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Homework 1

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1 Conformal invariance of the Maxwell action for $D = 4$

(a) Consider a classical abelian gauge field A_μ on $D = 4$ dimensionnal Minkowski spacetime. Under an infinitesimal conformal transformation, spacetime undergoes the transformation $\tilde{x}^\mu = f(x) = x^\mu + \xi^\mu(x)$ where $\xi^\mu(x)$ is a smal deformation. We want to calculate the effect of this transformation on the gauge field A_μ. The starting point is that we expect A_μ to transform as a tensor under the Lorenz transformation subgroup of the conformal group. This implies that A_μ is a primary operator and we denote its scaling dimension Δ. The transformed field $\tilde{A}_\mu$ at $\tilde{x}$ is related to the original field A_μ at x by an internal rotation, scaling, and special conformal transformation. The rotation operation acts on the components A_μ through its spin 1 representation which is the defining representation of rotations. The scaling and special conformal transformation act together through the multiplication of A_μ by the Jacobian factor $|\partial x/\partial \tilde{x}|_x^{\Delta/D}$. Finally, translations act trivially internally. This can be summarized with the relation $\tilde{A}_\mu(\tilde{x}) = |\partial x/\partial \tilde{x}|_x^{\Delta/D} R_\mu^\nu A_\nu(x)$ where R_μ^ν is the matrix associated with the part of $\xi^\mu(x)$ that does not change the metric components (after the Weyl and diffeomorphism transformations). With this in mind, we calculate the jacobian of the infinitesimal transformation to be

$$\left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right|_x = \left|\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right|_x^{-1} = |\delta_\nu^\mu + \partial_\nu \xi^\mu|_x^{-1} \approx |e^{-\partial_\nu \xi^\mu}|_x = e^{-\mathrm{Tr}\,\partial_\nu \xi^\mu(x)} = 1 - \partial_\mu \xi^\mu(x) + O(\xi^2).$$

The matrix $R_\mu^\nu(x)$ can be extracted by dividing the matrix $(\partial x/\partial \tilde{x})_x$ by a factor $\Omega(x)$ such that we extract the "metric component preserving" operation. To find this factor we consider the effect on the metric of $\Omega^{-1}(x)(\partial x/\partial \tilde{x})_x$. We can write the "metric component preserving" property as

$$\Omega^{-2}(x)\left(\frac{\partial x^\mu}{\partial \tilde{x}^\sigma}\right)_x \left(\frac{\partial x^\nu}{\partial \tilde{x}^\rho}\right)_x \eta_{\mu\nu} = \eta_{\sigma\rho}.$$

Since $\Omega(x)$ is a factor, we can extract it by taking the determinant on both sides of the previous relation to get

$$\det(\eta)\Omega(x)^{-2D}\left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right|_x^2 = \det(\eta) \iff \Omega(x) = \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right|_x^{-\frac{1}{D}}.$$

This result can be intuitively understood from the fact the Jacobian measures D-volume rescaling. Since we want metric components (associated with distances) to be preserved by the rescaled transformation, we need to divide by the D-root of the jacobian. The matrix $R_\mu^\nu(x)$ provided by the rescaling is given by

$$\begin{aligned} R_\mu^\nu(x) = \frac{1}{(1 - \partial_\sigma \xi^\sigma(x) + O(\xi^2))^{1/D}}\left(\frac{\partial x^\nu}{\partial \tilde{x}^\mu}\right)_x &= (1 + \partial_\sigma \xi^\sigma(x)/D + O(\xi^2))(\delta_\nu^\mu + \partial_\mu \xi^\nu(x) + O(\xi^2))^{-1} \\ &= \delta_\nu^\mu(1 + \partial_\sigma \xi^\sigma(x)/D) - \partial_\mu \xi^\nu(x) + O(\xi^2). \end{aligned}$$

