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HOMEWORK 2 : LINEARIZED GRAVITY

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1 Linearized field equations

Weak gravitationnal effects can be modeled as a perturbation of the flat Minkowski metric η. On the level of manifolds, this perturbation can be seen as a diffeomorphism $\phi : M \to M'$ mapping flat spacetime M into a weakly curved manifold M'. A global coordinate chart $\psi : M \to \mathbb{R}^4$ on the flat spacetime can be converted to a coordinate chart ψ' on the disformed manifold as $\psi' = \psi \circ \phi^{-1} : M' \to \mathbb{R}^4$. Taking the coordinates on M to be cartesian, we work with the inherited coordinates on M' as a staring point. In these coordinates, the full metric $g_{\mu\nu}$ can be Taylor expanded in a small parameter λ as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(\lambda^2)$ where $h_{\mu\nu}$ is the perturbation depending linearly on λ. For all the following calculations, we drop the $O(\lambda^2)$ but keep in mind that everything represents a first order expansion in λ.

To write the fisrt order contribution to the Einstein equations arising from this perturbation, we first compute the inverse metric. Expanding it in λ around the inverse minkowski metric, we have $g_{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu}$ and

$$\delta_\rho^\nu = g_{\rho\mu} g^{\mu\nu} = \eta_{\rho\mu}\eta^{\mu\nu} + h_{\rho\mu}\eta^{\mu\nu} + \eta_{\rho\mu} f^{\mu\nu} \iff f_\rho{}^\nu = -h_\rho{}^\nu \iff f^{\rho\nu} = -h^{\rho\nu}.$$

Then the expansion of the Christoffel symbols read

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(g_{\mu\rho,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) = \frac{1}{2}(\eta^{\sigma\rho} - h^{\sigma\rho})(h_{\mu\rho,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho}) = \frac{1}{2}\eta^{\sigma\rho}(h_{\mu\rho,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho})$$

because $\eta_{\mu\nu,\rho} = 0$ in cartesian coordinates. The Riemann tensor can now be expressed as

$$\begin{aligned} R^\rho{}_{\sigma\mu\nu} &= \Gamma^\rho{}_{\nu\sigma,\mu} - \Gamma^\rho{}_{\mu\sigma,\nu} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} \\ &= \Gamma^\rho{}_{\nu\sigma,\mu} - \Gamma^\rho{}_{\mu\sigma,\nu} = \frac{1}{2}\eta^{\rho\lambda}(h_{\nu\lambda,\sigma\mu} + h_{\lambda\sigma,\nu\mu} - h_{\nu\sigma,\lambda\mu}) - \frac{1}{2}\eta^{\rho\lambda}(h_{\mu\lambda,\sigma\nu} + h_{\lambda\sigma,\mu\nu} - h_{\mu\sigma,\lambda\nu}) \\ &= \frac{1}{2}\eta^{\rho\lambda}(h_{\nu\lambda,\sigma\mu} - h_{\nu\sigma,\lambda\mu} - h_{\mu\lambda,\sigma\nu} + h_{\mu\sigma,\lambda\nu}). \end{aligned}$$

Contracting the ρ and μ indices, we get the following Ricci tensor:

$$\begin{aligned} R_{\sigma\nu} &= \frac{1}{2}\eta^{\mu\lambda}(h_{\nu\lambda,\sigma\mu} - h_{\nu\sigma,\lambda\mu} - h_{\mu\lambda,\sigma\nu} + h_{\mu\sigma,\lambda\nu}) = \frac{1}{2}(h_\nu{}^\mu{}_{,\sigma\mu} - h_{\nu\sigma,\lambda}{}^\lambda - h^\lambda{}_{\lambda,\sigma\nu} + h^\mu{}_{\sigma,\mu\nu}) \\ &= \frac{1}{2}(h_\nu{}^\mu{}_{,\sigma\mu} + h^\mu{}_{\sigma,\mu\nu} - \Box h_{\nu\sigma} - h_{,\sigma\nu}), \quad h = h^\lambda{}_\lambda \end{aligned}$$

where we used the fact raising indices with $g^{\mu\nu}$ for tensors proportionnal to λ reduces to contracting them with $\eta^{\mu\nu}$ at first order in λ (the $-h^{\mu\nu}$ term only contributes to second order). Contracting the remaining indices (with the Minkowski) metric yields the Ricci scalar

$$R = \eta^{\sigma\nu} R_{\sigma\nu} = \frac{1}{2}(h^{\sigma\mu}{}_{,\sigma\mu} + h^{\sigma\mu}{}_{,\mu\sigma} - \Box h^\nu{}_\nu - h_{,\nu}{}^\nu) = h^{\sigma\mu}{}_{,\sigma\mu} - \Box h.$$

