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# Homework 2

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# Contents

# 1 Dynamics on the tangent bundle

(a) We are interested in the description of the dynamics of a set of particles with the language of vector bundles. Our starting point is to take the allowed positions $\mathbf{q}$ to constitute a smooth $n$-manifold $Q$. At each point, $\mathbf{q}$, the tangent space $T_{\mathbf{q}}Q$ is the vector space of directional derivatives $\mathbf{v}$ along trajectories going through $\mathbf{q}$. These derivatives are identified with the velocities allowed at $\mathbf{q}$. The complete description of dynamics is provided by the tangent bundle $TQ$ containing the pairs $(\shortparallel, \gtrapprox)$ describing all instantaneous configurations of the system.

To use the usual analysis of dynamics we use coordinate charts on $Q$ given by the coordinate functions $\{q^i\}_{i=1}^n$. A coordinate chart on $TQ$ can be constructed by appending the components of vectors in the coordinate basis induced by $q^i$ at $\mathbf{q}$ to the coordinates produced by $q^i$. The maps $\{v^i\}_{i=1}^n$ returning the the vector components at $\mathbf{q}$ can be expressed with the dual coordinate basis $\mathrm{d}q^i_{\mathbf{q}}$ through the relation $v^i(\mathbf{q},\mathbf{v}) = \mathrm{d}q^i_{\mathbf{q}}(\mathbf{v})$.

The dynamics of the system is represented by a Lagrangian smooth function $L : TQ \to \mathbb{R}$. The legender transform associate to $L$ is the map between $TQ$ and the cotangent bundle $T^\star Q$ given by $\mathbf{F}L : (\mathbf{q},\mathbf{v}) \mapsto (\mathbf{q}, DL_{\mathbf{q}}(\mathbf{v}))$ where $DL_{\mathbf{q}} : \mathbf{v} \in T_{\mathbf{q}}Q \mapsto \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}q^i_{\mathbf{q}} \in T^\star_{\mathbf{q}}Q$ (with the coordinate representation $\hat{L} = L \circ ((q^i)^{-1}, (v^i)^{-1})$ and $\hat{q}^i = q^i(\mathbf{q},\mathbf{v})$ and $\hat{v}^i = v^i(\mathbf{q},\mathbf{v})$).

Since the Legendre transform provides a smooth map between $TQ$ and $T^\star Q$, we can use it to pull back the canonical symplectic structure on $T^\star Q$ and bring it to $TQ$. This structure is provided by the symplectic potential 1-form $\theta = p_i \mathrm{d}q^i \in T^\star T^\star Q$ where $p_i$ are coordinate functions forming a chart $T^\star Q$ when combined with $q^i$. More precisely, the $p_i$ functions give the components of covectors $\mathbf{p}$ at point $\mathbf{q}$ trough the relation $p_i(\mathbf{q},\mathbf{p}) = \left.\frac{\partial}{\partial q^i}\right|_{\mathbf{q}}(\mathbf{p})$.

The pullback $\theta_L = \mathbf{F}L^\star(\theta) \in T^\star TQ$ of $\theta$ is both linear and commutes with exterior derivatives. Using these properties we can calculate $\theta_L$ by first calculating the pullback of $q^i$ as functions over $TQ$ and then taking the exterior derivative. At $(\mathbf{q},\mathbf{v}) \in TQ$, we have

$$\mathbf{F}L^\star q^i(\mathbf{q},\mathbf{v}) = q^i \circ \mathbf{F}L(\mathbf{q},\mathbf{v}) = q^i(\mathbf{q}, DL_{\mathbf{q}}(\mathbf{v})) = q^i(\mathbf{q},\mathbf{p})$$

and applying an exterior derivatives leads to $\mathbf{F}L^\star \mathrm{d}q^i_{\mathbf{q},\mathbf{p}} = \mathrm{d}(\mathbf{F}L^\star q^i) = \mathrm{d}q^i_{\mathbf{q},\mathbf{v}}$. We note that while $\mathrm{d}q^i \in T^\star Q$ can be evaluated at $\mathbf{q}$, the new $\mathrm{d}q^i$ obtained here is constructed from a function over the bundle $TQ$ and is therefore evaluated at $\mathbf{q},\mathbf{v}$. Then we evaluate the pullback of the functions $p_i$ at $(\mathbf{q},\mathbf{v}) \in TQ$ to be

$$\mathbf{F}L^\star p_i(\mathbf{q},\mathbf{v}) = p_i \circ \mathbf{F}L(\mathbf{q},\mathbf{v}) = p_i(\mathbf{q}, DL_{\mathbf{q}}(\mathbf{v})) = \left.\frac{\partial}{\partial q^i}\right|_{\mathbf{q}} DL_{\mathbf{q}}(\mathbf{v}) = \frac{\partial \hat{L}}{\partial v^j}(\hat{q},\hat{v}) \left.\frac{\partial}{\partial q^i}\right|_{\mathbf{q}} \mathrm{d}q^j_{\mathbf{q}} = \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v}).$$

