8

Fermi-Walker Transport

In § 2.4 we investigated parallel transport of a vector along an arbitrary worldline $x^\mu(s)$. The motivation was that we should be able to compare, at different places along the orbit, the vectors associated with a point mass, such as the speed or the spin. The vectors are supposed to be known along the orbit, and we compare the vector $\boldsymbol{A}$ with $\boldsymbol{A}'$, obtained by parallel transport, see Fig. 2.4. If these two do not coincide we say that the vector has intrinsically changed due to influences other than gravity. The actual change of the vector $\boldsymbol{A}$ along the worldline is a matter of studying the dynamics. We know that the 4-velocity u^μ is by definition tangent vector and $u^\mu u_\mu = 1$, but the change of the spin vector for example depends on the applied torque. Here we analyse a seemingly innocuous question: a spinning top moves along a worldline that is not a geodesic, i.e. the top experiences an acceleration, but there are no external torques. How does the spin axis behave? The result will be used to derive the *Thomas precession* of the electron and the *geodesic precession* of a gyroscope.

8.1 Transport of accelerated vectors

A test mass moves along its worldline W due to gravity and other forces, and $x^\mu(s)$ is determined by eq. (3.60), see Fig. 8.1. Now imagine that the test mass carries orthonormal unit vectors, the 4-velocity u^μ and n_i^μ ($i = 1, 2, 3$). In the local rest-frame $u^\mu = (1, 0, 0, 0)$. The $n_i^\mu = (0, \boldsymbol{n}_i)$ are spacelike, $n_i^\mu n_{j\mu} = -\delta_{ij}$ and $u^\mu n_{i\mu} = 0$. The unit vectors $\boldsymbol{n}_i$ may be thought of as defined by the spin axes of ideal precession-free gyroscopes (no external torques). Having defined the physical situation in the rest-frame, we now seek a mathematical description of the change or 'transport' of u^μ and n_i^μ, or rather of A^μ (a linear combination of u^μ and the n_i^μ) along $x^\mu(s)$ in an arbitrary reference frame. We surmise that the transport law is a generalisation of parallel transport, and try to achieve our goal with an extra term in (2.28). Accordingly, we define the following operator on $x^\mu(s)$:

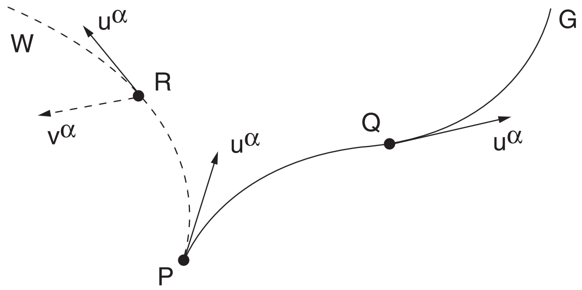

Fig. 8.1. Introducing Fermi-Walker transport. If there is only gravity, a test mass with initial 4-velocity u^α in P moves on a unique geodesic G, but in the presence of additional non-gravitational forces it moves on a non-geodesic worldline W. The 4-velocity $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}s$ is always tangent to G and to W, and $u^\alpha u_\alpha = 1$ (as always). Parallel transport $\mathrm{D}A^\alpha/\mathrm{D}s = 0$ along G carries $u^\alpha(P)$ over into $u^\alpha(Q)$ because G is a geodesic. But parallel transport along W produces some $v^\alpha(R) \neq u^\alpha(R)$. We seek a generalised (Fermi-Walker) transport law $\delta A^\alpha/\delta s = 0$ that carries u^α over into itself and preserves the value of the inner product $A^\alpha B_\alpha$ of two vectors along an arbitrary worldline.

$$\frac{\delta A^\mu}{\delta s} \equiv \frac{\mathrm{D}A^\mu}{\mathrm{D}s} - K^\mu{}_\alpha A^\alpha . \tag{8.1}$$

