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Homework 2

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Contents

1 Dynamics on the tangent bundle

(a) We are interested in the description of the dynamics of a set of particle with the language of vector bundles. Our starting point is to take the allowed positions $\mathbf{q}$ to constitute a smooth n-manifold Q. At each point $\mathbf{q}$, the tangent space $T_\mathbf{q}Q$ is the vector space of directional derivatives $\mathbf{v}$ along trajectories going trought $\mathbf{q}$. These derivatives are identified with the velocities allowed at $\mathbf{q}$. The complete description of dynamics is provided by the tangent bundle TQ containing the pairs $(\shortparallel, \succapprox)$ describing all instantaneous configurations of the system.

To use the usual analysis of dynamics we use coordinate charts on Q given by the coordinate functions $\{q^i\}_{i=1}^n$. A coordinate chart on TQ can be constructed by appending the components of vectors in the coordinate basis induced by q^i at $\mathbf{q}$ to the coordinates produced by q^i. The maps $\{v^i\}_{i=1}^n$ returning the the vector components at $\mathbf{q}$ can be expressed with the dual coordinate basis $\mathrm{d}q^i_\mathbf{q}$ through the relation $v^i(\mathbf{q},\mathbf{v}) = \mathrm{d}q^i_\mathbf{q}(\mathbf{v})$.

The dynamics of the system is represented by a Lagrangian smooth function $L : TQ \to \mathbb{R}$. The legender transform associate to L is the map between TQ and the cotangent bundle $T^\star Q$ given by $\mathbf{F}L : (\mathbf{q},\mathbf{v}) \mapsto (\mathbf{q}, DL_\mathbf{q}(\mathbf{v}))$ where $DL_\mathbf{q} : \mathbf{v} \in T_\mathbf{q}Q \mapsto \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}q^i_\mathbf{q} \in T^\star_\mathbf{q}Q$ (with the coordinate representation $\hat{L} = L \circ ((q^i)^{-1}, (v^i)^{-1})$ and $\hat{q}^i = q^i(\mathbf{q},\mathbf{v})$ and $\hat{v}^i = v^i(\mathbf{q},\mathbf{v})$).

Since the Legender transform provides a smooth map between TQ and $T^\star Q$, we can use it to pullback the canonical symplectic structure on $T^\star Q$ and bring it to TQ. This structure is provided by the symplectic potential 1-form $\theta = p_i \mathrm{d}q^i \in T^\star T^\star Q$ where p_i are coordinate functions forming a chart $T^\star Q$ when combined with q^i. More precisely, the p_i functions give the components of covectors $\mathbf{p}$ at point $\mathbf{q}$ trough the relation $p_i(\mathbf{q},\mathbf{p}) = \left.\frac{\partial}{\partial q^i}\right|_\mathbf{q}(\mathbf{p})$.

The pullback $\theta_L = \mathbf{F}L^\star(\theta) \in T^\star TQ$ of θ is both linear and commutes with exterior derivatives. Using these properties we can calculate θ_L by first calculating the pullback of q^i as functions over TQ and then take the exterior derivative. At $(\mathbf{q},\mathbf{v}) \in TQ$, we have

$$\mathbf{F}L^\star q^i(\mathbf{q},\mathbf{v}) = q^i \circ \mathbf{F}L(\mathbf{q},\mathbf{v}) = q^i(\mathbf{q}, DL_\mathbf{q}(\mathbf{v})) = q^i(\mathbf{q},\mathbf{p})$$

and applying an exterior derivatives leads to $\mathbf{F}L^\star \mathrm{d}q^i_{\mathbf{q},\mathbf{p}} = \mathrm{d}(\mathbf{F}L^\star q^i) = \mathrm{d}q^i_{\mathbf{q},\mathbf{v}}$. We note that while $\mathrm{d}q^i \in T^\star Q$ can be evaluated at $\mathbf{q}$, the new $\mathrm{d}q^i$ obtained here is constructed from a function over the bundle TQ and is therfore evaluated at $\mathbf{q},\mathbf{v}$. Then we evaluate the pullback of the functions p_i at $(\mathbf{q},\mathbf{v}) \in TQ$ to be

