Pierre-Antoine Graham

# HOMEWORK 1

Gang Xu
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# Contents

# 1 The Poincaré Algebra

(a) The Poincaré is the group of transformation of Minkowski space that preserves the spacetime interval between all events. This group contains spacetime translations and Lorentz transformation (boosts and rotations). In a coordinate system where events hapenning at $x$ with four-coordinate $x^\mu$, translation by a constant four-vector $a$ with components $a^\mu$ reads $x' = x + a$ ($x^{\mu'} = x^\mu + a^\mu$). The lorentz transformation $\Lambda$ with components $\Lambda^\nu{}_\mu$ act as $x' = \Lambda x$ ($x^{\mu'} = \Lambda^\mu{}_\nu x^\nu$, following the matrix multiplication convention $x^\nu$ can be written as a column with $\nu$ as a row index and $\Lambda^\mu{}_\nu$ as a square matrix with $\mu$ row index and $\nu$ column index). We want to find the characteristic of the unitary operator $U$ representing Poincaré transformation near the identity $\delta$ (with components $\delta^\mu{}_\nu$). To do this, we write the first order Taylor expansions $\Lambda = \delta + \omega + O(\omega^2)$ and $a = \varepsilon$ (exact even for large $\varepsilon$) with respect to an infinitesimal Lorentz shift $\omega$ with components $\omega_{\mu\nu}$ (combining infinitesimal rotation angles and boost angles) and translation $\varepsilon$ with components $\varepsilon^\mu$. The first order in $\omega$ and $\varepsilon$ expansion of the unitary is $U(\delta + \omega, \varepsilon) = \mathbf{1} + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\varepsilon_\mu P^\mu + O(\omega^2, \varepsilon^2)$ where $J^{\mu\nu}$, $P^\mu$ are the hermitian matrices generating the Poincaré transformation. Since $\Lambda = \delta + \omega$ is a Lorentz transformation, we have that it preserves space-time intervals. The spacetime interval between events $x$ and $y$ is $(x_\mu - y_\mu)(x^\mu - y^\mu) = x_\mu x^\mu + y_\mu y^\mu - 2y_\mu x^\mu$. Since the first two terms are themselves spacetime intervals between $x$, $y$, and 0, they are individually preserved by a Poincaré transformation. This forces the invariance of the lorentzian product $y_\mu x^\mu$ for any $x, y$ under Poincaré transformations ($x^\mu y_\mu = x^{\mu'} y'_\mu$).

In general, we can Taylor expand $x^{\mu'} y'_\mu$ around $x^\mu y_\mu$ in powers of $\omega$ as

$$x^{\mu'} y'_\mu = (x^\mu + \omega^{\mu\sigma}x_\sigma + O(\omega^2))(y_\mu + \omega_{\mu\nu}y^\nu + O(\omega^2)) = x^\mu y_\mu + \omega_{\mu\nu}x^\mu y^\nu + \omega^{\mu\sigma}x_\sigma y_\mu + O(\omega^2)$$

but we know that length is preserved, and the unique Taylor series stops at $O(1)$ forcing all other orders to vanish. This implies that $0 = \omega_{\mu\nu}x^\mu y^\nu + \omega_{\mu\nu}x^\nu y^\mu = (\omega_{\mu\nu} + \omega_{\nu\mu})x^\mu y^\nu$. This finally implies that $\omega_{\mu\nu} = -\omega_{\nu\mu}$ since it is true for any $x, y$ which shows $\omega_{\nu\mu}$ is antisymmetric.

(b) The unitary $U$ is meant to represent a Poincaré symmetry transformation of the quantum states of a module Hilbert space. In quantum mechanics, A symmetry transformation preserves all the statistical properties of observables $O$. This is equivalent to saying that for any two states $|\phi\rangle$ and $|\psi\rangle$ the quantities $\langle\phi| O |\psi\rangle$ are left unchanged by the symmetry. We have the following transformation of states $|\psi'\rangle = U|\psi\rangle$ and $\langle\phi'| = \langle\phi| U^\dagger$. The transformed $O'$ operator is such that

$$\langle\phi| O |\psi\rangle = \langle\phi'| O' |\psi'\rangle = \langle\phi| U^\dagger O' U |\psi\rangle, \quad \forall \langle\phi|, |\psi\rangle \iff O = U^\dagger O' U \iff O' = UOU^\dagger.$$

