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Homework 1

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Contents

1 Cartan in a FLRW universe

(a) The Friedmann-Lemaitre-Robinson-Walker (FLRW) metric two-form describes a spacetime with spacelike foliation in homogeneous and isotropic hypersurfaces. In a coordinate chart with coordinates $x^\mu = \{t,\theta,\phi,r\}$ making the isotropy and foliation manifest, this metric reads

$$g_{\mu\nu}\underline{d}x^\mu \otimes \underline{d}x^\nu \equiv \underline{d}t \otimes \underline{d}t - a^2(t)\left(\frac{\underline{d}r \otimes \underline{d}r}{1-kr^2} + r^2\left(\underline{d}\theta \otimes \underline{d}\theta + \sin^2\theta \underline{d}\phi \otimes \underline{d}\phi\right)\right)$$

where $\{\underline{d}x^\mu\}_{\mu=0}^3 = \{\underline{d}t, \underline{d}\theta, \underline{d}\phi, \underline{d}r\}$ are the coordinate on-forms dual to the vector basis $\underline{e}_a = \{\partial_t, \partial_\theta, \partial_\phi, \partial_r\}$, $a(t) > 0$ is the scale factor and $k = 0, -1, 1$ gives the sign of the curvature of the spacelike hypersurfaces (respectively flat, Anti-de Sitter, de Sitter). In what follows, the tensor products are implicit. At every point in our chart, we define an orthonormal basis of one-forms $\underline{\omega}^a = c^a_\mu \underline{d}x^\mu$ such that $g_{\mu\nu}\underline{d}x^\mu \underline{d}x^\nu = \eta_{ab}\underline{\omega}^a\underline{\omega}^b$ where η_{ab} is the Minkowski metric components with signature $(+,-,-,-)$. We can write

$$\begin{aligned} g_{\mu\nu}\underline{d}x^\mu \underline{d}x^\nu &= \underline{d}t\underline{d}t - \left(\frac{a(t)\underline{d}r}{\sqrt{1-kr^2}}\right)\left(\frac{a(t)\underline{d}r}{\sqrt{1-kr^2}}\right) - \left(a(t)r\underline{d}\theta\right)\left(a(t)r\underline{d}\theta\right) - (a(t)r\sin\theta\underline{d}\phi)(a(t)r\sin\theta\underline{d}\phi) \\ &= \underline{\omega}^0\underline{\omega}^0 - \underline{\omega}^1\underline{\omega}^1 - \underline{\omega}^2\underline{\omega}^2 - \underline{\omega}^3\underline{\omega}^3 \end{aligned}$$

where $\{\underline{\omega}^a\}_{a=0}^3 = \{\underline{d}t,\ a(t)r\underline{d}\theta,\ a(t)r\sin\theta\underline{d}\phi,\ \frac{a(t)}{\sqrt{1-kr^2}}\underline{d}r\}$ is shown to satisfy the orthonormality condition. We note that the resulting choice of basis is unique up to a local Lorentz transformation (which preserves orthonormality).

(b) To calculate the connection one-forms $\underline{\theta}^a{}_b$, we use the orthonormal basis found in (a) and Cartan's first structure equation for vanishing torsion to get

$$\underline{\theta}^a{}_b \wedge \underline{\omega}^b = -\underline{d}\underline{\omega}^a = \begin{cases} -\partial_\mu(1)\ \underline{d}x^\mu \wedge \underline{d}t \\ -\partial_\mu(a(t)r)\ \underline{d}x^\mu \wedge \underline{d}\theta \\ -\partial_\mu(a(t)r\sin\theta)\ \underline{d}x^\mu \wedge \underline{d}\phi \\ -\partial_\mu\left(\frac{a(t)}{\sqrt{1-kr^2}}\right)\ \underline{d}x^\mu \wedge \underline{d}r \end{cases} = \begin{cases} 0 \\ -a'(t)r\underline{d}t \wedge \underline{d}\theta - a(t)\underline{d}r \wedge \underline{d}\theta \\ -a'(t)r\sin\theta\underline{d}t \wedge \underline{d}\phi - a(t)\sin\theta\underline{d}r \wedge \underline{d}\phi - a(t)r\cos\theta\underline{d}\theta \wedge \underline{d}\phi \\ -\frac{a'(t)}{\sqrt{1-kr^2}}\underline{d}t \wedge \underline{d}r - [\cdots]\underline{d}r \wedge \underline{d}r \end{cases}$$

