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Homework 1

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Contents

1 Hodge star operator and vector calculus

We are interested in the spaces $\Omega^k(M)$ of k-forms over a smooth manifold M of dimension n equiped with a pseudo-Riemanian metric tensor g represented as $g=\gamma_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j$ in the frame field induced by the coordinate maps x_i over a open subset $U\subset M$. The Hodge star operator constitutes a linear map $\star:\Omega^k(M)\to\Omega^{n-k}(M)$. In a local frame given by $\mathrm{d}x^i$, its action is specified by

$$\star 1=\sqrt{|\det\gamma|}\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n,\quad \mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}=\frac{1}{(n-k)!}\sqrt{\det\gamma}(\gamma^{-1})^{i_1j_1}\cdots(\gamma^{-1})^{i_1j_1}\epsilon_{j_1\cdots j_n}\mathrm{d}x^{j_{k+1}}\wedge\cdots\wedge\mathrm{d}x^{j_n}\quad \&\quad \star(f\alpha)=f\star\alpha$$

where ϵ is the levi-civita symbol, $f\in\Omega^0(M)$ and $\alpha\in\Omega^k(M)$.

(a) For now, we treat a cartesian frame field $\mathrm{d}x^1=\mathrm{d}x,\mathrm{d}x^2=\mathrm{d}y,\mathrm{d}x^3=\mathrm{d}z$ in three dimensionnal euclidean space by replacing γ_{ij} by δ_{ij}. The Hodge dual of each frame field calculated as follows

$$\star\,\mathrm{d}x=\frac{\sqrt{|1|}}{(3-1)!}(\delta^{-1})^{11}\varepsilon_{123}\mathrm{d}x^2\wedge\mathrm{d}x^3+\frac{\sqrt{|1|}}{(3-1)!}(\delta^{-1})^{11}\varepsilon_{132}\mathrm{d}x^3\wedge\mathrm{d}x^2=\frac{1}{2}(+1)\mathrm{d}x^2\wedge\mathrm{d}x^3+\frac{1}{2}(-1)(-1)\mathrm{d}x^2\wedge\mathrm{d}x^3=\mathrm{d}x^2\wedge\mathrm{d}x^3$$
$$\star\,\mathrm{d}y=\frac{\sqrt{|1|}}{(3-1)!}(\delta^{-1})^{22}\varepsilon_{231}\mathrm{d}x^3\wedge\mathrm{d}x^1+\frac{\sqrt{|1|}}{(3-1)!}(\delta^{-1})^{22}\varepsilon_{213}\mathrm{d}x^1\wedge\mathrm{d}x^3=\frac{1}{2}(+1)\mathrm{d}x^3\wedge\mathrm{d}x^1+\frac{1}{2}(-1)(-1)\mathrm{d}x^3\wedge\mathrm{d}x^1=\mathrm{d}x^3\wedge\mathrm{d}x^1$$
$$\star\,\mathrm{d}z=\frac{\sqrt{|1|}}{(3-1)!}(\delta^{-1})^{33}\varepsilon_{312}\mathrm{d}x^1\wedge\mathrm{d}x^2+\frac{\sqrt{|1|}}{(3-1)!}(\delta^{-1})^{33}\varepsilon_{321}\mathrm{d}x^2\wedge\mathrm{d}x^1=\frac{1}{2}(+1)\mathrm{d}x^1\wedge\mathrm{d}x^2+\frac{1}{2}(-1)(-1)\mathrm{d}x^1\wedge\mathrm{d}x^2=\mathrm{d}x^1\wedge\mathrm{d}x^2$$

