimally

$$\delta \left\langle \theta_1^{A_1}(x_1) \theta_2^{A_2}(x_2) \dots \theta_n^{A_n}(x_n) \right\rangle \tag{274}$$

$$= \sum_{i=1}^n \left\langle \theta_1^{A_1}(x_1) \theta_2^{A_2}(x_2) \dots \delta\theta_i^{A_i}(x_i) \dots \theta_n^{A_n}(x_n) \right\rangle = 0. \tag{275}$$

Transforming the fields explicitly

$$\left\langle \theta_1^{A_1}(\tilde{x}_1)\theta_2^{A_2}(\tilde{x}_2)\dots\theta_n^{A_n}(\tilde{x}_n)\right\rangle \tag{276}$$

$$= \left|\frac{\partial x}{\partial \tilde{x}}\right|^{\Delta_1/d}\dots\left|\frac{\partial x}{\partial \tilde{x}}\right|^{\Delta_n/d} L^{A_1}_{B_1}(R)\dots L^{A_n}_{B_n}(R)\left\langle \theta_1^{A_1}(x_1)\dots\theta_n^{A_n}(x_n)\right\rangle \tag{277}$$

14.1 One Point Correlation Function

We can now calculate the most simple expectation value, for Scalar Operators $\theta(x)$:

1. Translational invariance requires that the expectation value is a constant $\langle\theta_\Delta(x)\rangle = c$. And Lorentz invariance follows from this.

2. Scale invariance implies that $\langle\tilde{\theta}(\tilde{x})\rangle = \langle\theta(\tilde{x})\rangle$ with $\tilde{x} = \lambda x$ and $\tilde{\theta}(\tilde{x}) = \lambda^{-\Delta}\theta(x)$. Together with translation invariance this implies that $\lambda^{-\Delta}c = c$. So $c = 0$ if $\Delta \neq 0$, if $\Delta = 0$ then it is possible that $\langle\theta_{\Delta=0}\rangle = c$ but the only primary with $\Delta = 0$ is the identity (at least for unitary theories).

Non-trivial one point functions appear in systems with boundaries, it's left as an exercise to calculate what is the effect of having a CFT only in the upper half space to the one point function.

14.2 Two Point Correlation Function

As before, we can use the required invariances to restrict the form of the two point correlation function

1. From translations and rotations, we find

$$\langle\theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\rangle = f(|x_{12}|) \tag{278}$$

where $x_{12} = x_1 - x_2$ and $|x_{12}|^2 = (x_1 - x_2)^2$. Now for dilatations

$$\begin{aligned}
\langle\tilde{\theta}_{\Delta_1}(\tilde{x}_1)\tilde{\theta}_{\Delta_2}(\tilde{x}_2)\rangle &= \langle\theta_{\Delta_1}(\tilde{x}_1)\theta_{\Delta_2}(\tilde{x}_2)\rangle &(279)\\
\lambda^{-(\Delta_1+\Delta_2)}\langle\theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\rangle &= \langle\theta_{\Delta_1}(\tilde{x}_1)\theta_{\Delta_2}(\tilde{x}_2)\rangle &(280)\\
\lambda^{-(\Delta_1+\Delta_2)}f(|x_{12}|) = &= f(|\tilde{x}_{12}|) = f(\lambda|x_{12}|) &(281)\\
f(|x_{12}|) &= \frac{c_{12}}{|x_{12}|^{\Delta_1+\Delta_2}} &(282)
\end{aligned}$$

Thus we have

$$\langle\theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\rangle = \frac{c_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} \tag{283}$$

where c_{12} is a constant. This is in sharp contrast to the correlators for a massive (hence non-conformal) theories which behave like $e^{-m|x_1-x_2|}$ for large distances.

2. The special conformal transformation K_μ is given by

$$K_\mu = IP_\mu I \tag{284}$$

where the inversion is $I:\ x^\mu \to \tilde{x}^\mu = \frac{x^\mu}{x^2}$. For inversion we have

$$\frac{\tilde{x}_1^2\tilde{x}_2^2}{(\tilde{x}_1-\tilde{x}_2)^2} = \frac{1}{(x_1-x_2)^2} \tag{285}$$

We can save work by requiring invariance under inversion. The operators transforms under inversions as

$$\tilde{\theta}_\Delta(\tilde{x}) = \frac{1}{(\tilde{x}^2)^\Delta}\theta_\Delta(x) \tag{286}$$

This tells us

$$\langle\tilde{\theta}_{\Delta_1}(\tilde{x}_1)\tilde{\theta}_{\Delta_2}(\tilde{x}_2)\rangle = \frac{\langle\theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\rangle}{\left(\tilde{x}_1^2\right)^{\Delta_1}\left(\tilde{x}_2^2\right)^{\Delta_2}} \tag{287}$$

$$\langle\theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\rangle = \frac{c_{12}}{|x_{12}|^{\Delta_1+\Delta_2}} \tag{288}$$

Calculating and rearranging this final expression we find

$$\frac{(\tilde{x}_1^2)^{\Delta_1}(\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_{12}|^{\Delta_1+\Delta_2}} = \left[\frac{(\tilde{x}_1)^2(\tilde{x}_2)^2}{|\tilde{x}_{12}|^2}\right]^{\frac{\Delta_1+\Delta_2}{2}} \tag{289}$$

this is true iff $\Delta_1=\Delta_2$.

Hence we conclude

$$\langle\theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\rangle = \begin{cases} 0, & \Delta_1\neq\Delta_2 \\ \frac{c_{12}}{|x_{12}|^{2\Delta_1}}, & \Delta_1=\Delta_2 \end{cases} \tag{290}$$

Another (more painful) way to derive this last constraint is to use explicitly the form of a special conformal transformation

$$\tilde{x}^\mu = \frac{x^\mu+b^\mu x^2}{1+2b\cdot x+b^2x^2} \tag{291}$$

14.3 Three Point Correlation Function

1. Similarly for the three point function, using rotation and translation the correlator can only depend on the distances between coordinates

$$\langle \theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\theta_{\Delta_3}(x_3)\rangle = \frac{1}{|x_{12}|^{2a}|x_{13}|^{2b}|x_{23}|^{2c}} \tag{292}$$

invariance under scale transformation tells us that

$$a + b + c = \frac{\Delta_1 + \Delta_2 + \Delta_3}{2} \tag{293}$$

.