$\mathrm{D}/\mathrm{D}s$ is the operator (2.26) for parallel transport. We lower the index on the right hand side of (8.1) by multiplying with $g_{\nu\mu}$. The result is $\mathrm{D}A_\nu/\mathrm{D}s - K_\nu{}^\alpha A_\alpha$ (see exercise), and thus we define for covariant vectors

$$\frac{\delta A_\nu}{\delta s} \equiv \frac{\mathrm{D}A_\nu}{\mathrm{D}s} - K_\nu{}^\alpha A_\alpha , \tag{8.2}$$

where $\mathrm{D}A_\nu/\mathrm{D}s$ is now given by (2.27). The transport law would then be

$$\frac{\delta A^\mu}{\delta s} = 0 \qquad \text{or} \qquad \frac{\delta A_\nu}{\delta s} = 0 , \tag{8.3}$$

for contravariant and covariant vectors, respectively. With the help of (8.1) and (2.26) we obtain

$$\frac{\delta A^\mu}{\delta s} \equiv \frac{\mathrm{d}A^\mu}{\mathrm{d}s} - (K^\mu{}_\nu - \Gamma^\mu_{\nu\sigma} u^\sigma)\, A^\nu = 0 . \tag{8.4}$$

This is the explicit form of the so called *Fermi-Walker transport law* for a contravariant vector. In order to be able to handle tensors of higher rank we define for two vectors X and Y, conform relation (2.44):

$$\frac{\delta}{\delta s} XY = \frac{\delta X}{\delta s} Y + X \frac{\delta Y}{\delta s} . \tag{8.5}$$

We now proceed to determine the tensor $K^{\mu\nu}$. The inner product $A^\mu B_\mu$ of two vectors A^μ and B^μ (i.e. two linear combinations of u^μ and the

n_i^μ) is constant in the local rest-frame. But $A^\mu B_\mu$ is scalar and therefore one and the same constant in all frames. This implies according to (2.47) that $\mathrm{D}A^\mu B_\mu/\mathrm{D}s = \mathrm{d}A^\mu B_\mu/\mathrm{d}s = 0$, though $\mathrm{D}A^\mu/\mathrm{D}s$ and $\mathrm{D}B^\mu/\mathrm{D}s$ in general do not vanish since they are not parallel-transported. We elaborate $0 = \delta(A^\mu B_\mu)/\delta s \equiv (\delta A^\mu/\delta s)B_\mu + A^\mu(\delta B_\mu/\delta s)$:

$$\begin{aligned} 0 &= A^\mu \frac{\mathrm{D}B_\mu}{\mathrm{D}s} + B_\mu \frac{\mathrm{D}A^\mu}{\mathrm{D}s} - A^\mu K_\mu{}^\alpha B_\alpha - B_\mu K^\mu{}_\alpha A^\alpha \\ &= \frac{\mathrm{D}}{\mathrm{D}s}(A^\mu B_\mu) - K^{\mu\alpha} A_\mu B_\alpha - K^{\mu\alpha} A_\alpha B_\mu \\ &= -(K^{\mu\alpha} + K^{\alpha\mu}) A_\mu B_\alpha \ . \end{aligned} \tag{8.6}$$

It follows that $K^{\mu\nu}$ must be antisymmetric, $K^{\mu\alpha} = -K^{\alpha\mu}$. It seems natural to expect that $K^{\mu\alpha}$ depends on the 4-velocity, and therefore we try

$$K^{\mu\nu} = a^\mu u^\nu - u^\mu a^\nu \ , \tag{8.7}$$

for a certain vector a^μ. A component of a^μ parallel to u^μ does not contribute to (8.7), so we may impose without restriction that

$$a^\mu u_\mu = 0 \ , \tag{8.8}$$

and then we also have that

$$K^{\mu\nu} u_\nu = a^\mu \ . \tag{8.9}$$

The unknown vector a^μ may be found by requiring that u^μ obey the transport law $\delta u^\mu/\delta s = 0$. With the help of (8.1), (8.8) and (8.12) we get:

$$0 = \frac{\mathrm{D}u^\mu}{\mathrm{D}s} - (a^\mu u_\alpha - u^\mu a_\alpha)\, u^\alpha = \frac{\mathrm{D}u^\mu}{\mathrm{D}s} - a^\mu \ , \tag{8.10}$$

because of (8.8) and $u_\alpha u^\alpha = 1$. Consequently:

$$a^\mu = \frac{\mathrm{D}u^\mu}{\mathrm{D}s} \ . \tag{8.11}$$

By comparing with (3.60) we see that a^ν is equal to the non-inertial acceleration f^μ of P divided by $m_0 c^2$.