(b) Using the metric we can associate a vector $\alpha^\sharp$ to each one-from α such that $g(\alpha^\sharp,v)=v(\alpha)$ for all vector fields v. Consider the expansions $\alpha=\alpha_x\mathrm{d}x+\alpha_y\mathrm{d}y+\alpha_z\mathrm{d}z$, $\alpha^\sharp=\alpha_x^\sharp\frac{\partial}{\partial x}+\alpha_y^\sharp\frac{\partial}{\partial y}+\alpha_z^\sharp\frac{\partial}{\partial z}$ and $v=v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}$. If the components of g are δ_{ij}, then $g(\alpha^\sharp,v)=\alpha_x^\sharp v_x+\alpha_y^\sharp v_y+\alpha_z^\sharp v_z$ and $\alpha(v)=\alpha_x v_x+\alpha_y v_y+\alpha_z v_z$. Since these two expressions are equal for all v by definition of $\sharp$, we conclude $\alpha_x^\sharp=\alpha_x,\ \alpha_y^\sharp=\alpha_y,\ \alpha_z^\sharp=\alpha_z$ for the cartesian euclidean metric. We have a correspondance between one-forms and vectors which can be used to recover vector calculus from diferential form operations.

- Suppose f is a smooth function then $\mathrm{d}f=\partial_x f\,\mathrm{d}x+\partial_y f\,\mathrm{d}y+\partial_z f\,\mathrm{d}z$ which maps to the vector $\mathrm{d}f^\sharp=\partial_x f\frac{\partial}{\partial x}+\partial_y f\frac{\partial}{\partial y}+\partial_z f\frac{\partial}{\partial z}$ which corresponds to the usual notion of a gradient ∇f.
- For a one-form $\alpha=\alpha_x\mathrm{d}x+\alpha_y\mathrm{d}y+\alpha_z\mathrm{d}z$, the exterior derivative reads

$$\begin{aligned}\mathrm{d}\alpha&=(\partial_x\alpha_x\mathrm{d}x+\partial_y\alpha_x\mathrm{d}y+\partial_z\alpha_x\mathrm{d}z)\wedge\mathrm{d}x+(\partial_x\alpha_y\mathrm{d}x+\partial_y\alpha_y\mathrm{d}y+\partial_z\alpha_y\mathrm{d}z)\wedge\mathrm{d}y+(\partial_x\alpha_z\mathrm{d}x+\partial_y\alpha_z\mathrm{d}y+\partial_z\alpha_z\mathrm{d}z)\wedge\mathrm{d}z\\&=\partial_y\alpha_x\mathrm{d}y\wedge\mathrm{d}x+\partial_z\alpha_x\mathrm{d}z\wedge\mathrm{d}x+\partial_x\alpha_y\mathrm{d}x\wedge\mathrm{d}y+\partial_z\alpha_y\mathrm{d}z\wedge\mathrm{d}y+\partial_x\alpha_z\mathrm{d}x\wedge\mathrm{d}z+\partial_y\alpha_z\mathrm{d}y\wedge\mathrm{d}z\\&=(\partial_x\alpha_y-\partial_y\alpha_x)\mathrm{d}x\wedge\mathrm{d}y+(\partial_z\alpha_x-\partial_x\alpha_z)\mathrm{d}z\wedge\mathrm{d}x+(\partial_y\alpha_z-\partial_z\alpha_y)\mathrm{d}y\wedge\mathrm{d}z.\end{aligned}$$

  From the property $\star\star\alpha=(-1)^{1(3-1)}\mathrm{sgn}(\det(\delta_{ij}))=\alpha$ (for one-forms), applying $\star$ to the results found in (a) should bring us back to the expression on which $\star$ was applied to obtain them. Then it follows that

$$\begin{aligned}\star\mathrm{d}\alpha&=(\partial_x\alpha_y-\partial_y\alpha_x)\star\mathrm{d}x\wedge\mathrm{d}y+(\partial_z\alpha_x-\partial_x\alpha_z)\star\mathrm{d}z\wedge\mathrm{d}x+(\partial_y\alpha_z-\partial_z\alpha_y)\star\mathrm{d}y\wedge\mathrm{d}z\quad\text{linearity of }\star\text{ and }\star(\text{coeff }\alpha)=\text{coeff}\star\alpha\\&=(\partial_x\alpha_y-\partial_y\alpha_x)\mathrm{d}z+(\partial_z\alpha_x-\partial_x\alpha_z)\mathrm{d}y+(\partial_y\alpha_z-\partial_z\alpha_y)\mathrm{d}x.\end{aligned}$$

