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HOMWORK 2

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Quantum Field Theory I

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1 The commutator/anti-commutator

The construction of the spinor field starts with creation (and associated annihilation) operators $(b_p^s)^\dagger, (c_p^s)^\dagger$ respectively creating a spinor and an anti-spinor. They act on the free vacuum $|0\rangle$ to create respective eigenstates $|\mathbf{p}, s\rangle$ and $|\overline{\mathbf{p}}, s\rangle$. They are eigenstates of space translation generators with eigenvalues \mathbf{p} and intrinsic z rotation generators with eigenvalues s . Using the Casimir operator (mass m times identity or four-norm of the translation generators) of the Poincare group, we can use the eigenvalues \mathbf{p} to extract the energy p^0 to form the four-momentum p .

The creation and annihilation operators can be linearly combined to form spinor components of creation and annihilation fields at four-position x respectively denoted $\psi_\ell^+(x)$ and $\psi_\ell^-(x)$. The combination is done with coefficients $u_\ell^s(\mathbf{p})e^{-ip \cdot x}$ and $v_\ell^s(\mathbf{p})e^{+ip \cdot x}$ with momentum integrated with the Lorentz invariant measure dV_p and spin summed. It reads

$$\psi_\ell^-(x) = \sum_s \int dV_p u_\ell^s(\mathbf{p}) e^{-ip \cdot x} b_p^s \quad \& \quad \psi_\ell^+(x) = \sum_s \int dV_p v_\ell^s(\mathbf{p}) e^{+ip \cdot x} (c_p^s)^\dagger.$$

The coefficient of the decomposition of the fields are chosen so that the Poincare transformations of $(b_p^s)^\dagger, (c_p^s)^\dagger$ are consistent with a representation of the Poincare group acting on the spinor component space. More precisely, this imposes relations between the spinor index transformation of the coefficients and their spin index transformation inherited from operator transformations. Linear independence of the annihilation and creation operators (they create/annihilate a full basis of the Hilbert space) provides a relation for every pair \mathbf{p}, s and each element of the Lorentz group acting on the pair. Progress is made by restricting this relation to coefficients evaluated at $\mathbf{p} = 0$ (the same relation links $\mathbf{p} = 0$ to non-zero \mathbf{p} through consistency of boost Poincare transformations) and to rotation transformations for these *rest* coefficients. The fact the $\mathbf{p} = 0$ vector transforms trivially under rotation leaves all the interesting transformation properties to the spin index. We get the relations

$$\sum_{s'} u_{\ell'}^{s'} J_{s's}^j(R) = \sum_\ell \mathcal{J}_{\ell'\ell}(R) u_\ell^{s'} \quad \& \quad \sum_{s'} v_{\ell'}^{s'} J_{s's}^j(R) = - \sum_\ell \mathcal{J}_{\ell'\ell}(R) v_\ell^{s'}$$

where \mathbf{J}^j is the matrix representing the effect of rotations on the spin j index and \mathcal{J} is the matrix representing the effect of rotations on the spinor index. Here \mathbf{J}^j is restricted to be the irreducible representation of complex dimension $2j + 1$ of $SU(2)$ (because the created particles have spin the same j). Using matrix notation, the relation for $u_{m,\pm}^s$ becomes $\mathbf{U} \mathbf{J}^j = \mathcal{J} \mathbf{U}$ with components $[\mathbf{U}] = u_\ell^s(0)$.

By Schur's lemma, \mathbf{U} is either a square matrix or the representation \mathcal{J} is reducible. The spinor representation of the Lorentz group (and the associated \mathcal{J}) is reducible so we can't see \mathbf{U} as a square matrix at this stage. However, we can find a basis (Weyl representation) where \mathcal{J} is block diagonal with blocks $\mathcal{J}_+, \mathcal{J}_-$. The relation for \pm blocks reads $\mathbf{U}_\pm \mathbf{J}^j = \mathcal{J}_\pm \mathbf{U}_\pm$ where \mathbf{U}_\pm restricts the \mathbf{U} matrix to the blocks of the reducible representation \mathcal{J} .

Now, Schur's Lemma implies that \mathbf{U}_\pm are square matrices and this forces $2(2j + 1) = 4 \iff j = 1/2$ consistent with spin $1/2$ transformation for our particles. Furthermore, in the Weyl representation, we get $\mathbf{U}_\pm \sigma = \sigma \mathbf{U}_\pm$ where σ is any Pauli matrix. The only matrix U_\pm that commutes with all the Pauli matrices is a constant c_\pm multiple of the identity. Restoring the internal indices of \mathbf{U}_\pm (s spin index and m internal block index) leads to $u_{m,\pm}^s = c_\pm \delta_{m,s}$ and similar considerations lead to $v_{m,\pm}^s = -i d_\pm (\sigma_2)_{ms}$ where d_\pm is another proportionality constant.