One might object that expression (8.7) is not the most general choice, and that

$$K^{\mu\nu} = a^\mu u^\nu - u^\mu a^\nu + H^{\mu\nu} \tag{8.12}$$

with antisymmetric $H^{\mu\nu}$ would also satisfy the requirements. We now show that $H^{\mu\nu} = 0$ implies the absence of any rotation of spatial vectors in the local rest-frame, hence absence of external torques. To that end we study the change of a purely spatial vector n^μ in the local rest-frame, where $n^\mu = (0, \boldsymbol{n})$ and $u^\mu = (1, 0, 0, 0)$, so that $n^\mu u_\mu = g_{\mu\nu} n^\mu u^\nu = \eta_{\mu\nu} n^\mu u^\nu = 0$, as before. The Christoffel symbols are also zero, and Fermi-Walker transport $\delta n^\mu / \delta s = 0$ implies

$$\frac{\mathrm{d}n^\mu}{\mathrm{d}s} = (a^\mu u_\nu - u^\mu a_\nu)\, n^\nu = -\, u^\mu a_\nu n^\nu \ . \tag{8.13}$$

It follows that $\mathrm{d}n^i / \mathrm{d}s = 0$: the instantaneous rate of change of the spatial part of n^μ is zero, so that there is no instantaneous rotation (but there would be one if $H^{\mu\nu} \neq 0$).

This completes the derivation of the *Fermi-Walker transport law* (8.4), with $K^{\mu\nu}$ given by (8.7), (8.11) and $u^\mu = \mathrm{d}x^\mu / \mathrm{d}s$. It is a differential equation specifying the change of an accelerated vector A^μ on which no torques are exerted in the local rest-frame. We note the following:

(1). The middle term on the right hand side of (8.4) is of special-relativistic origin. In SR the Γ's are zero (in rectangular co-ordinates) but $K^{\mu\nu} \neq 0$. This term is responsible for the Thomas precession.

(2). The last term in (8.4) is a general-relativistic effect. If the only force is gravity, then x^μ is a geodesic $\rightarrow$ $\mathrm{D}u^\mu / \mathrm{D}s = 0$ $\rightarrow$ $a^\mu = 0$ $\rightarrow$ $K^{\mu\nu} = 0$. And in that case eq. (8.4) is identical to parallel transport. One of the consequences is the geodesic precession. Any additional (non-inertial) force causes an extra Thomas-like precession.

Exercise 8.1: We are using a spacelike unit vector n^μ with $n^\mu n_\mu = -1$. Negative lengths, how is that again?

Hint: Very simple. For example, in the local rest-frame $n^\mu = (0, n^1, n^2, n^3)$ and $n_\mu = \eta_{\mu\nu} n^\nu = (0, -n^1, -n^2, -n^3)$. The value of the scalar $n^\mu n_\mu = -|\boldsymbol{n}|^2 = -1$ is invariant.

Exercise 8.2: Prove the statement between (8.1) and (8.2).

Hint: § 2.6: $g_{\nu\mu} \mathrm{D}A^\mu / \mathrm{D}s = g_{\nu\mu} A^\mu{}_{:\sigma}\, u^\sigma = (g_{\nu\mu} A^\mu)_{:\sigma}\, u^\sigma = A_{\nu:\sigma}\, u^\sigma = \mathrm{D}A_\nu / \mathrm{D}s$. Furthermore, $g_{\nu\mu} K^\mu{}_\alpha A^\alpha = K_{\nu\alpha} A^\alpha = K_\nu{}^\alpha A_\alpha$.

Exercise 8.3: Show that $a^\mu u_\mu$ is indeed zero.