$$\mathbf{F}L^\star p_i(\mathbf{q},\mathbf{v}) = p_i \circ \mathbf{F}L(\mathbf{q},\mathbf{v}) = p_i(\mathbf{q}, DL_\mathbf{q}(\mathbf{v})) = \left.\frac{\partial}{\partial q^i}\right|_\mathbf{q} DL_\mathbf{q}(\mathbf{v}) = \frac{\partial \hat{L}}{\partial v^j}(\hat{q},\hat{v}) \left.\frac{\partial}{\partial q^i}\right|_\mathbf{q} \mathrm{d}q^j_\mathbf{q} = \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v}).$$

Combining these results with the linearity of the pullback, we get $\theta_L(\mathbf{q},\mathbf{v}) = \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}q^i_{\mathbf{q},\mathbf{v}}$.

(b) Using again the commutation of pullback and exterior derivative, we obtain the pullback at $(\mathbf{q},\mathbf{v})$ of the symplectic form $\omega = -\mathrm{d}\theta$ by $\mathbf{F}L$ as follows:

$$\begin{aligned}\omega_L(\mathbf{q},\mathbf{v}) &= (\mathbf{F}L^\star\omega)(\mathbf{q},\mathbf{v}) = -(\mathbf{F}L^\star \mathrm{d}\theta)(\mathbf{q},\mathbf{v}) = -\mathrm{d}(\mathbf{F}L^\star\theta)(\mathbf{q},\mathbf{v}) = -\mathrm{d}\left(\frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}q^i_{\mathbf{q},\mathbf{v}}\right)\\ &= \underbrace{-\frac{\partial \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v})}_{B}\mathrm{d}v^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}q^i_{\mathbf{q},\mathbf{v}} + \underbrace{\frac{1}{2}\left(\frac{\partial \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) - \frac{\partial \hat{L}}{\partial q^j \partial v^i}(\hat{q},\hat{v})\right)}_{A}\mathrm{d}q^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}q^i_{\mathbf{q},\mathbf{v}}.\end{aligned}$$

(c) This 2-form is a section on $T^\star TQ$ and we now determine under which condition on L it becomes a symplectic 2-form. In a local basis $\mathrm{d}x^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}x^i_{\mathbf{q},\mathbf{v}}$ with $\{x^i\}_{i=1}^{2n} = \{q^1 \cdots q^n, v^1 \cdots v^n\}$, a symplectif 2-form must be given by $\omega_{i,j}\mathrm{d}x^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}x^i_{\mathbf{q},\mathbf{v}}$ with $\omega_{j,i}$ having non-vanishing determinant as a matrix. Here we have the matrix

$$[\omega_{i,j}] = \begin{pmatrix} A & B \\ -B & 0 \end{pmatrix} \implies \det[\omega_{i,j}] = -\det\begin{pmatrix} B & A \\ 0 & -B \end{pmatrix} = -\det\left[\frac{\partial \hat{L}}{\partial v^j \partial v^i}\right]^2.$$

As long as the determinant of $\left[\frac{\partial \hat{L}}{\partial v^j \partial v^i}\right]^2$ does not vanish the 2-form considered will be non-degenerate. Since ω_L was computed by taking an exterior derivative of a potential, it is exact forcing it to be closed and symplectic if $\left[\frac{\partial \hat{L}}{\partial v^j \partial v^i}\right]$ is regular.

