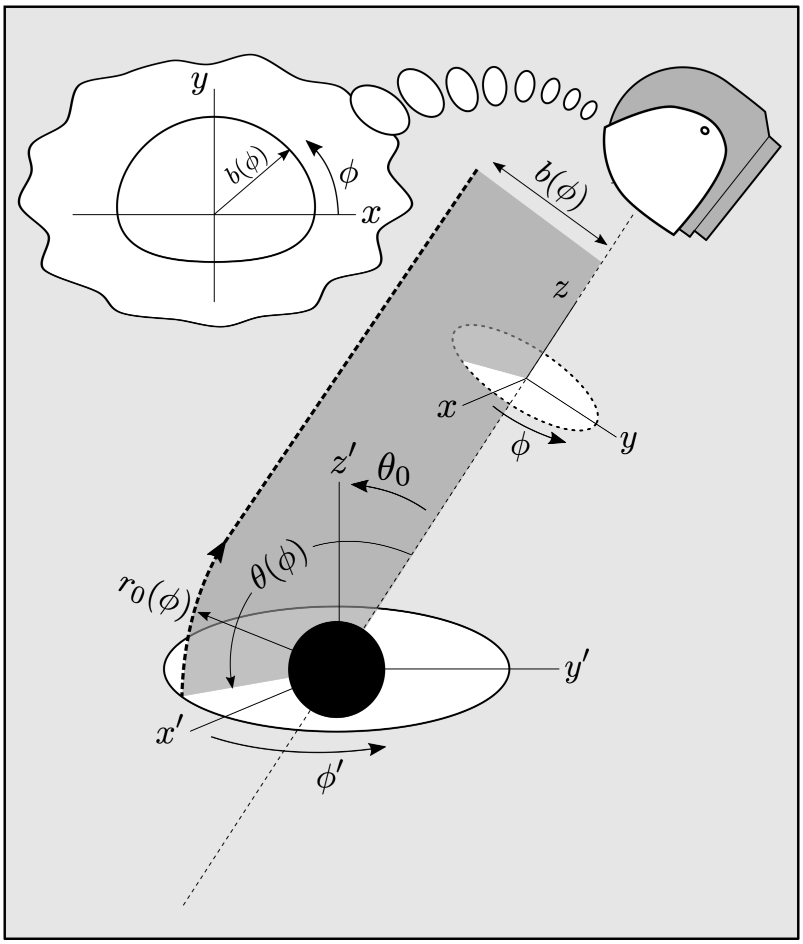
# Interstellar's Ring

Main deadline: Friday 16 Feb @ 9PM https://www.dropbox.com/request/FRfY2bY4BarH8nPE1Tvg
Late deadline: Friday 2 Mar @ 9PM https://www.dropbox.com/request/vHunykXN9qGO7pFXesgr
Pass-or-fail deadline: Friday 16 Mar @ 9PM – `contact Aldo & Dan`

The goal of this homework is to reproduce the distorted shape of the accretion disk around a static black as it would appear to a faraway observer. It comes with a Mathematica notebook that you will fill in and complete.



# 1 Setup

The geometrical setup for the calculation—which you will perform using Mathematica—is the one described in the previous drawing which represents the outer region of a Schwarzschild black hole. More precisely it represents the region $r' > 2GM$ in a $t' = const$ slice of a mass-$M$ Schwarzschild black hole, by mapping its Schwarzschild coordinate description into $\mathbb{R}^3$.

We denote the Schwarzschild coordinates describing the black hole frame by $(t', r', \theta', \phi')$. They asymptote at infinity to the usual $(t', x', y', z') = (t', r' \sin\theta' \cos\phi', r' \sin\theta' \sin\phi', \cos\theta')$ Minkowskian coordinates. In the Schwarzschild coordinate system the accretion disk lies on the $\theta' = \frac{\pi}{2}$ plane at radius $r' = a$. The main reference frame, denoted $(t, x, y, z)$, is a Minkowskian frame aligned with the (asymptotic!) observer. We choose $t = t'$ thus meaning that the observer is not moving w.r.t. to the black hole, and henceforth forgo all reference to the time coordiante from now on. In space, the two (asymptotic) Minkowskian reference frames are rotated with respect to one another by an angle $\theta_0$ around the $x = x'$ axis so that the observer lies along the $z$ axis of the new frame. For simplicity we call this new frame the observer's frame. Since the two reference frames are rotated w.r.t. one another, we have $r = r'$; this means that the origins of the two reference frames are taken to coincide (even though, to avoid clutter, in the figure the origins of the two reference frames have been off-set).

