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Homework 2: Monopoles

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(a) We are interested in the relation between the global properties of a manifold M and the structure of diffrential forms taking values on its cotangent bundle $T^\star M$ at each point of M.

Poincaré's lemma on $M = \mathbb{R}$**:** Let ω be a p-form ($p \in \{0,1\}$) constructed from the cotengent space $T^\star M$ of M. Then $\mathrm{d}\omega = 0$ (ω is closed) implies $\omega = \mathrm{d}\lambda$ (ω is closed) where λ is a $(p-1)$-form (0-form).

Proof: On $\mathbb{R}$, we can use the identity map as a global coordinate chart. The induced basis on 1-forms is $\{\mathrm{d}x\}$ (a smooth frame field) and any 1-forms can be written as $\omega = g\mathrm{d}x$ with $g \in C^\infty(\mathbb{R})$. Suppose now that ω is closed: we have $0 = \mathrm{d}\omega = \partial_x g \mathrm{d}x \wedge \mathrm{d}x = 0, \forall g \in C^\infty(\mathbb{R})$ (ω being a 1-form is not restrictive, but would be for $\mathbb{R}^n$ with $n > 1$). Then we take the 0-form $\lambda = G$ where G is any primitive of g ($G(x)$ exists because g is smooth) and apply an exterior derivative to get $\mathrm{d}\lambda = g\mathrm{d}x$. Because there are no $(0-1)$-forms there is no need to check the lemma for 0-forms.

Counterexample: Consider the circle smooth manifold $\mathbb{S}^1 \subset \mathbb{R}^2$ (embeded as $\{x^2 + y^2 = 1 | (x, y) \in \mathbb{R}^2\}$ for simplicity). It takes at least two charts to cover this manifold and, although on individual charts all closed 1-forms are exact (charts make the manifold look like $\mathbb{R}$ locally), this property is lost globally. Choose the chart map $\theta = \arctan_2$ sending points (x, y) on the circle to their angle with the x axis excluding the point $(1,0)$ so that the domain is open. With this chart we have the coordinate induced one form frame field $\mathrm{d}\theta$ which we use to construct the closed form $\omega = \mathrm{d}\theta$. On $(0, 2\pi)$, this form is exact since we have a 0-form $\lambda = F \in C^\infty((0, 2\pi))$ such that $\omega = \mathrm{d}\lambda = \partial_\theta F \mathrm{d}\theta = \mathrm{d}\theta$ forcing $F = \theta + c$, $c \in \mathbb{R}$ since F has to be a primitive of 1 in the variable θ. The function F is smooth on the chart, but can never be extended to s smooth function over $\mathbb{S}^1$ globally. Indeed, 0 and 2π being identified, a continuous function on $\mathbb{S}^1$ should be consistant at the excluded point $(0,1)$ and this would requiere $\lim_{\theta \to 0^+}(\theta + c) = \lim_{\theta \to 2\pi^-}(\theta + c)$ which is impossible. Therefore there is a closed form on $\mathbb{S}^1$ that is not exact.

(b) Let $F^{(2)}$ be a 2-form on the 2-sphere $\mathbb{S}^2$. Suppose $F^{(2)}$ is globally exact implying there is a 1-form ω such that $F^{(2)} = \mathrm{d}\omega$. Then we can use Stokes theorem in combination with the fact $\mathbb{S}^2$ has no boundary to write $g = \frac{1}{4\pi}\int_{\mathbb{S}^2} F^{(2)} = \frac{1}{4\pi}\int_{\partial\mathbb{S}^2} \mathrm{d}\omega = 0$.

