Homework 2

*Submit a **single file** (pdf or zip) online using the dropbox submission link for the appropriate deadline. Acknowledge any references you use as well as any other students with whom you collaborate.*

1 Gaussian model for $T < T_c$

The Gaussian model you have seen in the lecture notes only makes sense for $T > T_c$. Here you will derive a version of it that works on the other side of the phase transition as well.

We'll work with $H = 0$ for the sake of simplicity, but the approach can be generalised. Let's start with the exact partition function

$$Z = \sqrt{\det\left(\frac{2\beta A^2 B}{\pi}\right)} \int_{\mathbb{R}^N} \mathrm{d}^N \phi \, e^{-S(\phi)} \tag{1}$$

with

$$S(\phi) = \frac{\beta A^2}{2} \phi^{\mathrm{t}} B \phi - \sum_i \ln(\cosh(\beta A (B\phi)_i)). \tag{2}$$

The goal is to evaluate the integral by expanding S to quadratic order as

$$S(\phi) \approx S(\psi) + \frac{1}{2} \sum_{i,j} (\phi - \psi)_i (\phi - \psi)_j \frac{\partial^2 S}{\partial \phi_i \, \partial \phi_j}(\psi) \tag{3}$$

where ψ is a minimum of S, making Z a Gaussian integral.

Note: you can use any results from the mean field theory lecture notes without proof.

(a) Start by showing that

$$\sum_j B_{ij} = B_0 \tag{4}$$

using the lattice Fourier transform of B. As we have done in the lectures, we will interpret B_0 as the critical temperature, i.e.,

$$B_0 = k_{\mathrm{B}} T_{\mathrm{c}}. \tag{5}$$

(b) Assuming that we are on a periodic lattice one can use a symmetry argument to show that the minima must happen at points ψ that are the same for every lattice site[1], i.e.,

$$\psi_i = \bar{\psi}. \tag{6}$$

Show that, if we introduce $M = A\bar{\psi}$, ψ is a stationary point of S if and only if

$$M = \tanh\left(\frac{T_{\mathrm{c}} M}{T}\right), \tag{7}$$

which is the equation from mean field theory! In particular, you already know how M behaves near $T = T_c$ (both above and below).

[1] I won't ask you to prove this since it took me a while to figure it out myself, but if you are interested the idea is that S is invariant under cyclic permutations of lattice sites under each lattice axis, so there cannot be a preferred direction for the minima.

(c) Show that, as long as T is close to T_c (but $T \neq T_\mathrm{c}$), the Hessian matrix $\frac{\partial^2 S}{\partial\phi_i\,\partial\phi_j}(\psi)$ is positive definite (assuming we pick $M \neq 0$ for $T < T_\mathrm{c}$). Hence is ψ a minimum point and the Gaussian integral is well-defined.

Hint: Use the fact that

$$\operatorname{sech}(\operatorname{arctanh}(M)) = \sqrt{1-M^2} \tag{8}$$

and look at the eigenvalues of the Hessian matrix, remembering that $B_0 \geq B_k > 0$.

(d) Write the approximate expression for $S(\phi)$ on both sides of the phase transition. Does the result match what you already knew from the lecture notes?

(e) Compute the average spin

$$\langle\sigma_i\rangle = A\langle\phi_i\rangle_S \approx A\frac{\int_{\mathbb{R}^N} \mathrm{d}^N\phi\, e^{-\frac{1}{2}(\phi-\psi)^\mathrm{t}\partial^2 S(\psi)(\phi-\psi)}\phi_i}{\int_{\mathbb{R}^N} \mathrm{d}^N\phi\, e^{-\frac{1}{2}(\phi-\psi)^\mathrm{t}\partial^2 S(\psi)(\phi-\psi)}} \tag{9}$$

and use it to compute the average magnetisation. What is the critical exponent β?

(f) This whole thing looks very similar to using Laplace's method to compute the partition function in mean field theory. Does this approximation give the exact result in the limit $N \to \infty$?