(d) Now supposing $\left[\frac{\partial \hat{L}}{\partial v^j \partial v^i}\right]$ is regular, we have built a symplectic 1-form ω_L on $\Omega_2(TQ)$. In order to use it to describe dynamics we need a Lagrangian vector fields of which the integral curves are the trajectories of the set of particles on Q. This vector field is defined trough the energy function $E : (\mathbf{q},\mathbf{v}) \mapsto (DL_\mathbf{q}(\mathbf{v}))(\mathbf{v}) - L(\mathbf{q},\mathbf{v})$. To get this energy as a function of coordinate $\hat{q},\hat{v}$ we use the coordinate function q^i, v^i (regrouped in a chart map ϕ with ϕ^{-1} which return a pair $\phi^{-1}_\mathbf{q}(\hat{q},\hat{v}) = \mathbf{q}$ and $\phi^{-1}_\mathbf{v}(\hat{q},\hat{v}) = \mathbf{v} \in T_\mathbf{q}Q$) to write

$$\hat{E}(\hat{q},\hat{v}) = (E \circ \phi^{-1})(q^i(\mathbf{q},\mathbf{v}), v^i(\mathbf{q},\mathbf{v})) = \left(\frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}q^i_\mathbf{q}\right)(\mathbf{v}) - L \circ \phi^{-1}(q^i(\mathbf{q}), v^i(\mathbf{v})) = \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})v^i - \hat{L}(\hat{q},\hat{v})$$

where we used $DL_\mathbf{q}(\mathbf{v}) = DL_{\phi^{-1}_\mathbf{q}}\phi^{-1}_\mathbf{v} \circ (\hat{q},\hat{v}) = \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}q^i_\mathbf{q}$ and applied it to $\mathbf{v}$. By definition, the action of $DL_\mathbf{q}(\mathbf{v})$ on $\mathbf{v}$ extracts the v^i component of $\mathbf{v}$ in the coordinate basis. Stricly speaking, to properly precompose with ϕ^{-1}, we should have considered

$\mathrm{d}q^i_{\phi_{\mathbf{q}^{-1}}} \circ \phi_{\mathbf{v}}^{-1}(\hat{q},\hat{v})$ where $\mathrm{d}q^i_{\phi_{\mathbf{q}^{-1}}} \circ \phi_{\mathbf{v}}^{-1} = \mathrm{d}\hat{q}^i$ is the pullback by the coordinate chart on Q of the 1-form basis (indeed, we can interpret ϕ_v as a pushforward of vectors on $TQ \to T\mathbb{R}^n$ since it maps the tangent vector to a curve to the tangent vector of the image of the curve by $\phi_{\mathbf{q}}$ by construction).

(e) From the energy function and symplectic form ω_L, we can define the Lagrangian vector field X_E (section over TTQ) by the relation $\omega_L(X_E, \bullet) = \mathrm{d}E$. To use this definition, we work with the decomposition $X_E = X_E^i \left.\frac{\partial}{\partial x^i}\right|_{\mathbf{x}} = X_{E,q}^i \left.\frac{\partial}{\partial q^i}\right|_{\mathbf{q},\mathbf{v}} + X_{E,v}^i \left.\frac{\partial}{\partial v^i}\right|_{\mathbf{q},\mathbf{v}}$ in the coordinate basis of TTQ. We also evaluate the coordinate representation of exterior derivative of E to obtain

$$\begin{aligned}
\mathrm{d}\hat{E} &= \frac{\partial^2 \hat{L}}{\partial v^i \partial q^j}(\hat{q},\hat{v}) v^i \mathrm{d}\hat{q}^j + \frac{\partial^2 \hat{L}}{\partial v^i \partial v^j}(\hat{q},\hat{v}) v^i \mathrm{d}\hat{v}^j + \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v}) \mathrm{d}\hat{v}^i - \frac{\partial \hat{L}}{\partial v^j}(\hat{q},\hat{v}) \mathrm{d}\hat{v}^j - \frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v}) \mathrm{d}\hat{q}^j \\
&= \frac{\partial^2 \hat{L}}{\partial v^i \partial q^j}(\hat{q},\hat{v}) v^i \mathrm{d}\hat{q}^j + \frac{\partial^2 \hat{L}}{\partial v^i \partial v^j}(\hat{q},\hat{v}) v^i \mathrm{d}\hat{v}^j - \frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v}) \mathrm{d}\hat{q}^j.
\end{aligned}$$