2. For special conformal transformations we again just check invariance under inversions

$$\langle \tilde{\theta}_{\Delta_1}(\tilde{x}_1)\tilde{\theta}_{\Delta_2}(\tilde{x}_2)\tilde{\theta}_{\Delta_3}(\tilde{x}_3)\rangle = \langle \theta_{\Delta_1}(\tilde{x}_1)\theta_{\Delta_2}(\tilde{x}_2)\theta_{\Delta_3}(\tilde{x}_3)\rangle \tag{294}$$

$$\frac{\langle \theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\theta_{\Delta_3}(x_3)\rangle}{(\tilde{x}_1^2)^{\Delta_1}(\tilde{x}_2^2)^{\Delta_2}(\tilde{x}_3^2)^{\Delta_3}} = \frac{1}{|\tilde{x}_{12}|^{2a}|\tilde{x}_{13}|^{2b}|\tilde{x}_{23}|^{2c}} \tag{295}$$

Where we can substitute in Eq. (292), and use the 'master formula' given by

$$\frac{1}{|x_{ij}|^2} = \frac{\tilde{x}_i^2\tilde{x}_j^2}{|\tilde{x}_{ij}|^2} \tag{296}$$

to look at powers of $\tilde{x}_i^2$. This gives us the system

$$\Delta_1 - 2a - 2b = 0 \tag{297}$$

$$\Delta_2 - 2a - 2c = 0 \tag{298}$$

$$\Delta_3 - 2b - 2c = 0 \tag{299}$$

which has solutions

$$2a = \Delta_1 + \Delta_2 - \Delta_3, \quad 2b = \Delta_1 + \Delta_3 - \Delta_2, \quad 2c = \Delta_2 + \Delta_3 - \Delta_1 \tag{300}$$

Thus finally we arrive at our solution

$$\langle \theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\theta_{\Delta_3}(x_3)\rangle = \frac{c}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_3+\Delta_2-\Delta_1}} \tag{301}$$

14.4 Three points

Three points x_1, x_2, x_3 can always be mapped to $(0, 1, \infty)$: First we can use special conformal transformation $\tilde{x}^\mu = \frac{x^\mu + b^\mu x^2}{1+2b\cdot x+b^2x^2}$ generated by $K_\mu = IP_\mu I$ to map x_3 to ∞, then we use P_μ to map x_2 to 0. Then we use rotation $M_{\mu\nu}$ to map x_1 to $(\lambda, 0, \ldots, 0)$, and last use dilatation D to map it to $(1, 0, \ldots, 0)$.

14.5 Four Point Correlation Function

Now we are running out of luck, given four points there exists "conformally invariant" ratios (crossratios or anharmonic ratios). So now we can add any function of these ratios to the correlation function, conformal symmetry doesn't completely fix the correlation functions anymore.

For four points the crossratios are

$$\eta_1 = \frac{|x_{12}||x_{34}|}{|x_{13}||x_{24}|}; \quad \eta_2 = \frac{|x_{12}||x_{34}|}{|x_{14}||x_{23}|}. \tag{302}$$

The four point function will be an unknown function of these crossratios times a factor that is constrained by conformal invariance

$$\langle \theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\theta_{\Delta_3}(x_3)\theta_{\Delta_4}(x_4)\rangle = F(\eta_1, \eta_2)\prod_{i<j}|x_{ij}|^{\gamma_{ij}} \tag{303}$$

where for a fixed i,

$$\sum_{j\neq i}\gamma_{ij} = -2\Delta_i. \tag{304}$$

Our task now is around the study of the functions F, they have many interesting properties and can be highly non-trivial, for example they can be hyper-geometric functions. OPE in CFT conformal symmetry $F(u, v)$ is determined by the CFT data, Δ_I, c_{IJK}.This data has an important consistency condition, known as crossing symmetry. This makes it possible to measure the critical exponents of 3D Ising CFT with pure thought.

14.6 2 Point Function of currents, J_μ

If the CFT has a global symmetry, not associated with the spacetime, then there are important operators given by the conserved currents which are primary vector operators.

$$\langle j_\mu(x) j_\nu(y)\rangle = \frac{I_{\mu\nu}(x-y)}{(x-y)^{2\Delta}}$$

The one point function vanishes trivially by Lorentz invariance. The two point function must be as before conformal invariant

$$\langle \tilde{J}_\mu(\tilde{x}_1)\tilde{J}_\nu(\tilde{x}_2)\rangle = \langle J_\mu(\tilde{x}_1) J_\nu(\tilde{x}_2)\rangle \tag{305}$$

By translation, rotation and dilation invariance the general form of the 2 point function of conserved current is

$$\langle J_\mu(\tilde{x}_1) J_\nu(\tilde{x}_2)\rangle = \frac{\alpha_{\mu\nu}(\tilde{x}_1 - \tilde{x}_2)}{|\tilde{x}_{12}|^{2\Delta}}. \tag{306}$$

where $\alpha_{\mu\nu}$ is a tensor with scaling dimension zero.

Under an inversion the current transforms as

$$\tilde{J}_\mu(\tilde{x}) = \frac{I^\nu{}_\mu(x)}{\tilde{x}^{2\Delta}} J_\nu(x) \tag{307}$$

so the two point function transforms as

$$\langle \tilde{J}_\mu(\tilde{x}_1)\tilde{J}_\nu(\tilde{x}_2)\rangle = \frac{I_\mu{}^\rho(x_1)}{(\tilde{x}_1^2)^\Delta}\frac{I_\nu{}^\sigma(x_2)}{(\tilde{x}_2^2)^\Delta}\langle J_\rho(x_1) J_\sigma(x_2)\rangle \tag{308}$$

where

$$I_\mu{}^\rho(x) = (\tilde{x}^2)^\Delta \frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\rho_\mu - 2\frac{x^\rho x_\mu}{x^2} \tag{309}$$

This yields

$$\frac{\alpha_{\mu\nu}(\tilde{x}_{12})}{|\tilde{x}_{12}|^{2\Delta}} = \frac{1}{(\tilde{x}_1^2)^\Delta(\tilde{x}_2^2)^\Delta}\frac{I_\mu{}^\rho(x_1)\alpha_{\rho\sigma}(x_1-x_2) I_\mu{}^\sigma(x_2)}{|x_{12}|^{2\Delta}} \tag{310}$$

Which in turns means that the tensor $\alpha_{\mu\nu}$ obeys

$$\alpha(\tilde{x}_1 - \tilde{x}_2) = I^T(x_1)\alpha(x_{12}) I(x_2) \tag{311}$$

and the solution is

$$\alpha(x_{12}) = I(x_{12}). \tag{312}$$

This follows that the inversion tensor obeys the identity

$$I(\tilde{x}-\tilde{y}) = I^T(x)I(x-y)I(y). \tag{313}$$

which is the same as the transformation law for the holonomy of the connection.