We note that $R_\mu^\nu(x)$ will represent a rotation if $\partial_\sigma \xi^\sigma(x) = 0$ (bring the conformal Killing equation to the normal Killing equation with a rotation isometry as its solution). If $\partial_\sigma \xi^\sigma(x) \neq 0$, the rescaled transformation contains a special conformal transformation. The special conformal transformation as a Weyl transformation does not preserve distances but can be combined with a diffeomorphism to preserve the initial components of the metric. With these results, we can write the effect of the infinitesimal transformation as

$$\begin{aligned} \tilde{A}_\mu(\tilde{x}) &= (1 - \partial_\rho \xi^\rho(f^{-1}(\tilde{x})) + O(\xi^2))^{\Delta/D}(A_\mu(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma \xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu \xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2)) \\ &= \left(1 - \frac{\Delta}{D}\partial_\rho \xi^\rho(f^{-1}(\tilde{x})) + O(\xi^2)\right)(A_\mu(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma \xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu \xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2)) \\ &= A_\mu(f^{-1}(\tilde{x})) - A_\mu(f^{-1}(\tilde{x}))\frac{\Delta}{D}\partial_\sigma \xi^\sigma(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma \xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu \xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2). \end{aligned}$$

Since $\xi(f^{-1}(\tilde{x}))$ is already first order in ξ, the only term contribution to its expansion around $\xi = 0$ at $O(\xi)$ is $\xi(\tilde{x})$. To go further, we expand $f^{-1}(\tilde{x})$ at first order in $\xi(\tilde{x})$ with the ansatz $f^{-1}(\tilde{x})^\nu = \tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x})$ (the first term of this ansatz is justified by noticing the transformation reduces to identity at $\xi = 0$). From $f(f^{-1}(\tilde{x})) = \tilde{x}$, we find

$$\tilde{x}^\nu = \tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x}) + \xi(\tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x})) + O(\xi^2) \implies B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x}) + \xi^\nu(\tilde{x}) = 0, \quad \forall \xi(\tilde{x}) \implies B_\mu^\nu(\tilde{x}) = -\delta_\mu^\nu.$$

Using this result, we can expand $A_\mu(f^{-1}(\tilde{x}))$ as

$$A_\mu(f^{-1}(\tilde{x})) = A_\mu(\tilde{x}^\nu - \xi^\nu(\tilde{x}) + O(\xi^2)) = A_\mu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2)$$

Combining this expression with the internal transformation at first order in ξ, we get

$$\begin{aligned} \tilde{A}_\mu(\tilde{x}) &= \left(1 - \frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) + \partial_\sigma \xi^\sigma(\tilde{x}) - \partial_\mu \xi^\nu(\tilde{x})\right)(A_\mu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x})) + O(\xi^2) \\ &= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta - 1}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\partial_\mu \xi^\nu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2) \end{aligned}$$

with $\xi(f^{-1}(\tilde{x})) = \xi(\tilde{x}) + O(\xi^2)$. This result can be simplified by using the conformal killing equation $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2\eta_{\mu\nu}\partial_\sigma \xi^\sigma / D$ as follows:

$$\begin{aligned}
\tilde{A}_\mu(\tilde{x}) &= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta-1}{D}\partial_\sigma\xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\left(\frac{1}{2}\partial_\mu\xi^\nu(\tilde{x}) + \frac{1}{2}\partial_\mu\xi^\nu(\tilde{x})\right) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2) \\
&= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta-1}{D}\partial_\sigma\xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\left(\frac{1}{2}\partial_\mu\xi^\nu(\tilde{x}) - \frac{1}{2}\partial_\nu\xi^\mu(\tilde{x}) + \delta^\nu_\mu\partial_\sigma\xi^\sigma(\tilde{x})\frac{1}{D}\right) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2) \\
&= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma\xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\underbrace{\left(\frac{1}{2}\partial_\mu\xi^\nu(\tilde{x}) - \frac{1}{2}\partial^\nu\xi_\mu(\tilde{x})\right)}_{M_\mu{}^\nu} - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2).
\end{aligned}$$

From this transformed gauge field, we calculate the transformation of gauge field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ to $\tilde{F}_{\mu\nu}$. We start by writting the transformation law of the derivatives used to construct $F_{\mu\nu}$. The chain rule yields

$$\tilde{\partial}_\mu \equiv \frac{\partial}{\partial\tilde{x}^\mu} = \left(\frac{\partial f^{-1}(\tilde{x})^\nu}{\partial\tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} = \left(\frac{\partial\tilde{x}^\nu - \xi^\nu(\tilde{x})}{\partial\tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} = \left(-\frac{\partial\xi^\nu(\tilde{x})}{\partial\tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} + \left(\frac{\partial}{\partial x^\mu}\right)_{\tilde{x}} \equiv -\partial_\mu\xi^\nu(\tilde{x})\partial_\nu + \partial_\mu.$$

