Fourteen Lectures in
CLASSICAL PHYSICS

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Contents

Chapter 1

The action principle and Lagrangian mechanics

1.1 Setup

The configuration of a mechanical system is defined by assigning values to n "*generalized*" *coordinates* $(q^1, \ldots, q^n)$. Abstractly, these coordinates identify points in the *configuration space* Q of the system.

The change of a generalized coordinate in time defines a generalized velocity, v^i. The set of all possible pairs of generalized coordinates and velocities defines the *tangent space* $\mathrm{T}Q$ *of configurations space* Q:

$$(q^i, v^i) \in \mathrm{T}Q.$$

Example 1.1.1 (Particle)**.** The configuration space of a (free) particle moving is $Q = \mathbb{R}^3$, since its position is given by the assignment of its position in $\mathbb{R}^3$, i.e. by the assignment of 3 values of the Cartesian coordinates $(q^1, q^2, q^3) = (x, y, z)$. We could describe the particle in terms of spherical coordinates as well: this, of course, does not affect the nature of the "abstract" configuration space Q. The velocity of a particle is a vector in $\mathbb{R}^3$, therefore $\mathrm{T}Q = \mathbb{R}^3 \times \mathbb{R}^3$. $\lozenge$

Example 1.1.2 (Pendulum)**.** The configuration of a two-dimensional mathematical pendulum is its position on a circle of fixed radius, i.e. $Q = S^1$, and is best described by the assignment of an angle $q = \theta \in S^1$. The pendulum's configuration could also be given by the assignment of its position on the plane, $(x, y) \in \mathbb{R}^2$. This description of the pendulum would be redundant: although there is a way to deal with these redundant descriptions, in these notes we will always assume that our choice of coordinates is optimal. The velocity of a pendulum is described by a real number ranging from minus to plus infinity, therefore $\mathrm{T}Q = S^1 \times \mathbb{R} \ni (\theta, v)$. Notice that although the configuration of a pendulum is compact, the space of velocities is not. This is a general result; velocities are always unbounded. $\lozenge$

Example 1.1.3 (Double pendulum)**.** The configuration space of the double pendulum (Figure 1.1) is $Q = S^1 \times S^1$, and thus $\mathrm{T}Q = \mathrm{T}S^1 \times \mathrm{T}S^1 \simeq S^1 \times S^1 \times \mathbb{R} \times \mathbb{R}$. $\lozenge$

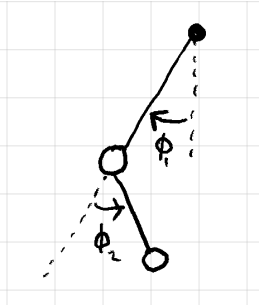

Figure 1.1: The double pendulum

A *history* or *trajectory* of a mechanical system over Q is a map

$$\gamma : \mathbb{R} \to Q, \qquad t \mapsto q^i = \gamma^i(t).$$

where $t \in \mathbb{R}$ is the time parameter. At each time along the trajectory, the system has the velocity

$$v^i = \dot{\gamma}^i(t) \doteq \frac{d\gamma^i(t)}{dt}.$$

The space of all histories over Q is therefore

$$\mathcal{H}_Q = C^\infty(\mathbb{R}, Q).$$

The *Lagrangian function* of a mechanical system over Q is then a function

$$L : \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}, \quad (q^i, v^i, t) \mapsto L(q^i, v^i, t).$$

Given a Lagrangian function and a history, we can compute the value of the *action functional*[1] associated to a certain time interval $[t_0, t_1]$:

$$S : \mathbb{R} \times \mathbb{R} \times \mathcal{H}_Q \mapsto \mathbb{R}, \qquad S(t_0, t_1, \gamma) = \int_{t_0}^{t_1} dt\ L(q^i(t), \dot{q}^i(t), t).$$

Often, for brevity (but a little sloppily), one writes $q^i(t)$ instead of γ and, instead of the above,

$$S[q(t)] = \int_{t_0}^{t_1} dt\ L(q^i(t), \dot{q}^i(t), t).$$

[1] Function*al*: a function of a function, e.g. S depends on $\gamma \in C^\infty(\mathbb{R}, Q)$.

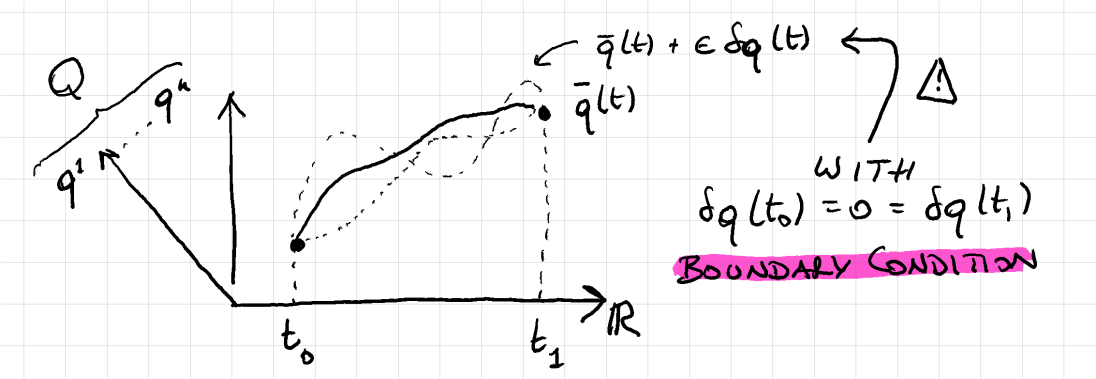

Figure 1.2: Hamilton's action principle.

1.2 Hamilton's action principle

The dynamics of any[2] *physical system is fully characterized by a Lagrangian function. Its physically realized histories are extrema of the associated action functional at fixed boundary condition (in time).*

Let us denote a physically realized history with an over-bar, $\bar{q}^i(t)$. Being an extremum of the action functional means that small (i.e. first order) variations of $\bar{q}^i(t)$ do not change the value of the action functional—as long as all the varied histories have the same boundary conditions as $\bar{q}(t)$. In other words, the first *functional derivative* of the action with respect to the history of the system vanishes at a physically realized trajectory.

Denoting by ϵ a first order infinitesimal, we write a first order variation of $\bar{q}^i(t)$ compatible with the time boundary conditions:

$$q(t) = \bar{q}^i(t) + \epsilon\delta q^i(t), \qquad \delta q(t_0) = 0 = \delta q^i(t_1),$$

with $\delta q^i(t)$ to be understood as a "unique symbol", i.e. "δ" has here no operational meaning on its own. See Figure 1.2.

Mathematically, *Hamilton's action principle* then states that a physically realized history is characterized by the condition:

$$S[\bar{q}^i(t) + \epsilon\delta q^i(t)] = S[\bar{q}^i(t)] + O(\epsilon^2) \quad \forall \delta q(t) \;:\; \delta q(t_0) = 0 = \delta q^i(t_1).$$

This is often evocatively written as the vanishing of the first functional derivative of S at $\bar{q}^i(t)$:

$$\left.\frac{\delta S}{\delta q^i(t)}\right|_{\bar{q}(t)} = 0.$$

(Note that this notation is not perfect, because it does not make explicit which boundary conditions have to be used in taking the functional derivative.)

[2] This has, of course, to be taken with a grain of salt.

We can now turn Hamilton's action principle into an explicit set of equations, the *Euler–Lagrange equations of motion*, that a physical history must satisfy.

To do so, we need first to compute the velocity of the perturbed history:

$$\dot{q}^i = \frac{d}{dt}(\bar{q}^i + \epsilon\delta q^i) \doteq \dot{\bar{q}}^i + \epsilon\delta\dot{q}^i, \quad \text{where} \quad \delta\dot{q}^i(t) \doteq \frac{d}{dt}\delta q^i(t).$$

Note: here, we have *defined* the symbol "$\delta\dot{q}^i(t)$" as the time derivative of $\delta q(t)$. With this, we compute:

$$\begin{aligned} L\left(q(t),\dot{q}(t),t\right) &= L\left(\bar{q}(t)+\epsilon\delta q(t),\dot{\bar{q}}(t)+\epsilon\delta\dot{q}(t),t\right) \\ &= L\left(\bar{q}(t),\dot{\bar{q}}(t),t\right) + \\ &\quad + \epsilon\sum_i \left.\frac{\partial L(q,v,t)}{\partial q^i}\right|_{\left(\bar{q}(t),\dot{\bar{q}}(t),t\right)} \delta q^i(t) + \left.\frac{\partial L(q,v,t)}{\partial v^i}\right|_{\left(\bar{q}(t),\dot{\bar{q}}(t),t\right)} \delta\dot{q}^i \\ &\quad + O(\epsilon^2). \end{aligned}$$

For short, this can be written:

$$L(\bar{q}+\epsilon\delta q,\dot{\bar{q}}+\epsilon\delta\dot{q},t) = L(\bar{q},\dot{\bar{q}},t)+\epsilon\delta L(\bar{q},\dot{\bar{q}},t)+O(\epsilon^2), \quad \delta L \doteq \sum_i\left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial\dot{q}^i}\delta\dot{q}^i\right).$$

And thus, the variation of the action becomes:

$$S[\bar{q}(t)+\epsilon\delta q] - S[\bar{q}(t)] = \epsilon\delta S[\bar{q}(t)] + O(\epsilon^2)$$

where

$$\begin{aligned} \delta S[\bar{q}(t)] &= \int_{t_0}^{t_1} dt\ \delta L(\bar{q}(t),\dot{\bar{q}}(t),t) \\ &= \int_{t_0}^{t_1} dt\ \sum_i \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial\dot{q}^i}\frac{d}{dt}\delta q^i \\ &= \int_{t_0}^{t_1} dt\ \sum_i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial\dot{q}^i}\right)\delta q^i + \left[\sum_i \frac{\partial L}{\partial\dot{q}^i}\delta q^i\right]_{t_0}^{t_1}. \end{aligned}$$

Hamilton's action principle then states that, if $\bar{q}(t)$ is a physical history, then $\delta S[\bar{q}(t)]$ vanishes whenever the variation $\delta q^i(t)$ satisfies the boundary condition at $t = t_0$ and t_1. These boundary conditions mean that the last term in the expression above vanishes, whereas from the arbitrariness of $\delta q(t)$ for $t \in (t_0, t_1)$, we obtain that Hamilton's action principle implies that—and is in fact equivalent to the statement that—a physical history $\bar{q}(t)$ must satisfy the *Euler–Lagrange* (EL) *equations of motion* (there are n of them, one for each value of i):

$$\text{Hamilton's action principle} \iff \left.\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\right|_{\bar{q}(t)} = 0.$$

Often, to denote that a quantity is evaluated at a physical history in a more compact manner, one uses a "hatted equality symbol", as in:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \hat{=} 0.$$

Remark 1.2.1. Since—and this is an axiom, see below for its relation to Newton's second law—the Lagrangian depends at most on velocities and not accelerations, jerks, or higher derivatives, the EL equations are at most second order (ordindary) differential equations in t, i.e. they might contain first and second time derivatives of $q(t)$ but no higher derivatives. As such they can be solved in terms of e.g. the *initial conditions*

$$\big(q^i(t_0), \dot{q}^i(t_0)\big) = \big(q^i_0, v^i_0\big)\,.$$

However, note that the action principle is not formulated in terms of initial conditions for q and $\dot{q}$, but in terms of boundary conditions for q (only)! This does not change the number of conditions imposed, but in general the existence and uniqueness of a solution to the EL equations at fixed boundary conditions are not guaranteed, except possibly on a very short, i.e. infinitesimal, time interval $[t_0, t_1]$. However, note that the action principle is not formulated in terms of initial conditions for q and $\dot{q}$, but in terms of boundary conditions for q (only)! This does not change the number of conditions imposed, but in general the existence and uniqueness of a solution to the EL equations at fixed boundary conditions are not guaranteed, except possibly on a very short, i.e. infinitesimal, time interval $[t_0, t_1]$. Finally, let us mention that there is a beautiful and deep connection—first fully understood by Feynmann—between Hamilton's action principle, which is formulated in terms of boundary conditions, and quantum mechanical transition amplitudes $\langle\psi(q,t_1)|\psi(q,t_0)\rangle$. We will (briefly) come back on this point in Chapter 6 on the Hamilton–Jacobi equation. ♢

Remark 1.2.2 (Newton's second law)**.** Consider Newton's second law for a particle moving in a conservative force field

$$m\vec{a} = \vec{F}(\vec{x}) = -\vec{\nabla}V(\vec{x}).$$

It is easy to show that this is the Euler–Lagrange equations for the following Lagrangian:

$$L(\vec{x},\vec{v},t) = \frac{1}{2}m|\vec{v}|^2 - V(\vec{x})$$

where we have set $q \equiv \vec{x} \in \mathbb{R}^3 = Q$. This is readily generalized to a set of N interacting particles moving in an external (possibly time-dependent) potential undergoing mutual pair-interactions, as in:

$$L(\vec{x}_\alpha,\vec{v}_\alpha,t) = \sum_{\alpha=1}^{N}\frac{1}{2}m_\alpha|\vec{v}_\alpha|^2 - V_{\text{ext}}(\vec{x}_\alpha,t) - \sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{N} V_{\text{pair}}(|\vec{x}_\alpha - \vec{x}_\beta|),$$

or to even more general systems (like the rigid body in an external gravitational field) via the formula:

$$L = T - V$$

where T stands for "kinetic energy"' and V for "potential energy". $\lozenge$

1.3 Ambiguity in the Lagrangian function

Now, a very important observation: two Lagrangians L and $\tilde{L}$ that are distinct from each other by a total time derivative yield the same EL equations of motion, and therefore describe the same dynamical system.

In other words, given any function

$$\ell : Q \times \mathbb{R} \to \mathbb{R}, \quad (q,t) \mapsto \ell(q,t),$$

(note: a dependence on the velocity v is not allowed, see below), the following two Lagrangians are dynamically equivalent:

$$L(q,v,t) \sim \tilde{L}(q,v,t) = L(q,v,t) + \frac{\partial \ell(q,t)}{\partial t} + \sum_i \frac{\partial \ell(q,t)}{\partial q^i} v^i.$$

Indeed,

$$\begin{aligned} \tilde{S}[q(t)] &= \int dt\ \tilde{L}(q,\dot{q},t) = \int_{t_0}^{t_1} dt\ L(q,\dot{q},t) + \frac{\partial \ell}{\partial t} + \frac{\partial \ell}{\partial q}\dot{q} \\ &= S[q(t)] + \int_{t_0}^{t_1} dt\ \frac{d\ell}{dt} \\ &= S[q(t)] + \ell(q(t_1),t_1) - \ell(q(t_0),t_0) \end{aligned}$$

and, in view of the boundary conditions imposed on $q(t)$ by Hamilton's action principle, the last term does not partake in the derivation of the EL equations of motion (this is why no dependence of ℓ on v is allowed here).

Exercise 1.3.1. Write down the Euler–Lagrange equations for $\tilde{L} = L + \partial_t \ell + \sum_i v^i \partial_{q^i} \ell$ and show that they do not depend on ℓ. $\lozenge$

1.4 Cyclical coordinates

Assume that $L(q^i, v^i, t)$ does *not* depend on the coordinate q^1:

$$\frac{\partial L}{\partial q^1} = 0.$$

A coordinate q^1 with this property is said *cyclical*.

Then, the EL equation for q^1 reads:

$$\frac{\partial L}{\partial q^1} = 0 \implies \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^1} \hat{=} 0$$

i.e.

$$q^1 \text{ is cyclical } \implies p_1 \doteq \frac{\partial L}{\partial \dot{q}^1} \text{ is conserved.}^3$$

This quantity p_1 is called the *momentum conjugate to* q^1, and generalizes the idea of linear momentum.

Example 1.4.1 (Particle mechanics)**.** If the Lagrangian of a particle in $\mathbb{R}^3$ does not depend on the coordinate $q^1 = x$, then the x-component of its linear momentum, $p_1 = mv_x$, is conserved. In a standard situation where the kinetic term does not depend on the positions, but only on the velocities, the independence of the Lagrangian of $q^1 = x$ means that $\partial_x V = 0$, and therefore the Newton–Euler–Lagrange equations of motion along the direction $\hat{x}$ read:

$$m\vec{a}_x = \vec{F} \cdot \hat{x} = -\partial_x V(\vec{x}) = 0.$$

The statement made in this section is, however, more general than this, and applies to cases where a time dependence in the "mass" might depend on the position of the particle. ♢

The statement made in this section is an elementary version of Noether's theorem, studied in the following chapter.

[3] Along a physical trajectory.

Chapter 2

Noether theorem

2.1 Setup

Consider a function s of the form (note that the two spaces $\mathrm{T}Q$ in the equation below will play quite different roles):

$$\mathrm{T}Q \times \mathbb{R} \to \mathrm{T}Q, \quad (q, v, t) \mapsto (q, s(q, v, t)) \,.$$

which we interpret as defining a particular infinitesimal variation of any history $q(t)$ (Figure 2.1):

$$q(t) \rightsquigarrow q(t) + \epsilon \tilde{\delta}_s q(t), \quad \tilde{\delta}_s q(t) \doteq s\left(q(t), \dot{q}(t), t\right).$$

Note that contrary to the variations δq from the previous chapters, the variation $\tilde{\delta}_s q$ does *not* satisfy any boundary conditions.

The associated variation of the velocity of the history $q(t)$ is:

$$\dot{q}(t) \rightsquigarrow \dot{q}(t) + \epsilon \frac{d}{dt} \tilde{\delta}_s q(t) \doteq \dot{q}(t) + \epsilon \tilde{\delta}_s \dot{q}(t),$$

or, fully explicitly,

$$\tilde{\delta}_s \dot{q}(t) \doteq \frac{d}{dt} \tilde{\delta}_s q(t) = \left(\frac{\partial s}{\partial t} + \sum_i \dot{q}^i \frac{\partial s}{\partial q^i} + \ddot{q}^i \frac{\partial s}{\partial v^i} \right) \Bigg|_{(q(t), \dot{q}(t), t)} .$$

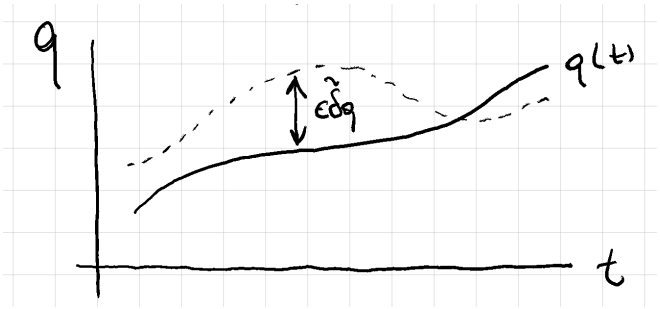

Figure 2.1: A sketch of the variation $\tilde{\delta}_s q$ around a history $q(t)$.

2.2 Noether's first theorem

The function s is said an (*infinitesimal*) *symmetry* if it leaves the action invariant up to a boundary term or, equivalently, if there exists a function $R_s : \mathcal{H}_Q \times \mathbb{R} \to \mathbb{R}$ such that for any history $q(t)$,[1]

$$L(q + \epsilon\tilde{\delta}_s q, \dot{q} + \epsilon\tilde{\delta}_s \dot{q}, t) = L(q, \dot{q}, t) + \epsilon \frac{d}{dt} R_s(q(t), \dot{q}(t), \dots, t) + O(\epsilon^2).$$

More, succinctly we write

$$s \text{ is an infinitesimal symmetry if } \tilde{\delta}_s L = \frac{d}{dt} R_s.$$

Note: it is crucial that the derivative on the right-hand side is a total time derivative, and not a partial one.

Remark 2.2.1. Notice, also, that R_s is not quite the same as $\ell = \ell(q, t)$ which we encountered when we studied the ambiguities in the Lagrangian of a system. This is because R_s can depend on time derivatives of q (typically, up to the first, see the example about time translations below). ♢

cross refs

We can now formulate Noether's first theorem.

Theorem 2.2.2 (Nother)**.** *If $s : \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}$ is an infinitesimal symmetry of L, then the quantity*

$$Q_s(t) \doteq \sum_i \frac{\partial L}{\partial \dot{q}^i} \tilde{\delta}_s q^i - R_s$$

is conserved along any physical history, i.e.

$$\frac{dQ_s}{dt} \hat{=} 0.$$

Q_s is then said the Noether charge *associated to the symmetry s.*

Before proving the theorem, we provide some remarks and a few examples.

Remark 2.2.3. If s is an infinitesimal symmetry of L, then it is also an infinitesimal symmetry of $L' = L + \frac{d}{dt}\ell$. It is then easy to verify, using its definition, that the (conserved) Noether charge Q_s is the same in the two cases. (Do it!) ♢

Remark 2.2.4. There are different formulations of Noether's theorem. Some of them involve changes in the definition of the time variable while demanding exact invariance of the action (or Lagrangian)—rather than invariance up to boundary terms.[2] The version given here provides the simplest possible proof, and generalizes nicely to field theories and to more abstract methods (e.g. the covariant phase space method) used in theoretical physics. The possible disadvantage of this formulation is that "abstractly" it is at time confusing what it means to compute R_s. The idea

[1] The dots denote a potential depedence of R_s on higher time derivatives of the history $q(t)$.

[2] Cf. David Kubiznak's lecture notes.

is that R_s is computed by explicitly inserting the variation $\tilde{\delta}_s q(t)$ in the Lagrangian, bunching up terms, and checking that it gives a total time derivative. Importantly, in doing these computations we are *not* allowed to use the equations of motion (the relation between $\tilde{\delta}_s L$ and $\frac{d}{dt}R_s$ is written in terms of an exact, and not "hatted", equality). The examples below should clarify its meaning. $\Diamond$

Remark 2.2.5. Recalling the definition of the (generalized) momentum $p_i = p_i(q, v, t)$ given at the end of the previous chapter, that is

$$p_i \doteq \frac{\partial L}{\partial \dot{q}^i},$$

the Noether charge Q_s can be re-written

$$Q_s \doteq \sum_i p_i \tilde{\delta}_s q^i - R_s. \qquad \Diamond$$

2.3 Examples

3.1 Cyclical coordinate This is the example we saw at the end of last chapter. In the language adopted here, it can be formulated as follows. Let L be such that $\partial L/\partial q^1 = 0$, and consider then $s^i(q, v, t) = \delta^i_0$ i.e.

$$\tilde{\delta}_s q^i(t) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}.$$

Then, obviously, $\tilde{\delta}_s L = 0$, which means we can choose $R_s = 0$. Thus, we conclude that $Q_s = \partial L/\partial \dot{q}^1 = p_1$ is conserved. Notice that the variation $\tilde{\delta}_s q$ does *not* vanish at $t = t_0$ or t_1.

3.2 Spatial translations Let's now consider the case of a set of N interacting particle described by the Lagrangian

$$L = \sum_\alpha \frac{1}{2} m_\alpha |\dot{\vec{x}}_\alpha|^2 + \sum_{\alpha<\beta} V(\vec{x}_\alpha - \vec{x}_\beta).$$

Such a Lagrangian is invariant (i.e. $\tilde{\delta}_s L = 0$, and $R_s = 0$) under global translations, i.e. under any transformation $\vec{s}_\alpha = \vec{a} \in \mathbb{R}^3$ i.e.

$$\tilde{\delta}_s \vec{x}_\alpha = \vec{a} \quad \forall \alpha,$$

or $\tilde{\delta}_s \vec{x}_\alpha = \vec{a}$. Note that this implies $\tilde{\delta}_s \dot{\vec{x}}_\alpha = \vec{0}$. Then the following charge is conserved:

$$Q_s = \sum_\alpha \sum_i \frac{\partial L}{\partial \dot{x}^i} \tilde{\delta}_s x^i_\alpha = \sum_\alpha m \dot{\vec{x}}_\alpha \cdot \vec{a}.$$

From the arbitrariness of $\vec{a}$, we deduce that it is the total momentum $\vec{P} = \sum_\alpha m\dot{\vec{x}}_\alpha$ to be here conserved as a vector quantity.

Finally, note that through the following change of variables,[3]

$$\{\vec{x}_\alpha\}_{\alpha=1}^N \mapsto (\vec{X}, \{\vec{z}_\alpha\}_{\alpha=1}^{N-1}) \quad \text{where} \quad \begin{cases} \vec{X} \doteq \dfrac{\sum_\alpha m_\alpha \vec{x}_\alpha}{\sum_\alpha m_\alpha} \\ \vec{z}_\alpha \doteq \vec{x}_\alpha - \vec{X} \end{cases}$$

we can write the Lagrangian L in a form that is manifestly independent of $\vec{X}$—i.e. in a form that make the centre of mass position into a cyclical coordinate. It is then clear that the associated momentum, that can be shown to be equal to $\vec{P}$, is conserved (do it!).

3.3 Time translations This example, although superficially similar to the previous ones of spatial translations, is in fact quite different from it. This is because time is not a coordinate on configuration space.[4]

reference in footnote

Time translations relate two histories $q(t)$ and $\tilde{q}(t)$ as per

$$\tilde{q}(t) = q(t+T).$$

Infinitesimally, $\tilde{q}(t) = q(t+\epsilon) = q(t) + \epsilon\dot{q}(t) + O(\epsilon^2)$. We thus define

$$s^i(q, v, t) = v^i$$

and

$$\tilde{\delta}_s q(t) = \dot{q}(t) \quad \text{and} \quad \tilde{\delta}_s \dot{q}(t) = \ddot{q}(t).$$

This is a crucial example in which s depends not only on the configuration of the system, q, but its velocity, v. Such symmetries are sometimes called "hidden symmetries".

Consider now a Lagrangian $L = L(q, v, t)$ evaluated at a history $q(t)$ (left implicit in the following equation) and let's compute the effect of the above transformation:

$$\tilde{\delta}_s L = \frac{\partial L}{\partial q}\tilde{\delta}_s q + \frac{\partial L}{\partial \dot{q}}\tilde{\delta}_s \dot{q} = \frac{dL}{dt} - \frac{\partial L}{\partial t}.$$

[3]The N variables $\vec{z}_\alpha$ are not independent of each other, for they satisfy the following linear relation among them

$$\sum_{\alpha=1}^N m_\alpha \vec{z}_\alpha = \vec{0}.$$

Therefore, we can choose to drop one of them, e.g. $\vec{z}_N$ as proposed here above, and thus be left with $N-1$ independent variables which, together with $\vec{X}$ give a total of N independent variables, i.e. as many as we started with.

[4]There are formulations where time can be promoted to a configuration space variable. The ensuing mechanical system is called the "parametrized particle". A nicer example is the relativistic particle discussed in Section 10.2.

Thus, if $L = L(q, v, t)$ does not depend explicitly on the time variable t we see that time translation is a symmetry transformation for $R_s = L$:

$$\frac{\partial L}{\partial t} = 0 \implies \tilde{\delta}_s L = \frac{d}{dt} L.$$

Applying Noether's theorem, we hence find that the following quantity is conserved:

$$Q_s = \sum_i \frac{\partial L}{\partial \dot{q}^i} \tilde{\delta}_s q^i - R_s = \sum_i p_i \dot{q}^i - L.$$

In a mechanical system where $L = T - V$ and $T = \sum_\alpha \frac{1}{2} m_\alpha |\dot{\vec{x}}_\alpha|^2$, it is immediate to see that Q_s is the total mechanical energy

$$Q_s = T + V.$$

This is why, given a Lagrangian L, the following quantity is called the (*generalized*) *energy* of the system:

$$E \doteq \sum_i p_i \dot{q}^i - L.$$

As we just proved, the generalized energy is conserved if L is time-independent.

3.4 Rotations As a last example consider again the Lagrangian

$$L = \sum_\alpha \frac{1}{2} m_\alpha |\dot{\vec{x}}_\alpha|^2 + \sum_{\alpha<\beta} V(|\vec{x}_\alpha - \vec{x}_\beta|).$$

It is manifestly invariant under rotations of the $\vec{x}_\alpha$ about an arbitrary axis $\vec{a}$, which infinitesimally reads

$$\vec{s}_\alpha = \vec{a} \times \vec{x}_\alpha,$$

i.e. $\tilde{\delta}_s \vec{x}_\alpha = \vec{a} \times \vec{x}_\alpha$ etc., and $\tilde{\delta}_s L = 0$. Thus, we can take $R_s = 0$ and thus find out that the Noether charge is the total angular momentum $\vec{M}$ of the system:

$$Q_s = \sum_\alpha \sum_i \frac{\partial L}{\partial \dot{x}^i_\alpha} \tilde{\delta}_s x^i_\alpha = \sum_\alpha \vec{p}_\alpha \cdot (\vec{a} \times \vec{x}_\alpha) = \vec{a} \cdot \sum_\alpha \vec{x}_\alpha \times \vec{p}_\alpha \doteq \vec{a} \cdot \vec{M}.$$

Now, we can ask: can we find coordinates such that $\vec{M}$ is the conjugate momentum to a triple of cyclical coordinates? (Recall the example of spatial translation symmetry: there, the total momentum is the momentum associated to the cyclical coordinate determining the position of the centre of mass of the system.) The answer is no.

To clarify the matter, let us introduce a different system where this impossibility can be made quite plausible.

The new system is a particle moving in a central potential (e.g. a planet around the Sun):

$$L = \frac{1}{2} m |\dot{\vec{x}}|^2 - V(|\vec{x}|).$$

This Lagrangian is manifestly invariant under rotations. We might think that spherical coordinates are then ideally suited to discover the conservation of angular momentum. We thus introduce spherical coordinates,

$$(x, y, z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta), \quad (r, \theta, \varphi) \in \mathbb{R}_{>0} \times [0, \pi] \times S^1,$$

so that, after some algebra (do it!), we find

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) - V(r).$$

Here, the only cyclical coordinate is φ, and the associated Noether charge is its momentum

$$p_\varphi = mr^2\sin^2\theta\dot{\varphi} = \hat{z}\cdot\vec{M}.$$

This reflects the fact that spherical (and cylindrical) coordinates make it possible to write rotations around the z-axis as translations in the azimuthal angle φ. However, rotations around any other axis take complicated and rather obscure expressions.

(A possible way to realize that all components of the angular momentum vector are conserved is to observe that spherical coordinates are written with respect to an *arbitrary* choice of z-axis.)

There are in fact no coordinates systems in which all three rotations can be written as translations of three independent (cyclical) variables. Ultimately, this is because these three translations would commute with each other, whereas rotations do not—in other words, the rotation group is non-Abelian.

2.4 Proof of Noether's first theorem

The idea to prove Noether's theorem is to write the variation $\tilde{\delta}_s L$ in two different ways. First, we have the explicit computation of

$$\tilde{\delta}_s L = \frac{d}{dt}R_s,$$

which in particular makes no use of the equations of motion. Second, we write it in such a way as to put into evidence the EL equations of motion:

$$\begin{aligned}\tilde{\delta}_s L &= \sum_i \left(\frac{\partial L}{\partial q^i}\tilde{\delta}_s q^i + \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\tilde{\delta}_s q^i\right) \\ &= \sum_i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)\tilde{\delta}_s q^i + \frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{q}^i}\tilde{\delta}_s q^i \hat{=} \frac{d}{dt}\sum_i p_i\tilde{\delta}_s q^i.\end{aligned}$$

Subtracting the previous two equations and rearranging, we conclude. □

First Tutorial

Euler–Lagrange equations

Consider a generalized Lagrangian that depends not only on the system's generalized coordinates, their velocities, and time, but also on their accelerations:

$$\widetilde{L}(q, \dot{q}, \ddot{q}, t).$$

1. Formulate an action principle for $\widetilde{L}$ and derive the appropriate Euler–Lagrange equations. Of which order are such equations? What are the additional requirements at $t = t_0$ and t_1?

2. Consider now the special case in which the Lagrangian $\widetilde{L}(q, \dot{q}, \ddot{q}, t)$ can be written as

   $$\widetilde{L}(q, \dot{q}, \ddot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt}\ell(q, \dot{q}, t).$$

   Of which order are the Euler–Lagrange equations in this case? Is the action principle you formulated in the previous part well-posed for this type of Lagrangian?

3. Discuss a viable action principle for $\widetilde{L}(q, \dot{q}, \ddot{q}, t)$ defined as in the previous part when

   $$\ell(q, \dot{q}, t) = -\sum_i \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}^i} q^i.$$

4. You will soon learn that the Einstein–Hilbert Lagrangian density for the gravitational field is

   $$\mathcal{L}_{\mathrm{EH}} = R\sqrt{g},$$

   where the Ricci scalar is a function of the metric and its first and second derivatives, i.e. $R = R(g, \partial g, \partial^2 g)$. Of which order do you naively expect the corresponding Einstein equations to be? In fact, you will see in the Relativity course that the Einstein equations are second-order equations for the metric. Why do you think this will be the case?

Another formulation of Noether's theorem

In the main text of these lecture notes, we always defined a symmetry as a transformation of the history $q(t)$. We did so, even in the case of time-translations, which we viewed as a map from a history $q(t)$ to a time-translated one $q'(t) = q(t+T)$ (active perspective), and not as a transformation of the time parameter $t \mapsto t' = t + T$ that leaves the history invariant $q'(t') = q(t)$ (passive perspective). We now address Noether's theorem from the latter perspective.

Consider a joint transformation of the time parameter t and of the history $q(t)$:

$$t \mapsto t' = t'(t) \quad \text{and} \quad q(t) \mapsto q'(t'),$$

which infinitesimally we write as

$$t \mapsto t' = t + \epsilon \bar{\delta}_s t(t) \quad \text{and} \quad q(t) \mapsto q'(t') = q(t) + \epsilon \bar{\delta}_s q(t).$$

where both $\bar{\delta}_s t$ and $\bar{\delta}_s q$ are understood as stand-alone symbols denoting a particular function of time.

Remark 2.4.1. We introduced the notation $\bar{\delta}_s$ to distinguish it from the variations δq and $\widetilde{\delta}_s q$ introduced in the main text, both of which are quite different in nature. However, you should feel free to use a simple δ to lighten up the notation when performing your computations. ♢

1. Consider the infinitesimal time element dt (used e.g. when writing an integral) and show that under the above transformation it changes as follows:

$$\bar{\delta}_s dt = \frac{d\bar{\delta}_s t}{dt} dt.$$

2. Show that under the above transformation, the velocity $\dot{q}(t)$ changes as follows:

$$\dot{q}(t) \mapsto \dot{q}'(t') = \dot{q}(t) + \epsilon \bar{\delta}_s \dot{q}(t), \quad \text{with} \quad \bar{\delta}_s \dot{q}(t) = \frac{d}{dt} \bar{\delta}_s q(t) - \dot{q} \frac{d}{dt} \bar{\delta}_s t.$$

   Note: contrary to what happens for the variations $\widetilde{\delta}_s q$, here $\bar{\delta}_s \dot{q} \neq \frac{d}{dt} \bar{\delta}_s q(t)$.

3. Consider a function $L(q, v, t)$, and use it to define a functional of a history $q(t)$ and time t:

$$L(q(t), \dot{q}(t), t).$$

   and its variation $\bar{\delta}_s L$ under the above infinitesimal joint transformation of t and of the history $q(t)$,

$$L(q'(t'), \dot{q}'(t'), t') = L(q(t), \dot{q}(t), t) + \epsilon \bar{\delta}_s L(q(t), \dot{q}(t), t) + O(\epsilon^2).$$

   Show that it is given by the following expression:

$$\bar{\delta}_s L = \frac{dL}{dt} \bar{\delta}_s t + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \bar{\delta}_s q - \frac{\partial L}{\partial \dot{q}} \dot{q} \bar{\delta}_s t \right) - (\bar{\delta}_s q - \dot{q} \bar{\delta}_s t) \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right).$$

4. Given a history $q(t)$, define the Noether charge

$$\bar{Q}_s(t) \doteq \frac{\partial L}{\partial \dot{q}} \bar{\delta}_s q - \left(\frac{\partial L}{\partial \dot{q}} \dot{q} - L \right) \bar{\delta}_s t.$$

Interpreting L as a Lagrangian, introduce the action $S[q(t)] = \int_{t_0}^{t_1} dt\, L(q(t), \dot{q}(t), t)$, and show that if the action is invariant under the joint variation $(\bar{\delta}_s t, \bar{\delta}_s q(t))$, then the Noether charge $\bar{Q}_s$ is a constant of motion:

$$\bar{\delta}_s S = 0 \implies \frac{d}{dt} \bar{Q}_s \hat{=} 0 \qquad \text{(Noether theorem)}$$

i.e. that $\bar{Q}_s$ is conserved along any history that satisfies the Euler–Lagrange equations of motion.

5. Note that the Noether charge $\bar{Q}_s$ can be written as[5]

$$\bar{Q}_s(t) = \sum_i p_i \bar{\delta}_s q^i - E \bar{\delta}_s t,$$

where p_i is the canonical momentum associated to q^i and E is the generalized energy. Conclude from this expression that if q^i is a cyclical variable, then p_i is conserved as a Noether charge, and if L is independent of time, then E is conserved as a Noether charge.

6. Let $L(\vec{x}, \dot{\vec{x}})$ be a Lagrangian for a point particle that is invariant under rotations around the z-axis. Use a change of variables to cylindrical coordinates to identify a cyclical variable and show that the following quantity is its conjugate momentum and therefore a Noether charge:

$$M_z = \hat{z} \cdot (\vec{x} \times \vec{p}), \quad p_i \doteq \frac{\partial L}{\partial \dot{x}^i}$$

What does it represent physically?

Remark 2.4.2. Although not apparent, this version of the Noether theorem is equivalent to the one given in the text. Roughly speaking we traded the opportunity of transforming t (while leaving the action exactly invariant), for the weaker demand of requiring invariance of the action only up to a boundary term R_s. ♢

Bonus: a sneaky recipe for the Noether charge

Here we give a "sneaky recipe" for how to extract the Noether charge by turning a *global* infinitesimal symmetry into a *local* infinitesimal variation.[6] We follow here the version of the Noether theorem given in the main text.

Here is the recipe:

[5] Here we also reintroduce the possibility of having multiple generalized coordinates q^i.

[6] See e.g. Section 7.3 of Weinberg, The Quantum Theory of Fields, Vol. 1, Cambridge University Press (2005).

1. Let
$$q(t) \mapsto q'(t) = q(t) + \epsilon \widetilde{\delta}_s q(t)$$
be such that, *for ϵ an infinitesimal constant*, it leaves the action functional invariant, $\widetilde{\delta}_s L = \frac{d}{dt} R_s$.

2. Consider now the same transformation of t and $q(t)$ written at the previous point, but replace the infinitesimal constant ϵ with an arbitrary infinitesimal function $\epsilon f(t)$, *that vanishes at the boundary* of the time interval, i.e.
$$\widetilde{\delta}_s q(t) \rightsquigarrow \delta_s^f q(t) = f(t) \widetilde{\delta}_s q(t) \quad \text{and} \quad f(t_0) = 0 = f(t_1).$$
The variation of the action under the new transformation will not vanish anymore. Repeat the steps used to prove Noether's theorem to show that now
$$\delta_s^f S = \int_{t_0}^{t_1} dt \; Q_s \dot{f},$$
with $Q_s = \frac{\partial L}{\partial \dot{q}} \widetilde{\delta}_s q - R_s$ the Noether charge.

3. Show that the above equality implies that $Q_s(t)$ is conserved on-shell of the equations of motion.

Chapter 3

Hamiltonian mechanics

3.1 Prelude

The Euler-Lagrange equations, like Newton's equation, are second order ODEs (ordinary differential equations). How do one goes to solve them?

The standard way to approach this problem is to double the variables and rewrite the 2nd order ODEs as 1st order ODEs, which can be solved by a standard method—at least in principle, or perturbatively. Later we will see what is the *geometrical* structure underpinning this solution method.

$$\text{2nd order for } q(t)\text{: } \ddot{q}^i = f^i(q,\dot{q}) \;\rightsquigarrow\; \text{1st order for } (q,v)(t)\text{: } \begin{cases} \dot{q} = v^i \\ \dot{v}^i = f^i(q,v) \end{cases}$$

For a Lagrangian system there is a particularly nice way to introduce 1st order ODEs: this is the Hamiltonian formalism.

