# Appendix A

# Differential forms

## A.1 Defintions and Cartan's calculus

We recall the main objects and identities of Cartan's calculus [Nak03, Ch.5].

Let $M$ be a $n$-dim manifold and $(x^1, \dots, x^n)$ be coordinates on it. We denote *coordinate indices*—rather than abstract indices—by the letters $(\mu, \nu, \dots)$.

1. Differential forms, $\boldsymbol{\alpha} \in \Omega^k(M)$ $(k \leq n)$:

$$\boldsymbol{\alpha}(x) = \frac{1}{k!}\alpha_{\mu_1\cdots\mu_k}(x)dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$$

where $\alpha_{\mu_1\cdots\mu_k}$ is understood to be completely antisymmetric:

$$\alpha_{\mu_1\cdots\mu_k} = \alpha_{[\mu_1\cdots\mu_k]} \doteq \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \mathrm{sign}(\sigma)\alpha_{\sigma(\mu_1)\cdots\sigma(\mu_k)}.$$

where $\mathfrak{S}_k$ is the permutation group of $k$ elements. (Abstractly differential $k$-forms over $M$ are defined as sections of the completely-antisymmetrized tensor-product-bundle of $k$ cotangent bundles over $M$.)

2. Wedge product of differential forms, $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$:

$$\boldsymbol{\alpha}\wedge\boldsymbol{\beta} = \frac{1}{k!l!}\alpha_{\mu_1\cdots\mu_k}\beta_{\nu_1\cdots\nu_l}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\wedge dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_l} = (-1)^{kl}\boldsymbol{\beta}\wedge\boldsymbol{\alpha}.$$

or

$$(\alpha \wedge \beta)_{\mu_1\cdots\mu_{k+l}} = \frac{(k+l)!}{k!l!}\alpha_{[\mu_1\dots\mu_k}\beta_{a_{k+1}\cdots\mu_{k+l}]}$$

3. Exterior derivative (De Rham differential) $d : \Omega^k(M) \to \Omega^{k+1}(M)$:

$$d\boldsymbol{\alpha}(x) = \frac{1}{k!}\partial_\mu \alpha_{\mu_1\cdots\mu_k}(x) dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$$

   that is:

$$(d\alpha)_{\mu_1\cdots\mu_{k+1}}(x) = (k+1)\partial_{[\mu_1}\alpha_{\mu_2\cdots\mu_{k+1}]}(x)$$

   Crucially:

$$\text{(Graded) Leibniz :} \quad d(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = d\boldsymbol{\alpha} \wedge \boldsymbol{\beta} + (-1)^k \boldsymbol{\alpha} \wedge d\boldsymbol{\beta}.$$

$$\text{Nilpotence :} \quad d^2 = 0 \quad \text{i.e.} \quad d(d\boldsymbol{\alpha}) \equiv 0.$$

4. If $f : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto f(x)$ is smooth, then one has the following identity involving the determinant of its Jacobian:

$$df^1 \wedge \cdots \wedge df^n = \mathsf{det}\,(\partial_\mu f^\nu)\, dx^1 \wedge \cdots \wedge dx^n.$$

   The sign of the determinant of the Jacobian is positive iff $f$ defines an orientation-preserving diffeomorphism.

5. Vector fields $X \in \mathfrak{X}(M)$ (where $\partial_\mu \equiv \partial/\partial x^\mu$):

$$X(x) = X^\mu(x)\partial_\mu.$$

   (Abstractly, vector fields are defined as sections of the tangent bundle, and $k$-multivector fields as sections of the completely-antisymmetrized tensor-product-bundle of $k$ copies of the tangent bundle. For $k = 2$ one calls them bivector fields.)

6. Interior product $i : \mathfrak{X}(M) \times \Omega^{k+1}(M) \to \Omega^k(M)$:

$$i_X\boldsymbol{\alpha}(x) = \frac{1}{k!} X^\nu \alpha_{\nu\mu_1\cdots\mu_k}(x) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}.$$

   Notice this can be deduced from $i_X dx^\mu = X^\mu$ and

$$i_X(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = i_X\boldsymbol{\alpha} \wedge \boldsymbol{\beta} + (-1)^p \boldsymbol{\alpha} \wedge i_X\boldsymbol{\beta}, \qquad \boldsymbol{\alpha} \in \Omega^p(M).$$

   (Abstractly the interior product is nothing else than the natural pairing between sections of the tangent and cotangent bundles.)

7. Lie derivative of differential forms $L : \mathfrak{X}(M) \times \Omega^k(M) \to \Omega^k(M)$

$$L_X \boldsymbol{\alpha} = i_X d\boldsymbol{\alpha} + d i_X \boldsymbol{\alpha};$$

This is called *Cartan's (magic) formula*—and it is extremely useful. Check that this definition is compatible with that of a Lie derivative of the covariant $k$-tensor $\alpha_{\mu_1 \cdots \mu_k}$ (where the hat $\hat{\bullet}$ indicates omission):

$$(L_X \alpha)_{\mu_1 \cdots \mu_k}(x) = X^\nu(x) \partial_\nu \alpha_{\mu_1 \cdots \mu_k}(x) + \sum_i \alpha_{\mu_1 \cdots \hat{\mu}_i \nu \cdots \mu_k}(x) \partial_{\mu_i} X^\nu(x)$$

8. Lie derivative of vector fields $L : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$:

$$L_X Y \equiv [X, Y]$$

is a Lie bracket (antisymmetric, Jacobi),

$$[X, Y] = -[Y, X] \quad \text{and} \quad [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0,$$

defined in coordinates as:

$$(L_X Y)^\mu(x) = X^\nu(x) \partial_\nu Y^\mu(x) - Y^\nu(x) \partial_\nu X^\mu(x).$$

9. Together with Cartan's formula, the following are useful:

$$\begin{aligned} [L_X, L_Y] \boldsymbol{\alpha} &= L_{[X,Y]} \boldsymbol{\alpha} \\ L_X i_Y \boldsymbol{\alpha} &= i_{[X,Y]} \boldsymbol{\alpha} + i_Y L_X \boldsymbol{\alpha} \\ d L_X \boldsymbol{\alpha} &= L_X d \boldsymbol{\alpha} \\ i_X i_Y \boldsymbol{\alpha} &= -i_Y i_X \boldsymbol{\alpha} \end{aligned}$$

Using a *graded* commutator,[1] such that $i_X$ and $d$ have degree -1 and 1 respectively, and $L_X$ has degree 0, one can summarize these identities as follows:

$$[d, d] = 0, \tag{A.1a}$$

$$[L_X, L_Y] = L_{[X,Y]}, \tag{A.1b}$$

$$[i_X, i_Y] = 0, \tag{A.1c}$$

$$[d, L_X] = 0, \tag{A.1d}$$

$$[d, i_X] = L_X, \tag{A.1e}$$

$$[L_X, i_Y] = i_{[X,Y]}. \tag{A.1f}$$

[1] If $A$ has degree $|A|$ and $B$ has degree $|B|$, then their graded commutator is

$$[A, B] \doteq AB - (-1)^{|A||B|} BA.$$

I.e. there is a minus by default (it's still a commutator!) corrected by the "cost" of passing $A$ across $B$.