where the subscripts indicate that a partial derivative with respect to x^μ should be precomposed with $x = f^{-1}(x^\mu)$ to yield a function dependent on the left-hand side variable $\tilde{x}$. Now we can calculate the transformed field strength at first order in ξ to be

$$\begin{aligned}
\tilde{F}_{\mu\nu} &= \tilde{\partial}_\mu\tilde{A}_\nu - (\mu\leftrightarrow\nu) \\
&= \left(-\partial_\mu\xi^\rho(\tilde{x})\partial_\rho + \partial_\mu\right)\left(A_\nu(\tilde{x}) - A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\sigma\xi^\sigma(\tilde{x}) - A_\lambda(\tilde{x})M_\nu{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x})\right) - (\mu\leftrightarrow\nu) \\
&= \partial_\mu A_\nu(\tilde{x}) - (\partial_\mu\xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda\xi^\lambda(\tilde{x})\right) - \partial_\mu\left(A_\lambda(\tilde{x})M_\nu{}^\lambda\right) - \partial_\mu\left(\xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x})\right) - (\mu\leftrightarrow\nu) \\
&= \partial_\mu A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda\xi^\lambda(\tilde{x})\right) - \partial_\mu A_\lambda(\tilde{x})\partial_\nu\xi^\lambda(\tilde{x}) - A_\lambda(\tilde{x})\partial_\mu\partial_\nu\xi^\lambda(\tilde{x}) - \xi^\lambda(\tilde{x})\partial_\lambda\partial_\mu A_\nu(\tilde{x}) - 2(\partial_\mu\xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - (\mu\leftrightarrow\nu) \\
&= \partial_\mu A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda\xi^\lambda(\tilde{x})\right) - (\partial_\mu A_\lambda(\tilde{x}))M_\nu{}^\lambda - A_\lambda(\tilde{x})\partial_\mu M_\nu{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda\partial_\mu A_\nu(\tilde{x}) - 2(\partial_\mu\xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - (\mu\leftrightarrow\nu) \\
&= F_{\mu\nu}(\tilde{x}) - F_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\lambda\xi^\lambda(\tilde{x}) - A_{(\nu}(\tilde{x})\frac{\Delta}{D}\partial_{\mu)}\partial_\lambda\xi^\lambda(\tilde{x}) - (\partial_{(\mu}A_\lambda(\tilde{x}))M_{\nu)}{}^\lambda - A_\lambda(\tilde{x})\partial_{(\mu}M_{\nu)}{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda F_{\mu\nu}(\tilde{x}) - 2(\partial_{(\mu}\xi^\lambda(\tilde{x}))\partial_\lambda A_{\nu)}(\tilde{x}) \\
&= F_{\mu\nu}(\tilde{x}) - F_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\lambda\xi^\lambda(\tilde{x}) - A_{(\nu}(\tilde{x})\frac{\Delta}{D}\partial_{\mu)}\partial_\lambda\xi^\lambda(\tilde{x}) - (\partial_{(\mu}A_\lambda(\tilde{x}))M_{\nu)}{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda F_{\mu\nu}(\tilde{x}) - 2(\partial_{(\mu}\xi^\lambda(\tilde{x}))\partial_\lambda A_{\nu)}(\tilde{x})
\end{aligned}$$

where we simplified further by expliciting

$$2\partial_{(\mu}M_{\nu)}{}^\lambda = \partial_\mu\partial_\nu\xi^\lambda(\tilde{x}) - \partial_\mu\partial^\lambda\xi_\nu(\tilde{x}) - \partial_\nu\partial_\mu\xi^\lambda(\tilde{x}) - \partial_\nu\partial^\lambda\xi_\mu(\tilde{x}) = 0.$$

We note that the transformation law of $F_{\mu\nu}$ involves A_μ homogeneously which is an example of mixing of CFT fields under the transformation of a descendant.