(c) Following the result of the previous item, we take $O = U(\delta + \omega, \varepsilon)$ and compute the operator $O'$ associated to the general $U(\Lambda, a)$ unitary Poincaré transformation representing the combined Lorentz and translation transformation $T(\Lambda, a)$. With the same notation, we write $T(\delta + \omega, \varepsilon)$ to reference the infinitesimal Poincaré transformation. Because the representation is a homomorphism, we have

$$O' = U(\Lambda, a)U(T(\delta + \omega, \varepsilon))U^\dagger(\Lambda, a) = U(T(\Lambda, a)T(\delta + \omega, \varepsilon)T^{-1}(\Lambda, a)).$$

To make this expression more precise, we look for $\Lambda'$ and $a'$ such that $T^{-1}(\Lambda, a) = T(\Lambda', a')$. Acting with the identity on an arbitrary four-vector $x$ leads to

$$\begin{aligned} x &= T^{-1}(\Lambda, a)T(\Lambda, a)x = T(\Lambda', a')T(\Lambda, a)x = \Lambda'(\Lambda x + a) + a' \\ &= \Lambda'\Lambda x + \Lambda' a + a', \quad \forall x \iff \Lambda'\Lambda = 1 \,\&\, \Lambda' a + a' = 0 \end{aligned}$$

With these relations in hand, the action on $x$ of the product Poincaré transformation represented by $O'$ is expanded as follows:

$$\begin{aligned} T(\Lambda, a)T(\delta + \omega, \varepsilon)T^{-1}(\Lambda, a)x &= T(\Lambda, a)T(\delta + \omega, \varepsilon)(\Lambda^{-1}x - \Lambda^{-1}a) \\ &= T(\Lambda, a)((\Lambda^{-1}x - \Lambda^{-1}a) + \omega(\Lambda^{-1}x - \Lambda^{-1}a) + \varepsilon) \\ &= T(\Lambda, a)((\Lambda^{-1} + \omega\Lambda^{-1})x - \omega\Lambda^{-1}a - \Lambda^{-1}a + \varepsilon) \\ &= ((\delta + \Lambda\omega\Lambda^{-1})x - \Lambda\omega\Lambda^{-1}a - a + \Lambda\varepsilon + a) = T(\delta + \Lambda\omega\Lambda^{-1}, -\Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon)x. \end{aligned}$$

This relation holds for all $x$ and we can finally write $O' = U(\delta + \Lambda\omega\Lambda^{-1}, -\Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon)$.

(d) Combining the hermitian generator expansion of $U(\delta+\omega,a)$ (we omit Landau order notation in the next calculations) given in item (a) to the expression for the transformed operator $O'$ of item (c), we find