$$= \begin{cases} 0 \\ \frac{a'(t)}{a(t)}\underline{\omega}^1 \wedge \underline{\omega}^0 + \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1 \wedge \underline{\omega}^3 \\ \frac{a'(t)}{a(t)}\underline{\omega}^2 \wedge \underline{\omega}^0 + \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2 \wedge \underline{\omega}^3 + \frac{1}{a(t)r}\cot\theta\underline{\omega}^2 \wedge \underline{\omega}^1 \\ \frac{a'(t)}{a(t)}\underline{\omega}^3 \wedge \underline{\omega}^0 \end{cases} = \begin{cases} \underline{\theta}^0{}_b \wedge \underline{\omega}^b \\ \underline{\theta}^1{}_b \wedge \underline{\omega}^b \\ \underline{\theta}^2{}_b \wedge \underline{\omega}^b \\ \underline{\theta}^3{}_b \wedge \underline{\omega}^b \end{cases}$$

Since the $\wedge$ product with $\underline{\omega}^b$ maps $\underline{\omega}^{c\neq b}$ to linearly independent two-forms, we can read the coefficients of $\underline{\omega}^{c\neq b}$ preceding the $\wedge$ product in the previous expressions. We have

$$\begin{cases} \underline{\theta}^0{}_1 = [\cdots]\underline{\omega}^1, \quad \underline{\theta}^0{}_2 = [\cdots]\underline{\omega}^2, \quad \underline{\theta}^0{}_3 = [\cdots]\underline{\omega}^3 \\ \underline{\theta}^1{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^1 + [\cdots]\underline{\omega}^0, \quad \underline{\theta}^1{}_2 = [\cdots]\underline{\omega}^2, \quad \underline{\theta}^1{}_3 = \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1 + [\cdots]\underline{\omega}^3 \\ \underline{\theta}^2{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^2 + [\cdots]\underline{\omega}^0, \quad \underline{\theta}^2{}_3 = \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2 + [\cdots]\underline{\omega}^3, \quad \underline{\theta}^2{}_1 = \frac{1}{a(t)r}\cot\theta\underline{\omega}^2 + [\cdots]\underline{\omega}^1 \\ \underline{\theta}^3{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^3 + [\cdots]\underline{\omega}^0, \quad \underline{\theta}^3{}_1 = [\cdots]\underline{\omega}^1, \quad \underline{\theta}^3{}_2 = [\cdots]\underline{\omega}^2 \end{cases}$$

where $[\cdots]$ terms represent the terms mapped to 0 by the $\wedge$ product from which information about $\underline{\theta}^a{}_b$ was read. From the first line we can also read $\underline{\theta}^0{}_{1,2,3} = [\cdots]\underline{\omega}^{1,2,3}$

To fully determine the one-forms components from these relations, we invoke the relation $\underline{\theta}_{ab} + \underline{\theta}_{ba} = \underline{d}g_{ab}$ where $\underline{\theta}_{ba} = g_{bc}\underline{\theta}^c{}_a$. Recalling that in our orthonormal basis $g_{ab} = \eta_{ab}$, we get the antisymmetry relation $\underline{\theta}_{ab} + \underline{\theta}_{ba}$. It follows that $\underline{\theta}^a{}_a = 0$, $\forall a$ and we can use it to determine [...]. Making the relation between $\underline{\theta}^b{}_a$ and $\underline{\theta}^a{}_b$ more explicit yields