(b) For a D-dimensionnal spacetime, the Maxwell action reads

$$S = \int \mathrm{d}^D x\sqrt{|g|}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \int \mathrm{d}^D x\sqrt{|g|}\; g^{\mu\sigma}g^{\nu\rho}\frac{1}{4}F_{\mu\nu}F_{\sigma\rho}.$$

where g is the metric (which we suppose conformally flat). We aim to apply the results found in (a) to determine when this action gains conformal symmetry. Under a conformal transformation given by the killing vector $\xi^\mu(x)$ and the scaling $\Omega(x) = 1 + \partial_\mu\xi^\mu(x)/D + O(\xi^2)$ of the metric components, we have

$$g_{\nu\rho}(x) = \Omega(f(x))^{-2}\tilde{g}_{\nu\rho}(f(x)) = \Omega(\tilde{x})^{-2}\tilde{g}_{\nu\rho}(\tilde{x}) \quad \text{Defining property of a conformal transformation}$$

$$|g|(x) = \Omega(f(x))^{-2D}|\tilde{g}|(f(x)), \quad g^{\nu\rho}(x) = \Omega(f(x))^{+2}\tilde{g}^{\nu\rho}(f(x)) = \Omega(\tilde{x})^2\tilde{g}^{\nu\rho}(\tilde{x}), \quad \mathrm{d}^D x\sqrt{|g|} = \mathrm{d}^D\tilde{x}\;\Omega(\tilde{x})^{-D}\sqrt{|\tilde{g}|(\tilde{x})}$$

Without loss of generality, we take the target metric $\tilde{g}$ to be the Minkowski metric. Inverting the result found in (a) for the transformation of the gauge field, we write

$$\begin{aligned}
A_\mu(x) = |\partial x/\partial\tilde{x}|_{\tilde{x}}^{-\Delta/D}(R^{-1})^\nu_\mu\tilde{A}_\nu(\tilde{x}) &= \tilde{A}_\mu(\tilde{x}) + \tilde{A}_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma\xi^\sigma(\tilde{x}) - \tilde{A}_\mu(\tilde{x})\partial_\sigma\xi^\sigma(\tilde{x})\frac{1}{D} + \tilde{A}_\nu(\tilde{x})\partial_\mu\xi^\nu(\tilde{x}) + O(\xi^2) \\
&= \tilde{A}_\mu(\tilde{x}) + \tilde{A}_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma\xi^\sigma(\tilde{x}) + \frac{1}{2}\tilde{A}_\nu(\tilde{x})\left(\partial_\mu\xi^\nu(\tilde{x}) - \partial^\nu\xi_\mu(\tilde{x})\right) + O(\xi^2).
\end{aligned}$$

Then, with the derivative $(\partial_\mu)_{\tilde{x}} = \tilde{\partial}_\mu\xi^\nu(\tilde{x})\tilde{\partial}_\nu + \tilde{\partial}_\mu$, the field strength transforms as

$$\begin{aligned}
F_{\mu\nu} = \partial_\mu A_\nu(x) - (\mu\leftrightarrow\nu) &= \left(\tilde{\partial}_\mu\xi^\lambda(\tilde{x})\tilde{\partial}_\lambda + \tilde{\partial}_\mu\right)\left(\tilde{A}_\nu(\tilde{x}) + \tilde{A}_\nu(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_\sigma\xi^\sigma(\tilde{x}) + \tilde{A}_\lambda(\tilde{x})M_\nu{}^\lambda\right) - (\mu\leftrightarrow\nu) \\
&= \tilde{\partial}_\mu\left(\tilde{A}_\nu(\tilde{x}) + \tilde{A}_\nu(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_\sigma\xi^\sigma(\tilde{x}) + \tilde{A}_\lambda(\tilde{x})M_\nu{}^\lambda\right) + \tilde{\partial}_\mu\xi^\lambda(\tilde{x})\tilde{\partial}_\lambda\tilde{A}_\nu(\tilde{x}) - (\mu\leftrightarrow\nu) \\
&= \tilde{F}_{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\sigma\xi^\sigma(\tilde{x}) + \tilde{A}_{(\nu}(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_{\mu)}\tilde{\partial}_\sigma\xi^\sigma(\tilde{x}) + \tilde{\partial}_{(\mu}(\tilde{A}_\lambda(\tilde{x}))M_{\nu)}{}^\lambda + \tilde{\partial}_{(\mu}\xi^\lambda(\tilde{x})\tilde{\partial}_\lambda\tilde{A}_{\nu)}(\tilde{x})
\end{aligned}$$

