Notes for Haldane Psdudopotential on Spherical Geometry

Contents

I. SPHERE GEOMETRY

In the spherical geometry, the two-dimensional sheet containing electrons is wrapped around the surface of a sphere, and a perpendicular magnetic field is generated by placing a Dirac magnetic monopole at the center of the sphere. The spherical geometry is compact, i.e. it does not have edges, which makes it suitable for an investigation of the bulk properties. In particular, filled Landau levels are unambiguously defined. The spherical geometry has been instrumental in establishing the validity of the theory of the FQHE, and provides the cleanest proofs for many properties, which was first introduced by Haldane[1].

A. Landau levels on sphere

We consider a sphere of radius R with a flux $2s_0\phi_0 = 4\pi R^2 B$ extending radially outward through the surface. The flux corresponds to a magnetic field

$$\mathbf{B} = \frac{2s_0\phi_0}{4\pi R^2}\hat{r} \tag{1}$$

which and is produced by the vector potential

$$\mathbf{A} = -\frac{2s_0\phi_0}{4\pi R}\cot\theta\vec{\phi}, \tag{2}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial}{\partial\phi}A_\theta]\vec{r} + \frac{1}{r}[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{\partial}{\partial r}(rA_\phi)]\vec{\theta} + \frac{1}{r}[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r]\vec{\phi} \tag{3}$$

$2s_0$ must be an integer due to Dirac's monopole quantization condition.

The kinetic energy operator is given by

$$H_0 = \frac{\hbar^2}{2mR^2}|\mathbf{\Lambda}|^2, \tag{4}$$

$$\mathbf{\Lambda} = \mathbf{R}\times(-i\nabla + e\mathbf{A}) \tag{5}$$

Next we consider the sphere with a fixed radius R.

$$\mathbf{\Lambda} = \frac{\mathbf{R}}{R}\times[-i(\vec{\theta}\frac{\partial}{\partial\theta}\vec{\phi}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}) - s_0\cot\theta\vec{\phi}] \tag{6}$$

$$= -i(\vec{\phi}\frac{\partial}{\partial\theta} - \vec{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + Q\cot\theta\vec{\theta}) \tag{7}$$

where we used $\vec{r}\times\vec{\theta}=\vec{\phi}, \vec{r}\times\vec{\phi}=-\vec{\theta}$, and $\nabla=\vec{r}\frac{\partial}{\partial r}+\vec{\theta}\frac{\partial}{r\partial\theta}+\vec{\phi}\frac{\partial}{r\sin\theta\partial\phi}$.

And

$$|\mathbf{\Lambda}|^2 \overset{!}{=} -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+(s_0\cot\theta+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi})^2 \tag{8}$$

To fully exploit the spherical symmetry of the problem, it is useful to introduce an angular momentum operator. It may be verified that

$$[\Lambda_i,\Lambda_j]=i\varepsilon_{ijk}(\Lambda_k-s_0\vec{r}_k), \tag{9}$$
$$[\Lambda_i,\Omega_j]=i\varepsilon_{ijk}\vec{r}_k \tag{10}$$

Here, the subscripts i,j,k stand for x,y,z of the Cartesian coordinate system. These relations make it straightforward to see that the operator

$$\mathbf{L}=\mathbf{\Lambda}+s_0\vec{r} \tag{11}$$

satisfies the angular momentum algebra

$$[L_i,L_j]=i\varepsilon_{ijk}L_k \tag{12}$$

The raising and lowering operators are defined as

$$L_\pm=L_x+\pm iL_y, [L_z,L_\pm]=\pm L_z \tag{13}$$

where explicit expressions can be obtained for the angular momentum operator:

$$L_z=-i\frac{\partial}{\partial\phi}, L_\pm=e^{i\pm\phi}[\pm\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}+\frac{s_0}{\sin\theta}] \tag{14}$$

Noting that $\mathbf{\Lambda}\cdot\mathbf{\Omega}=\mathbf{\Omega}\cdot\mathbf{\Lambda}=0$, we have

$$|\mathbf{\Lambda}|^2=|\mathbf{L}-s_0\vec{r}|^2=L^2-s_0^2 \tag{15}$$

The Hamiltonian, therefore, commutes with the angular momentum operators. Because $[L^2,L_z]=0$, we choose eigenfunctions that simultaneously diagonalize $H,\hat{L}^2,\hat{L}_z$. These eigenfunctions are called "monopole harmonics", denoted by $Y_{l,m}^{(2s_0)}$.

$$\hat{L}^2Y_{l,m}^{(s_0)}=l(l+1)Y_{l,m}^{(2s_0)}, \tag{16}$$
$$\hat{L}_zY_{l,m}^{(s_0)}=mY_{l,m}^{(2s_0)} \tag{17}$$

The eigenvalues of $|\Lambda|^2$ are $l(l+1)-s_0^2$, giving the energy eigenvalues

$$E_{s_0,l,m}=\frac{\hbar^2}{2mR^2}[l(l+1)-s_0^2]=\hbar\omega_c\frac{l(l+1)-s_0^2}{2|s_0|} \tag{18}$$
$$=\hbar\omega_c[n+\frac{1}{2}+\frac{n(n+1)}{2|s_0|}] \tag{19}$$

where we used $2s_0\phi_0=4\pi R^2B$, and $l=s_0+n$ (n hence labels the Landau levels).