These identities are known as *Cartan's calculus*, and you must not forget them!

## A.2 Pullback & Pushforward

Consider a smooth map:

$$\varphi : M_1 \to M_2.$$

Then, define the *pull-back* of a function:

$$\varphi^* : C^\infty(M_2) \to C^\infty(M_1), \quad f_2 \mapsto f_1 = \varphi^* f_2 \doteq f_2 \circ \varphi.$$

This allows one to introduce the *tangent map* between tangent spaces,

$$d\varphi_{x_1} : T_{x_1}M_1 \to T_{\varphi(x_1)}M_2, \quad v_1 \mapsto v_2 = d\varphi_{x_1} v_1 \text{ defined by } (d\varphi_{x_1} v_1)(f_2) \doteq v_1(\varphi^* f_2).$$

and similarly the *pushforward* of vector fields:[2]

$$\varphi_* : \mathfrak{X}(M_1) \to \mathfrak{X}(M_2), \quad X_1 \mapsto X_2 = \varphi_* X_1 \text{ defined by } (\varphi_* X_1)(f_2) = X_1(\varphi^* f_2).$$

By dualization of the tangent map one also obtains the *pullback* of a differential form:

$$\varphi^* : \Omega^k(M_2) \to \Omega^k(M_1), \quad \boldsymbol{\alpha}_2 \mapsto \boldsymbol{\alpha}_1 = \varphi^* \boldsymbol{\alpha}_2,$$

defined by the following identity valid for any collection of tangent vectors $v_1^i \in T_{x_1}M_1$:

$$(\varphi^* \boldsymbol{\alpha}_2(x_1))(v_1^1, \cdots, v_1^k) = \boldsymbol{\alpha}_2(x_2)(d\varphi_{x_1} v_1^1, \cdots, d\varphi_{x_1} v_1^k), \quad \text{with} \quad x_2 = \varphi(x_1).$$

Of course, whenever both sides of the equality are well defined,

$$i_{X_1} \varphi^* \boldsymbol{\alpha}_2 = i_{\varphi_* X_1} \boldsymbol{\alpha}_2.$$

In coordinates, $x_2^\nu = \varphi^\nu(x_1^\mu)$ and

$$(\varphi_* X_1)^\nu(\varphi(x_1)) = \left( \frac{\partial \varphi^\nu}{\partial x_1^\mu} X_1^\mu \right)(x_1).$$

Moreover

$$(\varphi^* \alpha_2)_{\mu_1 \cdots \mu_k}(x_1) = \left( \frac{\partial \varphi^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \varphi^{\nu_k}}{\partial x^{\mu_k}} (\alpha_2)_{\nu_1 \cdots \nu_k} \right)(\varphi(x_1)),$$

[2] We neglect possible global issues, and notice that $\varphi_* X_1$ is only defined on the image of $\varphi$.

or

$$\frac{1}{k!}(\varphi^*\alpha_2)_{\mu_1\cdots\mu_k}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k} = \frac{1}{k!}(\alpha_2)_{\nu_1\cdots\nu_k}\circ\varphi\; d\varphi^{\nu_1}\wedge\cdots\wedge d\varphi^{\nu_k}.$$

Importantly, notice how the differential commutes with the pullback:

$$d^{(1)}\varphi^*\boldsymbol{\alpha}_2 = \varphi^*(d^{(2)}\boldsymbol{\alpha}_2)$$

where here we have emphasized on which manifold each differential is defined.

## A.3 Hodge dual

In this section we distinguish *abstract* indices $(a, b, \dots)$ from *coordinate* indices $(\mu, \nu, \dots)$

From now on we work on an orientable manifold $M$ of dimension $n$,

$$\dim(M) = n,$$

and all forms are defined on $M$ unless otherwise specified. We also equip $M$ with a metric $g_{ab}$. Its signature and the (square root of the absolute value of the) determinant of the metric $g_{ab}$ are denoted by

$$s \doteq \mathrm{sign}(g_{\mu\nu}) \quad \text{and} \quad \sqrt{g} \equiv \sqrt{|\det g_{\mu\nu}|}.$$

(Note: the signature is a coordinate-independent notion, the determinant isn't).

In a coordinate patch we introduce the completely antisymmetric *symbol*

$$\sigma_{\mu_1\dots\mu_n} = \sigma_{[\mu_1\dots\mu_n]}, \qquad \text{with} \qquad \begin{cases} \sigma_{01\dots(n-1)} = 1, \\ \sigma^{\mu_1\dots\mu_n} \equiv \sigma_{\mu_1\dots\mu_n}. \end{cases}$$

This symbol is *not* a tensor as the last of the equations above makes very clear.

In a coordinate basis, the Levi-Civita (covariant) *tensor* over $(M, g_{ab})$ is given by

$$\boxed{\epsilon_{\mu_1\dots\mu_n} \doteq \sqrt{g}\sigma_{\mu_1\dots\mu_n}}$$

so that its contravariant counterpart is

$$\epsilon^{\mu_1\dots\mu_n} \doteq g^{\mu_1\nu_1}\dots g^{\mu_n\nu_n}\epsilon_{\nu_1\dots\nu_n} = \frac{(-1)^s}{\sqrt{g}}\sigma^{\mu_1\dots\mu_n}.$$

Note that the second equality is a consequence of the usual definition of the determinant of any matrix:

$$g^{\mu_1\nu_1}\dots g^{\mu_n\nu_n}\sigma_{\mu_1\dots\mu_n} = \frac{1}{\mathsf{det}\, g}\sigma^{\nu_1\dots\nu_n} = \frac{(-1)^s}{(\sqrt{g})^2}\sigma^{\nu_1\dots\nu_n}, \tag{A.2}$$

It is important to check that $\epsilon_{\mu_1\dots\mu_n}$ does indeed transform as a tensor under (orientation preserving) changes of coordinates $x^\mu \mapsto (x')^{\mu'}(x)$:[3]

$$\begin{aligned}\epsilon'_{\mu'_1\dots\mu'_n} &= \sqrt{g'}\sigma_{\mu'_1\dots\mu'_n} \\ &= \left|\mathsf{det}\left(\frac{\partial(x')^{\sigma'}}{\partial x^\sigma}\right)\right|^{-1}\sqrt{g}\sigma_{\mu'_1\dots\mu'_n} = \pm\frac{\partial x^{\mu_1}}{\partial x'^{\mu'_1}}\cdots\frac{\partial x^{\mu_n}}{\partial x'^{\mu'_n}}\sqrt{g}\sigma_{\mu_1\dots\mu_n} \\ &= \pm\frac{\partial x^{\mu_1}}{\partial x'^{\mu'_1}}\cdots\frac{\partial x^{\mu_n}}{\partial x'^{\mu'_n}}\epsilon_{\mu_1\dots\mu_n}\end{aligned} \tag{A.3}$$

where the plus holds for orientation preserving coordinate transformations (the first line holds by definition of the Levi-Civita tensor in the primed coordinate system; the second follows from the transformation of the determinant $\mathsf{det}(g_{\mu\nu}) = \mathsf{det}(\partial_\mu(x')^{\nu'})^2\mathsf{det}(g'_{\mu'\nu'})$ and the identity of equation (A.2); finally the last line follows again from the definition of the Levi-Civita tensor in the non-primed coordinate system).

