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Homework 2

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Contents

1 Gaussian model for $T < T_c$

(a) The Ising model can be formulated as a statistical field theory on a lattice using the Hubbard–Stratonovich transformation. The auxiliary field takes values ϕ_i (organized in a vector ϕ) on sites i at position x_i of a square lattice containing a total of N sites. The connectivity of the Ising interaction is encoded in a translationaly symmetric matrix with elements $B_{ij} = B(x_i - x_j)$. At zero magnetic field, the theory is parameterized by inverse temperature β and a positive constant A. The statistical field theory is given by the following partition function and action:

$$Z = \sqrt{\det\left(\frac{2\beta A^2 B}{\pi}\right)} \int_{\mathbb{R}^N} \mathrm{d}^N \phi \, e^{-S(\phi)}, \quad S(\phi) = \frac{\beta A^2}{2} \phi^{\mathrm{t}} B \phi - \sum_i \ln\left(\cosh\left(\beta A (B\phi)_i\right)\right).$$

We are interested in the model resulting from a gaussian approximation of $S(\phi)$ around its minimizing field configuration given at each site by ψ_j. Up to second order in deviations form this minimum, the expansion of the discrete field reads

$$S(\phi) \approx S(\psi) + \frac{1}{2} \sum_{i,j} (\phi - \psi)_i (\phi - \psi)_j \frac{\partial^2 S}{\partial \phi_i \partial \phi_j}(\psi).$$

Because B encodes translationaly symmetric nearest neighbour interaction, B_{ij} can be expressed [1] as a Fourier sum on a single crystal momentum k. We have

$$B_{ij} = \frac{1}{N} \sum_k B_k e^{-k\cdot(x_i - x_j)} \implies \sum_i B_{i,j} = \frac{1}{N} \sum_k B_k e^{-k\cdot x_j} \left(\sum_i e^{-k\cdot x_i}\right) = \frac{1}{N} \sum_k B_k e^{-k\cdot x_j} N \delta_{k,0} = B_0.$$

where we used the exponential sum representation of the $\delta_{k,0}$ [1]. We note that B_0 is related to the critical temperature of the gaussian model by $B_0 = k_B T_c$.

(b) Using the fact $S(\phi)$ is a function of N real variables ϕ_i, we have that its minimal is realized for $\phi_i = \psi_i$ such that the derivatives $\left.\frac{\partial S}{\partial \phi}\right|_\psi$ simultaneously vanish. This corresponds to

$$\begin{aligned} 0 = \left.\frac{\partial S}{\partial \phi_k}\right|_\psi &= \frac{\beta A^2}{2} \sum_{i,j} \psi_j B_{ij} \delta_{ik} + \frac{\beta A^2}{2} \sum_{i,j} \delta_{jk} B_{ij} \psi_i - \sum_i \tanh\left(\beta A (B\psi)_i\right) \sum_j \beta A B_{ij} \delta_{jk} \\ &= \frac{\beta A^2}{2} \sum_j \psi_j B_{kj} + \frac{\beta A^2}{2} \sum_i B_{ik} \psi_i - \sum_i \tanh\left(\beta A (B\psi)_i\right) \beta A B_{ik} \\ &= \beta A^2 \sum_i \psi_i B_{ki} - \beta A \sum_i \tanh\left(\beta A (B\psi)_i\right) B_{ik} \quad \text{(using (a) and } B_{ij} = B_{ji}\text{)}. \end{aligned}$$

Since the lattice is translational symmetric, the minimizing field is uniform with $\psi_i = \overline{\psi}$. This allows us to express the minimization condition in terms of the average field $\overline{\psi}$ as

$$0 = \beta A^2 \overline{\psi} B_0 - \beta A \sum_i \tanh\left(\beta A \overline{\psi} B_0\right) B_{ik} = \beta A^2 \overline{\psi} B_0 - \beta A \tanh\left(\beta A \overline{\psi} B_0\right) B_0 \iff A\overline{\psi} = \tanh\left(\beta A \overline{\psi} B_0\right)$$

Denoting $M = A\overline{\psi}$ and using $B_0 = k_B T_c$ we recover the familiar mean field theory self-consistency relation $M = \tanh\left(\frac{T_c}{T} M\right)$.

