Monopoles

Main deadline: Friday 2 Feb @ 9PM https://www.dropbox.com/request/HKc7VYMgNnGHCBF8w8Pm
Late deadline: Friday 9 Feb @ 9PM https://www.dropbox.com/request/1o71e9Uf64HEyf5XdRAl
Pass-or-fail deadline: Friday 23 Feb @ 9PM – `contact Aldo & Dan`

1 Dirac

a) A p-form ω is said to be *closed* if $d\omega = 0$, and it is said to be *exact* if there exists a $(p-1)$-form λ such that $\omega = d\lambda$. Prove that all exact forms are closed. Note also that all closed forms on $\mathbb{R}^n$ are exact, a fact known as Poincaré's lemma. Prove Poincaré's lemma for $n = 1$. Provide a counterexample to the statement "all closed forms are exact (over an arbitrary manifold)". [Hint: consider a 1-form on the circle S^1. Explain as clearly as you can what is going on. What is the difference between $\mathbb{R}^1$ and S^1?]

Note: Since every (small enough) neighbourhood of a point on a manifold is diffeomorphic to $\mathbb{R}^n$, Poincaré's lemma tells that *locally* (i.e. in a small enough chart) every closed form is exact. However, the counterexample you found tells that this result *cannot* be extended globally, i.e. beyond "small enough neighbourhoods".

b) Consider a 2-form[1] $F^{(2)}$ on a 2-sphere S^2. Employ Stokes theorem to show that when it is (everywhere, i.e. globally) exact we find that the following quantity vanishes:

$$g = \frac{1}{4\pi} \int_{S^2} F^{(2)}. \tag{1}$$

c) Let us now turn our attention to Minkowski space $\mathbb{R}^{1,3}$ where, in the standard spherical coordinates (t, r, θ, ϕ) over $\mathbb{R}^3$, we consider the specific 2-form

$$F^{(4)} = Q \sin\theta \; d\theta \wedge d\phi, \tag{2}$$

for Q a real constant. Show that $F^{(4)}$ is a solution to Maxwell's equations:

$$dF^{(4)} = 0, \quad d * F^{(4)} = 0. \tag{3}$$

[1] I am using this heavy notation to distinguish forms on different manifolds. You can drop the $\bullet^{(n)}$ notation in your solutions, unless it helps you keep track of things. See the next footnote.

d) To better picture this solution, go to the Cartesian coordinates (t, x, y, z) and show that, $F^{(4)}$ corresponds to a (static) *magnetic monopole* with electric (E) and magnetic (B) fields

$$E^i = 0, \quad B^i \sim \frac{x^i}{r^3}. \tag{4}$$

Find the exact expression for B and compare to the standard electric "monopole" (Coulomb) solution.

e) Let's now drop the time direction (things are static anyway). We can thus identify $F^{(4)}$ with a 2-form $F^{(3)}$ over $\mathbb{R}^3$ (at fixed time).[2] Consider, in $\mathbb{R}^3$, an S^2 centred around the magnetic monopole. We can also identify $F^{(3)}$ with a 2-form $F^{(2)}$ on this 2-sphere (at fixed radius). Show that the integral (1) yields $g = Q$. Lay out a careful argument to deduce from this that (i) $F^{(2)}$ is not exact (globally, on S^2), and that therefore (ii) $F^{(3)}$ (and $F^{(4)}$) cannot be defined globally on $\mathbb{R}^3$ (or $\mathbb{R}^4$, respectively) either. Specify the actual domain of definition of $F^{(3)}$ and $F^{(4)}$.

f) Let us now focus on the 2-sphere. Let $U_\pm$ be the sets of all points of S^2 except the south pole (the north pole, respectively). Construct 1-forms $A_\pm^{(2)}$ that are regular on $U_\pm$ respectively so that, in each domain, $F^{(2)}|_{U_\pm} = dA_\pm^{(2)}$.

Back to $\mathbb{R}^3$, just as before for $F^{(n)}$ we can think of $A_\pm^{(2)}$ as defining gauge potentials $A_\pm^{(3)}$ for $F^{(3)}$.[3] Argue that when doing physics with the gauge potential $A^{(3)}$ as a "fundamental field", the magnetic monopole is necessarily accompanied by a string-like singularity that starts from the position of the monopole and goes out to infinity. This singularity is called the *Dirac string*.

Note: the singularity of $F^{(3)}$ and $A^{(3)}$ are of different dimensionality.

g) Denoting the fundamental electric charge by e, show that, on the overlap $U_- \cap U_+$, the potentials A_- and A_+ are related by a U(1)-valued gauge transformation γ,

$$A_- = A_+ + \frac{1}{ie}\gamma^{-1}d\gamma, \tag{5}$$

which is *single-valued* over $U_- \cap U_+$ *only if* a certain relationship between e and Q is satisfied. What is this relationship?