Finally, we impose that the parity transformation of the spinor fields be compatible with the parity transformation of its constituent creation/annihilation operator (inherited from the parity transformation of the states). This constraint allows to make c_\pm, d_\pm more precise by reducing them to sign factors b_u, b_v (with $b_u^2 = b_v^2 = 1$). Explicitly, we can write

$$\begin{aligned} u^{(1/2)}(0) &= \begin{pmatrix} 1 & 0 & b_u & 0 \end{pmatrix}^T, & u^{(-1/2)}(0) &= \begin{pmatrix} 0 & 1 & 0 & b_u \end{pmatrix}^T \\ v^{(1/2)}(0) &= \begin{pmatrix} 0 & 1 & 0 & b_v \end{pmatrix}^T, & v^{(-1/2)}(0) &= \begin{pmatrix} 1 & 0 & b_v & 0 \end{pmatrix}^T. \end{aligned}$$

- (a) The next step in the construction of the spinor field consists in combining the annihilation and creation field to produce a total field $\psi_\ell(x) = \kappa\psi_\ell^-(x) + \lambda\psi_\ell^+(x)$ with $\kappa, \lambda \in \mathbb{C}$. Causality requires that the commutator/anti-commutator (respectively denoted $[\circ, \circ]_-$ and $[\circ, \circ]_+$) of the field at x with its adjoint at y vanishes for spacelike $x - y$. The commutator/anti-commutator is computed in general by using the fundamental commutator/anti-commutator

$$[b_p^s, (b_q^{s'})^\dagger]_\pm = 2E_q(2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) \delta_{ss'} \quad \& \quad [c_p^s, (c_q^{s'})^\dagger]_\pm = 2E_q(2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) \delta_{ss'}$$

with all other combinations having $[\circ, \circ]_\pm = 0$. We have the expansion

$$\begin{aligned} & [\kappa\psi_\ell^-(x) + \lambda\psi_\ell^+(x), \kappa^*(\psi_\ell^-(y))_{\ell'}^\dagger + \lambda^*(\psi_\ell^+(y))_{\ell'}^\dagger]_\pm \\ &= [\kappa\psi_\ell^-(x), \kappa^*(\psi_{\ell'}^-(y))^\dagger]_\pm + [\kappa\psi_\ell^-(x), \lambda^*(\psi_{\ell'}^+(y))^\dagger]_\pm + [\lambda\psi_\ell^+(x), \kappa^*(\psi_{\ell'}^-(y))^\dagger]_\pm + [\lambda\psi_\ell^+(x), \lambda^*(\psi_{\ell'}^+(y))^\dagger]_\pm \\ &= |\kappa|^2 [\psi_\ell^-(x), (\psi_{\ell'}^-(y))^\dagger]_\pm + |\lambda|^2 [\psi_\ell^+(x), (\psi_{\ell'}^+(y))^\dagger]_\pm + \sim [[b, c]_\pm + [c^\dagger, b^\dagger]_\pm] \\ &= |\kappa|^2 \sum_s \sum_{s'} \int \int dV_p dV_q u_l^s(\mathbf{p}) u_{l'}^{s'*}(\mathbf{q}) e^{-ip \cdot x + iq \cdot y} [b_p^s, (b_q^{s'})^\dagger] + |\lambda|^2 \sum_s \sum_{s'} \int \int dV_p dV_q v_l^s(\mathbf{p}) v_{l'}^{s'*}(\mathbf{q}) e^{+ip \cdot x - iq \cdot y} [(c_p^s)^\dagger, c_q^{s'}] \\ &= |\kappa|^2 \sum_s \int dV_p u_l^s(\mathbf{p}) u_{l'}^{s*}(\mathbf{p}) e^{-ip \cdot (x-y)} \pm |\lambda|^2 \sum_s \int dV_p v_l^s(\mathbf{p}) v_{l'}^{s*}(\mathbf{p}) e^{+ip \cdot (x-y)} \\ &= \int dV_p (|\kappa|^2 N_{\ell\ell'}(\mathbf{p}) e^{-ip \cdot (x-y)} \pm |\lambda|^2 M_{\ell\ell'}(\mathbf{p}) e^{+ip \cdot (x-y)}), \quad N_{\ell\ell'}(\mathbf{p}) = \sum_s u_l^s(\mathbf{p}) u_{l'}^{s*}(\mathbf{p}), \quad M_{\ell\ell'}(\mathbf{p}) = \sum_s v_l^s(\mathbf{p}) v_{l'}^{s*}(\mathbf{p}). \end{aligned}$$