  Finally, using $\sharp$ this result is mapped to the vector with components resulting from the vector product $\nabla\times\alpha^\sharp$. Indeed, we can write $(\star\mathrm{d}\alpha)^\sharp=\nabla\times\alpha^\sharp$.
- Now consider again a one-form $\alpha=\alpha_x\mathrm{d}x+\alpha_y\mathrm{d}y+\alpha_z\mathrm{d}z$. This time, we start by applying $\star$ followed by an exterior derivative to obtain

$$\begin{aligned}&\mathrm{d}\star\alpha=\mathrm{d}(\alpha_x\mathrm{d}y\wedge\mathrm{d}z+\alpha_y\mathrm{d}z\wedge\mathrm{d}y+\alpha_z\mathrm{d}x\wedge\mathrm{d}y)\\&=(\partial_x\alpha_x\mathrm{d}x+\partial_y\alpha_x\mathrm{d}y+\partial_z\alpha_x\mathrm{d}z)\wedge\mathrm{d}y\wedge\mathrm{d}z+(\partial_x\alpha_y\mathrm{d}x+\partial_y\alpha_y\mathrm{d}y+\partial_z\alpha_y\mathrm{d}z)\wedge\mathrm{d}z\wedge\mathrm{d}x+(\partial_x\alpha_z\mathrm{d}x+\partial_y\alpha_z\mathrm{d}y+\partial_z\alpha_z\mathrm{d}z)\wedge\mathrm{d}x\wedge\mathrm{d}y\\&=\partial_x\alpha_x\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z+\partial_y\alpha_y\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}x+\partial_z\alpha_z\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y=(\partial_x\alpha_x+\partial_y\alpha_y+\partial_z\alpha_z)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z.\end{aligned}$$

  Now using the fact $\star$ is an involution in our space (see previous •), we have $1=\star\star1=\star\sqrt{\det(\delta_{ij})}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z=\star\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$ leading to

$$\star\mathrm{d}\star\alpha=\star(\partial_x\alpha_x+\partial_y\alpha_y+\partial_z\alpha_z)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z=\partial_x\alpha_x+\partial_y\alpha_y+\partial_z\alpha_z$$

  which is directly the scalar result obtained when taking the divergence $\nabla\cdot\alpha^\sharp$.
- Finally, consider two one forms $\alpha=\alpha_x\mathrm{d}x+\alpha_y\mathrm{d}y+\alpha_z\mathrm{d}z$ and $\beta=\beta_x\mathrm{d}x+\beta_y\mathrm{d}y+\beta_z\mathrm{d}z$. The $\star$ of their $\wedge$ product reads

$$\begin{aligned}\star(\alpha\wedge\beta)&=\star((\alpha_x\mathrm{d}x+\alpha_y\mathrm{d}y+\alpha_z\mathrm{d}z)\wedge(\beta_x\mathrm{d}x+\beta_y\mathrm{d}y+\beta_z\mathrm{d}z))\\&=\star(\alpha_x\beta_y\mathrm{d}x\wedge\mathrm{d}y+\alpha_x\beta_z\mathrm{d}x\wedge\mathrm{d}z+\alpha_y\beta_x\mathrm{d}y\wedge\mathrm{d}x+\alpha_y\beta_z\mathrm{d}y\wedge\mathrm{d}z+\alpha_z\beta_x\mathrm{d}z\wedge\mathrm{d}x+\alpha_z\beta_y\mathrm{d}z\wedge\mathrm{d}y)\\&=(\alpha_x\beta_y-\alpha_y\beta_y)\star\mathrm{d}x\wedge\mathrm{d}y+(\alpha_z\beta_x-\alpha_x\beta_z)\star\mathrm{d}z\wedge\mathrm{d}x+(\alpha_y\beta_z-\alpha_z\beta_y)\star\mathrm{d}y\wedge\mathrm{d}z\\&=(\alpha_x\beta_y-\alpha_y\beta_y)\mathrm{d}z+(\alpha_z\beta_x-\alpha_x\beta_z)\mathrm{d}y+(\alpha_y\beta_z-\alpha_z\beta_y)\mathrm{d}x\end{aligned}$$

  which matched the component of a vector product and gives the relation $(\star(\alpha\wedge\beta))^\sharp=\alpha^\sharp\times\beta^\sharp$.