Hint: $a^\mu u_\mu = \frac{1}{2} u_\mu \mathrm{D}u^\mu/\mathrm{D}s + \frac{1}{2} u_\mu \mathrm{D}u^\mu/\mathrm{D}s = \frac{1}{2}\mathrm{D}(u_\mu u^\mu)/\mathrm{D}s = 0$. This last step requires that $u_\mu \mathrm{D}u^\mu/\mathrm{D}s = u^\mu \mathrm{D}u_\mu/\mathrm{D}s$. See previous exercise for inspiration.

8.2 Thomas precession

This is a problem from SR, and the qualitive explanation has already been given in § 1.1. An electron moves in a circular orbit in the x^1, x^2 plane. Spacetime is flat and we use Cartesian co-ordinates so that all Γ's are zero. According to (8.4), Fermi-Walker transport of the spin vector s^μ is described by

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\tau} = c\, K^\mu{}_\nu s^\nu \ , \tag{8.14}$$

because $\mathrm{d}/\mathrm{d}s = (1/c)\mathrm{d}/\mathrm{d}\tau$. To determine $K^{\mu\nu}$ we analyse the circular motion of the electron and take

$$x^1 = r\cos\omega\tau \ ; \qquad x^2 = r\sin\omega\tau \ ; \qquad x^3 = 0 \ , \tag{8.15}$$

from which

$$\left.\begin{aligned} u^1 &= c^{-1}\,\mathrm{d}x^1/\mathrm{d}\tau = -(\omega r/c)\sin\omega\tau \ ; \\ u^2 &= (\omega r/c)\cos\omega\tau \ ; \\ u^3 &= 0 \ . \end{aligned}\right\} \tag{8.16}$$

Here ω is the orbital frequency measured in the proper time of the electron; u^0 can be obtained from $1 = u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = (u^0)^2 - (u^1)^2 - (u^2)^2$:

$$u^0 = \sqrt{1 + (\omega r/c)^2} = \text{constant} \ , \tag{8.17}$$

and this serves to find the relation between proper time τ and laboratory time t, because $u^0 = \gamma = 1/\sqrt{1-\beta^2}$, see (3.23). Therefore $\omega\tau = \omega t/\gamma \equiv \Omega t$, where Ω = orbital frequency in laboratory time:

$$\left.\begin{aligned} &\gamma = \sqrt{1 + (\omega r/c)^2} \ ; \qquad \Omega = \omega/\gamma \ ; \\ &\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{\Omega}\frac{\mathrm{d}}{\mathrm{d}t} \ . \end{aligned}\right\} \tag{8.18}$$

Since the Γ's are zero, we infer from (8.11) and (2.26) that $a^\mu = \mathrm{D}u^\mu/\mathrm{D}s = c^{-1}\mathrm{d}u^\mu/\mathrm{d}\tau$. We may now write (8.14) as:

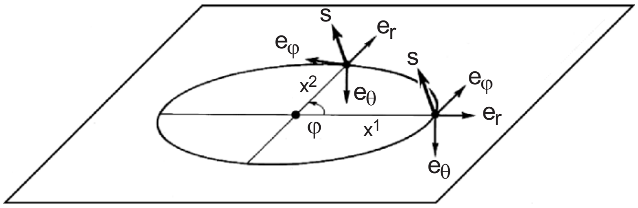

Fig. 8.2. Geodesic precession of the vector $\boldsymbol{s}$ analysed in the equatorial plane $\theta = \pi/2$ of the rotating reference frame $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\varphi$.

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\tau} = c\left(a^\mu u_\nu - u^\mu a_\nu\right) s^\nu = -u^\mu \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} s^\nu ; \tag{8.19}$$

$$u_\nu s^\nu = 0 , \tag{8.20}$$

because we know that $u_\nu s^\nu$ is constant (Fermi-Walker transport), and that $u^\mu = (1, 0, 0, 0)$ and $s^\mu = (0, \boldsymbol{s})$ in the local rest-frame, so that $u_\nu s^\nu = \eta_{\nu\alpha} u^\alpha s^\nu = 0$. Because $u_\nu s^\nu$ is invariant (8.20) holds in any frame. Since $u^3 = 0$ we conclude from (8.19) that $\mathrm{d}s^3/\mathrm{d}\tau = 0$, or