Next we proceed to obtain the explicit expression for the single particle eigenstate, $Y_{l,m}^{(s_0)}$. A complete, orthogonal basis of the states spanning the lowest Landau level ($n=0, l=s_0$) is given by

$$\psi_m^l(u,v)=u^{l+m}v^{l-m}, \tag{20}$$
$$u=\cos\frac{\theta}{2}e^{i\phi/2}, v=\sin\frac{\theta}{2}e^{-i\phi/2} \tag{21}$$

with $m=-l,-l+1,...,l$. And we obtain the monopole harmonics functions

$$Y_{l,m}^{(s_0)} \overset{!}{=} [\frac{(2l+1)!}{4\pi(l-m)!(l+m)!}]^{1/2}v^{l-m}u^{l+m} \tag{22}$$

B. Two-body interaction

We need the Hamiltonian like this,

$$H = \frac{1}{2} \sum_{m_1,m_2,m_3,m_4=-s}^{s} a^\dagger_{m_1,n} a^\dagger_{m_2,n} a_{m_3,n} a_{m_4,n} \delta_{m_1+m_2,m_3+m_4} \langle s, m_1; s, m_2 | V | s, m_3; s, m_4 \rangle \tag{23}$$

where the n is Landau level index and $s = s_0 + n$ is orbital momentum (s_0 is the magnetic monopole placed in the center of sphere). Next we consider the lowest Landau level with $n = 0$ case. The matrix element is

$$\langle s, m_1; s, m_2 | V | s, m_3; s, m_4 \rangle = \int d\Omega_1 \int d\Omega_2 \overline{Y}^{(s_0)}_{s,m_1}(\mathbf{r}_1) \overline{Y}^{(s_0)}_{s,m_2}(\mathbf{r}_2) V(\mathbf{r}_1, \mathbf{r}_2) Y^{(s_0)}_{s,m_3}(\mathbf{r}_2) Y^{(s_0)}_{s,m_4}(\mathbf{r}_1) \tag{24}$$

where $Y^{(s_0)}_{s,m}(\mathbf{r})$ is monopole harmonics functions.

If the potential V is a function of $|\mathbf{r}_1 - \mathbf{r}_2|$ (as is the case for the Coulomb potential), it can expanded in Legendre polynomials,

$$V(|\mathbf{r}_1 - \mathbf{r}_2|) = \sum_{k=0}^{\infty} U_k(\mathbf{r}_1, \mathbf{r}_2) P_k(\cos\theta_{12}) \tag{25}$$

and

$$U_k = \frac{1}{2} \int_0^\pi d\theta V(r_{12}) P_k(\cos\theta) \sin\theta. \tag{26}$$

So the interaction can be rewritten in terms of a new set of parameters, U_k. The U_k are unitless coefficients that defines the potential. Let us show several examples here:

- For coulomb potential $V(r) = \frac{1}{r}$, we define the chord distance between two points on a sphere is given by

$$V(r) = V(|\mathbf{r}_1 - \mathbf{r}_2|) = \frac{1}{2R|\sin\frac{\theta_1-\theta_2}{2}|} = \frac{1}{R\sqrt{2 - 2\cos\theta_{12}}} = \frac{1}{R}\sum_n P_n(\cos\theta_{12}) \tag{27}$$

  , where we need the formula for Legendre polynomials $P_n(x)$:

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0} P_n(x) t^n. \tag{28}$$

  So we have $U_k = 1$.

- For short-ranged potential $V(r) = \delta(\mathbf{r})$, we can use the expansion

$$\delta(\Omega_a - \Omega_b) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{l,m}(\Omega_a) Y_{l,m}(\Omega_b) = \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta_{ab}) \tag{29}$$

  , where we used $P_l(\cos\theta_{ab}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\Omega_a) Y_{lm}(\Omega_b)$ (Y_{lm} is sphere harmonics function). Thus we have $U_k = 2k + 1$ for short-ranged potentials.

- For short-ranged potential $V(r) = \nabla^2\delta(\mathbf{r})$, we use the expansion

$$\nabla^2_a \delta(\Omega_a - \Omega_b) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \nabla^2_a Y^*_{l,m}(\Omega_a) Y_{l,m}(\Omega_b) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-l(l+1)) Y^*_{l,m}(\Omega_a) Y_{l,m}(\Omega_b) = \sum_{l=0}^{\infty} (-l(l+1))(2l+1) P_l(\cos\theta \tag{30}$$

  , where we used $P_l(\cos\theta_{ab}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\Omega_a) Y_{lm}(\Omega_b)$ (Y_{lm} is sphere harmonics function). Thus we have $U_k = -k(k+1)(2k+1)$ for short-ranged potentials.

Next we insert the potential form Eq. 25 into the matrix element, we have

$$\begin{aligned}
\langle s,m_1;s,m_2|V|s,m_3;s,m_4\rangle &= \int d\Omega_1 \int d\Omega_2 \overline{Y}^{(s_0)}_{s,m_1}(\mathbf{r}_1)\overline{Y}^{(s_0)}_{s,m_2}(\mathbf{r}_2)V(\mathbf{r}_1,\mathbf{r}_2)Y^{(s_0)}_{s,m_3}(\mathbf{r}_2)Y^{(s_0)}_{s,m_4}(\mathbf{r}_1)\\
&= \int d\Omega_1 \int d\Omega_2 \overline{Y}^{(s_0)}_{s,m_1}(\Omega_1)\overline{Y}^{(s_0)}_{s,m_2}(\Omega_2)[\sum_k U_k \frac{4\pi}{2k+1}\sum_{m=-k}^{k} Y^*_{km}(\Omega_1)Y_{km}(\Omega_2)]Y^{(s_0)}_{s,m_3}(\Omega_2)Y^{(s_0)}_{s,m_4}(\Omega_1)\\
&= \sum_k U_k \frac{4\pi}{2k+1}\sum_{m=-k}^{k} \times \int d\Omega_1 \overline{Y}^{(s_0)}_{s,m_1}(\Omega_1)\overline{Y}_{km}(\Omega_1)Y^{(s_0)}_{s,m_4}(\Omega_1)\int d\Omega_2 \overline{Y}^{(s_0)}_{s,m_2}(\Omega_2)Y_{km}(\Omega_2)Y^{(s_0)}_{s,m_3}(\Omega_2) \qquad (31)
\end{aligned}$$