(c) Now working in Minkowski space $\{\eta,\ \mathbb{R}^{1,3}\}$ with mostly + signature in the coordinate chart (t, r, θ, ϕ) (this order for the variables provides the notion of positive orientation of a basis) built from spherical coordinates on $\mathbb{R}^3$, we have the 2-form $F^{(4)} = Q\sin(\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\phi$ with $Q \in \mathbb{R}$. We want to determine if $F^{(4)}$ satisfies Maxwell's equations $\mathrm{d}F^{(4)} = 0, \quad \mathrm{d}\star F^{(4)} = 0$. We have $\mathrm{d}F^{(4)} = Q\cos(\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi = 0$. To evaluate the Hodge dual of $F^{(4)}$, we first calculate

$$
\begin{aligned}
\star\,\mathrm{d}\theta\wedge\mathrm{d}\phi &= \sqrt{|r^4\sin^2\theta|}\frac{1}{2!}\frac{1}{2!}\epsilon^{\theta\phi}{}_{rt}\mathrm{d}r\wedge\mathrm{d}t + \frac{1}{2!}\frac{1}{2!}\epsilon^{\theta\phi}{}_{tr}\mathrm{d}t\wedge\mathrm{d}r - \frac{1}{2!}\frac{1}{2!}\epsilon^{\phi\theta}{}_{rt}\mathrm{d}r\wedge\mathrm{d}t - \frac{1}{2!}\frac{1}{2!}\epsilon^{\phi\theta}{}_{tr}\mathrm{d}t\wedge\mathrm{d}r \\
&= r^2|\sin\theta|\eta^{\theta\theta}\eta^{\phi\phi}\left(\frac{1}{2!}\frac{1}{2!}\epsilon_{\theta\phi rt}\mathrm{d}r\wedge\mathrm{d}t + \frac{1}{2!}\frac{1}{2!}\epsilon_{\theta\phi tr}\mathrm{d}t\wedge\mathrm{d}r - \frac{1}{2!}\frac{1}{2!}\epsilon_{\phi\theta rt}\mathrm{d}r\wedge\mathrm{d}t - \frac{1}{2!}\frac{1}{2!}\epsilon_{\phi\theta tr}\mathrm{d}t\wedge\mathrm{d}r\right) \\
&= r^2|\sin\theta|\frac{1}{r^4\sin^2\theta}\frac{1}{2!}\frac{1}{2!}\left((-1)\,\mathrm{d}r\wedge\mathrm{d}t + (+1)\,\mathrm{d}t\wedge\mathrm{d}r - (+1)\,\mathrm{d}r\wedge\mathrm{d}t - (-1)\,\mathrm{d}t\wedge\mathrm{d}r\right) = \mathrm{d}t\wedge\mathrm{d}r
\end{aligned}
$$

and it follows that $\mathrm{d}\star F^{(4)} = \mathrm{d}(Q/r^2\ (\mathrm{d}t\wedge\mathrm{d}r)) = -Q/r^3\ (\mathrm{d}r\wedge\mathrm{d}t\wedge\mathrm{d}r) = 0$ where the absolute value was ignored because $\theta \in (0, 2\pi)$ making $\sin(\theta) > 0$.

(d) We can convert the form $F^{(4)}$ to cartesian coordinates with the relations

$$
\phi = \arctan_2(y, x), \quad \theta = \arctan_2\left(z, \sqrt{x^2+y^2}\right) \implies \mathrm{d}\phi = \frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2}, \quad \mathrm{d}\theta = \frac{\sqrt{x^2+y^2}\mathrm{d}z - (x\mathrm{d}x + y\mathrm{d}y)\frac{z}{\sqrt{x^2+y^2}}}{r^2}
$$

leading to

$$
\begin{aligned}
F^{(4)} = Q\sin(\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\phi &= Q\frac{\sqrt{x^2+y^2}}{r}\left(\frac{\sqrt{x^2+y^2}\mathrm{d}z - (x\mathrm{d}x + y\mathrm{d}y)\frac{z}{\sqrt{x^2+y^2}}}{r^2}\right)\wedge\left(\frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2}\right) \\
&= Q\frac{1}{r^3}\left(\mathrm{d}z\wedge(-y\mathrm{d}x + x\mathrm{d}y) - (x^2\mathrm{d}x\wedge\mathrm{d}y - y^2\mathrm{d}y\wedge\mathrm{d}x)\frac{z}{x^2+y^2}\right) = Q\frac{1}{r^3}\ (-y\mathrm{d}z\wedge\mathrm{d}x - x\mathrm{d}y\wedge\mathrm{d}z - z\mathrm{d}x\wedge\mathrm{d}y).
\end{aligned}
$$

We note the electric field components (associated to $\mathrm{d}x^i\wedge\mathrm{d}t$) all vanish and we only have a magnetic field (associated to $\mathrm{d}x^i\wedge\mathrm{d}x^j$). The magnetic field has the same from has an electric monopole (inverse square law multiplies by a unit "vector").