$$\begin{cases} b \text{ spacelike} \implies \underline{\theta}^b{}_a = \eta^{bc}\underline{\theta}_{ca} = (-1)\underline{\theta}_{ba} = \underline{\theta}_{ab} \implies \begin{cases} a \text{ spacelike} \implies \underline{\theta}^b{}_a = -\underline{\theta}^a{}_b \\ a \text{ timelike} \implies \underline{\theta}^b{}_a = \underline{\theta}^a{}_b \end{cases} \\ b \text{ timelike} \implies \underline{\theta}^b{}_a = \eta^{bc}\underline{\theta}_{ca} = \underline{\theta}_{ba} = -\underline{\theta}_{ab} \implies \begin{cases} a \text{ spacelike} \implies \underline{\theta}^b{}_a = \underline{\theta}^a{}_b \\ a \text{ timelike} \implies \underline{\theta}^b{}_a = -\underline{\theta}^a{}_b \quad \text{never happens } (a \neq b) \end{cases} \end{cases}$$

Comparing $\underline{\theta}^a{}_b$ with $\underline{\theta}^b{}_a$, we finally see

$$[\cdots]\underline{\omega}^1 = \underline{\theta}^0{}_1 = \underline{\theta}^1{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^1 + [\cdots]\underline{\omega}^0 \iff \underline{\theta}^1{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^1, \quad \underline{\theta}^0{}_1 = \frac{a'(t)}{a(t)}\underline{\omega}^1$$

$$[\cdots]\underline{\omega}^2 = \underline{\theta}^0{}_2 = \underline{\theta}^2{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^2 + [\cdots]\underline{\omega}^0 \iff \underline{\theta}^2{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^2, \quad \underline{\theta}^0{}_2 = \frac{a'(t)}{a(t)}\underline{\omega}^2$$

$$[\cdots]\underline{\omega}^3 = \underline{\theta}^0{}_3 = \underline{\theta}^3{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^3 + [\cdots]\underline{\omega}^0 \iff \underline{\theta}^3{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^3, \quad \underline{\theta}^0{}_3 = \frac{a'(t)}{a(t)}\underline{\omega}^3$$

$$[\cdots]\underline{\omega}^2 = \underline{\theta}^1{}_2 = -\underline{\theta}^2{}_1 = -\frac{1}{a(t)r}\cot\theta\underline{\omega}^2 - [\cdots]\underline{\omega}^1 \iff \underline{\theta}^1{}_2 = -\frac{1}{a(t)r}\cot\theta\underline{\omega}^2, \quad \underline{\theta}^2{}_1 = \frac{1}{a(t)r}\cot\theta\underline{\omega}^2$$

$$[\cdots]\underline{\omega}^2 = \underline{\theta}^3{}_2 = -\underline{\theta}^2{}_3 = -\frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2 - [\cdots]\underline{\omega}^3 \iff \underline{\theta}^3{}_2 = -\frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2, \quad \underline{\theta}^3{}_2 = \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2$$

$$[\cdots]\underline{\omega}^1 = \underline{\theta}^3{}_1 = -\underline{\theta}^1{}_3 = -\frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1 - [\cdots]\underline{\omega}^3 \iff \underline{\theta}^3{}_1 = -\frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1, \quad \underline{\theta}^1{}_3 = \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1$$

(c) The curvature two-forms are obtained from the connection one-forms calculated above with the relation $\underline{R}^a{}_b = \underline{d}\underline{\theta}^a{}_b + \underline{\theta}^a{}_c \wedge \underline{\theta}^c{}_b$. Using $H = a'(t)/a(t)$, $A = \frac{1}{a(t)r}\sqrt{1-kr2}$ and $B = \frac{1}{a(t)r}\cot\theta$ the connection one form can be organised as

$$[\underline{\theta}^a{}_b] = \begin{pmatrix} 0 & H\underline{\omega}^1 & H\underline{\omega}^2 & H\underline{\omega}^3 \\ H\underline{\omega}^1 & 0 & B\underline{\omega}^2 & A\underline{\omega}^1 \\ H\underline{\omega}^2 & -B\underline{\omega}^2 & 0 & A\underline{\omega}^2 \\ H\underline{\omega}^3 & -A\underline{\omega}^1 & -A\underline{\omega}^2 & 0 \end{pmatrix}$$