Combining all the previous results, the linearised Einstein tensor can be written as

$$G_{\sigma\nu} = R_{\sigma\nu} - \frac{1}{2}\eta_{\sigma\nu} R = \frac{1}{2}(h_\nu{}^\mu{}_{,\sigma\mu} + h^\mu{}_{\sigma,\mu\nu} - \Box h_{\nu\sigma} - h_{,\sigma\nu} - \eta_{\sigma\nu} h^{\rho\mu}{}_{,\rho\mu} + \eta_{\sigma\nu}\Box h).$$

We define $\bar{h}_{\sigma\nu} = h_{\sigma\nu} - \frac{1}{2}\eta_{\sigma\nu} h$ with trace $\bar{h} = \eta^{\sigma\nu}\bar{h}_{\sigma\nu} = h - \frac{4}{2}h = -h$. With this in mind, the perturbation can be written as $h_{\sigma\nu} = \bar{h}_{\sigma\nu} + \frac{1}{2}\eta_{\sigma\nu}(-\bar{h})$. Substitution of this form in the Einstein tensor leads to

$$\begin{aligned} G_{\sigma\nu} &= \frac{1}{2}(h_\nu{}^\mu{}_{,\sigma\mu} + h^\mu{}_{\sigma,\mu\nu} - \Box h_{\nu\sigma} - h_{,\sigma\nu} - \eta_{\sigma\nu} h^{\rho\mu}{}_{,\rho\mu} + \eta_{\sigma\nu}\Box h) \\ &= \frac{1}{2}(\bar{h}_\nu{}^\mu{}_{,\sigma\mu} - \frac{1}{2}\bar{h}_{,\sigma\nu} + \bar{h}^\mu{}_{\sigma,\mu\nu} - \frac{1}{2}\bar{h}_{,\sigma\nu} - \Box\bar{h}_{\sigma\nu} + \frac{1}{2}\eta_{\sigma\nu}\Box\bar{h} + \bar{h}_{,\sigma\nu} - \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu} + \frac{1}{2}\eta_{\sigma\nu}\Box\bar{h} - \eta_{\sigma\nu}\Box\bar{h}) \\ &= \frac{1}{2}(\bar{h}_\nu{}^\mu{}_{,\sigma\mu} + \bar{h}^\mu{}_{\sigma,\mu\nu} - \Box\bar{h}_{\sigma\nu} - \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu}) \end{aligned}$$

with

$$\begin{gathered} h_\nu{}^\mu{}_{,\sigma\mu} = \bar{h}_\nu{}^\mu{}_{,\sigma\mu} - \frac{1}{2}\eta_\nu{}^\mu\bar{h}_{,\sigma\mu} = \bar{h}_\nu{}^\mu{}_{,\sigma\mu} - \frac{1}{2}\bar{h}_{,\sigma\nu}, \quad h^\mu{}_{\sigma,\mu\nu} = \bar{h}^\mu{}_{\sigma,\mu\nu} - \frac{1}{2}\eta^\mu{}_\sigma\bar{h}_{,\mu\nu} = \bar{h}^\mu{}_{\sigma,\mu\nu} - \frac{1}{2}\bar{h}_{,\sigma\nu} \\ \Box h_{\sigma\nu} = \bar{h}_{\sigma\nu} - \frac{1}{2}\eta_{\sigma\nu}\Box\bar{h}, \quad h_{,\sigma\nu} = -\bar{h}_{,\sigma\nu}, \quad \eta_{\sigma\nu} h^{\rho\mu}{}_{,\rho\mu} = \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu} - \frac{1}{2}\eta_{\sigma\nu}\eta^{\rho\mu}\bar{h}_{,\rho\mu} = \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu} - \frac{1}{2}\eta_{\sigma\nu}\Box\bar{h}. \end{gathered}$$

Finally, the relation between the Einstein tensor and the stress-energy tensor $T_{\mu\nu}$ is provided by Einstein equations. We take $T_{\mu\nu}$ to be of the order of λ consistently with the weak field on almost flat space ($T_{\mu\nu}$ has no zeroth order contribution) assumptions. The perturbation satisfies the equation

$$\frac{1}{2}(\bar{h}^\mu{}_{\nu,\sigma\mu} + \bar{h}^\mu{}_{\sigma,\nu\mu} - \Box\bar{h}_{\sigma\nu} - \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu}) = \frac{1}{2}(\bar{h}^\mu{}_{(\nu,\sigma)\mu} - \Box\bar{h}_{\sigma\nu} - \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu}) = 8\pi G T_{\sigma\nu}$$

with gravitationnal coupling strength G.