- (b) Using the explicit expressions for $u^s(0)$ and $v^s(0)$, we can evaluate $N_{\ell\ell'}(0)$ (components of a matrix N) as follows:

$$\begin{aligned} N(0) &= \begin{pmatrix} 1 & 0 & b_u & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ b_u \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & b_u \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ b_u \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & b_u & 0 \\ 0 & 0 & 0 & 0 \\ b_u & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_u \\ 0 & 0 & 0 & 0 \\ 0 & b_u & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & b_u & 0 \\ 0 & 0 & 0 & b_u \\ b_u & 0 & 0 & 0 \\ 0 & b_u & 0 & 0 \end{pmatrix} = 1 + b_u \beta \end{aligned}$$

$$\text{with } \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- (c) To obtain the \mathbf{p} dependance of the coefficients u^s and v^s , we can apply the spinor representation $\mathbf{D}(L(p))$ (with matrix components $D_{\ell'\ell}$) of the Lorentz boost $L(p)$ from the reference four-momentum to the four-momentum p with three-momentum \mathbf{p} . For u^s and u^{s*} , this corresponds to the relations

$$\begin{aligned} u_{\ell'}^s(\mathbf{p}) &= \sum_\ell D_{\ell'\ell}(L(p)) u_\ell^s(0) \iff u^s(\mathbf{p}) = \mathbf{D}(L(p)) u^s(0), \\ u_{\ell'}^{s*}(\mathbf{p}) &= \sum_\ell D_{\ell'\ell}^\dagger(L(p)) u_\ell^{s*}(0) \iff (u^{s*}(\mathbf{p}))^T = (u^{s*}(0))^T \mathbf{D}^\dagger(L(p)). \end{aligned}$$

- (d) Using the definition of $N(\mathbf{p})$ and the previous boost transformation of the u^s vector, we have

$$N(\mathbf{p}) = \sum_s u^s(\mathbf{p}) (u^{s*}(\mathbf{p}))^T = \sum_s \mathbf{D} u^s(0) (u^{s*}(0))^T \mathbf{D}^\dagger = \mathbf{D}(L(p)) (1 + b_u \beta) \mathbf{D}^\dagger(L(p)).$$

- (e) To compute the action of $\mathbf{D}(L(p))$ on $N(0)$, we use expand it to first order in Lorentz group generators $\mathbf{J}^{\mu\nu}$ ($\mathbf{J}^{0\nu}$ are the boost generators) with real expansion coefficients $\omega_{\mu\nu}$ as $\mathbf{D}(L(p)) = 1 + i\omega_{\mu\nu}\mathbf{J}^{\mu\nu} + O(\omega^2)$. In the spinor representation emerging from the clifford algebra of γ^μ matrices, we have $\mathbf{J}^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$. We note that $\beta = \gamma^0$ and $\beta(\gamma^\mu)^\dagger\beta = \gamma^\mu$ to write

$$(\mathbf{J}^{\mu\nu})^\dagger = +\frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^\dagger = +\frac{i}{4}(\beta\gamma^\nu\beta\beta\gamma^\mu\beta - \beta\gamma^\mu\beta\beta\gamma^\nu\beta) = -\frac{i}{4}\beta(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\beta = \beta\mathbf{J}^{\mu\nu}\beta.$$

Then, we can relate $\mathbf{D}^\dagger(L(p))$ and $\mathbf{D}^{-1}(L(p)) = 1 - i\omega_{\mu\nu}\mathbf{J}^{\mu\nu} + O(\omega^2)$ with

$$\mathbf{D}^\dagger(L(p)) = 1 - i\omega_{\mu\nu}(\mathbf{J}^{\mu\nu})^\dagger + O(\omega^2) = \beta^2 - i\omega_{\mu\nu}\beta\mathbf{J}^{\mu\nu}\beta + O(\omega^2) = \beta\mathbf{D}^{-1}(L(p))\beta.$$

- (f) Applying the previous result to the expression for $N(\mathbf{p})$, we get

$$N(\mathbf{p}) = \mathbf{D}(L(p))(1 + b_u\beta)\beta\mathbf{D}^{-1}(L(p))\beta = \mathbf{D}(L(p))\beta\mathbf{D}^{-1}(L(p))\beta + b_u\beta.$$