Combining these results with the linearity of the pullback, we get $\theta_L(\mathbf{q},\mathbf{v}) = \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}q^i_{\mathbf{q},\mathbf{v}}$.

(b) Using again the commutation of pullback and exterior derivative, we obtain the pullback at $(\mathbf{q},\mathbf{v})$ of the symplectic form $\omega = -\mathrm{d}\theta$ by $\mathbf{F}L$ as follows:

$$\begin{aligned}\omega_L(\mathbf{q},\mathbf{v}) &= (\mathbf{F}L^\star \omega)(\mathbf{q},\mathbf{v}) = -(\mathbf{F}L^\star \mathrm{d}\theta)(\mathbf{q},\mathbf{v}) = -\mathrm{d}(\mathbf{F}L^\star \theta)(\mathbf{q},\mathbf{v}) = -\mathrm{d}\left(\frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}q^i_{\mathbf{q},\mathbf{v}}\right) \\ &= \underbrace{-\frac{\partial \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v})}_{B}\mathrm{d}v^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}q^i_{\mathbf{q},\mathbf{v}} + \underbrace{\frac{1}{2}\left(\frac{\partial \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) - \frac{\partial \hat{L}}{\partial q^j \partial v^i}(\hat{q},\hat{v})\right)}_{A}\mathrm{d}q^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}q^i_{\mathbf{q},\mathbf{v}}.\end{aligned}$$

(c) This 2-form is a section on $T^\star TQ$ and we now determine under which condition on $L$ it becomes a symplectic 2-form. In a local basis $\mathrm{d}x^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}x^i_{\mathbf{q},\mathbf{v}}$ with $\{x^i\}_{i=1}^{2n} = \{q^1 \cdots q^n, v^1 \cdots v^n\}$, a symplectif 2-form must be given by $\omega_{i,j}\mathrm{d}x^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}x^i_{\mathbf{q},\mathbf{v}}$ with $\omega_{j,i}$ having non-vanishing determinant as a matrix. Here we have the matrix

$$[\omega_{i,j}] = \begin{pmatrix} A & B \\ -B & 0 \end{pmatrix} \implies \det[\omega_{i,j}] = -\det\begin{pmatrix} B & A \\ 0 & -B \end{pmatrix} = -\det\left[\frac{\partial \hat{L}}{\partial v^j \partial v^i}\right]^2.$$

As long as the determinant of $\left[\frac{\partial \hat{L}}{\partial v^j \partial v^i}\right]^2$ does not vanish the 2-form considered will be non-degenerate. Since $\omega_L$ was computed by taking an exterior derivative of potential, it is exact forcing it to be closed and symplectic if $\left[\frac{\partial \hat{L}}{\partial v^j \partial v^i}\right]$ is regular.

(d) Now supposing $\left[\frac{\partial \hat{L}}{\partial v^j \partial v^i}\right]$ is regular, we have built a symplectic 1-form $\omega_L$ on $\Omega_2(TQ)$. In order to use it to describe dynamics we need a Lagrangian vector field of which the integral curves are the trajectories of the set of particles on $Q$. This vector field is defined trough the energy function $E : (\mathbf{q},\mathbf{v}) \mapsto (DL_{\mathbf{q}}(\mathbf{v}))(\mathbf{v}) - L(\mathbf{q},\mathbf{v})$. To get this energy as a function of coordinate $\hat{q},\hat{v}$ we use the coordinate function $q^i, v^i$ (regrouped in a chart map $\phi$ with $\phi^{-1}$ which return a pair $\phi^{-1}_{\mathbf{q}}(\hat{q},\hat{v}) = \mathbf{q}$ and $\phi^{-1}_{\mathbf{v}}(\hat{q},\hat{v}) = \mathbf{v} \in T_{\mathbf{q}}Q$) to write