Hence, having checked that the Levi-Civita tensor is indeed a tensor, we can readily denote it in the abstract index notation:

$$\epsilon_{a_1\dots a_n}.$$

For the $n$-dimensional manifold $(M, g_{ab})$, its volume form is given precisely by its (covariant) Levi-Civita tensor:

$$\boldsymbol{\epsilon} \doteq \frac{1}{n!}\epsilon_{\mu_1\dots\mu_n}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_n} = \frac{1}{n!}\sqrt{g}\sigma_{\mu_1\dots\mu_n}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_n}.$$

It is often useful to express the contraction of the Levi-Civita tensor with itself in terms of antisymmetrized Kronecker deltas:[4]

$$\epsilon_{a_1\dots a_p c_{p+1}\dots c_n}\epsilon^{b_1\dots b_p c_{p+1}\dots c_n} = (-1)^s p!(n-p)!\delta^{[b_1}_{a_1}\delta^{\cdots}_{\cdots}\delta^{b_p]}_{a_p}. \tag{A.4}$$

[3]Orientation preserving changes of coordinates are such that the determinant of their Jacobian is positive. See point 4 of Appendix A.1.

[4]This immediately follows from the corresponding identity between the antisymmetric symbols (of course no signature appears here):

$$\sigma_{\mu_1\dots\mu_p\gamma_{p+1}\dots\gamma_n}\sigma^{\nu_1\dots\nu_p\gamma_{p+1}\dots\gamma_n} = p!(n-p)!\delta^{[\nu_1}_{\mu_1}\delta^{\cdots}_{\cdots}\delta^{\nu_p]}_{\mu_p}.$$

In case we have a (nondegenerate) metric $g_{ab}$ at our disposal, we can define the *Hodge-star operator*, which defines an isomorphism between $p$-forms and $(n-p)$-forms:

$$\star : \Omega^p(M) \to \Omega^{n-k}(M).$$

Starting with a $p$-form $\boldsymbol{\alpha}$,

$$\boldsymbol{\alpha} = \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p},$$

its Hodge dual takes the form

$$\boxed{\star\,\boldsymbol{\alpha} = \frac{1}{(n-p)!}\frac{1}{p!}\epsilon^{\gamma_1\ldots\gamma_p}{}_{\mu_{p+1}\ldots\mu_n}\alpha_{\gamma_1\ldots\gamma_p} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_n}.}$$

Notice that it is the tensor $\epsilon_{\mu_1\ldots\mu_n} = \sqrt{g}\sigma_{\mu_1\ldots\mu_n}$ (with its indices lowered and raised) that is featured in the above definition.

We now compute the square of the Hodge-star operator:

$$\star\star\boldsymbol{\alpha} = \frac{1}{p!}\frac{1}{(n-p)!}\frac{1}{p!}\epsilon^{\mu_{p+1}\ldots\mu_n}{}_{\nu_1\ldots\nu_p}\epsilon^{\gamma_1\ldots\gamma_p}{}_{\mu_{p+1}\ldots\mu_n}\alpha_{\gamma_1\ldots\gamma_p} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p},$$

which, using (A.4), is

$$\begin{aligned}\star\star\boldsymbol{\alpha} &= (-1)^s(-1)^{p(n-p)}\frac{1}{p!}\delta^{[\gamma_1}_{\nu_1}\delta^{\cdots}_{\cdots}\delta^{\gamma_p]}_{\nu_p}\alpha_{\gamma_1\ldots\gamma_p} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p} \\ &= (-1)^s(-1)^{p(n-p)}\frac{1}{p!}\alpha_{\nu_1\ldots\nu_p} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p},\end{aligned}$$

so that

$$\boxed{\star\star\boldsymbol{\alpha} = (-1)^s(-1)^{p(n-p)}\boldsymbol{\alpha}.}$$

In the following we will denote the coordinate components of $\star\boldsymbol{\alpha}$ with a tilde—up to a choice of sign introduced for later convenience:

$$\star\boldsymbol{\alpha} = \frac{1}{(n-p)!}(-1)^s(-1)^{p(n-p)}\tilde{\alpha}_{\mu_1\ldots\mu_{n-p}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{n-p}},$$

i.e.

$$\tilde{\alpha}_{\mu_1\ldots\mu_{n-p}} = (-1)^s(-1)^{p(n-p)}\frac{1}{p!}\epsilon^{\gamma_1\ldots\gamma_p}{}_{\mu_1\ldots\mu_{n-p}}\alpha_{\gamma_1\ldots\gamma_p}$$

Since differential forms are tensors, and so is $\epsilon_{a_1\ldots a_n}$, we can also adopt an abstract-index notation:

$$\tilde{\alpha}_{a_1\ldots a_{n-p}} = (-1)^{s+p(n-p)}\frac{1}{p!}\epsilon^{c_1\ldots c_p}{}_{a_1\ldots a_{n-p}}\alpha_{c_1\ldots c_p} = (-1)^{s+p(n-p)}(\star\alpha)_{a_1\ldots a_{n-p}}.$$

To push this notation a little bit further we can even introduce an abstract-index contraction between forms and multivector fields, even when not all indices are contracted. For example, for $X^a$ a vector field we can denote

$$i_X \boldsymbol{\epsilon} \equiv X^a \boldsymbol{\epsilon}_a,$$

or for $Z^{ab} = Z^{[ab]}$ a bivector field, e.g. $Z^{ab} = X^a Y^b - Y^a X^b$, we can denote the contraction (mind the order of the indices and of the contractions)

$$\frac{1}{2!} Z^{ab} \boldsymbol{\epsilon}_{ab} = X^a Y^b \boldsymbol{\epsilon}_{ab} = i_Y i_X \boldsymbol{\epsilon}.$$

(Note: not every bivector is of this "simple" form, but every bivector is a linear combination of simple bivectors.)

**Remark A.3.1.** Although in a given coordinate chart one can write e.g.

$$\boldsymbol{\epsilon}_\mu = i_{\partial_\mu} \boldsymbol{\epsilon} = \frac{1}{(n-1)!} \epsilon_{\mu\mu_2 \ldots \mu_n} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_n},$$

or similarly $\boldsymbol{\epsilon}_{\mu\nu} = i_{\partial_\nu} i_{\partial_\mu} \boldsymbol{\epsilon}$, it is not possible to define $\boldsymbol{\epsilon}_a$ or $\boldsymbol{\epsilon}_{ab}$ as abstract tensors themselves. So the above notation has to be taken with a grain of salt. We mention also an important equivalent definition for the contraction $\boldsymbol{\epsilon}_{\mu_1 \ldots \mu_p}$:

$$\boldsymbol{\epsilon}_{\mu_1 \ldots \mu_p} = g_{\mu_1 \nu_1} \ldots g_{\mu_p \nu_p} \star \left( dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p} \right). \tag{A.5}$$

$\Diamond$

Using the obvious generalization of this notation for the contraction of arbitrary multivector fields with $\boldsymbol{\epsilon}$, we can write the Hodge dualization as

$$\star \boldsymbol{\alpha} = \frac{1}{p!} \alpha^{a_1 \ldots a_p} \boldsymbol{\epsilon}_{a_1 \ldots a_p}$$

or, conversely, using the formula above for the square of the Hodge-star operator,

$$\boxed{\boldsymbol{\alpha} = \frac{1}{(n-p)!} \tilde{\alpha}^{a_1 \ldots a_{n-p}} \boldsymbol{\epsilon}_{a_1 \ldots a_{n-p}},}$$

*without any extra sign.*

**Remark A.3.2** (Removing the tilde?)**.** Unless $p = n/2$, removing the tilde from the notation generates no particular confusion, since in that case $\alpha_{a_1 \ldots a_p}$ and $\tilde{\alpha}_{a_1 \ldots a_{n-p}}$ have a different number of indices and can therefore be told apart. Care must be taken, however, in the physically common case of $n = 4$ and $p = 2$. $\Diamond$

In the following we specialize to certain common cases, and provide some useful identities.