To deal with this integral, we explicitly substitute the monopole harmonic with $s_0 = 0$ for the spherical harmonics: $Y_{lm} = Y^{s_0=0}_{lm}$. Then we use the result from Ref.[4]:

$$\int d\Omega Y^{(Q_1)}_{s_1,m_1}(\Omega)Y^{(Q_2)}_{s_2,m_2}(\Omega)Y^{(Q_3)}_{s_3,m_3}(\Omega) = (-)^{s_1+s_2+s_3}[\frac{(2s_1+1)(2s_2+1)(2s_3+1)}{4\pi}]^{1/2}\begin{pmatrix} s_1 & s_2 & s_3\\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} s_1 & s_2 & s_3\\ Q_1 & Q_2 & Q_3\end{pmatrix} \qquad (32)$$

where the round brackets are 3j symbols, and it is nonzero only if $Q_1 + Q_2 + Q_3 = 0$ and $m_1 + m_2 + m_3 = 0$. And the relation is also need: $\overline{Y}^{s_0}_{l,m} = (-)^{s_0+m}Y^{-s_0}_{l,-m}$.

Under this useful integrals, we can integrate over the angular coordinates $\Omega_{1,2}$, and we reach

$$\begin{aligned}
H &= \frac{1}{2}\sum_{m_1,m_2,m_3,m_4=-s}^{s} a^\dagger_{m_1,n}a^\dagger_{m_2,n}a_{m_3,n}a_{m_4,n}\delta_{m_1+m_2,m_3+m_4}\langle s,m_1;s,m_2|V|s,m_3;s,m_4\rangle\\
\langle s,m_1;s,m_2|V|s,m_3;s,m_4\rangle &= \sum_{k=0}^{k_{max}} U_k(-)^{2s_0+m_2+m_4+4s+2k}[(2s+1)]^2\times\\
&\begin{pmatrix} s & k & s\\ -m_1 & m_1-m_4 & m_4\end{pmatrix}\begin{pmatrix} s & k & s\\ -m_2 & m_2-m_3 & m_3\end{pmatrix}\begin{pmatrix} s & k & s\\ -s_0 & 0 & s_0\end{pmatrix}\begin{pmatrix} s & k & s\\ -s_0 & 0 & s_0\end{pmatrix} \qquad (33)
\end{aligned}$$

Here, for a general Wigner 3j coefficient, $\begin{pmatrix} s_1 & s_2 & s_3\\ m_1 & m_2 & m_3\end{pmatrix}$, it is non-zero only when $m_1 + m_2 + m_3 = 0$ and when s_1, s_2, s_3 together satisfy the triangle inequality, $|s_1 - s_2| \le s_3 \le s_1 + s_2$. As such, the sum over m collapses, and $m = m_1 - m_3 = m_4 - m_2$.

However, this result is not shown in the literature. A more popular form is to express this formula in a *pair pseudopotentials*, which is widely used in the community of quantum Hall effect.

C. Haldane's derivation

The pair pseudopotential $V^n(l)$ is defined as the interaction energy of a pair of electrons as a function of their pair angular momentum l. Although the pseudopotential is a function only of the pair angular momentum, $V^n(l)$ actually contains all of the correlative behaviors of any many-body system, and can be used in the place of the two-body matrix elements calculated above to perform the same calculations. The pseudopotential is defined on the Haldane sphere by using standard angular momentum coupling to expand the monopole harmonics into a coupled basis,

$$|s,m_1;s,m_2\rangle = \sum_l |s,s;l,m_1+m_2\rangle\langle s,s;l,m_1+m_2|s,m_1;s,m_2\rangle$$

This is actually a momentum coupling transformation, and the coefficient $\langle s,s;l,m_1+m_2|s,m_1;s,m_2\rangle$ is the ordinary Clebsch-Gordan coefficient.

If we expand both the initial and final state vectors in the coupled angular momentum basis, we can rewrite the two-body matrix element in the following form:

$$\begin{aligned}\langle s,m_1;s,m_2|V(r)|s,m_3;s,m_4\rangle &= \sum_{l,l'}\langle s,m_1;s,m_2|s,s;l,m_1+m_2\rangle\langle s,s;l',m_3+m_4|s,m_3;s,m_4\rangle \times \\ &\quad \langle s,s;l,m_1+m_2|V(\mathbf{r}_1-\mathbf{r}_2)|s,s;l',m_3+m_4\rangle\end{aligned}$$

In this expression, the pseudopotential for particles in a single Landau level is evaluated from the matrix element from

$$\langle s,s;l,m|V(r_1-r_2)|s,s;l',m'\rangle\delta_{m=m_1+m_2}\delta_{m'=m_3+m_4}$$

For particles in the n th Landau level, that is, when $s=s_0+n$ matrix element gives the pair pseudopotential.

In order to reduce the following expression slightly, we need lots of integral relations and results. We omit these tedious process, and reach the finial result:

$$\langle s,s;l,m|V(|r_1-r_2|)|s,s;l',m'\rangle = V_s^{n=0}(l)\delta_{l,l'} = \delta_{l,l'}\sum_{k=0}^{2s}U_k(-)^{2s_0+l}(2s+1)^2\begin{Bmatrix} l & s & s\\ k & s & s\end{Bmatrix}\begin{pmatrix} s & k & s\\ -s_0 & 0 & s_0\end{pmatrix}^2 \tag{34}$$

where $s=s_0+n$ is the shell angular momentum of the n-th Laudan level, l is the relative angular momentum between two particles. Here, $\begin{pmatrix} s & k & s\\ -s_0 & 0 & s_0\end{pmatrix}$ is Wigner 3j coefficient and $\begin{Bmatrix} l & s & s\\ k & s & s\end{Bmatrix}$ is Wigner 6j coefficient. Please note that the expression of $V_s^n(l)$ does not depend on the relative angular momentum m,m'.