3.2 Legendre transform

Given a Lagrangian system, $L = L(q,v,t)$, the naive starting point to replace the 2nd order EL ODEs with a set of first order equations is to to write

$$\begin{cases} \dot{q}^i = v^i \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial v^i} = \dfrac{\partial L}{\partial q^i} \end{cases}$$

However, this is a little ugly, because we do not have a nice expression for $\dot{v}^i$.

An expression that appears nicely, here, inside a time derivative is that for the generalized momentum

$$p_i = p_i(q,v,t) \doteq \frac{\partial L(q,v,t)}{\partial v^i}.$$

What if we were able to use p_i as an independent variable, instead of v^i?

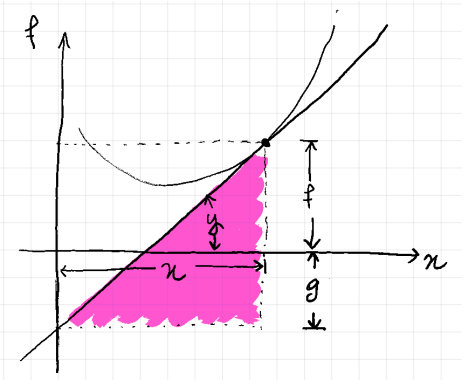

Figure 3.1: The Legendre transform $g(y)$ of the function $f(x)$.

This is the job of the *Legendre transform*:[1] indeed, given a (convex) function $x \mapsto f(x)$ the Legendre transform creates a function that "contains the same information" but as a function of $y = \frac{df}{dx} \mapsto g(y)$.

Note that for this to be possible at all, we need the map $x \mapsto \frac{df}{dx}$ to be 1-to-1, i.e. df/dx to be monotonous, i.e. $\frac{d^2 f}{dx^2} > 0$ (or < 0 but of definite sign), that is f (or $-f$) to be convex.

Since $x \mapsto y(x) = \frac{df}{dx}$ is 1-to-1, denote its inverse by $y \mapsto x(y)$. The new function of y that we are looking for, i.e. the *Legendre transform* $g(y)$ of $f(x)$, is defined as

$$g(y) \doteq yx(y) - f(x(y)).$$

change figure
$s \rightsquigarrow y$

To understand its meaning, consider the colored triangle in Figure 3.1, and write the defining equation implicitly as:

$$xy = f(x) + g(y).$$

Note that this expression is completely symmetric under $(x, f) \leftrightarrow (y, g)$. This means that the Legendre transform is an involution, i.e. applied twice it gives back the original function. In particular:

$$\frac{dg}{dy} = \frac{d}{dy}\Big(x(y)y - f(x(y))\Big) = x(y) + \frac{dx(y)}{dy}y - \underbrace{\frac{df}{dx}}_{=y}\frac{dx(y)}{dy} = x(y).$$

3.3 Hamilton's equations

Recall that our goal was to write the Euler–Lagrange equations in terms as first order equations, and that we thought it would be a good idea to try and express these first

[1] See Zia et al., Making Sense of the Legendre Transform, American Journal of Physics 77, 614–622 (2009), arxiv:0806.1147.

order equations in terms of q and $p \doteq \partial L/\partial v$, instead of q and v. Since p is the derivative of L with respect to v, the information encoded in the Lagrangian function $L(q, v, t)$ has to be repackaged into a new function of (q, p, t). We saw in the previous section that this is the job accomplished by the Legendre transform.

The Hamiltonian function $H(q, p, t)$ is defined as the Legendre transform of $L(q, v, t)$ with respect to the variable v:

$$H(q, p, t) = pv - L(q, v, t), \quad p \doteq \frac{\partial L}{\partial v},$$

where in the right-hand side of the above equation v is everywhere understood as the inverse function $v = v(q, p, t)$ of $p = \frac{\partial L}{\partial v}$.[2]

Often, for short, the above expression is written as

$$H(q, p, t) = p\dot{q} - L(q, \dot{q}, t), \quad p \doteq \frac{\partial L}{\partial \dot{q}}.$$

Now notice the following two properties: since the Legendre transform concerns only v, and not q, we have that

$$\frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}.$$

Moreover, due to the involutive character of the Legendre transform, we have that the "inverse" of the definition $p = \partial L/\partial v$ is given by

$$v = \frac{\partial H}{\partial p}.$$

Combining these properties of the Legendre transform, we can finally translate the Euler–Lagrange equations

$$\begin{cases} \dot{q} = v \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial v} = \dfrac{\partial L}{\partial q} \end{cases}$$

to the Hamiltonian picture, to obtain:

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

This equation are symmetric and elegant in $q \leftrightarrow p$, up to a crucial sign. From the next chapter will start exploring the geometrical and algebraic properties of this hiding behind this formulation of mechanics (these relate to the mathematical branches known as symplectic and Poisson geometry). First, however, we turn to a series of remarks, examples, as well as to the formulation of a action principle for the Hamiltonian formalism.

[2] This is the analogue of the inverse function $x(y)$ of $y = df/dx$ from the previous section.

Remark 3.3.1 (Convexity)**.** For the Legendre transform to be applicable, one needs to be dealing with a convex function. The Lagrangian function is quite generally quadratic in the velocities, and is therefore a convex function in v. ♢

Example 3.3.2 (Particle mechanics)**.** Consider the particle's Lagrangian

$$L(\vec{q},\vec{v},t)=\frac{1}{2}m|\vec{v}|^2+V(\vec{q}).$$

Then

$$p_i=\frac{\partial L}{\partial v^i}=mv \iff \vec{v}=\frac{\vec{p}}{m}.$$

Thus, denoting $\vec{v}\equiv\vec{p}/m$, we find:

$$H(\vec{q},\vec{p},t)=\vec{p}\cdot\vec{v}-L(\vec{q},\vec{v},t)=\frac{|\vec{p}|^2}{m}-\frac{1}{2}m|\vec{v}|^2|+V(\vec{q})=\frac{|\vec{p}|^2}{2m}+V(\vec{q}). \qquad ♢$$

3.4 The first order action principle

Consider a configuration space $Q\ni q^i$. While the coordinates (q^i,v^i) identify points in the tangent bundle $\mathrm{T}Q$, the coordinates (q^i,p_i) identify points in the *cotangent bundle* $P=\mathrm{T}^*Q$. Let us denote points in phase space by

$$z^I=(q^i,p_i)\in P=\mathrm{T}^*Q.$$

Hamilton's action principle, which we could symbolically summarize as

$$\delta S[\bar{q}(t)]=0,$$

is eminently based on configurations space and gives rise to second order ODEs. Consider now the following action functional:

$$S_{1st}[z(t)]=\int_{t_0}^{t_1}dt\ \sum_i p_i(t)\dot{q}^i(t)-H\left(q^i(t),p_i(t),t\right).$$

Then, the variation of S_{1st} is given by (we omit the sum over repeated indices):

$$\begin{aligned}\delta S_{1st}[z(t)]&=\int_{t_0}^{t_1}dt\ \sum_i \dot{q}^i\delta p_i+p_i\delta\dot{q}^i-\frac{\partial H}{\partial q^i}\delta q^i-\frac{\partial H}{\partial p_i}\delta p_i\\&=\int_{t_0}^{t_1}dt\sum_i\left(\dot{q}^i-\frac{\partial H}{\partial p_i}\right)\delta p_i-\left(\dot{p}_i+\frac{\partial H}{\partial q^i}\right)\delta p_i+\Big[p_i\delta q_i\Big]_{t_0}^{t_1}\end{aligned}$$

Thus, we see that we can formulate the following action principle in phase space, which yields the Hamilton equations of motion:

The dynamics of any[3] *physical system is fully characterized by a Hamiltonian function over phase space. Its physically realized histories are extrema of the associated action functional at fixed boundary condition (in time) for the configuration variables.*

Note that only boundary conditions in the configuration variables q^i need to be imposed:

$$\delta q^i(t_0) = 0 = \delta q^i(t_1),$$

while the boundary values of the momentum variables, $p_i(t_0)$ and $p_i(t_f)$, are left unconstrained.

Since the equations of motion resulting from this phase space action principle are first order ODEs, sometimes one refers to the action principle in the Hamiltonian picture as a *first-order action principle*.

[3]This has, of course, to be taken with a grain of salt.

Chapter 4

Phase space structures

4.1 The geometrical meaning of Hamilton's equations of motion

Consider a physical history in phase space, i.e. a history $\bar{z}^I(t) = (\bar{q}^i(t), \bar{p}_i(t))$ that satisfies Hamilton's equation of motion:

$$\begin{cases} \bar{q}^i(t+\epsilon\delta t) = \bar{q}^i(t) + \epsilon\delta t\dot{\bar{q}}^i(t) + O(\epsilon^2) = \bar{q}^i(t) + \epsilon\delta t\dfrac{\partial H}{\partial p}(\bar{q}(t), \bar{p}(t), t) + O(\epsilon^2) \\ \bar{p}_i(t+\epsilon\delta t) = \bar{p}_i(t) + \epsilon\delta t\dot{\bar{p}}_i(t) + O(\epsilon^2) = \bar{q}(t) - \epsilon\delta t\dfrac{\partial H}{\partial q^i}(\bar{q}(t), \bar{p}(t), t) + O(\epsilon^2) \end{cases}$$

In other words, the partial derivatives of the Hamiltonian $\partial H/\partial q^i$ and $-\partial H/\partial p_i$ define a "velocity field" X_H over phase space $P \ni z^I$:

$$z^I = \begin{pmatrix} q^i \\ p_i \end{pmatrix}, \qquad X_H^I(q,p) = \begin{pmatrix} -\partial H/\partial p_i \\ \partial H/\partial q^i \end{pmatrix}, \qquad \dot{\bar{z}}^I(t) = X_H^I(\bar{z}).$$

In analogy to fluid mechanics where solving the dynamics means finding the trajectories of the fluid molecules as given by a velocity field satisfying certain equations, solving the Hamilotnian equations of motion means integrating the velocity field X_H to find the trajectory of the system in phase space. With this picture in mind, the phase space "velocity field" X_H is generally called the *Hamiltonian flow of* H.

Since the velocity field X_H is obtained by partial derivative of the Hamiltonian, we would like to think of H as a sort of "potential" for the velocity field. However, contrary to the standard potential which is given by a gradient, the relationship between X_H and H involves some peculiar minus signs in front of *some* of the partial derivatives. We will now introduce a formalism that will, in particular, encode these twisted gradients.

4.2 Poisson brackets

Consider a function F over phase space:

$$F : P \times \mathbb{R} \to \mathbb{R}, \quad (z^I, t) \equiv (q^i, p_i, t) \mapsto F(z^I, t) \equiv F(q^i, p_i, t)$$

Example 4.2.1. A particular, and fundamental, example of such a function is the Hamiltonian function $H(q, p, t)$ itself. ◊

One can then evaluate F on a physical history $\bar{z}(t)$, obtaining the function of time

$$f_{\bar{z}} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto f_{\bar{z}}(t) = F(\bar{z}^I(t), t).$$

We then ask: how does the value $f_{\bar{z}}$ of F on the physical history $\bar{z}^I(t)$ evolve in time? The answer is

$$\begin{aligned}
\frac{df_{\bar{z}}}{t} &\doteq \frac{d}{dt} F(\bar{z}^I(t), t) \\
&= \left(\frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q^i} \dot{q}^i + \frac{\partial F}{\partial p_i} \dot{p}_i \right)_{(\bar{q}(t), \bar{p}(t), t)} \\
&= \left(\frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q^i} \right)_{(\bar{q}(t), \bar{p}(t), t)}
\end{aligned}$$

This expression suggests the introduction of the following map, called the *Poisson bracket*:

$$\begin{aligned}
\{\cdot, \cdot\} : C^\infty(P) \times C^\infty(P) &\to C^\infty(P) \\
(F, G) \qquad &\mapsto \{F, G\} \doteq \sum_i \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i}
\end{aligned}$$

This bracket, in other words, "eats" two functions over phase space, F and G, and "spits out" a third function over phase space, denoted $\{F, G\}$, which is built out of the partial derivatives of the original function with respect to the configuration and momentum variables, q^i and p_i.

The Poisson bracket $\{\cdot, \cdot\}$ satisfies—and it is in fact abstractly defined by—the following four properties (here $F, G, H \in C^\infty(P)$ and $a \in \mathbb{R}$ are completely arbitrary):

1. *skew symmetry*: $\{F, G\} + \{G, F\} = 0$;
2. *(bi)-linearity*: $\{aF + G, \cdot\} = a\{F, \cdot\} + \{G, \cdot\}$;
3. *Jacobi*: $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$;
4. *Leibniz*: $\{FG, \cdot\} = F\{G, \cdot\} + \{F, \cdot\}G$.

Remark 4.2.2. A bracket that satisfies the formal properties (1–3) is called a Lie bracket. ◊

With this notation, the Hamiltonian time evolution of a function $F(z,t)$ takes the form:

$$\frac{df_{\bar{z}}}{dt} = \left(\frac{\partial F}{\partial t} + \{F, H\}\right)\Big|_{(\bar{z}(t),t)} \tag{4.1}$$

For short, this is often written as

$$\dot{F} = \partial_t F + \{F, H\}.$$

This equation is exactly equivalent to Hamilton's equations of motion.

The following statement, which we saw in Chapter 2 as a rather nontrivial application of Noether's first theorem to time translation invariant systems, admits a 1-line proof in the Hamiltonian formalism:

Theorem 4.2.3. *The Hamiltonian function $H(q^i, p_i, t)$ in a constant of motion iff $\partial_t H = 0$ iff $\partial_t L = 0$*

Proof. Expressing Hamilton's equation of motion via the Poisson bracket and making use of its skew symmetry:

$$\dot{H} = \partial_t H + \{H, H\} = \partial_t H = -\partial_t L,$$

where the last equality follows from the definition of the Hamiltonian itself as the Legendre transform of L with respect to v. The theorem follows. □

We conclude this section with a remark:

Remark 4.2.4. The Poisson bracket between phase space coordinates (q^i, p_i) reads:

$$\{q^i, q^j\} = 0 = \{p_i, p_j\} \quad \text{and} \quad \{q^i, p_j\} = \delta^i_j,$$

where δ^i_j is equal to 1 if $i = j$ and to zero otherwise. This is fully analogous to the Heisenberg commutation relation found in quantum mechanics:

$$[\hat{q}^i, \hat{q}^j] = 0 = [\hat{p}_i, \hat{p}_j] \quad \text{and} \quad [\hat{q}^i, \hat{p}_j] = \delta^i_j \, \mathrm{i}\hbar \mathbb{1}.$$

Similarly, Hamilton's equation in the form $\dot{F} = \partial_t F + [F, H]$ are analogous to Eherenfest's theorem:

$$\mathrm{i}\hbar \frac{d}{dt}\langle \hat{F} \rangle = \mathrm{i}\hbar \langle \partial_t \hat{F} \rangle + \langle [\hat{F}, \hat{H}] \rangle,$$

where angular brackets are a short-hand notation for the expectation value of an operator with respect to a given state $\psi \in \mathcal{H}$, i.e. $\langle \hat{F} \rangle \equiv \langle \psi | \hat{F} | \psi \rangle$, for $\mathcal{H}$ the appropriate Hilbert space. ◊ double check

4.3 Canonical transformations

The previous definitions rely on the choice of special coordinates q^i and p_i. In terms of the Poisson brackets, what is special about these coordinates is that they satisfy

$$\{q^i, q^j\} = 0 = \{p_i, p_j\} \quad \text{and} \quad \{q^i, p_j\} = \delta^i_j.$$

What if, however, certain phase space functions are best expressed in terms of other coordinates, (Q^i, P_i)? When are we allowed to write the original Poisson bracket in terms of the new variables?

Let's compute: say

$$Q^i = Q^i(q,p) \quad \text{and} \quad P_i = P_i(q,p),$$

and denote

$$f(q,p) \doteq F(Q(q,p), P(q,p)) \quad \text{and} \quad g(q,p) \doteq G(Q(q,p), P(q,p))$$

Then, using Leibniz's rule,

$$\frac{\partial f}{\partial q^i} = \sum_a \frac{\partial Q^a}{\partial q^i}\frac{\partial F}{\partial Q^a} \quad \text{etc}$$

one can show (do it!) that

$$\begin{aligned}\{f,g\}_{(q,p)} &\doteq \sum_i \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \\ &= \sum_{k,l} \{Q^k, Q^l\}_{(p,q)} \frac{\partial F}{\partial Q^k}\frac{\partial G}{\partial Q^l} + \{P^k, P^l\}_{(p,q)} \frac{\partial F}{\partial P^k}\frac{\partial G}{\partial P^l} \\ &\qquad + \{Q^k, P^l\}_{(p,q)} \left(\frac{\partial F}{\partial Q^k}\frac{\partial G}{\partial P^l} - \frac{\partial F}{\partial P^k}\frac{\partial G}{\partial Q^l}\right)\end{aligned}$$

where we relabelled certain indices Therefore, we deduce that

$$\{f,g\}_{(q,p)} = \{F,G\}_{(Q,P)} \quad \forall F, G \iff \begin{cases} \{Q^i, Q^j\}_{(q,p)} = 0 = \{P_i, P_j\}_{(q,p)}, \\ \{Q^i, P_j\}_{(p,q)} = \delta^i_j. \end{cases}$$

A set of coordinates that satisfy these properties is said *canonical*. A change in canonical coordinates, $(q,p) \mapsto (Q,P)$, is said a *canonical transformation*.

Remark 4.3.1. In view of (4.1), the Hamiltonian equations of motion takes the same form in all sets of *canonical* coordinates. ◊

Consider now an infinitesimal change of variable:

$$Q^i = q^i + \epsilon \delta q^i \quad \text{and} \quad P_i = p_i + \epsilon \delta p_i$$

where $\delta q^i = \delta q^i(q,p)$ etc.

Then,

$$\{Q^i, P_j\} = \cdots = \delta^i_j + \epsilon \left(\frac{\partial \delta q^i}{\partial q^j} + \frac{\partial \delta p_j}{\partial p_i} \right) + O(\epsilon^2).$$

and thus we deduce that the above is an infinitesimal *canonical* change of variables iff

$$\{Q^i, P_j\} = \delta^i_j \iff \frac{\partial \delta q^i}{\partial q^j} + \frac{\partial \delta p_j}{\partial p_i} = 0.$$

A large family of solutions to this equation is given by variations of the form[1]

$$\begin{cases} \delta q^i = \dfrac{\partial F}{\partial p_i} = \{q^i, F\}, \\ \delta p_i = -\dfrac{\partial F}{\partial p_i} = \{p_i, F\}. \end{cases} \quad \text{for some} \quad F \in C^\infty(P).$$

Comparing with the definition of a Hamiltonian flow given at the beginning of this chapter, we see that this result tells us that: *any infinitesimal canonical transformations is the Hamiltonian flows for a function F over phase space, and viceversa..*

Remark 4.3.2 (Time evolution as a canonical transformation)**.** If the Hamiltonian function is *time independent*, $\partial_t H = 0$, we can chose $F = H$ and conclude that *time evolution with respect to a time-independent Hamiltonian defines a canonical transformation.* More formally, if we denote $\bar{z}^I(t) = (\bar{q}^i(q,p,t), \bar{p}_i(q,p,t))$ the (unique) solution to Hamilton's equations of motion with initial conditions $\bar{z}^I(t=0) = (q^i, p_i)$, then the map

$$\left(q^i, p_i\right) \mapsto \left(Q^i, P_i\right) = \left(\bar{q}^i(q,p,T), \bar{p}_i(q,p,T)\right),$$

is a canonical transformation for any value of T—please take some time to unpack this statement. This observation is at the basis of the Hamilton–Jacobi theory. ♢

Theory of finite generating functions?

4.4 Liouville's theorem

We conclude this chapter with the statement of an important result:

Theorem 4.4.1 (Liouville)**.** *Canonical transformations preserve the Liouville phase space volume:*

$$d^{2n}\mu_L \doteq dq^1 dp_1 \ldots dq^n dp_n,$$

where (q^i, p_i) is (any) set of canonical. In particular, time evolution with respect to a time-independent Hamiltonian preserves teh Liouville volume form.

Proof, for $n = 1$ only. Consider a canonical transformation $(q,p) \mapsto (Q,P)$, then:

$$dQdP = \left| \det \left(\frac{\partial (QP)}{\partial (q,p)} \right) \right| dqdp = |\det \{Q,P\}_{(q,p)}| dqdp = dqdp. \qquad \square$$

[1] This equation is an integrability condition that tells us that such a function $F(q,p)$ always exists provided no global, i.e. topological, obstruction arises.

A 1-line proof of Liouville's theorem for arbitrary n is available when one adopts the formalism of symplectic geometry in terms of differential forms.

Second Tutorial

Poisson brackets & angular momentum

1. Consider a point particle in $\mathbb{R}^3$ described by Cartesian coordinates $\vec{x}$, with their conjugate momenta $\vec{p}$. Its angular momentum is defined as $\vec{L} \doteq \vec{x} \times \vec{p}$, or, in components

$$L_i = \sum_{j,k=1}^{3} \epsilon_{ijk} x^j p^k \,.$$

   Here ϵ_{ijk} stands for the totally skew symmetric symbol with $\epsilon_{123} = 1$. Show that

$$\{L_i, L_j\} = \sum_{k=1}^{3} \epsilon_{ijk} L_k \,, \quad \{\vec{L}^2, L_i\} = 0 \,.$$

   [*Hint*: You can use either the explicit expression of the canonical bracket or, better, its Leibniz property.]

2. Let $H(q,p)$ be the Hamiltonian. Recall why a function $Q(q,p)$ over phase space is called a constant of motion if and only if $\{Q, H\} = 0$. Show that when we have two such constants, Q_1 and Q_2, their Poisson bracket,

$$Q_3 = \{Q_1, Q_2\} \,,$$

   is also a constant of motion. Conclude that whenever two components of the angular momentum are conserved, so is the third one.

Canonical transformations

Consider a transformation of the coordinates by a rotation $R^i{}_j \in \mathrm{SO}(3)$,

$$x^i \to x'^i = \sum_{j=1}^{3} R^i{}_j x^j .$$

1. How do their conjugate momenta $p_i = \frac{\partial L}{\partial \dot{x}^i}$ transform under this transformation if the Lagrangian of the theory has rotational symmetry?

2. Show that this transformation of (x^i, p_i) is a canonical transformation. [*Hint:* you may for example check the fundamental Poisson brackets or look at the Hamilton's equations.]

Canonical transformations from generating functions

Consider a Hamiltonian system with first order Lagrangian $L_{1\text{st}} = p\dot{q} - H(q,p)$, as well as a function $F(q,Q,t)$. Let

$$L'_{1\text{st}} = L_{1\text{st}} - \frac{d}{dt}F.$$

In this tutorial we will denote partial derivates as in $F_x \equiv \frac{\partial}{\partial x}F$.

1. Argue that, for any fixed value of Q, the Lagrangians L and L' give equivalent action principles under the condition $\delta q(t_0) = 0 = \delta q(t_1)$.

2. Now, switch perspective and let Q be itself a function $Q(q,p,t)$. Vary the primed Lagrangian and find an equation involving a partial derivative of F that implicitly determines the function $Q(q,p,t)$ in such a way that $L'_{1\text{st}}$ gives a well-defined action principle under the condition $\delta Q(t_0) = 0 = \delta Q(t_1)$.

3. We now want to interpret Q as a new canonical variable and we ask what function $P(q,p,t)$ defines a canonical change of variables, $(q,p) \rightsquigarrow (Q,P)$. Use the new action principle for $L'_{1\text{st}}$ to argue that

$$P = -\frac{\partial F}{\partial Q}$$

and that the Hamiltonian for the new variables is related to the old one by

$$H'(Q,P) = H(q,p) + \frac{\partial F}{\partial t}.$$

4. Using the the answers to the previous two points, show that an infinitesimal variation of the new variables is related to an infinitesimal variation of the old ones and dt by

$$dQ = -(F_{Qq})^{-1}\left(dp + F_{qq}dq + F_{qt}dt\right)$$
$$dP = -F_{QQ}(F_{Qq})^{-1}dp + (F_{Qq} - F_{QQ}(F_{Qq})^{-1}F_{qq})dq + (F_{Qt} - F_{QQ}(F_{Qq})^{-1}F_{qt})dt$$

Use these expressions to explicitly prove that the (implicit) map $(q,p) \mapsto (Q,P)$ defined by the above relations is a canonical transformation, i.e. that

$$\{Q,P\}_{(q,p)} = 1.$$

The function F is called the *generating function* of the canonical transformation $(q,p) \mapsto (Q,P)$.

5. In the main text we discussed how the Hamiltonian equations of motion define an infinitesimal canonical transformation. Here we ask what is the finite version of this canonical transformation. To avoid confusion, we rename the initial set of variables (q_0, p_0) and use the symbol $q_t = q(q_0, p_0, t)$ and $p_t = p(q_0, p_0, t)$ to denote the value of position and momentum at time t along the history that had initial conditions $q_0 = q(q_0, p_0, 0)$ and $p_0 = p(q_0, p_0, 0)$. Then, we want to see if the following transformation is canonical

$$(q_0, p_0) \mapsto (q_t, p_t).$$

Show that in the new variables, time evolution reads:

$$\dot{q}_t = \partial_t q_t \quad \text{and} \quad \dot{q} = \partial_t q_t,$$

and deduce that this means that $H'(q_t, p_t, t) = 0$. Conclude that the corresponding generating function, denoted $F = \mathcal{S}(q_0, p_t, t)$, satisfies the following equation:

$$\frac{\partial \mathcal{S}}{\partial t} + H\left(q, \frac{\partial \mathcal{S}}{\partial q}\right) = 0.$$

This is known as the *Hamilton–Jacobi* equation, and $\mathcal{S}$ as *Hamilton's principal function*. We will encounter this equation again later in the course.

Chapter 5

Symmetries in phase space

5.1 Hamiltonian flows

Recall Noether's theorem (Chapter 2). There, we considered a function s of the form

$$\mathrm{T}Q \times \mathbb{R} \to \mathrm{T}Q, \quad (q, v, t) \mapsto (q, s(q, v, t))$$

to define an infinitesimal variation of an history $q(t)$:

$$q(t) \mapsto q(t) + \epsilon \tilde{\delta}_s q(t).$$

This variation was said a symmetry if it left the action invariant up to boundary terms,

$$\tilde{\delta}_s L = \frac{d}{dt} R_s.$$

Then, we proved Noether's theorem stating that to all symmetry s there corresponds a conserved quantity Q_s:

$$Q_s = \sum_i p_i \tilde{\delta}_s q^i - R_s, \qquad \frac{dQ_s}{dt} \hat{=} 0,$$

where $\hat{=}$ defines an equality that holds along any solution of the equations of motion.

In the Hamiltonian picture we have seen that a time-independent function on phase space, $Q(q, p)$, is conserved iff $\{Q, H\} = 0$. We now ask: to what phase space transformation does the conserved quantity Q corresponds to?

To answer the previous question we go back to the beginning of Chapter 4, where to the Hamiltonian H we associated a "velocity field" X_H according to the formula:

$$z^I = \begin{pmatrix} q^i \\ p_i \end{pmatrix}, \qquad X_H^I(q, p) = \begin{pmatrix} -\partial H/\partial p_i \\ \partial H/\partial q^i \end{pmatrix}, \qquad \dot{\bar{z}}^I(t) = X_H^I(\bar{z}).$$

Now, it is clear that the only characteristic of these equations that is peculiar to having chosen the Hamiltonian function H, is the fact that X_H^I encodes the *time*

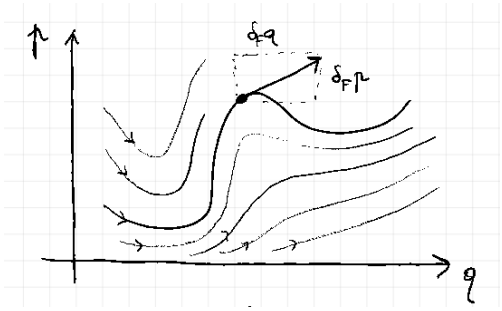

Figure 5.1: The Hamiltonian flow X_F of $F \in C^\infty(P)$.

evolution along a physical history: all other formulas could be written for a generic function over phase space, $H \rightsquigarrow Q$.

Given a function over phase space, $F \in C^\infty(P)$, define the infinitesimal variations $(q^i, p_i) \mapsto (q^i + \epsilon\tilde{\delta}_Q q^i, p_i + \epsilon\tilde{\delta}_Q p_i)$, by

$$\begin{cases} \tilde{\delta}_Q q^i = \{q^i, Q\} = \dfrac{\partial Q}{\partial p_i} \\ \tilde{\delta}_Q q^i = \{p_i, Q\} = -\dfrac{\partial Q}{\partial q^i} \end{cases}$$

If Q is a time-independent function over phase space, $Q \in C^\infty(P)$, *and* if Q Poisson-commute with the Hamiltonian, $\{Q, H\} = 0$, then we call Q a *(Hamiltonian) charge* and $(\tilde{\delta}_Q q^i, \tilde{\delta}_Q p_i)$ the *(Hamiltonian) symmetry generated by the charge* Q.

As in the case of the Hamiltonian function and time evolution, we can define a "generalized velocity field", that is a vector field,over phase space—that is the *Hamitltonian vector field*, or *Hamiltonian flow*, of Q:[1]

$$z^I = \begin{pmatrix} q^i \\ p_i \end{pmatrix}, \qquad X_Q^I(q,p) = \begin{pmatrix} \tilde{\delta}_Q q^i \\ \tilde{\delta}_Q p_i \end{pmatrix} = \begin{pmatrix} -\partial Q/\partial p_i \\ \partial Q/\partial q^i \end{pmatrix}.$$

change $F \rightsquigarrow Q$

See figure 5.1.

5.2 Properties of Hamiltonian flows

The previous definition of a Hamiltonian flow X_Q suggests in fact a more concise way of writing it in terms of the "collective" phase space coordinate z^I and the Poisson bracket $\{\cdot,\cdot\}$:

$$X_Q^I(z) = \{z^I, Q\},$$

[1] A vector field is the smooth assignment of an arrow at each point of phase space; thinking of the arrows as instantenous velocities in a fluid, we can define the trajectories followed by a fluid particle: these trajectories defines a family of lines over phase space called the *(integral) flow of the vector field*.

or, even more succinctly (we used the skew symmetry of the bracket):

$$X_Q = -\{Q, \cdot\}$$

In other words, we are re-interpreting the Poisson bracket as a map from functions over phase space to vector fields over phase space; in symbols:

$$\{\cdot, \cdot\} : C^\infty(P) \to \mathfrak{X}(P), \quad Q \mapsto X_Q = -\{Q, \cdot\}.$$

This makes sense because a vector field is essentially a derivation, and derivations are simply objects that eat functions and spit out functions while satisfying Leibniz's rule (which the Poisson bracket satisfies, see Section 4.2).

To make this more explicit, consider a generic vector field

$$X \in \mathfrak{X}(P),$$

which at $z \in P$ is an arrow of component $X^I(z)$ in the deriction z^I. Then, the vector field X defines a derivation via:

$$X : C^\infty(P) \to C^\infty(P), \quad f \mapsto X(f) \quad \text{defined by} \quad X(f)(z) = X^I(z)\frac{\partial f}{\partial z^I}.$$

Finally, in the specific case of the Hamiltonian vector field associated to Q, we have:

$$X_Q(f) = X_Q^I \frac{\partial f}{\partial z^I} = \{f, Q\}.$$

Given two vector fields, we can define a third one by taking their *Lie bracket*:

$$[\cdot, \cdot] : \mathfrak{X}(P) \times \mathfrak{X}(P) \mapsto \mathfrak{X}(P), \quad (X, Y) \mapsto [X, Y]$$

defined by its action on an arbitrary function on phase space, $f \in C^\infty(P)$ as follows:

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Exercise 5.2.1. Show that, setting $Z = [X, Y]$, in components it reads:

$$Z^I = \sum_J X^J \partial_J Y^I - Y^J \partial_J X^I \quad \text{where} \quad \partial_J \equiv \frac{\partial}{\partial z^J}. \qquad \Diamond$$

It is easy to verify that this is a Lie bracket in the sense of Section 4.2, i.e. that (formulas are understood to hold $\forall X, Y, Z \in \mathfrak{X}(P)$ and $a \in \mathbb{R}$):

1. it is skew-symmetric; $[X, Y] = [Y, Z]$;
2. it is (bi-)linear: $[aX + Y, \cdot] = a[X, \cdot] + [Y, \cdot]$;
3. it satisfies Jacobi, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

A natural question to ask is whether the Lie bracket of two *Hamiltonian* vector fields is itself Hamiltonian, i.e. whether for any $F, G \in C^\infty(P)$, there exists a $K \in C^\infty(P)$ such that $[X_F, X_G] = X_K$?

Let's compute! Consider an arbitrary function $f \in C^\infty(P)$, and write

$$\begin{aligned}[X_F, X_G](f) &= X_F(X_G(f)) - X_F(X_F(f)) = X_F(\{f, G\}) - X_G(\{f, F\}) \\ &= \{\{f, G\}, F\} - \{\{f, F\}, G\} = -\{\{G, f\}, F\} - \{\{f, F\}, G\} \\ &\overset{(*)}{=} \{\{F, G\}, f\} = -\{f, \{F, G\}\} = -X_{\{F,G\}}(f).\end{aligned}$$

where we used multiple times that the Poisson bracket skew-symmetric and, in the $(*)$-equality, that it also satisfies the Jacobi identity. Thus, we conclude with the fundamental result:

$$[X_F, X_G] = -X_{\{F,G\}}$$

In terms of symmetries, this result tells us that the commutator of two Hamiltonian infinitesimal symmetries (the left-hand side) is also a Hamiltonian symmetry (the right-hand side). This fact has far-reaching properties that highly constrain the physical theories we work with.

To provide one more layer of abstraction, let us introduce the concept of Lie algebra: a *Lie algebra* is a vector space V equipped with a Lie bracket (i.e. a skew, bilinear map $V \times V \to V$ that satisfies the Jacobi identity).

Both $(C^\infty(P), \{\cdot, \cdot\})$ and $(\mathfrak{X}(P), [\cdot, \cdot])$ are Lie algebras, and we just proved that the subspace of *Hamiltonian vector fields* $\mathfrak{X}_{\text{Ham}}(P) = \mathfrak{X}(P)$—i.e. the subspace of vector fields over P that are of the form $X_Q = -\{Q, \cdot\}$ for some $Q \in C^\infty(P)$—is in fact a sub-Lie algebra of $(\mathfrak{X}(P), [\cdot, \cdot])$, and one that is anti[2]-homomorphic to the Poisson algebra $(C^\infty(P), \{\cdot, \cdot\})$.

Remark 5.2.2. Therefore, we have that if Q is a symmetry (i.e. $\{Q, H\} = 0$), then $[X_Q, X_H] = 0$. This means that the symmetry flow commutes with the time-evolution flow, i.e. that two configurations that are related by the symmetry $\tilde{\delta}_Q z^I$ at time $t = t_0$ will evolve into two configurations related by the same symmetry at any subsequent (or precedent) time. ♢

Exercise 5.2.3. Consider the free 1d particle, i.e. $P = \mathbb{R}^2 \ni (q, p)$ and $H = \frac{1}{2}p^2$. Let $F(q, p) = p$ and $G(q, p) = q$. Which one is a symmetry? Draw X_H, X_F, and X_G, and use these drawings to deduce whether $[X_F, X_H]$ and/or $[X_G, X_H]$ vanish. Check the result by computing $\{F, H\}$ and $\{G, H\}$. [*Hint*: drawing the flow lines will not be enough, you will need to draw some of the arrows representing the direction *and magnitude* of X_H^I, X_F^I, and X_G^I.] ♢

[2]The map $Q \mapsto X_Q$ is an *anti*-homomorphism because although it maps the bracket on $C^\infty(P)$ to the bracket on $\mathfrak{X}(P)$, the two brackets are related via a *minus* sign.

5.3 Hamiltonian symmetries and Noether's theorem

We conclude this discussion on symmetries by looking at the first-order action principle and its invariance properties under a Hamiltonian symmetry transformation.

Consider the first oder action

$$S_{1\text{st}}[z(t)] = \int_{t_0}^{t_1} dt\ L_{1\text{st}}(z,\dot{z},t), \qquad L_{1\text{st}}(z,\dot{z},t) = \sum_i p_i\dot{q}^i - H(q,p,t).$$

and let $Q(q,p)$ be a Hamiltonian symmetry, i.e. $\{Q,H\}=0$.

Then, under the infinitesimal symmetry's flow,

$$z^I \mapsto z^I + \epsilon X_Q^I = z^I + \epsilon\{z^I, Q\}$$

we have that (we omit all terms of order ϵ^k, $k\geq 2$):[3]

$$\begin{aligned}
L_{1\text{st}}(z,\dot{z},t) &\mapsto \sum_i \Big(p_i + \epsilon\{p_i,Q\}\Big)\frac{d}{dt}\Big(q^i + \epsilon\{q^i,Q\}\Big) - H(z+\epsilon\{z,Q\}) \\
&= L_{1\text{st}}(z,\dot{z},t) + \epsilon\left(\sum_i \{p_i,Q\}\dot{q}^i + p_i\frac{d}{dt}\{q^i,Q\} - \sum_I \frac{\partial H}{\partial z^I}\{z^I,Q\}\right) \\
&= L_{1\text{st}}(z,\dot{z},t) + \epsilon\sum_i\left(\{p_i,Q\}\dot{q}^i - \dot{p}_i\{q^i,Q\} + \frac{d}{dt}\Big(p_i\{q^i,Q\}\Big)\right) \\
&= L_{1\text{st}}(z,\dot{z},t) + \epsilon\sum_i\left(-\frac{\partial Q}{\partial q^i}\dot{q}^i - \dot{p}_i\frac{\partial Q}{\partial p_i} + \frac{d}{dt}\Big(p_i\{q^i,Q\}\Big)\right)
\end{aligned}$$

where in the third step we used the fact that the Poisson bracket satisfies Leibniz's rule to deduce $\sum_I \frac{\partial H}{\partial z^I}\{z^I,Q\} = \{H,Q\} = 0$. Finally, since $\partial_t Q = 0$, we deduce:

$$L_{1\text{st}}(z,\dot{z},t) \mapsto L_{1\text{st}}(z,\dot{z},t) + \epsilon\frac{d}{dt}\Big(-Q + p_i\{q^i,Q\}\Big)$$

Therefore, a Hamiltonian symmetry changes the first order action by a boundary term (total time derivative) and therefore is a symmetry also in the sense explored in the context of Noether's theorem, i.e. that is

$$\tilde{\delta}_Q L_{1\text{st}} = \frac{d}{dt}R_Q \quad \text{with} \quad R_Q = -Q + p_i\{q^i,Q\}$$

Rearranging, we find the same formula we had found when we formulate Noether's theorem:

$$Q = p_i\tilde{\delta}_Q q^i - R_Q \quad \text{where} \quad \tilde{\delta}_Q q^i \equiv \{q^i,Q\}.$$

In other words, we have used the Hamiltonian picture to reverse-engineered Noether's theorem for the first-order action principle.

[3]Summation over i go from $i=1$ to n, while summations over I go from $I=1$ to $2n$ since the coordinates z^I comprise both the coordinates q^i and p_i.

