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Homework 2: Monopoles

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Contents

1 Dirac

(a) We are interested in the relation between the global properties of a manifold M and the structure of diffrential forms taking values on its cotangent bundle $T^{\star}M$ at each point of M.

Poincaré's lemma on $M = \mathbb{R}$**:** Let ω be a p-form ($p \in \{0,1\}$) constructed from the cotengent space $T^{\star}M$ of M. Then $\mathrm{d}\omega = 0$ (ω is closed) implies $\omega = \mathrm{d}\lambda$ (ω is closed) where λ is a $(p-1)$-form (0-form).

Proof: On $\mathbb{R}$, we can use the identity map as a global coordinate chart. The induced basis on 1-forms is $\{\mathrm{d}x\}$ (a smooth frame field) and any 1-forms can be written as $\omega = g\mathrm{d}x$ with $g \in C^{\infty}(\mathbb{R})$. Suppose now that ω is closed: we have $0 = \mathrm{d}\omega = \partial_x g \mathrm{d}x \wedge \mathrm{d}x = 0, \forall g \in C^{\infty}(\mathbb{R})$ (ω being a 1-form is not restrictive, but would be for $\mathbb{R}^n$ with $n > 1$). Then we take the 0-form $\lambda = G$ where G is any primitive of g ($G(x)$ exists because g is smooth) and apply an exterior derivative to get $\mathrm{d}\lambda = g\mathrm{d}x$. Because there are no $(0-1)$-forms there is no need to check the lemma for 0-forms.

Counterexample: Consider the circle smooth manifold $\mathbb{S}^1 \subset \mathbb{R}^2$ (embeded as $\{x^2 + y^2 = 1 | (x,y) \in \mathbb{R}^2\}$ for simplicity). It takes at least two charts to cover this manifold and, although on individual charts all closed 1-forms are exact (charts make the manifold look like $\mathbb{R}$ locally), this property is lost globally. Choose the chart map $\theta = \arctan_2$ sending points (x,y) on the circle to their angle with the x axis excluding the point $(1,0)$ so that the domain is open. With this chart we have the coordinate induced one form frame field $\mathrm{d}\theta$ which we use to construct the closed form $\omega = \mathrm{d}\theta$. On $(0,2\pi)$, this form is exact since we have a 0-form $\lambda = F \in C^{\infty}((0,2\pi))$ such that $\omega = \mathrm{d}\lambda = \partial_\theta F \mathrm{d}\theta = \mathrm{d}\theta$ forcing $F = \theta + c$, $c \in \mathbb{R}$ since F has to be a primitive of 1 in the variable θ. The function F is smooth on the chart, but can never be extended to s smooth function over $\mathbb{S}^1$ globally. Indeed, 0 and 2π being identified, a continuous function on $\mathbb{S}^1$ should be consistant at the excluded point $(0,1)$ and this would requiere $\lim_{\theta\to 0^+}(\theta + c) = \lim_{\theta \to 2\pi^-}(\theta + c)$ which is impossible. Therefore there is a closed form on $\mathbb{S}^1$ that is not exact.

(b) Let $F^{(2)}$ be a 2-form on the 2-sphere $\mathbb{S}^2$. Suppose $F^{(2)}$ is globally exact implying there is a 1-form ω such that $F^{(2)} = \mathrm{d}\omega$. Then we can use Stokes theorem in combination with the fact $\mathbb{S}^2$ has no boundary to write $g = \frac{1}{4\pi} \in F^{(2)} = \frac{1}{4\pi}\int_{\partial\mathbb{S}^2} \mathrm{d}\omega = 0$.

(c) Now working in Minkowski space $\mathbb{R}^{1,3}$ in the coordinate chart (t,r,θ,ϕ) built from spherical coordinates on $\mathbf{R}^3$, we have the 2-form $F^{(4)} = Q\sin(\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi$ with $Q \in \mathbb{R}$. We want to determine if $F^{(4)}$ satisfies Maxwell's equations $\mathrm{d}F^{(4)} = 0, \quad \mathrm{d} \star F^{(4)} = 0$. We have $\mathrm{d}F^{(4)} = Q\cos(\theta)\,\mathrm{d}\theta \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi = 0$. To evaluate the Hodge dual of $F^{(4)}$

(d)

(e)

(f)

(g)

(h)

(i)

2 Taub-NUT, or the gravitomagnetic monopole

(a)

(b)

(c)

(d)