To sum up, the hamiltonian written by pseudopotential will be

$$\begin{aligned}H &= \frac{1}{2}\sum_{m_1,m_2,m_3,m_4=-s}^{s} a^\dagger_{m_1,n}a^\dagger_{m_2,n}a_{m_3,n}a_{m_4,n}\delta_{m_1+m_2,m_3+m_4}\langle s,m_1;s,m_2|V|s,m_3;s,m_4\rangle\\ &\langle s,m_1;s,m_2|V|s,m_3;s,m_4\rangle =\\ &\sum_{l,l'}\langle s,m_1;s,m_2|s,s;l,m_1+m_2\rangle\langle s,s;l',m_3+m_4|s,m_3;s,m_4\rangle\times\langle s,s;l,m_1+m_2|V(\mathbf{r}_1-\mathbf{r}_2)|s,s;l',m_3+m_4\rangle\\ &=\sum_l V_l\sqrt{2l+1}(-1)^{m_1+m_2}\begin{pmatrix} s & s & l\\ m_1 & m_2 & -m_1-m_2\end{pmatrix}\sqrt{2l+1}(-1)^{m_3+m_4}\begin{pmatrix} s & s & l\\ m_3 & m_4 & -m_3-m_4\end{pmatrix}\\ &=\sum_l V_l(2l+1)\begin{pmatrix} s & s & l\\ m_1 & m_2 & -m_1-m_2\end{pmatrix}\begin{pmatrix} s & s & l\\ m_3 & m_4 & -m_3-m_4\end{pmatrix}\end{aligned} \tag{35}$$

Let us calculate the pseudopotential using Eq. 34.

- For short-ranged interaction $V(r)=\delta(\mathbf{r})$, $U_k=2k+1$, $V_s(l)=\begin{cases}\frac{(2s+1)^2}{(4s+1)}, l=2s\\ 0, l\neq 2s\end{cases}$

- For short-ranged interaction $V(r) = \nabla^2\delta(\mathbf{r})$, $U_k = -k(k+1)(2k+1)$, $V_s(l) = \begin{cases}-\frac{(1+s)^2(2s+1)^4((2s)!)^4}{s(4s+1)(2(s+1)!)^2((2s-1)!)^2}<0, l=2s\\ \frac{4s(1+s)^2(2s+1)^4((2s)!)^4}{(4s-1)(2(s+1)!)^2}>0, l=2s-1\\ 0, others\end{cases}$

- For coulomb interaction $V(r) = 1/r$, $U_k = 1$, $V_s^n(l) = \frac{2^{-2-4s}(2s+1)(2s+1)!((8s-1)/2)!}{((4s-1)/2)!(((4s+1)/2)!)^2}$

For coulomb interaction $V(r) = 1/r$, the above equation has a analytical form, first derived by Fano, for the lowest Landau level, $n = 0$:

$$V_s^{n=0}(l) = \frac{\begin{pmatrix} 4s_0 - 2l \\ 2s_0 - l \end{pmatrix} \begin{pmatrix} 4s_0 + 2l + 2 \\ 2s_0 + l + 1 \end{pmatrix}}{\begin{pmatrix} 4s_0 + 2 \\ 2s_0 + 1 \end{pmatrix}^2} \tag{36}$$

, which is widely used for calculation in first Laudau level.

Haldane defines the rotational invariant the interaction as

$$H = \sum_{i<j} \sum_{L} V_{L,2} \; P_{ij}^{L} \tag{37}$$

where

$$P_{ij}^L = |L; i, j\rangle \langle L; i, j| \tag{38}$$

is a projection operator that projects particles i and j to a state of relative angular momentum L (within the lowest Landau level). In Eq. 37,

$$\begin{aligned} V_{L,2} &= \langle L; i, j|V(\vec{r}_i - \vec{r}_j)|L; i, j\rangle & (39) \\ &= \langle L|V|L\rangle & (40) \end{aligned}$$

is known as the pseudopotential coefficient, and we have added the subscript 2 here to indicate that this is a two-body interaction. In Eq. 40 we have written this matrix element in a convenient shorthand, since by symmetry between particles, $V_{L,2}$ is independent of which i and j is chosen.

In the discussion of pseudopotential, the disk geometry is usually used (which is also rotational symmetric). In this geometry, one can choose symmetric Landau gauge:

$$\vec{A} = (By, -Bx, 0) \tag{41}$$

Thus the eigenstates are:

$$\phi_{n,m} = \frac{(a^\dagger)^n}{\sqrt{n!}} \frac{(b^\dagger)^m}{\sqrt{m!}} |0> \tag{42}$$

where $\hat{a^\dagger}$ is the raising operator for Landau index and $\hat{b^\dagger}$ is the raising operator for orbit index. Specially, for the lowest Landau level, we have

$$\psi_{n=0,m} = C_m z^m e^{-\frac{|z|^2}{4}} \tag{43}$$

where we set $l_0 = 1$. C_m is the normalized factor.

Here we consider the state of two electrons in a magnetic field. Without the Coulomb interaction the wave function of relative motion is:

$$\psi_{n=0,M,m}(Z, z) = \frac{Z^M e^{-\frac{|Z|^2}{2l^2}} z^m e^{-\frac{|z|^2}{8l^2}}}{2\pi l^2 \sqrt{2^{M+m}} M! m!} \tag{44}$$

where $Z = \frac{z_1+z_2}{2}$ and $z = z_1 - z_2$ and m is relative momentum.