[2]Technically, there is a "fixed time" embedding map $i_T : \mathbb{R}^3 \hookrightarrow \mathbb{R}^4$, $(r, \theta, \phi) \mapsto (T, r, \theta, \phi)$. Then, the pullback operation sends k-forms on $\mathbb{R}^4$ into k-forms on $\mathbb{R}^4$, i.e. $i_T^* : \Omega^k(\mathbb{R}^4) \hookrightarrow \Omega^k(\mathbb{R}^3)$, $F^{(3)} \mapsto i_T^* F^{(4)}$. In the case of the embedding i_T, all we need to do is to evaluate $F^{(4)}$ at $t = T$ (T being a fixed constant, this means in particular setting $dt \mapsto dT \equiv 0$. In the case of $F^{(n)}$ above, which does not depend on t, the identification looks rather trivial!

[3]Basically, we set $A_\pm^{(3)}(r, \theta, \phi) = A_\pm^{(2)}(\theta, \phi)$. See the previous footnote for a more rigorous mathematical explanation.

Congratulations, you have just derived *Dirac's quantization condition*: the existence of one magnetic monopole somewhere in the universe would imply the quantization of the electric charge!

h) Show that it is *impossible* to find a (globally-defined on $U_- \cap U_+$, single-valued) function λ such that $\gamma = e^{i\lambda}$, i.e. we *cannot* write $A_- = A_+ + \frac{1}{e} d\lambda$ globally over $U_- \cap U_+$. This implies that the gauge group of electrodynamics in the presence of magnetic monopoles cannot be $G = (\mathbb{R}, +)$ but it must be $G = \mathrm{U}(1)$. Note that this is true even though the two groups have the same Lie algebra, i.e. even though they yield the same *infinitesimal* gauge transformations $\delta_\xi A = d\xi$.

i) Argue that Dirac's quantization condition that makes $\gamma : U_- \cap U_+ \to \mathrm{U}(1)$ single-valued also makes the Dirac string "unobservable", i.e. that its position can be moved around by means of a gauge transformations.

2 Taub-NUT, or the gravitomagnetic monopole

Consider the following vacuum solution of Einstein's equations due to Taub and Newman-Unti-Tamburino (NUT):

$$ds^2 = f(dt + 2A_\sigma)^2 - \frac{dr^2}{f} - (r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2), \tag{6a}$$

where

$$f = \frac{r^2 - 2mr - n^2}{r^2 + n^2}, \qquad A_\sigma = n(\cos\theta + \sigma)d\phi. \tag{6b}$$

We call A_σ the *gravitomagnetic potential.*

a) Show that when the NUT charge $n = 0$, we recover the Schwartzschild solution with mass m.

b) When $n \neq 0$ the solution drastically changes. i) First, there is no curvature singularity at $r = 0$ anymore. ii) Comparing (6) to the magnetic potential for the Dirac monopole in the previous problem, we may suspect that n is some kind of *gravitomagnetic mass.*

   Show that a choice $\sigma = \pm 1$ makes the metric along the south/north half of the polar axis regular, while the other axis possesses a *Misner string singularity* analogous to the Dirac String. [Hint: focus on the $(r, t) = const$ surfaces.] What happens for $\sigma = 0$?

c) Show that σ can be eliminated by means of a "large coordinate transformation"

$$t \mapsto t_\sigma := t - 2n\sigma\phi. \tag{7}$$

That is, any value of σ can be (locally)[4] reabsorbed into the definition of a shifted time coordinate t_σ. Define $t_\pm$ so that the south/north axis are regular and show that in the overlap region we must have $t_+ = t_- 4n\phi$. Thus deduce that, in order for the Misner string to be unobservable (like the Dirac string), time must be periodic:

$$t \sim t + 8\pi n. \tag{8}$$

Do you see any problem with this choice? Do you see any connection with the *Kaluza-Klein monopole* solution of Gross–Perry–Sorkin discussed in the lecture?

d) To further confirm that the Misner string with $\sigma = \pm 1$ is physical, consider the rotational Killing vector $k = \partial_\phi$ and show that the (renormalized) Komar angular momentum of the spacetime, defined as[5]

$$J = -\frac{1}{16\pi}\int_{S^2_\infty} *d(k^\flat - k^\flat|_{m=0}), \tag{9}$$

is given by

$$J = 3nm\sigma. \tag{10}$$

Whence, we conclude that the Misner string encodes a nontrivial angular momentum of the Taub-NUT spacetime, whose sign depends on the choice of σ.

[4] See below!

[5] Here, S^2_∞ is the sphere of constant infinite radius, $r = \infty$. Moreover, the 1-form $k^\flat = g_{ab}k^a dx^b$ associated to the Killing vector $k^a\partial_a = \partial_\phi$ depends on the parameter m entering the Taub-NUT metric. Here, $k^\flat|_{m=0}$ stands for that 1-form $k^\flat$ computed for the metric with $m = 0$. This subtraction is necessary since $\int_{S^2_\infty} *dk^\flat$ gives an infinite value.