The conjugation of the γ^μ matrices by the $\mathbf{D}(L(p))$ can be expressed as $\mathbf{D}(L(p))\gamma^\mu\mathbf{D}^{-1}(L(p)) = L_\rho^\mu(p)\gamma^\rho$ where $L_\rho^\mu(p)$ are the components of the boost (to \mathbf{p}) matrices. Since $\beta = \gamma^0$, it follows that

$$\mathbf{D}(L(p))\beta\mathbf{D}^{-1}(L(p)) = \mathbf{D}(L(p))\gamma^0\mathbf{D}^{-1}(L(p)) = L^0_\rho(p)\gamma^\rho.$$

The effect of $\mathbf{D}(L(p))$ conjugaison on γ^μ is the same as the effet of a basis transformation for a reference frame. This allows to interpret $1\gamma^0 + 0\gamma^i$ as the the dual four-velocity of a rest observer. The active boost $L(p)$ sends it to the dual four-velocity $L^0_\mu(p)\gamma^\mu = p_\mu\gamma^\mu/m$ (associated to the three-velocity \mathbf{p}/m).

- (g) The final expressions for $N(\mathbf{p})$ and $M(\mathbf{p})$ (obtained replacing b_u by b_v) can be written as

$$N(\mathbf{p}) = \mathbf{D}(L(p))\beta\mathbf{D}^{-1}(L(p))\beta + b_u\beta = \frac{1}{m}(p_\mu\gamma^\mu + b_um)\beta \quad \& \quad M(\mathbf{p}) = \frac{1}{m}(p_\mu\gamma^\mu + b_vm)\beta.$$

- (h) Defining $D(u) = \int dV_{\mathbf{p}} e^{-ip\cdot u}$, we have $i\partial_\mu D(u) = \int dV_{\mathbf{p}} p_\mu e^{-ip\cdot u}$ where ∂_μ differenciates with respect to u . With this in mind, the commutator/anti-commutator of spinor field components becomes

$$\begin{aligned} [\psi_\ell(x), \psi_{\ell'}^\dagger(y)]_\pm &= \int dV_{\mathbf{p}} (|\kappa|^2 N_{\ell\ell'}(\mathbf{p})e^{-ip\cdot(x-y)} \pm |\lambda|^2 M_{\ell\ell'}(\mathbf{p})e^{+ip\cdot(x-y)}) \\ &= \int dV_{\mathbf{p}} \left(|\kappa|^2 \frac{1}{m}(p_\mu\gamma^\mu + b_um)\beta e^{-ip\cdot(x-y)} \pm |\lambda|^2 \frac{1}{m}(p_\mu\gamma^\mu + b_vm)\beta e^{+ip\cdot(x-y)} \right) \\ &= |\kappa'|^2 (i\gamma^\mu\partial_\mu + b_um)\beta D(u)|_{u=x-y} \pm |\lambda'|^2 (i\gamma^\mu\partial_\mu + b_vm)\beta D(u)|_{u=y-x}. \end{aligned}$$

where $\kappa' = \kappa/\sqrt{m}$ and $\lambda' = \lambda/\sqrt{m}$. The evaluation of $D(u)$ is taken after the differenciation (it is really an evaluation of the derivative).

2 Causality condition

- (a) For $y - x$ spacelike, we can find a reference frame where the time component of $x - y$ vanishes. In this reference frame, we can use the change of variable $\mathbf{p} = -\mathbf{p}'$ (which switches the bounds of integration) to get

$$D(u) = \int dV_{\mathbf{p}} e^{-ip\cdot u} = - \int dV_{-\mathbf{p}'} e^{ip'\cdot u} = \int dV_{\mathbf{p}'} e^{ip'\cdot(-u)} = D(-u), \quad dV_{-\mathbf{p}'} = \frac{-d^3p}{(2\pi)^3 2p^0(-\mathbf{p}')} = -dV_{\mathbf{p}'}$$

where \mathbf{u} are the non-vanishing space components of u . Since $D(u)$ is even for spacelike separation, we have

$$\partial_\mu D(u)|_{u=u'} = \frac{\partial}{\partial v^\mu} D(v) \Big|_{v=u'} = \frac{\partial}{\partial v^\mu} D(-v) \Big|_{v=u'} = -\frac{\partial}{\partial u^\mu} D(u) \Big|_{u=-u'} = -\partial_\mu D(u)|_{u=-u'}$$

implying the derivatives are odd at spacelike separations. With this parity property of the derivatives, the derivative terms of commutator/anti-commutator of spinor field components read

$$[\psi_\ell(x), \psi_{\ell'}^\dagger(y)]_{\pm, \partial} = |\kappa'|^2 i\gamma^\mu \partial_\mu D(u)|_{u=x-y} \beta \pm |\lambda'|^2 i\gamma^\mu \partial_\mu D(u)|_{u=y-x} \beta = (|\kappa'|^2 \mp |\lambda'|^2) i\gamma^\mu \partial_\mu D(u)|_{u=x-y} \beta.$$