- **$\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta}$, $\boldsymbol{\alpha}$ 1-form, $\boldsymbol{\beta}$ 2-form**

There exists a vector field $X^a$ such that

$$\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta} = X^a \boldsymbol{\epsilon}_a,$$

meaning that the components satisfy

$$\frac{(n-1)!}{(n-2)!}\alpha_{[a_2}(\star\beta)_{a_3\ldots a_n]} = X^a \epsilon_{aa_2\ldots a_n}.$$

Multiplying both sides by $\epsilon^{a_1\ldots a_n}$, we get

$$\begin{aligned}(n-1)\alpha_{a_2}\frac{1}{2}\epsilon_{aba_3\ldots a_n}\beta^{ab}\epsilon^{a_1\ldots a_n} &= (-1)^s X^a (n-1)!\delta^{a_1}_a \\ \alpha_{a_2}\delta^{[a_1}_a\delta^{a_2]}_b\beta^{ab} &= X^{a_1},\end{aligned}$$

so that

$$\boxed{\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta} = \alpha_b\beta^{ab}\boldsymbol{\epsilon}_a.} \tag{A.6}$$

- **$d(\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta})$, $\boldsymbol{\alpha}$ 1-form, $\boldsymbol{\beta}$ 2-form**

Since all top-forms are proportional to each other, we have

$$d(\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta}) = d(X^a\boldsymbol{\epsilon}_a) = f\boldsymbol{\epsilon},$$

for $f$ some scalar (rank-0 tensor). Now, we compute[5]

$$\begin{aligned}d(X^a\boldsymbol{\epsilon}_a) &= \frac{1}{(n-1)!}\partial_\nu(X^\mu\sqrt{g})\sigma_{\mu\mu_2\ldots\mu_n}dx^\nu \wedge dx^{\mu_2} \wedge \ldots dx^{\mu_n} \\ &= (\nabla_\nu X^\mu)\frac{1}{(n-1)!}\sqrt{g}\sigma_{\mu\mu_2\ldots\mu_n}dx^\nu \wedge dx^{\mu_2} \wedge \ldots dx^{\mu_n} \\ &= (\nabla_\gamma X^\gamma)\frac{1}{n!}\sqrt{g}\sigma_{\mu_1\ldots\mu_n}dx^{\mu_1} \wedge \ldots dx^{\mu_n}\end{aligned}$$

(the last step can be performed by observing that $dx^\nu \wedge dx^{\mu_2} \wedge \ldots dx^{\mu_n} = \sigma^{\nu\mu_2\ldots\mu_n}dx^1 \wedge \ldots \wedge dx^n$). Whence

$$f = \nabla_a X^a = \nabla_a(\beta^{ab}\alpha_b),$$

and

$$\boxed{d(\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta}) = \nabla_a(\beta^{ab}\alpha_b)\boldsymbol{\epsilon}.} \tag{A.7}$$

[5] $\nabla$ is the *Levi-Civita* connection.

- **$d\boldsymbol{\alpha}$, $\boldsymbol{\alpha}$ $(n-1)$-form & $d\star\boldsymbol{\beta}$, $\boldsymbol{\beta}$ 1-form**

As a consequence of the previous demonstration, we also infer that

$$\boxed{d\boldsymbol{\alpha} = (\nabla_a \tilde{\alpha}^a)\boldsymbol{\epsilon},}$$

where $\alpha^a$ is defined as above by $\boldsymbol{\alpha} = \tilde{\alpha}^a \boldsymbol{\epsilon}_a$. If we start with the 1-form $\boldsymbol{\beta} = \beta_\mu dx^\mu$, then $\star\boldsymbol{\beta} = \beta^a \boldsymbol{\epsilon}_a$. Hence we also have

$$\boxed{d(\star\boldsymbol{\beta}) = (\nabla_a \beta^a)\boldsymbol{\epsilon}.} \tag{A.8}$$

- **$d\star d\varphi$, $\varphi$ 0-form**

From (A.8), we immediately deduce that

$$\boxed{d\star d\varphi = \Box\varphi\,\boldsymbol{\epsilon}.},$$

where the symbol $\Box \equiv \nabla_a \nabla^a$ is the Laplace-d'Alambert operator.

- **$\boldsymbol{\alpha}\wedge\star\boldsymbol{\beta}$, $\boldsymbol{\alpha},\boldsymbol{\beta}$ p-forms**

An important identity, that we leave as an exercise for the reader, is

$$\boxed{\boldsymbol{\alpha}\wedge\star\boldsymbol{\beta} = \frac{1}{p!}\alpha_{a_1\ldots a_p}\beta^{a_1\ldots a_p}\boldsymbol{\epsilon}.}$$

- **$d\star\boldsymbol{\beta}$, $\boldsymbol{\beta}$ 2-form; & $d\boldsymbol{\alpha}$, $\boldsymbol{\alpha}$ $(n-2)$-form**

For $\boldsymbol{\beta} = \frac{1}{2}\beta_{\mu\nu}dx^\mu \wedge dx^\nu$ one has

$$\boxed{d\star\boldsymbol{\beta} = \nabla_b \beta^{ab}\boldsymbol{\epsilon}_a.}$$

while for $\boldsymbol{\alpha} = \frac{1}{2}\tilde{\alpha}^{ab}\boldsymbol{\epsilon}_{ab}$ one has

$$\boxed{d\boldsymbol{\alpha} = \nabla_b \tilde{\alpha}^{ab}\boldsymbol{\epsilon}_a.}$$

An efficient way to prove the last formula is to first notice that, as a consequence of (A.5) and (A.6),

$$dx^\gamma \wedge \boldsymbol{\epsilon}_{\mu\nu} = \delta^\gamma_\nu \boldsymbol{\epsilon}_\mu - \delta^\gamma_\mu \boldsymbol{\epsilon}_\nu.$$

Therefore, if we leverage the property that every (torsionless) connection is equivalent when acting on forms[6], we can represent $d$ by the Levi-Civita connection (so that $d\boldsymbol{\epsilon}_{ab} \sim \nabla\sqrt{g} = 0$) and infer that

$$d\boldsymbol{\alpha} = \frac{1}{2}d\tilde{\alpha}^{\mu\nu}\wedge\boldsymbol{\epsilon}_{\mu\nu} + \frac{1}{2}\tilde{\alpha}^{\mu\nu}d\boldsymbol{\epsilon}_{\mu\nu} = \frac{1}{2}\nabla_\gamma\tilde{\alpha}^{\mu\nu}dx^\gamma\wedge\boldsymbol{\epsilon}_{\mu\nu} = \nabla_\nu\tilde{\alpha}^{\mu\nu}\boldsymbol{\epsilon}_\mu.$$

[6] See the beginning of Chapter 3 of Wald's book for instance.