Thus we arrive at

$$\langle J_\mu(x_1)J_\nu(x_2)\rangle = \frac{I_{\mu\nu}(x_1-x_2)}{|x_1-x_2|^{2\Delta}}. \tag{314}$$

14.7 2 Point Function of $T_{\mu\nu}$

The other important operator for any CFT is the stress-energy tensor. Again using translation, rotation and dilation invariance the general form of the two point function is

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle = \frac{c_T}{|x-y|^{2D}}\alpha_{\mu\nu,\rho\sigma}(x-y) \tag{315}$$

where c_T is the central charge of the CFT and recall that the stress energy always has scaling dimension $\Delta = D$. The tensor

$$\alpha_{\mu\nu,\rho\sigma}(x) = \frac{1}{2}\left(I_{\mu\rho}(x)I_{\nu\sigma}(x) + I_{\mu\sigma}(x)I_{\nu\rho}(x)\right) - \frac{1}{D}\delta_{\mu\nu}\delta_{\rho\sigma} \tag{316}$$

has the same indice symmetries as the tensors on the LHS and is traceless on the first two and last two indices.

The central charge c_T is one way of classifying CFTs, if the values of the central charge for two theories differ they cannot be the same. For free field theories $c_T = \frac{1}{\pi^4}(n_S + 2n_F + 4n_V)$ where n_S is the number of scalars, n_F the number of fermions and n_V is the number of vectors. As we just saw the central charge can be computed from the two point function of the stress energy tensor, it is left as an exercise to verify the central charge for a free theory given above.

Specially in $D = 2$, c captures lots of physical system, we can compute c from flat space correlator or by coupling to gravity $\eta_{\mu\nu} \to g_{\mu\nu}$. c is interpreted as a conformal anomaly. We have seen $T^\mu_\mu = 0$ in a CFT in flat space. In a curved background, $T^\mu_\mu \neg 0 \propto c$.

$$\partial^\mu j^D_\mu = T^\mu_\mu$$

We can write down five different structures for $\langle T_{\mu\nu}(p)T_{\rho\sigma}(q) \propto \delta(p+q)$ based on Lorentz invariance and Bose symmetry $\mu\nu \leftrightarrow \rho\sigma$ using covariant quantities $\eta_{\mu\nu}, P_\mu$. After imposing energy momentum conservation, there are only 2 independent structures left.

$$\langle T_{\mu\nu}(p)T_{\rho\sigma}(-p)\rangle = \left(\Pi_{\mu\rho}\Pi_{\nu\sigma} + \Pi_{\mu\sigma}\Pi_{\nu\rho} - \frac{2}{d-1}\Pi_{\mu\nu}\Pi_{\rho\sigma}\right) F(p^2)\nabla_\nu \quad +\Pi_{\mu\nu}\Pi_{\rho\sigma}G(p^2) \tag{317}$$

where $\Pi_{\mu\nu} = p_\mu p_\nu - p^2\eta_{\mu\nu}$ and we have $p^\mu \Pi_{\mu\nu} = 0$. In $D = 2$, the first term just vanishes, and we have

$$\langle T_{\mu\nu}(p)T_{\rho\sigma}(-p)\rangle = \Pi_{\mu\nu}\Pi_{\rho\sigma}G(p^2)$$

In 2d we can define

$$\tilde{p}_\mu = \epsilon_{\mu\lambda}p^\lambda$$

then

$$\Pi_{\mu\nu} \propto \tilde{p}_\mu \tilde{p}_\nu$$

$$p^\mu \tilde{p}_\mu = 0$$

In 2d, we have the following relationship,

$$\epsilon_{\mu\alpha}\epsilon_{\nu\beta} = \eta_{\mu\nu}\eta_{\alpha\beta - \eta_{\mu\beta}\eta_{\alpha\nu}}$$

In D=2, we have

$$\langle T_{\mu\nu}(p)T_{\rho\sigma}(-p)\rangle = c\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\rho\tilde{p}_\sigma\frac{1}{p^2}$$

$$\eta^{\mu\nu}\tilde{p}_\mu\tilde{p}_\nu = p^2$$

where we used dimensional analysis $0 = 4 - 2 + [F]$ to conclude that $[F] = -2$. Now Fourier transform, we have

$$\langle T^\mu_\mu(x)T_{\rho\sigma}(0)\rangle = c(\partial_\rho\partial_\sigma - \partial^2\eta_{\rho\sigma})$$

It is traceless unless at the conincidental point.

We can also see this by couple $T^\mu_\mu(0)$ to curved metric.

$$\langle T^\mu_\mu(0)\rangle_{h_{\mu\nu}} = \langle T^\mu_\mu(0)e^{\int h^{\rho\sigma}T_{\rho\sigma}}\rangle = \int dx T^\mu_\mu(0)T_{\rho\sigma}(x)h^{\rho\sigma}(x)\nabla_\nu \quad = c(\partial_\rho\partial_\sigma - \Box\eta_{\rho\sigma})h^{\rho\sigma}(x)\nabla_\nu \tag{318}$$

It is traceless unless at the coincidental point. Thus we have

$$T^\mu_\mu = c(\partial_\rho \partial_\sigma - \Box \eta_{\rho\sigma}) h^{\rho\sigma}(x) = cR|_{\eta_{\mu\nu}+h_{\mu\nu}}$$

This gives me

$$T^\mu_\mu = cR$$

The conformal anomalies in $2d$ and $4d$ (not in odd dimensions, as by dimension analysis Ricci scalar R has dimension 2. but there is still something like "c" to provide the order in CFT theories.) provides an order in CFT theories. A UV CFT will follow to an IR CFT but not the other way around. And we have $c_{UV} > c_{IR}$. Especially in $D = 2$, c contains lots of physics. c is interpreted as a conformal anomaly, controls high E behavior of $\rho(E)$, is useful in thermodynamics. Related to entanglement entropy. For $D = 4$, we have

$$T^\mu_\mu = c\mathrm{Weyl}^2 + a\mathrm{Euler}$$

a is what govern the CFT flow: $a_{UV} > a_{IR}$. This can be measured by 3 point function in flat space. $T^\mu_\mu = 0$ is a CFT in flat space time.