$$\hat{E}(\hat{q},\hat{v}) = (E \circ \phi^{-1})(q^i(\mathbf{q},\mathbf{v}), v^i(\mathbf{q},\mathbf{v})) = \left(\frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}q^i_{\mathbf{q}}\right)(\mathbf{v}) - L \circ \phi^{-1}(q^i(\mathbf{q}), v^i(\mathbf{v})) = \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\hat{v}^i - \hat{L}(\hat{q},\hat{v})$$

where we used $DL_{\mathbf{q}}(\mathbf{v}) = DL_{\phi^{-1}_{\mathbf{q}}}\phi^{-1}_{\mathbf{v}} \circ (\hat{q},\hat{v}) = \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}q^i_{\mathbf{q}}$ and applied it to $\mathbf{v}$. By definition, the action of $DL_{\mathbf{q}}(\mathbf{v})$ on $\mathbf{v}$ extracts the $v^i$ component of $\mathbf{v}$ in the coordinate basis. Strictly speaking, to properly precompose with $\phi^{-1}$, we should have considered

$\mathrm{d}q^i_{\phi_{\mathbf{q}^{-1}}} \circ \phi_{\mathbf{v}}^{-1}(\hat{q},\hat{v})$ where $\mathrm{d}q^i_{\phi_{\mathbf{q}^{-1}}} \circ \phi_{\mathbf{v}}^{-1} = \mathrm{d}\hat{q}^i$ is the pullback by the coordinate chart on $Q$ of the 1-form basis (indeed, we can interpret $\phi_v$ as a pushforward of vectors on $TQ \to T\mathbb{R}^n$ since it maps the tangent vector to a curve to the tangent vector of the image of the curve by $\phi_{\mathbf{q}}$ by construction).

(e) From the energy function and symplectic form $\omega_L$, we can define the Lagrangian vector field $X_E$ (section over $TTQ$) by the relation $\omega_L(X_E,\bullet) = \mathrm{d}E$. To use this definition, we work with the decomposition $X_E = X_E^i \left.\frac{\partial}{\partial x^i}\right|_{\mathbf{x}} = X^i_{E,q} \left.\frac{\partial}{\partial q^i}\right|_{\mathbf{q},\mathbf{v}} + X^i_{E,v} \left.\frac{\partial}{\partial v^i}\right|_{\mathbf{q},\mathbf{v}}$ in the coordinate basis of $TTQ$. We also evaluate the coordinate representation of the exterior derivative of $E$ to obtain

$$\begin{aligned}\mathrm{d}\hat{E} &= \frac{\partial^2 \hat{L}}{\partial v^i \partial q^j}(\hat{q},\hat{v})\hat{v}^i \mathrm{d}\hat{q}^j + \frac{\partial^2 \hat{L}}{\partial v^i \partial v^j}(\hat{q},\hat{v})\hat{v}^i \mathrm{d}\hat{v}^j + \frac{\partial \hat{L}}{\partial v^i}(\hat{q},\hat{v})\mathrm{d}\hat{v}^i - \frac{\partial \hat{L}}{\partial v^j}(\hat{q},\hat{v})\mathrm{d}\hat{v}^j - \frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v})\mathrm{d}\hat{q}^j \\ &= \frac{\partial^2 \hat{L}}{\partial v^i \partial q^j}(\hat{q},\hat{v})\hat{v}^i \mathrm{d}\hat{q}^j + \frac{\partial^2 \hat{L}}{\partial v^i \partial v^j}(\hat{q},\hat{v})\hat{v}^i \mathrm{d}\hat{v}^j - \frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v})\mathrm{d}\hat{q}^j.\end{aligned}$$

We note that the form on $T^\star TQ$ is obteined from this coordinate representation by replacing the basis $\mathrm{d}\hat{v}^j, \mathrm{d}\hat{q}^j$ by its pullback $\mathrm{d}q^i_{\mathbf{q},\mathbf{v}}, \mathrm{d}v^i_{\mathbf{q},\mathbf{v}}$ to $T^\star TQ$ by the chart coordinate maps. Applying our symplectic form to $X_E$ in its first entry, we find

