PERIMETER SCHOLARS INTERNATIONAL

Classical Physics HW2
Fall 2023
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Relativistic composition of velocities and the Thomas precession

Due: Monday September 25th @ 9PM
https://www.dropbox.com/request/4F9OZTfRFJjlPnK2RHIS
(deadline and resubmission policy available in the course outline)

Late deadline: Monday October 9th @ 9PM
https://www.dropbox.com/request/lzzzIGEJvlC9g0Min9tH

Pass/fail deadline: formal request needed by Monday October 23th

1 Composition of velocities

a) In class we have shown that the special relativistic formula for the composition of velocities in a 1+1d world is

$$v' = \frac{\pm v \pm u}{1 \pm \frac{uv}{c^2}},$$

where v and v' are the velocities of a moving object with respect to the frames K and K', respectively, and u is the velocity at which the frame K' moves with respect to the frame K.

Fix the 3 signs in the formula above by physical considerations, i.e. by considering limiting situations in which the result is clear.

b) We are now going to consider a 3+1d world, where motions need not be *collinear*. Denote $\mathbf{u}$ the velocity of the frame $K'$with respect to the frame K, and $\mathbf{v}$ ($\mathbf{v}'$, resp.) the velocity of the object of interest in the frame K (and K', resp.).

In each frame, write down the infinitesimal displacement of the object after a time dt, resp. dt', then use the Lorentz transformation formula to deduce that:

$$\mathbf{v}_{\parallel} = \frac{\mathbf{v}'_{\parallel} + \mathbf{u}}{1 + \frac{\mathbf{u}\cdot\mathbf{v}'}{c^2}} \quad \text{and} \quad \mathbf{v}_{\perp} = \gamma(u)^{-1} \frac{\mathbf{v}'_{\perp}}{1 + \frac{\mathbf{u}\cdot\mathbf{v}'}{c^2}}. \tag{1}$$

where $\bullet_{\parallel}$ and $\bullet_{\perp}$ denote the components parallel and perpendicular to $\mathbf{u}$.

c) Show that a light-beam moving at speed c in the direction of angle θ' with respect to $\mathbf{u}$ in the frame K', will move in K also at speed c but in a direction of angle θ with respect to $\mathbf{u}$, where:

$$\cos\theta = \frac{\frac{u}{c} + \cos\theta'}{1 + \frac{u}{c}\cos\theta'}.$$

This is a case of light aberration.

d) Introduce the symbol $\oplus$ to denote the (nonlinear!) "addition" of velocities in special relativity, and show that

$$\mathbf{v} \doteq \mathbf{u} \oplus \mathbf{v}' = \frac{1}{1 + \frac{\mathbf{u}\cdot\mathbf{v}'}{c^2}} \left(\mathbf{u} + \mathbf{v}' + \frac{\gamma(u)}{1 + \gamma(u)} \frac{\mathbf{u}}{c} \times \left(\frac{\mathbf{u}}{c} \times \mathbf{v}' \right) \right). \tag{2}$$

Prove that this "addition" operation fails to be linear, commutative, and even associative—that is

$$\lambda(\mathbf{u} \oplus \mathbf{v}') \neq (\lambda\mathbf{u}) \oplus (\lambda\mathbf{v}'), \quad \mathbf{u} \oplus \mathbf{v}' \neq \mathbf{v}' \oplus \mathbf{u}, \quad \text{and} \quad (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} \neq \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}).$$

[Hint: work at the lowest nontrivial order in $1/c$].

e) Prove (1) using the fact that the 4-velocity is a Lorentz-contravariant 4-vector.

f) The non-associativity of this operation is particularly remarkable: it means that the composition of velocities does not even form a group! To which property of the 3+1d Lorentz Lie algebra (and/or Lorentz Lie group) is this fact related? Justify.

2 Frame Dragging and Thomas precession

(You can set $c = 1$, if you want.)

Consider now two moving frames K' and K'' with respect to a laboratory frame K respectively at velocities $\mathbf{v}$ and $\mathbf{u} = \mathbf{v} + \delta\mathbf{v}$.

There are two "natural" ways to go from K' to K'', that is either directly (blue) or via the lab frame (magenta):

$$\begin{tikzcd} K' \arrow[rr, "B(\delta\mathbf{v}')"] \arrow[dr, "B(-\mathbf{v})"'] & (?) & K'' \\ & K \arrow[ur, "B(\mathbf{v}+\delta\mathbf{v})"'] & \end{tikzcd} \tag{3}$$

Here, the symbol $B(\mathbf{u})$ denotes a boost of velocity $\mathbf{u}$. Recall that by definition

$$B(\mathbf{u}) \doteq \exp(-\boldsymbol{\psi} \cdot \boldsymbol{K}) \quad \text{where} \quad \begin{cases} \boldsymbol{\psi} \propto \mathbf{u} \\ \mathrm{th}|\boldsymbol{\psi}| = |\mathbf{u}|/c \end{cases}$$

and $\boldsymbol{K} \doteq (K_1, K_2, K_3)$ is the vector of boost generators, each K_i a 4×4 matrix. Similarly, we will denote

$$R(\boldsymbol{\theta}) \doteq \exp(-\boldsymbol{\theta} \cdot \boldsymbol{J})$$

a rotation of angle $\theta = |\boldsymbol{\theta}|$ around the axis $\boldsymbol{n} = \boldsymbol{\theta}/|\boldsymbol{\theta}|$, with $\boldsymbol{J} \doteq (J_1, J_2, J_3)$ the vector of boost generators, each J_i a 4×4 matrix.