In a curved background $T^\mu_\mu \neq 0$. Partition function $Z_{CFT}[S^D]$ measures the quantity we can extract things that we care in curved space.

We can study $\delta \log Z[S^D]$ under $r \to r + dr$, In $D = 3$, there is no conformal anomaly, but there is still $F_{UV} > F_{IR}$ that controls the RG flow.

$$\delta \log Z[g_{\mu\nu}] = \int dx \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} \propto \frac{\delta r}{r} \int dx \sqrt{g} T^\mu_\mu$$

In $D = 2$,

$$\delta \log Z \sim c\frac{\delta r}{r} \int_{S^2} dx \sqrt{g} R \sim c\frac{\delta r}{r}$$

and

$$T^\mu_\mu = cR \to Z[S^2] \propto r^c$$

In $D = 4$,

$$T^\mu_\mu = c\mathrm{Weyl}^2 + a\mathrm{Euler}$$

we have

$$Z[S^4] \propto r^a$$

In $D = 3$,

$$Z[S^3]|_{\text{finite}} = e^{-F}$$

15 Hilbert space of CFT's

15.1 Review of Hilbert space in QFT

Just imposing conformal symmetry will give us bounds on the dimensions of operators of unitary CFT. *A priori*, the dimension of an operator is arbitrary, could be any positive or negative number. But as we will see, unitarity will give a lower bound of the operators [11]. First we will review the Hilbert spaces of QFTs. In a unitary QFT, the symmetries are realized by unitary operators acting on the Hilbert space. In CFT, we will realize the action of the conformal group unitarily in the Hilbert space. In QFT, we can define the Hilbert space $\mathcal{H}$. We take the spacetime and foliate with leaves, with a vector field moves us between leaves. On each of the leaf, we define a Hilbert space on that leaf, we can evolve state from one leaf to the next with an evolution operator that is conjugated to the spacetime operator that moves geometrically between the leaves. We have $U|\psi_1\rangle = |\psi_2\rangle$. See figure at the end of the notes.

Some familiar examples: Minkowski space $R^{1,D}$ foliated with time slice x^0, the operator moves us up and down is the Hamiltonian $U = e^{ix^0 P_0}$. See figure at the end of the notes. When high energy particles are colliding, it is useful to use foliation by light-cones of $R^{1,D}$ and the operator is the light-cone Hamiltonian $U = e^{ix_- P^-}$. See figure at the end of the notes.

15.2 Radial Quantization

There is a very useful slicing in CFT known as radial quantization. This is also useful for massive theories in which the evolution operator will depend on the scale(like Hamiltonian that explicitly depends on time). We will foliate our Euclidean space R^D

[11] Pedro will discuss more about unitarity constraints.

with concentrating S^{D-1} spheres(see figure at end of notes). We can use the metric

$$dx^\mu dx_\mu = dr^2 + r^2 d\Omega_{D-1} \tag{319}$$

The dilatation operator will evolve states between the spheres: $U = r^{i\hat{D}} = e^{i\hat{D}\log r}$. $\hat{D}$ plays the role of Hamiltonian. Rotations $M_{\mu\nu}$ and particular combination of translation and special conformal transformation (Together form the $SO(D)$ generators) preserve the spheres. We need the scale dimension Δ of the dilatation operator and the representation of $SO(D)$ to describe an operator (i.e. the same information we need to determine how fields transform under a conformal transformation.)

States are labeled by $|\Delta, A\rangle$ where Δ is the scaling dimension and A is the $SO(D)$ label, we have

$$\hat{D}|\Delta, A\rangle = -i\Delta|\Delta, A\rangle \tag{320}$$

$$M_{\mu\nu}|\Delta, A\rangle = (\Sigma^R_{\mu\nu})_{AB}|\Delta, B\rangle \tag{321}$$

where $(\Sigma^R_{\mu\nu})_{AB}$ are irreducible representations of $SO(D)$.

CFT has a conformal invariant vacuum state. This is a state annihilated by all conformal generators.

$$\hat{D}|0\rangle = 0 \tag{322}$$

$$M_{\mu\nu}|0\rangle = 0 \tag{323}$$

This is expected and it is analogous to that we have a Poincaré invariant vacuum in a Poincaré invariant theory.

Now we want to construct representations of the conformal algebra on the Hilbert space. We have the following commutator relationship:

$$[\hat{D}, P_\mu] = -iP_\mu \tag{324}$$

$$[\hat{D}, K_\mu] = iK_\mu \tag{325}$$

We find that P_μ will raise the energy and K_μ will lower the energy, as

$$\hat{D}(P_\mu|\Delta, A\rangle) = -i(\Delta+1)P_\mu|\Delta, A\rangle \tag{326}$$

$$\hat{D}(K_\mu|\Delta, A\rangle) = -i(\Delta-1)K_\mu|\Delta, A\rangle \tag{327}$$

In a unitary CFT, if we want the scale dimension to be bounded below (analogous of requiring Hamiltonian the energy to be bounded below). We recover the primary condition for the operators. In other words, there exists a highest weight state that has the lowest energy such that $K_\mu|\Delta, A\rangle = 0$ for a primary state. We can then act on the primary state with arbitrary number of P_μ's and obtain the descendant state. $P_{\mu 1} \ldots P_{\mu n}|\Delta, A\rangle$. This representation is infinitely dimensional.