With this expression, we find

$$\begin{aligned}
\omega_L(X_E, \bullet) &= \left(X_{E,q}^k \left.\frac{\partial}{\partial q^k}\right|_{\mathbf{q},\mathbf{v}} + X_{E,v}^k \left.\frac{\partial}{\partial v^k}\right|_{\mathbf{q},\mathbf{v}} \right) \left(-\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \mathrm{d}v^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}q^i_{\mathbf{q},\mathbf{v}} + \frac{1}{2} \left(\frac{\partial \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) - \frac{\partial \hat{L}}{\partial q^j \partial v^i}(\hat{q},\hat{v}) \right) \mathrm{d}q^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}q^i_{\mathbf{q},\mathbf{v}} \right) \\
&= -X_{E,v}^i \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \mathrm{d}q^j_{\mathbf{q},\mathbf{v}} + X_{E,q}^i \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \mathrm{d}v^j_{\mathbf{q},\mathbf{v}} + X_{E,q}^j \frac{1}{2} \left(\frac{\partial \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) - \frac{\partial \hat{L}}{\partial q^j \partial v^i}(\hat{q},\hat{v}) \right) \mathrm{d}q^i_{\mathbf{q},\mathbf{v}}
\end{aligned}$$

Comparing this result with the coordinate expression for $\mathrm{d}E$, linear independance leads to the relations

$$\begin{aligned}
&\frac{\partial^2 \hat{L}}{\partial v^i \partial v^j}(\hat{q},\hat{v}) v^i = X_{E,q}^i \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \implies X_{E,q}^i = v_i \quad []^{-1} \text{ exists because } \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \text{ is regular} \\
&-X_{E,v}^i \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) + v^i \frac{\partial^2 \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) = X_{E,v}^i \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) + v^i \frac{1}{2} \left(\frac{\partial \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) - \frac{\partial \hat{L}}{\partial q^j \partial v^i}(\hat{q},\hat{v}) \right) = v^i \frac{\partial^2 \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) - \frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v}) \\
&\implies X_{E,v}^i \left(\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \right) = \left(\frac{v^i}{2} \frac{\partial^2 \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) + \frac{v^i}{2} \frac{\partial^2 \hat{L}}{\partial q^j \partial v^i}(\hat{q},\hat{v}) - \frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v}) \right) \quad []^{-1} \text{ exists because } \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \text{ is regular}
\end{aligned}$$

(f) We now consider curve $\gamma : U \subset \mathbb{R} \to TQ$ given in the coordinate chart by $\hat{q}(t)$ and $\hat{v}(t)$. This curve represents a trajectory if it is an integral curve of X_E: composing the tangent vector to the curve $\frac{\mathrm{d}\gamma}{\mathrm{d}t}$ at the point $\gamma(t)$ with the coordinate functions should return the components of X_E associated to the point. An integral cuve has to satisfy

$$\begin{aligned}
\frac{\mathrm{d}\gamma}{\mathrm{d}t}(q^i) &= \frac{\mathrm{d}}{\mathrm{d}t}\hat{q}^i(t) = X_E(q^i) = X_{E,q}^i = \hat{v}^i(t) \\
\frac{\mathrm{d}\gamma}{\mathrm{d}t}(v^i) &= \frac{\mathrm{d}}{\mathrm{d}t}\hat{v}^i(t) = X_E(v^i) = X_{E,v}^i = \left[\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \right]^{-1} \left(-\frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v}) \right)
\end{aligned}$$

The second equation can be cast in the usual form of the Euler-Lagrange equations with Leibniz's rule in the following way

$$-\frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v}) = \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \frac{\mathrm{d}}{\mathrm{d}t}\hat{v}^i(t)$$

2 Acknowledgement

I worked on this assignment on my own.