2 Maxwell's equations

To express Maxwell's equations with differential forms, we use the following representations of the electric field E, magnetic field B, current density J

$$B = B_x \mathrm{d}y \wedge \mathrm{d}z + B_y \mathrm{d}z \wedge \mathrm{d}x + B_z \mathrm{d}x \wedge \mathrm{d}y, \quad E = E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z, \quad J = J_x \mathrm{d}x + J_y \mathrm{d}y + J_z \mathrm{d}z.$$

and we take the charge density ρ to be a zero-form.

(a) The representation of B as a two-form is motivated by the fact magnetic field components behave as pseudo-vectors under full inversion $(x, y, z) \mapsto (-x, -y, -z)$. To verify that the two-form representation is consistent with this property, we perform a change basis $\{\mathrm{d}x^i\}_{i=1}^3$ to the basis $\{\mathrm{d}\tilde{x}^i\}_{i=1}^3$ generated by fully inverted spatial coordinates. We have $\mathrm{d}\tilde{x}^i = \mathrm{d}(-x) = -\mathrm{d}x^i$ and $\mathrm{d}\tilde{x}^i \wedge \mathrm{d}\tilde{x}^j = (-1)^2 \mathrm{d}x^i \wedge \mathrm{d}x^j$. This implies that at the level of fully spatial two-forms, the full inversion of space leaves the components invariant as expected for a magnetic field.

(b) To connect with the usual vector form of Maxwell's equations, we notice that the usual electric $\tilde{E}$, magnetic $\tilde{B}$ and current density $\tilde{J}$ vector fields are related to the forms given above by $\tilde{E} = E^\sharp$, $\tilde{B} = (\star B)^\sharp = (B_x \mathrm{d}x + B_y \mathrm{d}y + B_z \mathrm{d}z)^\sharp$, $\quad \tilde{J} = J^\sharp$. Using results derived from problem 1, (b) we can use the original set of Maxwell equations to write

$$0 = \nabla \cdot \tilde{B} = \nabla \cdot (\star B)^\sharp = \star \mathrm{d} \star (\star B) = \star \mathrm{d}B \iff \mathrm{d}B = 0 \quad (\star \text{ is an involution})$$

$$\mu_0 \rho = \nabla \cdot \tilde{E} = \nabla \cdot E^\sharp = \star \mathrm{d} \star E$$

$$0 = \frac{\partial \tilde{B}}{\partial t} + \nabla \times \tilde{E} = \frac{\partial (\star B)^\sharp}{\partial t} + (\star \mathrm{d}E)^\sharp \iff 0 = \frac{\partial \star B}{\partial t} + (\star \mathrm{d}E) \iff 0 = \frac{\partial B}{\partial t} + \mathrm{d}E \quad (\star \text{ is an involution, } \sharp \text{ is invertible})$$

$$\mu_0 \tilde{J} = \mu_0 J^\sharp = -\frac{\partial \tilde{E}}{\partial t} + \nabla \times \tilde{B} = -\frac{\partial E^\sharp}{\partial t} + \nabla \times (\star B)^\sharp = -\frac{\partial E^\sharp}{\partial t} + (\star \mathrm{d} \star B)^\sharp \iff \mu_0 J = -\frac{\partial E^\sharp}{\partial t} + (\star \mathrm{d} \star B)^\sharp \quad (\sharp \text{ is invertible})$$

where μ_0 is the magnetic permeability (the equations are written in $c = 1$ units)

(c) By Poincaré's lemma, every closed form on $\mathbb{R}^n$ is exact. Maxwell's equations tell us that B is a closed two-form implying B is also exact. Then, there exists a one-form A such that $B = \mathrm{d}A$. Inserting this result in Faraday's law, we get

$$0 = \frac{\partial B}{\partial t} + \mathrm{d}E = \frac{\partial \mathrm{d}A}{\partial t} + \mathrm{d}E = \mathrm{d}\left(\frac{\partial A}{\partial t} + E\right).$$

Since $-\frac{\partial A}{\partial t} + E$ is a closed one-form it is also exact and there must exist zero-form $-\phi$ such that $-\mathrm{d}\phi = \frac{\partial A}{\partial t} + E \iff E = -\mathrm{d}\phi - \frac{\partial A}{\partial t}$.