To see the scaling dimension is analogous to energy, we can use Weyl transformation to see it in a different geometry. We write the metric of R^D as

$$dx^\mu dx_\mu = dr^2 + r^2 d\Omega_{D-1} = r^2\left(\frac{dr^2}{r^2} + d\Omega_{D-1}\right) \tag{328}$$

Now we can make the coordinate transformation $r = e^\tau$, $(dr)^2 = e^{2\tau}d\tau^2$. The metric then is

$$\frac{dx^\mu dx_\mu}{r^2} = d\tau^2 + d\Omega_{D-1} \tag{329}$$

The theory is conformal, then the geometry is a cylinder(see diagram at the end of the notes). The past corresponds to the lower part of the cylinder and smaller circles in the original space. The future corresponds to the higher part of the cylinder and bigger circle in the original space. The dilatation operator is given by

$$\hat{D} = r\frac{\partial}{\partial r} = \frac{\partial}{\partial \tau} \tag{330}$$

which is the Hamiltonian in the cylinder geometry. The state $|in\rangle$ at $\tau = -\infty$ in the cylinder geometry corresponds a state created by the operator at the origin in the original space $O(0)|0\rangle$. This is the state-operator correspondence. It is miraculous that any state in CFT can be written down in term of a local operator at the origin.

15.3 Constraints from unitarity

If the state only have non-negative norm, can we get something interesting?[12] It should be clear that states with different energy are orthogonal. We can only mix states with

[12]We should also emphasize non-unitary CFTs are also important.

the same energy. The first non-trivial constraint comes from level 1 state:

$$||P_\mu|\Delta, A\rangle||^2 \geq 0 \tag{331}$$

Let us look at the following matrix,

$$T_{A\nu,B\mu} =< A|P_\nu^\dagger P_\mu|B > \tag{332}$$

It needs to have non-negative eigenvalues.

Now we define the conjugate of P_ν as follows. The in-state is inside the unit sphere and the out-state is outside the unit sphere and to connect them, we use inversion. So the conjugate of an operator is constructed by conjugate by inversion.

$$O^\dagger = IOI \tag{333}$$

We have studied this already, and we have the following relationship,

$$IP_\mu I = K_\mu \tag{334}$$

$$IK_\mu I = P_\mu \tag{335}$$

$$I\hat{D}I = -\hat{D} \tag{336}$$

$$IM_{\mu\nu}I = M_{\mu\nu} \tag{337}$$

On the cylinder geometry, we have $O^\dagger(\tau) = O(-\tau)$, where

$$O(\tau) = e^{H\tau}O(0)e^{-H\tau} \tag{338}$$

$$O(\tau)^\dagger = e^{-H\tau}O(0)e^{H\tau} = O(-\tau) \tag{339}$$

This is time reversal, using our coordinate transformation to go back to the original picture, it is inversion. This shows that $\hat{D}$ is anti-Hermitian. Thus we have

$$T_{A\nu,B\mu} =< A|K_\nu P_\mu|B > \tag{340}$$

by using the most non-trivial commutator

$$[P_\mu, K_\nu] = -2i(\eta_{\mu\nu}\hat{D} + M_{\mu\nu}) \tag{341}$$

and the fact $|A\rangle$ is primary. So we have, in the orthonormal base,

$$T_{A\nu,B\mu} = 2i\left(\Delta\delta_{AB}\delta_{\mu\nu} + \langle A|M_{\mu\nu}|B\rangle\right) \tag{342}$$

In other words, the sum of Δ and the smallest eigenvalue of $(\Sigma^R_{\mu\nu})^{AB} = \langle A|M_{\mu\nu}|B\rangle$ is non-negative. We will compute this $(\Sigma^R_{\mu\nu})_{AB}$ by using a trick in quantum mechanics. Let us rewrite it in the following way,

$$(\Sigma^R_{\mu\nu})^{AB} = \frac{1}{2}(\Sigma_{\mu\nu})^{AB}_{\alpha\beta}(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu) \tag{343}$$

$$= \Sigma_R \cdot V_{vector} \tag{344}$$

As $\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu$ is the vector representation of Lorentz group. Thus we can have

$$\Sigma_R \cdot V = \frac{1}{2}((\Sigma_R + \Sigma_V)^2 - \Sigma_R^2 - \Sigma_V^2) \tag{345}$$

$$= \frac{1}{2}(C_2(R\otimes V) - C_2(R) - C_2(V)) \tag{346}$$

where C_2 is the Casimir. This is the same trick used in quantum mechanics dealing with spin-orbital coupling. The conclusion is that

$$\Delta \geq \mathrm{Min}(C_2(R\otimes V) - C_2(R) - C_2(V)) \tag{347}$$

First let us look at $D = 3$. The group is $SO(3)$, we label the representation with spin j, and vector representation with $j = 1$ and we have

$$j\otimes 1 = j-1\oplus j\oplus j+1 \qquad C_2(j) = j(j+1) \tag{348}$$

and thus we have $\Delta \geq j+1$ in $D = 3$ so the scalar operator has dimension at least 1. For $D = 4$, as $SO(4)\sim SU(2)\times SU(2)$, we label any representation with (j_1, j_2) and the vector representation is given by $(\frac{1}{2}, \frac{1}{2})$. We get

$$\Delta \geq j_1 + j_2 + 2 \quad \text{if} \quad j_1 j_2 \neq 0 \tag{349}$$

$$\Delta \geq j_1 + j_2 + 1 \quad \text{if} \quad j_1 j_2 = 0 \tag{350}$$

For any general dimension, at level 1, the conclusion is that scalars have $\Delta \geq 0$, spinors have $\Delta \geq \frac{D-1}{2}$, and vectors have $\Delta \geq D-1$. The constraints stop at level 1 for spinors and the vectors. If we study the problem at level 2 for scalars $||P_\mu P_\nu|\Delta, A\rangle||^2 \geq 0$ and we have $\Delta \geq \frac{D-2}{2}$, which is the dimension of the free scalar field. The constraint stops at level 2 for scalars.

What can we say when the bound is saturated(the norm is 0)? For scalar, at level 1, we have $P_\mu|\Delta = 0\rangle = 0 \Leftrightarrow [P_\mu, O(0)]|0\rangle$. $O(0) \propto \mathbf{1}$ because identity is the only operator that has dimension 0.

At level 2, the scalar operator with $\Delta = \frac{D-2}{2}$ must obey the free wave equation $P^2 O(0) = 0$. When spinor hits the unitary bound at level 1, $\Delta = \frac{D-1}{2}$ and we find the spinor satisfies Dirac equation.