## A.4 Stokes theorem

In this last part, we focus on Stokes's theorem in codimension 0 and 1, i.e. for boundaries of codimension 1 and 2.

Keeping the same notation as in the main text, we will be interested in a manifold $M$ whose boundary may have a spacelike piece $\Sigma$ and a timelike piece $B$. We denote their respective unit normals by $n^a$ and $s^a$. Using the notation $\boldsymbol{\epsilon}_\Sigma$ and $\boldsymbol{\epsilon}_{\partial\Sigma}$ for the induced volume forms on $\Sigma$ and its boundary $\partial\Sigma$, $n^a$ and $s^a$ satisfy the following set of relations:

$$\begin{aligned} n_a n^a &= -1, & \iota_\Sigma^* \boldsymbol{\epsilon}_a &= n_a \boldsymbol{\epsilon}_\Sigma, & \iota_\Sigma^* \boldsymbol{\epsilon}_n &= -\boldsymbol{\epsilon}_\Sigma, \\ s_a s^a &= 1, & \iota_B^* \boldsymbol{\epsilon}_a &= s_a \boldsymbol{\epsilon}_B, & \iota_B^* \boldsymbol{\epsilon}_s &= \boldsymbol{\epsilon}_B. \end{aligned}$$

Note that in these equations $\iota_\Sigma^* \boldsymbol{\epsilon}_a$ is an object of mixed nature: its form part is pulled-back to $\Sigma$, and its tensor index is normal to the surface. Superficially this might seem to contradict what we stated earlier about $\boldsymbol{\epsilon}_a$ *not* being a tensor. The reason why there is no contradiction is that in the above formula the form part of $\boldsymbol{\epsilon}_\mu$ has been pulled-back to $\Sigma$; in other words we are not trying to describe $\boldsymbol{\epsilon}_a$ as a "bulk" $(\text{top}-1)$-form on $M$, but rather just as "boundary" top-form on $\Sigma$—times a spacetime vector that now the pullback forces to point in the normal direction to $\Sigma$ itself.

**Remark A.4.1** (In coordinates)**.** In coordinates: suppose that $\Sigma$ is determined by the vanishing of the coordinate[7] $x^0$, i.e. $\Sigma = \{x^0 = 0\}$ in some coordinate system (any other fixed value of the coordinate would do). Then $\iota_\Sigma^* dx^0 = 0$ and therefore the only component of $\boldsymbol{\epsilon}_\mu$ that survives the pullback is the one for $\mu = 0$, i.e.

$$\iota_\Sigma^* \boldsymbol{\epsilon}_\mu = \delta_\mu^0 \iota_\Sigma^* \boldsymbol{\epsilon}_0.$$

Moreover, we note that the unit co-normal to $\Sigma$ is given by

$$n_\mu = \frac{1}{\sqrt{|g^{00}|}} \delta_\mu^0 \doteq N \delta_\mu^0$$

and that $\sqrt{h} \doteq N^{-1}\sqrt{g}$ is the induced volume element on $\Sigma$.[8] The quantity ref fnt
$N$ is called the *lapse* function. From this one deduces that

$$\iota_\Sigma^* \boldsymbol{\epsilon}_\mu = N^{-1} n_\mu \iota_\Sigma^* \boldsymbol{\epsilon}_0 = n_\mu \boldsymbol{\epsilon}_\Sigma \quad \text{and} \quad \iota_\Sigma^*(i_n \boldsymbol{\epsilon}) = \iota_\Sigma^*(n^\mu \boldsymbol{\epsilon}_\mu) = -\boldsymbol{\epsilon}_\Sigma,$$

where $\boldsymbol{\epsilon}_\Sigma = \frac{1}{(n-1)!}\sqrt{h}\sigma_{\mu_1\dots\mu_{n-1}} dx^1 \wedge \ldots \wedge dx^{n-1}$ is the induced (Levi-Civita) volume form intrinsic to $\Sigma$. $\Diamond$

[7] Wrt the previous sections we have here relabelled our coordinates so that they start at 0 rather than 1.

[8] See the chapter on the 3+1 split and the ADM decomposition in either [Wal84], or [MTW73], or probably most simply in [**PoissonToolkit**].

## Codimension 0 to 1

We now turn to Stokes's theorem over $M$. Consider a $(n-1)$-form $\boldsymbol{\alpha} = \tilde{\alpha}^a \boldsymbol{\epsilon}_a$. Then, using the identities from the previous section we see that—and considering a spacetime such that $B = \emptyset$ (generalization is obvious)—

$$\begin{aligned}\int_M d\boldsymbol{\alpha} = \int_M \nabla_a \tilde{\alpha}^a \boldsymbol{\epsilon} = \int_{\partial M} n_a \tilde{\alpha}^a \boldsymbol{\epsilon}_\Sigma \\ = \int_{\partial M} (-\tilde{\alpha}^a n_a \iota_\Sigma^* \boldsymbol{\epsilon}_n) = \int_{\partial M} (-n_a n^b + h_a{}^b) \tilde{\alpha}^a \iota_\Sigma^* \boldsymbol{\epsilon}_b = \int_{\partial M} \iota_\Sigma^* (\tilde{\alpha}^a \delta_a^b \boldsymbol{\epsilon}_b) \\ = \int_{\partial M} \iota_\Sigma^* \boldsymbol{\alpha}.\end{aligned}$$

Comparing the first and last line we see that both form and tensor notations fit nicely together, in part thanks to our seemingly unusual sign convention for defining $\tilde{\alpha}$. In the second line, we inserted the identity $\delta_b^a = -n^a n_b + h^a{}_b$, where $h^a{}_b$ is the projector on $T\Sigma \hookrightarrow T_\Sigma M$. Note that the minus sign in $\iota_\Sigma^* \boldsymbol{\epsilon}_n = -\boldsymbol{\epsilon}_\Sigma$ is necessary for consistency of the above formulas. It is the same sign appearing in the projector to $T\Sigma$. It is morally the same sign appearing when going from $n_\mu dx^\mu = dt$ to $n^\mu \partial_\mu = -\partial_t$ in Minkowski (that's also why no such sign would appear on a timelike boundary $B$).

## Codimension 1 to 2

We now move up one codimension. Let's start the discussion from the geometry of the corner surface $C = \Sigma \cap B$ within spacetime.