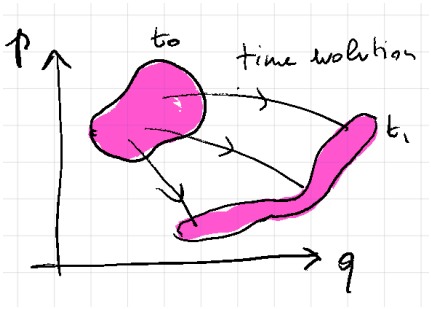

Figure 5.2: The Liouville theorem states that the two shaded regions have the same Liouville volume, no matter how much the Hamilotnian flow "scrambles" their shape.

5.4 Liouville theorem, revisited

Recall Liouville theorem: it stated that the Liouville volume element is invariant under time evolution (if $\partial_t H = 0$) or any other infinitesimal canonical transformation of the form $\delta z^I = \{z^I, F\}$. In terms of Hamiltonian flows, this can be restated as follows: a given region of phase space preserves its Liouville volume when dragged along a Hamiltonian flow. See Figure 5.2.

Third Tutorial

Hamilton's function

This exercise is a warm-up for the following lecture.

1. Consider the motion of a free particle of mass m. Write down the action and the corresponding equations of motion. Solve these equations and write the solution in terms of *variational data* q_1, t_1, q_0, t_0. Calculate the corresponding 'action'. Hence show that the Hamilton's function reads

$$S(q_1, t_1, q_0, t_0) = \frac{m(q_1 - q_0)^2}{2(t_1 - t_0)} \,.$$

2. The Hamilton's function contains all the information about the motion. In particular, verify that for our example

$$\frac{\partial S}{\partial q_1} = p_1 \,, \qquad \frac{\partial S}{\partial q_0} = -p_0 \,.$$

   (In our case of course we have $p_0 = p_1$.) Invert the latter relation to find the solution of equations of motion as a function of *initial data*. Show also that

$$\frac{\partial S}{\partial t_1} = -E \,.$$

3. Show that for an harmonic oscillator with frequency ω we have

$$S(q_1, t_1, q_0, t_0) = m\omega \frac{(q_0^2 + q_1^2)\cos[\omega(t_1 - t_0)] - 2q_0 q_1}{2\sin[\omega(t_1 - t_0)]} \,.$$

Chapter 6

The Hamilton–Jacobi equation

6.1 Setup: Hamilton's function

In this chapter we study yet another formulation of classical mechanics.

Consider Hamilton's action principle, which states that a physical history $\bar{q}^i(t)$ is an extremum of the action functional $S[q(t)]$ evaluated at fixed boundary conditions:

$$\delta S[\bar{q}(t)] = 0 \quad \text{with} \quad S[q(t)] = \int_{t_0}^{t_1} dt\ L(q, \dot{q}, t) \quad \text{and} \quad \delta q^i(t_0) = 0 = \delta q^i(t_1).$$

Note: we will assume that this solution is unique—e.g. for small, possibly infinitesimal, time interval $t_1 - t_2$.

We want now to ask the following questions: what are the properties of the action functional evaluated at a solution of the equations of motion, $S[\bar{q}(t)]$?

Since $\bar{q}(t)$ is uniquely determined by the chosen boundary values

$$\bar{q}^i(t_0) = q_0^i \quad \text{and} \quad \bar{q}^i(t_1) = q_1^i$$

we introduce the following, more explicit notation:

$$S[\bar{q}(t)] \doteq \mathcal{S}(q_0, t_0; q_1, t_1).$$

The function $\mathcal{S}(q_0, t_0; q_1, t_1)$ is called *Hamilton's (characteristic) function*.

The natural question therefore consists in investigating what is the dependence of Hamilton's function on its four arguments.

6.2 Variation with respect to the position

We start investigating the dependence of Hamilton's function on the final position q_1^i, if the values of (q_0^i, t_0) *and* of t_1 are kept fixed:

$$\mathcal{S}(q_0, t_0; q_1 + \epsilon\xi, t_1) = S[\bar{q}(t) + \epsilon\delta_\xi \bar{q}(t)]$$

where $\bar{q}_\xi(t) = \bar{q}(t) + \epsilon\delta_\xi\bar{q}(t)$ is the solution of the Euler-Lagrange equations defined by the boundary conditions

$$\bar{q}^i_\xi(t_0) = q^i_0 \quad \text{and} \quad \bar{q}^i_\xi(t_1) = q^i_1 + \epsilon\xi^i,$$

which mean that in particular

$$\delta_\xi\bar{q}^i(t_0) = 0 \quad \text{and} \quad \delta_\xi\bar{q}^i(t_1) = \xi^i.$$

With this we compute:

$$\begin{aligned}
\mathcal{S}(q_0, t_0; q_1 + \epsilon\xi, t_1) &= \int_{t_0}^{t_1} dt\ L(\bar{q} + \epsilon\delta_\xi\bar{q}, \dot{\bar{q}} + \epsilon\delta_\xi\dot{\bar{q}}, t) \\
&= \int_{t_0}^{t_1} dt\ L(\bar{q}, \dot{\bar{q}}, t) + \epsilon\sum_i \left(\frac{\partial L}{\partial q^i}\bigg|_{\bar{q}} \delta_\xi\bar{q}^i + \frac{\partial L}{\partial \dot{q}^i}\bigg|_{\bar{q}} \delta_\xi\dot{\bar{q}}^i \right) + O(\epsilon^2) \\
&= \mathcal{S}(q_0, t_0; q_1, t_1) + \epsilon\int_{t_0}^{t_1} dt\ \sum_i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \right)\bigg|_{\bar{q}} \delta_\xi\bar{q}^i \\
&\qquad + \epsilon\left[\sum_i \frac{\partial L}{\partial q^i}\bigg|_{\bar{q}} \delta_\xi\bar{q}^i \right]_{t_0}^{t_1} + O(\epsilon^2) \\
&= \mathcal{S}(q_0, t_0; q_1, t_1) + \epsilon\sum_i \bar{p}_i(t_1)\xi^i + O(\epsilon^2)
\end{aligned}$$

where in the last step we used that $\bar{q}(t)$ is a solution of the Euler–Lagrange equations of motion as well as the above results for $\delta_\xi\bar{q}(t_0) = 0$ and $\delta_\xi\bar{q}(t_1) = \xi_1$. We also introduced the following notation for the on-shell value of the momentum:

$$\bar{p}_i(t) \doteq \frac{\partial L}{\partial v^i}\bigg|_{(\bar{q}(t), \dot{\bar{q}}(t), t)}.$$

Thus, from the above result, we deduce the first important result of this chapter, that is:

$$\frac{\partial \mathcal{S}}{\partial q^i_1} = \bar{p}_i(t_1).$$

In words: the partial derivative of the Hamilton's function with respect to the final boundary condition is given by the on-shell value of the final momentum. Similarly, one can show that (do it! Be careful to the minus sign):

$$\frac{\partial \mathcal{S}}{\partial q^i_0} = -\bar{p}_i(t_0).$$

More formally, we can write:

$$\frac{\partial \mathcal{S}}{\partial q^i_1} = p^1_i \quad \text{and} \quad \frac{\partial \mathcal{S}}{\partial q^i_0} = -p^0_i$$

6.3 Variation with respect to time

We are thus left to investigate the dependence of $\mathcal{S}$ on the times t_0 and t_1. We start with the latter:

$$\mathcal{S}(q_0, t_0; q_1, t_1 + \epsilon) = S[\bar{q}(t) + \epsilon\delta_\tau\bar{q}(t)]$$

where $\bar{q}_\tau(t) = \bar{q}(t) + \epsilon\delta_\tau\bar{q}(t)$ is the solution of the Euler–Lagrange equations of motion with boundary conditions:

$$\begin{cases} \bar{q}_\tau(t_0) = q_0 = \bar{q}(t_0) \\ \bar{q}_\tau(t_1 + \epsilon) = q_1 = \bar{q}(t_1) \end{cases}$$

From the first expression we get immediately, $\delta_\tau\bar{q}(t_0) = 0$, to unpack the second expression we first compute:

$$\bar{q}_\tau(t_1 + \epsilon) = \bar{q}(t_1 + \epsilon) + \epsilon\delta_\tau\bar{q}(t_1 + \epsilon) = \bar{q}(t_1) + \epsilon\left(\dot{\bar{q}}(t_1) + \delta_\tau\bar{q}(t_1)\right) + O(\epsilon^2)$$

And thus, find:

$$\begin{cases} \delta_\tau\bar{q}(t_0) = 0 \\ \delta_\tau\bar{q}(t_1) = -\dot{\bar{q}}(t_1) + O(\epsilon) \end{cases}$$

We can now compute the variation of Hamilton's function—note the change in the extrema of integration:

$$\begin{aligned} \mathcal{S}(q_0, t_0; q_1, t_1 + \epsilon) &= \int_{t_0}^{t_1+\epsilon} dt\ L(\bar{q} + \epsilon\delta_\tau\bar{q}, \dot{\bar{q}} + \epsilon\delta_\tau\dot{\bar{q}}, t) \\ &= \int_{t_0}^{t_1} dt\ L(\bar{q}, \dot{\bar{q}}, t) + \epsilon L(\bar{q}(t_1), \dot{\bar{q}}(t_1), t_1) \\ &\qquad + \epsilon\sum_i \left(\frac{\partial L}{\partial q^i}\bigg|_{\bar{q}} \delta_\tau\bar{q}^i + \frac{\partial L}{\partial \dot{q}^i}\bigg|_{\bar{q}} \delta_\tau\dot{\bar{q}}^i] \right) + O(\epsilon^2) \\ &= \mathcal{S}(q_0, t_0; q_1, t_1) + \epsilon\left(L(\bar{q}(t_1), \dot{\bar{q}}(t_1), t_1) - \sum_i \bar{p}_i(t_1)\dot{\bar{q}}^i(t_1) \right) \\ &\qquad + O(\epsilon^2) \end{aligned}$$

where in the second line the first term of order ϵ comes from the change in the extrema of integration, while to obtain the third line we skipped here a couple of steps completely analogous to those performed in the derivation of $\partial\mathcal{S}/\partial q_1^i$, with the sole difference that the variation of the final boundary condition is now found to be $\delta_\tau\bar{q}(t_1) = -\dot{\bar{q}}(t_1)$.

We conclude by noting that the quantity in parenthesis in the last line of the above computation is nothing else than the value at $t = t_1$ of minus the energy of the system (see the Noether's theorem for time-translation invariance in Section 3.3), i.e.

$$\frac{\partial\mathcal{S}}{\partial t_1} = -E(\bar{q}(t_1), \dot{\bar{q}}(t_1), t_1)$$

Similarly, one can show (do it!)

$$\frac{\partial \mathcal{S}}{\partial t_0} = E(\bar{q}(t_0), \dot{\bar{q}}(t_0), t_0)$$

6.4 The Hamilotn–Jacobi equation

Noting that the generalized energy is nothing else than Hamilton's function written in terms of q and $\dot{q}$ instead of q and p, we can summarize the results of the previous section as follows:

$$\frac{\partial \mathcal{S}}{\partial q_1^i} = p_i^1 \quad \text{and} \quad \frac{\partial \mathcal{S}}{\partial t_1} = -H(q_1, p^1, t_1).$$

This equations can be combined to give the *Hamilton–Jacobi equation*:[1]

$$\frac{\partial \mathcal{S}}{\partial t_1} + H\left(q_1, \frac{\partial \mathcal{S}}{\partial q_1}, t_1\right) = 0.$$

Remark 6.4.1 (Schrödinger)**.** If the Hamilton–Jacobi equation looks vaguely familiar, but you had not seen it before, it is probably because it reminds you of Schrödinger's equation:

$$-\mathrm{i}\hbar \frac{\partial}{\partial t}\Psi + H\left(\hat{q}, \hat{p}, t\right)\Psi = 0$$

in the position representation where $\Psi = \Psi(q,t)$, $\hat{q}^i = q^i$, and $\hat{p}_i = -\mathrm{i}\hbar\frac{\partial}{\partial q^i}$. To make the resemblance precise, write Ψ as the product of a real function ρ and a phase factor,

$$\Psi(q,t) = \rho(q,t)e^{\frac{\mathrm{i}}{\hbar}\varphi(q,t)},$$

plug this expression into Schrödinger's equation, and keep only the leading-order terms in $\hbar$ to obtain the following eikonal approximation to the wave equation (a.k.a. the classical or leading-order WKB approximation):

$$\frac{\partial \varphi}{\partial t} + H\left(q, \frac{\partial \varphi}{\partial q}, t\right) + O(\hbar) = 0,$$

which is precisely the Hamilton–Jacobi equation for $\mathcal{S}(q,t;q_0,t_0) = \varphi(q,t)$ provided that one choses the phase convention for Ψ such that $\varphi(q,t_0) = 0$.

This relationship between the Schrödinger and Hamilton–Jacobi equations is fascinating because it provides an explicit way to derive the classical limit of quantum mechanics, and thus suggests a precise correspondence between the classical and quantum

[1] Similarly, for the initial data, one obtains the time-reversed equation (notice that upon time reversal, $(q,p,t) \mapsto (q,-p,-t)$):

$$\frac{\partial \mathcal{S}}{\partial t_0} - H\left(q_0, -\frac{\partial \mathcal{S}}{\partial q_0}, t_0\right) = 0.$$

worlds mediated by a correspondence between the wave function and the (on-shell) action funcftional. A full and "conceptually clear" explanation[2] of this correspondence is achieved in Feynmann's path-integral formulation of quantum mechanics. Through the path integral formulation of quantum mechanics and quantum field theory, to this day the Hamilton–Jacobi equation plays a central role in approaches to Quantum Gravity, including the AdS/CFT duality (holography) and the spin-foam formulation of Loop Quantum Gravity. It also plays a role in the Hartle–Hawking hypothesis of quantum cosmology. ♢

Remark 6.4.2. For the geometrical meaning of Hamilton–Jacobi theory see Sections 6.5, 7.9, and 8.2 of Marsden and Ratiu's book.[3] ♢

Remark 6.4.3. Traditionally, the Hamilton–Jacobi equation is derived by studying the theory of "generating functions" for canonical transformations:[4] Hamilton's generating function is nothing else than the generating function of the canonical transformation given by time evolution from $t = t_0$ to $t = t_1$, cf. Remark 4.3.2. Despite its complexity, the Hamilton–Jacobi equation can also be a powerful tool to solve the equations of motion, in particular for integrable systems where coordinates can be found so that $\mathcal{S}$ "separates" into a sum of functions each dependent on only one coordinate.[5] ♢

Remark 6.4.4 (Mechanics and geometrical optics)**.** Hamilton–Jacobi theory has strong analogies to the theory of geometrical optics studied in particular by Hamilton. It is fascinating to read the account of this analogy and its historical development in Lanczos[6] in light of Remark 6.4.1 on Schrödinger's equation for (wave) quantum mechanics. ♢

[2]Whether the path integral representation of quantum mechanics is considered "conceptually clear" is a subjective matter.

[3]Marsden and Ratiu, *Introduction to Mechanics and Symmetry*, 2nd edition (1991).

[4]See e.g. David Kubiznak's PSI lecture notes. See also Marsden and Ratiu, *ibidem*, as well as Chapters VII and VIII of Lanczos, *The Variational Principles of Mechanics*, Dover Publications (1986) [a reprint of the 1970's fourth edition of the 1949 original published by the University of Toronto Press].

[5]See David Kubiznak's PSI lecture notes, as well as Lanczos, *ibidem*, in particular Sections VIII.3 and VIII.4.

[6]*Ibidem*, in particular Section VIII.8.

Chapter 7

Special relativity: kinematics

> *The* principle of relativity*, according to which the laws of physical phenomena must be the same for a stationary observer as for an observer carried along in a uniform motion of translation; so that we have not and cannot have any means of discerning whether or not we are carried along in such a motion.*
>
> Poincaré (1904)[1]

> *If a co-ordinate system K be so chosen that when referred to it the physical laws hold in their simplest forms, these laws would be also valid when referred to another system of co-ordinates K' which is subjected to an uniform translational motion relative to K. We call this postulate "*The* Special Relativity Principle*". By the word special, it is signified that the principle is limited to the case, when K' has uniform translatory motion with reference to K, but the equivalence of K and K' does not extend to the case of non-uniform motion of K' relative to K.*
>
> Einstein (1916)[2]

See also, Appendix A for a relevant and famous quote from Galileo's *Dialogue Concerning the Two Chief World Systems* (1632).

7.1 Lorentz boosts

Consider, to start, a 1 dimensional world (e.g. a straight river) described in one *inertial*[3] reference frame K (the river's bank) by a spatial coordinate x and one time

[1] Chapter 7 of Poincaré, The Foundations of Science, The Value of Science, (1904).

[2] Part A.1 of Einstein, The Foundation of the Generalised Theory of Relativity (1916).

[3] We will not struggle with defining what an inertial frame is. This is a thorny philosophical problem: *The weakness of the principle of inertia lies in this, that it involves an argument in a circle: a mass moves without acceleration if it is sufficiently far from other bodies; we know that it is sufficiently far from other bodies only by the fact that it moves without acceleration.* (Einstein, The Meaning of Relativity, 1921, p.33).)

coordinate t. Assume that the result of any physical experiment does not depend on where along the river it is performed, i.e. whether it is performed at position x or at $x + a$, nor on when it is performed, i.e. whether it is performed at time t or $t + T$. Consider now a second reference frame K' moving with respect to the first one with constant velocity v (a ship flowing down the river). Call the space and time coordinates in this reference frame x' and t'. We call K and K'

How are the reference frames K and K' related? I.e. how are the coordinates (x', t') expressed in terms of (x, t)?

The relativity principle together with space and time's homogeneity and isotropy (i.e. translation and rotation invariance) necessarily implies that all the inertial frames are related either by Galilean transformations, or by Lorentz transformations with some universal constant "c".[4] Moreover, Galilean transformation are obtained from the Lorentz transformations by taking the limit of infinitely large c.

If the origin of the two inertial reference frames K and K' are chosen to coincide, the *Lorentz transformation law* between $(x, t) \mapsto (x', t')$ reads:

$$\begin{cases} x' = \gamma(v)\Big(x - vt\Big) \\ t' = \gamma(v)\left(t - \dfrac{v}{c^2}x\right) \end{cases} \quad \text{where} \quad \gamma(v) \doteq \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{(Lorentz)}.$$

Note that one has to demand $|v| < c$ for the Lorentz factor $\gamma(v)$, and hence the entire set of transformations, to be well defined. This transformation is also called a *(Lorentz) boost*. In the limit $c \to \infty$, the Lorentz transformation law reduces to the Galilean transformation law:

$$\begin{cases} x' = x - vt \\ t' = t \end{cases} \qquad \text{(Galileo)}.$$

This means that for relative velocities v much smaller than c, the Galilean transformation law is a good approximation to the Lorentz transformation law.

Remark 7.1.1 (Carrollian transformation)**.** There is another limit (more technically, "contraction") of the Lorentz transformations, the Carrollian limits[5], with applications in various domains of forefront theoretical physics, such as gravitation, supersymmetry and string theory.[6] It is obtained by sending $c \to 0$ instead of infinity. Since $|v| < c$, sending c to zero means that also v must go to zero—but how fast? As fast as c^2 so that the Carrollian limit is obtained by sending $c \to 0$ while keeping $b = v/c^2$ finite. Thus the Carrollian transformations read

$$\begin{cases} x' = x \\ t' = t - bx \end{cases} \qquad \text{(Carroll)}.$$

[4] See e.g. LévyLeblond, One more derivation of the Lorentz transformation, Am. J. Phys. 44, 271 (1976), and references therein.

[5] LévyLeblond, Une nouvelle limite non-relativiste du groupe de Poincaré, Annales de l'institut Henri Poincaré. Section A, Physique Théorique, Volume 3 (1965).

[6] Cf. LévyLeblond On the unexpected fate of scientific ideas: An archeology of the Carroll group (2022).

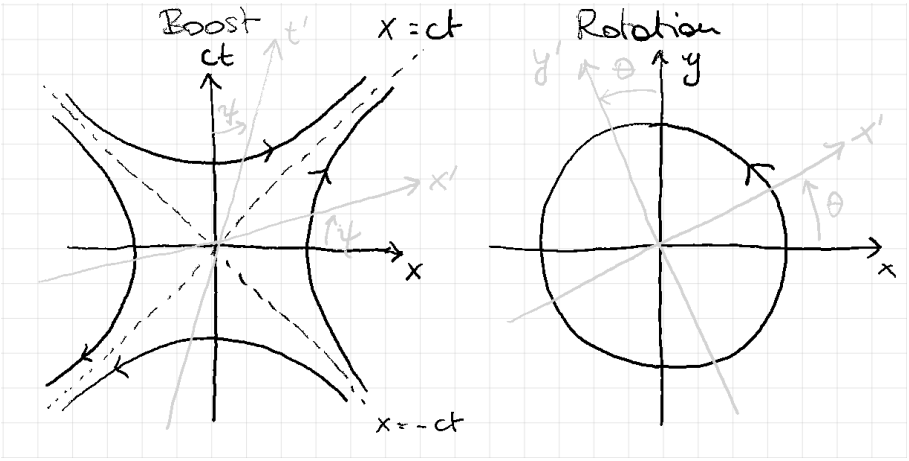

Figure 7.1: Boost vs. rotations.

Note that, in opposition to the Galilean case, in a "Carrollian world" space is absolute, but time is relative. Also, in such a world causality is "frozen", meaning that all points in space are causally disconnected from each other. ◊

We noted that $-1 < v/c < 1$. We can thus perform the following change of variable:

$$\text{th}\psi \doteq \frac{v}{c}$$

where $\psi \in \mathbb{R}$ is called the *rapidity*, and $\text{th}\psi = (e^{\psi} - e^{-\psi})/(e^{\psi} + e^{-\psi})$ is the hyperbolic tangent function, which is a smooth monotonic map from $\mathbb{R}$ to $(-1, 1)$. Then, a little algebra (do it!) shows that

$$\gamma(v) = \frac{1}{\text{ch}\psi} \leq 1$$

and thus

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda(\psi) \begin{pmatrix} ct \\ x \end{pmatrix} \quad \text{where} \quad \Lambda(\psi) \doteq \begin{pmatrix} \text{ch}\psi & -\text{sh}\psi \\ -\text{sh}\psi & \text{ch}\psi \end{pmatrix}.$$

where the hyperbolic sine and cosine are defined as $\text{sh}\psi = (e^{\psi} - e^{-\psi})/2$ and $\text{ch}\psi = (e^{\psi} + e^{-\psi})/2$. They satisfy the identity:[7]

$$\text{ch}^2\psi - \text{sh}^2\psi = 1.$$

Therefore, it is clear that Lorentz boosts are hyperbolic analogues of 2d rotations, see Figure 7.1:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R(\theta) \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{where} \quad R(\theta) \doteq \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

There is a lot we can learn from Figure 7.1.

[7] By $\text{ch}^2\psi$ we mean $(\text{ch}\psi)^2$, and *not* $\text{ch}(\text{ch}\psi)$, etc.

First, comparing the orbits of a point along all possible boosts $\Lambda(\psi)$ and rotation $R(\theta)$, we see that the first draw hyperbolae and the second circles. These points are characterized by the quantities

$$s^2 = -(ct)^2 + x^2 \quad \text{and} \quad \ell^2 = x^2 + y^2$$

being left invariant, respectively. In the next chapter we will use this observation to investigate the most general Lorentz transformation in 3 spatial dimensions.

Note: the quantity s^2, called the *interval*, should be understood as a symbol in itself, and not as the square of a quantity s, for s^2 clearly fails to be positive definite:

$$\text{the interval } s^2 \text{ is said } \begin{cases} \text{spacelike if } s^2 > 0, \\ \text{lightlike if } s^2 = 0, \\ \text{timelike if } s^2 < 0. \end{cases}$$

The set of points at null interval from the origin is called the *lightcone*, and it is the trajectory of entities (photons) that start at the origin $(ct, x) = (0, 0)$ and travel at speed $|v| = c$ (the speed of light) in an arbitrary direction.

From the left hand side of Figure 7.1, it is clear that the notion of simultaneity, i.e. what happens at a given time, is frame dependent. Consider for example the notion of the present instant, i.e. of all events that happen at "time equal zero". This set of events is different in the frame K, where is given by the x-axis ($t = 0$), and in the frame K' where it is given by the x'-axis ($t' = 0$).

Although the notion of "present" in the special relativistic context is blurred, refined notions of "past" and "future" survive. In fact the set of points that lie above (respectively, below) the lightcone will lie above (resp. below) the lightcone will do so in *all* frames that share the same origin O. These regions are called the *future* (resp. *past*) of the *event* O. See Figure 7.2.

Since all physical entities must move at velocities less than c (otherwise the boost required to transform to their rest frame would be ill defined, see $\gamma(v)$), it turns out that O can be reached only by physical observers lying in its past, and can reach only physical observers lying in its future. For this reason, the union of the past and future of O is called the *causal domain* of O, and comprises all spacetime points that can be connected by exchanges of physical entities with O.

The causal domain of O is the set of points that lie at a timelike interval from O. It is the disjoint union of the future of O—i.e. the set of points at timelike interval from O with $ct > ct_O$—and the past of O—i.e. the set of points at timelike interval from O with $ct < ct_O$.

7.2 Composition of velocities

We now want to investigate the following question: how do two subsequent Lorentz boosts, $(x, t) \to (x', t') \to (x'', t'')$, compose?

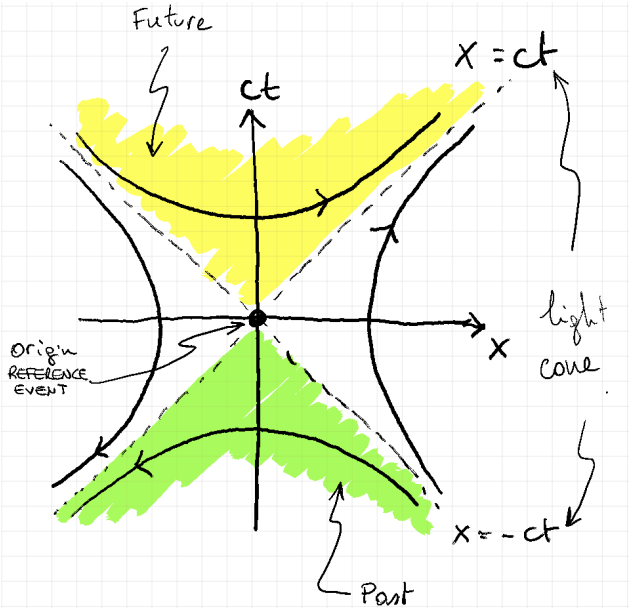

Figure 7.2: Lightcone structure.

It is a matter of a simple (hyperbolic) trigonometric computation to show that rapidities, as Euclidean angles, are additive—viz.:

$$\Lambda(\psi_2)\Lambda(\psi_1) = \Lambda(\psi_3) \quad \text{with} \quad \psi_3 = \psi_1 + \psi_2.$$

If rapidities are additive, what about velocities?

$$v_3 = \mathrm{th}\psi_3 = \frac{\mathrm{sh}(\psi_1+\psi_2)}{\mathrm{ch}(\psi_1+\psi_2)} = \frac{\mathrm{sh}\psi_1\mathrm{ch}\psi_2 + \mathrm{ch}\psi_1\mathrm{sh}\psi_2}{\mathrm{ch}\psi_1\mathrm{ch}\psi_2 + \mathrm{sh}\psi_1\mathrm{sh}\psi_2} = \frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}.$$

Considering the reference frame (x'', t'') as the rest frame of an object moving at velocity v_2 in the reference frame K', we see that this formula also describes the velocity, $v = v_3$, of an object moving at velocity $v' = v_2$ in K' as measured by an observer at rest in the frame K, with K' moving with respect to K at velocity $u = v_1$:

$$v = \frac{v'+u}{1+\frac{uv'}{c^2}} \quad \text{or} \quad v' = \frac{v-u}{1-\frac{uv}{c^2}}$$

It is immediate to see that $\pm c$ is a fixed point of the transformation $v \mapsto v'$, for any value of $|u| < c$. This means that *in Lorentzian relativity the velocity c is absolute, i.e. it is measured to be the same in all inertial reference frames.*

Remark 7.2.1. The result that the Lorentz transformations can be deduced solely from the principle of relativity curiously suggests that the existence of a (large) absolute speed c could have been discovered much before the advent of Maxwell equations. ◊

HW/tutorial: composition of velocities in dim > 1?

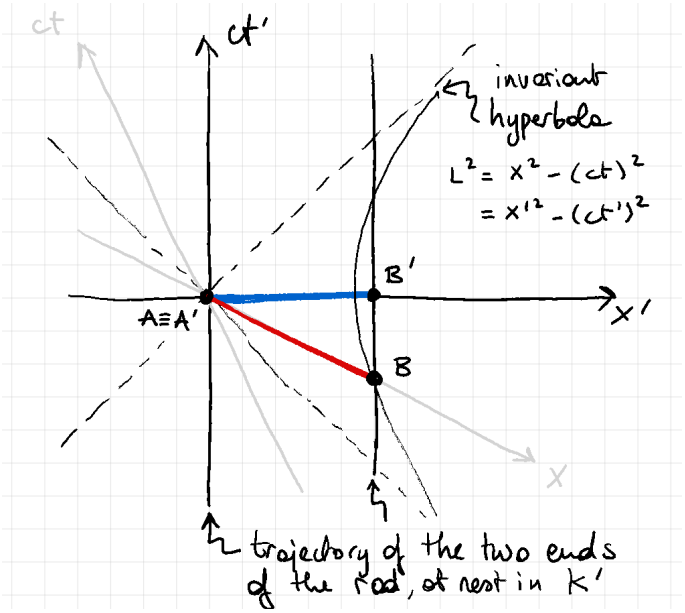

Figure 7.3: Length contraction.

7.3 Length contraction

Consider two inertial reference frames K and K' moving with relative velocity u (for simplicity we keep working in 1 spatial dimensions). A rod lies at rest in the reference frame K' where it is found to be of length ℓ. This is called the *proper length* of the rod. How long does the rod appear to an observer at rest in the frame K?

To see what to expect we start by working graphically,[8] see Figure 7.3. We choose the spacetime origins of K' and K to coincide with one of the two ends of the rod at a given instant, $A \equiv A'$. (Whereas the spatial origin of K' will keep coinciding with the end of the rod, this will of course not be true for the spatial origin of K). Consider now the other end of the rod: in the reference frame K', at the instant $t' = 0$, it will find itself at the event B'; while, in the reference frame K, at the instant $t = 0$ it will find itself at B. Due to the lack of invariance of simultaneity $B \not\equiv B'$. We drew in blue a "spacetime snapshot" of the rod at $t' = 0$ in the frame K', and in red a spacetime snapshot of the rod at $t = 0$ in the frame K.

The length of the rod in a reference frame is defined as the (absolute value of the) difference of its spatial coordinates at a given time. Thus, in the reference frame K we have $L = x_B$, while in its rest frame K' we have $L = x'_{B'} \equiv \ell$. Notice that these are the spatial coordinates of *different* events (B vs. B') in different coordinate systems ((ct, x) vs. (ct', x')).

Before proceeding to compute L and L', we want to use the graphical calculus to see which one will be longer and which one will be shorter. To compare distances on

[8] For a beautiful little reference on graphical calculus in special relativity, see Bais, *Very Special Relativity: An Illustrated Guide*, Harvard University Press (2007).

the hyperbolic plane depicted in Figure 7.3, we cannot rely on the "paper distance".[9] In fact, instead of along circles, intervals of constant hyperbolic length are reported along hyperbolae. This shows that

$$L < L' \equiv \ell.$$

Therefore we expect that an object in motion will appear shorter than its proper length. This is called the *Lorentz-FitzGerald length contraction*.

To compute how much shorter it will appear, we use the figure to see that the coordinates in the frame K' of the point B are:

$$x'_B = \ell \quad \text{and} \quad ct'_B = -\frac{u}{c}\ell = -\text{th}\psi\ \ell.$$

Hence, since $t_B = 0$ and

$$-(ct'_B)^2 + (x'_B)^2 = -(ct_B)^2 + (x_B)^2,$$

we deduce:

$$L^2 = x_B^2 = (1 - \text{th}^2\psi)\ell^2 = \frac{\ell^2}{\text{ch}^2\psi} = \gamma(u)^2\ell^2$$

And finally:

$$L = \frac{\ell}{\gamma(u)}.$$

Remark 7.3.1. In this 1-dimensional problem, the rod is lying parallel to the direction of motion. This setup yields Lorentz's contraction. However, if the rod was sitting transverse to the direction of motion, its length would result unaltered by the boost. We will prove this fact in a turnaround way at the end of the next section. ♢

Remark 7.3.2. For a different and rather unorthodox viewpoint on Lorentz's contraction, see Bell *How to teach special relativity*,[10] as well as its criticism by Brown and Pooley.[11] ♢

7.4 Time dilation

We now consider the ticking of a clock at rest in the moving frame K', and ask how its ticking is perceived by an observer at rest in the frame K, if K' moves with respect

[9] According to this *incorrect* criterion, we would conclude that the rod appears longer in the frame K (red interval) compared to the frame K' (blue interval).

[10] Bell, *How to teach special relativity*, Progress in Scientific Cul- ture, Vol 1 (1976), reprinted in Bell, *Speakable and unspeakable in quantum mechanics.* Cambridge, England: Cambridge University Press (1987).

[11] Brown and Pooley, he origin of the spacetime metric: Bell's 'Lorentzian pedagogy' and its significance in general relativity, Physics meets Philosophy at the Planck Scale, C. Callender and N. Huggett (eds.), Cambridge University Press (2001) [arXiv:9908048], and Minkowski space-time: a glorious non-entity, Philosophy and Foundations of Physics, Vol. 1 (2006) [arXiv:0403088].

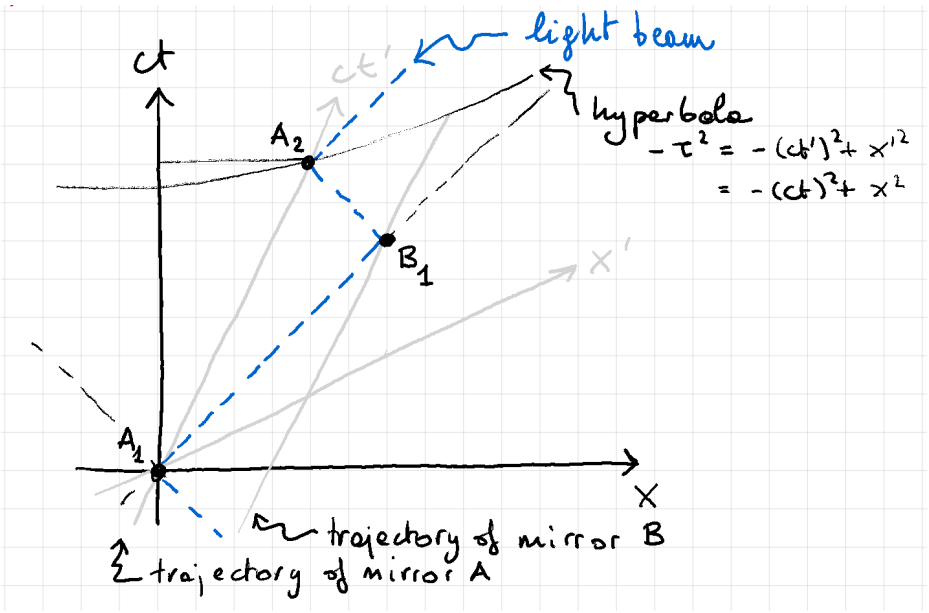

Figure 7.4: Time dilation.

to K at velocity u. First, however, we need to clarify what *is* the "ticking of a clock". The ticking of a clock corresponds to the counting of the cycles of a periodic physical process. So, to analyze a clock's ticking, we need first to "design" a clock.

In the rest frame K', our clock of choice is made of two parallel planar mirrors, A' and B' lying a (proper) distance ℓ from each other, with a light beam traveling back and forth between them: each tick of the clock is then given by the photon bouncing off mirror A'.

How this process appears in the frames K and K' is represented in Figure 7.4. There we see the two mirrors moving at rest in the moving frame K', with the light beam traveling back and forth between them drawn in blue: notice that, the speed of light being the same in all frame, the light beam moves at speed c (i.e. along lightlike lines at 45°) in both the resting and moving frames. The time measured in the rest frame of the clock, called *proper time* τ is the time interval between two consecutive bounces of the light beam on the mirror A:

$$\tau = T' \doteq t'_{A2} - t'_{A1} = t'_{A2}.$$

In the frame K, on the other hand, the time interval between these two events is given by

$$T = t_{A2} - t_{A1} = t_{A2}.$$

Following the constant-interval hyperbola, we see that:

$$T > T' = \tau,$$

that is proper time is the shortest measured time interval between two events. Or, conversely, the ticking of a moving clock appears slower, a phenomenon known as *Lorentz-FitzGerald time dilation*.

To quantify the extent of this dilation, we recall that

$$-(ct_{A2})^2 + (x_{A2})^2 = -(ct'_{A2})^2 + (x'_{A2})^2,$$

and since $x'_{A2} = 0$ (resting frame condition), while $x_{A2} = ut_{A2}$ (this is the distance travelled by the mirror A between two consecutive ticks). Replacing, we readily find:

$$T = \gamma(u)\tau.$$

The same result could be found in a more pedestrian way by computing the time that it takes for the light to travel between mirror A and B and back:

$$T = \Delta t_1 + \Delta t_2, \quad \text{where} \quad \Delta t_2 = t_{A2} - t_{B1} \text{ and } \Delta t_1 = t_{B1} - t_{A1}$$

and, since the light beams moves at the speed of light c but the mirror moves at speed u, we have that:

$$\begin{cases} c\Delta t_1 = L + u\Delta t_1 \\ c\Delta t_2 = L - u\Delta t_2 \end{cases} \implies \begin{cases} \Delta t_1 = L/(c-u) \\ \Delta t_2 = L/(c+u) \end{cases}$$

Thus

$$T = \Delta t_1 + \Delta t_2 = \frac{2cL}{c^2 - u^2} = \gamma(u)^2 \frac{L}{c} = \gamma(u)\tau,$$

where in the last step we recalled the extent of Lorentz's contraction of lengths, $L = L'/\gamma(u)$, together with the fact that $\tau = T' = L'/c$ due to the constancy of the speed of light.

* * *

We shall now use this result and Pythagoras's theorem to show that transverse lengths do not contract nor dilate.

Consider the following setup in the clock's rest frame K': instead of having just two parallel mirrors at (proper) distance $L' = \ell$ in the direction of motion, we consider a Michelson's interferometer with two orthogonal but otherwise identical arms, one of which extends in the direction of the clock's motion, that join at a beam splitter. In the rest frame the light beams travelling the two arms leave and return to the beam splitter concomitantly. Said differently, the interferometer registers constructive interference. These are frame-*in*dependent facts.

We shall now see that this is possible only if the transverse arm does not undergo any contraction or dilation, i.e. $L_\perp = L'_\perp \equiv \ell$. Indeed, the time of travel on the orthogonal arm as observed by the two observers can be computed using Pythagoras's theorem as follows, see Figure 7.5:

$$c\tau = cT'_\parallel = cT'_\perp = 2L'_\perp = 2\ell \quad \text{and} \quad cT_\perp = \sqrt{4L_\perp^2 + (uT_\perp)^2}$$

However we know from the previous discussion that $T_\perp$ must be equal to $T_\parallel = \gamma(u)\tau$, for the observer in K must also see the two light beams come back at the beamsplitter concomitantly. Thus, we conclude that

$$L_\perp = \ell.$$

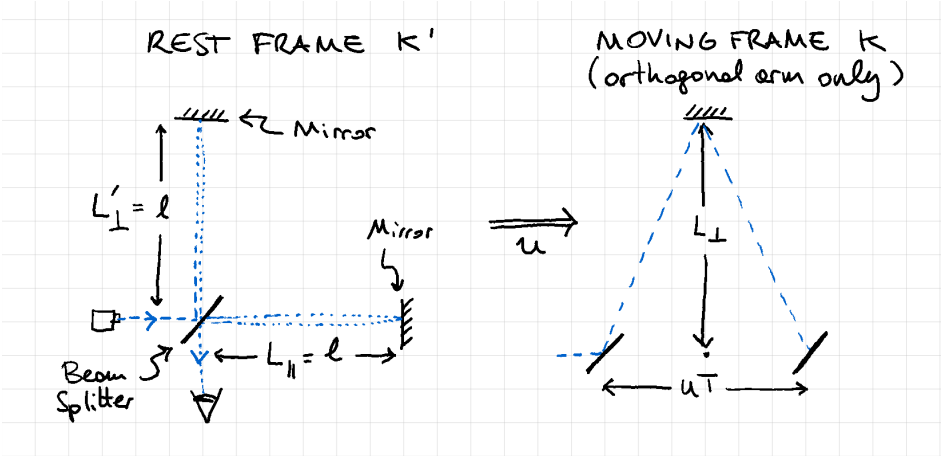

Figure 7.5: The Michelson–Moreley interferometer.

