Homework 1: Poincaré Group and Poincaré Algebra

Due October 16th.

By the end of this homework, you will be able to tell your friends what we mean by the MASS of a particle.

Q: What should you study if the power is out?

1 What is that smell?

You are about to leave PI at the end of the day. But a pungent smell arouses your curiosity. You trace the smell to the kitchen.

Enter the kitchen.

Pay extra attention to the multiplication table of the Poincaré group (including inversion). They are useful for the next problem. Good luck!

2 The Poincaré Algebra

Now we are ready to find the Poincaré algebra.

We want to study the following infinitesimal group element:

$$U(\delta+\omega,\epsilon) = \mathbf{1} + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\epsilon_\mu P^\mu \tag{1}$$

where δ has matrix element $\delta^\mu_{\ \nu}$. The goal is to find the Poincaré algebra: commutator relationship satisfied for the generators/operators $J^{\mu\nu}$ and P^μ.

a) By demanding $\Lambda = \delta + \omega$ is a Lorentz transformation, show that $\omega_{\mu\nu}$ is anti-symmetric.

b) Show that under a symmetrical transformation, not only the state transforms, but also the operator transforms, $O' = UOU^\dagger$. [1]

c) Let us study how the operator $U(\delta+\omega,\epsilon)$ transforms under a transformation of $T(\Lambda, a)$. In other words, show that $U(\Lambda,a)U(\delta+\omega,\epsilon)U^\dagger(\Lambda,a) = U(\delta + \Lambda\omega\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)$.

d) Now expand $U(\delta+\omega,\epsilon)$ as in equation (1), and show that $J^{\mu\nu}$ and P^μ transform in the following way

$$\begin{aligned} U(\Lambda,a)J^{\mu\nu}U^\dagger(\Lambda,a) &= \Lambda_\rho^{\ \mu}\Lambda_\sigma^{\ \nu}(J^{\rho\sigma} + a^\rho P^\sigma - a^\sigma P^\rho) \\ U(\Lambda,a)P^\mu U^\dagger(\Lambda,a) &= \Lambda_\rho^{\ \mu}P^\rho. \end{aligned} \tag{2}$$

e) Now expand $\Lambda = \delta + \omega$, $a = \epsilon$ again(Hint: Expand the momentum one first!), and show that Poincaré algebra is given by

$$\begin{aligned} i[J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu} \\ i[P^\mu, J^{\rho\sigma}] &= \eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho \\ [P^\mu, P^\nu] &= 0. \end{aligned} \tag{3}$$

f) Take $\vec{J} = (J^{23}, J^{31}, J^{12})$, and show that they indeed satisfy the commutator relationship of angular momentum[2],

$$[J_i, J_j] = -i\epsilon_{ijk}J_k. \tag{4}$$

g) Show that $P^2 \equiv P^\mu P_\mu = -(P^0)^2 + (P^1)^2 + (P^2)^2 + (P^3)^2$ commutes with all the generators. This is what we define to be $-M^2$ of a particle.

[1] Hint: if the transformation is symmetric, we have $\langle\psi|O|\phi\rangle = \langle\psi'|O'|\phi'\rangle$.

[2] The unsettling minus sign comes from the fact we are in a mostly minus signature and we have been very sloppy with our indices. With index properly positioned, the angular momentum Lie algebra should be $[J^a, J^b] = i\epsilon^{ab}_c J^c$.

3 Acknowledgement

Who did you collaborate with on this homework assignment?

A: The group theory. As the groups come with their own generators.