(d) We now go back to minkowski space (with $-+++$ signature) with coordinates $(x^0, x^1, x^2, x^3) = (x, y, z, t)$ associated to the one-form frame field $\{\mathrm{d}x^\mu\}_{\mu=0}^3$. We define the two-form $F = B + E \wedge \mathrm{d}t$ and combine the current and charge densities into a single one-form $J = -\rho \star \mathrm{d}t + J_x \star \mathrm{d}x + J_y \star \mathrm{d}y + J_z \star \mathrm{d}z$. To write Maxwell's equations in terms of these new objects, we need to determine the effect of $\star$ and its relation to vector calculus in Minkowski space. We start by calculating

$$\begin{aligned}
\star \mathrm{d}t = &\frac{\sqrt{|-1|}}{(4-1)!}(\gamma^{-1})^{00}\varepsilon_{0123}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{\sqrt{|-1|}}{(4-1)!}(\gamma^{-1})^{00}\varepsilon_{0132}\mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^2 \\
&+ \frac{\sqrt{|-1|}}{(4-1)!}(\gamma^{-1})^{00}\varepsilon_{0213}\mathrm{d}x^2 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \frac{\sqrt{|-1|}}{(4-1)!}(\gamma^{-1})^{00}\varepsilon_{0231}\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 \\
&+ \frac{\sqrt{|-1|}}{(4-1)!}(\gamma^{-1})^{00}\varepsilon_{0312}\mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \frac{\sqrt{|-1|}}{(4-1)!}(\gamma^{-1})^{00}\varepsilon_{0321}\mathrm{d}x^3 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^1 \\
&= (\gamma^{-1})^{00}\mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^2 = -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}$$

For the next calculations, we use the fact $\star$ acting on a $\wedge$ product of basis forms will yield the $\wedge$ product of the basis forms absent of the original product with a sign. The order of the resulting product is such that concatenating it with the original product on the left will produce an even permutation of $txyz$. To fully determine the sign factor, we add multiply by -1 if one of the one-forms in the original product is $\mathrm{d}t$. With this in mind, we can write

$$\star \mathrm{d}x = -\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z \quad (xtyz \text{ odd}), \quad \star \mathrm{d}y = \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}z \quad (ytxz \text{ even}), \quad \star \mathrm{d}z = -\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \quad (ztxy \text{ odd})$$

$$\star(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}t \wedge \mathrm{d}z \quad (xytz \text{ even}), \quad \star(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}t \wedge \mathrm{d}x \quad (yztx \text{ even}), \quad \star(\mathrm{d}z \wedge \mathrm{d}x) = \mathrm{d}t \wedge \mathrm{d}y \quad (zxty \text{ even})$$

$$\star(\mathrm{d}t \wedge \mathrm{d}x) = (-1)\mathrm{d}y \wedge \mathrm{d}z \quad (txyz \text{ even}), \quad \star(\mathrm{d}t \wedge \mathrm{d}y) = (-1)\mathrm{d}z \wedge \mathrm{d}x \quad (tyzx \text{ even}) \quad \star(\mathrm{d}t \wedge \mathrm{d}z) = (-1)\mathrm{d}x \wedge \mathrm{d}y \quad (tzxy \text{ even})$$

$$\star 1 = \sqrt{|\det(\gamma)|}\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

These relations are completed with $\star \star \alpha = (-1)^{k(4-k)}(-1)\alpha$.