$$\frac{\mathrm{d}s^3}{\mathrm{d}t} = 0 . \tag{8.21}$$

Apparently, the z-component of the spin is constant. The behaviour of s^0 follows from (8.20): $0 = \eta_{\nu\sigma} u^\sigma s^\nu = u^0 s^0 - u^1 s^1 - u^2 s^2 \rightarrow s^0 = (u^1 s^1 + u^2 s^2)/u^0$. However, s^0 has no physical meaning – its 'function' is to ensure that $u_\nu s^\nu$ and $s^\nu s_\nu$ are constant. The physics is in the behaviour of s^1 and s^2. With (8.16) and $u_i = \eta_{i\nu} u^\nu = -u^i$ we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} s^1 \\ s^2 \end{pmatrix} = \frac{\omega^3 r^2}{c^2}\begin{pmatrix} \sin\omega\tau\cos\omega\tau & \sin^2\omega\tau \\ -\cos^2\omega\tau & -\sin\omega\tau\cos\omega\tau \end{pmatrix}\begin{pmatrix} s^1 \\ s^2 \end{pmatrix} . \tag{8.22}$$

Express this in laboratory time with (8.18):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} s^1 \\ s^2 \end{pmatrix} = (\gamma^2 - 1)\Omega\begin{pmatrix} \sin\Omega t\cos\Omega t & \sin^2\Omega t \\ -\cos^2\Omega t & -\sin\Omega t\cos\Omega t \end{pmatrix}\begin{pmatrix} s^1 \\ s^2 \end{pmatrix} . \tag{8.23}$$

Exercise 8.4: Verify that the solution of (8.23) with initial values $s^1(0) = s$ and $s^2(0) = 0$ is given by

$$\begin{aligned} s^1 &= \tfrac{1}{2}s\left[(1+\gamma)\cos(1-\gamma)\Omega t + (1-\gamma)\cos(1+\gamma)\Omega t\right] ; \\ s^2 &= \tfrac{1}{2}s\left[(1+\gamma)\sin(1-\gamma)\Omega t + (1-\gamma)\sin(1+\gamma)\Omega t\right] . \end{aligned} \tag{8.24}$$

Expand for $\beta \ll 1$:

$$\begin{aligned} s^1 &\simeq \quad s\left[\cos\tfrac{1}{2}\beta^2\Omega t - \tfrac{1}{4}\beta^2\cos 2\Omega t\right] ; \\ s^2 &\simeq -s\left[\sin\tfrac{1}{2}\beta^2\Omega t + \tfrac{1}{4}\beta^2\sin 2\Omega t\right] . \end{aligned} \tag{8.25}$$

Verify that the first terms in (8.24) and (8.25) correspond to a rotation of the spin vector with a frequency

$$\Omega_{\text{Thomas}} = (\gamma - 1)\,\Omega_{\text{orbit}} \simeq \tfrac{1}{2}\beta^2\,\Omega_{\text{orbit}} \quad , \tag{8.26}$$

with $\beta \simeq \omega r/c \simeq \Omega r/c \ll 1$. The sense of the rotation is opposite to the orbital rotation. Both second terms in (8.25) describe a small, fast modulation that averages to zero.

8.3 Geodesic precession

In § 4.4 we analysed the motion of a test mass moving in the Schwarzschild metric, and found, among other things, that the orbit precesses. This precession of the perihelium is not the only GR effect. If the test mass behaves as a vector, as for example a gyroscope, the (spin) vector will also perform a precession, even when no torque is exerted. We shall now derive this so-called *geodesic precession*. Because the body moves along a geodesic we have that $K^{\mu\nu} = 0$, in which case (8.4) reduces to the equation for parallel transport:

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\tau} + c\,\Gamma^\mu_{\nu\sigma}\,u^\sigma s^\nu = 0 \; . \tag{8.27}$$

Here s^ν is the unit vector along the spin axis. The following analysis is a sequel of § 4.3, and we shall employ the notation we used there. The 4-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}s$ is given by:[1]

[1] For the geodesic precession the rotation of the Earth is irrelevant. So although Fig. 8.3 suggests otherwise, the satellite may be taken move on the equator r, $\theta =$ constant of the Schwarzschild metric.