and the second term in the curvature two-forms can be expressed as a matrix multiplication where the elementwise multiplication is a $\wedge$. We have

$$\begin{aligned}
&[\underline{\theta}^a{}_c \wedge \underline{\theta}^c{}_b] \\
&= \begin{pmatrix} 0 & H\underline{\omega}^1 & H\underline{\omega}^2 & H\underline{\omega}^3 \\ H\underline{\omega}^1 & 0 & B\underline{\omega}^2 & A\underline{\omega}^1 \\ H\underline{\omega}^2 & -B\underline{\omega}^2 & 0 & A\underline{\omega}^2 \\ H\underline{\omega}^3 & -A\underline{\omega}^1 & -A\underline{\omega}^2 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & H\underline{\omega}^1 & H\underline{\omega}^2 & H\underline{\omega}^3 \\ H\underline{\omega}^1 & 0 & B\underline{\omega}^2 & A\underline{\omega}^1 \\ H\underline{\omega}^2 & -B\underline{\omega}^2 & 0 & A\underline{\omega}^2 \\ H\underline{\omega}^3 & -A\underline{\omega}^1 & -A\underline{\omega}^2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & H\underline{\omega}^2 \wedge (-B\underline{\omega}^2) + H\underline{\omega}^3 \wedge (-A\underline{\omega}^1) & H\underline{\omega}^1 \wedge (B\underline{\omega}^2) + H\underline{\omega}^3 \wedge (-A\underline{\omega}^2) & 0 \\ A\underline{\omega}^1 \wedge (H\underline{\omega}^3) & 0 & H\underline{\omega}^1 \wedge (H\underline{\omega}^2) + A\underline{\omega}^1 \wedge (-A\underline{\omega}^2) & H\underline{\omega}^1 \wedge (H\underline{\omega}^3) \\ -B\underline{\omega}^2 \wedge (H\underline{\omega}^1) + A\underline{\omega}^2 \wedge (H\underline{\omega}^3) & H\underline{\omega}^2 \wedge (H\underline{\omega}^1) + A\underline{\omega}^2 \wedge (-A\underline{\omega}^1) & 0 & H\underline{\omega}^2 \wedge (H\underline{\omega}^3) - B\underline{\omega}^2 \wedge (A\underline{\omega}^1) \\ 0 & H\underline{\omega}^3 \wedge (H\underline{\omega}^1) & H\underline{\omega}^3 \wedge (H\underline{\omega}^2) - A\underline{\omega}^1 \wedge (B\underline{\omega}^2) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (HA)\underline{\omega}^1 \wedge \underline{\omega}^3 & (HB)\underline{\omega}^1 \wedge \underline{\omega}^2 - (HA)\underline{\omega}^3 \wedge \underline{\omega}^2 & 0 \\ (HA)\underline{\omega}^1 \wedge \underline{\omega}^3 & 0 & (H^2 - A^2)\underline{\omega}^1 \wedge \underline{\omega}^2 & (H^2)\underline{\omega}^1 \wedge \underline{\omega}^3 \\ (HB)\underline{\omega}^1 \wedge \underline{\omega}^2 - (HA)\underline{\omega}^3 \wedge \underline{\omega}^2 & -(H^2 - A^2)\underline{\omega}^1 \wedge \underline{\omega}^2 & 0 & (H^2)\underline{\omega}^2 \wedge \underline{\omega}^3 - (AB)\underline{\omega}^2 \wedge \underline{\omega}^1 \\ 0 & -(H^2)\underline{\omega}^1 \wedge \underline{\omega}^3 & -((H^2)\underline{\omega}^2 \wedge \underline{\omega}^3 - (AB)\underline{\omega}^2 \wedge \underline{\omega}^1) & 0 \end{pmatrix}
\end{aligned}$$