Remark 7.4.1. Here we are following Einstein's reasoning by using the constancy of the speed of light to deduce facts about the Lorentz transformations.[12] Conversely one could have used the absence of interference in a moving Michelson's interferometer (Michelson and Moreley, 1887) to deduce the Lorentz-FitzGerald contraction. ◊

[12] Actually, we have not introduced Maxwell's equations and light just yet. We have however deduced the Lorentz transformations from the principle of relativity. These come with a free parameter c that we have proven represents an absolute speed that is measured to be the same in all reference frames. Assuming that something can move at this speed, we then used it to build a clock and an interforemeter to deduce a formula for time dilation as well as the absence of a transverse length contractions or dilation. Later we will see that Maxwell's equation are compatible with the principle of relativity if and only if c is the speed of an electromagnetic wave, i.e. c is the speed of light.

Fourth Tutorial

beginning of T3 missing, check the following

Kepler problem: Laplace–Runge–Lenz vector

So far we have considered the motion in an arbitrary spherically symmetric potential $V(r)$ and showed that such a motion is completely integrable. Let us now specify to the *Kepler problem*, i.e. consider a potential

$$V = -\frac{k}{r}\,.$$

In this case, the motion admits three additional integrals of motion encoded in the so called *Laplace–Runge–Lenz vector*

$$\vec{A} = \vec{p}\times\vec{L} - mk\frac{\vec{r}}{r}\,,$$

making the motion *"maximally superintegrable"*.

1. To find a phase space vector field X_A generated by the Laplace–Runge–Lenz vector, we shall work in Cartesian coordinates. In those we have

   $$A^i = x^i p^2 - p^i x\cdot p - mk\frac{x^i}{r}\,,\quad r=\sqrt{x^2}\,.$$

   Each component generates a corresponding vector field

   $$X_{A^i} = \{\cdot, A^i\} = \big(2x^i p^k - \delta^i_k x\cdot p - p^i x^k\big)\frac{\partial}{\partial x^k} - \Big(\delta^i_k p^2 - p^i p_k - mk\delta^i_k\frac{1}{r} + mk\frac{x^i x^k}{r^3}\Big)\frac{\partial}{\partial p^k}\,.$$

   Their action on a Hamiltonian

   $$H = \frac{p^2}{2m} - \frac{k}{r}\,,$$

   is given by

   $$\begin{aligned}\mathcal{L}_{A^i}H &= \{H, A^i\}\\ &= \big(2x^i p^k - \delta^i_k x\cdot p - p^i x^k\big)\frac{\partial H}{\partial x^k} - \Big(\delta^i_k p^2 - p^i p_k - mk\delta^i_k\frac{1}{r} + mk\frac{x^i x^k}{r^3}\Big)\frac{\partial H}{\partial p^k}\\ &= -\big(2x^i p^k - \delta^i_k x\cdot p - p^i x^k\big)\frac{kx_k}{r^3} - \Big(\delta^i_k p^2 - p^i p_k - mk\delta^i_k\frac{1}{r} + mk\frac{x^i x^k}{r^3}\Big)\frac{p_k}{m}\\ &= 0,\end{aligned}$$

and the Hamiltonian is preserved, i.e., X_{A^i} define symmetries.

2. Consider next the canonical projection of $X_{A^i}, \pi^*(X_{A^i})$. This 'kills' all the directions $\frac{\partial}{\partial p^k}$. So we get

$$\pi^*(X_{A^i}) = \left(2x^i p^k - \delta^i_k x \cdot p - p^i x^k\right) \frac{\partial}{\partial x^k} .$$

Since this depends on p, such an action is not well defined on the configuration space. For this reason the vector $\vec{A}$ is called a *"hidden symmetry"*. The fact that hidden symmetry miraculously emerges for a realistic spherically symmetric potential, given by the Newton's or Coulomb's laws, whereas it is absent for general $V = V(r)$ is in my opinion very interesting (even more that it survives in general relativity). Is the nature trying to tell us something here?

Chapter 8

The Lorentz group and algebra

8.1 Definition of group and Lie algebra

A *group* is a set G, together with a map $G \times G \to G$, $(g_1, g_2) \mapsto g = g_2 \cdot g_1$, called multiplication, such that

1. the multiplication is associative, i.e.
$$g_3 \cdot (g_2 \cdot g_1) = (g_3 \cdot g_2) \cdot g_1 \quad \forall g_1, g_2, g_3 \in G;$$
2. there exists a (unique) element $1 \in G$ such that
$$1 \cdot g = g \cdot 1 = g \quad \forall g \in G;$$
3. every elements $g \in G$ has an inverse $g^{-1} \in G$ such that
$$g^{-1} \cdot g = g \cdot g^{-1} = 1.$$

Note: multiplication needs not be commutative, i.e. $g_2 \cdot g_1$ needs not be equal to $g_1 \cdot g_2$. The dot denoting the multiplication will often be omitted.

A group whose elements and multiplication depend smoothly on a continuous set of parameters is called a Lie group (this is of course a rough working definition).

A *Lie algebra* is a (real) vector space A, together with a map $[\cdot, \cdot] : A \times A \to A$, $(a_1, a_2) \mapsto a = [a_1, a_2]$, called the Lie bracket, such that

1. it is bi-linear;
2. skew-symmetric;
3. satisfies the Jacobi identity:
$$\text{cycl}[[a_1, a_2], a_3] \equiv [[a_1, a_2], a_3] + [[a_2, a_3], a_1] + [[a_3, a_1], a_2] = 0$$

The Lie algebra captures the "infinitesimal" properties of a Lie group around the identity element, with $g \approx 1 + \epsilon a + O(\epsilon^2)$. We will discuss how this arises and what this means in the specific context of the Lorentz (and rotation) groups.

8.2 1+1d Minkowski metric and Lorentz group

For the time being, we stay in a world with only one spatial dimension, as we did so far, even though, as we will soon see, the following statements generalize to arbitrary dimensions. Since we have one space and one time coordinate, we will call this world *1+1 dimensional*.

Lorentz boosts, like 2d rotations, form a *group*, indeed:

$$\Lambda(\psi_2)\cdot\Lambda(\psi_1)=\Lambda(\psi_1+\psi_2),\quad \Lambda(\psi)^{-1}=\Lambda(-\Psi),\quad 1=\Lambda(0);$$

$$R(\theta_2)\cdot R(\theta_1)=R(\theta_1+\theta_2),\quad R(\theta)^{-1}=R(-\theta),\quad 1=R(0).$$

In this section we want to answer: how can we abstractly characterize the Lorentz group in 1+1 dimensions?

Let's consider the rotation group as inspiration: it can be characterized as the group of all linear (i.e. matricial) transformation of $\mathbb{R}^2$ that

E1. leave the Euclidean norm of a vector invariant,

E2. have unit determinant.

This group is called the special orthogonal group in 2 dimensions, i.e.

$$\text{Group of 2d rotations} = \mathrm{SO}(2).$$

Here: 2 is the dimension of the underlying real vector space, S stands for "special" and refers to condition E2, and O stands for "orthogonal" and refers to condition E1.

Proof. Let $\mathbb{R}^2 \ni X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$ and denote $|X|^2 \doteq X^T X = X^1X^1 + X^2X^2$ the (square of the) Euclidean norm of X. Then, write the most generic linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Condition E2 means that $ad - bc = 1$, whereas condition E1 demands that for all X, $|X|^2 = |RX|^2$ i.e.

$$(X^1)^2 + (X^2)^2 = (a^2 + c^2)(X^1)^2 + (b^2 + d^2)(X^2)^2 + 2(ab + cd)X^1X^2.$$

Thus, we deduce the further conditions $a^2 + c^2 = b^2 + d^2 = 1$, $ac + bd = 0$. Hence, from the first of these conditions, we see that $|a|, |c| \leq 1$ and there exists a θ such that $a = \cos\theta$ and $c = -\sin\theta$. Similarly, there must exist a θ' such that $d = \cos\theta'$ and $b = -\sin\theta'$. Thus, the remaining two conditions $ac + bd = 0$ and $ad - bc = 1$ can be rewritten as $\sin(2\theta) + \sin(2\theta') = 0$ and $\cos(\theta + \theta') = 1$. These then imply that $\theta' = -\theta + 2\pi k$ for some integer k. Hence,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \qquad \square$$

To give a similar characterization of the Lorentz group we need to find what are Lorentz boosts leaving invariant. In the Lorentzian context, it is common to adopt a convention where entries of vectors start at the index 0:

$$X = \begin{pmatrix} X^0 \\ X^1 \end{pmatrix}.$$

This is because these entry plays a special role.

Since the hyperbolic trigonometric functions satisfy $\mathrm{ch}^2\psi - \mathrm{sh}^2\psi = 1$, with a minus rather than a plus as for rotations, it is natural to introduce the two-dimensional *Minkowski metric* of signature[1] (1,1),

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus, in analogy with the Euclidean norm $|X|^2 = X^T X$, also the *Minkowski (pseudo-)norm*:[2]

$$|X|_\eta^2 = X^T \eta X = -(X^0)^2 + (X^1)^2.$$

The term $(X^0)^2$ takes a different sign. Importantly, the quantity $|X|_\eta^2$ is *not* to be understood as the square of a real number but just as a symbol in itself. This is because its sign is not necessarily positive definite. We will come back to this point shortly.

The Lorentz group can be abstractly characterized as the group of linear transformations of $\mathbb{R}^2$ that

M1. leave the Minkwoski norm invariant,

M2. do not flip the orientation of spacetime. i.e. have unit determinant,

M3. do not flip the direction of time ($\Lambda^0{}_0 > 0$).

This group is denoted

$$\text{Group of 1+1d Lorentz transformations} = \mathrm{SO}^+(1,1).$$

With reference to conditions M2 and M3 respectively, this should be called the *1+1d proper orthochronus Lorentz group*. However, we will not adopt this language, and simply call it the 1+1d Lorentz group.[3]

The rationale behind the notation is as follows: $\mathrm{O}(1,1)$ denotes the group of "orthogonal" transformations that preserve the Minkowski norm of signature (1,1).

[1] A metric of signature (p,q) is a real, invertible, symmetric matrix with p negative and q positive eigenvalues. Note that the Euclidean metric is (in Cartesian coordinates) the identity matrix and has a positive definite signature.

[2] The word "pseudo" is sometimes added to emphasize that the Minkowski "norm" is not positively definite and therefore, technically, it is not a norm.

[3] Similarly, the term Lorentz group sometimes refers to condition M1 alone, i.e. to the group $\mathrm{O}(1,1)$.

The S refers to "propriety"[4] of i.e. condition M2, whereas the superscript $\bullet^+$ refers to "orthochronicity" i.e. condition M3.

Remark 8.2.1. Any element of $O(1,1)$ can be written as the composition of an element of $SO^+(1,1)$ with a combination of the following discrete transformations, known as *time and parity reversal* transformations, respectively:

$$\mathcal{T} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \Diamond$$

Exercise 8.2.2. Show that the elements of $\mathrm{SO}^+(1,1)$ are of the form $\Lambda(\psi)$. $\Diamond$

8.3 1+3d Minkowski metric and Lorentz group

This abstract formulation has the advantage of readily generalizing to arbitrary dimensions. In 1+3d we have that the Minkowski metric reads

$$\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and thus, through conditions M1-3, we have that

$$\text{1+3d (proper orthochronus) Lorentz group} = \mathrm{SO}^+(1,3).$$

Remark 8.3.1. In 1+3 dimensions, $O(1,3)$ can be written as the composition of an element of $SO^+(1,3)$ with a combination of the *time and parity reversal* transformations, respectively:

$$\mathcal{T} = \begin{pmatrix} -1 & \\ & 1_{3\times 3} \end{pmatrix} \quad \text{and} \quad \mathcal{P} = \begin{pmatrix} 1 & \\ & -1_{3\times 3} \end{pmatrix}$$

The same formula with $1_{3\times 3}$ replaced by $1_{d\times d}$ holds in 1+d dimensions, when d is *odd*. $\Diamond$

Exercise 8.3.2. With reference to the previous remark: why and how should those formulas be modified when d is even? (The answer lies in the mirror!) $\Diamond$

The space 4-dimensional real vector space that constitutes space time is called *Minkoski space* and is denoted $\mathbb{R}^{1,3}$ to emphasize the signature of its metric. Its points, $X = (ct, \vec{x})$ are called *events*. The "Lorentzian distance" between two events $X_1, X_2 \in \mathbb{R}^{1,3}$ is called the *spacetime interval*, and is denoted (here, $\Delta X \doteq X_2 - X_1$):

$$\Delta s^2 = \Delta X^T \eta \Delta X = -c^2(t_2 - t_1)^2 + |\vec{x}_2 - \vec{x}_1|^2.$$

Once again, Δs^2 is not the square of Δs but a symbol in itself that reminds us that this expression is quadratic in ΔX, even though it is not positive definite.

[4] S stands for "special", and refers to the unit determinant condition as in the case of rotations. In the context of the Lorentz group, this property is instead called "propriety".

Remark 8.3.3. The group of 3d rotations, SO(3) is a subgroup of $\mathrm{SO}^+(1,3)$. In fact, $\mathrm{SO}^+(1,3)$ contains many inequivalent subgroups all isomorphic to SO(3): one per choice of reference frame. I.e. consider the "standard" SO(3) subgroup given by Lorentz transformations of the form:

$$\rho(R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}.$$

Then, for any fixed choice of $\Lambda \in \mathrm{SO}^+(1,3)$, the set of matrices $\rho_\Lambda(R) = \Lambda\rho(R)\Lambda^{-1}$ form a subgroup of $\mathrm{SO}^+(1,3)$ isomorphic to SO(3), but inequivalent to the "standard" one. Physically, this says that there is a natural group of rotations attached to any choice of rest frame, but these groups are not equivalent because Lorentz boosts mix space and time and thus do not preserve the notion of what a "rotation" is. In the following, we will often abuse notation and denote $\rho(R)$ simply with the symbol R. $\Diamond$

The sign of the Minkowski norm of a vector—which we showed is a quantity independent of the chosen frame of reference—provides the following classification of vectors:

$$X \text{ is said } \begin{cases} \text{spacelike if } |X|_\eta^2 > 0, \\ \text{timelike if } |X|_\eta^2 < 0, \\ \text{lightlike if } |X|_\eta^2 = 0. \end{cases}$$

Timelike vectors can moreover be *future*- or *past*-pointing (cf. condition M3) if $X^0 > 0$ or $X^0 < 0$ respectively.

Given an event $X_O \in \mathbb{R}^{1,3}$, the set of events separated from it by a lightlike interval is called the *lightcone of* X_O. The set of events separated from it by a timelike interval is called the *causal domain of* O. If $X - X_O$ is a future- (resp. past-)pointing timelike vector, then X is said to be in the future (resp. past) of X_O.

c being the speed of light, the future lightcone of X_O can be seen as the spacetime trajectory of an isotropic sphere of photons shined in all directions at the spacetime event X_O.

See Figure 8.1.

8.4 The Lorentz algebra

In 1+1d we had an explicit form for the most general Lorentz transformation. We would now like to know how to write the most general Lorentz transformation in 1+3d.

Let's go back to 1+1d, and let's expand the trigonometric functions in a Taylor

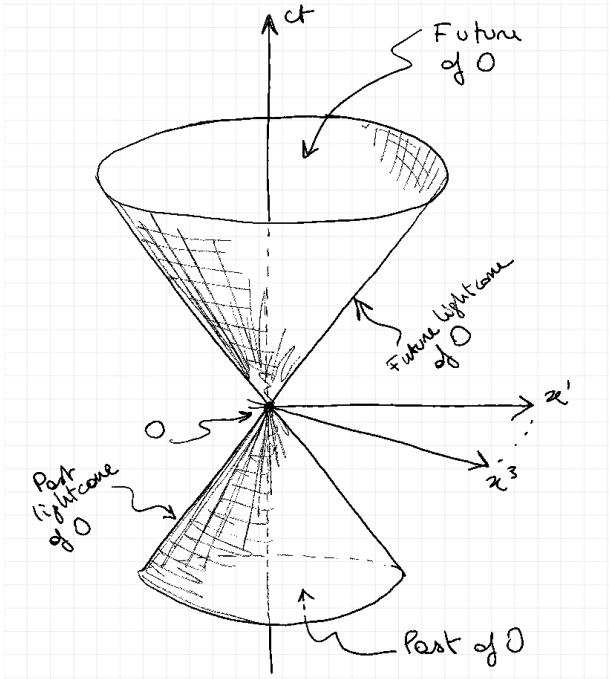

Figure 8.1: Lightcone structure.

series around the identity transformation:

$$\begin{aligned}\Lambda(\psi) &= \begin{pmatrix} \mathrm{ch}\psi & -\mathrm{sh}\psi \\ -\mathrm{sh}\psi & \mathrm{ch}\psi \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\psi \\ -\psi & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2!}\psi^2 & 0 \\ 0 & \frac{1}{2!}\psi^2 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{3!}\psi^3 \\ -\frac{1}{3!}\psi^3 & 0 \end{pmatrix} + \dots\end{aligned}$$

Introduce

$$K \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and note that

$$K^n = \begin{cases} K & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Then, recalling the formula for a (matrix) exponential, $\exp M = \sum_{n=0}^{\infty} \frac{M^n}{n!}$, we obtain

$$\Lambda(\psi) = \exp(-\psi K).$$

Exercise 8.4.1. Show that in 2d:

$$R(\theta) = \exp(-\theta J) \quad \text{with} \quad J \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Diamond$$

To interpret these formulas, recall that the exponential of M can also be written as the limit $\lim_{n\to\infty} \left(1 + \frac{1}{n}M\right)^n$. This way, we see that the Lorentz transformation

$$\Lambda = \exp(-\psi K) = \lim_{n\to\infty} \left(1 - \frac{\psi}{n}K\right)^n.$$

can be thought of as the composition of $n \to \infty$ infinitesimal rotations each of magnitude $-\frac{1}{n}\psi$. Hence, the matrix K is called the *boost generator*, for it characterizes the infinitesimal boost that gives rise to Λ.

Can we generalize this understanding to higher dimensions? In other words, if Λ is the exponential of a certain infinitesimal transformation λ, then, we ask, which properties must λ satisfy?

Recall that a Lorentz transformation $\Lambda \in \mathrm{SO}^+(1,3)$ satisfies in particular the following properties:

$$\Lambda^T \eta \Lambda = \eta \quad \text{and} \quad \det\Lambda = 1.$$

Replacing in here the expression $\Lambda = \exp(-\epsilon\lambda) = 1 - \epsilon\lambda + O(\epsilon^2)$, we find that:[5]

$$\lambda^T = -\eta\lambda\eta \quad \text{and} \quad \mathrm{tr}(\lambda) = 0.$$

Note that the condition on the determinant of Λ, which is nonlinear in its entries, translates into a condition on its trace, which is instead linear in its entries and thus much simpler to solve. The most general solution to these equations is of the form:

$$\lambda = \begin{pmatrix} 0 & \psi_1 & \psi_2 & \psi_3 \\ \psi_1 & & & \\ \psi_2 & & r & \\ \psi_3 & & & \end{pmatrix} \quad \text{with} \quad r^T = -r.$$

Exercise 8.4.2. Show that an infinitesimal 3d rotation r, i.e. a matrix such that $R = \exp(-\epsilon r) \in \mathrm{SO}(3)$, must satisfy $r^T = -r$. In fact, this is not only a necessary but also a sufficient condition for r to be an infinitesimal rotation. $\lozenge$

We therefore introduce the following *six generators of the Lorentz group*:

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so that any[6] Lorentz transformation can be written as:

$$\Lambda(\vec{\psi}, \vec{\theta}) = \exp\Big(-\vec{\psi} \cdot \vec{K} - \vec{\theta} \cdot \vec{J}\Big),$$

[5] It is important to remember the following classical identity of linear algebra: $\det(1 - \epsilon M) = 1 - \epsilon\,\mathrm{tr}(M) + O(\epsilon^2)$. You can find more information on this by seeking information on the characteristic polynomial $p_M(t) = \det(t1 - M)$ of a matrix M.

[6] That this is actually the case is a rather nontrivial fact.

for some $\vec{\psi}, \vec{\theta} \in \mathbb{R}^3$, i.e.

$$\Lambda(\vec{\psi}, \vec{\theta}) = \exp\Big(-\lambda(\vec{\psi}, \vec{\theta})\Big) \quad \text{for} \quad \lambda(\vec{\psi}, \vec{\theta}) = \begin{pmatrix} 0 & \psi_1 & \psi_2 & \psi_3 \\ \psi_1 & 0 & \theta_3 & -\theta_2 \\ \psi_2 & -\theta_3 & 0 & \theta_1 \\ \psi_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}.$$

The Lorentz transformations of the form $\Lambda(\vec{\psi}, \vec{0})$ is a *boost of rapidity* $\psi = |\vec{\psi}|$ *in the direction* $\vec{n}_b \doteq \vec{\psi}/\psi$. The Lorentz transformations of the form $\Lambda(\vec{0}, \vec{\theta})$ are *rotations of angle* $\theta = |\vec{\theta}|$ *around the axis* $\vec{n}_r \doteq \vec{\theta}/\theta$. Rotations form the "standard" SO(3) subgroup of $\mathrm{SO}^+(1,3)$.

Moreover, one has that

$$\Lambda(\vec{\psi}, \vec{\theta})^{-1} = \Lambda(-\vec{\psi}, -\vec{\theta}).$$

Since computing $\exp(-\psi^1 K_1)$, $\exp(-\theta^2 J_2)$ etc is extremely simple—and in fact we have already done it!—it is natural to ask whether $\Lambda(\vec{\psi}, \vec{\theta})$ is the product of these more basic building blocks. In other words, is $\Lambda(\vec{\psi}_2, \vec{\theta}_2)\Lambda(\vec{\psi}_1, \vec{\theta}_1)$ equal to $\Lambda(\vec{\psi}_1+\vec{\psi}_2, \vec{\theta}_1+\vec{\theta}_2)$? The answer is *absolutely not!* This is in general not even the case when $\vec{\theta}_1 = \vec{\theta}_2 = \vec{0}$, nor when $\vec{\psi}_1 = \vec{\psi}_2 = \vec{0}$, nor when $\vec{\psi}_1 = \vec{\theta}_2 = \vec{0}$.

The extent to which the previous ansatz was false is quantified by the Baker–Campbell–Hausdorff (BCH) formula: given matrices A and B,

$$\exp(A)\exp(B) = \exp\Big(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] + \dots\Big)$$

where the bracket denotes the commutator:

$$[A,B] \doteq AB - BA.$$

Hence, we see that the commutator between its generators encodes crucial information about the Lorentz group. In fact, it encodes all the information there is to know up to global issues on the topology of the group.

These commutators are:

$$\begin{aligned} [J_i, J_j] &= -\sum_k \epsilon_{ij}{}^k J_k, \\ [J_i, K_j] &= -\sum_k \epsilon_{ij}{}^k K_k, \\ [K_i, K_j] &= +\sum_k \epsilon_{ij}{}^k J_k. \end{aligned}$$

Since the commutator is a bilinear operation between matrices, from these expressions we can compute the commutator between any two infinitesimal Lorentz transformations.

Exercise 8.4.3. Setting $\lambda_1 = \vec{\psi}_1 \cdot \vec{K} + \vec{\theta}_1 \cdot \vec{J}$ and similarly for λ_2, show that:

$$[\lambda_1, \lambda_2] = (-\vec{\theta}_1 \times \vec{\psi}_2 + \vec{\theta}_2 \times \vec{\psi}_1) \cdot \vec{K} + (-\vec{\theta}_1 \times \vec{\theta}_2 + \vec{\psi}_1 \times \vec{\psi}_2) \cdot \vec{J} \qquad \Diamond$$

More abstractly, we want to think of $(\vec{K}, \vec{J})$ as an abstract basis of a 6-dimensional real vector space that we shall denote $\mathfrak{so}(1,3)$, and of the 6-tuples $(\vec{\psi}, \vec{\theta})$ as the components of a vector in this space. Then, with this understanding the commutator defines a bilinear, skew-symmetric map on $\mathfrak{so}(1,3)$ that moreover satisfies the Jacobi identity—this is in fact an immediate consequence of the associativity of matrix products, $(AB)C = A(BC)$. Therefore, the vector space $\mathfrak{so}(1,3)$ equipped with $[\cdot,\cdot]$ is Lie algebra, which we will call the *Lorentz Lie algebra*.

Remark 8.4.4. Boost and rotations aligned along the same axis commute with each other, since each affects orthogonal planes in $\mathbb{R}^{1,3}$. E.g. whereas a rotation around the z-axis affects the (xy)-plane while leaving invariant the (ct, z) plane, the converse is true for a boost along the z-axis. Using the BCH formula this means that

$$\Lambda(\psi\hat{n}, \theta\hat{n}) = \Lambda(\psi\hat{n}, \vec{0})\Lambda(\vec{0}, \theta\hat{n}) = \Lambda(\vec{0}, \theta\hat{n})\Lambda(\psi\hat{n}, \vec{0}). \qquad \Diamond$$

Remark 8.4.5. The bracket $[J_i, K_j] = -\sum_k \epsilon_{ij}{}^k K_k$ is the infinitesimal version of the following identity which holds for any rotation $R = \exp(-\vec{\varphi} \cdot \vec{J})$:

$$R(\vec{\psi} \cdot \vec{K})R^{-1} = (R\vec{\psi}) \cdot \vec{K}.$$

From this it follows that

$$R\Lambda(\vec{\psi}, \vec{0})R^{-1} = \Lambda(R\vec{\psi}, \vec{0}).$$

For the same reasons, it is also true that:

$$R\Lambda(\vec{\psi}, \vec{\theta})R^{-1} = \Lambda(R\vec{\psi}, R\vec{\theta}). \qquad \Diamond$$

Exercise 8.4.6. Prove the three statements made in the previous remark. [*Hint*: for the first one it will be useful to notice the following expression for the components of the generators of rotation, $(J_i)^j{}_k = -\epsilon_{ijk}$, whereas for the second one use the definition of the exponential of a matrix involving a infinite series expansion.] $\Diamond$

Remark 8.4.7. Rotations are a subgroup of the Lorentz group, and indeed infinitesimal rotations are a subalgebra of the Lorentz algebra. This is *not* the case for boosts, which do not form a subalgebra of the Lorentz Lie algebra. Indeed, the Lie bracket of two boosts fails to be itself a boost, and rather gives a rotation. This mathematical fact is at the basis of the physical phenomenon known as Thomas's precession. $\Diamond$ tutorial? HW?

Exercise 8.4.8. Denote $\mathfrak{sl}_{\mathbb{C}}(2)$ the Lie algebra of the group $\mathrm{SL}_{\mathbb{C}}(2)$ (the special linear group in 2 complex dimensions, i.e. the group of 2×2 complex matrices of unit determinant). Find the generators of $\mathfrak{sl}_{\mathbb{C}}(2)$ seen as a *real* Lie algebra of dimension 6, and show that $\mathfrak{sl}_{\mathbb{C}}(2)$ is isomorphic to the Lorentz algebra $\mathfrak{so}(1,3)$. At the bottom of this exercise lies the fact that SU(2) is to SO(3) what $\mathrm{SL}_{\mathbb{C}}(2)$ is to $\mathrm{SO}^+(1,3)$. [*Hint*: use the Pauli matrices and the imaginary unit to find a nice basis of $\mathfrak{sl}_{\mathbb{C}}(2)$ as a real vector space of dimension 6.] $\Diamond$

Chapter 9

Relativistic notation

9.1 Tensors and the Einstein–Ricci notation

Let V be an n-dimensional real vector space, and let $\{e_i \in V\}_{i=1}^n$ be a basis. This means that for all $v \in V$ there exists a unique n-tuple of numbers $\{v^i \in \mathbb{R}\}_i^n$ such that

$$v = \sum_{i=1}^{n} v^i e_i.$$

The dual space V^* of V is defined as the set of linear functions over V, that is

$$V^* \doteq \mathrm{Lin}(V, \mathbb{R}).$$

The evaluation of an element $\alpha \in V^*$ at a given $v \in V$ is often denoted with an angled bracket,

$$\langle \alpha, v \rangle \doteq \alpha(v).$$

This is because, as we will now see, there is a certain "symmetry" between elements of V and V^* which this notation emphasizes.

Indeed, V^* is itself a vector space, since—as it is easy to check—for all $a \in \mathbb{R}$ and $\alpha, \beta \ni V^*$, we have that $a\alpha + \beta \in V^*$ as well. Its dimension is also n. Of all bases of V^*, given $\{e_i\}$ we can always chose a *dual* basis $\{f^i\}$,

$$\alpha = \sum_{i=1}^{n} \alpha_i f^i,$$

that is fully characterized by the property that

$$\langle f^i, e_j \rangle = \delta^i_j \doteq \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

whence

$$\langle \alpha, v \rangle = \sum_{i,j=1}^{n} \alpha_i v^j \langle f^i, e_j \rangle = \sum_{i=1}^{n} \alpha_i v^i.$$

If elements of V are called vectors, elements of V^* are sometimes called dual vectors or *co*-vectors—-this even though they are vectors as well, just in a different space (in mechanics, the velocity $v^i \in \mathrm{T}_q Q$ is a vector whereas the momentum $p_i \in \mathrm{T}_q^* Q$ is a covector).

In the Einstein–Ricci[1] notation a vector v is denoted by the n-tuple of its components v^i, and similarly a covector α by α_i. Notice the position of the indices: component-indices in V are up, while component-indices in V^* are down; and conversely for the indices of the basis elements. The reason for this convention is that we can then unambiguously omit the sum symbol in expressions with a repeated up-down index, as in:

$$v = v^i e_i, \quad \alpha = \alpha_i f^i, \quad \langle \alpha, v \rangle = \alpha_i v^i.$$

This notation is called Einstein's index convention, or Einstein's notation. It is extremely convenient when dealing with higher rank tensors, i.e. objects with multiple indices.

9.2 Changes of basis

Consider an invertible linear transformation $M : V \to V$. The transpose of M, denoted M^T, is the unique linear transformation $M^T : V^* \to V^*$ such that for all $\alpha \in V^*$ and $v \in V$,

$$\langle M^T \alpha, v \rangle = \langle \alpha, M v \rangle.$$

Given $\{e_i\}$ we can use M to define a new basis of V, i.e.

$$e'_i = M e_i.$$

It is easy to show that then the new dual basis is $f'^i = (M^T)^{-1} f^i$, where $(M^T)^{-1} = (M^{-1})^T$ (check it!).

The components of the new basis with respect to the old one are

$$M^i{}_j \doteq \langle f^i, M e_j \rangle = \langle f^i, e'_i \rangle.$$

The components of v in the new basis then read:

$$v'^i = \langle f'^i, v \rangle = \langle (M^T)^{-1} f^i, v \rangle = \langle f^i, M^{-1} v \rangle = (M^{-1})^i{}_j v^j.$$

Note that they transform with respect to the inverse matrix when compared to the basis elements. Similarly,

$$\alpha'_i = \langle \alpha, e'_i \rangle = \langle \alpha, M e_i \rangle = \alpha_i M^i{}_j \equiv M^i{}_j \alpha_i.$$

In view of these equations, upper indices are called "contravariant" and lower indices are called "covariant". Hence, vectors v^i can be called "contravariant vectors", while co-vectors α_i can be called "covariant vectors".

[1]Named after Ricci-Curbastro and Einstein, this notation was developed and refined also by Levi-Civita (a student of Ricci-Curbastro's) and Schouten.

Remark 9.2.1. The last equality in the previous equation is there to emphasize that we do not care in which order we write the components: since we know how they should be contracted just by looking at the repeated-index structure. This is a great advantage of the index notation, especially when we will start using objects with many indices. ◊

9.3 Tensors

A linear map $A : V \to V$ can be written in components as $A^i{}_j = \langle f^i, Ae_i \rangle$. In the new basis, A reads (check it!):

$$A'^i{}_j = \langle f'^i, Ae'_j \rangle = (M^{-1})^i_k M^l{}_j A^k{}_l.$$

Thus $A^i{}_j$ is an object with both covariant and contravariant indices.

This suggests that the linear map $A : V \to V$ can also be seen as an element of $V \otimes V^*$, i.e. as

$$A = A^i{}_j e_i \otimes f^j.$$

From this perspective, the expression, $w = Av$, which in components reads

$$w^i = A^i{}_j v^j,$$

can also be thought of as the result of the following contraction of an element of V with one of $V \otimes V^*$:

$$w = A^i{}_j e_i \otimes \langle f^j, v \rangle = (A^i{}_j v^j) e_i.$$

Note that, as the index structure and the Einstein's convention suggest, there is no other way to contract v^j with $A^i{}_j$, which reflects the fact that there is only one natural pairing of elements of V with an element of $V \otimes V^*$ (i.e. we cannot just pair elements of V among themselves).

An object with an arbitrary number of covariant and contravariant indices (i.e. possibly zero, one, or more indices) is called a tensor, and the number of its indices its rank. Tensors of rank 0 are called scalars.

Using the Einstein–Ricci index notation, it easy to deal with tensor of arbitrary rank, such as (mind the order of the indices: it is very important!)

$$T \in V \otimes V \otimes V^* \otimes V^* \otimes V^* \otimes V, \quad T = T^{ij}{}_{klm}{}^n \; e_i \otimes e_j \otimes f^k \otimes f^l \otimes f^m \otimes e_n,$$

and their contractions (notice that we have also contracted different indices in the same tensor),

$$U^j{}_{kp} = T^{ij}{}_{kln}{}^n S_i{}^l{}_p,$$

and changes of basis:

$$T'^{i'j'}{}_{k'l'm'}{}^{n'} = (M^{-1})^{i'}{}_i (M^{-1})^{j'}{}_j (M^{-1})^{n'}{}_n M^k{}_{k'} M^l{}_{l'} M^m{}_{m'} T^{ij}{}_{klm}{}^n.$$

Remark 9.3.1. Note that, being summed over, the name of a repeated index is completely immaterial, but not so for the remaining "free indices":

$$U^j{}_{kp} = T^{ij}{}_{kln}{}^n S_i{}^l{}_p \equiv T^{aj}{}_{kbc}{}^c S_a{}^b{}_p. \qquad \Diamond$$

Exercise 9.3.2 (Important!). Check that we can perform an index contraction in any base we like, without affecting the result, e.g.

$$U'^{j'}{}_{k'p'} = T'^{i'j'}{}_{k'l'n'}{}^{n'} S'_{i'}{}^{l'm'}{}_{p'} = (M^{-1})^{j'}{}_j M^k{}_{k'} M^p{}_{p'} U^j{}_{kp}.$$

In particular expressions that are index-free, i.e. obtained by a full contraction of indices, are invariant under a change of basis, e.g.

$$U'^i{}_{ij} v'^j = U^p{}_{pq} v^q$$

In particular, the quantity $A^i{}_i$ is called the *trace* of $A^i{}_j$. $\Diamond$

In the previous exercise, we have shown that indices that are contracted and "disappear" need not be kept track of when it comes to a change of basis, and that changes of basis and contractions are operations that commute with each other. This is what makes this notation powerful.

It also allows for a so-called "abstract index notation",[2] where the indices do not refer to a specific basis but are simply used to keep track of the nature of tensors and their contractions.

9.4 Gradients

Consider a function $f \in C^\infty(V)$ (which does not need to be linear). A vector $v \in V$ then defines a direction derivative, according to the following formula valid at all $x \in V$:

$$v : C^\infty(V) \to C^\infty(V), \quad f \mapsto v(f) \ : \ v(f)(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon v) - f(x)}{\epsilon}.$$

Denoting:

$$\partial_i f \doteq e_i(f),$$

we have that

$$v(f) = v^i \partial_i f.$$

Since $v(f) = v^i \partial_i f$ is independent of the chosen basis, and since v^i is a contravariant tensor, we conclude that at each $x \in V$, $\partial_i f(x)$ is a covariant vector. We call $\partial_i f$ the *gradient of* f.

The above formulas readily generalize to v a vector field over V, i.e. an assignment of a vector $v(x) = v^i(x) e_i$ at each $x \in V$, so that

$$v(f)(x) = v^i(x) \partial_i f(x).$$

[2] E.g. Wald, General Relativity, University of Chicago Press (1984).

9.5 Tensors in relativity

In special relativity, spacetime is a vector space[3] $V = \mathbb{R}^4$, whose elements we call *events*, and which is equipped with a Lorentzian metric η, i.e. a bilinear, symmetric, positive definite, map $V \otimes V \to \mathbb{R}$ of signature $(1, 3)$. We thus consider a "Cartesian" basis (e_0, e_1, e_2, e_3) of spacetime which is orthonormal with respect to η, that is:[4]

$$\eta(e_0, e_0) = -1, \quad \eta(e_0, e_i) = 0, \quad \text{and} \quad \eta(e_i, e_j) = \delta_{ij} \quad \forall i, j \in \{1, 2, 3\}.$$

We adopt the following Ricci–Einstein convention for spacetime tensors: the index μ goes from 0 to 3, and the index i goes from 1 to 3, so that, thus:

$$X = X^\mu e_\mu = X^0 e_0 + X^i e_i,$$

and we write

$$\eta_{\mu\nu} \doteq \eta(e_\mu, e_\nu).$$

The metric η can also be seen as an element of $V^* \otimes V^*$, as in $\eta = \eta_{\mu\nu} f^\mu \otimes f^\nu$. Most importantly, it can also be seen as a map $V \to V^*$:

$$X \mapsto \eta(X) = (\eta_{\mu\nu} X^\mu) f^\nu.$$

This map plays a crucial role, and appear so often that we will leave it implicit, denoting the covector $\eta(X)$ simply as X_μ

$$X_\mu \doteq \eta_{\mu\nu} X^\nu.$$

Similarly, we denote the inverse map $\eta^{-1} : V^* \to V$ by $\eta^{\mu\nu}$, i.e.

$$\eta^{\mu\nu} \eta_{\nu\rho} = \delta^\mu_\rho,$$

and thus we can consistently write

$$X^\mu = \eta^{\mu\nu} X_\nu.$$

Remark 9.5.1. *Numerically*, one has the following relation between the components of X^μ and X_μ, notice the minus sign affecting the time component:

$$X^0 = -X_0 \text{ while } X^1 = X_1,\ X^2 = X_2,\ X^3 = X_3. \qquad \Diamond$$

Summarizing: η and η^{-1} are very special maps from $V \to V^*$ and $V^* \to V$ that play a structural role in the theory of special relativity. For this reason they are given an operationally privileged status, i.e. they correspond to special tensors $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ which when contracted with other tensor "raise and lower" its indices, as in

$$T^{\mu\nu}{}_\rho = \eta^{\mu\mu'} \eta_{\rho\rho'} T_{\mu'}{}^{\nu}{}_{\rho'}.$$

[3] More precisely, it is an affine space, i.e. it is a "vector space without a pre-set origin". In the following we assume that a reference event has been chosen as an origin.