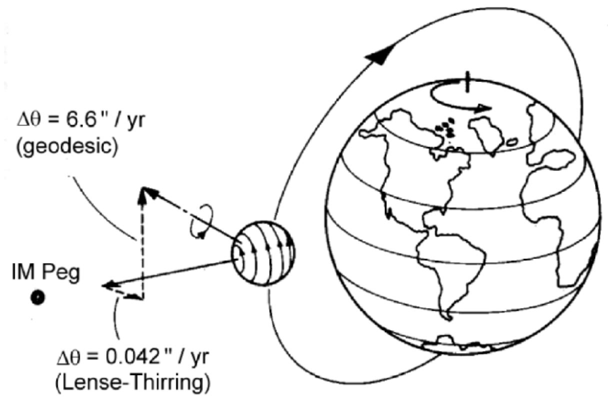

Fig. 8.3. A gyroscope orbiting a rotating mass like the Earth and moving only under the influence of gravity should exhibit a geodesic precession and a Lense-Thirring precession. The experiment is now in progress in the Gravity Probe B satellite, launched in April 2004 into a polar orbit of 640 km altitude. The star IM Pegasi (HR 8703) serves as the pointing reference. See text for details. Adapted from: *Near Zero*, J.D. Fairbank et al. (eds.) Freeman & Co (1988).

$$\begin{aligned} u^\mu &= (c\dot{t},\, \dot{r},\, \dot{\theta},\, \dot{\varphi}) \;=\; (c\dot{t},\, 0,\, 0,\, h/r^2) \\ &= \left(\left\{ 1 - \frac{3r_\mathrm{s}}{2r} \right\}^{-1/2},\, 0,\, 0,\, \frac{1}{r^2} \left\{ \frac{r r_\mathrm{s}/2}{1 - 3r_\mathrm{s}/2r} \right\}^{1/2} \right) . \end{aligned} \tag{8.28}$$

At the second = sign we choose a circular orbit: r = constant and $\theta = \pi/2$, and we have used (4.34) as well. The last expression in (8.28) follows immediately from (4.32) and (4.45). Next we write out (8.27) explicitly, and obtain the following equations (see exercises):

$$s^0 = \frac{\sqrt{r r_\mathrm{s}/2}}{1 - r_\mathrm{s}/r}\, s^3 \; ; \tag{8.29}$$

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} = \frac{c}{r} \sqrt{r r_\mathrm{s}/2}\, \sqrt{1 - 3r_\mathrm{s}/2r}\, s^3 \; ; \tag{8.30}$$

$$\frac{\mathrm{d}s^2}{\mathrm{d}\tau} = 0 \; ; \tag{8.31}$$

$$\frac{\mathrm{d}s^3}{\mathrm{d}\tau} = -\frac{c}{r^3} \left(\frac{r r_\mathrm{s}/2}{1 - 3r_\mathrm{s}/2r} \right)^{1/2} s^1 \; . \tag{8.32}$$

Take $\mathrm{d}/\mathrm{d}\tau$ of (8.32) and eliminate $\mathrm{d}s^1/\mathrm{d}\tau$ with (8.30):

$$\frac{\mathrm{d}^2 s^3}{\mathrm{d}\tau^2} + \frac{c^2 r_\mathrm{s}}{2r^3}\, s^3 = 0 \,, \tag{8.33}$$

and it is easy to verify that the same equation holds for s^1. The solution with initial value $s^3(0) = 0$ is:

$$\left.\begin{aligned} s^3 &= s^\varphi = -\, s \sin\omega\tau \,; \\ s^1 &= s^r = sr\sqrt{1 - 3r_\mathrm{s}/2r}\, \cos\omega\tau \,; \\ s^2 &= s^\theta = \text{constant} \,, \end{aligned}\right\} \tag{8.34}$$

where

$$\omega = c\left(\frac{r_\mathrm{s}}{2r^3}\right)^{1/2} = \left(\frac{GM}{r^3}\right)^{1/2} . \tag{8.35}$$