$$\begin{aligned}\omega_L(X_E,\bullet) &= \left(X^k_{E,q} \left.\frac{\partial}{\partial q^k}\right|_{\mathbf{q},\mathbf{v}} + X^k_{E,v} \left.\frac{\partial}{\partial v^k}\right|_{\mathbf{q},\mathbf{v}}\right)\left(-\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v})\mathrm{d}v^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}q^i_{\mathbf{q},\mathbf{v}} + \frac{1}{2}\left(\frac{\partial^2 \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) - \frac{\partial^2 \hat{L}}{\partial q^j \partial v^i}(\hat{q},\hat{v})\right)\mathrm{d}q^j_{\mathbf{q},\mathbf{v}} \wedge \mathrm{d}q^i_{\mathbf{q},\mathbf{v}}\right) \\ &= -X^i_{E,v}\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v})\mathrm{d}q^j_{\mathbf{q},\mathbf{v}} + X^i_{E,q}\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v})\mathrm{d}v^j_{\mathbf{q},\mathbf{v}} \\ &+ X^j_{E,q}\frac{1}{2}\left(\frac{\partial^2 \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) - \frac{\partial^2 \hat{L}}{\partial q^j \partial v^i}(\hat{q},\hat{v})\right)\mathrm{d}q^i_{\mathbf{q},\mathbf{v}} - X^i_{E,q}\frac{1}{2}\left(\frac{\partial \hat{L}}{\partial q^j \partial v^i}(\hat{q},\hat{v}) - \frac{\partial^2 \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v})\right)\mathrm{d}q^j_{\mathbf{q},\mathbf{v}} \quad \text{(reindex and cancel)} \\ &= -X^i_{E,v}\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v})\mathrm{d}q^j_{\mathbf{q},\mathbf{v}} + X^i_{E,q}\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v})\mathrm{d}v^j_{\mathbf{q},\mathbf{v}}\end{aligned}$$

Comparing this result with the expression for $\mathrm{d}E$, linear independence leads to the relations

$$\begin{aligned}&\frac{\partial^2 \hat{L}}{\partial v^i \partial v^j}(\hat{q},\hat{v})\hat{v}^i = X^i_{E,q}\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \implies X^i_{E,q} = \hat{v}_i \quad []^{-1} \text{ exists because } \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \text{ is regular} \\ &-X^i_{E,v}\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) = \hat{v}^i\frac{\partial^2 \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) - \frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v}) \quad \text{can solve with } []^{-1} \text{ exists because } \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v}) \text{ is regular.}\end{aligned}$$

(f) We now consider curve $\gamma : U \subset \mathbb{R} \to TQ$ given in the coordinate chart by $\hat{q}(t)$ and $\hat{v}(t)$. This curve represents a trajectory if it is an integral curve of $X_E$: composing the tangent vector to the curve $\frac{\mathrm{d}\gamma}{\mathrm{d}t}$ at the point $\gamma(t)$ with the coordinate functions should return the components of $X_E$ associated to the point. An integral curve has to satisfy

$$\begin{aligned}&\frac{\mathrm{d}\gamma}{\mathrm{d}t}(q^i) = \frac{\mathrm{d}}{\mathrm{d}t}\hat{q}^i(t) = X_E(q^i) = X^i_{E,q} = \hat{v}^i(t) \\ &\frac{\mathrm{d}\gamma}{\mathrm{d}t}(v^i) = \frac{\mathrm{d}}{\mathrm{d}t}\hat{v}^i(t) = X_E(v^i) = X^i_{E,v} \implies -\frac{\mathrm{d}}{\mathrm{d}t}\hat{v}^i(t)\left(\frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v})\right) = \left(\hat{v}^i\frac{\partial^2 \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) - \frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v})\right)\end{aligned}$$

The second equation can be cast in the usual form of the Euler-Lagrange equations with Leibniz's rule in the following way

$$\frac{\partial \hat{L}}{\partial q^j}(\hat{q},\hat{v}) = \frac{\partial^2 \hat{L}}{\partial v^j \partial v^i}(\hat{q},\hat{v})\frac{\mathrm{d}}{\mathrm{d}t}\hat{v}^i(t) + \hat{v}^i\frac{\partial^2 \hat{L}}{\partial q^i \partial v^j}(\hat{q},\hat{v}) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \hat{L}}{\partial v^j}(\hat{q},\hat{v}).$$

# 2 Acknowledgement

I worked on this assignment on my own.

# References

[1] Giuseppe Sellaroli. *Mathematical physics (core course), Lecture 7*. PSI 2023/2024.