(e) Since the monopole field is static, we drop the time direction by mapping $F^{(4)}$ to the two-form $F^{(3)}$ in the cotangent bundle over $\mathbb{R}^3$ on a fixed time slice. Going further we can map $F^{(3)}$ on the cotangent bundle over $\mathbb{S}^2$ (embeded in $\mathbb{R}^3$ as a sphere of radius 1) to get the two-form $F^{(2)}$. To characterize the two-form $F^{(2)}$, we evaluate the integral given in (b) as

$$
g = \frac{Q}{4\pi}\int_{\mathbb{S}^2}\sin(\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\phi = \frac{Q}{4\pi}\int_{\mathbb{S}^2}\sin(\theta)\,\mathrm{d}\theta(e_\theta)\wedge\mathrm{d}\phi(e_\phi) = Q
$$

with e_ϕ, e_θ the dual vector basis to $\mathrm{d}\phi, \mathrm{d}\theta$. More formally, this integration on $V \subset \mathbb{S}^2$ is brought to an integral in $U \subset \mathbb{R}^2$ on the pullback of $F^{(2)}$ by a diffeomorphism mapping U to V. A convenient choice of diffeomorphism is the coordinate chart already used to write $F^{(2)}$. Under this diffeomrophism, $\mathrm{d}\theta$ and $\mathrm{d}\phi$ are mapped to the exterior derivatives of the coordinate fucntions θ, ϕ over U (the exterior derivative of the projection map on each axis which are aslo named $\mathrm{d}\theta$ and $\mathrm{d}\phi$). This allows us to use regular borns of integration where θ and ϕ range from 0 to π and 0 to 2π respectively and use the coordinate representation of the two-form components. Since **exact** $\implies$ **vanishing of** g as shown in (b), we have **non vanishing of** g $\implies$ **not exact** and $F^{(2)}$ is not exact.

One could say that $g = \frac{1}{4\pi}\int_{\mathbb{S}^2=\partial\text{Ball}} F^{(2)} = \frac{1}{4\pi}\int_{\text{Ball}} \mathrm{d}F^{(2)} = 0$ forming a contradiction with $F^{(2)}$ not being exact. The solution to this problem can be seen with result (c) where $F^{(2)}$ is shown to be ill-defined at the origin. Therfore we need to puncture $\mathbb{R}^3$ by removing the origin from the domain of definition of $F^{(3)}$ creating a second boundary restoring the result $0 = \frac{1}{4\pi}\int_{\partial\text{Ball+puncture}} F^{(2)}$. Normally the set added to the boundary would be of zero measure, but comparing with the usual treatement of electric monopoles, we get that a dirac delta at the puncture point will change the value of g from 0 to Q.

(f) Stereographic projections provide maps from $U_+ = \mathbb{S}^2 -$ North pole (projecting from the north pole) and $U_- = \mathbb{S}^2 -$ South pole (projecting to the south pole) to all of $\mathbb{R}^2$. Expressed in the cartesian coordinates of the embeding space of $\mathbb{S}^2$ in $\mathbb{R}^3$, the associated coordinate maps $\varphi_\pm$ are

$$\varphi_\pm : (x,y,z) \mapsto (u_\pm, v_\pm) = \left(\frac{x}{1\mp z}, \frac{y}{1\mp z}\right).$$

This form can be obtained by looking at a cut of the sphere in a zw-plane containing the z axis. In this plane, we look for the intersection $u_\pm, v_\pm$ of a line passing trough the relevant pole and the point x,y,z with the xy-plane. In the section plane, the line is given by points of coordinates w_l, z_l such that $z_l = 1 - \frac{1+z}{w} w_l$ (North pole) or $z_l = -1 + \frac{1-z}{w} w_l$ (South pole). The intersection with the xy-plane is given by $u_\pm = \frac{w}{1\mp z}\frac{x}{w}$ and $v_\pm = \frac{w}{1\mp z}\frac{y}{w}$ (w coordinate projected on the x and y axis respectively).