The geodesic precession is a consequence of the fact that the precession frequency ω is a little smaller than the orbital frequency, which is equal to

$$\frac{2\pi}{\Delta\tau} = c\left(\frac{r_\mathrm{s}}{2r^3}\right)^{1/2}\left(1 - \frac{3r_\mathrm{s}}{2r}\right)^{-1/2} . \tag{8.36}$$

Here we have used expression (4.46) for the orbital period $\Delta\tau$. After each orbit the spin vector has rotated over an angle of

$$\omega\Delta\tau = 2\pi\sqrt{1 - 3r_\mathrm{s}/2r} \,. \tag{8.37}$$

The spin vector precesses about an axis orthogonal to the orbital plane, but the major part of the precession is caused by the fact that the reference frame itself rotates over an angle of 2π, see Fig. 8.2. When viewed from a non-rotating frame the precession angle per orbit equals

$$\delta\psi = 2\pi\left(1 - \sqrt{1 - 3r_\mathrm{s}/2r}\right) \simeq \frac{3\pi r_\mathrm{s}}{2r} \,. \tag{8.38}$$

Actually, we must still transform to co-ordinate time, but that gives rise to a correction of higher order. The precession has the same sense of rotation as the orbit. The physical origin of the precession is that a vector that is parallel transported constantly changes its direction, due to the curvature of space-time, see § 2.4. This is visible as a small secular angular rotation. The effect of geodesic precession has been observed in the binary pulsar PSR 1913+16.[2] What if the central object rotates? In that case its exterior metric is replaced by the Kerr metric (in good approximation), and frame-dragging (§ 6.5) induces an additional precession, called the Lense-Thirring effect. The LAGEOS satellites have confirmed the Lense-Thirring effect due to the rotation of the Earth with a precision of 10%.[3]

[2] Weisberg, J.M. and Taylor, J.H., *Ap. J. 576* (2002) 942.
[3] Ciufolini, I. and Pavlis, E.C., *Nature 431* (2004) 958.

Fig. 8.4. Inside view of a gyroscope of Gravity Probe B and its housing. The rotor has a diameter of 3.8 cm, and is made of fused quartz coated with niobium. Image credit: Don Harley.

8.4 Gravity Probe B

The technology for high-precision measurements of the geodesic precession and the Lense-Thirring effect has been developed in the USA from the beginning of the 1960s. The outcome of this long development programme, the longest in NASA's history to date[4], is Gravity Probe B, launched on April 20, 2004, see Fig. 8.3. The satellite carries 4 precision gyroscopes. The geodesic precession is only $6.6''$ per year, and the Lense-Thirring precession is much smaller: $0.04''$ per year. The gyros consist of quartz rotors coated with superconducting niobium, suspended in an electrostatic field, see Figs. 8.4 and 8.5. The rotation (about 70 Hz) induces a London magnetic moment that generates a magnetic dipole field aligned with the spin axis. Its direction, and hence the orientation of the spin axis can be measured with high precision.[5] There are many experimental complications. For example, any parasitic torque will cause the gyroscope to precess, and any non-inertial acceleration induces an extra Thomas precession. By using a drag-free satellite that literally follows the inertial motion of one of the the gyroscopes, the residual acceleration will

[4] For the programmatic and scientific issues involved see Reichhardt, T., *Nature 426* (2003) 380.

[5] For more details see *Near Zero*, J.D. Fairbank et al. (eds.), Ch. 6.1 – 6.3 (Freeman & Co 1988); for theoretical aspects see Will (1993) p. 208; Gravity Probe B website: http://einstein.stanford.edu/

Fig. 8.5. Gravity Probe B carries four gyroscopes, mounted in a single quartz bloc, a prototype of which is shown here. The pointing telescope (not shown) is attached to the flange at the lower end. The whole unit is placed in a much larger helium dewar. Image credit: Gravity Probe B, Stanford University.

be at the 10^{-11}g level. The gyroscopes have a pointing stability of better than 5×10^{-4} arcseconds over a period of a year!

