

Pierre-Antoine Graham

HOMEWORK 2

Eduardo Martín-Martínez, Bindiya Arora
Quantum Information

Perimeter Institute for Theoretical Physics
March 30, 2024

Contents

1 Back to basics: quantum circuits

In what follows, we evaluate the matrix expressions representing a quantum circuit unitary acting on a sequence of qubit input. We work in the computational basis $\{|0\rangle, |1\rangle\}$ and use the notation X, Y, Z for the Pauli gates in this basis.

- (a) First we consider the conjugation of a CNOT by two CNOT with control and target qubit reversed:

$$\begin{array}{c} 1 \text{ --- } \oplus \text{ --- } \bullet \text{ --- } \oplus \text{ ---} \\ | \\ 2 \text{ --- } \bullet \text{ --- } \oplus \text{ --- } \bullet \text{ ---} \end{array} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which exchanges the qubits ($|00\rangle \rightarrow |00\rangle, |01\rangle \rightarrow |10\rangle, |10\rangle \rightarrow |01\rangle, |11\rangle \rightarrow |11\rangle$) and constitutes a SWAP gate. The matrix expression for the reversed CNOT was obtained by writing its action on the computational basis which reads $|00\rangle \rightarrow |00\rangle, |01\rangle \rightarrow |11\rangle, |10\rangle \rightarrow |10\rangle, |11\rangle \rightarrow |01\rangle$.

- (b) Then we calculate the matrix expression of the entanglement-generating circuit

$$\begin{array}{c} 1 \text{ --- } \oplus \text{ ---} \\ | \\ 2 \text{ --- } [H] \text{ --- } [R_{\pi/4}] \text{ --- } \bullet \text{ ---} \end{array} = (1_1 \otimes |0\rangle \langle 0|_2 + X_1 \otimes |1\rangle \langle 1|_2) R_{\pi/4,2} \left(1_1 \otimes \frac{1}{\sqrt{2}} (X_2 + Z_2) \right) \\ = (1_1 \otimes |0\rangle \langle 0|_2 + X_1 \otimes e^{i\pi/4} |1\rangle \langle 1|_2) \left(1_1 \otimes \frac{1}{\sqrt{2}} (X_2 + Z_2) \right) \\ = \frac{1}{\sqrt{2}} (1_1 \otimes |0\rangle_2 \langle 0|_2 + |1\rangle_2 \langle 1|_2 + X_1 \otimes e^{i\pi/4} |1\rangle_2 \langle 0|_2 - |1\rangle_2 \langle 1|_2) = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ e^{i\pi/4} & -e^{i\pi/4} \end{pmatrix} \right) \\ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & -e^{i\pi/4} \\ 0 & 0 & 0 & 0 \\ e^{i\pi/4} & -e^{i\pi/4} & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & -e^{i\pi/4} \\ 0 & 0 & 1 & 1 \\ e^{i\pi/4} & -e^{i\pi/4} & 0 & 0 \end{pmatrix}.$$

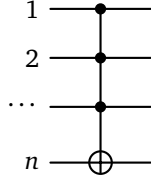
If we set the phases to 1, we recover the Bell state mapping $|00\rangle \rightarrow (|00\rangle + |11\rangle)/\sqrt{2}$, $|01\rangle \rightarrow (|00\rangle - |11\rangle)/\sqrt{2}$, $|10\rangle \rightarrow (|01\rangle + |10\rangle)/\sqrt{2}$ and $|11\rangle \rightarrow (-|01\rangle + |10\rangle)/\sqrt{2}$.

- (c) Finally, we calculate the matrix expression associated with a three-qubit circuit as follows:

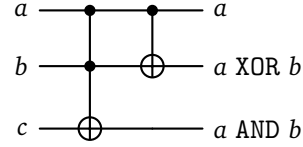
$$\begin{array}{c} 1 \text{ --- } [H] \text{ --- } \bullet \text{ ---} \\ | \\ 2 \text{ --- } \oplus \text{ --- } [H] \text{ --- } \bullet \text{ ---} \\ | \\ 3 \text{ --- } [R_{\pi/4}] \text{ --- } \oplus \text{ ---} \end{array} = \text{TOFFOLI } R_{\pi/4,3} H_2 (|0\rangle \langle 0|_1 \otimes |1\rangle_2 + |1\rangle \langle 1|_1 \otimes X_2) H_1 \\ = \text{TOFFOLI } R_{\pi/4,3} \frac{1}{\sqrt{2}} (|0\rangle_1 \langle 0|_1 + |1\rangle_1 \langle 1|_1) \otimes H_2 + |1\rangle_1 \langle 0|_1 \otimes H_2 X_2 \\ = \text{TOFFOLI } R_{\pi/4,3} \frac{1}{2} \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \right) \\ = \text{TOFFOLI } R_{\pi/4,3} \frac{1}{2} \left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \right) \\ = \text{TOFFOLI } R_{\pi/4,3} \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1_3 & 0 & 0 & 0 \\ 0 & 1_3 & 0 & 0 \\ 0 & 0 & 1_3 & 0 \\ 0 & 0 & 0 & X_3 \end{pmatrix} R_{\pi/4,3} \begin{pmatrix} 1_3 & 1_3 & 1_3 & 1_3 \\ 1_3 & -1_3 & 1_3 & -1_3 \\ 1_3 & 1_3 & -1_3 & -1_3 \\ -1_3 & 1_3 & 1_3 & -1_3 \end{pmatrix}.$$