[4] This basis is of course *not* unique, see below.

Similarly, we introduce the metric δ_{ij} and its inverse δ^{ij} to raise and lower the spatial indices only. In particular, the Lorentzian norm of a vector is

$$\begin{aligned} |X|^2_\mu &\doteq \eta_{\mu\nu}X^\mu X^\nu = X^\mu X_\mu = \eta^{\mu\nu}X_\mu X_\nu \\ &= X^0X_0 + X^iX_i = -(X^0)^2 + \delta_{ij}X^iX^j \\ &= -(X^0)^2 + X^1X^1 + X^2X^2 + X^3X^3. \end{aligned}$$

So far we have worked in a fixed basis $\{e_\mu\}$ of spacetime. This basis was special in that it was taken to be orthonormal with respect to η. Consider an invertible linear map $\Lambda : V \to V$, and use it to define a new basis $e'_\mu = \Lambda e_\mu$. The new basis is also orthonormal iff η is Λ-invariant i.e. iff (check it!)

$$\eta_{\mu'\nu'} = \eta'_{\mu'\nu'} \doteq \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\eta_{\mu\nu}.$$

It is important to realize that our convention of using the same symbol to denote X^μ and $X_\mu = \eta_{\mu\nu}X^\nu$ is consistent only as long as we restrict to changes of basis $\Lambda^\mu{}_\nu$ that satisfy the property above (cf. the left- and right-most terms in this string of equalities with the previous equations):

$$\Lambda^\mu{}_{\mu'}X_\mu = X'_{\mu'} = \eta_{\mu'\nu'}X'^{\nu'} = \eta_{\mu'\nu'}(\Lambda^{-1})^{\nu'}{}_\nu X^\nu.$$

Of course, such Λ's are nothing else than the $\mathrm{O}(1,3)$ (Lorentz) transformations we introduced in the previous section. As already stated, in these notes we will only deal with the subset of proper orthochronus Lorentz transformations $\Lambda \in \mathrm{SO}^+(1,3)$.

This state of affairs is often summarized as stating that the index notation introduced here, with $X_\mu = \eta_{\mu\nu}X^\nu$, is "Lorentz covariant". In other words, it guarantees that all expressions and manipulations done using it are automatically valid in any orthonormal basis of Minkowski's space.

Vectors and co-vectors that transform under a Lorentz transformation as

$$V^\mu \mapsto (\Lambda^{-1})^\mu{}_\nu V^\nu \quad \text{and} \quad A_\mu \mapsto \Lambda^\nu{}_\mu A_\nu$$

are generically called *4-vectors*. (One usually does not say "4-tensors", but the same idea could be applied there.)

Exercise 9.5.2. Show that

$$(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu$$

Note the importance of the index position! To avoid confusion, in the following we will avoid the notation on the right hand side of this equality. ◊

We conclude this section by revisiting the Lorentz algebra in relativistic notation. Let us introduce the symbol $J_{\alpha\beta}$

$$J_{\alpha\beta} \doteq \begin{cases} 0 & \text{if } (\mu,\nu) = 0 \\ -K_i & \text{if } (\mu,\nu) = (0,i) \\ K_i & \text{if } (\mu,\nu) = (i,0) \\ \epsilon_{ij}{}^k J_k & \text{if } (\mu,\nu) = (i,j) \end{cases}$$

The Lorentz algebra then reads

$$[J_{\alpha\beta}, J_{\gamma\delta}] = -\eta_{\alpha\gamma} J_{\beta\delta} + \eta_{\beta\gamma} J_{\alpha\delta} - \eta_{\beta\delta} J_{\alpha\gamma} + \eta_{\alpha\delta} J_{\beta\gamma}.$$

Remark 9.5.3. The symbol $J_{\alpha\beta}$ is a skew-symmetric matrix whose entries are the 6 independent generators of the Lorentz algebras, (K_i, J_i), each a 4×4 matrix itself. Making explicit the indices of these matrices, we leave to the reader to check that:

$$(J_{\alpha\beta})^{\mu}{}_{\nu} = \delta^{\mu}_{\alpha}\eta_{\beta\nu} - \delta^{\mu}_{\beta}\eta_{\alpha\nu}.$$

The 4×4 matrices $(K_i)^{\mu}{}_{\nu}$ and $(J_i)^{\mu}{}_{\nu}$ can be found in Section 8.4. Before doing the computation, note that the matrix expression for the (K_i, J_i) given in Section 8.4 correspond to the Lorentz tensors $(K_i)^{\mu}{}_{\nu}$ and $(J_i)^{\mu}{}_{\nu}$ *with one spacetime index up and one down*. This is because these tensors are built as maps from the vector space $V = \mathbb{R}^{1,3}$ to itself (with no dualization involved), i.e. as elements of $V \otimes V^*$.

The above equation for the Lie bracket of two generators of the Lorentz algebra is an equality between 4×4 matrices labelled by fixed-values of the $(\alpha\beta)$ and $(\gamma\delta)$ indices. Making their matrix indices explicit, the left hand side reads:

$$[J_{\alpha\beta}, J_{\gamma\delta}]^{\mu}{}_{\nu} \doteq (J_{\alpha\beta})^{\mu}{}_{\rho}(J_{\gamma\delta})^{\rho}{}_{\nu} - (J_{\gamma\delta})^{\mu}{}_{\rho}(J_{\alpha\beta})^{\rho}{}_{\nu}.$$

$\Diamond$

Exercise 9.5.4. Check that this expression is correct using the explicit form of $(J_{\alpha\beta})^{\mu}{}_{\nu}$ given above. $\Diamond$

The 6 rank-2 tensors $((K_i)^{\mu}{}_{\nu}, (J_i)^{\mu}{}_{\nu})$ are maps from the vector space $V = \mathbb{R}^{1,3}$ to itself (with no dualization involved), i.e. they are elements of $V \otimes V^*$. If we were to dualize one of the vector spaces, e.g. by raising the index $\bullet_{\nu}$ by contraction with the inverse metric, we would obtain 6 rank-2 Lorentz tensors which can be shown to be *skew-symmetric* (check this statement!), i.e.

$$(K_i)^{\mu\nu} + (K_i)^{\nu\mu} = 0 = (J_i)^{\mu\nu} + (J_i)^{\nu\mu}.$$

These 6 skew symmetric rank-2 tensors are best organized in a rank-4 tensor which is skew symmetric in both pair of indices, i.e.

$$(J_{\alpha\beta})^{\mu\nu} = 2\delta^{\mu}_{[\alpha}\delta^{\nu}_{\beta]}.$$

Note that the 6 $((K_i)^{\mu\nu}, (J_i)^{\mu\nu})$, i.e. the 6 $((J_{\alpha\beta})^{\mu\nu})$, constitute a basis of the space of skew-symmetric rank-2 tensors $\lambda^{\mu\nu}$. Therefore, since the $((K_i)^{\mu}{}_{\nu}, (J_i)^{\mu}{}_{\nu})$ are the generators of the Lorentz algebra, we conclude that the Lorentz algebra $\mathfrak{so}(1,3)$ is isomorphic to the 6 dimensional vector space of skew-symmetric rank-2 tensors $\lambda^{\mu\nu}$. In symbols, one writes

$$\mathfrak{so}(1,3) \simeq V \wedge V \simeq V^* \wedge V^*, \qquad V \doteq \mathbb{R}^{1,3}.$$

Proof 1. In fact, a generic infinitesimal Lorentz transformation λ, that is a generic element $\lambda \in \mathfrak{so}(1,3)$, can be decomposed over the basis above as (cf. Section 8.4):

$$\lambda = \frac{1}{2}\lambda^{\alpha\beta} J_{\alpha\beta} = \lambda^{i0} K_i + \widetilde{\lambda}^i J_i,$$

where $\lambda^{\alpha\beta} = \lambda^{[\alpha\beta]}$ is skew-symmetric and thus has 6 independent entries which we organize into two 3-vectors λ^{0i} and $\widetilde{\lambda}^k \doteq \epsilon^i{}_{jk}\lambda^{jk}$. Note that these 6 entries are the components of λ in the basis (K_i, J_i) of the six-dimensional vector space which is the Lorentz algebra $\mathfrak{so}(1,3)$. Now, each λ is itself a 4×4 matrix and adopting the index notation the above expression becomes:

$$\lambda^\mu{}_\nu = \frac{1}{2}\lambda^{\alpha\beta}(J_{\alpha\beta})^\mu{}_\nu = \lambda^{i0}(K_i)^\mu{}_\nu + \widetilde{\lambda}^i(J_i)^\mu{}_\nu.$$

Raising the index $\bullet^\nu$, we would find

$$\lambda^{\mu\nu} = \frac{1}{2}\lambda^{\alpha\beta}(J_{\alpha\beta})^{\mu\nu} = \lambda^{i0}(K_i)^{\mu\nu} + \widetilde{\lambda}^i(J_i)^{\mu\nu}.$$

If view of the previous observation saying that the 6 $((K_i)^{\mu\nu}, (J_i)^{\mu\nu})$, i.e. the 6 $((J_{\alpha\beta})^{\mu\nu})$, constitute a basis of the space of skew-symmetric rank-2 tensors, we conclude that to each $\lambda \in \mathfrak{so}(1,3)$ there corresponds a skew symmetric rank-2 tensor $\lambda^{\mu\nu}$ and viceversa. □

Proof 2. A more conceptual proof of this statement goes as follows. First, recall that $\Lambda \in \mathrm{SO}(1,3)$ if and only if $\Lambda^T\eta\Lambda = \eta$ and $\det\Lambda = 1$. From this it follows that, at the infinitesimal level, $\lambda \in \mathfrak{so}(1,3)$ if and only if $\lambda^T\eta + \eta\lambda = 0$ and $\operatorname{tr}\lambda = 0$. Using the index notation introduced above, this means that

$$\lambda^\mu{}_{\mu'}\eta_{\mu\nu'} + \lambda^\nu{}_{\nu'}\eta_{\mu'\nu} = 0 \quad \text{and} \quad \delta^\nu{}_\mu\lambda^\mu{}_\nu = 0$$

that is, after removing the primes from the indices:

$$\lambda_{\nu\mu} + \lambda_{\mu\nu} = 0,$$

as we wanted to show.

Finally, note that the traceless condition is implied by the antisymmetry. This can be seen as follows: if $\Lambda^T\eta\Lambda = \eta$ then $(\det\Lambda)^2 = 1$ which implies $\det\Lambda = \pm 1$. Hence, if $\Lambda = 1 + \epsilon\lambda + O(\epsilon^2)$ is an infinitesimal Lorentz transformation, we get $\det\Lambda = 1 + \epsilon\operatorname{tr}\lambda + O(\epsilon^2) = \pm 1$ and clearly the only viable choice is the one corresponding to $\det\Lambda = 1$, i.e. $\operatorname{tr}\lambda = 0$ (in fact $\det\Lambda = -1$ would require $-2/\epsilon \to \infty$). This observation is closely related to the fact that the space of infinitesimally generated Lorentz transformations is $\mathrm{SO}^+(1,3) \subset \mathrm{SO}(1,3)$ and not $\mathrm{O}(1,3)$. □

Fifth Tutorial

Chapter 10

Relativistic mechanics

10.1 Prelude

In non-relativistic physics, dynamics is encoded in Newton's second[1] law:

$$\frac{d\vec{p}}{dt} = \vec{F} \quad \text{where} \quad \vec{p} = m\vec{v},$$

which is invariant under Galilean transformations, for $\vec{x} \mapsto \vec{x}' = \vec{x} - \vec{u}t$ and $t \mapsto t' = t$ implies $\vec{v}(t) \mapsto \vec{v}'(t) = \vec{v}(t) - \vec{u}$ and $\vec{a}(t) \mapsto \vec{a}'(t) = \vec{a}(t)$. I.e. accelerations and therefore forces are frame independent in Newtonian physics, i.e. they are invariant under Galilean boosts.

None of these statement holds true in relativistic physics: a new, Lorentz covariant, formulation of mechanics is needed.

Starting from this section we will work in units in which

$$c = 1.$$

10.2 Spacetime trajectories and proper time

The trajectory of a particle in a frame K can be described in terms of its position at a given time $\vec{x}(t)$. However, upon a change of frame both $\vec{x}$ *and* t are affected by the Lorentz transformation. This makes this description rather inconvenient for a relativistic treatment. Said differently, the expression $x^i(x^0)$ is clearly not Lorentz covariant.

We therefore switch perspective and think of the trajectory of a particle as a curve in Minkowski space $\mathbb{R}^{1,3}$. This curve is called the particle's *world line*. Mathematically it is described as map:

$$\gamma^\mu : \mathbb{R} \to \mathbb{R}^{1,3}, \quad \lambda \mapsto x^\mu = \gamma^\mu(\lambda)$$

[1]Newton's third law of action and reaction is left unchanged in the relativistic realm, and so is the principle of inertia i.e. Newton's first law.

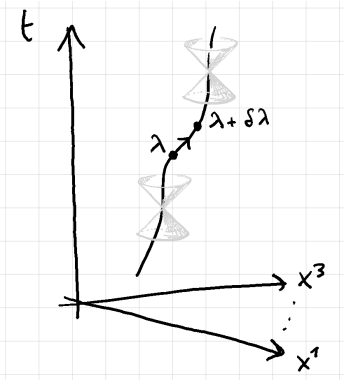

Figure 10.1: A particle's world line.

with

$$\frac{d\gamma^0}{d\lambda} > 0,$$

which ensures that the particle is moving "forward in time". This condition is preserved by the action of the (proper, *orthochronus*) Lorentz group.

Finally, if we want the trajectory to be causal, we need to require that its velocity is at all time smaller than the speed of light. This condition corresponds to asking that the curve lies inside (any one of) the lightcones centered around (any one of) the points of worldine itself. Mathemtically, this means that the interval between any two points along the curve is timelike, i.e.

$$|\gamma^\mu(\lambda_2) - \gamma^\mu(\lambda_1)|^2_\eta < 0.$$

Infinitesimally, we get the necessary and sufficient condition:

$$\left|\frac{d\gamma^\mu}{d\lambda}\right|^2_\eta < 0.$$

This is also a Lorentz-invariant condition. See Figure 10.1.

Here λ is an arbitrary, frame-independent, parameter along the curve. Since $d\gamma^0/d\lambda > 0$ in any frame, the map $\lambda \mapsto t = \gamma^0(\lambda)$ is invertible and one can, in principle, always convert this description to the previous one.

Physically, we can think of λ as a "lousy clock"[2] that ticks forward in time, but not necessarily in a regular way. For this reason we have to make sure that none of our expressions ultimately depends on the rate of ticking of this lousy clock, i.e. on the particular choice of parameter λ. Mathematically, this means that all our formulas must be invariant under the change $\lambda \mapsto \lambda' = f(\lambda)$ as long as $df/d\lambda > 0$.

Here, we described the particle's worldline in terms of an arbitrary parameter λ. The same could be done for a curve in space. In the case of a curve in space, however, we know that a natural parameter along the curve exists: its arc-length. The analogue

[2]From David Kubiznak's PSI lectures.

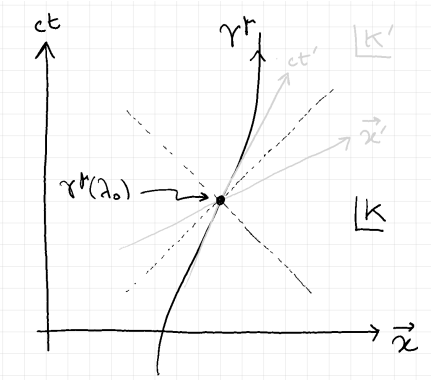

Figure 10.2: A particle's instantaneous rest frame.

of the arc-length for the worldline is its "Minkowskian arc-length". If we consider two neighbouring points $\gamma^\mu(\lambda)$ and $\gamma^\mu(\lambda+\delta\lambda)$, the "Minkowskian arc-length" $c\delta\tau$ is given by the square root of minus[3] the *interval* δs^2 between them, i.e.

$$c\delta\tau = \sqrt{-\delta s^2}, \quad \delta s^2 = \eta_{\mu\nu}\delta\gamma^\mu\delta\gamma^\nu \quad \text{where} \quad \delta\gamma^\mu = \frac{d\gamma^\mu}{d\lambda}\delta\lambda + O(\delta\lambda^2).$$

To interpret this quantity, let us go into one given frame and write it as:

$$\delta\tau = \sqrt{(\delta t)^2 - (\delta\vec{x})^2}.$$

Consider then the frame K' in which the particle is *instantaneously* at rest at $\lambda = \lambda_0$. This frame will a priori be different for different λ's, see Figure 10.2. Then, *at* $\lambda = \lambda_0$, *in the instantaneous rest frame*, we have $\delta\vec{x}'(\lambda_0) = 0$ and thus:

$$\delta\tau(\lambda_0) = \delta t'(\lambda_0) \quad \text{(instantaneous rest frame } K' \text{ at } \lambda = \lambda_0).$$

Therefore, we see that the physical interpretation of the frame-invariant $\delta\tau(\lambda)$ is the instantaneous *proper time interval* between two neighbouring point along the worldline.

Thus, the analogue for a worldline of the arc-length of a curve in space, is the proper time along the worldline. At infinitesimal level, we have:

$$d\tau = \sqrt{-\eta_{\mu\nu}\frac{d\gamma^\mu}{d\lambda}\frac{d\gamma^\nu}{d\lambda}}d\lambda,$$

and hence, choosing an arbitrary origin λ_0 at which $\tau(\lambda_0) = 0$:

$$\tau(\lambda) = \int_{\lambda_0}^{\lambda} \sqrt{-\eta_{\mu\nu}\frac{d\gamma^\mu}{d\lambda}\frac{d\gamma^\nu}{d\lambda}}d\lambda.$$

Exercise 10.2.1. Check that this expression is invariant under reparametrizations, i.e. under the change $\lambda \mapsto \lambda' = f(\lambda)$. Take proper care of the extrema of integration. $\Diamond$

[3]The minus is necessary in view of the causality condition above.

Manifestly, $d\tau/d\lambda > 0$, and therefore the proper time itself can serve—as we desired—as a parameter along the worldline. Note that parametrizing a worldline with respect to its proper time requires knowing the worldline itself, in other words it is not a parametrization that it is easy to give a priori! It is characterized by the fact that (check it!)

$$\eta_{\mu\nu}\frac{d\gamma^\mu}{d\tau}\frac{d\gamma^\nu}{d\tau} = -1.$$

10.3 Four-velocity and four-momentum

If we choose $\lambda = \tau$, then derivatives along the worldline γ^μ with respect to τ are denoted with a dot. The quantity $\dot\gamma^\mu$ goes under the name of *4-velocity*, it has always norm 1:

$$u^\mu \doteq \dot\gamma^\mu, \qquad u^\mu u_\mu = -1.$$

With respect to an arbitrary parametrization it reads:

$$u^\mu(\lambda) = \frac{d\gamma^\mu}{d\tau} = \frac{d\gamma^\mu/d\lambda}{d\tau/d\lambda} = \frac{d\gamma^\mu/d\lambda}{\sqrt{-\eta_{\mu\nu}\frac{d\gamma^\mu}{d\lambda}\frac{d\gamma^\nu}{d\lambda}}}.$$

Since τ is a frame-independent quantity, then it is clear that u^μ transforms like a contravariant vector under changes of inertial frame, i.e. u^μ is a 4-vector.

Exercise 10.3.1. Consider a scalar $f : \mathbb{R}^{1,3} \to \mathbb{R}$. Show that its derivative along the worldine with respect to the proper time is:

$$\dot f \doteq \frac{df(\gamma(\tau))}{d\tau} = u^\mu \partial_\mu f\big|_{\gamma^\mu}.$$

Argue that this is also a scalar quantity, i.e. that it is invariant under changes of inertial frame. ♢

Exercise 10.3.2. Consider an inertial frame K. In this frame a particle's worldline will read $\gamma^\mu(\lambda) = \big(t(\lambda), \vec x(\lambda)\big)$. Defining the particle velocity in the frame K as usual, i.e. as $\vec v \doteq d\vec x/dt$, show that

$$d\tau = \frac{dt}{\gamma(v)} \quad \text{and} \quad u^\mu = \big(\gamma(v), \gamma(v)\vec v\big).$$

where we have set $v = \sqrt{\delta_{ij}v^iv^j}$. What is the instantenous 4-velocity of a particle in its instantaneous rest-frame? The answer can be interpreted as the statement that "in its own rest frame a particle moves towards the future at speed 1".

Finally, check the following statement: we could have as well defined the 4-velocity of a particle at rest as $u^\mu = (1, \vec 0)$, and deduced its form above by asking that u^μ transforms as a contravariant vector under Lorentz transformations. [*Hint*: perform a boost of rapidity ψ in the direction $\hat z$ to $u^\mu = (1, \vec 0)$ and finally apply a rotation to deduce the general formula above.] ♢

Given the notion of 4-velocity we introduce that of *4-momentum* as:

$$p^\mu = mu^\mu,$$

where m is the *mass*. Since u^μ is a 4-vector itself, for p^μ to be a 4-vector too, we need m to be a scalar, i.e. a frame-independent quantity:

$$p^\mu p_\mu = -m^2.$$

Remark 10.3.3 (Rest mass?)**.** Sometimes, one is lead to write (we temporarily reinstate the factors of c):

$$p^\mu = (m_v c, m_v \vec{v}) \quad \text{where} \quad m_v \doteq \gamma(v) m.$$

and thus to distinguish the "rest mass" $m = m_0$ from the "inertial mass" m_v. According to this notation, the "inertial mass" increases with velocity to the point that it explodes as $v \to c$ in a way that makes it impossible to accelerate massive bodies to or beyond the speed of light. The issue with this intuitive picture is that neither m_v nor $(1, \vec{v})$ are Lorentz-covariant objects, which makes this notation frame-dependent. We will thus refrain from using this language and notation. For us there will be only one mass, the (rest) mass m, which is the same in all frames. ◊

As clarified by the previous remark, the spatial component p^i of the 4-momentum is a relativistic version of the Newtonian linear momentum, but what is the physical meaning of its time component p^0? To answer this question it is convenient to reinstate the factors of c and do a "post-Newtonian" expansion in $v/c \ll 1$:

$$cp^0 = mc^2\gamma(v) = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = mc^2 + \frac{1}{2}mv^2 + O\left(\frac{v^4}{c^4}\right).$$

Thus, we see that in the limit of small velocities, the time-component of the 4-momentum is the Newtonian kinetic energy of the particle, $\frac{1}{2}mv^2$, offset by a "rest energy"

$$E_0 = mc^2.$$

We conclude that the 4-momentum of a particle is a 4-vector whose time and spatial components in a given frame are the (kinetic) energy of the particle and its linear momentum, respectively. For this reason p^μ is sometimes called the *energy-momentum vector*.

Importantly, in relativity, the (kinetic) energy of a particle at rest cannot be set to vanish, but is constrained by Lorentz covariance to be proportional to the mass of the particle, $E_0 = mc^2$. This fact as deep consequences, especially in quantum field theory: e.g. a particle of mass m which is at rest can decay into two moving particles of masses m_1 and m_2 without violating energy-momentum conservation, as long as $m \geq m_1 + m_2$.

10.4 Relativistic second and third law

The relativistic second law of dynamics takes the form

$$\frac{dp^\mu}{d\tau} = F^\mu$$

where F^μ is a force 4-vector. The prototypical example of such a force 4-vector is given by the Lorentz force, which we will soon see.

In Newtonian physics, the third law of dynamics—or the law of action and reaction—combined with the second law, gives the conservation of linear momentum in a system on which no *external* forces act. The same is true in relativistic physics, where 4-momentum is conserved. For example, in collision theory we have that the total incoming and outgoing 4-momenta are the same:

$$\sum_{\alpha=1}^{N} p^\mu_{\alpha,\text{in}} = \sum_{\beta=1}^{M} p^\mu_{\beta,\text{out}}.$$

This relativist law encompasses the conservation of both (relativistic) linear momentum *and* energy. Its form is dictated by the necessity of conserving linear momentum *in all inertial frames*.

What is remarkable about this law is that it holds also for inelastic collisions, e.g. even when the number of incoming and outgoing "particles" changes because two of them stuck to each other. The reason why this is so remarkable is that in Newtonian physics energy *fails* to be conserved in inelastic collisions.

In relativistic physics this is possible because in an inelastic collision the mass of the outgoing particle is different than the sum of the masses of the incoming particles. This is a time-reversed version of the decay phenomenon described at the end of the previous section.

In Newtonian physics the lack of (kinetic) energy conservation in an inelastic collision is explained to be compatible with the first law of thermodynamics (a.k.a. energy conservation) by saying that the remaining energy is dissipated into heat, i.e. it is lost at a macroscopic level but conserved if the excitations of the microscopic constituents of the outgoing particles are taken into account. This explanation is perfectly correct and combined with the relativistic conservation of energy-momentum states that *internal heat contributes to the mass of an macroscopic object*, or more catchily that "heat weighs".

10.5 Action principle for the relativistic particle

We want now to find an action principle for the relativistic particle. Consider the motion of a free particle: since its equations of motion, $\dot{p}^\mu = 0$, are Lorentz covariant, we expect its action principle to be Lorentz invariant (otherwise Lorentz-breaking quantities would appear in the equations of motion). To be able to write the action elegantly and in a manifestly Lorentz-invariant manner, we better describe its

trajectory in terms of its worldline $\gamma^\mu(\lambda)$, instead of its $\vec{x}(t)$ (see the discussion at the beginning of this chapter). However, if want to write the action in terms of the particle's worldline $\gamma^\mu(\lambda)$, we need to make sure that the action is invariant under the change of parametrization of the particle worldline, since the choice of parametrization must be physically immaterial. As we discussed in the section about proper time, the most natural function(al) of the worldline that is both Lorentz and reparametrization invariant is its "Minkowskian arc-length". Thus, we guess that the particle's action should be

$$S[\gamma(\lambda)] = -m \int_{\lambda_0}^{\lambda_1} d\lambda \, \sqrt{-\eta_{\mu\nu} \frac{d\gamma^\mu}{d\lambda} \frac{d\gamma^\nu}{d\lambda}}.$$

Before computing the equations of motion and checking that this is indeed a correct guess, we notice that this action is nothing else than the total proper time along the worldline, i.e. symbolically:

$$S[\gamma] = -m \int_\gamma d\tau.$$

In other words, the action principle states that the physical trajectory of a (free) particle is given by the worldline that maximizes the total elapsed proper time between fixed initial and final position $x_0^\mu = \gamma^\mu(\lambda_0)$ and $x_1^\mu = \gamma^\mu(\lambda_1)$. At time this is principle has been dubbed the *law of cosmic laziness*.

Remark 10.5.1. In Euclidean geometry, where a worldline is replaced by a curve on (possibly curved geometry, such as the surface of a pear), the analogue of the worldline of maximal proper time is the curve of minimal length, or geodesics. In general relativity, the motion of a free (test) particle over a curved spacetime is described by a geodesic equation. ♢

In this section we denote with a prime derivatives with respect to λ:[4]

$$\gamma'^\mu \doteq \frac{d\gamma^\mu}{d\lambda}$$

Then, the action and its Lagrangian read:

$$S[\gamma] = -m \int_{\lambda_0}^{\lambda_1} d\lambda \; L(\gamma(\lambda), \gamma'(\lambda), \lambda) \quad \text{where} \quad L(\gamma(\lambda), \gamma'(\lambda), \lambda) = \sqrt{-\gamma'^\mu \gamma'_\mu}.$$

The Euler–Lagrange equations of motion then are

$$\frac{d}{d\lambda} \frac{\partial L}{\partial \gamma'^\mu} - \frac{\partial L}{\partial \gamma^\mu} = 0$$

but since all "position coordinates" γ^μ are cyclical, we deduce that the corresponding conjugate momentum π_μ is conserved:

$$\frac{\partial L}{\partial \gamma^\mu} = 0 \implies \frac{d\pi_\mu}{d\lambda} = 0 \quad \text{where} \quad \pi_\mu \doteq \frac{\partial L}{\partial \gamma'^\mu}.$$

[4]We always reserve the dot for derivatives with respect to proper time.

But what is π_μ? A short computation (do it!) gives us:

$$\pi_\mu = \frac{m\gamma'_\mu}{\sqrt{-\gamma'^\nu\gamma'_\nu}} = mu_\mu.$$

Therefore,

$$\pi_\mu \equiv p_\mu,$$

and the Euler–Lagrange equations of motion tell us that the 4-momentum p_μ is conserved along the trajectory (since $\eta_{\mu\nu}$ is a constant, so is p^μ).

Exercise 10.5.2 (The Lorentz force). Consider the following action, where the particle is now coupled, via a coupling constant $q \in \mathbb{R}$, to a "background" (i.e. non-dynamical) 4-vector field $A_\mu(x)$ living over spacetime:

$$S[\gamma(\lambda)] = -\int_{\lambda_0}^{\lambda_1} d\lambda \ \left(m\sqrt{-\gamma'^\mu\gamma'_\mu} + qA_\mu(\gamma(\lambda))\gamma'^\mu\right)$$

Show that this action is Lorentz and reparametrization invariant, and argue that it can be written as:

$$S[\gamma] = -\int_\gamma d\tau \ \ (m + qA_\mu\dot{\gamma}^\mu) = -m\int_\gamma d\tau - q\int_\gamma A_\mu d\gamma^\mu$$

Next, compute the corresponding Euler–Lagrange equations of motion and show that they read:

$$\dot{p}_\mu = -qF_{\mu\nu}(\gamma)u^\nu \quad \text{where} \quad F_{\mu\nu}(x) \doteq \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x).$$

The quantity $F^\mu \doteq qF^{\mu\nu}u_\nu$ is called the *Lorentz force*. Show that the Lorentz force does not affect the mass of the particle, i.e.[5]

$$\dot{m} = 0.$$

Finally, note that the equations of motion are invariant under the "gauge" transformation $A_\mu(x) \mapsto A_\mu(x) + \partial_\mu\xi(x)$ for any scalar function $\xi(x)$. How does the gauge transformation of A_μ affect the action? Could have we expected that the equations of motion would have been invariant under gauge transformations of A_μ? $\Diamond$

[5] *Hint*: recall that $m^2 = -p_\mu p^\mu$.

Sixth Tutorial

Chapter 11

Electrodynamics and relativity

11.1 Vector calculus: a summary

Recall that if x^i are Cartesian coordinates, then

$$(\vec{\nabla} f)_i \equiv \partial_i f, \quad (\vec{\nabla} \cdot \vec{V})^i \equiv \epsilon^{ij}{}_k \partial_j V^k \quad \text{and} \quad \vec{\nabla} \cdot \vec{V} \equiv \partial_i V^i.$$

It readily follows that

$$\vec{\nabla} \times \vec{\nabla} f \equiv 0 \equiv \vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{V}) \quad \text{and} \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \Delta^2 \vec{V},$$

where $\Delta \equiv \nabla^2 \equiv \partial_i \partial^i \equiv \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator. In fact, in $\mathbb{R}^3$, a "converse" of the first two equations is also true:

- if $\vec{\nabla} \cdot \vec{V} = 0$, then there exists a f such that $\vec{V} = \vec{\nabla} f$, unique up to a constant $f(x) \mapsto f(x) + c$;
- if $\vec{\nabla} \times \vec{V} = 0$, then there exists a $\vec{A}$ such that $\vec{V} = \vec{\nabla} \times \vec{A}$, unique up to a gradient $\vec{A} \mapsto \vec{A} + \vec{\nabla}\xi$.

Finally, let R be a region of $\mathbb{R}^3$ and denote $\vec{s}$ the outgoing unit normal to its boundary ∂R. Then, the divergence theorem states that

$$\int_R d\mathrm{vol}_R \; \vec{\nabla} \cdot \vec{V} = \int_{\partial R} d\mathrm{vol}_{\partial R} \; \vec{s} \cdot \vec{V}.$$

where $d\mathrm{vol}_R$ and $d\mathrm{vol}_{\partial R}$ are the volume and area elements of R and ∂R; in particular, in Cartesian coordinates, $d\mathrm{vol}_R = dx^1 dx^2 dx^3$. The quantity on the right hand side of the last equation is called the *flux* of $\vec{V}$ through ∂R, $\Phi_{\partial R}(\vec{V})$.

11.2 The Maxwell equations and charge conservation

The Maxwell equations in vectorial differential notation, when written in Gaussian units (we temporarily reinstates the factors of c), read:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \text{(Gauss)}$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \text{(Faraday)}$$
$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\left(4\pi\vec{j} + \frac{\partial \vec{E}}{\partial t}\right) \qquad \text{(Ampère–Maxwell)}$$

where $\vec{E}$ is the electric field, $\vec{B}$ the magnetic field, ρ the electric charge density, and $\vec{j}$ the electric current density.

The Maxwell equations are complemented and completed by the law that describes the motion of a particle of charge q in the presence of magnetic and electric fields, i.e. by the formula for the Lorentz force

$$\vec{F}_L = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad \text{(Lorentz)}.$$

Taking the divergence of the Ampère–Maxwell law, and plugging in the Gauss law, we find the crucial *continuity equation* for the electric charge:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0.$$

The continuity equation is equivalent to the law of charge conservation. Consider a region $R \subset \mathbb{R}^3$ with boundary ∂R. The (total, or net) electric charge contained in R at time t is

$$Q_R(t) = \int_R d\mathrm{vol}_R \; \rho(t, \vec{x}).$$

Using the continuity equation and the divergence theorem, we obtain:

$$\frac{dQ_R}{dt} = \int_R d\mathrm{vol}_R \; \frac{\partial \rho}{\partial t} = -\int_R d\mathrm{vol}_R \; \vec{\nabla} \cdot \vec{j} = -\int_{\partial R} d\mathrm{vol}_{\partial R} \; \vec{s} \cdot \vec{j} \equiv -\Phi_{\partial R}(\vec{j}).$$

This states that the rate of change of the electric charge enclosed in R is equal to the instantaneous flux of the electric current through ∂R. Finally, integrating in t, we obtain:

$$\Delta Q_R \equiv Q_R(t_1) - Q_R(t_0) = -\int_{t_0}^{t_1} dt \; \Phi_{\partial R}(\vec{j}).$$

The continuity equation is nothing else than this conservation, or "balance", equation over an infinitesimal volume.

We conclude by deriving the wave equations for $(\vec{E}, \vec{B})$. Taking the curl of the Faraday law and replacing the Gauss and Ampère–Maxwell law, we find:

$$0 = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right) = 4\pi \vec{\nabla} \rho - \Delta \vec{E} + \frac{1}{c^2} \frac{\partial}{\partial t} \left(4\pi \vec{j} + \frac{\partial \vec{E}}{\partial t} \right)$$

that is:

$$-\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \Delta \vec{E} = 4\pi \vec{\nabla} \rho + 4\pi \frac{\partial \vec{j}}{\partial t}.$$

Similarly, one obtains:

$$-\frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} + \Delta \vec{B} = \frac{4\pi}{c} \vec{\nabla} \times \vec{j}.$$

In the absence of source ($\rho = 0 = \vec{j}$), i.e. in vacuum, these are equations for waves propagating at the speed c.

11.3 Relativistic notation

We now want to rewrite the Maxwell equations in a relativistic form, i.e. in terms of spacetime tensors. First we introduce the 4-dimensional differential operator:

$$\nabla_\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right)$$

which by definition is equal to ∂_μ when one chooses Cartesian coordinates. Its "square" is the *wave operator*, or dAlambertian,

$$\Box \equiv \nabla_\mu \nabla^\mu \equiv -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \Delta.$$

Next, we introduce the 4-current:

$$j^\mu = (c\rho, \vec{j}),$$

whence the continuity equation reads:

$$\nabla_\mu j^\mu = 0.$$

Consider now a spacetime region $M = R \times [t_0, t_1] \subset \mathbb{R}^{1,3}$, with R as above (Figure 11.1), and denote by n^μ the outgoing normal at its boundary ∂M. The boundary ∂M has three distinct component, i.e. $R \times \{t_0\}$, $R \times \{t_1\}$ and $\partial R \times [t_0, t_1]$. At each of these components the normal reads:

$$n^\mu|_{R \times \{t_0\}} = -\delta^\mu_0, \quad n^\mu|_{R \times \{t_1\}} = \delta^\mu_0, \quad n^\mu|_{\partial R \times [t_0, t_1]} = \delta^\mu_i s^i.$$

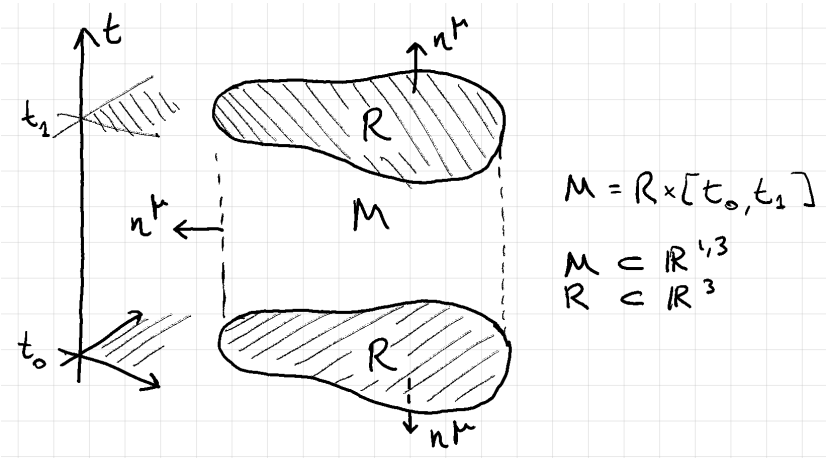

Figure 11.1: The bounded spacetime region $M = R \times [t_0, t_1]$.

It turns out that the divergence theorem applies in any dimensions and signature. This is because it is nothing else than a multidimensional application of the fundamental theorem of calculus. Hence, we find:

$$0 = \int_M d\mathrm{vol}_M \ \nabla_\mu j^\mu = \int_{\partial M} d\mathrm{vol}_{\partial M} n_\mu j^\mu$$

And since ∂M is the union of three distinct components, we obtain:

$$\begin{aligned} 0 &= \int_{\partial M} d\mathrm{vol}_{\partial M} \ n_\mu j^\mu \\ &= \int_R d\mathrm{vol}_R \ j^0(t_1) - \int_R d\mathrm{vol}_R \ j^0(t_0) + c \int_{t_0}^{t_1} dt \int_{\partial R} d\mathrm{vol}_{\partial R} \ s_i j^i \\ &= c \left(Q(t_1) - Q(t_0) + \int_{t_0}^{t_1} dt \ \Phi_{\partial R}(\vec{j}) \right). \end{aligned}$$

which is the balance equation detailed at the end of the previous section.

