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Homework 2: Linearized gravity

Due: Friday 9pm, November 3, 2023

1 Linearized field equations

In the absence of gravity the spacetime is flat. The weak gravitational field can be described (in appropriate coordinates) by the following spacetime metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)\,, \tag{1}$$

where $\eta_{\mu\nu}$ is the standard Minkowski metric and $h_{\mu\nu}$ stands for its perturbation (a small deviation from flatness). Show that when expanded to linear order in $h_{\mu\nu}$ the linearized Einstein equations read

$$G^{(1)}_{\mu\nu} = -\frac{1}{2}\partial^\kappa\partial_\kappa\bar{h}_{\mu\nu} + \partial^\kappa\partial_{(\mu}\bar{h}_{\nu)\kappa} - \frac{1}{2}\eta_{\mu\nu}\partial^\kappa\partial^\delta\bar{h}_{\kappa\delta} = 8\pi G T_{\mu\nu}\,, \tag{2}$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \quad h = h^\alpha_\alpha\,. \tag{3}$$

[*Hint*: Use the study text as a reference for all problems.]

2 Let's simplify our lives

Let us now use the fact that gravity is a gauge theory and choose a gauge where the above horrible formula simplifies considerably. Gauge freedom in general relativity corresponds to the group of diffeomorphisms (coordinate transformations). This means that the two metrics related by a diffeomorphism represent the same spacetime geometry.

a) Show that an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)\,, \tag{4}$$

induces the *gauge freedom*

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\,, \tag{5}$$

similar to electromagnetic gauge freedom $A_\mu \to A_\mu + \partial_\mu \xi$.

b) Show that it is possible to choose the functions $\xi_\mu(x)$ in such a way that the transformed quantity $\bar{h}_{\mu\nu}$ obeys the *De Donder gauge* condition

$$\partial^\mu \bar{h}_{\mu\nu} = 0\,, \tag{6}$$

and that in this gauge the linearized Einstein's equations simply read, $\Box = \partial^\kappa \partial_\kappa$,

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}\,. \tag{7}$$

3 Gravitomagnetism

Let us consider the linearized Einstein's equations together with the geodesic equation

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}\,, \quad \partial^\mu \bar{h}_{\mu\nu} = 0\,, \qquad \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,. \tag{8}$$

a) In the *Newtonian limit* the energy density dominates all other components of $T_{\mu\nu}$, i.e., $T_{00} = \rho$ while all other components are negligible. Show that by identifying $\phi = -\frac{1}{4}\bar{h}_{00}$ and for slow motion and static field we recover the Newtonian gravity:

$$\nabla^2 \phi = 4\pi\rho\,, \quad \frac{d^2 \vec{x}}{dt^2} = -\nabla\phi\,. \tag{9}$$

Find all components of the metric in this case.

b) Let us now go further and assume that the stresses of the energy momentum tensor are still negligible but the energy flux is not (we allow the sources to move and expand to linear order in their velocity $\vec{v}$). So we set $T_{0\mu} = -J_\mu$

while all other components vanish. We further set $A_\mu = -\frac{1}{4}\bar{h}_{\mu 0}$. Show that for slowly changing fields, $\partial_t A_\mu \approx 0$, we now recover

$$\Box A_\mu = -4\pi J_\mu \,, \quad \partial_\mu A^\mu = 0 \,, \qquad \frac{d^2\vec{x}}{dt^2} = \vec{E} - 4\vec{v} \times \vec{B} \,, \tag{10}$$

where E and B are determined through A_μ in a familiar way. So we have recovered the Maxwell's equations and the Lorentz force (with $q = m$, and apart from the minus 4 factor in the magnetic force). Thus, you have shown that the linearized gravity predicts that the motion of masses produces gravitomagnetic effects similar to those of electromagnetism.