(c) The hessian matrix of $S(\phi)$ at the minimizing field configuration is given by

$$\begin{aligned} \left.\frac{\partial^2 S}{\partial \phi_k \partial \phi_m}\right|_\psi &= \beta A^2 \sum_i \delta_{im} B_{ki} - \beta A \sum_i \mathrm{sech}^2\left(\beta A (B\psi)_i\right) B_{ik} \beta A \sum_p B_{ip} \delta_{mp} \\ &= \beta A^2 B_{km} - (\beta A)^2 \sum_i \mathrm{sech}^2\left(\beta A (B\psi)_i\right) B_{ik} B_{im} \\ &= \beta A^2 B_{km} - (\beta A)^2 \mathrm{sech}^2\left(\beta A B_0 \overline{\psi}\right) (B^2)_{km} = \beta A^2 B_{km} - (\beta A)^2 \mathrm{sech}^2\left(\beta B_0 M\right) (B^2)_{km}. \end{aligned}$$

Using the mean field condition on M, the hessian becomes

$$\left.\frac{\partial^2 S}{\partial \phi_k \partial \phi_m}\right|_\psi = \beta A^2 B_{km} - (\beta A)^2 \mathrm{sech}^2\left(\tanh^{-1}(M)\right) (B^2)_{km} = \beta A^2 B_{km} - (\beta A)^2 (1 - M^2) (B^2)_{km}.$$

To show the resulting matrix is positive definite we can move to the eigenbasis of B_{ij} where B_{ij} is diagonal with eigenvalues B_k [1]. The hessian is diagonal in this eigenbasis and its eigenvalues are $H_k = B_k(\beta A^2 - (\beta A)^2 (1 - M^2) B_k)$. More can be said by noting that for nearest neighbour positive coupling, the eigenvalues B_{ij} are ordered as $B_0 \geq B_k \geq 0$. From this ordering, the positive definiteness of the hessian matrix reduces to $\beta A^2 - (\beta A)^2 (1 - M^2) B_k > 0$. Then, because $- < \tanh\left(\frac{T_c}{T} M\right) < 1$, we have $1 - M^2 > 0$ leading to the refinement $H_k > 1 - \beta(1 - M^2) B_0 > 0$ (the coefficient of B_k is negative and H_k is lower bounded by H_0). For a deviation t from $T_c = B_0/k_B$, $\beta = (1-t)/B_0$ and $H_k > 1 - (1-t)(1 - M^2) = (1-t)M^2 + t > 0$. We finally see that positive definiteness is satisfied on both sides of the critical temperature because of the structure of the solution of the mean field condition on M. Indeed

$$\begin{cases} M = 0, \quad t > 0 \\ M \sim \sqrt{-3t}, \quad t < 0 \quad \text{(tutorial 3)} \end{cases} \implies M^2 > -\frac{t}{1-t} \sim -t \quad (t \to 0).$$

(d) The correction to $S(\phi)$ provided by the hessian takes the form

$$\frac{1}{2}\sum_{k,m}(\phi-\psi)_k(\phi-\psi)_m\frac{\partial^2 S}{\partial\phi_k\partial\phi_m}(\psi)=\begin{cases}\frac{A^2}{2}\phi_k\phi_m(\beta B_{km}-\beta^2(B^2)_{km}), & T>T_c\\ \frac{A^2}{2}(\phi-M/A)_k(\phi-M/A)_m(\beta B_{km}-\beta^2\mathrm{sech}^2(\beta B_0 M)(B^2)_{km}), & T<T_c\end{cases}$$

$$=\begin{cases}\frac{1}{2}A\phi_k A\phi_m(\beta B_{km}-\beta^2(B^2)_{km}), & T>T_c\\ \frac{1}{2}(A\phi-M)_k(A\phi-M)_m(\beta B_{km}-\beta^2\mathrm{sech}^2(\beta B_0 M)(B^2)_{km}), & T<T_c\end{cases}$$

(e)

(f)

2 Acknowledgement

References

[1] *Statistical Physics, lecture notes, week 4, Emilie Huffman and Giuseppe Sellaroli*. 2023.