$$
\begin{aligned}
O' &= U(\Lambda,a)U(\delta+\omega,\varepsilon)U^{\dagger}(\Lambda,a) \qquad (\star)\\
&= \mathbf{1}+\frac{i}{2}\omega_{\mu\nu}U(\Lambda,a)J^{\mu\nu}U^{\dagger}(\Lambda,a)+i\varepsilon_{\mu}U(\Lambda,a)P^{\mu}U^{\dagger}(\Lambda,a)\\
&= \mathbf{1}+\frac{i}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu}J^{\mu\nu}+i(-\Lambda\omega\Lambda^{-1}a+\Lambda\varepsilon)_{\mu}P^{\mu}\\
&= \mathbf{1}+\frac{i}{2}(\Lambda_{\mu}{}^{\rho}\omega_{\rho\sigma}\Lambda_{\nu}{}^{\sigma})J^{\mu\nu}+i(-\Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}a^{\nu}\omega_{\rho\sigma}+\Lambda_{\mu}{}^{\nu}\varepsilon_{\nu})P^{\mu}\\
&= \mathbf{1}+\frac{i}{2}(\Lambda_{\rho}{}^{\mu}\omega_{\mu\nu}\Lambda_{\sigma}{}^{\nu})J^{\rho\sigma}+i(-\Lambda_{\rho}{}^{\mu}\Lambda_{\sigma}{}^{\nu}a^{\sigma}\omega_{\mu\nu}P^{\rho}+\varepsilon_{\mu}\Lambda_{\nu}{}^{\mu}P^{\nu})\\
&= \mathbf{1}+\frac{i}{2}(\Lambda_{\rho}{}^{\mu}\omega_{\mu\nu}\Lambda_{\sigma}{}^{\nu})J^{\rho\sigma}+i\varepsilon_{\mu}\Lambda_{\nu}{}^{\mu}P^{\nu}+\frac{i}{2}(-\Lambda_{\rho}{}^{\mu}\Lambda_{\sigma}{}^{\nu}a^{\sigma}\omega_{\mu\nu}P^{\rho}+\Lambda_{\rho}{}^{\mu}\Lambda_{\sigma}{}^{\nu}a^{\sigma}\omega_{\nu\mu}P^{\rho})\\
&= \mathbf{1}+\frac{i}{2}(\Lambda_{\rho}{}^{\mu}\omega_{\mu\nu}\Lambda_{\sigma}{}^{\nu})J^{\rho\sigma}+i\varepsilon_{\mu}\Lambda_{\rho}{}^{\mu}P^{\rho}+\frac{i}{2}(-\Lambda_{\rho}{}^{\mu}\Lambda_{\sigma}{}^{\nu}a^{\sigma}\omega_{\mu\nu}P^{\rho}+\Lambda_{\sigma}{}^{\nu}\Lambda_{\rho}{}^{\mu}a^{\rho}\omega_{\mu\nu}P^{\sigma})
\end{aligned}
$$

where we expanded the result of item (c) at $O(\omega,\varepsilon)$ in the third line and used antisymmetry of $\omega_{\mu\nu}$ in the second last line. To obtain the component representation of the previous result, the vector/matrix multiplication was written and then converted to the appropriated index structure:

$$
\begin{aligned}
&(-\Lambda\omega\Lambda^{-1}a+\Lambda\varepsilon)^{\mu}=-\Lambda^{\mu}{}_{\rho}\omega^{\rho}{}_{\sigma}(\Lambda^{-1})^{\sigma}{}_{\nu}a^{\nu}+\Lambda^{\mu}{}_{\nu}\varepsilon^{\nu}\\
&\iff (-\Lambda\omega\Lambda^{-1}a+\Lambda\varepsilon)_{\mu}=-\Lambda_{\mu}{}^{\rho}\omega_{\rho\sigma}(\Lambda^{-1})^{\sigma}{}_{\nu}a^{\nu}+\Lambda_{\mu}{}^{\nu}\varepsilon_{\nu}=-\Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}a_{\nu}\omega_{\rho\sigma}+\Lambda_{\mu}{}^{\nu}\varepsilon_{\nu}\\
&(\Lambda\omega\Lambda^{-1})^{\mu}{}_{\nu}=\Lambda^{\mu}{}_{\rho}\omega^{\rho}{}_{\sigma}(\Lambda^{-1})^{\sigma}{}_{\nu}\\
&\iff (\Lambda\omega\Lambda^{-1})_{\mu\nu}=\eta_{\mu\lambda}(\Lambda\omega\Lambda^{-1})^{\lambda}{}_{\nu}=\Lambda_{\mu}{}^{\rho}\omega_{\rho\sigma}(\Lambda^{-1})^{\sigma}{}_{\nu}=\Lambda_{\mu}{}^{\rho}\omega_{\rho\sigma}\Lambda_{\nu}{}^{\sigma}.
\end{aligned}
$$

The inverse transformation components could be related to the direct components because the Lorentz matrices preserve the Lorentzian product of arbitrary $x,y$. Indeed, this property implies

$$
\begin{aligned}
&\eta_{\rho\sigma}x^{\rho}y^{\sigma}=\eta_{\mu\nu}(\Lambda^{\mu}{}_{\rho}x^{\rho}\Lambda^{\nu}{}_{\sigma}y^{\sigma}),\ \forall x,y \iff \eta_{\rho\sigma}=\eta_{\mu\nu}(\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma})\\
&\iff \eta_{\rho\sigma}(\Lambda^{-1})^{\sigma}{}_{\lambda}=\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}(\Lambda^{-1})^{\sigma}{}_{\lambda}=\eta_{\mu\lambda}\Lambda^{\mu}{}_{\rho} \iff (\Lambda^{-1})^{\nu}{}_{\lambda}=\eta^{\nu\rho}\eta_{\rho\sigma}(\Lambda^{-1})^{\sigma}{}_{\lambda}=\eta_{\mu\lambda}\eta^{\nu\rho}\Lambda^{\mu}{}_{\rho}=\Lambda_{\lambda}{}^{\nu}.
\end{aligned}
$$