(e) If we express E and B with the potentials $A = A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z$ and $\phi = -A_t$, F becomes

$$\begin{aligned}
F &= \mathrm{d}A + \left(-\mathrm{d}\phi - \frac{\partial A}{\partial t}\right) \wedge \mathrm{d}t \\
&= (\partial_x A_y - \partial_y A_x)\mathrm{d}x \wedge \mathrm{d}y + (\partial_z A_x - \partial_x A_z)\mathrm{d}z \wedge \mathrm{d}x + (\partial_y A_z - \partial_z A_y)\mathrm{d}y \wedge \mathrm{d}z \\
&+ \partial_x A_t \mathrm{d}x \wedge \mathrm{d}t + \partial_y A_t \mathrm{d}y \wedge \mathrm{d}t + \partial_z A_t \mathrm{d}z \wedge \mathrm{d}t - (\partial_t A_x \mathrm{d}x + \partial_t A_y \mathrm{d}y + \partial_t A_z \mathrm{d}z) \wedge \mathrm{d}t \\
&= (\partial_x A_y - \partial_y A_x)\mathrm{d}x \wedge \mathrm{d}y + (\partial_z A_x - \partial_x A_z)\mathrm{d}z \wedge \mathrm{d}x + (\partial_y A_z - \partial_z A_y)\mathrm{d}y \wedge \mathrm{d}z(\partial_t A_x - \partial_x A_t)\mathrm{d}t \wedge \mathrm{d}x + (\partial_t A_y - \partial_y A_t)\mathrm{d}t \wedge \mathrm{d}y + (\partial_t A_z - \partial_z A_t)\mathrm{d}t \wedge \mathrm{d}z
\end{aligned}$$

We see the equality of coefficients of $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ with components of the covariant Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(f) The Faraday tensor formulation of Maxwell's equations reads $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$ and $\partial_{(\mu} F_{\nu\sigma)} = 0$. We first notice that

$$\partial_{(\mu} F_{\nu\sigma)} = 0 \iff dF = \partial_\sigma F_{\mu\nu} \mathrm{d}x^\sigma \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu/2 = 6\partial_{(\sigma} F_{\mu\nu)} \mathrm{d}x^\sigma \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu/2 = 0.$$

Then we can explicitly verify that

$$\begin{aligned} \star F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu &= F_{01} \star \mathrm{d}x^0 \wedge \mathrm{d}x^1 + F_{02} \star \mathrm{d}x^0 \wedge \mathrm{d}x^2 + F_{03} \star \mathrm{d}x^0 \wedge \mathrm{d}x^3 + F_{12} \star \mathrm{d}x^1 \wedge \mathrm{d}x^2 + F_{23} \star \mathrm{d}x^2 \wedge \mathrm{d}x^3 + F_{31} \star \mathrm{d}x^3 \wedge \mathrm{d}x^1 \\ &= -F_{01} \mathrm{d}x^2 \wedge \mathrm{d}x^3 - F_{02} \mathrm{d}x^3 \wedge \mathrm{d}x^1 - F_{03} \mathrm{d}x^1 \wedge \mathrm{d}x^2 + F_{12} \mathrm{d}x^0 \wedge \mathrm{d}x^3 + F_{23} \mathrm{d}x^0 \wedge \mathrm{d}x^1 + F_{31} \mathrm{d}x^0 \wedge \mathrm{d}x^2 \end{aligned}$$

and apply an exterior derivative to find

$$\begin{aligned} \mathrm{d} \star F = &-\partial_0 F_{01} \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \partial_0 F_{02} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 - \partial_0 F_{03} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 - \partial_1 F_{12} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \partial_2 F_{23} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \partial_3 F_{31} \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &+ \partial_1 F_{10} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \partial_2 F_{20} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \partial_3 F_{30} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \partial_2 F_{21} \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 - \partial_3 F_{32} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \partial_1 F_{13} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &= (\partial_1 F_{10} + \partial_2 F_{20} + \partial_3 F_{30}) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \cdots = -(\mu_0 J^0) \star \mathrm{d}t + \cdots = (\mu_0 J_0) \star \mathrm{d}t + \cdots \end{aligned}$$

We see that the -1 factors introduced by $\star$ essentially raise the indices of $F_{\mu\nu}$ and the exterior derivative contracts them with a partial derivative. We recover a three-form (J is represented as a three-form).

(g) The result found in (e) can be restated as $F = (\partial_\mu A_\nu - \partial_\nu A_\mu) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu/2 = \partial_\mu A_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}A$ by definition of the exterior derivative.

3 Acknowledgement

I worked on this assignment on my own.