Then, the first term in the curvature two-forms reads

$$[\underline{d}\theta^a{}_b] = \begin{pmatrix} 0 & H'\underline{d}t \wedge \underline{\omega}^1 + H\underline{d}\underline{\omega}^1 & H'\underline{d}t \wedge \underline{\omega}^2 + H\underline{d}\underline{\omega}^2 & H'\underline{d}t \wedge \underline{\omega}^3 + H\underline{d}\underline{\omega}^3 \\ +[\cdots] & 0 & (\partial_r B\underline{d}r + \partial_\theta B\underline{d}\theta + \partial_t B\underline{d}t) \wedge \underline{\omega}^2 + B\underline{d}\underline{\omega}^2 & (\partial_r A\underline{d}r + \partial_t A\underline{d}t) \wedge \underline{\omega}^1 + A\underline{d}\underline{\omega}^1 \\ +[\cdots] & -[\cdots] & 0 & (\partial_r A\underline{d}r + \partial_t A\underline{d}t) \wedge \underline{\omega}^2 + A\underline{d}\underline{\omega}^2 \\ +[\cdots] & -[\cdots] & -[\cdots] & 0 \end{pmatrix}$$

$$\begin{cases} H'\underline{d}t \wedge \underline{\omega}^1 + H\underline{d}\underline{\omega}^1 = H'\underline{\omega}^0 \wedge \underline{\omega}^1 + H^2\underline{\omega}^0 \wedge \underline{\omega}^1 + (HA)\underline{\omega}^3 \wedge \underline{\omega}^1 \\ H'\underline{d}t \wedge \underline{\omega}^2 + H\underline{d}\underline{\omega}^2 = H'\underline{\omega}^0 \wedge \underline{\omega}^2 + H^2\underline{\omega}^0 \wedge \underline{\omega}^2 + (HA)\underline{\omega}^3 \wedge \underline{\omega}^2 + (HB)\underline{\omega}^1 \wedge \underline{\omega}^2 \\ H'\underline{d}t \wedge \underline{\omega}^3 + H\underline{d}\underline{\omega}^3 = H'\underline{\omega}^0 \wedge \underline{\omega}^3 + H^2\underline{\omega}^0 \wedge \underline{\omega}^3 \\ (\partial_r B\underline{d}r + \partial_\theta B\underline{d}\theta + \partial_t B\underline{d}t) \wedge \underline{\omega}^2 + B\underline{d}\underline{\omega}^2 = -(AB)\underline{\omega}^3 \wedge \underline{\omega}^2 - \frac{\csc^2(\theta)}{r^2 a(t)^2}\underline{\omega}^1 \wedge \underline{\omega}^2 - (BH)\underline{\omega}^0 \wedge \underline{\omega}^2 + (BH)\underline{\omega}^0 \wedge \underline{\omega}^2 + (AB)\underline{\omega}^3 \wedge \underline{\omega}^2 + B^2\underline{\omega}^1 \wedge \underline{\omega}^2 \\ (\partial_r A\underline{d}r + \partial_t A\underline{d}t) \wedge \underline{\omega}^1 + A\underline{d}\underline{\omega}^1 = (-A^2 + \frac{k}{a(t)^2})\underline{\omega}^3 \wedge \underline{\omega}^1 - (HA)\underline{\omega}^0 \wedge \underline{\omega}^1 + (HA)\underline{\omega}^0 \wedge \underline{\omega}^1 + A^2\underline{\omega}^3 \wedge \underline{\omega}^1 \\ (\partial_r A\underline{d}r + \partial_t A\underline{d}t) \wedge \underline{\omega}^2 + A\underline{d}\underline{\omega}^2 = (-A^2 + \frac{k}{a(t)^2})\underline{\omega}^3 \wedge \underline{\omega}^2 - (HA)\underline{\omega}^0 \wedge \underline{\omega}^2 + (HA)\underline{\omega}^0 \wedge \underline{\omega}^2 + A^2\underline{\omega}^3 \wedge \underline{\omega}^2 + (AB)\underline{\omega}^1 \wedge \underline{\omega}^2 \end{cases}$$