At level 1, the vector operator hits the bound at $\Delta = D - 1$. We have $P_\mu|\Delta = D - 1 >= 0$, thus it is associated with a conserved current $\partial^\mu j_\mu = 0$.

16 Operator Product Expansion and Conformal Bootstrap

The idea is that two close operators close to each other look like one from far away,

$$O_i(x_1)O_j(x_2) = \sum_k f_{ijk}(x_{12}, \partial_2)O_k(x_2)$$

OPE should always be understood in the sense of correlation function

$$\langle O_i(x_1)O_j(x_2)\ldots\rangle$$

Under Lorentz $M_{\mu\nu}$ and dilatation D, we know that

$$f_{ijk}(x, \partial_2) \propto |x|^{\Delta_k - \Delta_i - \Delta_j}(1 + \alpha x^\mu \partial_\mu + \beta x^\mu x^\nu \partial_\mu \partial_\nu + \gamma x^2 \partial^2 + \ldots)$$

To find K_μ, all coefficients $\alpha, \beta, \gamma, \ldots$ are fixed by conformal symmetry. Let us look at 3 point correlation function

$$\langle O_i(x_1)O_j(x_2)O_k(x_3)\rangle = \sum_{k'} f_{ijk'}(x_12, \partial_k)\langle O_{k'}(x_1)O_k(x_3)\rangle$$

The left hand side is fixed to be

$$\frac{c_{ijk'}}{\cdots}$$

The right hand side the 2 point correlation function is fixed to be

$$\langle O_{k'}(x_1)O_k(x_3)\rangle = \frac{\delta_{\Delta k,\Delta k'}}{(x_2-x_3)^{2\Delta_k}}$$

Now expand both sides in $\frac{|x_{12}|}{|x_{23}|} < 1$. It should converge with a convergent radius of x_{23}. We have

$$f_{ijk} \propto c_{ijk}$$

Now consider 4 point correlation function,

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$

and use OPE on first pair of fields and then on the second pair of fields, this should give us

$$\sum_{O,O'} C_{\phi\phi O}C_{\phi\phi O'}c_a(x_{12},\partial_2)c_b(x_{34},\partial_4)\langle O^a(x_2)O^b(x_4)\rangle$$

where

$$\langle O^a(x_2)O^b(x_4)\rangle = \frac{I^{ab}(x_2-x_4)\delta_{OO'}}{x_{24}^{2\Delta_O}}$$

which equals to

$$\frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\sum_O C^2_{\phi\phi O}g_{\Delta_O,l_O}(x)$$

and

$$g_{\Delta_O,l_O}(x) = x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}c_a(x_{12},\partial_2)c_b(x_{34},\partial_4)\frac{I^{ab}(x_{24})}{x_{24}^{2\Delta_O}}$$

$$g(u,v) = \sum_O c^2_{\phi\phi O}g_{\Delta_O,l_O}(u,v)$$

$g_{\Delta_O,l_O}(u,v)$ are known as the conformal blocks, which are completely fixed by conformal symmetry. This is analogous to partial wave decomposition. The conformal blocks are the harmonics of $SO(D+1,1)$. For D even, they are products of hypergeometric functions. For D odd, they are known. The crossing symmetry refers to the idea that fusing operator pairs with 12 and 34 should yield the same result as fusing 13 and 24. It is represented by the following diagram. Note they are in no way related to Feynman diagrams. Now exchanging field 1 and 3 will lead to the following consequences,

$$v^{\Delta\phi}g(u,v) = u^{\Delta\phi}g(v,u)$$

or

$$\sum_O c^2_{\phi\phi O}(v^{\Delta_\phi} g_{\Delta_O,l_O}(u,v) - u^{\Delta_\phi} g_{\Delta_O,l_O}(v,u) = 0$$

This will constraint admissable CFT data (Δ_I, C_{IJK}). handful of CFT in IR.

In $D \geq 3$ crossing symmetry are all the constraints. 5 point function follows from crossing symmetry. In $D = 2$ crossing symmetry is not enough, in particular modular invariance plays a roll.

17 D>2 Conformal Field Theories

Before descending to two dimensional CFTs let's do a quick summary of the CFTs we have in higher dimensions

1. Free field theories, since they don't have a mass scale.

2. "Non-trivial CFTs" which can be obtained by flowing in the direction of a relevant operator until we reach a non-trivial fixed (Wilson-Fischer) point. The beta function in general is $\beta(\lambda) = -\epsilon\lambda + b_0(\lambda)$ and we cancel the classical contribution with the one loop correction.

3. Banks-Zaks trick. In the context of gauge theories $\beta(g) = -b_0 g^3 + b_1 g^5$; $b_0 > 0$ $b_1 > 0$ we can cancel the one-loop contribution with the two-loop contribution.

4. Supersymmetric field theories with a fine-tuned spectrum are CFTs, since fermions and bosons contribute with different signs to the beta function we have many SUSY CFTs.

17.1 Review

Constraints imposed by conformal symmetry on correlation functions of primaries.

- 1-point function $\langle\theta_1\rangle = 0$ unless $\theta_\Delta = \mathbb{I}$

- 2-point and 3-point functions: the position dependence is fixed $\langle\theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\theta_{\Delta_3}(x_3)\rangle \propto c_{123}$

- Greater than or four point functions depend on functions of the cross ratios using η_1 and η_2

All the correlation functions have a scalar denominator and a tensor numerator.

(a) The inversion tensor is given by

$$I_{\mu\nu}(x-y) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} \tag{351}$$

(b) The tensor part of the numerator is given by

$$Z^\mu = \frac{(x-z)^\mu}{(x-z)^2} - \frac{(y-z)^\mu}{(y-z)^2} \tag{352}$$

(c) So we have

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(y)\rangle = \frac{c\alpha_{\mu\nu,\rho\sigma}(x-y)}{|x-y|^{2D}} \tag{353}$$

where we determine α from the symmetry and transformation properties of $T_{\mu\nu}$

$$\alpha_{\mu\nu,\rho\sigma}(x) = \frac{1}{2}\left(I_{\mu\rho}(x)I_{\nu\sigma}(x) + I_{\mu\sigma}(x)I_{\nu\rho}(x)\right) - \frac{1}{D}\delta_{\mu\nu}\delta_{\rho\sigma} \tag{354}$$

17.2 4D CFTs: Wilson Fischer Fixed Points

Before we return to 2D CFTs, we will give some examples of a 4D CFT that we can fully calculate.