The contravariant equivalent of this result is given by

$$
\begin{aligned}
F^{\mu\nu} &= g^{\mu\sigma} g^{\nu\rho} F_{\sigma\rho} = \Omega(\tilde{x})^4 \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} F_{\sigma\rho} \\
&= \Omega(\tilde{x})^4 \left(\tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}^{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} (\tilde{A}_\lambda(\tilde{x})) M_{\rho)}{}^\lambda + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\rho)}(\tilde{x}) \right)
\end{aligned}
$$

Next, we calculate

$$
\begin{aligned}
F_{\mu\nu} F^{\mu\nu} &= \left(\tilde{F}_{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + O(\xi^2) \right) \\
&\quad \times \Omega(\tilde{x})^4 \left(\tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}^{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} (\tilde{A}_\lambda(\tilde{x})) M_{\rho)}{}^\lambda + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\rho)}(\tilde{x}) \right) \\
&= \Omega(\tilde{x})^4 \left(\tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) \frac{2\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) \right) \\
&= \Omega(\tilde{x})^4 \left(\tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) \frac{2\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + 4\tilde{F}^{\mu\nu} \tilde{A}_\nu(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\mu \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) \right)
\end{aligned}
$$

I realized this calculation only applies a passive transformation to the action and should not not change its value without necessarly corresponding to a symmetry. I would have to redo this calculation by applying an active conformal transformation to each element of the action.

2 Axial anomaly

(a) We consider a $D=2$-dimensionnal fermion field ψ with vector current $j^V_\mu = \overline{\psi}\gamma_\mu\psi$ where γ_μ are matrices forming a 2-dimensionnal clifford algebra. We are interested in the 2-point correlator of the vector current $\langle j^V_\mu(x_1) j^V_\nu(x_2)\rangle$. By translational symmetry, the 2-point function is forced to be a function of the relative coordinates $x = (x_1 - x_2)/2$. Translating by $-X_{12} = -(x_1+x_2)/2$, we can bring the midpoint of the x_1, x_2 segment to the origin without changing the value of the 2-point function. Explicitly, we have $\langle j^V_\mu(x_1) j^V_\nu(x_2)\rangle = \langle j^V_\mu(x) j^V_\nu(-x)\rangle$. This property allows us to expand the 2-point function with a Fourier transform with respect to x as

$$\begin{aligned} F[\langle j^V_\mu(x_1) j^V_\nu(x_2)\rangle](q) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2x e^{-iq\cdot x}\langle j^V_\mu(x) j^V_\nu(-x)\rangle &= \frac{1}{(2\pi)^2}\int \mathrm{d}^2x e^{-iq\cdot x}\langle \int \mathrm{d}^2k\, e^{+ik\cdot x} j^V_\mu(k)\int \mathrm{d}^2p\, e^{-ip\cdot x} j^V_\nu(p)\rangle \\ &= \frac{1}{2\pi}\langle \int \mathrm{d}^2k\, \mathrm{d}^2p\, \delta(-q+k-p) j^V_\mu(k)\, j^V_\nu(p)\rangle \\ &= \frac{1}{2\pi}\int \mathrm{d}^2p\, \langle\, j^V_\mu(q-p)\, j^V_\nu(p)\rangle \end{aligned}$$

where the fourier decomposition $j^V_\rho(x_i) = \frac{1}{2\pi}\int \mathrm{d}^2p\, e^{ip\cdot x_i} j^V_\rho(p)$ of the vector current was used. In what follows, we focus on the Fourier space 2-point functions $\langle\, j^V_\mu(-p)\, j^V_\nu(p)\rangle$ contribution to the $q=0$ Fourier component of the spacetime 2-point function. Lorentz invariance requires that $\langle\, j^V_\mu(q-p)\, j^V_\nu(p)\rangle$ is a sum of tensors (it can be extracted from a Fourier transform linearly combining tensor so it is a tensor). Furthermore, it only depends on components p_μ of p. The only tensors with two indices built can be constructed by combining the Minkowski metric $\eta_{\mu\nu}$, the components p_μ, the norm p^2 and the matrices γ^μ (we only need to include a term $\gamma_\mu\gamma_n u$ since the anticommutator $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ relates it to $\gamma_\nu\gamma_\mu$). The most general form for the Fourier space 2-point function consistent with Lorentz invariance reads