Using this two-body interaction, we can define the Haldane pseudopotential:

$$V_m^{(n)} = \frac{< n, m|V(r)|n, m >}{< n, m|n, m >} \tag{45}$$

A straight forward calculation leads to the following form of pseudopotential:

$$V_m^{(n)} = \int_0^\infty dqq[L_n(\frac{q^2}{2})]^2 L_m(q^2) e^{-q^2} V(q) \tag{46}$$

where n is Landau level index. In order to get this form, we need go to the Fourier form of $V(r) = \int qdqd\theta_q V(q) e^{i\mathbf{q}\cdot\mathbf{r}}$, so we have

$$V_m^{(n)} = \frac{< n,m|V(r)|n,m >}{< n,m|n,m >} = \frac{\int qdqV(q) \int d\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} < n,m|r >< r|n,m >}{\int d\mathbf{r} < n,m|r >< r|n,m >} \tag{47}$$

With the help of

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta e^{iqrcos\theta} = J_0(qr) \tag{48}$$

where J_0 is Bessel function, we obtain (we set $n = 0$ here),

$$V_m^{(0)} = \int qdqV(q) \frac{\int rdrr^{2m} e^{-\frac{r^2}{4}} J_0(qr)}{\int rdrr^{2m} e^{-\frac{r^2}{4}}} \tag{49}$$

where the $\theta \equiv \theta_q - \theta_r$ has been integral.

Next we need a integral formula, which can be find on website of *NIST Digital Library of Mathematical Functions* or other integral books:

$$\int_0^\infty J_\nu(bt) e^{-p^2t^2} t^{\mu-1} dt = \frac{(\frac{b}{2p})^\nu \Gamma(\frac{\mu+\nu}{2})}{2p^\mu} e^{-\frac{b^2}{4p^2}} \ {}_1F_1(\frac{\nu}{2} - \frac{\mu}{2} + 1; \nu + 1; \frac{b^2}{4p^2}) \tag{50}$$

where ${}_1F_1(a, b, z)$ is confluent super-function.

$$\int_0^\infty \alpha x^m e^{-\beta x^n} dx = \frac{\alpha}{n\beta^{\frac{m+1}{n}}} \Gamma(\frac{m+1}{n}) \tag{51}$$

We obtain

$$V_m^{(0)} = \int qdqV(q) \frac{\frac{\Gamma(m+1)}{2(1/2)^{2m+2}} e^{-q^2} \ {}_1F_1(-m; 1; q^2)}{\frac{\Gamma(m+1)}{2(1/2)^{2m+2}}} = \int qdqV(q) e^{-q^2} L_m(q^2) \tag{52}$$

where we notice the Laguerre poly $L_m(z) \equiv {}_1F_1(-m; 1; z)$.

Here we see the so-called pseudo-potential is actually the expansion defined by Laguerre polynomials. Using the orthothogal condition of Laguerre polynomials,

$$2 \int_0^\infty dxxe^{-x^2} L_m(x^2) L_n(x^2) = \delta_{mn}, \tag{53}$$

we can get

$$V(q) = 2 \sum_{m=0}^\infty V_m^{(0)} L_m(q^2) \tag{54}$$

$$V_m^{(0)} = \int qdqV(q) e^{-q^2} L_m(q^2) \tag{55}$$

Thus, we know that, the so-called "V_0" interaction relates to $V(q) = V_0^0$, a constant independent of q. The so-called "V_1" interaction relates to $V(q) = V_1^0 L_1(q^2) = V_1^0(1 - q^2)$.

Going back to real space, we know

$$1 = e^0 = \int d\mathbf{r}\delta(\mathbf{r})e^{i\mathbf{qr}} \tag{56}$$

Thus, writting

$$\delta(r) = \frac{1}{4\pi^2}\int d\mathbf{q}e^{i\mathbf{qr}} \tag{57}$$

and

$$\nabla^2\delta(\mathbf{r}) = \frac{1}{4\pi^2}\int d\mathbf{q} - q^2 e^{i\mathbf{qr}} \tag{58}$$

So we have

$$V_1(1 - q^2) \rightarrow V_1(1 + \nabla^2)\delta(\mathbf{r}) \tag{59}$$

II. DENSITY OPERATOR

In the case of lowest Landau level ($s = s_0, n = 0$), we define the annihilation operator $\hat{\psi}(\theta,\varphi)$ on the projected Landau level as

$$\hat{\psi}(\theta,\varphi) = \sum_{m=-s}^{s} \overline{Y}^{(s_0)}_{s,m}\hat{c}_m. \tag{60}$$

$\hat{c}_m$ stands for the annihilation operator of Landau orbital m, and it is independent of coordinates (θ,φ). Monopole harmonics $Y^{(s_0)}_{l,m}$ see Eq. 22.