An important variable in the following will be the inverse Schwarzschild radius, denote here by,

$$u := \frac{1}{r}. \tag{1}$$

In the figure, the thick dashed line represents the geodesic of a photon emitted from the outer edge of the accretion disk and moving towards the asymptotic observer. This means that this photon's geodesic goes out to infinity parallel to the $z$-axis. For clarity, we shaded in grey the plane in which both the $z$-axis and the photon's geodesic lie (recall: geodesics in a Schwarzschild background are planar!). The geodesic's impact parameter is denoted $b$, whereas the geodisic's periastron[1] is denoted $r_0$.

Note: we are parametrizing the geodesics (and thus their impact parameter and periastron) in terms of the observer's viewing azimuthal angle $\phi$. Indeed, this is because our final goal is to compute the shape of the ring as viewed by the observer, which is given by the function $b(\phi)$ as depicted in the top-left corner of the drawing.

# 2 Warm up

We start with a *qualitative* study of the geodesics in a Schwarzschild spacetime. Our aim is for now only to clarify what we should expect in the computations of the following sections.

[1] "Peri-astron" means "the closest distance to a (central) star".

Start by considering a *third*, arbitrary, set of spherical coordinates for the Schwarzschild spacetime, with radius $r$, longitudinal angle $\chi$, and azimuthal angle $\psi$. Recall that any geodesic on a Shwarzschild spacetime lies in a plane (as defined by the Schwarzschild coordinate system). We then adapt our coordinates system so that the null geodesics we will consider lie in the equatorial ($\chi = \pi/2$) plane. Later we will have to identify this coordinate system with one adapted to each geodesic emanating from the disk and reaching the observer. In the figure we shaded in dark grey the plane on which one such geodesic lies.

On the geodesics' plane we thus have polar coordinates $(r, \psi)$, centred around the black hole. In the following the index $A \in \{1, 2\}$ denotes one of two (near-by) geodesics lying in this preferential plane.

a) Identifying the page with the geodesics' plane, sketch the trajectory of two near-by geodesics of different impact parameter $b^{(A)}$, both having their "initial" asymptote pointing in the $\psi = 0$ direction. On each geodesic mark the periastron $r_0^{(A)}$ and the point at which $r^{(A)} = a$. Note that if $b^{(A)}$ is larger than a certain critical value, $r_0(b) > a$ and the geodesic never reaches such a point.

b) Sketch the functions $r = r^{(A)}(\psi)$ for each geodesics on a graph with $\psi$ on the $X$-axis and $r$ on the $Y$-axis. Identify on this graph the periastra $r_0^{(A)}$, as well as the *total deflection angle* $\Delta_{\mathrm{Tot}}\psi^{(A)}$ defined as the angle between the two asymptotes.

c) As we will see, the geodesic equations is best solved as a function $\psi = \psi(u)$, with $u = 1/r$. Sketch the two trajectories of the previous questions on the $(X, Y) = (u, \psi)$ plane, and identify the periastra $u_0^{(A)}$. Identify on this graph the *partial deflection angle* $\Delta\psi^{(A)}$ defined as the angle $\psi$ travelled between their initial asymptote and the position $r = a$, i.e. according to our conventions $\Delta\psi^{(A)} = \psi^{(A)}(u = 1/a)$. Is this question well posed? That is: does it have a unique answer? What should we pay attention to? [Note: This subtlety will play a role later on.]

## 3 Null geodesics in a Schwarzschild's spacetime

We now want to turn the qualitative analysis of the previous section, into actual equations and graphs.

a) Denote $u := 1/r$ the inverse Schwarzschild radial coordinate, and recall from the Relativity course[2] that the trajectory of a null geodesic in the plane $\chi = \pi/2$ with impact parameter

[2] See Wald [Ch.6].

$b = L/E$ can be describe via the equation (here $R_S = 2GM$ is the Schwarzschild radius):[3]

$$\begin{cases} \dot{u}^2 = E^2 b^2 u^4 U_{\text{eff}}(u) \\ \dot{\psi} = Ebu^2 \end{cases} \quad \text{where} \quad U_{\text{eff}}(u) = \frac{1}{b^2} - u^2 + R_S u^3 \tag{2}$$

Choose units in which $M = 1$ i.e. $R_S = 2$ (this fixes the only scale in our problem), and show that (whatever the units) the value of $L = Eb$ can always be reabsorbed into the definition of the affine parameter $\lambda$. Conclude that, therefore, w.l.o.g. we can set in the following

$$R_S \equiv 2, \qquad Eb \equiv 1. \tag{3}$$

Note: this is possible only because the geodesic is null!

