Homework 1

Due on Tuesday, January 30

*Submit a **single file** (pdf or zip) online using the dropbox submission link for the appropriate deadline. Acknowledge any references you use as well as any other students with whom you collaborate.*

In this homework you will learn about the Hodge star operator and see how differential forms on $\mathbb{R}^3$ are connected to vector calculus and how they can be used to describe Maxwell's equation in a compact form.

1 Hodge star operator and vector calculus

As we have seen $\Omega^k(M)$ and $\Omega^{n-k}(M)$ (where $n = \dim(M)$) have the same number of generators. If the manifold has a pseudo-Riemannian metric tensor g which in local coordinates has the form

$$g = \gamma_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j, \tag{1}$$

we can map $\Omega^k(M)$ to $\Omega^{n-k}(M)$ and vice versa using the *Hodge star operator*, which is defined in terms of the local frames as follows:

$$*1 = \sqrt{|\det \gamma|}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{2}$$

$$*(\mathrm{d}x^{i_1} \wedge \cdots \mathrm{d}x^{i_k}) = \frac{1}{(n-k)!}\sqrt{|\det \gamma|}\, G^{i_1 j_1} \cdots G^{i_k j_k} \varepsilon_{j_1 \cdots j_n} \mathrm{d}x^{j_{k+1}} \wedge \cdots \mathrm{d}x^{j_n} \tag{3}$$

$$f \in \Omega^0(M), \alpha \in \Omega^k(M) \implies *(f\alpha) = f{*\alpha} \tag{4}$$

where G is the inverse matrix of γ and

$$\varepsilon_{i_1 \cdots i_n} = \begin{cases} +1 & \text{if } (i_1, \dots, i_n) \text{ is an even permutation of } (1, \dots, n) \\ -1 & \text{if } (i_1, \dots, i_n) \text{ is an odd permutation of } (1, \dots, n) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

is the Levi–Civita symbol.

The Hodge star operator satisfies the property

$$\alpha \in \Omega^k(M) \implies *(*\alpha) = (-1)^{k(n-k)} \operatorname{sgn}(\det \gamma)\, \alpha. \tag{6}$$

For the next questions we will focus on $\mathbb{R}^3$ with the global chart $(x^1, x^2, x^3) = (x, y, z)$ and Euclidean metric

$$g = \delta_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j, \tag{7}$$

and use exterior derivative, wedge product, and Hodge operator to describe cross product, gradient, divergence, and curl in terms of differential forms.

(a) Show that

$$*\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z \tag{8}$$

$$*\mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}x \tag{9}$$

$$*\mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y \tag{10}$$

and use the properties of the Hodge operator to find its action on 2-forms and 3-forms.

We can identify 1-forms with vectors using the metric, so that the one form

$$\alpha = \alpha_x \mathrm{d}x + \alpha_y \mathrm{d}y + \alpha_x \mathrm{d}z \tag{11}$$

is equivalent to the vector field[1]

$$\alpha^\sharp = \alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_x \frac{\partial}{\partial z}. \tag{12}$$

(b) Show that

- if f is a smooth function then $\mathrm{d}f$ is equivalent to ∇f
- if $\alpha \in \Omega^1(M)$ then $*\mathrm{d}\alpha$ is equivalent to $\nabla \times \alpha^\sharp$
- if $\alpha \in \Omega^1(M)$ then $*\mathrm{d}*\alpha$ is equivalent to $\nabla \cdot \alpha^\sharp$
- if $\alpha, \beta \in \Omega^1(M)$ then $*(\alpha \wedge \beta)$ is equivalent to $\alpha^\sharp \times \beta^\sharp$

2 Maxwell's equations

We can use differential forms to compactly write Maxwell's equations, both in $\mathbb{R}^3$ and in Minkowski space. For simplicity we will choose units in which $c = 1$.

Let's start with $\mathbb{R}^3$ and define the differential forms

$$B = B_x \mathrm{d}y \wedge \mathrm{d}z + B_y \mathrm{d}z \wedge \mathrm{d}x + B_z \mathrm{d}x \wedge \mathrm{d}y \tag{13}$$
$$E = E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z \tag{14}$$
$$J = J_x \mathrm{d}x + J_y \mathrm{d}y + J_z \mathrm{d}z \tag{15}$$

which represent magnetic field, electric field, and electric current density, as well as the 0-form ρ representing charge density.

(a) The reason why B should be a 2-form is that it is not actually a vector, but rather a pseudo-vector. Show that (B_x, B_y, B_z) transforms as a pseudo-vector under the change of coordinates

$$(x, y, z) \mapsto (-x, -y, -z), \tag{16}$$

which we can interpret as a reflection of our reference frame.

(b) Show that Maxwell's equations are equivalent to

$$\begin{cases} \mathrm{d}B = 0 \\ \mathrm{d}E + \dfrac{\partial B}{\partial t} = 0 \end{cases} \quad \begin{cases} *\mathrm{d}*E = \mu_0 \rho \\ *\mathrm{d}*B - \dfrac{\partial E}{\partial t} = \mu_0 J. \end{cases} \tag{17}$$

(c) On $\mathbb{R}^n$ every closed form is exact. Show that Maxwell's equations imply that

$$B = \mathrm{d}A, \quad E = -\mathrm{d}\varphi - \frac{\partial A}{\partial t}, \tag{18}$$

for a 1-form A and a 0-form φ.

[1] The symbol $\alpha^\sharp$ is the coordinate-free notation for raising indices and making a covector into a vector. It is defined by requiring that $g(\alpha^\sharp, \cdot) = \alpha$, where g is the metric tensor.

We can make things more compact by moving to Minkowski space with global coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ and introducing the 2-form

$$F = B + E \wedge \mathrm{d}t. \tag{19}$$

and the *current 3-form*

$$J = -\rho *\mathrm{d}t + J_x*\mathrm{d}x + J_y*\mathrm{d}y + J_z*\mathrm{d}z. \tag{20}$$

We'll use the $(-{+}{+}{+})$ convention for the signature of the metric.

(d) Find out what the action of the Hodge operator is in Minkowski space.

Hint: use the formula for the double Hodge operator when you can! Also, be careful, cyclic permutations are odd in 4 dimensions.

(e) Show that F is the same object as the Electromagnetic tensor (in covariant form).

(f) Show that Maxwell's equations reduce to

$$\mathrm{d}F = 0, \quad \mathrm{d}{*F} = \mu_0 J. \tag{21}$$

The beauty of these equations is that they stay exactly the same if we replace Minkowski space with a curved spacetime, and we don't have to worry about covariant derivatives (all the information about the metric is contained in the Hodge operator). Moreover, they can be extended to other gauge theories.

(g) Show that $F = \mathrm{d}A$ and that A is exactly the four-potential (seen as a covariant tensor).