2 Let's simplify our lives

(a) Since coordinate transformations localy transform the metric components without changing the spacetime it describes, we can interpret them as gauge transformations on a tensor component field $g_{\mu\nu}$. To preserve the validity of our linearized expansion, we consider the effect of infinitesimal coordinate transformations $x^{\mu\prime}(x) = x^{\mu} - \xi^{\mu}(x)$ with ξ at order in λ. This ensures that a coordinate change preserve $\eta_{\mu\nu}$ at zeroth order and sends $h_{\mu\nu}$ to a perturbation in the range satisfying the linearized Einstein equations. The transformed components $h'_{\mu\nu}(x')$ will satisfy the equation and we recover a notion of linearized covariance. Relating the $g_{\mu\nu}(x)$ components and the gauge transformed components $g'_{\mu\nu}(x')$ at first order, we have

$$\begin{aligned}\eta_{\mu\nu} + h_{\mu\nu}(x) = g_{\mu\nu}(x) &= x^{\mu\prime}{}_{,\mu} x^{\nu\prime}{}_{,\nu} g'_{\mu\nu}(x'(x)) \\ &= (\delta^{\sigma}_{\mu} - \xi^{\sigma}{}_{,\mu}(x))(\delta^{\rho}_{\nu} - \xi^{\rho}{}_{,\nu}(x))(\eta_{\sigma\rho} + h'_{\sigma\rho}(x'(x))) \\ &= \eta_{\mu\nu} + h'_{\mu\nu}(x'(x)) - \delta^{\sigma}_{\mu}\eta_{\sigma\rho}\xi^{\rho}{}_{,\nu}(x) - \delta^{\rho}_{\nu}\eta_{\sigma\rho}\xi^{\sigma}{}_{,\mu}(x) \\ &= \eta_{\mu\nu} + h'_{\mu\nu}(x'(x)) - \xi_{\mu,\nu} - \xi_{\nu,\mu}\end{aligned}$$

Comparing the right and left hand sides of this expression yields $h'_{\mu\nu}(x'(x)) = h_{\mu\nu}(x) + \xi_{\mu,\nu}(x) + \xi_{\nu,\mu}(x)$. To bring the dependency of $h'_{\mu\nu}$ to x explicitly, we write the expansion $h'_{\mu\nu}(x'(x)) = h'_{\mu\nu}(x) + \xi^{\sigma}(x)h'_{\mu\nu,\sigma}(x)$ where the $\xi^{\sigma}(x)h'_{\mu\nu,\sigma}(x)$ term is second order in λ and does not contribute so $h'_{\mu\nu}(x'(x)) = h'_{\mu\nu}(x)$.

(b) Using the previous result, the gauge transformation of $\bar{h}_{\sigma\nu}$ to $\bar{h}'_{\sigma\nu}$ reads

$$\begin{aligned}\bar{h}'_{\mu\nu}(x) &= h_{\mu\nu}(x) + \xi_{\mu,\nu}(x) + \xi_{\nu,\mu}(x) - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}(h_{\sigma\rho}(x) + \xi_{\sigma,\rho}(x) + \xi_{\sigma,\rho}(x)) \\ &= \bar{h}_{\mu\nu}(x) + \xi_{\mu,\nu}(x) + \xi_{\nu,\mu}(x) - \eta_{\mu\nu}\xi_{\sigma,}{}^{\sigma}(x).\end{aligned}$$

Now we contract the μ index of $\bar{h}_{\mu\nu}$ with a derivative and get

$$\begin{aligned}\bar{h}'_{\mu\nu,}{}^{\mu}(x) &= \bar{h}_{\mu\nu,}{}^{\mu}(x) + \xi_{\mu,\nu}{}^{\mu}(x) + \xi_{\nu,\mu}{}^{\mu}(x) - \eta_{\mu\nu}{}^{\mu}\xi_{\sigma,}{}^{\sigma}(x) \\ &= \bar{h}_{\mu\nu,}{}^{\mu}(x) + \xi_{\mu,\nu}{}^{\mu}(x) + \xi_{\nu,\mu}{}^{\mu}(x) - \xi_{\sigma,\nu}{}^{\sigma}(x) \\ &= \bar{h}_{\mu\nu,}{}^{\mu}(x) + \Box\xi_{\nu}(x).\end{aligned}$$

Choosing ξ_{ν} to make $\bar{h}'_{\mu\nu,}{}^{\mu}(x)$ vanish constitutes a choice of gauge called the *De Donder gauge*. The coordinate transforms leading to this gauge are constrained by

$$\Box\xi_{\nu}(x) = -\bar{h}_{\mu\nu,}{}^{\mu}(x)$$

which is a wave equation with $-\bar{h}_{\mu\nu,}{}^{\mu}(x)$ sources for each ν. Given any starting $\bar{h}_{\mu\nu}$, we can compute the associated source and solve the wave quation to go to the De Donder gauge.

3 Gravitomagnetism

(a)

(b)

4 Acknowledgement