2 Quantum Adder

- (a) The TOFFOLI gate can be generalized to n qubits by increasing the number of control qubit to $n - 1$ conditioning a NOT operation on qubit n . The circuit corresponding to this generalization is presented in Fig. 1 (a).



(a) n -qubit generalisation of the TOFFOLI gate.



(b) Circuit for the addition of two single-bit numbers a and b .

Figure 1: Circuits for 2 (a) and 2 (b).

- (b) Given two single-digit binary numbers a and b , we can calculate $a + b$ with a quantum circuit by encoding them in input numbers in qubit states $|a\rangle$ and $|b\rangle$ where the digit forms the 0,1 label of a computation basis element. In other words, the classical bit adder algorithm can be implemented with a quantum circuit. This algorithm requires qubits for inputs a and b and a qubit initialized to $|0\rangle$ that will eventually be updated to store the carry-on of $a + b$ (if we add $1 + 1$ we get 10 which is represented here by having 1 stored in the carry on qubit, and 0 stored in the output state for the b qubit Hilbert space. The state of a is unchanged to allow for the reversibility of calculation and its implementation as a sequence of unitary operations). The Quantum circuit implementing the single is presented in Fig. 1 (b). On one hand, the TOFFOLI gate flips the carry-on c to 1 iff both a and b are initialized to 1. On the other hand, the CNOT gate replaces the value of b by $a \text{ XOR } b$ storing the first digit of the addition output in b .
- (c) If we consider adding 4 binary numbers a, b, c, d together, we need an additional carry-on qubit to represent the result since $1 + 1 + 1 + 1 = 100$ requires 3 qubits to describe its digits. In this case, we name the carry-on c_1 and c_2 . The circuit performing the addition of 4 single-bit numbers together is presented in Fig. 2 The first two layers of this circuit add a and b in the way explained in (b). At the

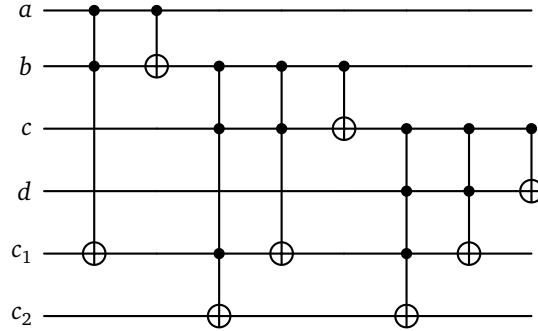


Figure 2: Circuit for the addition of four single bit numbers a, b, c, d . The output is stored in qubit d (first digit), c_1, c_2 (last digit)

second layer, the CNOT operation shifts the last digit output to the b qubit. The next group of operations uses the value stored in b, c, c_1 to add the updated b and c while taking the c_1 carry-on into account. The first gate of the third layer flips c_2 iff b, c, c_1 are all one (in our case this does not happen because $c_1 = 0$ implies $b = 0$). The fourth and fifth layers are copies of the operations described in (b) acting on b, c, c_1 . At the fifth layer, the CNOT operation shifts the last digit output to the c qubit. At layer six, a generalized TOFFOLI is applied to possibly flip the c_2 carry-on if c, d, c_1 are all 1 (which is a real possibility in this case). The seventh layer updated the value of the c_1 carry on from the values of c, d (if it was 1 and gets a 1 contribution from c, d , it is flipped to 0). The required carry-on to c_2 associated with this flip was already done by the sixth layer). At the last layer, the CNOT operation shifts the last digit output to the d qubit. The final output of the addition is stored in qubits d, c_1, c_2 .

3 Grover's algorithm on IBM composer ([Work in progress](#))

See the circuit <https://quantum.ibm.com/composer/files/ba6b78ecc0a4b80a7cbd758c6b52f13e0878ba3dc70e65f6986f254ce4d61395>.

4 Acknowledgement

Thanks to Jonathan for a discussion about the quantum adder.