We can impose that this group of term vanishes independantly of the other because they contain $\beta\gamma^\mu$ matrices and the remaining term is proportionnal to β . The γ^μ matrices have vanishing trace and we have orthogonality of derivative and non-derivative terms by $(\text{Tr}(\beta \times \beta\gamma^\mu) = \text{Tr}(1 \times \gamma^\mu) = 0)$. This contribution to the commutator (resp. anti-commutator) will vanish if $|\kappa'|^2 + |\lambda'|^2 = 0$ (resp. $|\kappa'|^2 - |\lambda'|^2 = 0$). Since $|\kappa'|^2, |\lambda'|^2 \geq 0$ either the field vanished everywhere and commutes at spacelike separations of anti-commutes with $|\kappa'| = |\lambda'| \iff |\kappa| = |\lambda|$. This means that any non-vanishing spinor field is quantized to fermions. For what follows, we choose the scale $|\kappa'| = |\lambda'| = 1$.

- (b) The spacelike vanishing of the remaining terms in what is now known to be an anti-commutator requires

$$0 = [\psi_\ell(x), \psi_{\ell'}^\dagger(y)]_{+, \dots} = (b_u + b_v) \beta m D(u)|_{u=y-x} \iff b_u + b_v = 0$$

where we used the even parity of $D(u)$ for spacelike u .

- (c) For what follows, we take $b_u = 1, b_v = -1$ and $\lambda = \kappa = 1$. We can express the total spinor field as

$$\psi(x) = \sum_s \int dV_{\mathbf{p}} \left(u^s(\mathbf{p}) e^{-ip \cdot x} b_{\mathbf{p}}^s + v^s(\mathbf{p}) e^{+ip \cdot x} (c_{\mathbf{p}}^s)^\dagger \right)$$

with

$$\begin{aligned} u^{(1/2)}(0) &= \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^T, & u^{(-1/2)}(0) &= \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}^T \\ v^{(1/2)}(0) &= \begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix}^T, & v^{(-1/2)}(0) &= \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix}^T. \end{aligned}$$

We note that $u^s(0)$ and $v^s(0)$ satisfy

$$\beta u^s(0) = (+1)u^s(0) \quad \& \quad \beta v^s(0) = (-1)v^s(0).$$

- (d) The conjugation of β by $\mathbf{D}(L(p))$ is now applied to $u^s(\mathbf{p})$ and $v^s(\mathbf{p})$ to get

$$\begin{aligned} \frac{1}{m} p_\mu \gamma^\mu u^s(\mathbf{p}) &= \mathbf{D}(L(p)) \beta \mathbf{D}^{-1}(L(p)) u^s(\mathbf{p}) = \mathbf{D}(L(p)) \beta u^s(\mathbf{0}) = \mathbf{D}(L(p)) u^s(\mathbf{0}) = u^s(\mathbf{p}) \iff (p_\mu \gamma^\mu - m) u^s(\mathbf{p}) = 0, \\ \frac{1}{m} p_\mu \gamma^\mu v^s(\mathbf{p}) &= \mathbf{D}(L(p)) \beta \mathbf{D}^{-1}(L(p)) v^s(\mathbf{p}) = \mathbf{D}(L(p)) \beta v^s(\mathbf{0}) = -\mathbf{D}(L(p)) v^s(\mathbf{0}) = -v^s(\mathbf{p}) \iff (p_\mu \gamma^\mu + m) v^s(\mathbf{p}) = 0. \end{aligned}$$

- (e) We finally apply the dirac equation operator to the spinor field expression obtained above. We have

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m) \psi(x) &= \sum_s \int dV_{\mathbf{p}} \left((i\gamma^\mu \partial_\mu - m) u^s(\mathbf{p}) e^{-ip \cdot x} b_{\mathbf{p}}^s + (i\gamma^\mu \partial_\mu - m) v^s(\mathbf{p}) e^{+ip \cdot x} (c_{\mathbf{p}}^s)^\dagger \right) \\ &= \sum_s \int dV_{\mathbf{p}} \left((\gamma^\mu p_\mu - m) u^s(\mathbf{p}) e^{-ip \cdot x} b_{\mathbf{p}}^s + (-\gamma^\mu p_\mu - m) v^s(\mathbf{p}) e^{+ip \cdot x} (c_{\mathbf{p}}^s)^\dagger \right) = 0 \end{aligned}$$

which show that the spinor field satisfies the Dirac equation.

3 Acknowledgement

I worked on my own for this assignment