In general the unit normals $n$ and $s$ to $\Sigma$ and $B$, respectively, have no reason to be mutually orthogonal, i.e. $n \cdot s \neq 0$. Therefore, if we want to express the normal to $\partial\Sigma$ in $T\Sigma$, in terms of a spacetime vector, this cannot be simply $s^a$—see the figure A.1. With this goal in mind we define the vectors $\tilde{n}^a$ and $\tilde{s}^a$ in $(TC)^\perp \subset T_C M$,[9]

$$\tilde{n}^a \doteq \frac{n^a - (n\cdot s) s^a}{\sqrt{1+(n\cdot s)^2}} \qquad \text{and} \qquad \tilde{s}^a \doteq \frac{s^a + (n\cdot s) n^a}{\sqrt{1+(n\cdot s)^2}},$$

so that

$$\tilde{s}\cdot n = 0, \qquad \tilde{n}\cdot s = 0, \qquad \tilde{s}\cdot\tilde{s} = 1, \qquad \tilde{n}\cdot\tilde{n} = -1.$$

With these vectors we can now efficiently write the resolution of the identity in terms of the normals to the corner and the projector $\gamma^a{}_b$ onto

[9] For $N \subset M$, we denote $T_N M \doteq \bigcup_{x\in N} T_x M$. In particular $T_C M \simeq TC \oplus_C (TC)^\perp$, with $(TC)^\perp := \{v \in T_C M : g(v,w) = 0\ \forall w \in TC\}$.

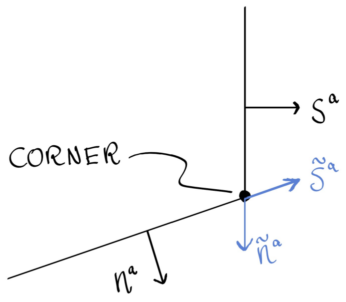


Figure A.1: Sketch of the normals to $\Sigma, B$ and $C$.

$TC \hookrightarrow T_C M$:

$$\begin{aligned}\delta^a_b &= -n^a n_b + h^a{}_b = -n^a n_b + \tilde{s}^a \tilde{s}_b + \gamma^a{}_b \\ &= -\tilde{n}^a \tilde{n}_b + s^a s_b + \gamma^a{}_b \\ &= \frac{-n^a n_b + s^a s_b}{1+(n\cdot s)^2} + \frac{n\cdot s}{1+(n\cdot s)^2}(n^a s_b + s^a n_b) + \gamma^a{}_b.\end{aligned} \tag{A.9}$$

Another important object is the corner's bi-normal $\varepsilon_{ab}$, defined by

$$\boxed{\varepsilon_{ab} := \tilde{n}_a s_b - s_a \tilde{n}_b = n_a \tilde{s}_b - \tilde{s}_a n_b = \frac{n_a s_b - s_a n_b}{\sqrt{1+(n\cdot s)^2}}.}$$

Its norm

$$\varepsilon_{ab}\varepsilon^{ab} = -2.$$

Similarly to the relations we found for the volume element on $\partial M$, the pullback of $\boldsymbol{\epsilon}_a$ and the boundary normal, in codimension-2 we similarly get

$$\iota^*_{\partial\Sigma}\boldsymbol{\epsilon}_{ab} = \varepsilon_{ab}\boldsymbol{\epsilon}_{\partial\Sigma}. \tag{A.10}$$

We can verify the consistency of these formulas with Stokes's theorem. We do the computation in two steps. First we compute the boundary integral of a $(n-2)$-form $\boldsymbol{\alpha} = \frac{1}{2}\tilde{\alpha}^{ab}\boldsymbol{\epsilon}_{ab}$:[10]

$$\begin{aligned}\int_{\partial\Sigma}\boldsymbol{\alpha} &= \int_{\partial\Sigma}\frac{1}{2}\tilde{\alpha}^{ab}\boldsymbol{\epsilon}_{ab} \\ &= \int_{\partial\Sigma}\frac{1}{2}\tilde{\alpha}^{ab}\delta^c_a\delta^d_b\boldsymbol{\epsilon}_{cd} = \int_{\partial\Sigma}\frac{1}{2}\tilde{\alpha}^{ab}\big(-n_a n^c + \tilde{s}_a\tilde{s}^c\big)\big(-n_b n^d + \tilde{s}_b\tilde{s}^d\big)\boldsymbol{\epsilon}_{cd} \\ &= \int_{\partial\Sigma}\tilde{\alpha}^{ab}\big(-n_a\tilde{s}_b n^c\tilde{s}^d\big)\boldsymbol{\epsilon}_{cd} = \int_{\partial\Sigma}\tilde{\alpha}^{ab}n_a\tilde{s}_b\boldsymbol{\epsilon}_{\partial\Sigma} = \int_{\partial\Sigma}\frac{1}{2}\tilde{\alpha}^{ab}\varepsilon_{ab}\boldsymbol{\epsilon}_{\partial\Sigma}.\end{aligned}$$

[10]The pullback of the integrand is implicit.

In the second line, we inserted the identity twice and used (A.9); next, we used the fact that $\gamma^c{}_a \iota^*_{\partial\Sigma} \boldsymbol{\epsilon}_{cd} = 0$ (i.e. the pullback of $\boldsymbol{\epsilon}_{cd}$ vanishes when contracted with vectors tangent to $\partial\Sigma$). In the third line, we took advantage of the anti-symmetry of $\alpha^{ab}$ and $\boldsymbol{\epsilon}_{cd}$ to get the final expression thanks to (A.10).

Next we compute the same quantity in terms of a bulk integral. Using the identities from the previous sections, we find:

$$\int_\Sigma d\boldsymbol{\alpha} = \int_\Sigma \nabla_b \tilde{\alpha}^{ab} \boldsymbol{\epsilon}_a = \int_\Sigma \big(\nabla_b \tilde{\alpha}^{ab}\big) n_a \boldsymbol{\epsilon}_\Sigma = \int_\Sigma \overline{\nabla}_i \big(n_a \tilde{\alpha}^{ai}\big) \boldsymbol{\epsilon}_\Sigma,$$

where $i, j, \ldots$ are tangent indices and $\overline{\nabla}_i$ is the intrinsic covariant derivative to $\Sigma$ (with respect to the induced metric $h_{ij}$). The last step is important and worth doing in details. First, introduce the "acceleration of $n^a$" i.e. $\mathrm{a}^a \doteq n^b \nabla_b n^a$ and note is tangent to $\Sigma$ since $n$ is normalized; with this, notice that

$$\big(\nabla_b \tilde{\alpha}^{ab}\big) n_a = \nabla_b \big(\tilde{\alpha}^{ab} n_a\big) - \tilde{\alpha}^{ab} \nabla_b n_a = \nabla_b \big(\tilde{\alpha}^{ab} n_a\big) - \tilde{\alpha}^{ab} n_{[a} \mathrm{a}_{b]},$$

which follows from the fact that $n_a$ is hypersurface-orthogonal, see the Remark below. Since $\bar{v}^b \equiv \tilde{\alpha}^{ab} n_a$ is a vector tangent to $\Sigma$, since $\bar{v}^b n_b = \tilde{\alpha}^{ab} n_a n_b \equiv 0$, we can use the relationship between bulk and boundary covariant derivatives, namely[11]

$$\overline{\nabla}_a \bar{v}^a = h^a{}_b \nabla_a \bar{v}^b = (\delta^a_b + n^a n_b) \nabla_a \bar{v}^b = \nabla_b \bar{v}^b - \mathrm{a}_b \bar{v}^b,$$

which directly implies

$$\nabla_b \big(\tilde{\alpha}^{ab} n_a\big) - \tilde{\alpha}^{ab} n_{[a} \mathrm{a}_{b]} = \overline{\nabla}_b \big(\tilde{\alpha}^{ab} n_a\big).$$

We conclude by combining these results and thus comparing the formulations of Stokes' theorem in terms of tensors and differential-forms:

$$\int_{\partial\Sigma} \boldsymbol{\alpha} = \int_\Sigma d\boldsymbol{\alpha} = \int_\Sigma \overline{\nabla}_b \big(\alpha^{ab} n_a\big) \boldsymbol{\epsilon}_\Sigma = \int_{\partial\Sigma} (\alpha^{ab} n_a) \tilde{s}_b \boldsymbol{\epsilon}_{\partial\Sigma} = \int_{\partial\Sigma} \frac{1}{2} \tilde{\alpha}^{ab} \varepsilon_{ab} \boldsymbol{\epsilon}_{\partial\Sigma},$$

which is indeed consistent with our previous computation.