Now, let us introduce the following rank-2 skew-symmetric tensor, called the *field strength* or *electromagnetic* tensor:

$$F_{\mu\nu} \doteq \begin{cases} 0 & \text{if } (\mu,\nu) = (0,0) \\ -E^i & \text{if } (\mu,\nu) = (0,i) \\ E^i & \text{if } (\mu,\nu) = (i,0) \\ \epsilon^{ij}{}_k B^k & \text{if } (\mu,\nu) = (i,j) \end{cases}$$

Then, with some patience, one can check that the Maxwell equations can be written as

$$\begin{aligned} &\nabla_\mu F^{\mu\nu} = -4\pi j^\nu && \text{(Gauss + Ampère–Maxwell)} \\ &\nabla_{[\mu} F_{\nu\rho]} = 0 && \text{(Faraday)} \end{aligned}$$

where the square bracket denotes complete skew-symmetrization:

$$A_{[\mu_1\cdots\mu_p]} = \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\mathrm{sign}(\sigma)A_{\mu_{\sigma(1)}\cdots\mu_{\sigma(p)}}$$

with $\mathfrak{S}_p$ is the group of permutations of p elements.[1]

Exercise 11.3.1. Show that, in view of the skew-symmetry of $F_{\mu\nu}$,

$$3\nabla_{[\mu}F_{\nu\rho]} = \text{cycl }\nabla_\mu F_{\nu\rho} \equiv \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu}. \qquad \lozenge$$

Remark 11.3.2 (Speed of light)**.** Note that it was possible to write the Maxwell equations in relativistic notation because we have assumed that the c entering the Lorentz transformations is the same as the c entering the Maxwell equations. Ultimately this identifies c with the speed of light. Note also that the signs of the Faraday and Ampère–Maxwell equations play a crucial role in this sense, as does the Maxwell's induced-current term (i.e. $\partial_t\vec{E}$) in the Moreover, note that the Ampère–Maxwell law. The fact that the Maxwell equations can be written in relativistic notation implies that they are Lorentz covariant, and therefore keep the same form in all inertial reference frames, *provided that* $F_{\mu\nu}$ transform under Lorentz transformations as a 4-tensor. The consequences of this fact are investigated in the next exercise. $\lozenge$

Exercise 11.3.3 (Change of reference frame). Consider an electrostatic field configuration with $\vec{B} = 0 = \vec{j}$ and $\partial_t\vec{E} = 0 = \partial_t\rho$. Perform a boost in the $\hat{z}$-direction of the 4-current and the electromagnetic tensor. Interpret the result, and argue that the mixing of $\vec{E}$ and $\vec{B}$ is necessary to ensure consistency of the Maxwell equations in all reference frames, especially in the light of the Ampère–Maxwell law. $\lozenge$

Remark 11.3.4 (Continuity and wave equations)**.** The continuity equation then follows from the skew-symmetry of $F^{\mu\nu}$:

$$0 \equiv \nabla_\mu\nabla_\nu F^{\mu\nu} = -4\pi\nabla_\nu j^\nu,$$

whereas the wave equations for $\vec{E}$ and $\vec{B}$ are obtained by taking the 4-divergence of the homogeneous Maxwell equation (prove it!):

$$0 = 2\nabla^\mu\nabla_{[\mu}F_{\nu\rho]} = \Box F_{\nu\rho} + 8\pi\nabla_{[\nu}j_{\rho]}. \qquad \lozenge$$

Remark 11.3.5. Written in terms of a skew-symmetric tensor of rank 2, the Maxwell equations immediately generalize to any spacetime dimension. However, only in 1+3d one has, in vacuum[2], conformal symmetry (whose study is beyond the scope of these notes), and electromagnetic duality.[3] The latter states that the vacuum Maxwell

is it?

[1] One also introduces the symmetrization

$$A_{(\mu_1\cdots\mu_p)} = \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}A_{\mu_{\sigma(1)}\cdots\mu_{\sigma(p)}}.$$

[2] I.e. in the absence of sources, $j^\mu = 0$.

equations are invariant under the exchange $\vec{E} \leftrightarrow \vec{B}$. If the number of spatial dimension is different than 3, then this cannot be the case since the number of components F_{0i} is different than the number of components F_{ij}. The relativistic expression of electromagnetic duality is the subject of the next exercise. $\Diamond$

Exercise 11.3.6. In 1+3d, introduce the *Levi–Civita symbol* $\epsilon^{\mu\nu\rho\sigma}$, as defined by

$$\epsilon^{\mu\nu\rho\sigma} = \epsilon^{[\mu\nu\rho\sigma]} \quad \text{and} \quad \epsilon^{0123} = 1.$$

Then, show that the Levi–Civita symbol is invariant under proper Lorentz transformations, $\mathrm{SO}(1,3)$. (How does it transform under the action of $\mathcal{O}(1,3)$?). In view of these properties, the Levi–Civita symbol is said to be a *(pseudo) 4-tensor*.

Next, define

$$\widetilde{F}_{\mu\nu} \doteq \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma},$$

and argue that it is the electromagnetic-dual electromagnetic tensor. Finally show that

$$\nabla_{[\mu}F_{\nu\rho]} = 0 \iff \nabla_\mu \widetilde{F}^{\mu\nu} = 0.$$

and argue that the vacuum Maxwell equations are self-dual.

What goes wrong in other spacetime dimensions? $\Diamond$

Recalling that the 4-momentum of a particle is given by $p^\mu = mu^\mu = m(\gamma(v), \gamma(v)\vec{v})$, and that the dot denotes the rate of change with respect to proper time, $d\tau = \gamma(v)^{-1}dt$, we have the equation of motion of a particle of charge q in an electromagnetic field is given, in relativistic notation, by:

$$\dot{p}^\mu = F_L^\mu \quad \text{with} \quad F_L^\mu \doteq -qF^{\mu\nu}u_\nu.$$

with F_L^μ the *Lorentz force*.

Exercise 11.3.7. Show (again) that the Lorentz force does not alter the (rest) mass m of the charge particle, $m^2 = -p^\mu p_\mu$, i.e.

$$\dot{m} = 0 \qquad \Diamond.$$

Exercise 11.3.8. Show that in a given inertial frame the relativistic Lorentz force equation given above is formally equivalent to the usual non-relativistic equation,

$$\frac{d\vec{p}}{dt} = \vec{F}_L \quad \text{with} \quad \vec{F}_L \doteq q(\vec{E} + \vec{v} \times \vec{B}),$$

provided that one corrects the notion of linear momentum according to $\vec{p} = \gamma(v)m\vec{v}$ (this is why sometimes a distinction between a velocity-dependent "inertial mass" $m_v = \gamma(v)m$ and a "rest mass" $m_0 = m$ is introduced; since these concepts are not Lorentz covariant, we refrain from using them in these notes). $\Diamond$

[3]Generalizing the duality to the presence of sources requires introducing magnetic monopoles and their currents. We will not address these questions here.

11.4 Electromagnetic potential

Let's focus on the homogenous Maxwell equations:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0.$$

From the first of these equations we know that there exists a *magnetic potential* $\vec{A}$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

Replacing this expression in the second of the equations above, we find:

$$\vec{\nabla} \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \right) = 0$$

from which we know that there exists an *electric potential* φ such that

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}.$$

One of the advantages of working with $(\varphi, \vec{A})$, instead of $(\vec{E}, \vec{B})$ is that we then have automatically solved half of the Maxwell equations.

Exercise 11.4.1. Show that the remaining Maxwell equations then read:

$$\Box \vec{A} = -\frac{4\pi}{c}\vec{j} + \vec{\nabla}\Gamma \quad \text{and} \quad \Box\varphi = -4\pi\rho - \frac{1}{c}\frac{\partial \Gamma}{\partial t}.$$

where

$$\Gamma = \vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t}. \qquad \Diamond$$

These equations can also be written as:

$$E_i = -\partial_i \varphi - \partial_0 A_i \quad \text{and} \quad B_i = \epsilon_i{}^{jk} \partial_j A_k$$

or, recalling that $E_i = F_{0i}$ and $B_i = \frac{1}{2}\epsilon_i{}^{jk} F_{jk}$, as

$$F_{0i} = \partial_i(-\varphi) - \partial_0 A_i \quad \text{and} \quad F_{ij} = \partial_i A_j - \partial_j A_i.$$

Finally, introducing the *4-potential*

$$A^\mu = (\varphi, \vec{A}),$$

we get

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu,$$

so that the non-homogenous Maxwell equations read (the homogenous ones are solved by construction):

$$-4\pi j_\nu = \nabla^\mu F_{\mu\nu} = \Box A_\nu - \nabla_\nu \nabla^\mu A_\mu.$$

Note that this reproduces the result of the last exercise, since $\Gamma = \nabla_\mu A^\mu$.

To conclude we observe that for all scalar function ξ, a *gauge transformation* of the 4-potential, i.e.

$$A_\mu \mapsto A_\mu + \nabla_\mu \xi,$$

leaves the field strength $F_{\mu\nu}$ invariant. Choosing the function ξ appropriately, i.e. as a solution of the equation[4] $\Box\xi = \nabla_\mu A^\mu$, we can always rewrite the Maxwell equations as a wave equation with sources for a 4-potential that satisfies the *Lorentz gauge condition*:

$$\begin{cases} \Box A_\mu = -4\pi j_\mu & \text{Maxwell eqs in Lorentz gauge} \\ \nabla_\mu A^\mu = 0 & \text{Lorentz gauge condition} \end{cases}$$

Notice that the two equations are compatible in view of the continuity equation $\nabla^\mu j_\mu = 0$.

[4]Details on how to solve this equation will be provided in the next section.

Chapter 12

The Poisson–Laplace and wave equations

12.1 The Poisson–Laplace equation

Electrostatics is the regime of the Maxwell equations in which $\vec{B} = 0 = \vec{j}$ and $\partial_t \equiv 0$. Then, the only equation that needs to be solved is the Gauss law:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho.$$

In the electrostatic regime, we choose without loss of generality an electromagnetic potential such that $\vec{A} = 0$ and $\partial_t \varphi = 0$. Then the Gauss law becomes a Poisson–Laplace equation

$$\Delta\varphi(x) = -4\pi\rho(x).$$

Often one is interested in solving this equation not so much in $\mathbb{R}^3$ but in a subregion $R \subset \mathbb{R}^3$ at the boundary of which (∂R) one imposes certain *boundary conditions*. Although they do not exhaust the realm of possibilities, two kinds of boundary conditions are usually considered for this equation: Dirichlet and Neumann boundary conditions. These give rise to the Dirichlet and Neumann *boundary value problems*, i.e. respetively

$$\begin{cases} \Delta\varphi(x) = -4\pi\rho(x) & \text{in } R \\ \varphi(x) = V(x) & \text{at } \partial R \end{cases} \qquad \text{(Dirichlet)}$$

and

$$\begin{cases} \Delta\varphi(x) = -4\pi\rho(x) & \text{in } R \\ \vec{\nabla}_s\varphi(x) = -f(x) & \text{at } \partial R \end{cases} \qquad \text{(Neumann)}$$

where $\vec{\nabla} \doteq \vec{s} \cdot \vec{\nabla}$ and $V(x)$ and $f(x)$ are fixed functions over ∂R in the same way as $\rho(x)$ is a fixed function over R.

The Dirichlet boundary value problems admit unique solution for all $V(x)$. The Neumann boundary value problem admits a unique solution up to an arbitrary addi-

tive constant ($\varphi(x) \mapsto \varphi(x)+c$) *provided* that $f(x)$ satisfies the integrability condition

$$\int_R d\mathrm{vol}_R\ \rho = \frac{1}{4\pi}\int_{\partial R} d\mathrm{vol}_{\partial R}\ f.$$

Remark 12.1.1 (Dirichlet condition)**.** If $V(x) = V_0$ is constant, then this means that $\vec{E}$ is normal to the boundary ∂R. Setting $V(x) = 0$ corresponds to the physical situation where ∂R is a grounded conducting boundary. ♢

Remark 12.1.2 (Neumann condition)**.** Fixing $f(x)$ means fixing the normal component of the electric field $\vec{s}\cdot\vec{E}$. To see that this integrability condition is a necessary condition for the existence of a solution to the Neumann boundary value problem, simply use the divergence theorem applied to $\Delta\varphi = \vec{\nabla}\cdot(\vec{\nabla}\varphi)$ to deduce:

$$-4\pi\int_R d\mathrm{vol}_R\ \rho = \int_R d\mathrm{vol}_R\ \Delta\varphi = \int_{\partial R} d\mathrm{vol}_{\partial R}\ \vec{\nabla}_s\varphi = \int_{\partial R} d\mathrm{vol}_{\partial R}\ f.$$

This has the following well known physical meaning: the total (or, "net") electric charge in R is equal to the electric flux across ∂R:

$$-4\pi Q_R = \Phi_{\partial R}(\vec{E}).$$

We call this the *(integrated)* Gauss law. ♢

Proof of uniqueness. We will not address the proof of existence of a solution, only that of its uniqueness. Let φ_1 and φ_2 be two solution of the Dirichlet (or Neumann) boundary value problems with the same sources and boundary conditions, and let $\Phi = \varphi_2 - \varphi_1$ be their difference. Then:

$$\text{in } R\text{: } \Delta\Phi(x) = 0 \text{ and at } \partial R\text{: } \begin{cases} \Phi(x) = 0 & \text{(Dirichlet)} \\ \vec{\nabla}_s\Phi(x) = 0 & \text{(Neumann).} \end{cases}$$

Consider the following expression, and integrate by parts to obtain

$$\int_R d\mathrm{vol}_R\ |\vec{\nabla}\Phi|^2 = -\int_R d\mathrm{vol}_R\ \Phi\Delta\Phi + \int_{\partial R} d\mathrm{vol}_{\partial R}\ \Phi\ \vec{\nabla}_s\Phi = 0.$$

However, being the integral of a positive expression, the left-most term vanishes iff $\vec{\nabla}\Phi = 0$ throughout R. But this is possible iff $\Phi(x) = c$ is constant. This concludes the uniqueness-up-to-constants statement in the Neumann case. However, in the Dirchlet case, we also know that $\Phi = 0$ at ∂R, and thus conclude that $\Phi = 0$ everywhere and thus the uniqueness of the solution of the corresponding boundary value problem. □

We will now argue that if we knew the solutions $G_D(y,x)$ and $G_N(y,x)$ to the

following Poisson–Laplace equations with point-sources,[1]

$$\begin{cases} \Delta_y G_D(y,x) = \delta^3(y-x) & \text{for } y \in R \\ G_D(y,x) = 0 & \text{for } y \in \partial R \end{cases}$$

and

$$\begin{cases} \Delta_y G_N(y,x) = \delta^3(y-x) & \text{for } y \in R \\ \vec{s} \cdot \vec{\nabla}_y G_N(y,x) = 0 & \text{for } y \in \partial R \end{cases}$$

then we would be able to write the (unique) solutions of all Dirichlet and Neumann boundary value problems over R, respectively as:

$$\varphi(x) = -4\pi \int_R d\text{vol}_R(y)\ \rho(y) G_D(y,x) + \int_{\partial R} d\text{vol}_{\partial R}(y)\ V(y)\ \vec{s} \cdot \vec{\nabla}_y G_D(y,x)$$

and

$$\varphi(x) = -4\pi \int_R d\text{vol}_R(y)\ \rho(y) G_N(y,x) + \int_{\partial R} d\text{vol}_{\partial R}(y)\ f(y) G_N(y,x).$$

The functions $G_{D/N}(x,y)$ are called the Dirichlet and Neumann *Green's function*, respectively.

Proof. Consider Green's identity, which is obtained using integration by parts twice, and is valid for all functions $\Phi(y)$ and $\Psi(y)$:

$$\int_R d\text{vol}_R(y)\ \Phi(y)\Delta\Psi(y) = \int_R d\text{vol}_R(y)\ \Psi(y)\Delta\Phi(y) + \int_{\partial R} d\text{vol}_{\partial R}(y)\ \Big(\Phi(y)\nabla_s\Psi(y) - \Psi(y)\nabla_s\Phi(y)\Big).$$

The sought results are found by replacing respectively:

$$\Phi(y) = \varphi(y) \quad \text{and} \quad \Psi(y) = G_{D/N}(y,x). \qquad \square$$

This is doubtless a simplification, because instead of having to solve the Poisson–Laplace equation one case at the time, we just need to solve it once and in the simplified case of a *point-like* source with *vanishing* Dirichlet and Neumann boundary

[1] $\delta^d(y-x)$ is Dirac's d-dimensional delta-"function". It is defined by the following property valid for any subregion $D \subset \mathbb{R}^d$:

$$\int_D d^d y\ f(x)\delta^d(y-x) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}.$$

Dirac's delta-function can be obtained as the infinitely sharp limit of a normal distribution centered at $x = y$:

$$\delta^3(y,x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{(2\pi\epsilon)^d}} \exp\left(-\frac{|\vec{y} - \vec{x}|^2}{2\epsilon}\right).$$

conditions. The question still remains on how to solve these simplified boundary value problems for the Green's functions $G_{D/N}$.

First we note that the Green's functions only depend on the shape of the region R. There are various methods to solve for them, notably the "method of images".[2] This method relies on (1) the general property of this kind of linear PDEs with sources whose most general solutions can be written as the sum of *a* special solution of the equation with sources plus the most general solution of the homogeneous equation, and (2) as far as the region R is concerned the potential $\varphi_{\text{im}}(x)$ associated to a charge distribution *outside* of the region R and computed via the chosen special solution of Green's equation $\Delta G_s(x,y) = \delta^3(x-y)$, constitutes a solution of the homogenous equation *within* R. Thus, the method of images consists in first finding *a* solution $G_s(y,x)$ of the equation with $\Delta_x G_s(x,y) = \delta^3(x-y)$, and then finding an appropraite "image" charge distribution[3] $\rho_{\text{im}}^{D/N}(x)$ supported in the complement of R, i.e. $\bar{R} = \mathbb{R}^3 \setminus R$, such that

$$G_{D/N}(y,x) = G_s(y,x) - 4\pi \int_R d\text{vol}_R(y)\; G_s(x,y)\rho_{\text{im}}^{D/N}(y)$$

is *the* Dirichlet/Neumann Green's function associated to the region R.

Exercise 12.1.3. Use symmetry consideration and the integrated Gauss law to deduce a solution $G_s(x,y)$ of the equation $\Delta_x G_s(x,y) = \delta^d(x-y)$ in d spatial dimensions and write down this solution explicitly in $d = 1, 2, 3$. [*Hint*: it might be useful to introduce the notation $\Omega^n(r)$ for the volume of the n-dimensional round sphere of radius r.] ♢

Exercise 12.1.4. Use the solution to the previous exercise and the method of images to find the Dirichlet Green's function for $R \subset \mathbb{R}^3$ the half-space $z > 0$. If you feel emboldened, do the same for R a ball of radius a centred around the origin of $\mathbb{R}^3$. [*Note:* The second problem is *much* harder than the first.] ♢

12.2 The wave equation

Consider now the wave equation with sources:[4]

$$\Box\varphi = -4\pi\rho$$

on the spacetime domain $M = \mathbb{R}^3 \times [t_0, t_1]$. Note that here we have removed any timelike boundary, by taking the entirety of space cross an finite time interval. The spatial hypersurface

$$\Sigma_t = \mathbb{R}^3 \times \{t\}$$

is called the *Cauchy hypersurface of time* t.

[2] See Section 1.10 and Chapter 2 of Jackson, Classical Electrodynamics, Wiley Ed., 3rd edition (1999).

[3] Usually this is a combination of a finite number of pointlike charges lying in $\bar{R}$.

[4] This is one of the "full" Maxwell equations when written in Lorentz gauge.

The wave equation, although superficially similar to the Poisson-Laplace equation, is very different from it. Most importantly, it admits a unique solution once boundary conditions are specified at *one* of its fixed-time boundary components, i.e. at either Σ_0 or Σ_1, but not at both at the same time because this would yield in general to an overdetermined problem. Conditions specified at Σ_0 are more common and are called *initial conditions*. The data that one needs to specify at a Cauchy surface Σ_t to determined the field everywhere else (given a source $\rho(x)$) is called the *Cauchy data*. In the case of the wave equation, the Cauchy data at Σ_0 (Σ_1) is given by the initial (final, resp.) configuration and time derivative. This leads us to the following *initial value problem*:[5]

$$\begin{cases} \Box\varphi(t,\vec{x}) = -4\pi\rho(t,\vec{x}) & \text{in } M \\ \varphi(t,\vec{x}) = \phi_0(\vec{x}) & \text{at } \Sigma_0 \\ \partial_t\varphi(t,\vec{x}) = \pi_0(\vec{x}) & \text{at } \Sigma_0 \end{cases}$$

where ρ, ϕ and π are fixed functions. (The "final value problem" takes the same form, with Σ_0 replaced by Σ_1.)

Remark 12.2.1 (Nomenclature)**.** Mathematically, the property distinguishing Δ from $\Box$ is that the former is an "elliptic" differential operator, while the latter is a "hyperbolic" one. ◊

Remark 12.2.2 (Propagating degrees of freedom)**.** Physically, one can think of the wave equation as controlling the evolution of a field configuration from Σ_0 to Σ_1, so that the latter is determined by the former. Said differently, the wave equation encodes the *propagation* of disturbances in spacetime. ◊

Exercise 12.2.3 (1+1 dimensions). Consider the homogeneous wave equation $\Box\varphi = 0$ in 1+1 dimensions. Perform the change of variable to light-cone coordinates $(t,x) \mapsto (u,v) = (t-x, t+x)$, also respectively known as retarded and advanced time, and show that its most general solution is of the form $\varphi(u,v) = \Phi_r(u) + \Phi_l(v)$, where "$r$" and "$l$" stand for right- and left-movers. Compare the field profiles at the times t_0 and t_1. ◊

One can apply the Green's function method to solve the wave equation with sources. For hyperbolic problems, Green's functions are often called *propagators*. We distinguish in particular the *retarded and advanced propagators* G_R and G_A, respectively defined by

$$\Box_y G_{R/A}(y,x) = \delta^4(y-x) \quad \text{and} \quad \begin{cases} G_R(y,x) = 0 & \text{if } y^0 > x^0, \\ G_A(y,x) = 0 & \text{if } y^0 < x^0. \end{cases}$$

[5] As opposed to a "boundary value problem".

Thanks to their property, for every $x \in M$ they give:

$$\varphi(x) = -4\pi \int_M d\mathrm{vol}_M(y)\ \rho(y) G_R(y,x) \\ + \int_{\Sigma_0} d\mathrm{vol}_\Sigma(\vec{y})\ \Big(\pi_0(\vec{y}) G_R\big((t_0,\vec{y}),x\big) - \phi_0(\vec{y}) \frac{\partial}{\partial y^0} G_R\big((t_0,\vec{y}),x\big)\Big).$$

and

$$\varphi(x) = -4\pi \int_M d\mathrm{vol}_M(y)\ \rho(y) G_A(y,x) \\ + \int_{\Sigma_1} d\mathrm{vol}_\Sigma(\vec{y})\ \Big(-\pi_1(\vec{y}) G_A\big((t_1,\vec{y}),x\big) + \phi_1(\vec{y}) \frac{\partial}{\partial y^0} G_A\big((t_1,\vec{y}),x\big)\Big).$$

Proof. Consider the 4-dimensional version of Green's theorem,

$$\int_M d\mathrm{vol}_M(y)\ \Phi(y) \Box \Psi(y) = \int_M d\mathrm{vol}_M(y)\ \Psi(y) \Box \Phi(y) \\ + \int_{\partial M} d\mathrm{vol}_{\partial M}(y)\ \Big(\Phi(y) n^\mu \nabla_\mu \Psi(y) - \Psi(y) n^\mu \nabla_\mu \Phi(y)\Big),$$

and replace $\Phi(y) = \varphi(y)$ and $\Psi(y) = G_{R/A}(y,x)$. In doing so, note that $\partial M = \Sigma_0 \cup \Sigma_1$ has two components and only one of them contributes to each one of the retarded and advanced cases. To get the sign right, note also that at the past boundary $\Sigma_0 \subset \partial M$, the time derivative is given by $\partial_t = -n^\mu \nabla_\mu$, since by definition n^μ is outgoing and ∂_t future pointing. □

Remark 12.2.4 (No-incoming-radiation)**.** Often, in physical applications, one is interested in the case where the sources are (adiabatically) turned on and off, i.e. vanish for $t < T_{\mathrm{on}}$ and $t > T_{\mathrm{off}}$, and where the field $\varphi(t,\vec{x})$ vanishes in the infinitely far past, i.e. for $t \to -\infty$. This is called the "no incoming radiation" condition. Under these assumptions, one can use the formula above involving the retarded propagator with $t_0 \to -\infty$ to deduce

$$\varphi(x) = -4\pi \int_{\mathbb{R}^4} d\mathrm{vol}_M(y)\ \rho(y) G_R(y,x).$$

Why, in this scenario, it is not appropriate (or at least extremely inconvenient) to use the advanced propagator? See the end of this chapter for more on this subject. ♢

12.3 Retarded, advanced, and Feynmann propagators

We now compute the retarded and advanced propagators. Let us start from

$$\Box_y G(y,x) = \delta^4(y-x).$$

Since this equation is invariant under translations, we shall look for solutions of the form

$$G(y,x) \equiv D(r) \quad \text{where} \quad r^\mu \doteq x^\mu - y^\mu$$

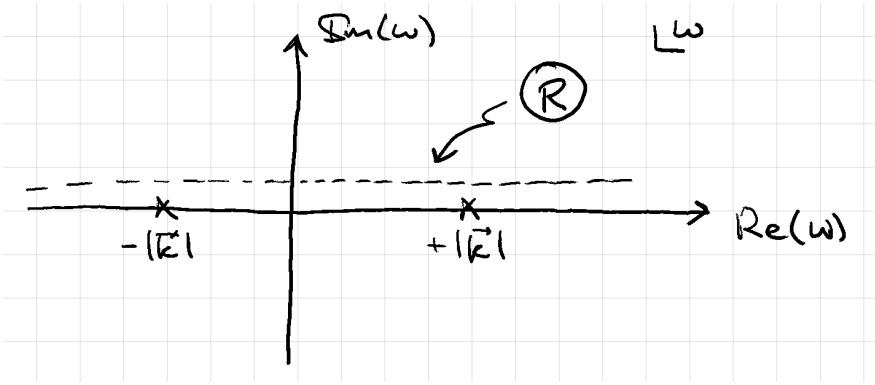

Figure 12.1: Over the poles: the retarded propagator

Remark 12.3.1. The choice of sign for r^μ is fixed as follows: when convoluted with a source $\rho(y)$ to give $\varphi(x)$ as per the formulas of the previous section, the 4-vector r^μ points from the source at y, to the point x at which we are computing its influence. $\Diamond$

Next, we Fourier transform $G(r)$:

$$D(r) = \int d^4k\ e^{\mathrm{i}k_\mu r^\mu} \hat{D}(k) \quad \text{and} \quad \hat{D}(k) = \frac{1}{2\pi^4} \int d^4k\ e^{-\mathrm{i}k_\mu r^\mu} D(r)$$

where $d^4\xi = d\xi^0 d\xi^1 d\xi^2 d\xi^3 = \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma} d\xi^\mu d\xi^\nu d\xi^\rho d\xi^\sigma$ is a Lorentz-invariant measure over $\mathbb{R}^{1,3}$ (prove it!).

Then, it follows that (show it!)

$$\Box D(r) = \delta^4(r) \iff \hat{D}(k) = -\frac{1}{(2\pi)^4} \frac{1}{k_\mu k^\mu}.$$

Writing

$$r^\mu = (t, \vec{r}) \quad \text{and} \quad k^\mu = (\omega, \vec{k})$$

we thus deduce:

$$D(r) \equiv D(t, \vec{r}) = \frac{1}{(2\pi)^4} \int d^3\vec{k} e^{\mathrm{i}\vec{k}\cdot\vec{r}} \int d\omega\ \frac{e^{-\mathrm{i}\omega t}}{\omega^2 - \vec{k}^2}.$$

Note that $\omega^2 - \vec{k}^2 = (\omega - |\vec{k}|)(\omega + |\vec{k}|)$. Thus, to perform the above integral in ω and hence compute $G(r)$, we need first to provide an appropriate prescription, or regularization, to deal with the poles at

$$\omega = \pm|\vec{k}|.$$

3.1 Over the poles: retarded propagator We start with the following prescription: in the complex ω plane, we avoid the poles by passing infinitesimally *over* them (Figure 12.1). We call the ensuing propagator $D_R(r)$.

To apply the residue theorem to the chosen counter in the complex ω plane, we need to close the integration contour. Then, we need to distinguish two cases:

- if $t < 0$, the exponential $e^{-\mathrm{i}\omega t}$ decays exponentially in the upper half plane; we thus define the contour $\gamma_R^{\uparrow}$ closing the dotted line in Figure 12.1 with a half circle at infinity in the *upper* half-plane that does not contribute to the integral. Now, since $\gamma_R^{\uparrow}$ does not encircle any poles, it vanishes:

$$\text{if } t < 0, \text{ then } \int_{\gamma_R^{\uparrow}} d\omega \ \frac{e^{-\mathrm{i}\omega t}}{\omega^2 - \vec{k}^2} = 0$$

- conversely, if $t > 0$, the exponential $e^{-\mathrm{i}\omega t}$ decays exponentially in the lower half plane; we thus define the contour $\gamma_R^{\downarrow}$ closing the dotted line in Figure 12.1 with a half circle at infinity in the *lower* half-plane that does not contribute to the integral. But now, $\gamma_R^{\downarrow}$ does indeed encircle the two poles *clockwise*, and therefore we obtain:

$$\text{if } t > 0, \text{ then } \int_{\gamma_R^{\downarrow}} d\omega \ \frac{e^{-\mathrm{i}\omega t}}{\omega^2 - \vec{k}^2} = -2\pi\mathrm{i} \ \mathrm{Res}\left(\frac{e^{-\mathrm{i}\omega t}}{\omega^2 - \vec{k}^2}\right) = -2\pi \frac{\sin\left(|\vec{k}|t\right)}{|\vec{k}|}.$$

Combining these two results, we obtain:

$$D_R(t,\vec{r}) = -\frac{\Theta(t)}{(2\pi)^3} \int d^3\vec{k} e^{\mathrm{i}\vec{k}\cdot\vec{r}} \ \frac{\sin\left(|\vec{k}|t\right)}{|\vec{k}|}, \quad \text{where} \quad \Theta = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

To perform this last integral we go to spherical coordinates, $\vec{k} = (\kappa, \theta, \psi)$, with $\psi \in [0, 2\pi)$ and $\theta \in [0, \pi]$ the azimuthal and polar angles around and from the direction $\vec{r}$ (we also denote $\rho = |\vec{r}|$):

$$\begin{aligned} D_R(t,\vec{r}) &= -\frac{\Theta(t)}{(2\pi)^2} \int_0^{\infty} d\kappa \ \kappa \sin(\kappa t) \int_0^{\pi} d\theta \ \sin\theta e^{\mathrm{i}\kappa\rho\cos\theta} \\ &= +\frac{\Theta(t)}{(2\pi)^2} \int_0^{\infty} d\kappa \ \kappa \sin(\kappa t) \left(\frac{e^{-\mathrm{i}\kappa\rho} - e^{\mathrm{i}\kappa\rho}}{\mathrm{i}\kappa\rho}\right) \\ &= -\frac{\Theta(t)}{8\pi^2\rho} \int_{-\infty}^{\infty} d\kappa \ \left(e^{\mathrm{i}\kappa t} - e^{-\mathrm{i}\kappa t}\right) e^{-\mathrm{i}\kappa\rho} \end{aligned}$$

and finally, using that that, in light of the step function $\Theta(t)$, the argument $t + \rho > 0$, we conclude:[6]

$$D_R(t,\vec{r}) = -\frac{1}{4\pi|\vec{r}|}\Theta(t)\delta(t - |\vec{r}|)$$

Hence we see that that $D_R(r) = G_R(y,x)$ is a *retarded propagator*, and that it is (distributionally) supported on the *future light cone*. Since this description is Lorentz-invariant, although the above writing is not, we expect that a manifestly Lorentz-

[6]Note: here, the step function $\Theta(t)$ is actually superfluous for it is subsumed by the delta function.

invariant expression for $D_R(r)$ exists. A "more" manifestly covariant expression is:[7]

$$D_R(r) = -\frac{1}{2\pi}\Theta(t)\delta(r_\mu r^\mu).$$

(Once again, although $\Theta(t)$ is not Lorentz invariant on its own, the combined expression above is.) Note that this expression breaks time-reversal symmetry and is only invariant under orthochronus Lorentz transformations.

3.2 Below the poles: advanced propagator I leave to you to check that the opposite regularization prescription, by which the poles are avoided by passing infinitesimally below them, gives instead the *advanced propagator*:

$$D_A(t,\vec{r}) = -\frac{1}{4\pi|\vec{r}|}\Theta(-t)\delta(t+|\vec{r}|) = -\frac{1}{2\pi}\Theta(-t)\delta(r_\mu r^\mu).$$

* * *

The contour prescription that yields the retarded propagator, which demands to pass infinitesimally *over* the poles, can be rephrased as the prescription of moving the poles infinitesimally *below* the real axis, which means performing the following replacement with ϵ a *positive* infinitesimal:

$$\omega^2 - \vec{k}^2 \overset{R}{\rightsquigarrow} (\omega - |\vec{k}| + \mathrm{i}\epsilon)(\omega + |\vec{k}| + \mathrm{i}\epsilon) = \omega^2 - \vec{k}^2 + 2\mathrm{i}\epsilon\omega + O(\epsilon^2).$$

whence we could have written:

$$D_R(r) \equiv D_R(t,\vec{r}) = \frac{1}{(2\pi)^4}\int d^3\vec{k}e^{\mathrm{i}\vec{k}\cdot\vec{r}}\int d\omega\ \frac{e^{-\mathrm{i}\omega t}}{\omega^2 - \vec{k}^2 + 2\mathrm{i}\epsilon\omega}.$$

Remark 12.3.2. In this expression, the factor of 2 next to ϵ is indeed inconsequential, since the expression is understood in the limit $\epsilon \to 0$; however the factor of ω does matter, because it affects the sign of the corrective term, see below. ♢

Fourier-transforming this expression back, we see that the retarded propagatori prescription corresponds to regularizing the wave equation by a term

$$\Box \overset{A}{\rightsquigarrow} \Box - 2\epsilon\partial_t = -\partial_t^2 - 2\epsilon\partial_t + \Delta$$

The regularized operator thus breaks time-reversal symmetry by introducing a damping term.

Can we introduce a regularization that does not break time reversal symmetry?

[7]Recall: if $f : \mathbb{R} \to \mathbb{R}$ has a unique zero that is located at $x = x_0$ and $f'(x_0) \neq 0$, then

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}.$$

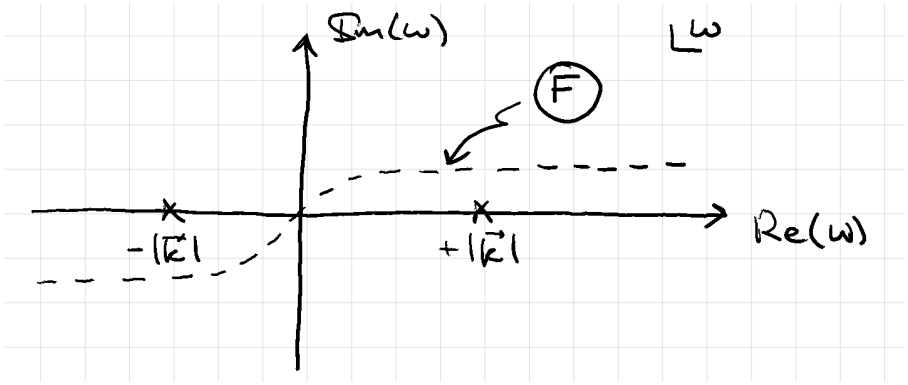

Figure 12.2: Slaloming around the poles: the Feynman propagator

3.3 Slaloming around the poles: Feynman propagator We now regularize the wave operator by adding to it an infinitesimal, *imaginary*, mass term $m^2 = -\mathrm{i}\epsilon$:

$$\Box \overset{F}{\rightsquigarrow} \Box + \mathrm{i}\epsilon \iff -k_\mu k^\mu \overset{F}{\rightsquigarrow} -k^\mu k_\mu + \mathrm{i}\epsilon,$$

whence one introduces the *Feynman propagator*:

$$D_F(r) = -\frac{1}{(2\pi)^4} \int d^4k \, \frac{e^{\mathrm{i}k_\mu r^\mu}}{k_\mu k^\mu - \mathrm{i}\epsilon}.$$

The Feynman propagator is indeed a Green's function for d'Alambertian operator $\Box$ that plays a pivotal role in Quantum Field Theory.[8] The question is: which boundary conditions are encoded in D_F? It turns out that the $-\mathrm{i}\epsilon$ prescription implies that, for $t \to +\infty$ (or $t \to -\infty$), the Feynman propagator $D_F(r) \equiv D_F(t, \vec{r})$ involves only positive frequencies (or only negative frequencies, respectively).

Exercise 12.3.3. Show that this statement is correct. To do so, first show that the Feynman prescription corresponds to the integration contour of Figure 12.2. Denote D_+ the contribution to the retarded propagator due to the pole of *positive* frequency, $\omega = |\vec{k}|$, by

$$\mathrm{i}D_+(r) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{k}}{2\omega_k} \, e^{\mathrm{i}\tilde{k}_\mu r^\mu}.$$

where $\tilde{k}^\mu \doteq (\omega_k, \vec{k}) \doteq (|\vec{k}|, \vec{k})$. Argue that $D_+(r)$ is Lorentz invariant. Next, show that the contribution to the retarded propagator due to the pole of *negative* frequency is equal to

$$D_-(r) = -D_+(-r) = D_+(r)^*,$$

[8] See the next exercise and cf. Sections 5.2 and 6.2 of Weinberg, The Quantum Theory of Fields, Vol. 1, Cambridge University Press (2005). To compare:

$$\Delta_+(x)|_{\text{there}} = \mathrm{i}D_+(r)|_{\text{here}}, \quad \Delta(x)|_{\text{there}} = \mathrm{i}D_{PJ}(r)|_{\text{here}}, \quad \text{and} \quad \Delta_F(x)|_{\text{there}} = -D_F(r)|_{\text{here}}.$$

where $\bullet^*$ denotes complex conjugation. Finally, show that:

$$\begin{aligned} D_R(r) &= \Theta(t)\,(D_+(r) - D_+(-r)) \\ D_A(r) &= D_R(-r), \\ D_F(r) &= \Theta(t)D_+(r) + \Theta(-t)D_+(-r). \end{aligned}$$

(Don't forget to come back to the question about the boundary conditions of D_F!) ♢

Remark 12.3.4. Note that the Feynman contour is homotopic (i.e. can be smoothly deformed without hitting the poles) to an integration along the imaginary axis, i.e. over imaginary frequencies. This observation is intimately related to the viability of the Wick rotation presescription. ♢

12.4 Liénard–Wiechert potential

We conclude this chapter with an applications of the above results to the Maxwell equations, that is we solve for the (retarded) electromagnetic potential $A_\mu(x)$ sourced by a moving point-particle with worldline γ^μ and charge q.

In other words, we look for the (retarded) solution of

$$\Box A^\mu(x) = -4\pi j^\mu(x) \quad \text{where} \quad j^\mu(x) = q\int d\lambda\ \frac{d\gamma^\mu}{d\lambda}\delta^4\big(x-\gamma(\lambda)\big).$$

It is convenient to rewrite this expression in terms of the particle's proper time τ and its 4-velocity $u^\mu = \dot\gamma^\mu$:

$$j^\mu(x) = q\int d\tau\ u^\mu(\tau)\delta^4\big(x-\gamma(\tau)\big).$$

Assuming a no-incoming-radiation initial condition, we find

$$\begin{aligned} A^\mu_R(x) &= -4\pi\int d^4y\ j^\mu(y)G_R(y,x) \\ &= -4\pi\int d^4y\ D_R(x-y)j^\mu(y) \\ &= 2\int d^4y\ \Theta(x^0-y^0)\delta\left((x-y)^2\right)j^\mu(y) \\ &= 2q\int d\tau\ \Theta\left(x^0-\gamma^0(\tau)\right)\delta\left((x-\gamma(\tau))^2\right)u^\mu(\tau). \end{aligned}$$

We already see from this expression that $A^\mu_R(x)$ is proportional to qu^μ computed at the point of the trajectory $y^R_x = \gamma(\tau^R_x)$ defined by the condition that x and y^R_x are separated by light-like interval with y^R_x lying in the past of x.