**Mathematica**: define the effective potential `Ueff[u_, b_]`. Plot `Ueff[u, b]` for `b` $=$ `4`, `3`$\sqrt{\texttt{3}}$, `7` in the range $-$`0.3` $<$ `u` $<$ `2`. What do you notice? Cf. Eq. 6.3.33 of Wald.

b) Recall that in a Schwarzschild spacetime we have: *stable* circular orbits for massive particles at $r > 6M$; *unstable* circular orbits for massive particles at $r > 3GM$; one *unstable* circular orbit for mass*less* particles at $r = 3GM$. What is the minimal theoretical size $a_{\text{min}}$ that the accretion disk can have, expressed in units of $GM$?

c) Find the periastron as a function of the impact parameter: $r_0 = r_0(b)$.

**Mathematica**: define the function `r0[b_]` and plot it for $b \in [3, 10]$.

> Hint: we can let Mathematica solve the equation for us: write `r0[b_] = x/. Solve[··· == 0, x][[#]]` where ··· denote the appropriate equation, which depends on $b$, that `x` must solve to give the periastron $r_0$ as a result. Note that since the equation is nonlinear, Mathematica will give more than one solution and we need to pick the right one, corresponding to the solution number # in the above code.

d) If the periastron of a given geodesic is $r_0 > a$ then the geodesic will never intersect the ring. Let us find to which critical value $b_{\text{crit}}$ of the impact parameter this corresponds to.

**Mathematica**: define the function `bcrit[a_]`.

e) The shape of the geodesic is given by the function $r(\psi)$. From the equations of motion above it is easiest to compute $\psi(r)$, or in fact $\psi(u)$, instead. The following expression computes the deflection angle between infinity and the position $r = 1/u$:[4]

$$\frac{d\psi}{du} = \frac{1}{\sqrt{U_{eff}(u)}} \implies \psi(0 \to u) = \int_0^u \frac{1}{\sqrt{U_{eff}(u')}} du' \tag{4}$$

[3] In a more standard notation: $E^2 U_{\text{eff}} = E^2 - 2V_{\text{eff}}$.

[4] In the notation $\psi(u_1 \to u_2)$ the quantities $u_1 \leq u_2$ denote the extrema of integration, and are not related to the direction of travel of the geodesic.

What is the relationship between this expression and question c) from the previous section?

**Mathematica**: Do a Parametric plot of the shape of the geodesic on the plane $\chi = \pi/2$; use the function `psiofu`$[u\_, b\_]$ and do the plot for the values $\mathtt{b} = 4, 3\sqrt{3}, 7$. What do you notice for the trajectory associated to $\mathtt{b} = 3\sqrt{3}$?

f) We are interested in geodesics originating from a precise location $(r, \psi) = (a, \psi_\phi)$ on the grey plane (recall that $\phi$ will eventually label our geodesics, see drawing). From question c) of the previous section, we recall that (if any) there are two possible partial deflection angles associated to the data $(a, \psi_\phi)$:

$$\Delta\psi_\phi \equiv \psi_\phi = \begin{cases} \psi(0 \to 1/a) \\ \psi(0 \to u_0) + \psi(1/a \to u_0) \end{cases} \tag{5}$$

Note that in the above equations, one must have $a > r_0$ i.e. $u_0 > 1/a$, otherwise no such geodesics exists. Ultimately, these expressions give us a partial deflection angle $\Delta\psi_\phi$ associated to a given starting radius $a$ and an impact parameter $b = b(\phi)$ (here, $\phi$ is, for now, just the abstract label assigned to a geodesic, see the figure).

**Mathematica**: Define two functions `Psi[b_,a_]` and `Psiplus[b_,a_]` corresponding to the two branches of the partial deflection angle $\Delta\psi_\phi$ as parametrized by the initial radial position and the final impact parameter of the geodesics labelled by $\phi$. Fix $a = 6$ and plot these two branches in a unique graph where $Y = \Delta\psi_\phi$ and $X = b$.