One way to construct such a CFT is to use "Wilson-Fischer" fixed points. They did this by trying to find the fixed points of a theory by starting with a $\lambda\phi^4$ theory and trying to identity a fixed point of the Renormalization Group by working in $D = 4 - \varepsilon$ dimensions. This dimensionality introduces a small mass dimension.

The Beta function is now given by

$$\beta(\lambda) = \lambda(-\varepsilon + \underbrace{|c|\lambda^2}_{\text{1-loop}}) \tag{355}$$

which gives us a non-trivial solution (fixed point) λ^*

Now they took $\varepsilon = 1$ and found valid results.

Similarly, we can consider $D = 2$, and the Lagrangian

$$\mathcal{L} = g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \tag{356}$$

in order to have non-trivial zeros of the β-function, we need to extrapolate to $D \to 2+\varepsilon$.

17.3 4D CFTs: Yang-Mills

We now consider a Yang Mills theory with matter. The β-function is given by

$$\beta(g) = -b_0 g^3 + b_1 g^5 \tag{357}$$

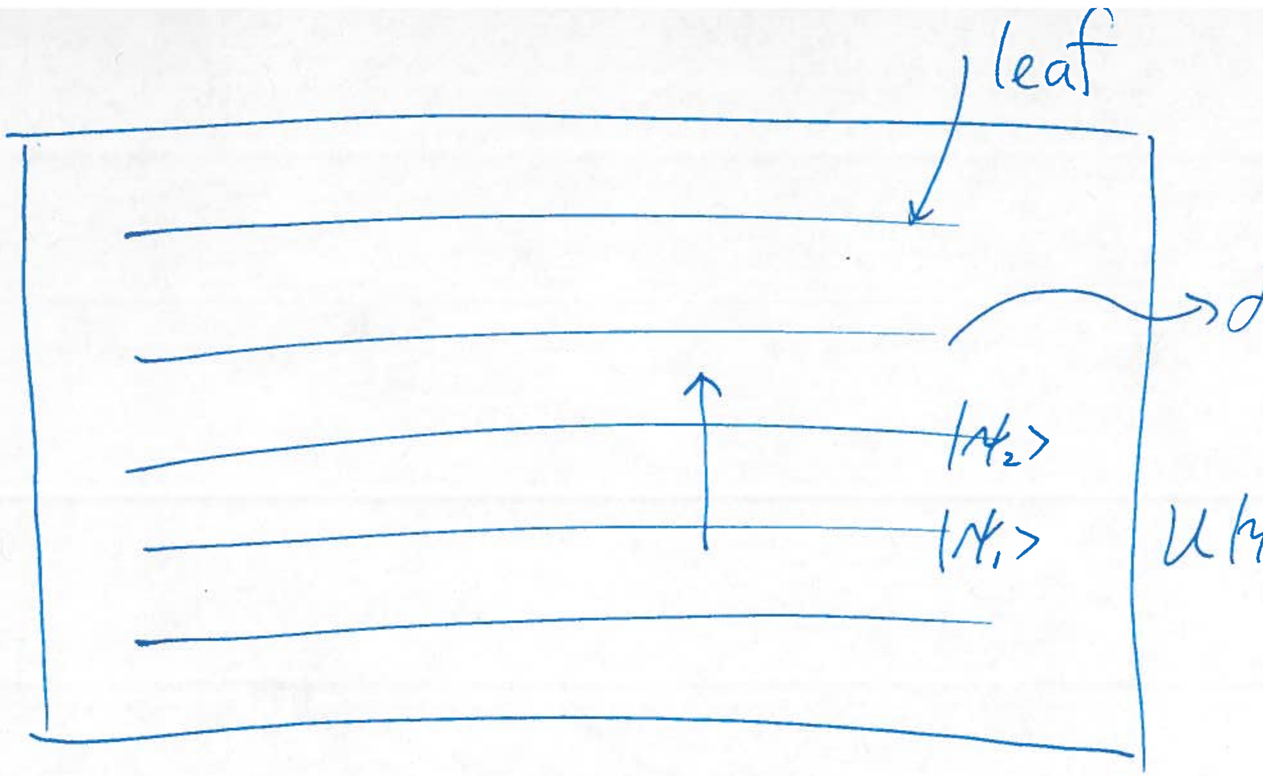
where $b_0, b_1 > 0$. There now exist non-trivial fixed points

$$\beta(g^*) = 0, \qquad g_*^2 = \frac{b_0}{b_1} << 1 \tag{358}$$

which is the Banks-Zaks condition.

17.4 4D CFTs: SuperSymmetric Gauge Theories

The other way to construct a 4D CFT is to use a Supersymmetric gauge theory. Here $\beta(g) = 0$ is the simplest example of $N = 4$ symmetry which is dual to AdS_5 Quantum Gravity.