$$= \begin{cases} (H' + H^2)\underline{\omega}^0 \wedge \underline{\omega}^1 + (HA)\underline{\omega}^3 \wedge \underline{\omega}^1 \\ (H' + H^2)\underline{\omega}^0 \wedge \underline{\omega}^2 + (HA)\underline{\omega}^3 \wedge \underline{\omega}^2 + (HB)\underline{\omega}^1 \wedge \underline{\omega}^2 \\ (H' + H^2)\underline{\omega}^0 \wedge \underline{\omega}^3 \\ -\left(\frac{\csc^2(\theta)}{r^2 a(t)^2} - B^2\right)\underline{\omega}^1 \wedge \underline{\omega}^2 \\ \frac{k}{a(t)^2}\underline{\omega}^3 \wedge \underline{\omega}^1 \\ \frac{k}{a(t)^2}\underline{\omega}^3 \wedge \underline{\omega}^2 + (AB)\underline{\omega}^1 \wedge \underline{\omega}^2 \end{cases}$$

Summing the two terms leads to

$$[\underline{R}^a{}_b] = \begin{pmatrix} 0 & (H' + H^2)\underline{\omega}^0 \wedge \underline{\omega}^1 & (H' + H^2)\underline{\omega}^0 \wedge \underline{\omega}^2 + (2HB)\underline{\omega}^1 \wedge \underline{\omega}^2 & (H' + H^2)\underline{\omega}^0 \wedge \underline{\omega}^3 \\ +[\cdots] & 0 & (H^2 - A^2 + B^2 - \frac{\csc^2(\theta)}{r^2 a(t)^2})\underline{\omega}^1 \wedge \underline{\omega}^2 & (H^2 - \frac{k}{a(t)^2})\underline{\omega}^1 \wedge \underline{\omega}^3 \\ +[\cdots] & -[\cdots] & 0 & (H^2 - \frac{k}{a(t)^2})\underline{\omega}^2 \wedge \underline{\omega}^3 \\ +[\cdots] & -[\cdots] & -[\cdots] & 0 \end{pmatrix}$$

(d) From each curvature two-form found above, we can extract the components of the Riemann tensor with $\underline{R}^a{}_b = \frac{1}{2}R^a{}_{bcd}\underline{\omega}^c \wedge \underline{\omega}^d$. To make these components more transparent we use the new notation $0 \to \hat{t}, 1 \to \hat{\theta}, 2 \to \hat{\phi}, 3 \to \hat{r}$ and the only non-vanishing components of the Riemann tensor (up to symmetry property of indices) are

$$2R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} = 2R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = 2R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = H' + H^2, \quad R^{\hat{t}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} = HB, \quad 2R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} = H^2 - A^2 + B^2 - \frac{\csc^2(\theta)}{r^2 a(t)^2}, \quad 2R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = 2R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = H^2 - \frac{k}{a(t)^2}.$$

We want to bring these components to the coordinate basis using $\{\underline{\omega}^a\}_{a=0}^3 = \{c^a \underline{d}x^a\}_{a=0}^3$ (no summation on a). The orthonormal indices $R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}$ refer to components in the $e_a \otimes \underline{\omega}^b \otimes \underline{\omega}^c \otimes \underline{\omega}^d$ with $e_a = e_a^\mu \partial_\mu$ being the dual vector basis element to the one-form $\underline{\omega}^a$. Since the coordinate one-forms are dual to the coordinate vector basis we have $\delta_a^b = \underline{\omega}^b(e_a) = e_a^\mu c^b \underline{d}x^b(\partial_\mu) = e_a^\mu c^b \delta_\mu^b = e_a^b c^b$ implies $e_a^b = \delta_a^b / c^b$ (no summation on b). With this in mind, we can write $e_a \otimes \underline{\omega}^b \otimes \underline{\omega}^c \otimes \underline{\omega}^d = \frac{c^b c^c c^d}{c^a} \partial_a \otimes \underline{d}x^b \otimes \underline{d}x^c \otimes \underline{d}x^d$ leading to the components

