Feynman Diagrams

3 October 2023

Deadline: Wednesday, 11 October 2023, 4 pm - submit here
Late deadline: Wednesday, 25 October 2023, 4 pm - submit here
Pass/fail deadline: Friday, 10 November 2023, 4 pm - submit here

You are encouraged to discuss the homework problems with your classmates.

Academic integrity requires that any solutions you submit are either your own or properly cited. If a classmate explains to you how to solve part of a problem, you should indicate this on your submission.

1 Two Real Scalars

Consider the following theory of two real scalar fields in four spacetime dimensions:

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \varphi)^2 - \frac{1}{2}\,m^2\varphi^2 + \frac{1}{2}\,(\partial_\mu \Phi)^2 - \frac{1}{2}\,M^2\Phi^2 - \frac{g}{2}\,\Phi\varphi^2\,. \tag{1}$$

For this homework ignore renormalization and use the Lagrangian as given.

a) Derive or guess the position-space Feynman rules for this theory. To distinguish Φ from φ, draw the Φ propagator with a dashed line.

b) Consider the three-point function

$$G(x,\, y,\, z) = \langle\Omega|\, \mathrm{T}\, \Phi(x)\varphi(y)\varphi(z)\, |\Omega\rangle\,. \tag{2}$$

Calculate this Green's function to leading order in g by drawing the Feynman diagrams, and writing down the associated analytic expressions in terms of the propagators in the coordinate representation. Do not compute the integrals.

c) Compute the Fourier transform $\tilde{G}(p_1, p_2, p_3)$ of the three point function $G(x, y, z)$ at leading order in g by:

   i) directly Fourier transforming your coordinate-space answer

   ii) drawing momentum-space diagrams and using momentum-space Feynman rules

   Are the Feynman rules for $\tilde{G}(p_1, p_2, p_3)$ the same as the Feynman rules for $i\mathcal{M}$ given in lecture? Do the same diagrams contribute?

d) Assume $M > 2m$. Use the LSZ reduction formula to find an analytic expression at next-to-leading order in g for the amplitude for a Φ particle of momentum $\mathbf{p}$ to decay to a pair of φ particles of momentum $\mathbf{k}_1$ and $\mathbf{k}_2$. You do not need to evaluate the integral.

e) Draw the *diagrams* (no analytic expressions needed) that contribute to:

   i) $\langle 0| \, \mathrm{T} \left(-i\frac{g}{2}\int d^4y\Phi(y)\varphi^2(y)\right)\left(-i\frac{g}{2}\int d^4z\Phi(z)\varphi^2(z)\right)\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\,|0\rangle$

   ii) the four-point function $G(x_1, x_2, x_3, x_4) \equiv \langle\Omega| \, \mathrm{T}\, \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\,|\Omega\rangle$ at order g^2

   iii) the $\varphi\varphi \to \varphi\varphi$ matrix element $i\mathcal{M}(\varphi(\mathbf{k}_1)\varphi(\mathbf{k}_2) \to \varphi(\mathbf{p}_1)\varphi(\mathbf{p}_2))$ at order g^2

f) Explain the relationship between parts i)-iii) of part e).

g) For at least one diagram from part (e) with a symmetry factor $S > 1$ explain the relationship between the number of symmetries of the Feynman diagram and the number of Wick contractions.

h) Who did you collaborate with on this homework assignment?