Homework 1: Conformal invariance of Maxwell theory, axial anomaly and OPE coefficients

10 March 2024

Due on evening of 24 March 2024
Please don't forget to indicate who did you collaborate with on this homework assignment

1 Conformal invariance of the Maxwell action for $D = 4$.

a) Determine how an abelian gauge field A_μ and the corresponding field strength $F_{\mu\nu}$ transform under an infinitesimal conformal transformation.

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x). \tag{1}$$

Keep in mind that A_μ should transform under finite conformal transformations like a *tensor density* and that under the scale transformation $\tilde{x} = \lambda x$ we expect

$$\tilde{A}_\mu(\tilde{x}) = \lambda^{-\Delta} A_\mu(x), \tag{2}$$

where Δ is the scaling dimension of the gauge field.

b) Recall that in D dimensions infinitesimal conformal transformations satisfy the *conformal Killing equation*

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D} \eta_{\mu\nu} \partial_\lambda \xi^\lambda. \tag{3}$$

Use this property to show that the Maxwell action is invariant under conformal transformations when $D = 4$.

2 Axial anomaly

This question lives in $D = 2$, not only because it is written on your computer screen.
The vector current is defined to be

$$j^V_\mu = \bar{\psi} \gamma_\mu \psi$$

where ψ is some fermion field.

a) Using first Poincare symmetry and Bose symmetry, and then energy-momentum conservation and scale invariance, what is the most general form of the two-point correlation function of the current operator[1]?

$$\langle j^V_\mu(p) j^V_\nu(-p)\rangle$$

Define the axial current to be

$$j^A_\mu = \epsilon_{\mu\nu} j^{V,\nu}$$

b) Use the properties of the $2d$ γ matrices to show that the classical conservation of the vector current (plus the equation of motion) implies the conservation of the axial current.

c) On the other hand, evaluate

$$p^\mu \langle j^A_\mu(p) j^V_\nu(-p)\rangle$$

and perform the Fourier transformation. You should realize that there is an axial anomaly: that is this is a correlator with the divergence of the axial current and it does not vanish.

d) Now as you have seen in class, let us couple this with a background gauge field,

$$\langle \partial_\mu j^{A,\mu}(0)\rangle_{A_\mu} \equiv \langle \partial_\mu j^{A,\mu}(0) e^{\int dx A^\mu j^V_\mu}\rangle$$

Evaluate this[2]. What is the physical meaning of the axial anomaly you find?

3 OPE coefficients from three point functions

Recall that the two-point and three-point functions of scalar operators in conformal theories can be constrained to be

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_{\Delta'}(y)\rangle = \frac{\delta_{\Delta,\Delta'}}{|x-y|^{2\Delta}} \tag{4}$$

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}} \tag{5}$$

a) The OPE expansion for two primary (scalar) fields is just the assumption that the operator algebra in CFT is closed under multiplication and all the fields are either primaries or descendants of primaries. Justify that for $D > 2$ it can be written as

$$\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) = \sum_{\Delta'} \sum_{\vec{k}\geq 0} \sum_{\vec{l}\geq 0} \beta^{\Delta',\vec{k},\vec{l}}_{\Delta_1,\Delta_2} |x_{12}|^{\Delta'-\Delta_1-\Delta_2} (x_{12})^{\vec{k}} \partial_{\vec{l}} \mathcal{O}_{\Delta'}(x_2) \tag{6}$$

where the first sum goes over all primary operators in the theory, and $\beta^{\Delta',\vec{k},\vec{l}}_{\Delta_1,\Delta_2}$ are just some numbers.

b) What a-priori constraints one can impose on multi-indices $\vec{k}, \vec{l}$? Show that all the indices $\vec{k}$ have to be contracted with indices $\vec{l}$ and the OPE can be written as

$$\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) = \sum_{\Delta'} \sum_{k\geq 0} \beta^{\Delta',k}_{\Delta_1,\Delta_2} |x_{12}|^{\Delta'-\Delta_1-\Delta_2} x_{12}^{\mu_1}...x_{12}^{\mu_k} \partial_{\mu_1}...\partial_{\mu_k} \mathcal{O}_{\Delta'}(x_2) \tag{7}$$

[1] Recall that the scaling dimension of a current operator is always only determined by the dimension of the space.

[2] To the first non-trivial order.

c) Compare exact expression for three-point function with the expansion for it, obtained by taking OPE of $\mathcal{O}_{\Delta_1}(x_1)$ and $\mathcal{O}_{\Delta_2}(x_2)$ in three point function and usage of the expression for two-point function afterward. Assume that all the primary operators have different conformal dimensions in this computation. Using this comparison, derive $\beta^{\Delta',1}_{\Delta_1,\Delta_2}$, $\beta^{\Delta',2}_{\Delta_1,\Delta_2}$.

d) (Optional) Consider four-point function of primary scalar fields. One can write two expressions for it: by taking OPEs of 1 and 2, 3 and 4, and then computing two-point functions, or taking OPEs of 1 and 3, 2 and 4, and then computing their correlation function. Notice that in the OPEs there might occur non-scalar fields, so the formulas might be more complicated then ones above, however the structure of the series will be the same. Because of associativity of OPEs this expansions should correspond to the same function. Find convergence domains of this two expansions. What is an intersection of those?