$$\begin{aligned}
R^t{}_{\phi t \phi} &= \frac{1}{2}(H' + H^2) a(t)^2 r^2 \sin^2(\theta) = \frac{1}{2}\ddot{a}(t) a(t) r^2 \sin^2(\theta), \\
R^t{}_{\theta t \theta} &= \frac{1}{2}\ddot{a}(t) a(t) r^2, \\
R^t{}_{r t r} &= \frac{1}{2}\ddot{a}(t) a(t) \frac{1}{1 - kr^2}, \\
R^t{}_{\phi \theta \phi} &= HB a(t)^3 r^3 \sin^2(\theta) = \dot{a} a(t) r^2 \sin^2(\theta), \\
R^\theta{}_{\phi\theta\phi} &= a(t)^2 r^2 \sin^2(\theta) \frac{1}{2}\left(H^2 - \frac{1 - kr^2}{a(t)^2 r^2} + \frac{\cot^2(\theta)}{a(t)^2 r^2} - \frac{\csc^2(\theta)}{r^2 a(t)^2}\right) = \sin^2(\theta)\frac{1}{2}\left(r^2 \dot{a}(t)^2 - 2 + kr^2\right), \\
R^\theta{}_{r\theta r} &= \frac{a(t)^2}{1 - kr^2}\frac{1}{2}\left(H^2 - \frac{k}{a(t)^2}\right) = \frac{1}{1 - kr^2}\frac{1}{2}\left(\dot{a}^2 - k\right), \\
R^\phi{}_{r\phi r} &= \frac{a(t)^2}{1 - kr^2}\frac{1}{2}\left(H^2 - \frac{k}{a(t)^2}\right) = \frac{1}{1 - kr^2}\frac{1}{2}\left(\dot{a}^2 - k\right).
\end{aligned}$$

and the Ricci scalar components are

$$\begin{aligned}
R_{\phi\phi} &= R^t{}_{\phi\phi t} + R^\theta{}_{\phi\phi\theta} = -\sin^2(\theta)\frac{1}{2}\left(r^2 \dot{a}(t)^2 - 2 + kr^2\right) - \frac{1}{2}\ddot{a}(t) a(t) r^2 \sin^2(\theta) \\
R_{tt} &= R^\phi{}_{tt\phi} + R^\theta{}_{tt\theta} + R^r{}_{ttr} = -g^{\phi\phi} g_{tt} R^t{}_{\phi t\phi} - g^{\theta\theta} g_{tt} R^t{}_{\theta t\theta} - g^{rr} g_{tt} R^t{}_{rtr} = \frac{3}{2}\ddot{a}(t) a(t) \\
R_{\theta\theta} &= R^\phi{}_{\theta\theta\phi} + R^r{}_{\theta\theta r} + R^t{}_{\theta\theta t} = -g^{\theta\theta} g_{\phi\phi} R^\theta{}_{\phi\theta\phi} - g^{\theta\theta} g_{rr} R^\theta{}_{r\theta r} - R^t_{\theta t\theta} = -\frac{a(t)^2}{2}\left(r^2 \dot{a}(t)^2 - 2 + kr^2\right) - \frac{a(t)^2}{2}\left(r^2 \dot{a}^2 - kr^2\right) - \frac{1}{2}\ddot{a}(t) a(t) r^2 \\
R_{rr} &= -R^\theta{}_{r\theta r} - R^\phi{}_{r\phi r} - R^t{}_{rtr} = -\frac{1}{1 - kr^2}\left(\dot{a}^2 - k\right) - \frac{1}{2}\ddot{a}(t) a(t) \frac{1}{1 - kr^2}
\end{aligned}$$

and the other components are 0 because the Riemann tensor has pairs of identical indices so making two of them different in the Ricci uncontracted index locations will leave the two others different and insignificant when the Riemann tensor is contracted.

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Thanks to Thomas for help verifying my answers for (b)