a) Compute the relative velocity $\delta\mathbf{v}'$ of K'' with respect to K'—for this, you can use the formula for the composition of velocities (2)—and show that at first order in $\delta\mathbf{v}$ this is

$$\delta\mathbf{v}' = \gamma^2 \delta\mathbf{v}_{\parallel} + \gamma\delta\mathbf{v}_{\perp},$$

where $\bullet_{\parallel}$ and $\bullet_{\perp}$ denote the parallel and orthogonal components with respect to $\mathbf{v}$, respectively.

b) The big question mark in the middle of the diagram above summarizes our question: is the diagram (3) commutative? I.e. do the two ways of getting to K'' map the frame in K' to the *same* frame in K''? Or in other words,

$$B(\mathbf{v} + \delta\mathbf{v})B(-\mathbf{v}) \overset{?}{\doteq} B(\delta\mathbf{v}').$$

Show that the answer is negative, and that—*at first order in* $\delta\mathbf{v}$—the two frames in K'' are related by an infinitesimal rotation of magnitude

$$\delta\boldsymbol{\theta} = \frac{\gamma(v)^2}{\gamma(v)+1}\frac{1}{c^2}\mathbf{v}\times\delta\mathbf{v}$$

or, in other words, that

$$B(\mathbf{v}+\delta\mathbf{v})B(-\mathbf{v}) = B(\delta\mathbf{v}')R(\delta\boldsymbol{\theta}) \equiv 1 + \delta\boldsymbol{\psi}'\cdot\boldsymbol{K} + \delta\boldsymbol{\theta}\cdot\boldsymbol{J},$$

where we neglected higher orders in $\delta\mathbf{v}$ and introduced the rapidity-vector $\delta\boldsymbol{\psi}'$ associated to the velocity-vector $\delta\boldsymbol{v}'$.

Note: you are (warmly) encouraged to use Mathematica, and to conveniently align your choice of axis!

c) Do the boosts form a subgroup of the Lorentz group? How is this question related to the previous result? Comment appealing

To deal with accelerated observers within the framework of special relativity, we have often to implicitly introduce a new axiom: that the "instantaneous rest frame" of an accelerating observer is boosted along its trajectory without rotating. In formulas, if we denote $K'(t)$ the instantaneous rest frame of an accelerating observer at time t, then our new axiom can be formulated as:

$$K'(t) \xrightarrow{B(\delta\mathbf{v}'(t))} K'' = K'(t+\delta t).$$

c) Argue, from the previous results, that the instantaneous rest frame of an accelerating observer on a circular obit undergoes a precession motion of angular velocity $\boldsymbol{\omega}_{\mathrm{T}}$ that you will compute in terms of the accelerating observer's velocity and acceleration in the lab frame.

d) Use this formula to show for an electron moving in a central potential $V(r)$, one has at leading order in $1/c$:

$$\boldsymbol{\omega}_{\mathrm{T}} \approx -\frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\mathbf{L}$$

where $\mathbf{L}$ is the electron's orbital angular momentum.

The angular velocity $\boldsymbol{\omega}_{\mathrm{T}}$ is named after Llewellyn Thomas—see the historical note below.

3 One more question...

Who did you collaborate with?

When handing in your assignment, you *must* acknowledge the contributions and support received from fellow PSIons on specific questions, as well as the resources (online or otherwise) you consulted to complete the homework. If you worked alone, say so, but do not leave this question unanswered—even if *what* you answer will not impact our evaluation of your assignment. Science is a collaborative endeavour and as scientists we have the ethical duty to give credit where credit is due.

Historical note

> In 1926 Uhlenbeck and Goudsmit introduced the idea of electron spin and showed that, if the electron had a g factor of 2, the anomalous Zeeman effect could be explained, as well as the existence of multiplet splittings. There was a difficulty, however, in that the observed fine structure intervals were only half the theoretically expected values. If a g factor of unity were chosen, the fine structure intervals were given correctly, but the Zeeman effect was then the normal one. The complete explanation of spin, including correctly the g factor and the proper fine structure interaction, came only with the relativistic electron theory of Dirac. But within the framework of an empirical spin angular momentum and a g factor of 2, Thomas showed in 1927 [at the age of 24!] that the origin of the discrepancy was a relativistic kinematic effect which, when included properly, gave both the anomalous Zeeman effect and the correct fine structure splittings. The Thomas precession, as it is called, also gives a qualitative explanation for a spin-orbit interaction in atomic nuclei and shows why the doublets are ”inverted” in nuclei.[1]

[1] From Chapter 11.8 J.D. Jackson, *Classical Electrodynamics*, 3rd edition (1998). Se also Chapter 11.11.