$$\langle\, j^V_\mu(-p)\, j^V_\nu(p)\rangle = F_1(p^2)\varepsilon_{\mu\nu} + F_2(p^2)\eta_{\mu\nu} + F_3(p^2)p_\mu p_\nu + F_4(p^2)\gamma_\mu\gamma_\nu + F_5(p^2)\gamma_\mu p_\nu + F_6(p^2)\gamma_\nu p_\mu$$

where the functions $F_i : \mathbb{R}\to\mathbb{C}$ provide full generality and $\varepsilon_{\mu\nu}$ is the 2-dimensionnal Levi-Civita tensor. Since the current operator follows a Bose statistic (they each contain an even number of fermion operators), we can exchange them without changing the value of the 2-point function. This property can be expressed as

$$\langle\, j^V_\mu(-p)\, j^V_\nu(p)\rangle = \langle\, j^V_\nu(p)\, j^V_\mu(-p)\rangle = -F_1(p^2)\varepsilon_{\mu\nu} + F_2(p^2)\eta_{\mu\nu} + F_3(p^2)(-p_\nu)(-p_\mu) + F_4(p^2)\gamma_\nu\gamma_\mu - F_5(p^2)\gamma_\nu p_\mu - F_6(p^2)\gamma_\mu p_\nu.$$

Subtracting this exchanged expression from the initial expression, we get the constraint

$$\begin{aligned} 0 &= 2F_1(p^2)\varepsilon_{\mu\nu} + F_4(p^2)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) + (F_6(p^2)+F_5(p^2))\gamma_\nu p_\mu + (F_6(p^2)+F_5(p^2))\gamma_\mu p_\nu, \forall p \\ &\implies F_4(p^2) = F_1(p^2) = 0, F_6(p^2) = -F_5(p^2) \end{aligned}$$

The current we are interested in is conserved as a result of the global symmetry $\psi \to e^{i\theta}\psi$, $\theta\in\mathbb{R}$. We note that applying the infinitesimal version of this symmetry transformation to the current leads to a vanishing variation $\delta j^\nu(x) = 0$. The ward identity corresponding to this symmetry reads

$$\langle \partial^\mu j_\mu(x_1) j_\nu(x_2)\rangle = \delta(x_1 - x_2)\langle \delta j_\nu(x_2)\rangle = 0.$$

We then compute the Fourier transformation with respect to x_1, x_2 to get

$$\begin{aligned} 0 = F[\langle \partial^\mu j^V_\mu(x_1) j^V_\nu(x_2)\rangle](q,p) &= \int \mathrm{d}^2x_1 e^{-iq\cdot x_1}\int \mathrm{d}^2x_2 e^{-ip\cdot x_2}\langle \partial^\mu j^V_\mu(x_1) j^V_\nu(x_2)\rangle \\ &= \langle (+iq^\mu)\int \mathrm{d}^2x_1 e^{-iq\cdot x} j^V_\mu(x_1)\int \mathrm{d}^2x_2 e^{-ip\cdot x} j^V_\nu(x_2)\rangle \quad \text{with integration by parts} \\ &= iq^\mu\langle j^V_\mu(q) j^V_\nu(p)\rangle \end{aligned}$$

At $q=-p$, we find $p^\mu\langle j^V_\mu(-p)\, j^V_\nu(p)\rangle = 0$ wich implies

$$\begin{aligned} 0 = p^\mu\langle\, j^V_\mu(-p)\, j^V_\nu(p)\rangle &= p^\mu\left(F_2(p^2)\eta_{\mu\nu} + F_3(p^2)p_\mu p_\nu\right) + F_5(p^2)p^\mu\left(\gamma_\mu p_\nu - \gamma_\nu p_\mu\right) \\ &= \left(F_2(p^2) + F_3(p^2)p^2\right)p_\nu + F_5(p^2)\left(p^\mu\gamma_\mu p_\nu - \gamma_\nu p^2\right), \forall p \\ &\implies \end{aligned}$$

(b)

(c)

(d)

3 OPE coefficients from three-point functions

(a)

(b)

(c)

(d)

4 Acknowledgement

Thanks to Thiago for a discussion about question 1 (b)