To perform the integral in τ we first observe that if the charged particle is massive and moves along a causal worldline (i.e. a timelike one), there is only one intersection

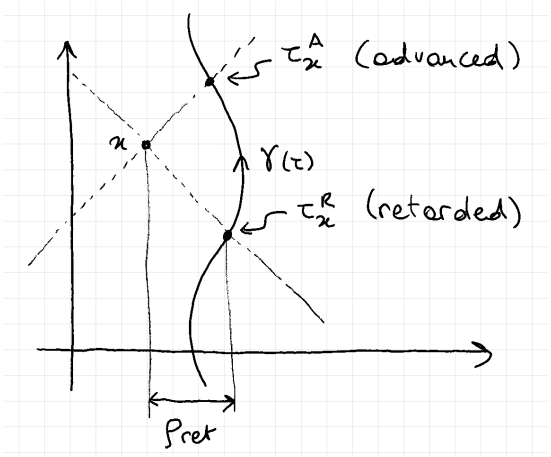

Figure 12.3: Spacetime diagram for the Liénard–Wiechert potentials.

of the past lightcone of x and the worldline of $\gamma(\tau)$, see Figure 12.3. Denote τ_x^R the corresponding proper time. Then[9]

$$\delta\left((x-\gamma(\tau))^2\right) = \frac{\delta(\tau-\tau_x^R)}{2|(x^\mu-\gamma^\mu(\tau_x^R))\dot\gamma_\mu(\tau_x^R)|}$$

whence we conclude:

$$A_R^\mu(x) = \left.\frac{qu^\mu(\tau)}{(x^\mu-\gamma^\mu(\tau))\,u_\mu(\tau)}\right|_{\tau=\tau_x^R}.$$

where we were able to drop the absolute value because both u^μ and $x^\mu-\gamma^\mu(\tau_x^R)$ are timelike, but with opposite time orientation.

Exercise 12.4.1 (Liénard–Wiechert potentials). Show that the expression for A_R^μ can be made even more explicit, i.e. it can be put into the form

$$\Phi_R(t,\vec x) = \left.\frac{q}{(1-\vec v\cdot\vec n)\,\rho}\right|_{\rm ret} \qquad \vec A_R(t,\vec x) = \left.\frac{q\vec v}{(1-\vec v\cdot\vec n)\,\rho}\right|_{\rm ret}$$

where $\vec r = \rho\vec n$, with $\vec n$ a unit vector, and $\vec v = d\vec\gamma/dt$ is the (Newtonian) velocity of the particle, while $\bullet|_{\rm ret}$ stands for the evaluation of all the time dependent quantity related to the particle at the retarded time τ_x^R as in

$$\vec r|_{\rm ret} \doteq \vec x - \vec\gamma(\tau_x^R).$$

These expressions are go under the name of (retarded) *Liénard–Wiechert potential*.[10] ♢

[9] See Footnote 7.

[10] See Section 14.1 of Jackson, Classical Electrodynamics, Wiley Ed., 3rd edition (1999).

The most general electromagnetic potential in the presence of the source j^μ is given by a superposition of a retarded Liénard–Wiechert potential and a solution A^μ_{in} of the homogenous wave equation:

$$A^\mu(x) = A^\mu_{\text{in}}(x) - 4\pi \int d^4y \; j^\mu(y) G_R(y,x)$$

We labelled the homogeneous solution by $\bullet|_{\text{in}}$ as "incoming" because in this expressioned we used the retarded potential (see Remark 12.2.4).

But the *same* electromagnetic potential A^μ can also be written in terms of an advanced Liénard–Wiechert potential and the corresponding *outgoing* homogenous solution:

$$A^\mu(x) = A^\mu_{\text{out}}(x) - 4\pi \int d^4y \; j^\mu(y) G_A(y,x)$$

Then, the net contribution to $A_\mu(t \to \infty, \vec{x})$ due to the radiation emitted by j^μ is encoded in the late time limit of *radiation field*

$$A^\mu_{\text{rad}} \doteq A^\mu_{\text{out}}(x) - A^\mu_{\text{in}}(x) = -4\pi \int d^4y \; j^\mu(y) D_{PJ}(x-y)$$

where we introduced the *Pauli–Jordan function*

$$D_{PJ}(r) \doteq D_A(r) - D_R(r).$$

The Pauli–Jordan function is *not* a Green's function, for it is a solution of the *homogenous* wave equation (why?).

Exercise 12.4.2. Show that the Pauli–Jordan function is the sum of the contributions of the two poles independently of the sign of t, i.e.[11]

$$D_{PJ} = D_+(r) - D_+(-r). \qquad \lozenge$$

[11] Cf. Footnote 8.

Seventh Tutorial

Chapter 13

Relativistic field theory

13.1 Preliminaries

The goal of this chapter is to provide a Lagrangian formulation of electrodynamics and, to a lesser extent, a Hamiltonian formulation too (a full fledged Hamiltonian analysis is beyond the scope of these notes). However, as a warm up, it is best to start with the analysis of a simpler field theory: that of the scalar field.

Let $M \subset \mathbb{R}^{1,3}$ be a region of spacetime which, for the sake of simplicity, we take of the form $M = \mathbb{R}^3 \times [t_0, t_1]$. Denote

$$\Sigma_t \doteq \mathbb{R}^3 \times \{t\}$$

the corresponding Cauchy surface at time t, with Σ_0 and Σ_1 the initial and final Cauchy surfaces comprising the boundary of M, $\partial M = \Sigma_0 \cup \Sigma_1$. Denote n^μ the timelike unit normal to ∂M. Note that

$$n^\mu|_{\Sigma_0} = -\delta^\mu_0 \qquad \text{and} \qquad n^\mu|_{\Sigma_1} = \delta^\mu_0.$$

In some limited cases we will be interested in a bounded spacetime region. We will then still call it M and for simplicity we will take it of the form $M = R \times [t_0, t_1]$ with $R \subset \mathbb{R}^3$, and $R_t = R \times \{t\} \subset \Sigma_t$. Then, $\partial M = R_0 \cup R_1 \times B$, where $B = \partial R \times [t_0, t_1]$. And $n^\mu|_B = \delta^\mu_i s^i$ is an outgoing spacelike unit vector.

13.2 Lagrangian scalar field theory

There are two main ways of thinking of field theory. One that is fully spacetime covariant and one better adapted to a Hamiltonian description with respect to a time variable t. Let us start from the fully covariant picture of a scalar field.

A real-valued scalar field is a function

$$\varphi : M \to \mathbb{R}$$

The term "scalar" means that φ is a function over spacetime that is invariant under Lorentz transformations: i.e. in changing from the inertial frame K to K' related by the Lorentz transformation Λ, the field φ transforms as:

$$\varphi(x) \mapsto \varphi'(x) = \varphi(\Lambda^{-1}x).$$

In the fully covariant picture, having a generalization of particle mechanics in mind, one thinks of $\mathbb{R}$ on the righthand side of the above equation—i.e. the space in which the scalar field is valued—as the "configuration space" Q of the scalar field, of M as a multidimensional generalization of the time interval, and of φ as a spacetime history of the scalar field.

Thus, we think of the scalar field as a "covariant history" from spacetime to configuration space $Q = \mathbb{R}$ and thus rewrite the equation above as:

$$\varphi : M \to Q, \quad x \mapsto q = \varphi(x), \quad \text{where} \quad Q = \mathbb{R}.$$

Remark 13.2.1. This viewpoint will allow us to readily generalize the scalar field to other types of fields by modifying our choice of configuration space Q. E.g. Q could be $\mathbb{C}$ in the case of a complex scalar field; or $\mathbb{R}^{1,3}$ if we are dealing with a vector field A_μ; or $\mathbb{C}^2$ in the case of spinorial fields ψ^A, etc. In general, Lorentz covariance requires more complex transformation properties of the fields, that involve transformations of the target space too, e.g.[1]

$$A_\mu(x) \mapsto A'_{\mu'}(x) = \Lambda^{\mu}{}_{\mu'} A_\mu(\Lambda^{-1}x).$$

If Q happens to be a curved manifold (such as a Lie group, $Q = \mathrm{SU}(2)$), one calls the resulting field theory a "non-linear σ-model". $\Diamond$

Note that with respect to the mechanical case the the "history" is not just a function of time, but of the 4 spacetime coordinates. Accordingly, next to the configuration variable q we have *four* derivatives v_μ describing "evolution" in both the time and space directions. These quantities, sometimes called the *first jets* generalize in a covariant manner the notion of velocity.

Once we are given a history φ, we have that at a spacetime point x, the field takes the value $q = \varphi(x)$ in configuration space, and its first jets take the vlaues $v_\mu = \partial_\mu \varphi(x)$.

Here we we summarize the parallel between particle mechanics and the scalar field. Note the very different roles played by space(time) in the particle and field cases:

COVARIANT PIC.	N **particles**	**scalar field**
independent variable	time, $t \in \mathbb{R}$	spacetime, $x^\mu \in M \subset \mathbb{R}^{1,3}$
configuration space	$Q = (\mathbb{R}^3)^N$	$Q = \mathbb{R}$
history	$\{\vec{\gamma}_\alpha(t)\}_{\alpha=1}^N$	$\varphi(x)$
1st derivative	velocity $\vec{v}_\alpha = \dot{\gamma}_\alpha$	1st jet $v_\mu = \partial_\mu \varphi$

[1] In the spinorial case $\mathrm{SO}^+(1,3)$ acting on A_μ is replaced by $\mathrm{SL}(2,\mathbb{C})$ acting on ψ^A, cf. Exercise 8.4.8, in the same way that one has to use SU(2) rather than SO(3) when deal with the rotation of particles with half-integer spin.

Next, we introduce the Lagrangian density, which is a scalar function of the configuration variable q, and the first jets v_μ, and of the spacetime point x:

$$\mathcal{L} : (q, v_\mu, x) \mapsto \mathcal{L}(q, v_\mu, x).$$

Typically, $\mathcal{L}$ is independent of x and—to ensure its scalar properties—of the form

$$\mathcal{L}(q, v_\mu, x) = -\frac{1}{2} v^\mu v_\mu - V(q).$$

The function $V(q)$ is called the potential, and we will come back to its shape and meaning after writing down the equations of motion.

Finally, we define the action functional by evaluating the Lagrangian over a history, and evaluating it over spacetime:

$$S[\varphi(x)] = \int_M d\mathrm{vol}_M \; \mathcal{L}\big(\varphi(x), \partial_\mu \varphi(x), x\big).$$

The equations of motion are computed from Hamilton's action principle, which states that a physical history $\bar{\varphi}$ is an extremum of the action functionals, at fixed boundary conditions over the configuration varibale:

$$\delta S[\bar{\varphi}(x)] = 0 \quad \text{with} \quad \delta\varphi|_{\partial M} = 0.$$

Here, $\partial M = \Sigma_0 \cup \Sigma_1$, and therefore we are taking variations at fixed values of the functions $\phi_0(\vec{x}) = \varphi(t_0, \vec{x})$ and $\phi_1(\vec{x}) = \varphi(t_1, \vec{x})$.

Following this action principle, we compute now the Euler–Lagrange equations of motion. We start by computing the most general variation of the action:

$$\begin{aligned}
S[\varphi(x) + \epsilon\delta\varphi(x)] &= \int_M d\mathrm{vol}_M \; \mathcal{L}\big(\varphi(x) + \epsilon\delta\varphi(x), \partial_\mu\varphi(x) + \epsilon\partial_\mu\delta\varphi(x), x\big) \\
&= S[\varphi(x)] + \epsilon \int_M d\mathrm{vol}_M \; \left(\frac{\partial \mathcal{L}}{\partial q}\delta\varphi(x) + \frac{\partial \mathcal{L}}{\partial v_\mu}\partial_\mu\delta\varphi(x)\right)\Big|_{(q,v_\mu,x)=(\varphi(x),\partial_\mu\varphi(x),x)} + O(\epsilon^2) \\
&= S[\varphi(x)] + \epsilon \int_M d\mathrm{vol}_M \; \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial v_\mu}\right)\Big|_{(q,v_\mu,x)=(\varphi(x),\partial_\mu\varphi(x),x)} \delta\varphi(x) \\
&\qquad + \epsilon \int_{\partial M} d\mathrm{vol}_{\partial M} \; n_\mu \left.\frac{\partial \mathcal{L}}{\partial v_\mu}\right|_{(q,v_\mu,x)=(\varphi(x),\partial_\mu\varphi(x),x)} \delta\varphi(x) + O(\epsilon^2)
\end{aligned}$$

or, more succinctly:

$$\delta S[\varphi(x)] = \int_M d\mathrm{vol}_M \; \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \int_{\partial M} d\mathrm{vol}_{\partial M} \; n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi.$$

Then, Hamilton's action principle tells us that the variation is taken at fixed boundary values, and so the second term drops, whereas from the arbitrariness of $\delta\varphi(x)$ in bulk, we deduce the following equations of motion:

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} - \left.\frac{\partial \mathcal{L}}{\partial \varphi}\right|_{\varphi=\bar{\varphi}} = 0.$$

Plugging in the explicit form of the Lagrangian, $\mathcal{L} = -\frac{1}{2}\eta^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi - V(\varphi)$, we then obtain:

$$-\partial_\mu(\partial^\mu\varphi(x)) + V'(\varphi(x)) = 0$$

where $V'(\varphi(x)) \doteq \frac{d}{dq}V|_{q=\varphi(x)}$. Rearranging:

$$\Box\varphi - V'(\varphi) = 0.$$

Note that if we were to find a *constant* solution of the equations of motion with $\varphi(x) = \varphi_0$, this would have to sit at an extremum of $V(q)$:

$$V'(\varphi_0) = 0.$$

A field $\varphi(x) = \varphi_0$ with this property is called a *(false/true) vacuum* if φ_0 is a (local/absolute) mimum of V, i.e. $V''(\varphi_0) > 0$. The minimum condition is, as usual, a stability requirement under small perturbations. Quantum mechanically, false vacua can spontaneously decay into true vacua via tunnelling.[2]

Taylor-expanding V around a vacuum solution, $\widetilde{\varphi} \doteq \varphi - \varphi_0$, we get:

$$V(\widetilde{\varphi}) = V_0 + \frac{1}{2}m^2\widetilde{\varphi}^2 + \frac{1}{3}\lambda\widetilde{\varphi}^3 + O(\widetilde{\varphi}^4).$$

and

$$\Box\widetilde{\varphi} - m^2\widetilde{\varphi} = \lambda\widetilde{\varphi}^2 + O(\widetilde{\varphi}^4).$$

The quadratic term (with this sign!) has the physical interpretation of a *mass* term for the (perturbative) excitations around the given vacuum. If V stops at the quadratic term the theory is said to be Gaussian, or *free*. The non-linear terms in the above equation are called "(self-)interactions" and are, most often, treated perturbatively. The equation of motion of the free scalar field is called the *Klein–Gordon equation*.

Remark 13.2.2. In field theory, the value of the (absolute) minimum of the potential has no physical meaning. However, this is not the case in general relativity where all energy "weighs" (principle of equivalence). In this theory V_0 contributes, at least *prima facie*, to the "vacuum energy", effectively shifting the value of the cosmological constant. ♢

Remark 13.2.3. In field theory, as in particle mechanics (cf. Section 1.3), the Lagrangian density of a physical system is not unique. In particular, it can be freely redefined by the addition of a total divergence of a function of the fields without affecting the equations of motion, nor the action principle:

$$\mathcal{L} \mapsto \mathcal{L} + \frac{d}{dx^\mu}\ell^\mu(\varphi, x) \quad \text{where} \quad \frac{d}{dx^\mu}\ell^\mu(\varphi, x) \doteq \left(\frac{\partial}{\partial x^\mu} + \partial_\mu\varphi\frac{\partial}{\partial\varphi}\right)\ell^\mu. \qquad ♢$$

[2]See Ruth Gregory's Gravitational Physics course, and the famous article by Coleman and De Luccia, Gravitational effects on and of vacuum decay, Phys. Rev. D 21, 3305 (1980).

13.3 Hamiltonian scalar field theory

To formulate the scalar field theory in a Hamiltonian language, we first need to revisit the Lagrangian formulation we gave in the previous section. The issue is that the Hamiltonian formulation of the theory is intrinsically about *time evolution* of the scalar field, whereas the formulation we gave above is "fully covariant", i.e. it doesn't truly distinguish a privileged time direction. Indeed, recall that in the case of particle mechanics, the Hamiltonian function $H(q,p,t)$ is precisely the generator of time evolution. This notion is not even relativistically invariant, meaning that each inertial frame K will have a different notion of simultaneity (see the chapter on special relativity) and thus of "time evolution". Therefore, in order to pass to the Hamiltonian picture, we first need to reformulate our Lagrangian field theory of the scalar field theory in a given inertial frame K as a theory of a "spatial configuration of the scalar field that evolves in time". For this reason, the new non-manifestly-covariant perspective is called the *3+1 formalism*.

At the beginning of this chapter, we have conveniently assumed that M is of the form $[t_0,t_1]\times\mathbb{R}^3$, and that is therefore foliated by Cauchy surfaces $\Sigma_t=\{t\}\times\mathbb{R}^3$. In other words our choice of M already breaks Lorentz invariance and is adapted to a particular inertial frame K, in which $M\ni x^\mu=(t,\vec{x})\in\mathbb{R}\times\mathbb{R}^3$.

In view of this switch of perspective it is convenient to rethink what we call "configuration" space. Since we want to think of a field $\varphi(x)=\varphi(t,\vec{x})$ as something that evolves in time only, we introduce

$$\mathcal{Q}=C^\infty(\mathbb{R}^3,Q)$$

so that $\varphi:[t_0,t_1]\times\mathbb{R}^3\to\mathbb{Q}$ can be thought alternatively as

$$\Phi:[t_0,t_1]\to\mathcal{Q},\quad t\mapsto\Phi(t)\quad\text{such that}\quad\Phi(t)(\vec{x})=\varphi(t,\vec{x}).$$

In practice there is *no* real need to introduce a new symbol Φ, we do it so here to emphasize this change in perspective.

Next, we need to re-interpret the (same) action functional in terms of the *time* integral of a Lagrangian function*al* L:

$$S[\varphi(t,\vec{x})]=S[\Phi(t)]=\int_{t_0}^{t_1}dt\;L(\Phi(t),\dot{\Phi}(t),t)$$

where

$$\begin{aligned}L(\Phi(t),\dot{\Phi}(t),t)&=\int_{\Sigma_t}d^3\vec{x}\;\frac{1}{2}\dot{\Phi}(t)\dot{\Phi}(t)-|\vec{\nabla}\Phi(t)|^2-V(\Phi(t))\\&\equiv\int_{\Sigma_t}d^3\vec{x}\;\frac{1}{2}\dot{\varphi}(t,\vec{x})\dot{\varphi}(t,\vec{x})-|\vec{\nabla}\varphi(t,\vec{x})|^2-V(\varphi(t,\vec{x})).\end{aligned}$$

Thus we see that 3+1 formalism is tailored to follow more closely the standard picture of particle mechanics:

3+1 FORMALISM	N **particles**	**scalar field**
independent variable	time, $t \in \mathbb{R}$	time, $t \in \mathbb{R}$
configuration space	$Q = (\mathbb{R}^3)^N$	$\mathcal{Q} = C^\infty(\mathbb{R}^3, Q)$
history	$\{\vec{\gamma}_\alpha(t)\}_{\alpha=1}^N$	$\Phi(t)(\vec{x})$
1st derivative	velocity $\vec{v}_\alpha = \dot{\gamma}_\alpha$	velocity $\dot{\Phi} = \partial_t \varphi$

Remark 13.3.1. Maybe the most important difference between the two columns is that the configuration space in the 3+1 field theory, $\mathcal{Q}$, is *infinite dimensional*, rather than finite dimensional. This reflects the idea that a field can be thought as one oscillator *per point* $\vec{x}$ of space. In this sense, there is a parallel between $\alpha \in \{1, \dots, N\}$ and $\vec{x} \in \mathbb{R}^3$. ♢

We can now proceed to perform the Legendre transform. First we introduce the momentum $\Pi(t)(\vec{x}) \equiv \pi(t, \vec{x})$:

$$\Pi(t) \doteq \frac{\delta L}{\delta \dot{\Phi}(t)} \overset{(*)}{=} \dot{\Phi}(t),$$

or, equivalently

$$\pi(t, \vec{x}) \doteq \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi(t, \vec{x}))} \overset{(*)}{=} \partial_t \varphi(t, \vec{x}),$$

where in both cases the step $(*)$ uses the specific form of the Lagrangian functional $L = \int_{t_0}^{t_1} dt\ \mathcal{L}$.

Remark 13.3.2. You can see the equation for π as a definition of what we mean by the previous equation for Π. As observed in the previous remark, the 3+1 configuration space $\mathcal{Q}$ is infinite dimensional and $\Phi(t)$ is a function of $\vec{x}$ itself. Conversely the covariant configuration space Q is just the real line, and $q = \varphi(x)$ is a number. This is why in the definition of Π we have to use a functional derivative, while in the definition of π we use a partial derivative. ♢

With this, the definition of the *Hamiltonian functional* of the scalar field as the Legendre transform of its Lagrangian functional reads:

$$H\big(\Phi(t), \Pi(t), t\big) \doteq \int_{\Sigma_t} d^3\vec{x}\ \Pi(t)\dot{\Phi}(t) - L(\Phi(t), \dot{\Phi}(t), t) \quad \text{with} \quad \Pi(t) \doteq \frac{\delta L}{\delta \dot{\Phi}(t)}$$

which can be equivalently, and most conveniently, expressed in terms of the *Hamiltonian density* $\mathcal{H}$,

$$H\big(\Phi(t), \Pi(t), t\big) = \int d^3\vec{x}\ \mathcal{H}\big(\varphi(t, \vec{x}), \pi(t, \vec{x}), (t, \vec{x})\big),$$

where

$$\mathcal{H}(\varphi, \pi, x) = \pi \partial_t \varphi - \mathcal{L}(\varphi, \partial_\mu \varphi, x) \quad \text{with} \quad \pi(t, \vec{x}) \doteq \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi(t, \vec{x}))}.$$

Plugging in the specific form of our Lagrangian density, we finally obtain:[3]

$$\mathcal{H}(\varphi, \pi, x) = \frac{1}{2}\pi^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + V(\varphi).$$

Thus, physically, the Hamiltonian $\mathcal{H}$ has the interpretation of an energy density of the scalar field, with $H = \int_{\Sigma_t} d^3\vec{x}\ \mathcal{H}$ the total energy at a given instant.

Using the general properties of the Legendre transform, we can rewrite the Euler–Lagrange equations of motion in Hamiltonian form (show it!)

$$\begin{cases} \dot{\Phi} = \dfrac{\delta H}{\delta \Pi} \\ \dot{\Pi} = -\dfrac{\delta H}{\delta \Phi} \end{cases} \iff \begin{cases} \partial_t \varphi = \dfrac{\partial \mathcal{H}}{\partial \pi} \\ \partial_t \pi = -\dfrac{\partial \mathcal{H}}{\partial \varphi} + \dfrac{d}{dx^i}\dfrac{\partial \mathcal{H}}{\partial(\partial_i \varphi)} \end{cases}$$

Note that they encode time evolution with respect to a given choice of time, and as equations are not manifestly Lorentz covariant.

Exercise 13.3.3. Show that the Hamiltonian equations of motion can also be deduced from the 1st order action principle:

$$\delta S_{1\text{st}}[\bar{\pi}, \bar{\varphi}] = 0 \quad \text{and} \quad \delta\Phi(t_0) = 0 = \delta\Phi(t_1).$$

for

$$S_{1\text{st}}[\varphi, \pi] = \int_M d\text{vol}_M\ \pi\partial_t\varphi - \mathcal{H}(\varphi, \pi, x).$$

Notice that the boundary conditions for the action principle are quite natural in this case, and say that the variations have to be taken at fixed initial/final values ϕ_0 and ϕ_1 of $\Phi(t)$. No conditions on π are imposed. ◊

We conclude this discussion by mentioning that we can also introduce a (functional) Poisson bracket:

$$\{\Phi(\vec{x}), \Phi(\vec{y})\} = 0 = \{\Pi(\vec{x}), \Pi(\vec{y})\} \quad \text{and} \quad \{\Phi(\vec{x}), \Pi(\vec{y})\} = \delta^3(\vec{x} - \vec{y}).$$

Exercise 13.3.4. Show that the Hamiltonian equations of motion can be written in terms of the functional Poisson bracket, and in particular that the time evolution along a physical history of a functional

$$F(\Phi(t), \Pi(t), t) \doteq \int_{\Sigma_t} d^3\vec{x}\ \mathcal{F}\big(\varphi(t, \vec{x}), \pi(t, \vec{x}), (t, \vec{x})\big)$$

is given by

$$\dot{F} = \partial_t F + \int_{\Sigma_t} d^3\vec{x}\ \{F, H\}.$$ ◊

[3]Of course this can also be equivalently written as

$$H(\Phi, \Pi, t) = \int_{\Sigma_t} d^3\vec{x}\ \frac{1}{2}\Pi^2 + \frac{1}{2}|\vec{\nabla}\Phi|^2 + V(\Phi).$$

13.4 Lagrangian formulation of electromagnetism

Recall the Maxwell equations (in unites where $c = 1$):

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= 4\pi\rho \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= 4\pi\vec{j} + \frac{\partial \vec{E}}{\partial t}
\end{aligned}$$

The first two equations have no time derivatives, while two remaining are first order.

The time derivative of the first two equations automatically vanishes if the electric charge is conserved and the other two Maxwell equations are satisfied,

$$\begin{aligned}
&\vec{\nabla} \cdot \partial_t \vec{E} - 4\pi\partial_t\rho = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B} - 4\pi\vec{j}) - 4\pi\partial_t\rho = -4\pi(\partial_t\rho + \vec{\nabla} \cdot \vec{j}), \\
&\vec{\nabla} \cdot \partial_t \vec{B} = -\vec{\nabla} \cdot \vec{\nabla} \times \vec{E} = 0.
\end{aligned}$$

This means that the first two equations are non-dynamical, but rather a *constraint* on the space of viable initial data.

Thus, to understand the dynamics, we focus on the second two equations. These being first order, we now look for a first-order action principle in the form:

$$\mathcal{L}_{1\text{st}}(\vec{E}, \vec{B}) \stackrel{?}{=} E^i \Omega_{ij} \dot{B}^j - \mathcal{H}(E, B).$$

where we expect

$$\mathcal{H}(E, B) = \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$$

to be the energy density of the electromagnetic field. We also introduced Ω_{ij}, a "matrix" that we need to fix by comparing the Euler–Lagrange equations of motion of $\mathcal{L}_{1\text{st}}$ with the third and fourth of Maxwell's equations (in vacuum). Following the same procedure and notation used at the end of the previous section, for $\mathcal{H}$ of the expected form we find the following Euler–Lagrange equation associated to the variation of our Ansatz for $\mathcal{L}_{1\text{st}}$ with respect to $\vec{E}$:

$$0 = \Omega_{ij}\partial_t B^j - E_i = 0.$$

Comparing with the corresponding Maxwell equation we find:

$$\partial_t B^j = (\Omega^{-1})^{ji} E_i = -\epsilon^{jkl}\partial_k E_l$$

which means that

$$(\Omega^{-1})^{ji} = \epsilon^{jik}\partial_k.$$

We thus have a puzzle: if $\Omega^{-1} = \vec{\nabla}\times$ is (minus) the curl differential operator, what is Ω itself?

Formally, we have no other option than saying that M is some integral operator such that:

$$\Omega : \vec{B} \mapsto \vec{A} \text{ such that } \vec{B} = \vec{\nabla} \times \vec{A}.$$

But once again, since the curl operator Ω^{-1} has a kernel, i.e. $\vec{A}$ of the form $\vec{\nabla}\xi$, we see that the Ω is ambiguous. This ambiguity in Ω is called the *gauge* ambiguity.

Now, looking at our tentative first order Lagrangian, we immediately realize that it is much easier to write it in terms of $(\vec{E}, \vec{A})$ rather than $(\vec{E}, \vec{B})$:

$$\mathcal{L}_{1\text{st}}(\vec{E}, \vec{A}) \stackrel{?}{=} \vec{E} \cdot \dot{\vec{A}} - \frac{1}{2}\vec{E}^2 - \frac{1}{2}|\vec{\nabla} \times \vec{A}|^2.$$

But what about the gauge ambiguity $\vec{A} \mapsto \vec{A} + \vec{\nabla}\xi$? It is easy to see that the Lagrangian functional $L_{1\text{st}} = \int_{\Sigma_t} d^3\vec{x}\, \mathcal{L}_{1\text{st}}$ is invariant under such a gauge transforamtions provided $\vec{\nabla} \cdot \vec{E} = 0$—which is one of the time-deriviative-free Maxwell equations! This interplay between the gauge ambiguity and the Gauss law/constraint is very deep—but unfortunately it also goes beyond the scope of these lecture notes.[4] To fix this issue, we introduce a so-called Lagrange-multiplier field, ϕ, which features linearly in the Lagrangian and whose sole role (for now at least) is to give a new equation of motion implementing the constraint $\vec{\nabla} \cdot \vec{E} = 0$:

sign of ϕ

$$\mathcal{L}'_{1\text{st}}(\vec{E}, \vec{A}, \varphi) = \vec{E} \cdot \dot{\vec{A}} - \varphi\vec{\nabla} \cdot \vec{E} - \frac{1}{2}\vec{E}^2 - \frac{1}{2}|\vec{\nabla} \times \vec{A}|^2.$$

In fact, by adding the spatial divergence $\vec{\nabla} \cdot (\varphi\vec{E})$, which is inconsequential since it drops[5] from the Lagrangian functional $L = \int_{\Sigma_t} d^3\vec{x}\, \mathcal{L}$, we can introduce the Lagrangian density $\mathcal{L}_{1\text{st}} = \mathcal{L}'_{1\text{st}} + \vec{\nabla} \cdot (\varphi\vec{E})$, i.e.

$$\mathcal{L}_{1\text{st}}(\vec{E}, \vec{A}, \varphi) = \vec{E} \cdot (\dot{\vec{A}} + \vec{\nabla}\varphi) - \frac{1}{2}\vec{E}^2 - \frac{1}{2}|\vec{\nabla} \times \vec{A}|^2.$$

This new Lagrangian density is can now be promoted to a fully gauge invariant quantity by demanding that also ϕ transforms under gauge transformations as per

$$(\varphi, \vec{A}) \mapsto (\varphi - \partial_t\xi, \vec{A} + \vec{\nabla}\xi).$$

Does it remind you of anything? More on this soon, cf. in particular Exercise 13.4.5.

Remark 13.4.1. This section comes with two crucial takeaways. First: in order to find a *local*[6] Lagrangian density yielding Maxwell equations as its equations of motion, one is forced to introduce the potential $\vec{A}$—which comes with a peculiar gauge ambiguity. Second: the Gauss law is intimately related to the gauge ambiguity itself. ♢

[4] See my Quantum Gravity course and lecture notes.

[5] In these treatments we always implicitly assume that the fields vanish "fast enough" at infinity.

[6] I.e. one that does not contain integral operators such as M_{ij}.

* * *

The main drawback of this Lagrangian—or at least of this way of writing it—is that it fails to be manifestly Lorentz invariant. Indeed, we know that Maxwell equations take a particularly simple fin terms of the electromagnetic 4-tensor $F_{\mu\nu}$:

$$\nabla^\mu F_{\mu\nu} = 0 \quad \text{and} \quad \nabla_{[\mu} F_{\nu\rho]} = 0.$$

How can we find a manifestly covariant Lagrangian density for the Maxwell equations?

Following the insights from the previous section we know that if we want a *local* Lagrangian density, we should look for a Lagrangian density written in terms of the potential $\vec{A}$. On the other hand, since we want a manifestly Lorentz invariant treatment we should in fact use the 4-potential A_μ instead. The 4-potential is a 4-vector field, transforming under Lorentz transformations as per Remark 13.2.1.

Next, we consider the problem of gauge ambiguity: the Maxwell equations are invariant under the gauge transformation

$$A_\mu \mapsto A_\mu + \partial_\mu \xi$$

and therefore, to guarantee this fact automatically, it is smart to look for a Lagrangian that does not depend on A_μ directly, but only by means of the gauge invariant combination

$$F_{\mu\nu}(A) \doteq \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Finally, since the Maxwell equations are linear in A_μ, we need to look for a Lagrangian that is quadratic in A_μ.

Combining all these ingredients together we are left with considering 3 potential contributions to our candidate Lorentz and gauge invariant Lagrangian for the Maxwell equations:

$$C^1(A) = F_{\mu\nu}F^{\mu\nu}, \quad C^2(A) = F_{\mu\nu}\widetilde{F}^{\mu\nu}, \quad C^3(A) = \widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu}.$$

where we recall that $\widetilde{F}^{\mu\nu} \doteq \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual electromagnetic tensor.

Exercise 13.4.2. Show that C^1 and C^3 are equal to each other (possibly up to a sign) and that C^2 is a total derivative. The rank-3 tensor $W_{\mu\nu\rho}(A) \doteq \frac{1}{2}A_{[\mu}F_{\nu\rho]}$ is called the (Abelian) Chern–Simons form. ♢

In light of the previous exercise and the fact that adding a total derivative of the Lagrangian does not alter the equations of motion,[7] we are left with the following candidate Lagrangian density for the Maxwell theory:

$$\mathcal{L}_{\text{Max}} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} \quad \text{where} \quad F_{\mu\nu}(A) \doteq \partial_\mu A_\nu - \partial_\nu A_\mu.$$

[7]This does not mean that the second term is completely irrelevant. It plays a particularly important role in topology, topological quantum field theories, and also in the non-Abelian generalization of electromagnetism that is used in quantum chromodynamics, where it is usually called the θ-term.

Remark 13.4.3. The normalization of the Lagrangian is fixed so as to get the Maxwell equations in the conventions used here when coupling the electromagnetic field to the charged matter degrees of freedom, see the next exercise. Its sign is fixed so as to get a positive energy density, see the next chapter. (Heuristically, we want the kinetic term, i.e. the square of the time derivatives of the fundamental fields, in this case $\frac{1}{2}\dot{\vec{A}}^2$, to appear with a positive sign.) ♢

Exercise 13.4.4. Show that the Euler–Lagrange equations of $\mathcal{L}_{\text{Max}}$ are the Maxwell equations. Use Exercise 10.5.2, as well as the results of Sections 11.3 and 12.4, to write the total, interacting, action functional that couples the Maxwell fields to a charged point particle. (Don't forget to write down the equations of motion and check their consistency with what you know!) ♢

Exercise 13.4.5. Show that if one sets $A^\mu \equiv (\varphi, \vec{A})$ and $\vec{E} = -\vec{\nabla}\varphi - \partial_t \vec{A}$, the 1st order Lagrangian density $\mathcal{L}_{\text{1st}}$ is numerically equivalent to $\mathcal{L}_{\text{Max}}$, and to the quantity $\frac{1}{2}(\vec{E}^2 - \vec{B}^2)$. Cf. with Section 11.4. ♢

Exercise 13.4.6. What happens if one tried to perform the Legendre transform of $\mathcal{L}_{\text{Max}}$ to find the Hamiltonian formulation of electromagnetism? [*Hint*: (1) What is the momentum conjugate to A_0? This means that this field should be left alone, it's just a Lagrange multiplier. (2) What is the momentum conjugate to $\vec{A}$? Express $\partial_t \vec{A}$ in terms of its conjugate momentum and compute the Hamiltonian density via a Legendre transform of $\mathcal{L}_{\text{Max}}$. Compare to $\mathcal{L}_{\text{1st}}$.] ♢

Eigth Tutorial

Chapter 14

Stress energy tensor and relativistic angular momentum tensor

14.1 Noether 1st theorem in field theory[1]

Consider a field theory in its (covariant) Lagrangian formulation that has configuration space Q.

For a scalar field $Q = \mathbb{R}$, for a vector field $Q = \mathbb{R}^{1,3}$, and for a spinor field $Q = \mathbb{C}^2$. In general we denote q^I points inside Q, $\bullet^I$ being a portmanteau for any label or index needed to characterize the configuration space, or the field $\varphi^I : M \to Q$.

Next, let s be a function of configuration space $q^I \in Q$, its first jet[2] v^I_μ, and the spacetime point x^μ, that is valued in the tangent of the configuration space, TQ:

$$s : \big(q^I, v^I_\mu, x^\mu\big) \mapsto \big(q^I, s^I(q, v, x)\big)\,.$$

We interpret this function as giving an infinitesimal variation of the history φ:

$$\varphi^I(x) \mapsto \varphi^I(x) + \epsilon\widetilde{\delta}_s\varphi^I(x), \qquad \widetilde{\delta}_s\varphi^I(x) \doteq s\big(\varphi^I(x), \partial_\mu\varphi^I(x), x\big).$$

If the infinitesimal transformation $\varphi \mapsto \varphi + \epsilon\widetilde{\delta}_s\varphi$ leaves $\mathcal{L}$ invariant up to a total divergence, i.e. if

$$\mathcal{L}(\varphi + \epsilon\widetilde{\delta}_s\varphi, \partial_\nu\varphi + \epsilon\partial_\nu\widetilde{\delta}_s\varphi, x) = \mathcal{L}(\varphi, \partial_\nu\varphi, x) + \epsilon\frac{d}{dx^\mu}R^\mu_s(\varphi, \partial_\nu\varphi, \ldots, x) + O(\epsilon^2)$$

or more succinctly if

$$\widetilde{\delta}_s\mathcal{L} = \frac{d}{dx^\mu}R^\mu_s,$$

then we say that s is an *infinitesimal symmetry* of $\mathcal{L}$.

[1] Cf. Chapter 2.

[2] Recall: this is a generalization of the velocity v^I of q^I in the context where there is not just a time direction, but 4 spacetime directions. Given a history $\varphi^I : M \to Q$, to each point of spacetime one can associate a configuration value $q^I = \varphi^I(x)$ and a first jet value $v^I_\mu = \partial_\mu\varphi^I(x)$.

Remark 14.1.1. Except for the sake of mathematical clarity, what one actually needs for computations is the quantity $\widetilde{\delta}_s\varphi(x)$, rather than the map s. ◊

In analogy with the particle mechanics, for notational convenience we introduce the symbol:

$$\pi^\mu_I \doteq \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^I)}.$$

Remark 14.1.2. In the previous section we were dealing with scalar fields where the index $\bullet^I$ is not necessary. When performing the Hamiltonian analysis of the scalar field we were led to introduce $\pi|_{\text{there}} = \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} = \pi^0|_{\text{here}}$. ◊

Theorem 14.1.3 (Noether)**.** *If s is an infinitesimal symmetry of $\mathcal{L}$, then the quantity*

$$J^\mu_s(x) \doteq \sum_I \pi^\mu_I \widetilde{\delta}_s \varphi^I - R^\mu_s$$

is a conserved current when computed at any physical history, i.e. at any history $\varphi(x)$ that solves the Euler–Lagrange equations. Symbolically, we write:

$$\nabla_\mu J^\mu_s \hat{=} 0.$$

J^μ_s is called the Noether current *associated to the symmetry s.*

Furthermore, if we denote $\Sigma_t = \{t\} \times \mathbb{R}^3$ the Cauchy surface at time t, then the quantity

$$Q_s(t) \doteq \int_{\Sigma_t} d^3\vec{x}\ J^0_s$$

is called the Noether charge *associated to the symmetry s, and*

$$\frac{dQ_s}{dt} \hat{=} 0.$$

The proof of the theorem is fully analogous to that done at the end of Chapter 2 in the case of particle mechanics: for this reason, it is left to the reader as an exercise. We also refer to Section 11.2 for the statements on charge conservation, and its generalization in terms of a balance equation.