The density operator $\hat{n}(\theta,\varphi) = \hat{\psi}^\dagger\hat{\psi}$ can be written as,

$$\hat{n}(\theta,\varphi) = \sum_{m_1,m_2} Y^{(s_0)}_{s,m_1}\overline{Y}^{(s_0)}_{s,m_2} c^\dagger_{m_1} c_{m_2}. \tag{61}$$

Next we write the Hamiltonian in terms of density operator $n_{l,m}$ in the angular momentum space, defined as,

$$n(\theta,\varphi) = \sum_{m_1,m_2=-s}^{s} \overline{Y}^{s_0}_{s,m_2} Y^{s_0}_{s,m_1} c^\dagger_{m_1} c_{m_2} = \sum_{l,m} n_{l,m} Y_{l,m}(\theta,\varphi). \tag{62}$$

Here $Y_{l,m}(\theta,\varphi) = Y^{(0)}_{l,m}$ is the spherical harmonics, with $m = -l, -l+1, \cdots, l$ and $l \in \mathbf{Z}$. $n_{l,m}$ can be obtained using spherical harmonic transformation,

$$\begin{aligned} n_{l,m} &= \int d\Omega\, \bar{Y}_{l,m}(\theta,\varphi) n(\theta,\varphi) \\ &= (2s+1)\sqrt{\frac{2l+1}{4\pi}} \sum_{m_1=-s}^{s} (-1)^{3s+m_1+l} \begin{pmatrix} s & l & s \\ m-m_1 & -m & m_1 \end{pmatrix} \begin{pmatrix} s & l & s \\ -s & 0 & s \end{pmatrix} c^\dagger_{m_1} c_{m_1-m} \\ &\overset{!}{=} (2s+1)\sqrt{\frac{2l+1}{4\pi}} \sum_{m_1=-s}^{s} (-1)^{3s+m_1+l}(-1)^{4s+2l} \begin{pmatrix} s & l & s \\ -m_1 & m & m_1-m \end{pmatrix} \begin{pmatrix} s & l & s \\ -s & 0 & s \end{pmatrix} c^\dagger_{m_1} c_{m_1-m}, \end{aligned} \tag{63}$$

where we used the integral in Eq. 32. Here $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner 3j-Symbol. To have the term $\begin{pmatrix} s & l & s \\ -m_1 & m & m_1-m \end{pmatrix}$ non-vanishing, we should have $l \leq 2s$. One can show that,

$$n^\dagger_{l,m} = (-1)^m n_{l,-m} \tag{64}$$

Any interaction can be straightforwardly (could be tedious although) written in the second quantized form using Landau orbital operators $c^\dagger_m, c_m$. For example, the density-density interaction $H_I = \int d^2\vec{r}_a d^2\vec{r}_b\, n(\vec{r}_a)U(\vec{r}_a - \vec{r}_b)n(\vec{r}_b)$ can be written as,

$$\begin{aligned} H_I &= \int d\Omega_a d\Omega_b n(\theta_a,\varphi_a) U(\theta_a,\varphi_a,\theta_b,\varphi_b) n(\theta_b,\varphi_b) \\ &= \sum_{m_1,m_2,m_3,m_4} V_{m_1,m_2,m_3,m_4}\, c^\dagger_{m_1} c_{m_4} c^\dagger_{m_2} c_{m_3}. \end{aligned} \tag{65}$$

Here V_{m_1,m_2,m_3,m_4} is

$$V_{m_1,m_2,m_3,m_4} = \int d\Omega_a d\Omega_b U(\theta_a,\varphi_a,\theta_b,\varphi_b) Y^{s_0}_{s,m_1}(\theta_a,\varphi_a)\overline{Y}^{s_0}_{s,m_4}(\theta_a,\varphi_a) Y^{s_0}_{s,m_2}(\theta_b,\varphi_b)\overline{Y}^{s_0}_{s,m_3}(\theta_b,\varphi_b) \tag{66}$$

which takes the form of Eq. 31.

Alternatively, the interaction can be expressed in a different way using density operators. Any rotationally invariant interaction $U(\theta_{ab}) = U(\vec{r}_a - \vec{r}_b)$ can be expanded using the Legendre polynomials (see Eq. 25),

$$\begin{aligned} U(\theta_{ab}) &= \sum_{l=0}^{l_{max}} U_l P_l(\cos\theta_{ab}) \\ &= \sum_{l=0}^{l_{max}} U_l \frac{4\pi}{2l+1} \sum_{m=-l}^{l} \bar{Y}^m_l(\Omega_a) Y^m_l(\Omega_b), \end{aligned} \tag{67}$$

The Legendre polynomial can be further written as

$$P_l(\cos\theta_{ab}) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}\bar{Y}_{l,m}(\Omega_a)Y_{l,m}(\Omega_b) \tag{68}$$

The interaction can be then written as

$$\begin{aligned}
H_I &= \int d\Omega_a d\Omega_b n(\theta_a,\varphi_a)U(\theta_{ab})n(\theta_b,\varphi_b) \\
&= \int d\Omega_a d\Omega_b \sum_{l_a,m_a,l_b,m_b} n_{l_a,m_a}Y_{l_a,m_a}(\Omega_a)\left[\sum_{l=0}^{l_{max}} U_l\frac{4\pi}{2l+1}\sum_{m=-l}^{l}\bar{Y}_{l,m}(\Omega_a)Y_{l,m}(\Omega_b)\right] n_{l_b,m_b}Y_{l_b,m_b}(\Omega_b) \\
&= \sum_{l=0}^{l_{max}} U_l\frac{4\pi}{2l+1}\sum_{m=-l}^{l}\sum_{l_a,m_a,l_b,m_b} n_{l_a,m_a}n_{l_b,m_b}\left(\int d\Omega_a Y_{l_a,m_a}(\Omega_a)\bar{Y}_{l,m}(\Omega_a)\right)\left(\int d\Omega_b Y_{l_b,m_b}(\Omega_b)Y_{l,m}(\Omega_b)\right) \\
&= \sum_{l=0}^{l_{max}} U_l\frac{4\pi}{2l+1}\sum_{m=-l}^{l}\sum_{l_a,m_a,l_b,m_b} n_{l_a,m_a}n_{l_b,m_b}\delta_{l,l_a}\delta_{m,m_a}(-1)^m\delta_{l,l_b}\delta_{-m,m_b} \\
&= \sum_{l=0}^{2s} U_l\frac{4\pi}{2l+1}\sum_{m=-l}^{l}(-1)^m n_{l,m}n_{l,-m} \\
&= \sum_{l=0}^{2s} U_l\frac{4\pi}{2l+1}\sum_{m=-l}^{l} n^{\dagger}_{l,m}n_{l,m}
\end{aligned} \tag{69}$$