Equality of the second and last lines of $(\star)$ for all $\omega,\varepsilon$ ensures that the tensors contracted with $\omega$ and $\varepsilon$ are equal. Regrouping terms proportional to $\omega_{\mu\nu}$ and $\varepsilon_{\mu}$ We have

$$
\begin{aligned}
U(\Lambda,a)J^{\mu\nu}U^{\dagger}(\Lambda,a) &= \Lambda_{\rho}{}^{\mu}\Lambda_{\rho}{}^{\nu}(J^{\rho\sigma}+a^{\rho}P^{\sigma}-a^{\sigma}P^{\rho}),\\
U(\Lambda,a)P^{\mu}U^{\dagger}(\Lambda,a) &= \Lambda_{\rho}{}^{\mu}P^{\rho}.
\end{aligned}
$$

(e) To extract the commutation relations of the generators $J^{\mu\nu},P^{\mu}$, we start by setting $\Lambda=\delta+\omega$, $a=\varepsilon$ in the final result of item (d). At first order in $\omega,\varepsilon$, we get

$$
\begin{aligned}
P^{\mu}+\eta^{\mu\sigma}\omega_{\rho\sigma}P^{\rho}=\Lambda_{\rho}{}^{\mu}P^{\rho}=U(\Lambda,a)P^{\mu}U^{\dagger}(\Lambda,a) &= \left(\mathbf{1}+\frac{i}{2}\omega_{\lambda\varepsilon}J^{\lambda\varepsilon}+i\varepsilon_{\nu}P^{\nu}\right)P^{\mu}\left(\mathbf{1}-\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}-i\varepsilon_{\sigma}P^{\sigma}\right)\\
&= P^{\mu}+\frac{i}{2}\left(\omega_{\lambda\varepsilon}J^{\lambda\varepsilon}P^{\mu}-P^{\mu}\omega_{\rho\sigma}J^{\rho\sigma}\right)+i(\varepsilon_{\nu}P^{\nu}P^{\mu}-P^{\mu}\varepsilon_{\nu}P^{\nu})\\
&= P^{\mu}+\frac{i}{2}\omega_{\rho\sigma}[J^{\rho\sigma},P^{\mu}]+i\varepsilon_{\nu}[P^{\nu},P^{\mu}].
\end{aligned}
$$

where $U^{\dagger}(\Lambda,a)=U((\delta+\omega)^{-1},-(\delta+\omega)^{-1}a)=U(\delta-\omega,-\varepsilon+\omega\varepsilon)=U(\delta-\omega,-\varepsilon)$ was used. Invoking the validity of the last set of equalities for all $\omega,\varepsilon$, we find:

$$
[P^{\nu},P^{\mu}]=0 \quad \& \quad \frac{i}{2}[J^{\rho\sigma},P^{\mu}]=\frac{1}{2}(\eta^{\mu\sigma}P^{\rho}-\eta^{\mu\rho}P^{\sigma}).
$$

with the antisymmetry of $\omega$ used to write

$$
2\eta^{\mu\sigma}\omega_{\rho\sigma}P^{\rho}=\eta^{\mu\sigma}\omega_{\rho\sigma}P^{\rho}-\eta^{\mu\sigma}\omega_{\sigma\rho}P^{\rho}=\eta^{\mu\sigma}\omega_{\rho\sigma}P^{\rho}-\eta^{\mu\rho}\omega_{\rho\sigma}P^{\sigma}.
$$

