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Homework 1

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Contents

1 Conformal invariance of the Maxwell action for $D = 4$

(a) Consider a classical abelian gauge field A_μ on $D = 3+1$ dimensionnal Minkowski spacetime. Under an infinitesimal conformal transformation, spacetime undergoes the transformation $\tilde{x}^\mu = f(x) = x^\mu + \xi^\mu(x)$ where $\xi^\mu(x)$ is a smal deformation. We want to calculate the effect of this transformation on the gauge field A_μ. The starting point is that we expect A_μ to transform as a tensor under the lorenz transformation subgroup of the conformal group. This implies that A_μ is a primary operator and we denote its scaling dimension Δ. The transformed field $\tilde{A}_\mu$ at $\tilde{x}$ is related to the original field A_μ at x by an internal rotation, scaling, and special conformal transformation. The rotation operation acts on the components A_μ through its spin 1 representation which is the defnining representation of rotations. The scaling and special conformal transformation act together through the multiplication of A_μ by the jacobian factor $|\partial x/\partial\tilde{x}|_x^{\Delta/D}$. Finally, translations act trivially internally. This can be summarized with the relation $\tilde{A}_\mu(\tilde{x}) = |\partial x/\partial\tilde{x}|_x^{\Delta/D} R_\mu^\nu A_\nu(x)$ where R_μ^ν is the rotation matrix associated with $\xi^\mu(x)$. With this in mind, we calculate the jacobian of the infinitesimal transformation to be

$$\left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|_x = \left|\frac{\partial\tilde{x}^\mu}{\partial x^\nu}\right|_x^{-1} = |\delta_\nu^\mu + \partial_\nu\xi^\mu|_x^{-1} \approx |e^{-\partial_\nu\xi^\mu}|_x = e^{-\mathrm{Tr}\partial_\nu\xi^\mu(x)} = 1 - \partial_\mu\xi^\mu(x) + O(\xi^2).$$

The rotation matrix $R_\mu^\nu(x)$ can be extracted by dividing the matrix $(\partial x/\partial\tilde{x})_x$ by its jacobian to extract the $SO(3)$ operation (we have a positive determinant transformation since it is infinitesimally close to identity). We have

$$R_\mu^\nu(x) = \frac{1}{1-\partial_\sigma\xi^\sigma(x)+O(\xi^2)}\left(\frac{\partial x^\nu}{\partial\tilde{x}^\mu}\right)_x = (1+\partial_\sigma\xi^\sigma(x)+O(\xi^2))(\delta_\nu^\mu + \partial_\mu\xi^\nu(x)+O(\xi^2))^{-1} = \delta_\nu^\mu(1+\partial_\sigma\xi^\sigma(x)) - \partial_\mu\xi^\nu(x) + O(\xi^2).$$

For a spacial conformal transformation, we have $\xi^\mu = -2x^\mu x_\lambda b^\lambda$ paramatrized by the transaltion vector b^λ around ∞. For this vector, we have

$$R_\mu^\nu(x) = \delta_\nu^\mu + \partial_\sigma\xi^\sigma(x) - \partial_\mu\xi^\nu(x) + O(\xi^2) = \delta_\nu^\mu$$

As expected for a special conformal transformation. With these results, we can write the effect of the infinitesimal transformation as

$$\begin{aligned}
\tilde{A}_\mu(\tilde{x}) &= (1-\partial_\rho\xi^\rho(f^{-1}(\tilde{x}))+O(\xi^2))^{\Delta/D}(A_\mu(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x})) - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2))\\
&= \left(1-\frac{\Delta}{D}\partial_\rho\xi^\rho(f^{-1}(\tilde{x}))+O(\xi^2)\right)(A_\mu(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x})) - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2))\\
&= A_\mu(f^{-1}(\tilde{x})) - A_\mu(f^{-1}(\tilde{x}))\frac{\Delta}{D}\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x})) - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2).
\end{aligned}$$

Since $\xi(f^{-1}(\tilde{x}))$ is already first order in ξ, the only term contribution to its expansion around $\xi = 0$ at $O(\xi)$ is $\xi(\tilde{x})$. To go further, we expand $f^{-1}(\tilde{x})$ at first order in $\xi(\tilde{x})$ with the ansatz $f^{-1}(\tilde{x})^\nu = \tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x})$ (the first term of this ansatz is justified by noticing the transformation reduces to identity at $\xi = 0$). From $f(f^{-1}(\tilde{x})) = \tilde{x}$, we find

$$\tilde{x}^\nu = \tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x}) + \xi(\tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x})) + O(\xi^2) \implies B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x}) + \xi^\nu(\tilde{x}) = 0, \quad \forall\xi(\tilde{x}) \implies B_\mu^\nu(\tilde{x}) = -\delta_\mu^\nu.$$