Therefore, the interaction Hamiltonian can be written as

$$H_I = \int d\Omega_a d\Omega_b n(\theta_a,\varphi_a)U(\theta_{ab})n(\theta_b,\varphi_b) = \sum_{l=0}^{2s} U_l\frac{4\pi}{2l+1}\sum_{m=-l}^{l} n^{\dagger}_{l,m}n_{l,m} \tag{70}$$

U_l can be obtained by inversely solving Eq. 34:

$$U_l = \frac{(-1)^{2s_0}}{(2s+1)^2}\begin{pmatrix} s & l & s \\ -s_0 & 0 & s_0 \end{pmatrix}^{-2}\sum_{l'=0}^{2s}\left[V_s(l')(-1)^{-l'}(2l'+1)(2l+1)\begin{Bmatrix} s & s & l \\ s & s & l' \end{Bmatrix}\right] \tag{71}$$

III. RELATION BETWEEN EQ. 35 AND EQ. 33

Here we try to bridge two different expressions Eq. 35 and Eq. 33

$$
\begin{aligned}
&\langle s,m_1;s,m_2|V(r)|s,m_3;s,m_4\rangle = \\
&\sum_{l,l'}\langle s,m_1;s,m_2|s,s;l,m_1+m_2\rangle\langle s,s;l',m_3+m_4|s,m_3;s,m_4\rangle \times \langle s,s;l,m_1+m_2|V(\mathbf{r}_1-\mathbf{r}_2)|s,s;l',m_3+m_4\rangle \\
&= \sum_l V_l\sqrt{2l+1}(-1)^{-m_1-m_2}\begin{pmatrix} s & s & l \\ m_1 & m_2 & -m_1-m_2\end{pmatrix}\sqrt{2l+1}(-1)^{-m_3-m_4}\begin{pmatrix} s & s & l \\ m_3 & m_4 & -m_3-m_4\end{pmatrix} \\
&= \sum_l V_l(2l+1)\begin{pmatrix} s & s & l \\ m_1 & m_2 & -m_1-m_2\end{pmatrix}\begin{pmatrix} s & s & l \\ m_3 & m_4 & -m_3-m_4\end{pmatrix}
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
&\langle s,m_1;s,m_2|V|s,m_3;s,m_4\rangle = (2s+1)^2\sum_{k=0}^{k_{max}} U_k(-)^{2s+m_2+m_4}\times \\
&\begin{pmatrix} s & k & s \\ -m_1 & m_1-m_4 & m_4\end{pmatrix}\begin{pmatrix} s & k & s \\ -m_2 & m_2-m_3 & m_3\end{pmatrix}\begin{pmatrix} s & k & s \\ -s & 0 & s\end{pmatrix}\begin{pmatrix} s & k & s \\ -s & 0 & s\end{pmatrix}
\end{aligned} \tag{73}
$$

Next we apply the condition of (if j_1, j_2, j_3 satisfy the triangle condition)

$$
(2j_3+1)\sum_{m_1,m_2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3'\end{pmatrix} = \delta_{j3,j3'}\delta_{m_3,m_3'} \tag{74}
$$

to the above to Equations, and get

$$
\begin{aligned}
&\sum_l V_l\frac{1}{2l+1}\delta_{l,l'}\delta_{l,l''}\delta_{m'=-m_1-m_2}\delta_{m''=-m_3-m_4} = \sum_{k=0}^{k_{max}} U_k(2s+1)^2\begin{pmatrix} s & k & s \\ -s & 0 & s\end{pmatrix}^2\times \\
&\sum_{m_1,m_2,m_3,m_4}(-)^{2s+m_2+m_4}\begin{pmatrix} s & k & s \\ -m_1 & m_1-m_4 & m_4\end{pmatrix}\begin{pmatrix} s & k & s \\ -m_2 & m_2-m_3 & m_3\end{pmatrix}\begin{pmatrix} s & s & l \\ m_1 & m_2 & -m_1-m_2\end{pmatrix}\begin{pmatrix} s & s & l \\ m_3 & m_4 & -m_3-m_4\end{pmatrix} \\
&\sum_{m_1,m_2,m_3,m_4}(-)^{2s+m_2+m_4}\begin{pmatrix} s & k & s \\ -m_1 & m_1-m_4 & m_4\end{pmatrix}\begin{pmatrix} s & k & s \\ -m_2 & m_2-m_3 & m_3\end{pmatrix}\begin{pmatrix} s & s & l \\ m_1 & m_2 & -m_1-m_2\end{pmatrix}\begin{pmatrix} s & s & l \\ m_3 & m_4 & -m_3-m_4\end{pmatrix}
\end{aligned} \tag{75}
$$

$$
\begin{aligned}
&V_l\frac{1}{2l+1} = \sum_{k=0}^{k_{max}} U_k(2s+1)^2(-)^{2s}\begin{pmatrix} s & k & s \\ -s & 0 & s\end{pmatrix}^2\times\sum_{m_1,m_2,m_3,m_4}(-)^{m_2+m_4} \\
&\begin{pmatrix} l & s & s \\ -m_1-m_2 & m_1 & m_2\end{pmatrix}(-1)^{2s+l}\begin{pmatrix} l & s & s \\ m_3+m_4 & -m_3 & -m_4\end{pmatrix}(-1)^{2s+k}\begin{pmatrix} k & s & s \\ m_1-m_4 & -m_1 & m_4\end{pmatrix}\begin{pmatrix} k & s & s \\ m_2-m_3 & m_3 & -m_2\end{pmatrix} \\
&= \sum_{k=0}^{k_{max}} U_k(2s+1)^2(-)^{2s+l+k}\begin{pmatrix} s & k & s \\ -s & 0 & s\end{pmatrix}^2\begin{Bmatrix} l & s & s \\ k & s & s\end{Bmatrix}
\end{aligned} \tag{76}
$$