The expanded transformation of $J^{\mu\nu}$ reads

$$\begin{aligned}
J^{\mu\nu} + \frac{i}{2}\omega_{\rho\sigma}[J^{\rho\sigma}, J^{\mu\nu}] + O(\varepsilon) &= \left(\mathbf{1} + \frac{i}{2}\omega_{\lambda\varepsilon}J^{\lambda\varepsilon} + i\varepsilon_\alpha P^\alpha\right) J^{\mu\nu} \left(\mathbf{1} - \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} - i\varepsilon_\sigma P^\sigma\right) \\
&= U(\Lambda, a) J^{\mu\nu} U^\dagger(\Lambda, a) \\
&= \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu (J^{\rho\sigma} + \varepsilon^\rho P^\sigma - \varepsilon^\sigma P^\rho) \\
&= (\delta_\rho{}^\mu + \omega_\rho{}^\mu)(\delta_\sigma{}^\nu + \omega_\sigma{}^\nu) J^{\rho\sigma} + O(\varepsilon) \\
&= J^{\mu\nu} + J^{\mu\sigma}\omega_\sigma{}^\nu + J^{\rho\nu}\omega_\rho{}^\mu + O(\varepsilon) \\
&= J^{\mu\nu} + \frac{1}{2}\omega_{\rho\sigma}\left(\eta^{\nu\sigma}J^{\mu\rho}\omega_{\rho\sigma} - \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\sigma}J^{\rho\nu} - \eta^{\mu\rho}J^{\sigma\nu}\right) + O(\varepsilon)
\end{aligned}$$

The expansion in $\varepsilon$ was not explicit because it only serves to determine $[J^{\rho\sigma}, P^\mu]$ which is already known at that point. Finally, using the fact $\eta$ is symmetric and $\omega$ arbitrary, we get

$$\frac{i}{2}[J^{\rho\sigma}, J^{\mu\nu}] = \frac{1}{2}(\eta^{\nu\sigma}J^{\mu\rho} - \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\sigma\mu}J^{\rho\nu} - \eta^{\rho\mu}J^{\sigma\nu}).$$

(f) We now define the angular momentum vector $\mathbf{J} = (J^{23}, J^{31}, J^{12}) \equiv (J^1, J^2, J^3)$. The commutation relation of these operators is given by

$$\begin{aligned}
[J^1, J^2] = [J^{23}, J^{31}] &= -i\left(\eta^{13}J^{32} - \eta^{12}J^{33} + \eta^{33}J^{21} - \eta^{23}J^{31}\right) = -i\left(\eta^{33}J^{21}\right) = -i(J^{12}) = -iJ^3 \\
[J^2, J^3] = [J^{31}, J^{12}] &= -i\left(\eta^{21}J^{13} - \eta^{23}J^{11} + \eta^{11}J^{32} - \eta^{31}J^{12}\right) = -i\left(\eta^{11}J^{32}\right) = -i\left(J^{23}\right) = -iJ^1 \\
[J^3, J^1] = [J^{12}, J^{23}] &= -i\left(\eta^{32}J^{21} - \eta^{31}J^{22} + \eta^{22}J^{13} - \eta^{12}J^{23}\right) = -i\left(\eta^{22}J^{13}\right) = -i\left(J^{31}\right) = -iJ^2
\end{aligned}$$

Combining these results with the antisymmetry of the commutator, we recover $\left[J^i, J^j\right] = -i\varepsilon^{ij}{}_k J^k$.

(g) We finally consider the commutation of the contraction $P^2 = P^\mu P_\mu$ with all other generators of the Poincaré group. Since $[P^\nu, P^\mu] = 0$ the $P^2$ commutes with all translation generators. For Lorentz transformation generators, we get

$$\begin{aligned}
[P^\mu P_\mu, J^{\rho\sigma}] &= P^\mu[P_\mu, J^{\rho\sigma}] + [P^\mu, J^{\rho\sigma}]P_\mu \\
&= P_\mu[P^\mu, J^{\rho\sigma}] + [P^\mu, J^{\rho\sigma}]P_\mu \\
&= -i\left(P_\mu\eta^{\mu\sigma}P^\rho - P_\mu\eta^{\mu\rho}P^\sigma\right) - i\left(\eta^{\mu\sigma}P^\rho P_\mu - \eta^{\mu\rho}P^\sigma P_\mu\right) \\
&= -i\left(P^\sigma P^\rho - P^\rho P^\sigma\right) - i\left(P^\rho P^\sigma - P^\sigma P^\rho\right) = 0
\end{aligned}$$

so $P^2$ commutes with all generators.

# 2 Acknowledgement

I worked on my own for this assignment.