## 4 The observer's frame

So far we have worked in a given geodesic's plane, i.e. in the grey plane of drawing. In the observer's frame each one of these planes is uniquely identified by a value of the azimuthal angle $\phi$. If we choose our polar coordinates on this plane so that $\psi = 0$ corresponds to the $z$-axis, and therefore so that the geodesics we discussed in the previous section asymptote to the $z$-axis, we see that the partial deflection angle $\psi_\phi$ is nothing else than the polar angle $\theta(\phi)$ drawn in the figure:

$$\psi_\phi = \theta(\phi). \tag{6}$$

The angle $\theta(\phi)$ is geometrically determined by rotating the (edge of the) accretion disk from the black-hole to the observer's frame, a rotation solely parametrized by the angle $\theta_0$.

a) Find the function the function $\theta(\phi)$ as parametrized by the rotation angle $\theta_0$.

Hint: start by writing down the parametrized circle on the equatorial plane $a(\cos\phi', \sin\phi', 0)$ and then rotate it by an angle $\theta_0$ around the $x$ axis. Compare the result with the observer's polar coordinates $a(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Solve for $\theta(\phi)$ and $\phi'(\phi)$—even though only the former is needed.

**Mathematica**: Use the analytical result for $\theta(\phi)$ to define the function `Theta[φ_,θ0_]`.

In defining this function, you will have to pay attention to (*i*) special values of $\phi$ where the formula is not defined, (*ii*) different branches of the function arctan which have to be patched together to obtain a continuous function. Plot the function `Theta[φ,θ0]`, e.g. at $\theta_0 = \frac{\pi}{3}$, to make sure everything is in order.

## 5 Drawing the observed disk

We henceforth fix

$$a = 6GM, \tag{7}$$

which is the innermost stable circular orbit and therefore the inner radius of any accretion disk.

Summarizing what we have achieved so far: (*i*) Each geodesic reaching the observer from the edge of the accretion disk is uniquely identified by the *observer*'s azimuthal angle of view $\phi$. (*ii*) In Section 1, we used the geodesic equation to find the partial deflection angle $\Delta\psi_\phi$ of one such geodesic as a function of the starting radius, which corresponds to the edge of the accretion disk $r = a$, and the final impact parameter $b$; in terms of $b$, this is a multivalued function defined for $b_{\min} < b < b_{crit}$; schematically, we have $\Delta\psi_\phi = \mathtt{Psi}[\mathtt{a\_}, \mathtt{b\_}]$. (*iii*) In Section 2, we observed that the partial deflection angle of a given geodesic reaching the observer is geometrically computable from a simple change of reference frame: it is given by the polar angle $\theta$ of a point on the (edge of the) accretion disk as parametrized by the *observer*'s azimuthal angle of view $\phi$ and the viewing angle $\theta_0$; this gave us the relation $\Delta\psi_\phi = \theta(\phi)$ as well as the function `Theta[φ_,θ0_]`.

Combining these results, we observe that the relation $\Delta\psi_\phi = \theta(\phi)$ in principle allows us to solve for the disk's apparent shape $b(\phi)$ in terms of the parameter $\theta_0$ (recall, we fixed $a = 6GM$). Once we know this function, we are done: we can draw the shape of the disk as perceived by the observer.

To compute the function $b(\phi)$, as parametrized by[5] $\theta_0$, i.e.

$$\mathtt{Impact}[\mathtt{phi\_}, \mathtt{theta0\_}], \tag{8}$$

we need to define the inverse of the partial deflection $\Delta\psi_\phi$ seen as a function of $b(\phi)$. At this point, one must recall (see question 3f) that although (*i*) the map $b(\phi) \mapsto \Delta\psi_\phi$ is not single-valued, (*ii*) the map $\Delta\psi_\phi \mapsto b$ *is* in fact single-valued—thus giving a sense to what we mean by "inverting the partial deflection as a function of $b$". In the Mathematica file, this inversion will be done numerically.

a) Since the partial deflection angle is a multivalued function, with two branches `Psi` and `Psiplus`, the invertion has to be done one branch at a time. Find the value of $b$ at which the two branches separate, call it `bmax` [Note: recall we have fixed $\mathtt{a} = 6$].

[5] Recall, we fixed $a = 6GM$.

b) Follow the next few steps which are provided to you. Their goal is to implement the numerical inversion. Use these steps to define the function `Impact[phi_, theta0_]` and plot it for $\theta_0 = \pi/3$.

c) Finally use `Impact[phi_, theta0_]` to plot the shape of the accretion disk as seen by the observer for different values of $\theta_0$, which progressively approach the value $\pi/2$ corresponding to the accretion plane. Comment the result. [Hint: for the plot to match what the observer actually sees, one needs to flip $\phi \mapsto -\phi$.]

## 6 Acknowledgments

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