Here use the formula:

$$
\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6\end{Bmatrix} = \sum_{x_{i=1,..,6}}(-1)^{\sum_{k=1}^{6} j_k-\sum_{k=1}^{6} x_k}\begin{pmatrix} j_1 & j_2 & j_3 \\ -x_1 & -x_2 & -x_3\end{pmatrix}\begin{pmatrix} j_1 & j_5 & j_6 \\ x_1 & -x_5 & x_6\end{pmatrix}\begin{pmatrix} j_4 & j_2 & j_6 \\ x_4 & x_2 & -x_6\end{pmatrix}\begin{pmatrix} j_4 & j_5 & j_3 \\ -x_4 & x_5 & x_3\end{pmatrix} \tag{77}
$$

by setting $j_1 = l, j_4 = k, j_2 = j_3 = j_5 = j_6 = s$ and $x_1 = -m_1-m_2 = -m_3-m_4, x_2 = m_3, x_3 = m_4, x_4 = m_2-m_3 =$

$-m_1+m_4, x_5=-m_1, x_6=m_2$, to do the summation

$$\sum_{m_{1-4}}(-)^{m_2+m_4}\begin{pmatrix} l & s & s \\ -m_1-m_2 & m_1 & m_2 \end{pmatrix}\begin{pmatrix} l & s & s \\ m_3+m_4 & -m_3 & -m_4 \end{pmatrix}\begin{pmatrix} k & s & s \\ m_1-m_4 & -m_1 & m_4 \end{pmatrix}\begin{pmatrix} k & s & s \\ m_2-m_3 & m_3 & -m_2 \end{pmatrix}$$
$$=\sum_{m_{1-6}}(-)^{m_2+m_4}\begin{pmatrix} l & s & s \\ m_5 & m_1 & m_2 \end{pmatrix}\begin{pmatrix} l & s & s \\ m_6 & -m_3 & -m_4 \end{pmatrix}\begin{pmatrix} k & s & s \\ m_1-m_4 & -m_1 & m_4 \end{pmatrix}\begin{pmatrix} k & s & s \\ m_2-m_3 & m_3 & -m_2 \end{pmatrix}$$
$$=\sum_{m_{1-6}}(-)^{2m_2+2m_4-2m_1}(-)^{-(m_4+m_2-2m_1)}\begin{pmatrix} l & s & s \\ m_5 & m_1 & m_2 \end{pmatrix}\begin{pmatrix} l & s & s \\ m_6 & -m_3 & -m_4 \end{pmatrix}\begin{pmatrix} k & s & s \\ m_1-m_4 & -m_1 & m_4 \end{pmatrix}\begin{pmatrix} k & s & s \\ m_2-m_3 & m_3 & -m_2 \end{pmatrix}$$
$$\overset{!}{=}\begin{Bmatrix} l & s & s \\ k & s & s \end{Bmatrix} \tag{78}$$

Similarly, we have

$$U_k=\frac{(-1)^{2s_0+k}(2k+1)^2}{(2s+1)^2}\begin{pmatrix} s & k & s \\ -s & 0 & s \end{pmatrix}^{-2}\sum_{l'=0}^{2s}\left[V_s(l')(-1)^{-l'}(2l'+1)\begin{Bmatrix} s & s & k \\ s & s & l' \end{Bmatrix}\right] \tag{79}$$

IV. RELATION BETWEEN EQ. 35 AND EQ. 65

$$
\begin{aligned}
H_I &= \int d\Omega_a d\Omega_b n(\theta_a,\varphi_a)U(\theta_a,\varphi_a,\theta_b,\varphi_b)n(\theta_b,\varphi_b) \\
&= \sum_{m_1,m_2,m_3,m_4} V_{m_1,m_2,m_3,m_4} c^\dagger_{m_1} c_{m_4} c^\dagger_{m_2} c_{m_3}, \\
V_{m_1,m_2,m_3,m_4} &= \int d\Omega_a d\Omega_b U(\theta_a,\varphi_a,\theta_b,\varphi_b) Y^{s_0}_{s,m_1}(\theta_a,\varphi_a)\overline{Y}^{s_0}_{s,m_4}(\theta_a,\varphi_a) Y^{s_0}_{s,m_2}(\theta_b,\varphi_b)\overline{Y}^{s_0}_{s,m_3}(\theta_b,\varphi_b) \\
&= \sum_l V_l(2l+1)\begin{pmatrix} s & s & l \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix}\begin{pmatrix} s & s & l \\ m_3 & m_4 & -m_3-m_4 \end{pmatrix}
\end{aligned} \tag{80}
$$

where V_l is Haldane pseudopotential.

For the interaction in real-space:

$$
U(\Omega_{ab}) = g_0\delta(\Omega_{ab}) + g_1\nabla^2\delta(\Omega_{ab}), \tag{81}
$$

we can write down that

$$
V_0 = g_0 - g_1, V_1 = g_1 \tag{82}
$$

because

$$
\delta(\Omega_{ab}) \to V_0 = 1, V_{l>0} = 0 \tag{83}
$$

$$
\nabla^2\delta(\Omega_{ab}) \to -V_0 = V_1 = 1, V_{l>1} = 0 \tag{84}
$$

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