To express $F^{(2)}$ in these new coordinates, we notice that $x = u_\pm(1\mp z)$ and $y = v_\pm(1\mp z)$. Since our sphere has radius 1, we have

$$u_\pm^2 + v_\pm^2 = (1-z^2)/(1\mp z)^2 = (1\pm z)/(1\mp z) \implies u_\pm^2 + v_\pm^2 \mp z(u_\pm^2 + v_\pm^2) = 1 \pm z \implies z = \pm\frac{1-u_\pm^2 - v_\pm^2}{1+u_\pm^2+v_\pm^2}$$

leading to $\mathrm{d}x = (1\mp z)\mathrm{d}u_\pm \mp u_\pm \mathrm{d}z$, $\mathrm{d}y = (1\mp z)\mathrm{d}v_\pm \mp v_\pm \mathrm{d}z$ and $\mathrm{d}z = A\mathrm{d}u_\pm + B\mathrm{d}v_\pm$ where

$$A = -\pm\frac{2u_\pm(1+u_\pm^2+v_\pm^2)}{(1+u_\pm^2+v_\pm^2)^2} - \pm\frac{2u_\pm(1-u_\pm^2-v_\pm^2)}{(1+u_\pm^2+v_\pm^2)^2} = \mp\frac{4u_\pm}{(1+u_\pm^2+v_\pm^2)^2}, \quad B = \mp\frac{4v_\pm}{(1+u_\pm^2+v_\pm^2)^2}.$$

We can also relate the two-form frame fields in cartesian coordinates to the $\mathrm{d}u\wedge\mathrm{d}v$ frame field as (omitting $\pm$ on u,v symbols from now on)

$$\begin{aligned}
\mathrm{d}x\wedge\mathrm{d}y &= ((1\mp z)\mathrm{d}u \mp u\mathrm{d}z)\wedge((1\mp z)\mathrm{d}v \mp v\mathrm{d}z) \\
&= (1\mp z)^2\mathrm{d}u\wedge\mathrm{d}v \mp (1\mp z)(Bv + Au)\mathrm{d}u\wedge\mathrm{d}v \\
&= (1\mp z)^2\mathrm{d}u\wedge\mathrm{d}v + 4(1\mp z)\frac{v^2+u^2}{(1+u^2+v^2)^2}\mathrm{d}u\wedge\mathrm{d}v \\
\mathrm{d}y\wedge\mathrm{d}z &= (1\mp z)A\mathrm{d}v\wedge\mathrm{d}u = \pm(1\mp z)\frac{4u}{(1+u^2+v^2)^2}\mathrm{d}u\wedge\mathrm{d}v \\
\mathrm{d}z\wedge\mathrm{d}x &= (1\mp z)B\mathrm{d}v\wedge\mathrm{d}u = \pm(1\mp z)\frac{4v}{(1+u^2+v^2)^2}\mathrm{d}u\wedge\mathrm{d}v
\end{aligned}$$

With these expressions we are ready to express $F^{(3)}$ in the $\mathrm{d}u$ and $\mathrm{d}v$ frame field (we omit the $\pm$ on u,v in what follows) as

$$\begin{aligned}
F^{(3)} &= Q\frac{1}{r^3}\,(-y\mathrm{d}z\wedge\mathrm{d}x - x\mathrm{d}y\wedge\mathrm{d}z - z\mathrm{d}x\wedge\mathrm{d}y) \\
&= -Q\left(z(1\mp z)^2 + 4z(1\mp z)\frac{v^2+u^2}{(1+u^2+v^2)^2} \pm (1\mp z)^2\frac{4u^2+4v^2}{(1+u^2+v^2)^2}\right) \\
&= -Q\left(z(1\mp z)^2 + 4(1\pm z)(1\mp z)\frac{v^2+u^2}{(1+u^2+v^2)^2}\right) \\
&= -Q(1\mp z)^2\left(\pm\frac{1-(u^2+v^2)^2}{(1+u^2+v^2)^2} + 4\frac{(u^2+v^2)^2}{(1+u^2+v^2)^2}\right).
\end{aligned}$$

(g)

(h)

(i)

2 Taub-NUT, or the gravitomagnetic monopole

(a)

(b)

(c)

(d)