**Remark A.4.2** (Hypersurface orthogonality). Let $\Sigma \hookrightarrow M$ be a codimension-1 submanifold, and denote the co-normal one-form $\boldsymbol{n} \doteq n_\mu dx^\mu$. Observe that the defining property of $\boldsymbol{n}$ is:[12] $\ker(\boldsymbol{n})|_\Sigma$ is the tangent distribution to $\Sigma$

[11] This can be written also as $\nabla_a \bar{v}^a = \overline{\nabla}_i \bar{v}^i + \mathrm{a}_i \bar{v}^i$.

[12] For the notation, see footnote 9.

in $T_\Sigma M$ (we are not demanding $n_a$ to be of unit norm, here). Then, the hypersurface orthogonality condition states that

$$\boxed{\iota_\Sigma^* d\boldsymbol{n} = 0}$$

or, equivalently,

$$\boxed{h_i{}^a h_j{}^b \nabla_{[a} n_{b]} = 0.}$$

This result follows from the fact that the Lie bracket of two vectors fields tangent to $\Sigma$ is itself a vector field tangent to $\Sigma$. Indeed: consider $\boldsymbol{n}$ to be the co-normal to $\Sigma$ and $X, Y, Z$ three vectors fields tangent to $\Sigma$ such that $[X, Y] = Z$; then, $X, Y, Z \in \ker(\boldsymbol{n})|_\Sigma$ i.e. $0 = i_X \boldsymbol{n} = i_Y \boldsymbol{n} = i_Z \boldsymbol{n}$, whence—using Cartan's calculus—it is immediate to show:

$$i_Y i_X \iota_\Sigma^* d\boldsymbol{n} = \iota_\Sigma^*(i_Y i_X d\boldsymbol{n}) = \iota_\Sigma^*(L_X i_Y \boldsymbol{n} - L_Y i_X \boldsymbol{n} - i_Z \boldsymbol{n}) = 0.$$

Since this must hold for all $X, Y$ tangent to $\Sigma$, we conclude $\iota_\Sigma^* d\boldsymbol{n} = 0$ as desired.

The hypersurface orthogonality condition allows us to study the properties of the rank-2 tensor $\nabla_a n_b$ in its "3+1" decomposition: inserting next to $\nabla_{[a} n_{b]}$ two copies of the resolution of the identity[13] $\delta_a^{a'} = -n_a n^{a'} + h_a{}^{a'}$, one sees that:

$$\nabla_a n_b = K_{ab} - n_a \mathrm{a_b} - \tfrac{1}{2} n_b \nabla_a n^2.$$

where:

$$K_{ab} \doteq h_a{}^{a'} h_b{}^{b'} \nabla_{a'} n_{b'} = K_{(ab)}$$

is a *symmetric* rank-2 tensor tangent to $\Sigma$ (this is the "extrinsic curvature" of $\Sigma$ in $M$);

$$\mathrm{a}_b \doteq h_b{}^{b'} \nabla_n n_{b'}$$

is the tangential acceleration of $n_a$; and where the last term vanishes if $n_a$ is of constant norm (in this case $\mathrm{a}_a$ coincides with the "total" acceleration of $n_a$). $\lozenge$

[13]This and the following formulas are adjusted to the case of $n$ timelike, i.e. to $\Sigma$ spacelike. Generalizations are straightforward.

# Appendix B

# Variation of the Einstein–Hilbert action

Consider the Einstein–Hilbert (EH) action

$$S = \frac{1}{16\pi G} \int_M \sqrt{g}\,(R - 2\Lambda)$$

Here $\Lambda$ is the cosmological constant, and $R$ the Ricci scalar curvature, defined by (we follow Wald's conventions [Wal84]):[1]

$$[\nabla_a, \nabla_b]v^c \doteq -R_{abd}{}^c v^d, \quad R_{ab} \doteq R_{acb}{}^c, \quad R \doteq g^{ab}R_{ab}.$$

Let's introduce a local coordinate system $x^A$. We want to think of the index structure of Christoffel symbols $\Gamma^C_{AB}$ as emphasized by the following (overzealous) use of the parenthesis:

$$\nabla_A v^C = \partial_A v^C + (\Gamma_A)^C{}_B v^B$$

Of course, we want to do so without forgetting the torsionless condition $\Gamma^C_{AB} = \Gamma^C_{BA}$. Then, with this in mind, it is much easier to remember the explicit formulas for $\Gamma$ and the Riemann tensor. First, the Christoffel symbol. It has the following general structure,

$$\Gamma = \frac{1}{2} g^{-1}\big(\partial g + \partial g - \partial g\big)$$

[1] Alternative versions of the defining equation of the Riemann curvature are:

$$[\nabla_a, \nabla_b]v^c \doteq R_{ab}{}^c{}_d v^d \quad \text{or} \quad [\nabla_a, \nabla_b]\omega_c \doteq R_{abc}{}^d \omega_d.$$

For the symmetry properties of the Riemann tensor, we refer to [Wal84, Ch.3.2].

and there is only one decent way to fill in the indices which is manifestly compatible with the symmetries. Second, the Riemann tensor: it is just a curvature of the connection $\Gamma$. This is emphasized both by its definition in terms of the commutator $\mathrm{Riem} = \nabla \wedge \nabla$ and in terms of the following formula

$$\mathrm{Riem} = \partial\Gamma - \partial\Gamma + \Gamma\Gamma,$$

which is in all akin to $F = \partial A - \partial A + AA$. Once again, (up to signs) there is only one way to fill in the indices which makes sense.[2]

These tricks are particularly useful if we want to compute the variation of the action, since they allow us to recover a couple of useful formulas quite easily.