leaf
$|\psi_2\rangle$
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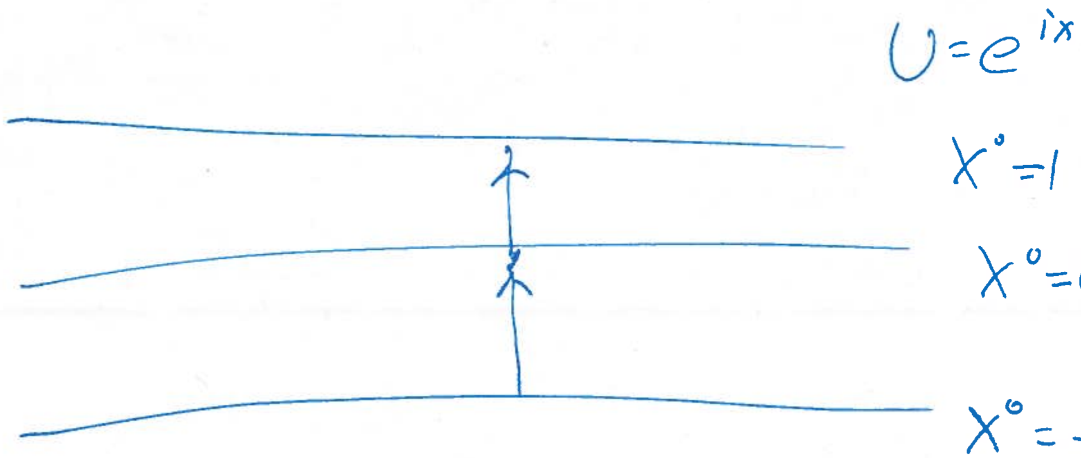
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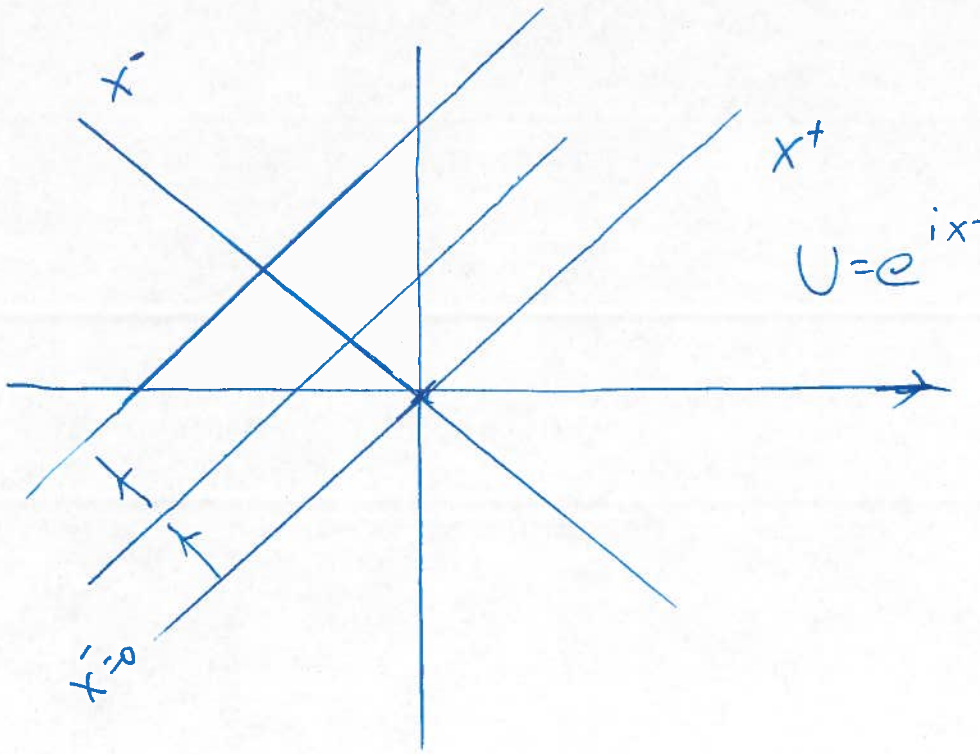
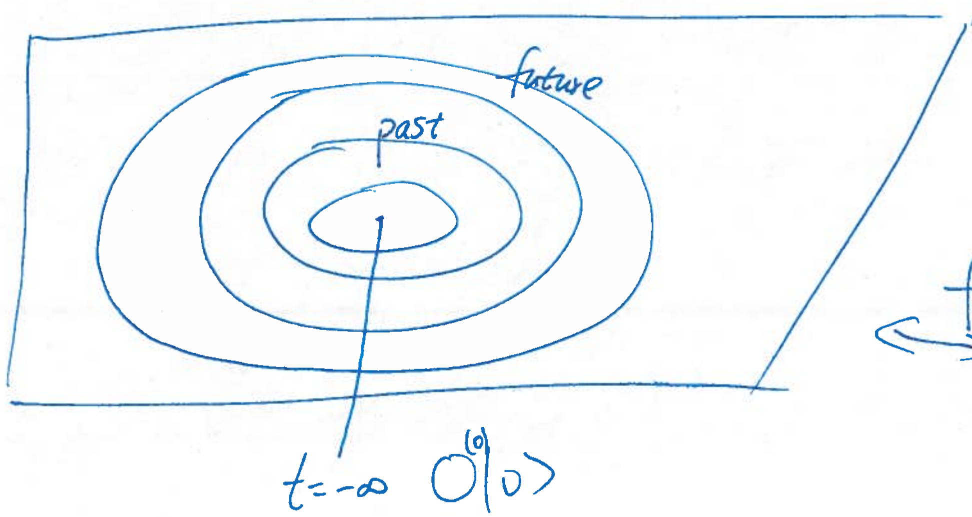
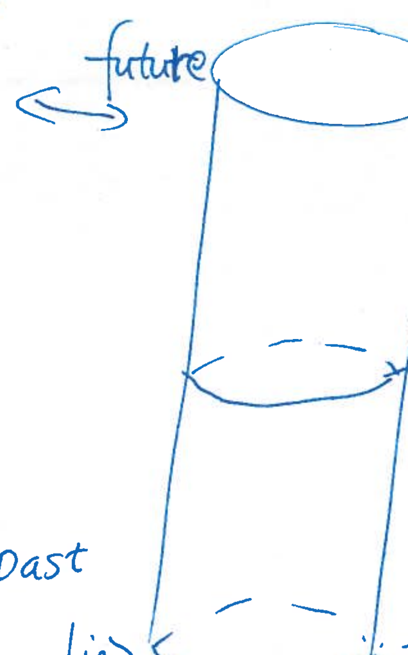

x^-
x^+
$x^-=0$


future
past
$t=-\infty$ $O^{(0)}|0\rangle$


future
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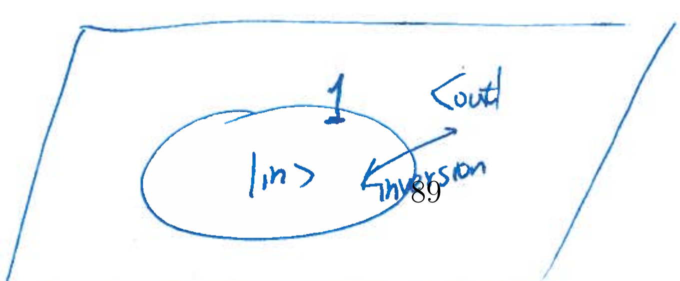
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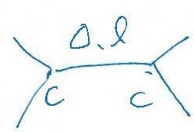
$\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle = \sum$ 

$= \sum$