Remark 14.1.4. If s is an infinitesimal symmetry of $\mathcal{L}$ then it is also an infinitesimal symmetry of $\mathcal{L}' = \mathcal{L} + \frac{d}{dx^\mu}\ell^\mu$. It is then easy to verify, using its definition, that the (conserved) Noether current J^μ is the same in the two cases. (Do it!) ◊

Remark 14.1.5 (Ambiguity in J^μ_s)**.** Let $\Psi^{\mu\nu} = \Psi^{[\mu\nu]}$ be a skew-symmetry function of the fields and their derivatives, and note that, if J^μ is a conserved current, then also

$$J'^\mu \doteq J^\mu + \frac{d}{dx^\nu}\Psi^{\nu\mu}$$

is a conserved current. In the case of the Noether current, this redefinition reflects an ambiguity intrinsic to the definition of R^μ_s. Indeed, what we can compute from the variation of the Lagrangian is only the *divergence* of R^μ_s. But since R^μ_s and

$$R'^\mu_s = R^\mu_s - \frac{d}{dx^\nu}\Psi^{\nu\mu}$$

have the same divergence, we cannot really distinguish one for the other. However, reassuringly, the Noether *charge* is not affected by this ambiguity (provided Ψ^{i0} falls off fast enough at infinity, i.e. as $\vec{x} \to \infty$):

$$Q'_s(t) = \int_{\Sigma_t} d^3\vec{x}\ J'^0_s = Q_s(t) + \int_{\Sigma_t} d^3\vec{x}\ \nabla_i \Psi^{i0} = Q_s(t),$$

where we used that $\partial_0 \Psi^{00} \equiv 0$ in view of the skew-symmetry of $\Psi^{\mu\nu}$.

In the literature, the quantity $\Psi^{\mu\nu}$ is sometimes called a "superpotential". ◊

1.1 Example: complex scalar field Consider a complex scalar field ($Q = \mathbb{C}$) with Lagrangian

$$\mathcal{L} = -\partial_\mu \bar{\varphi} \partial \partial^\mu \varphi - m^2 \bar{\varphi}\varphi - V(\bar{\varphi}\varphi).$$

where in this section the over-bar denotes complex conjugation, and $V : \mathbb{R}_{\geq 0} \to \mathbb{R}$.

Here, we either think of $Q = \mathbb{C} = \mathbb{R}^2$ and write $\varphi = \varphi^{\mathrm{re}} + \mathrm{i}\varphi^{\mathrm{im}}$ thus considering $\varphi^I = (\varphi^{\mathrm{re}}, \varphi^{\mathrm{im}})$ as two independent components of a real field, or instead deal with the fields φ and $\bar{\varphi}$ as we do with the variables z and $\bar{z}$ in complex analysis: we consider them as independent entities to be set equal to each other's complex-conjugate in the end. Following this second route, we calculate the Euler–Lagrange equations for $\bar{\varphi}$ and φ, finding respectively

$$\frac{\delta S}{\delta \bar{\varphi}} = \Box\varphi - m^2\varphi - V'(\bar{\varphi}\varphi)\varphi$$

and its complex-conjugate equation. This is of course a consequence of $\mathcal{L}$ being real, and therefore for our purposes it is enough to write only one complex equation of motion.

We also compute

$$\bar{\pi}^\mu \doteq \frac{\partial L}{\partial(\partial_\mu \bar{\varphi})} = \nabla^\mu \varphi \quad \text{and} \quad \pi^\mu \doteq \frac{\partial L}{\partial(\partial_\mu \varphi)} = \nabla^\mu \bar{\varphi}.$$

Next, we consider the following field variation:

$$\begin{pmatrix} \varphi(x) \\ \bar{\varphi}(x) \end{pmatrix} \mapsto \begin{pmatrix} \varphi(x) \\ \bar{\varphi}(x) \end{pmatrix} + \epsilon \begin{pmatrix} \widetilde{\delta}_s \varphi(x) \\ \widetilde{\delta}_s \bar{\varphi}(x) \end{pmatrix}, \quad \begin{pmatrix} \widetilde{\delta}_s \varphi(x) \\ \widetilde{\delta}_s \bar{\varphi}(x) \end{pmatrix} = \begin{pmatrix} \mathrm{i}\varphi(x) \\ -\mathrm{i}\bar{\varphi}(x) \end{pmatrix}.$$

Clearly, this is a symmetry of the Lagrangian, since

$$\widetilde{\delta}_s \mathcal{L} = 0.$$

Whence we set $R^\mu_s = 0$ and compute:

$$J^\mu_s = \pi^\mu \widetilde{\delta}_s \varphi + \bar{\pi}^\mu \widetilde{\delta}_s \bar{\varphi} - R^\mu_s = \nabla^\mu \varphi = \mathrm{i}\bar{\varphi}\nabla^\mu \varphi - \mathrm{i}\varphi \nabla^\mu \bar{\varphi} = \mathrm{Im}(\varphi \nabla^\mu \bar{\varphi}).$$

It is then easy to verify the validity of Noether's theorem (here "c.c." stands for "complex conjugate"):

$$\nabla_\mu J^\mu_s = \mathrm{i}\nabla_\mu \bar{\varphi}\nabla^\mu \varphi + \mathrm{i}\bar{\varphi}\Box\varphi + \text{c.c.} \mathrel{\hat{=}} \mathrm{i}\big(m^2 + V'(\bar{\varphi}\varphi)\big)\bar{\varphi}\varphi + \text{c.c.} = 0.$$

Remark 14.1.6. Here above we have admittedly shortcut the setup presented in the previous section, jumping directly to the history variation. If we had followed the more detailed setup, we would have ended up with the following. Let $Q = \mathbb{C}$ and introduce the map

$$\mathbb{R} \times Q \to Q, \quad (\alpha, q) \mapsto e^{\mathrm{i}\alpha} q$$

Considering the infinitesimal version of this map, $q \mapsto \mathrm{i}\epsilon q$, we would have then introduced the corresponding map s,

$$(q, v_\mu, x) \mapsto (q, s(q, v, x)) \doteq (q, \mathrm{i}q),$$

and the corresponding variation of the field's history φ:

$$\widetilde{\delta}_s \varphi(x) = s(\varphi, \partial_\mu \varphi, x) = \mathrm{i}\varphi(x).$$

Etc. ◊

14.2 The canonical stress energy tensor

One of the main results of Noether's first theorem in particle mechanics was the conservation of the linear and angular momenta, and of energy as a consequence of, respectively, spatial translation, spatial rotation, and time translation symmetry. What happens to these concepts in field theory?

We will start by studying spacetime translations $x^\mu \mapsto x^\mu + a^\mu$. Since the fields history is a function of x, note that their closest particle-mechanics analogue is time translations, and *not* spatial translations.

Consider a spacetime translation:

$$\varphi^I(x) \mapsto \varphi^I(x + a), \quad \text{where} \quad a^\mu \in \mathbb{R}^{1,3}.$$

Infinitesimally, this yields the following infinitesimal change in the history:

$$\widetilde{\delta}_s \varphi^I(x) = a^\mu \partial_\mu \varphi^I(x).$$

Exercise 14.2.1. Argue that the corresponding map s is $s^I(q, v, x) = a^\mu v^I_\mu$. Note that contrary to the example provided in the last section, this transformation depends on the 1st jet v_μ, rather than on the configuration variable q. ◊

Under this infinitesimal transformation *of the fields* the Lagrangian then transforms as follows:

$$\begin{aligned}
\widetilde{\delta}_s \mathcal{L}(\varphi, \partial_\mu \varphi, x) &= \sum_I \frac{\partial \mathcal{L}}{\partial \varphi^I} \widetilde{\delta}_s \varphi^I + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi^I)} \partial_\nu \widetilde{\delta}_s \varphi^I \\
&= \sum_I \frac{\partial \mathcal{L}}{\partial \varphi^I} a^\mu \partial_\mu \varphi^I + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi^I)} \partial_\nu (a^\mu \partial_\mu \varphi^I) \\
&= \sum_I \frac{\partial \mathcal{L}}{\partial \varphi^I} a^\mu \partial_\mu \varphi^I + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi^I)} a^\mu \partial_\mu (\partial_\nu \varphi^I) \\
&= a^\mu \left(\frac{d}{dx^\mu} - \frac{\partial}{\partial x^\mu} \right) \mathcal{L}
\end{aligned}$$

Therefore, if the Lagrangian density $\mathcal{L}(q^I, v^I_\mu, x^\mu)$ does not depend explicitly on the position x, then spacetime translations are a symmetry, since

$$\frac{\partial}{\partial x^\mu}\mathcal{L}(q^I, v^I_\mu, x^\mu) = 0, \implies \widetilde{\delta}_s\mathcal{L}(\varphi, \partial_\mu\varphi, x) = \frac{d}{dx^\mu}R^\mu_s \quad \text{for} \quad R^\mu_s = a^\mu\mathcal{L}.$$

The associated, conserved, Noether current then reads:

$$J^\mu_s = \sum_I \pi^\mu_I \widetilde{\delta}_s\varphi^I - R^\mu_s = \left(\pi^\mu_I\partial_\nu\varphi^I - \delta^\mu_\nu\mathcal{L}\right)a^\nu.$$

Since this current must be conserved for all values of $a^\mu \in \mathbb{R}^{1,3}$, it is convenient to introduce the following rank-2 tensor, called the *canonical stress-energy tensor*:

$$t^\mu{}_\nu \doteq \pi^\mu_I\partial_\nu\varphi^I - \delta^\mu_\nu\mathcal{L}.$$

As a consequence of Noether theorem for spacetime translations, the canonical stress-energy tensor is conserved "on-shell", i.e. one computed on a history φ that solves the equations of motion:

$$\nabla_\mu t^\mu{}_\nu \hat{=} 0.$$

The four different Noether charges associated to J^μ_s for each independent choice of a^μ form a Lorentz 4-vector that reads:

$$Q_s = a^\mu P_\mu$$

where

$$P_\mu \doteq \int_{\Sigma_t} d^3\vec{x}\; t^0{}_\mu.$$

This 4-vector is the *energy-momentum of the field*.

Remark 14.2.2 (Ambiguity in $t^\mu{}_\nu$)**.** As the Noether current carries an ambiguity $J^\mu \mapsto J^\mu + \nabla_\nu\Psi^{\nu\mu}$, the canonical stress-energy tensor carries the following ambiguity:

$$t^\mu{}_\nu \mapsto t^\mu{}_\nu + \nabla_\rho\Psi^{\rho\mu}{}_\nu, \qquad \text{with} \qquad \Psi^{\rho\mu}{}_\nu = \Psi^{[\rho\mu]}{}_\nu.$$

The energy-momentum vector P_μ is not affected by this ambiguity.

This observation leaves us with the suspicion that the canonical stress-energy tensor might *not* always be the best way to encode the energy-momentum content of the field. Later. tjos will indeed see that in electromagnetism $t^\mu{}_\nu$ is indeed not the best choice of stress-energy tensor. ♢

Remark 14.2.3 (Symmetry of $t_{\mu\nu}$)**.** Unless $\pi^\mu_I = \sum_J K^{\mu\nu}_{IJ}\partial_\nu\varphi^J$ for a symmetric matrix $K^{\mu\nu}_{IJ}$, the tensor $t_{\mu\nu}$ fails to be symmetric. This is sometimes presented as the issue that might affect $t^\mu{}_\nu$, and the reason why we might want to modify it by the addition of an appropriate superpotential. However, at this level, it is not yet clear why the symmetry of $t_{\mu\nu}$ should be important at all (it is very important if we want it to play the role of the source term for the Einstein's equations). We will come back to the question of the symmetry of $t_{\mu\nu}$ after having studied the problem of defining an angular momentum charge. ♢

Remark 14.2.4. A more serious issue with $t^\mu{}_\nu$ is that in electromagnetism it will not be gauge invariant. More on this topic, below. ♢

2.1 Example: the scalar field The most simple example is given by the scalar field $\varphi(x)$:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi).$$

In this case, as already observed, there is no index $\bullet^I$ to keep track of. Thus we have

$$\pi^\mu \doteq \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = -\partial^\mu\varphi,$$

whence

$$t^\mu{}_\nu = \pi^\mu\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L} = -\partial^\mu\varphi\partial_\nu\varphi + \frac{1}{2}\delta^\mu{}_\nu\partial^\rho\varphi\partial_\rho\varphi + \delta^\mu_\nu V(\varphi).$$

Finally:

$$P^0 \doteq \int_\Sigma d^3\vec{x}\; t^{00} = \int_\Sigma d^3\vec{x}\; \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + V(\varphi)$$

and

$$P^i \doteq \int_\Sigma d^3\vec{x}\; t^{0i} = \int_\Sigma d^3\vec{x}\; \dot{\varphi}\vec{\nabla}\varphi.$$

14.3 Angular momentum

Consider now the question of Lorentz invariance. Given a Lorentz transformation $\Lambda \in \mathrm{SO}^+(1,3)$, a field $\varphi^I(x)$ will in general transforms as follows:

$$\varphi^I(x) \mapsto D(\Lambda)^I{}_J\varphi^J(\Lambda^{-1}x).$$

Whence, under a infinitesimal Lorentz transformation $\lambda \in \mathfrak{so}(1,3)$, we have

$$\widetilde{\delta}_s\varphi^I(x) = -\lambda^\mu{}_\nu x^\nu\partial_\mu\varphi^I(x) + D(\lambda)^I{}_J\varphi^J(x).$$

Remark 14.3.1. If φ is a scalar field the index $\bullet^I$ is trivial (i.e. takes one value and is omitted) and thus the matrix $D(\Lambda)^I{}_J$ is trivial as well. On the other hand, if φ^I is a *contravariant* vector field φ^μ, the index I corresponds to a spacetime contravariant index μ and, the field transforms as:

$$\varphi^\mu(x) \mapsto \Lambda^\mu{}_\nu\varphi^\nu(\Lambda^{-1}x). \qquad \lozenge$$

Under this infinitesimal transformation *of the fields*, one finds after some algebra,

that the Lagrangian then transforms as follows:[3]

$$\begin{aligned}
\widetilde{\delta}_s\mathcal{L}(\varphi,\partial_\mu\varphi,x) &= \sum_I \frac{\partial\mathcal{L}}{\partial\varphi^I}\widetilde{\delta}_s\varphi^I + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\varphi^I)}\partial_\rho\widetilde{\delta}_s\varphi^I \\
&= -\sum_I \frac{\partial\mathcal{L}}{\partial\varphi^I}\lambda^\mu{}_\nu x^\nu\partial_\mu\varphi^I + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\varphi^I)}\lambda^\mu{}_\nu x^\nu\partial_\rho\partial_\mu\varphi^I \\
&\quad + \sum_I \frac{\partial\mathcal{L}}{\partial\varphi^I}D(\lambda)^I{}_J\varphi^J + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\varphi^I)}\partial_\rho(D(\lambda)^I{}_J\varphi^J) - \frac{\partial\mathcal{L}}{\partial(\partial_\rho\varphi^I)}\lambda^\mu{}_\rho\partial_\mu\varphi^I \\
&= \left(\frac{d}{dx^\mu} - \frac{\partial}{\partial x^\mu}\right)(-\lambda^\mu{}_\nu x^\nu\mathcal{L}) \\
&\quad + \sum_I \frac{\partial\mathcal{L}}{\partial\varphi^I}D(\lambda)^I{}_J\varphi^J + \pi_I^\rho D(\lambda)^I{}_J\partial_\rho\varphi^J - \pi_I^\rho\lambda^\mu{}_\rho\partial_\mu\varphi^I
\end{aligned}$$

Now let us assume that $\mathcal{L}(q^I, v^I_\mu, x^\mu)$ does not depend explicitly on the position x,

$$\frac{\partial}{\partial x^\mu}\mathcal{L}(q^I, v^I_\mu, x^\mu) = 0,$$

so that translations are a symmetry that yields a conserved (canonical) energy momentum tensor $t^\mu{}_\nu$.Then, the above becomes

$$\widetilde{\delta}_s\mathcal{L}(\varphi,\partial_\mu\varphi,x) = \frac{d}{dx^\mu}\left(\lambda^\mu{}_\nu x^\nu\mathcal{L}\right) + \sum_I \frac{\partial\mathcal{L}}{\partial\varphi^I}D(\lambda)^I{}_J\varphi^J + \pi_I^\rho D(\lambda)^I{}_J\partial_\rho\varphi^J - \pi_I^\rho\lambda^\mu{}_\rho\partial_\mu\varphi^I.$$

So, Lorentz transformations are a symmetry if and only if the sum above gives a total divergence. For simplicity, in the following we will assume that it vanishes (this will be the case in the examples at hand, e.g. in electromagnetism):

$$\text{assume that} \quad \mathcal{A}_s \doteq \sum_I \frac{\partial\mathcal{L}}{\partial\varphi^I}D(\lambda)^I{}_J\varphi^J + \pi_I^\rho D(\lambda)^I{}_J\partial_\rho\varphi^J - \pi_I^\rho\lambda^\mu{}_\rho\partial_\mu\varphi^I = 0$$

Remark 14.3.2. Notice that the condition $\mathcal{A}_s$ requires an interplay between the indices $\bullet^I$ and $\bullet_\mu$, and their transformation properties. In the case of a vector field φ^μ with Lagrangian of the form

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi^\nu\partial^\mu\varphi_\nu - V(\varphi^\mu\varphi_\mu),$$

we find

$$\mathcal{A}_s = -V'(\varphi^2)\varphi_\mu\lambda^\mu{}_\rho\varphi^\rho + \partial^\rho\varphi_\nu\lambda^\nu{}_\sigma\partial_\rho\varphi^\sigma - \partial^\rho\varphi_\nu\lambda^\mu{}_\rho\partial_\mu\varphi^\nu = 0,$$

since each term vanishes independently in view of the skew symmetry of $\lambda_{\mu\nu}$ (show it!). ◊

[3]Recall that $\partial_\rho x^\nu = \delta^\nu_\rho$ and $\lambda^\mu{}_\mu = 0$.

Exercise 14.3.3. Show that the statement of the previous remark holds true if one replaces the kinetic term with a term of form$\partial_{(\mu}\varphi_{\nu)}\partial^{(\mu}\varphi^{\nu)}$. ◊

From the assumption $\mathcal{A}_s = 0$, it follows that Lorentz transformations are a symmetry:

$$\widetilde{\delta}_s\mathcal{L}(\varphi, \partial_\mu\varphi, x) = \frac{d}{dx^\mu}R_s^\mu \quad \text{with} \quad R_s^\mu = -\lambda^\mu{}_\nu x^\nu \mathcal{L}.$$

The associated, conserved, Noether current then reads:

$$J_s^\mu = \sum_I \pi_I^\mu \widetilde{\delta}_s\varphi^I - R_s^\mu = -\lambda^\rho{}_\nu x^\nu t^\mu{}_\rho + \pi_I^\mu D(\lambda)^I{}_J\varphi^J$$

Since $\lambda = \frac{1}{2}\lambda^{\alpha\beta}J_{\alpha\beta}$, and $(J_{\alpha\beta})^\mu{}_\nu = 2\delta^\mu_{[\alpha}\eta_{\beta]\nu}$, we have

$$J_s^\mu = \frac{1}{2}\lambda^{\alpha\beta}\left(-2t^\mu{}_{[\alpha}x_{\beta]} + S^\mu{}_{\alpha\beta}\right) \quad \text{where} \quad S^\mu{}_{\alpha\beta} \doteq \pi_I^\mu D(J_{\alpha\beta})^I{}_J\varphi^J.$$

where we introduced the tesnor $S^\mu{}_{\alpha\beta}$ which we call—for reasons to be clarified soon—the *spin density current*. Now, owing to the fact that this current must be conserved on-shell of the equations of motion ($\nabla_\mu J_s^\mu \hat{=} 0$) for all choices of $\lambda^{\alpha\beta}$, we deduce the conservation of the following rank-3 tensor

$$\nabla_\mu M^\mu{}_{\alpha\beta} \hat{=} 0 \quad \text{where} \quad M^\mu{}_{\alpha\beta} \doteq -2t^\mu{}_{[\alpha}x_{\beta]} + S^\mu{}_{\alpha\beta}$$

Integrating over a Cauchy surface we find the following (conserved) Noether charges:

$$M_{\alpha\beta} \doteq \int_{\Sigma_t} d^3\vec{x}\ M^0{}_{\alpha\beta} = \int_{\Sigma_t} d^3\vec{x}\ \left(x_\alpha t^0{}_\beta - x_\beta t^0{}_\alpha\right) + \int_{\Sigma_t} d^3\vec{x}\ S^0{}_{\alpha\beta} \doteq L_{\alpha\beta} + S_{\alpha\beta}.$$

We will now discuss the interpretation of $M^\mu{}_{\alpha\beta}$ (and $M_{\alpha\beta}$, resp.) as a Lorentz-covariant generalization of the angular momentum density current (and total angular momentum, respectively).

Consider the spatial component $M_{ij} \doteq \epsilon_{ij}{}^k M_k$:

$$M_k = L_k + S_k, \quad \text{where} \quad \begin{cases} L_k = \int_{\Sigma_t} d^3\vec{x}\ \epsilon_k{}^{ij}x_i t^0{}_j \\ S_k = \int_{\Sigma_t} d^3\vec{x}\ \pi_I^0 D(J_k)^I{}_J\varphi^J \end{cases}$$

This is the conserved charge associated to an infinitesimal rotation generated by $J_k = \frac{1}{2}\epsilon_k{}^{ij}J_{ij}$, and as such we want to interpret it as a *total angular momentum*. Recalling that t_i^0 has the interpretation of an linear momentum density, we see that L_k has the interpretation of an *orbital angular momentum*. We call the remaining quantity, S_k, the *spin angular momentum*: note that it depends on the Lorentz-covariance properties of the field (in the index $\bullet^I$).

Remark 14.3.4. Note that using the notation adapted to the field-theoretic Hamiltonian picture, the orbital and spin contributions to the angular momentum $\vec{M} = \vec{L} + \vec{S}$ read

$$\begin{cases} \vec{L}(t) = \int_{\Sigma_t} d^3\vec{x}\ (\vec{x} \times \Pi_I\vec{\nabla}\Phi^I), \\ \vec{S}(t) = \int_{\Sigma_t} d^3\vec{x}\ \Pi_I D(\vec{J}\,)^I{}_J\Phi^J. \end{cases}$$

One can then check that $\vec{n}\cdot\vec{M}$ is the Poisson generator of rotations around the axis $\vec{n}$ (check it!):

$$\{\vec{n}\cdot\vec{M},\Phi^I\} = (\vec{n}\times\vec{x})\cdot\vec{\nabla}\Phi^I + D(\vec{n}\cdot\vec{J}\,)^I{}_J\Phi^J.$$

In this regards, recall also the expression of the total momentum and energy

$$\begin{cases}\vec{P} = \int_{\Sigma_t} d^3\vec{x}\ \Pi_I\vec{\nabla}\Phi^I,\\ E = \int_{\Sigma_t} d^3\vec{x}\ \Pi_I\dot{\Phi}^I.\end{cases}$$

The analogue formulas for the scalar field can be obtained from the above simply by removing the label $\bullet^I$, setting the spin contribution to zero, and replacing $(\Pi_I,\Phi^I)\rightsquigarrow(\dot\varphi,\varphi)$.[4] ♢

Consider next the mixed component $X^{\mathrm{i}}\doteq M^{i0}$:

$$X^i(t) = -tP^i + \int_{\Sigma_t} d^3\vec{x}\ \varepsilon x^i + S^{i0}.$$

where we denoted $\varepsilon\doteq t^{00}$ the energy density of the field. Thanks to $E=mc^2$, the quantity ε has also the interpretation of mass density, and therefore we see that the second term above has the interpretation of being the centre of mass of the system. The "spin" contribution offsets this value, whereas the first just encodes the law of inertia and guarantees that $\vec{X}(t)$ is a constant in time.

Remark 14.3.5. Summarizing, the conservation of $M^\mu{}_{\alpha\beta}$ is a covariant version of the conservation of the angular momentum of the system. In a given frame this generalized law encodes at the same time the conservation of the angular momentum itself, together with the law of inertia stating that the centre of mass of the system (possibly corrected by a "spin" contribution) moves with constant velocity in the direction of its total momentum P^i. ♢

We now come back to the question of symmetry of the canonical stress-energy tensor.

Computing the divergence of $M^\mu{}_{\alpha\beta}$, and using the fact that the canonical stress energy tensor is itself conserved on-shell, we obtain (recall, $\partial_\mu x_\beta=\eta_{\mu\beta}$):

$$\nabla_\mu M^\mu{}_{\alpha\beta}\,\hat{=}\,0 \iff t_{[\alpha\beta]}\,\hat{=}\,-\frac{1}{2}\nabla_\mu S^\mu{}_{\alpha\beta}.$$

That is, on-shell of the equations of motion, the skew symmetric part of the canonical stress energy tensor is the divergence of the spin density current.

Recall that given a certain Noether current one can always redefine it—without altering its conservation properties—by adding to it the divergence of an arbitrary (skew symmetric) "superpotential". In the case of the canonical stress-energy tensor, we can modify $t^\mu{}_\nu$ by the addition of the divergence of a "superpotential",

$$t^\mu{}_\nu \mapsto t^\mu{}_\nu + \nabla_\rho\Psi^{\rho\mu}{}_\nu \quad\text{where}\quad \Psi^{\rho\mu}{}_\nu = \Psi^{[\rho\mu]}{}_\nu.$$

[4] Here we used: $\Pi_I(t)(\vec{x})\equiv\pi^0_I(t,\vec{x}) = -\partial^0\varphi = \dot\varphi$.

Similarly, the angular momentum density current can be modified by

$$M^{\mu}{}_{\nu\rho} \mapsto M^{\mu}{}_{\nu\rho} + \nabla_\sigma \Psi^{\sigma\mu}{}_{\nu\rho} \quad \text{where} \quad \Psi^{\sigma\mu}{}_{\nu\rho} = \Psi^{[\sigma\mu]}{}_{\nu\rho}.$$

Inspire by the above observation we see that by introducing the superpotential

$$\Psi^{\rho\mu\nu} \doteq \frac{1}{2}\left(S^{\rho\mu\nu} - S^{\mu\rho\nu} - S^{\nu\rho\mu}\right)$$

we can define a new, conserved, stress energy tensor, known as the *Belinfante stress energy tensor*:

$$T^{\mu\nu} \doteq t^{\mu\nu} + \frac{1}{2}\nabla_\rho \left(S^{\rho\mu\nu} - S^{\mu\rho\nu} - S^{\nu\rho\mu}\right), \quad \nabla_\mu T^{\mu\nu} \hat{=} 0,$$

such that:

$$T^{\mu\nu} \hat{=} T^{(\mu\nu)} \quad \text{and} \quad M^{\mu\nu\rho} = -2T^{\mu[\nu}x^{\rho]} + \nabla_\sigma \Psi^{\sigma\mu\nu\rho}.$$

for a superpotential $\Psi^{\sigma\mu\nu\rho} = \Psi^{[\sigma\mu]\nu\rho}$.

Proof. Write $T^{\mu\nu} = T^{[\mu\nu]} + T^{(\mu\nu)}$ and observe that $T^{[\mu\nu]} \hat{=} 0$ readily follows from the computation of the skew symmetric part of $t^{[\mu\nu]}$ performed above. To prove the second equality we compute:

$$\begin{aligned}
\nabla_\sigma \left(S^{\sigma\mu[\nu|} - S^{\mu\sigma[\nu|} - S^{[\nu|\sigma\mu}\right) x^{|\rho]} &= \nabla_\sigma \left(S^{\sigma\mu[\nu}x^{\rho]} - S^{\mu\sigma[\nu}x^{\rho]} - S^{[\nu|\sigma\mu}x^{|\rho]}\right) \\
&\quad - \left(S^{[\rho|\mu|\nu]} - S^{\mu[\rho\nu]} - S^{[\nu\rho]\mu}\right) \\
&= \nabla_\sigma \Psi^{\sigma\mu\nu\rho} - \left(-S^{[\rho\nu]\mu} - S^{\mu\rho\nu} + S^{[\rho\nu]\mu}\right) \\
&= \nabla_\sigma \Psi^{\sigma\mu\nu\rho} - S^{\mu\nu\rho}
\end{aligned}$$

where $\Psi^{\sigma\mu\nu\rho}$ is manifestly skew symmetric in $(\sigma\mu)$, and thus obtain:

$$-2T^{\mu[\nu}x^{\rho]} = -2t^{\mu[\nu}x^{\rho]} - \nabla_\sigma \Psi^{\sigma\mu\nu\rho} + S^{\mu\nu\rho} = M^{\mu\nu\rho} - \nabla_\sigma \Psi^{\sigma\mu\nu\rho}. \qquad \square$$

Remark 14.3.6. (On-shell of the equations of motion for φ^I,) the Belinfante stress-energy tensor is the source term of General Relativity, where it is crucial that it is symmetric.[5] ♢

Remark 14.3.7. In electromagnetism, the Belinfante stress energy tensor is, on shell of the equations of motion, both symmetric (which we knew) and gauge-invariant. Providing a closer look onto these issues is the goal of the next session. ♢

[5]This was originally proved by Rosen, see Chapter 12 of Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, Wiley Ed. (1972).

14.4 Stress-energy and angular momentum in electromagnetism

In this section we apply the general formulas of the previous section to the electromagnetic field. We will find a series of surprises, e.g. the canonical stress energy tensor and the angular momentum density current will fail to be gauge invariant. These shortcomings can be overcome by using appropriate superpotential (and the equations of motion). In particular, we will see that the Belinfante stress energy tensor is on shell both symmetric (which we knew) and gauge invariant.

The Lagrangian of vacuum electrodynamics is:[6]

$$\mathcal{L}(A) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{where} \quad F_{\mu\nu} \doteq \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Invariance under translations is manifest since $\mathcal{L}$ does not depend explicitly on x; invariance under Lorentz transformations is also manifest owing to the fact that $\mathcal{L}$ is written in tensorial language; nonetheless it will be instructive to check it explicitly. We will do so shortly.

To compare to the previous sections, it is important to note that here:

$$\varphi^I(x) \equiv A_\mu(x),$$

so that we realize that $\bullet^I$ is a *covariant Lorentz index*: it will play a crucial role when we will study Lorentz symmetry.

Next, we compute π_I^μ:

$$\pi_I^\mu \doteq \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^I)} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}.$$

from which we can immediately deduce the canonical stress energy tensor:

$$t^\mu{}_\nu \doteq \pi_I^\mu \partial_\nu \varphi^I - \delta^\mu_\nu \mathcal{L} \equiv -F^{\mu\rho}\partial_\nu A_\rho - \frac{1}{4}\delta^\mu_\nu F_{\alpha\beta}F^{\alpha\beta}.$$

Remark 14.4.1. Recall that in the Hamiltonian treatment of a field theory, the relevant notion of canonical variable for the fields is given by the spatial configurations $\Phi^I(t)(\vec{x}) = \varphi^I(t,\vec{x})$. Their conjugate momenta, on the other hand, are $\Pi_I(t)(\vec{x}) = \pi_I^0(t,\vec{x})$. Thus, in electromagnetism we have:

$$\Phi^I \equiv A_\mu \qquad \text{and} \qquad \Pi_I \equiv F^{\mu 0}.$$

Note that the canonical momentum for A_0 identically vanishes:

$$\Pi_0 \equiv F^{00} \equiv 0.$$

[6]For simplicity we changed the normalization of the Lagrangian by a factor of 4π. That factor is convenient when coupling the electromagnetic field to matter.

This is equivalent to the statement that $A^0 = \varphi$ is a Lagrange multiplier for the Gauss constraint (see Section 13.4). As we already observed this is also tied to the gauge-ambiguity intrinsic to the description of electromagnetism in terms of a vector potential A, instead of the electric and magnetic fields. On a technical level, since $\Pi_0 \equiv 0$ means that the Legendre transform cannot be performed as usual, this equation also implies that the Hamiltonian analysis of electromagnetism is much more subtle than for other theories that do not have constraints and gauge symmetries. In sum, the innocent expression $\pi^\mu_I \equiv F^{\mu 0}$ hides in fact a fascinating Pandora's box that (un)fortunately we will not dare to open in these lecture notes. ♢

Exercise 14.4.2. Prove that $\nabla_\mu t^\mu{}_\nu \doteq 0$. ♢

Remark 14.4.3. The canonical stress energy tensor *fails* to be gauge invariant. Incidentally, it also fails to be symmetric. ♢

The lack of gauge invariance of $t^\mu{}_\nu$ can be fixed—at least on-shell—by adding to it the divergence of the superpotential (note that that the right hand side is indeed skew symmetric in $(\mu\rho)$):

$$\Psi^{\rho\mu}{}_\nu = F^{\mu\rho} A_\nu.$$

In fact, then one obtains

$$\widetilde{t}^\mu{}_\nu \doteq t^\mu{}_\nu + \nabla_\rho (F^{\mu\rho} A_\nu) = -F^{\mu\rho} F_{\nu\rho} - \frac{1}{4}\delta^\mu_\nu F_{\alpha\beta} F^{\alpha\beta} + (\nabla_\rho F^{\rho\mu}) A_\nu \doteq -F^{\mu\rho} F_{\nu\rho} - \frac{1}{4}\delta^\mu_\nu F_{\alpha\beta} F^{\alpha\beta}.$$

Notice that, on-shell, $\widetilde{t}_{\mu\nu}$ is not only gauge-invariant, but also symmetric. We will soon find out that this tensor equals the Belinfante stress energy tensor.

Next, we compute the transformation of the *co*vector field A_μ under a Lorentz transformation:

$$A_\mu(x) \mapsto \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1} x) = (\Lambda^{-1})^\nu{}_\mu A_\nu(\Lambda^{-1} x).$$

Infinitesimally:

$$\widetilde{\delta}_s A_\mu = -\lambda^\rho{}_\nu x^\nu \partial_\rho A_\mu - \lambda^\nu{}_\mu A_\nu,$$

i.e. $D(\lambda)^I{}_J \varphi^J \equiv \lambda_\mu{}^\nu A_\nu = -\lambda^\nu{}_\mu A_\nu$.

To make sure that we are doing things correctly, we check that indeed the Lagrangian is invariant under Lorentz transformations defined as above:

$$\begin{aligned} \mathcal{A}_s &\doteq \sum_I \frac{\partial L}{\partial \varphi^I} D(\lambda)^I{}_J + \pi^\rho_I D(\lambda)^I{}_J \partial_\rho \varphi^J - \pi^\rho_I \lambda^\mu{}_\rho \partial_\mu \varphi^I \\ &\equiv 0 + F^{\rho\sigma} \lambda^\tau{}_\sigma \partial_\rho A_\tau + F^{\rho\sigma} \lambda^\mu{}_\rho \partial_\mu A_\sigma = F^{\rho\sigma} F_{\rho\tau} \lambda^\tau{}_\sigma = 0 \end{aligned}$$

where the last step follows from the skew symmetry of $\lambda^{\tau\sigma}$.

From this we can compute the spin density current, by replacing $\lambda^\nu{}_\mu$ with a generator $(J_{\alpha\beta})^\nu{}_\mu = 2\delta^\nu_{[\alpha}\eta_{\beta]\mu}$:

$$S^\mu{}_{\alpha\beta} \doteq \pi^\mu_I D(J_{\alpha\beta})^I{}_J \varphi^J \equiv 2F^{\mu\rho}\delta^\nu_{[\alpha}\eta_{\beta]\rho} A_\nu = -2F^\mu{}_{[\alpha} A_{\beta]}$$

Exercise 14.4.4. Show that

$$2t_{[\mu\nu]} + \nabla_\rho S^\rho{}_{[\mu\nu]} = 0.$$

Note that in generality this equality holds only on-shell of the equations of motion. Here, instead, it holds off-shell as well. ◊

Remark 14.4.5. The spin density current, as the canonical stress energy tensor, also fails to be gauge invariant. ◊

The Belinfante stress-energy tensor then is equal to the $\widetilde{t}^{\mu\nu}$ introduced above:

$$\begin{aligned} T^{\mu\nu} &\doteq t^{\mu\nu} + \frac{1}{2}\nabla_\rho(S^{\rho\mu\nu} - S^{\mu\rho\nu} - S^{\nu\rho\mu}) \\ &= -F^{\mu\rho}\nabla^\nu A_\rho - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \nabla_\rho(-F^{\rho[\mu}A^{\nu]} + F^{\mu[\rho}A^{\nu]} + F^{\nu[\rho}A^{\mu]}) \\ &= -F^{\mu\rho}\nabla^\nu A_\rho - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \nabla_\rho(F^{\mu\rho}A^\nu) \\ &= -F^{\mu\rho}F^\nu{}_\rho - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + (\nabla_\rho F^{\mu\rho})A^\nu. \end{aligned}$$

Since physically we are interested only in on-shell quantities, i.e. in quantities evaluated on solutions of the equations of motion, for only these are physically realized, we can without further ado define a *gauge invariant, symmetric, stress energy tensor* which equals the Belinfante stress-energy tensor on-shell:

$$\bar{T}^{\mu\nu} \doteq -F^{\mu\rho}F^\nu{}_\rho - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \hat{=} T^{\mu\nu}.$$

This stress-energy tensor reproduces the expected expression for the energy momentum of the electromagnetic field

$$\bar{P}^0 = \int_\Sigma d^3\vec{x}\ \bar{T}^{00} = \int_\Sigma d^3\vec{x}\ \left(\vec{E}^2 + \frac{1}{2}(-\vec{E}^2 + \vec{B}^2)\right) = \int_\Sigma d^3\vec{x}\ \frac{1}{2}(\vec{E}^2 + \vec{B}^2) = \int_\Sigma d^3\vec{x}\ \mathcal{H}$$

and

$$\bar{P}^i = \int_\Sigma d^3\vec{x}\ \bar{T}^{0i} = \int_\Sigma d^3\vec{x}\ \left(E_j\epsilon^{ij}{}_k B^k\right) = \int_\Sigma d^3\vec{x}\ \mathcal{S}^i$$

where $\vec{\mathcal{S}} \doteq \vec{E} \times \vec{B}$ is the Poynting vector.

Through this, we automatically obtain a gauge invariant, conserved, angular momentum density current through the formula:

$$\bar{M}^{\mu\nu\rho} \doteq T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho.$$

To conclude let's compute the angular momentum and "centre of mass" components of this tensor:

$$\bar{M}^k \doteq \frac{1}{2}\epsilon^{kij}\int_\Sigma d^3\vec{x}\ M^0{}_{ij} = \int_\Sigma d^3\vec{x}\ \epsilon^k{}_{ij}x^i\mathcal{S}^j = \int_\Sigma d^3\vec{x}\ \left(\vec{x} \times \vec{\mathcal{S}}\right)^k$$

and

$$X^i \doteq \bar{M}^{i0} \int_{\Sigma_t} d^3\vec{x}\ M^{0i0} = -\bar{P}^i t + \int_{\Sigma_t} d^3\vec{x}\ \mathcal{H} x^i.$$

These expressions are manifestly gauge invariant. In particular the expression for $\bar{M}^k$ should be compared with that for the "canonical" angular momentum, $M^k = L^k + S^k$ (cf. the previous section), which naturally splits into orbital and spin contributions but fails to be gauge invariant.

double check, especially signs

Exercise 14.4.6. Show that

$$\vec{L}(t) = \int_{\Sigma_t} d^3\vec{x}\ \left(\vec{x} \times (E^j \vec{\nabla} A_j)\right) \quad \text{and} \quad \vec{S}(t) = \int_{\Sigma_t} d^3\vec{x}\ \vec{E} \times \vec{A}. \qquad \Diamond$$

Appendix A

Galileo's thought experiment

From Day 2 of Galileo, Dialogue Concerning the Two Chief World Systems, translated by Stillman Drake, University of California Press, 1953, pp. 186 - 187 (Second Day).

> *Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need to throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they*

will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other. The cause of all these correspondences of effects is the fact that the ship's motion is common to all the things contained in it, and to the air also. That is why I said you should be below decks; for if this took place above in the open air, which would not follow the course of the ship, more or less noticeable differences would be seen in some of the effects noted.

Galileo Galilei (1632)[1]

[1]Galileo, Dialogo sopra i due massimi sistemi del mondo tolemaico e copernicano (1632).