First, we recall that as a manifestation of the equivalence principle it is always possible to choose a system of coordinates in which $g = \eta$ and $\Gamma = 0$ *at a point*. This are the so-called Fermi normal coordinates (FNC). Then, in this coordinate system and at that point,

$$\mathbb{d}\Gamma \overset{\mathrm{FNC}}{=} \frac{1}{2}\eta^{-1}\big(\partial\mathbb{d}g + \partial\mathbb{d}g - \partial\mathbb{d}g\big).$$

However, we notice that $\mathbb{d}\Gamma$, being the difference between two connections, is a tensor, and as such must allow for a covariant formula. The only reasonable covariantization of the previous formula is

$$\mathbb{d}\Gamma = \frac{1}{2}g^{-1}\big(\nabla\mathbb{d}g + \nabla\mathbb{d}g - \nabla\mathbb{d}g\big).$$

Next, we turn to the Riemann tensor. A similar reasoning shows that

$$\mathbb{d}\mathrm{Riem} \overset{\mathrm{FNC}}{=} \partial\mathbb{d}\Gamma - \partial\mathbb{d}\Gamma \quad \rightsquigarrow \quad \mathbb{d}\mathrm{Riem} \overset{\mathrm{FNC}}{=} \nabla\mathbb{d}\Gamma - \nabla\mathbb{d}\Gamma$$

which is also a tensor. Also, we notice that this formula is generic for curvatures, as in $\mathbb{d}F = D\mathbb{d}A$ (where we used a form notation, $D = d + A$).

So, to compute $\mathbb{d}R_{ab}$ we are only left with putting things together, and contracting indices (no metric insertion is required for this!):

$$\mathbb{d}R_{ab} = \mathbb{d}R_{acb}{}^{c} = -\mathbb{d}R_{ac}{}^{c}{}_{b} = -\big(\nabla_a\mathbb{d}(\Gamma_c)^c{}_b - \nabla_c\mathbb{d}(\Gamma_a)^c{}_b\big) = \nabla_c\mathbb{d}\Gamma^c_{ab} - \nabla_a\mathbb{d}\Gamma^c_{bc}$$

and thus

$$g^{ab}\mathbb{d}R_{ab} = \nabla_c(g^{ab}\mathbb{d}\Gamma^c_{ab}) - \nabla^a\mathbb{d}\Gamma^c_{ac} = \nabla_a\big(g^{bc}\mathbb{d}\Gamma^a_{bc} - g^{ab}\mathbb{d}\Gamma^c_{bc}\big)$$

---

[2] To get the signs right, the best way is to write $-R_{ABD}{}^{C}$ as $(R_{AB})^C{}_D = \partial_A(\Gamma_B)^C{}_D + \ldots$ as suggested by the above formula for $\nabla = \partial + \Gamma$ and the definition $[\nabla_A, \nabla_B]v^C = (R_{AB})^C{}_D v^D$. The only thing to remember is that this index positioning is not the "natural one" used to define the Ricci tensor, which is $R_{ab} = R_{acb}{}^{c} = -(R_{ac})^c{}_b \ldots$ That's life.

is a total derivative.

Given

$$\mathbb{d}\Gamma^c_{ab} = \frac{1}{2} g^{cd}\big(\nabla_a \mathbb{d} g_{bd} + \nabla_b \mathbb{d} g_{da} - \nabla_d \mathbb{d} g_{ab}\big),$$

we compute

$$\mathbb{d}\Gamma^c_{bc} = \frac{1}{2} g^{cd}\nabla_b \mathbb{d} g_{cd} \quad \text{and} \quad g^{cd}\mathbb{d}\Gamma^b_{cd} = g^{be}\nabla^c \mathbb{d} g_{ec} - \frac{1}{2} g^{cd}\nabla^b \mathbb{d} g_{cd},$$

and thus find for the argument of the above total derivative:

$$g^{bc}\mathbb{d}\Gamma^a_{bc} - g^{ab}\mathbb{d}\Gamma^c_{bc} = g^{ab}g^{cd}\big(\nabla_c(\mathbb{d} g_{bd} - \nabla_b \mathbb{d} g_{cd}\big) = g^{ab}g^{cd}\big(\nabla_c \mathbb{d} g_{bd} - \nabla_b \mathbb{d} g_{cd}\big)$$

Before computing the variation of $S$, one last ingredient. Since $g_{ab}$ is symmetric its $\mathsf{det}$ is the product of its eigenvalues. The trace of a symmetric matrix is instead the sum of its eigenvalues. Thus:

$$\mathbb{d}\mathsf{det}(g) = \mathbb{d} e^{\operatorname{tr}\log(g)} = \mathsf{det}(g)\operatorname{tr}\big(g^{-1}\mathbb{d} g\big)$$

Hence,

$$\mathbb{d}\sqrt{g} = \frac{1}{2\sqrt{g}}\mathbb{d}\mathsf{det}(g_{ab}) = \frac{1}{2}\sqrt{g}g^{ab}\mathbb{d} g_{ab} = -\frac{1}{2}\sqrt{g}g_{ab}\mathbb{d}(g^{ab}).$$

Thus, setting

$$\mathcal{L} = \sqrt{g}\left(\frac{1}{2}R - \Lambda\right),$$

and piecing everything together, we find

$$\mathbb{d}\mathcal{L} = \frac{1}{2}\sqrt{g}\Big(\underbrace{R_{ab} - \frac{1}{2}R g_{ab}}_{\doteq G_{ab}} + \Lambda g_{ab}\Big)\mathbb{d}(g^{ab}) + \frac{1}{2}\sqrt{g}\nabla_a\Big(g^{bc}\mathbb{d}\Gamma^a_{bc} - g^{ab}\mathbb{d}\Gamma^c_{bc}\Big).$$

That is:

$$\mathbb{d}\mathcal{L} = -\frac{1}{2}\sqrt{g}(G^{ab} + \Lambda g^{ab})\mathbb{d} g_{ab} + \frac{1}{2}\sqrt{g}\nabla_a\big(g^{ab}g^{cd}\big(\nabla_c \mathbb{d} g_{bd} - \nabla_b \mathbb{d} g_{cd}\big)\big).$$

## B.1 Computing Noether's current

We have:

$$J^a(\xi) = \mathbb{i}_{\rho(\xi)}\Theta^a - R^a(\xi)$$

where $R^a(\xi) = \left(\frac{1}{2}R - \Lambda\right)\xi^a$.

Of course, the crux is the computation of the first term. To get the desired formula which makes appear the Einstein tensor, we need to rearrange terms so that the curvature tensor appears:

$$\begin{aligned}
\mathring{\imath}_{\rho(\xi)}\Theta^a &= \frac{1}{2}\big(\nabla_c\nabla^a\xi^c + \nabla_c\nabla^c\xi^a - 2\nabla^a\nabla_c\xi^c\big) \\
&= \frac{1}{2}g^{ab}\big(\nabla_c\nabla_b\xi^c + \nabla_c\nabla^c\xi_b - 2\nabla_b\nabla_c\xi^c\big) \\
&= -g^{ab}[\nabla_b, \nabla_c]\xi^c + \frac{1}{2}g^{ab}\nabla_c\big(\nabla^c\xi_b - \nabla_b\xi^c\big) \\
&= g^{ab}R_{bcd}{}^c\xi^d + \frac{1}{2}\nabla_c\big(\nabla^c\xi^a - \nabla^a\xi^c\big) \\
&= R^{ad}\xi_d + \frac{1}{2}\nabla_c\big(\nabla^c\xi^a - \nabla^a\xi^c\big).
\end{aligned}$$

Piecing all the terms together:

$$J^a(\xi) = \left(R^{ad} - \frac{1}{2}Rg^{ad} + \Lambda g^{ad}\right)\xi_d + \frac{1}{2}\nabla_c\big(\nabla^c\xi^a - \nabla^a\xi^c\big).$$