Using this result, we can expand $A_\mu(f^{-1}(\tilde{x}))$ as

$$A_\mu(f^{-1}(\tilde{x})) = A_\mu(\tilde{x}^\nu - \xi^\nu(\tilde{x}) + O(\xi^2)) = A_\mu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2)$$

Combining this expression with the internal transformation at first order in ξ, we get

$$\begin{aligned}
\tilde{A}_\mu(\tilde{x}) &= \left(1-\frac{\Delta}{D}\partial_\sigma\xi^\sigma(\tilde{x}) + \partial_\sigma\xi^\sigma(\tilde{x}) - \partial_\mu\xi^\nu(\tilde{x})\right)(A_\mu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x})) + O(\xi^2)\\
&= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta-D}{D}\partial_\sigma\xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\partial_\mu\xi^\nu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2)
\end{aligned}$$

with $\xi(f^{-1}(\tilde{x})) = \xi(\tilde{x}) + O(\xi^2)$. Form this transformed gauge field, we calculate the transformation of gauge field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ to $\tilde{F}_{\mu\nu}$. We start by writting the transformation law of the derivatives used to construct $F_{\mu\nu}$. The chain rule yields

$$\tilde{\partial}_\mu \equiv \frac{\partial}{\partial\tilde{x}^\mu} = \left(\frac{\partial f^{-1}(\tilde{x})^\nu}{\partial\tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} = \left(\frac{\partial\tilde{x}^\nu - \xi^\nu(\tilde{x})}{\partial\tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} = \left(\frac{\partial\xi^\nu(\tilde{x})}{\partial\tilde{x}^\mu}\right)_{\tilde{x}}\left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} + \left(\frac{\partial}{\partial x^\mu}\right)_{\tilde{x}} \equiv \partial_\mu\xi^\nu(\tilde{x})\partial_\nu + \partial_\mu.$$

Now we can calculate the transformed transformed field strength to be

$$\begin{aligned}
\tilde{F}_{\mu\nu} &= \tilde{\partial}_\mu\tilde{A}_\nu - (\mu\leftrightarrow\nu)\\
&= \left(\partial_\mu\xi^\rho(\tilde{x})\partial_\rho + \partial_\mu\right)\left(A_\nu(\tilde{x}) - A_\nu(\tilde{x})\frac{\Delta-D}{D}\partial_\sigma\xi^\sigma(\tilde{x}) - A_\lambda(\tilde{x})\partial_\nu\xi^\lambda(\tilde{x}) - \xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x})\right) - (\mu\leftrightarrow\nu)\\
&= \partial_\mu A_\nu(\tilde{x}) + \partial_\mu\xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta-D}{D}\partial_\lambda\xi^\lambda(\tilde{x})\right) - \partial_\mu\left(A_\lambda(\tilde{x})\partial_\nu\xi^\lambda(\tilde{x})\right) - \partial_\mu\left(\xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x})\right) - (\mu\leftrightarrow\nu)\\
&= \partial_\mu A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta-D}{D}\partial_\lambda\xi^\lambda(\tilde{x})\right) - \partial_\mu A_\lambda(\tilde{x})\partial_\nu\xi^\lambda(\tilde{x}) - A_\lambda(\tilde{x})\partial_\mu\partial_\nu\xi^\lambda(\tilde{x}) - \xi^\lambda(\tilde{x})\partial_\lambda\partial_\mu A_\nu(\tilde{x}) - (\mu\leftrightarrow\nu)\\
&= \partial_\mu A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta-D}{D}\partial_\lambda\xi^\lambda(\tilde{x})\right) - \partial_\mu A_\lambda(\tilde{x})\partial_\nu\xi^\lambda(\tilde{x}) - \xi^\lambda(\tilde{x})\partial_\lambda\partial_\mu A_\nu(\tilde{x}) - (\mu\leftrightarrow\nu)\\
&= F_{\mu\nu}(\tilde{x}) - F_{\mu\nu}(\tilde{x})\frac{\Delta-D}{D}\partial_\lambda\xi^\lambda(\tilde{x}) - A_{(\nu}(\tilde{x})\frac{\Delta-D}{D}\partial_{\mu)}\partial_\lambda\xi^\lambda(\tilde{x}) - \partial_{(\mu}A_\lambda(\tilde{x})\partial_{\nu)}\xi^\lambda(\tilde{x}) - \xi^\lambda(\tilde{x})\partial_\lambda F_{\mu\nu}(\tilde{x})
\end{aligned}$$

(b)

2 Axial anomaly

(a)
(b)
(c)
(d)

3 OPE coefficients from three point functions

(a)
(b)
(c)
(d)

4 Acknowledgement

References

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