In closing, we draw attention to two issues. The first is the fact that the precession angle (8.38) is independent of the spin rate of the gyroscope, and the same is true for the Lense-Thirring precession.[6] This is a reminder of the physics involved: both effects are a consequence of parallel transport of a vector in the Schwarzschild or Kerr metric. The nature of the vector is immaterial, and so is the existence of mass currents in the gyroscope. A gyroscope is for many reasons by far the best technical solution, but a non-rotating pencil would, as a matter of principle, also do very well – if one could eliminate all parasitic forces and moments.

[6] See Will (1993) p. 210.

The second issue is the pointing reference. Stellar parallaxes and proper motions are generally larger than the accuracy required for Gravity Probe B. Therefore the only suitable pointing references are quasars. Quasars are distant powerful radio sources that are believed to constitute the best available inertial reference frame. But quasars are too dim in visible light for the small pointing telescope (aperture 14 cm). Therefore a relatively bright star had to be found, that is also a strong radio point source, and located sufficiently close to a few reference quasars to permit measuring the relative positions with the method of Very Long Baseline Interferometry (VLBI). The outcome is IM Peg (HR 8703). The proper motion and parallax of IM Peg with respect to the quasars have been accurately measured in a VLBI programme extending over many years. In this way the orientation of the gyroscopes can ultimately be related to the quasar reference frame.

Exercise 8.5: Write down the explicit expression for the Christoffel symbols necessary to elaborate (8.27).

Hint: From (4.29): $2\nu = -2\lambda = \log(1 - r_\mathrm{s}/r)$; furthermore $\theta = \pi/2$. Result:

$$(4.10): \quad \Gamma^1{}_{00} = \frac{r_\mathrm{s}}{2r^2}(1 - r_\mathrm{s}/r) \; ; \qquad \Gamma^1{}_{33} = -r(1 - r_\mathrm{s}/r) \; .$$

$$(4.11): \quad \Gamma^2{}_{12} = \frac{1}{r} \; ; \qquad \Gamma^2{}_{33} = 0 \; .$$

$$(4.12): \quad \Gamma^3{}_{13} = \frac{1}{r} \; ; \qquad \Gamma^3{}_{23} = 0 \; .$$

Exercise 8.6: Show that $u^\mu s_\mu = 0$ holds here as well, just as in the case of Thomas precession. Use that to derive (8.29).

Hint: $0 = g_{\mu\nu} u^\mu s^\nu = g_{00} u^0 s^0 + g_{33} u^3 s^3$; use (4.29) and $\theta = \pi/2$.

Exercise 8.7: Prove now eqs. (8.30) to (8.32).

Hint: Insert the Γ's, and u^0 and u^3 from (8.28), and use (8.29).

Exercise 8.8: Show that a gyroscope in orbit around the Earth at an altitude of 650 km has a geodesic precession of $6.6''$ per year.

Hint: (8.38) + Keplerian orbit $\rightarrow$ $3(GM_\mathrm{a})^{3/2}/(2c^2 r^{5/2})$ rad s^{-1}, etc.

Exercise 8.9: We wish to compare the precession amplitudes along $\boldsymbol{e}_r$ and along $\boldsymbol{e}_\varphi$, see Fig. 8.2. But that is not possible as s^1 and s^3 in (8.34) have different dimensions. How is that?

Hint: Physical lengths follow from (3.7)! Amplitudes along the r-direction: $\mathrm{d}l_r^2 = -g_{rr}(s^r)^2 \simeq r^2(1 - r_\mathrm{s}/2r)s^2$; $\mathrm{d}l_\varphi^2 = -g_{\varphi\varphi}(s^\varphi)^2 = r^2 s^2$.

Exercise 8.10: Does a linearly accelerated electron experience any Thomas-like effect?

Hint: Take the 1-axis in the direction of the acceleration, then $u^2 = u^3 = 0$. According to (8.19) only s^0 and s^1 will change. To see what actually happens, assume that the electron experiences a constant acceleration a, and use that $x^1 = (c^2/a)\cosh(a\tau/c) + \text{const}$, $x^0 = ct = (c^2/a)\sinh(a\tau/c)$, see Rindler (2001), so that $u^0 = \cosh(a\tau/c)$ and $u^1 = \sinh(a\tau/c)$. Now solve (8.19).