Homework 2: 2D Conformal Field Theory and the 1D Transverse-Field Ising Chain

Main deadline: Friday, March 29

Submit ***a single file*** *(.pdf or zip) online using the PSI Portal.*
Attach your answer to question a and your Google Colab notebook with your comments to answer the other questions.
Acknowledge any references you use as well as any other students with whom you collaborate.

In Lecture 5 in this course, we have seen a very deep connection between the critical point of the 1D TFIM and the conformal symmetry in 2D. Conformal field theory (CFT) is ubiquitous in modern theoretical physics. It describes fixed points of the renormalization group flow, making it central to our understanding of quantum field theory. It is also a core component both of string theory and of the AdS/CFT correspondence of quantum gravity. In condensed matter, as well as in statistical mechanics, continuous phase transitions can often be understood in terms of an underlying CFT that describes their universal, long-distance/low-energy physics. Based on a previous work (See Ref [1]), in this homework we develop tools to investigate the emergence of conformal symmetry in the TFIM at criticality.

We first need to recall two well-known facts about CFTs in two spacetime dimensions. (i) On the plane, parameterized by a complex coordinate $z = x + iy$, a CFT contains infinitely many scaling operators $\varphi_\alpha(z)$. These are fields that transform covariantly under a rescaling of the plane by a factor $\lambda > 0$ or a rotation by an angle $\theta \in [0, 2\pi)$:

$$\begin{aligned} z \to \lambda z \quad &(\text{rescaling}) \quad \Leftrightarrow \quad \varphi_\alpha(z) \to \lambda^{-\Delta_\alpha}\, \varphi_\alpha(\lambda z), \\ z \to e^{\theta} z \quad &(\text{rotation}) \quad \Leftrightarrow \quad \varphi_\alpha(z) \to e^{-\theta S_\alpha}\, \varphi_\alpha(\lambda z), \end{aligned} \tag{1}$$

where Δ_α and S_α are the *scaling dimension* and *conformal spin* of $\varphi_\alpha(z)$. Scaling operators are organized into *conformal towers*, each consisting of a Virasoro *primary* operator and its *descendants*. (ii) The operator-state correspondence establishes that for each scaling operator φ_α there is an eigenstate $|\varphi_\alpha\rangle$ of the CFT Hamiltonian H^{CFT} on the circle, with energy gap and momentum given by

$$E_\alpha^{\scriptscriptstyle CFT} = \frac{2\pi}{L}\left(\Delta_\alpha - \frac{c}{12}\right), \qquad P_\alpha^{\scriptscriptstyle CFT} = \frac{2\pi}{L} S_\alpha, \tag{2}$$

where L is the length of the circle and c is the *central charge* of the CFT, which determines the Casimir energy. Recall that in tutorial 1, we have verified the $1/L$ scaling of the gap using exact diagonalization (ED). The scaling dimensions, conformal spins and *operator product expansion* (OPE) coefficients (three-point correlators) of the primary operators, together with the central charge, fully characterize the CFT and are referred to as *conformal data.* To simplify the discussion, in this Homework, we will not extract the conformal spins and will focus on the scaling dimensions Δ_α and the central charge c of the Ising CFT.

To illustrate these ideas, let us consider the 1D transverse field Ising model (TFIM) at the critical point with Hamiltonian

$$\hat{H} = -\sum_{i=1}^{L} Z_i Z_{i+1} - \sum_{i=1}^{L} X_i,$$

and periodic boundary conditions, $Z_{L+1} = Z_1$. Interestingly, the conformal symmetry at the Ising transition gives rise to an interesting structure where starting from a set of states called primary states $|\phi\rangle$. We can build the so-called descendants $|\phi'\rangle$ states using ladder operators to obtain the full spectrum of the 1D TFIM at criticality, similarly to how we can build the spectrum of the quantum Harmonic oscillator.

Ref. [1] demonstrated that the operators connecting the primary states of the 1D TFIM at criticality to their descendants correspond to the Fourier modes of the Hamiltonian $\hat{H}$ defined as:

$$\hat{H}_n = -\frac{N}{2\pi}\sum_{j=1}^{L} \exp\left(\mathrm{i}(j+1/2)n\frac{2\pi}{N}\right) Z_j Z_{j+1} - \frac{N}{2\pi}\sum_{j=1}^{L} \exp\left(\mathrm{i}jn\frac{2\pi}{N}\right) X_j.$$

Ref. [1] also defined the operators

$$\hat{O}_n = \frac{\hat{H}_n + \hat{H}_{-n}}{2},$$

which has lower finite-size effects and has a direct relationship with Virasoro operators $L_n^{\scriptscriptstyle CFT}$ and $\overline{L}_n^{\scriptscriptstyle CFT}$:

$$\hat{O}_n = \frac{L_n^{\scriptscriptstyle CFT} + \overline{L}_{-n}^{\scriptscriptstyle CFT} + L_{-n}^{\scriptscriptstyle CFT} + \overline{L}_n^{\scriptscriptstyle CFT}}{2} \qquad \text{for } n > 0.^1$$

[1] See Ref. [1] if curious about the details of Virasoro operators. You do not need to know the algebra of these operators to complete this Homework.

Let us first identify the primary states of the Ising CFT. Primary states are those states annihilated by all ladder operators that reduce the energy: $L_n^{\scriptscriptstyle CFT}|\varphi\rangle = \overline{L}_n^{\scriptscriptstyle CFT}|\varphi\rangle = 0$ for all $n > 0$. It is known that $L_{-1}^{\scriptscriptstyle CFT}, L_{-2}^{\scriptscriptstyle CFT}$ generate the subalgebra $L_n^{\scriptscriptstyle CFT}$ for $n < 0$ (and similar for $\overline{L}_n^{\scriptscriptstyle CFT}$, see Ref. [1]). Thus the previous condition is equivalent to:

$$|\varphi\rangle \text{ primary } \Leftrightarrow L_n^{\scriptscriptstyle CFT}|\varphi\rangle = \overline{L}_n^{\scriptscriptstyle CFT}|\varphi\rangle = 0, \quad n = 1, 2. \tag{3}$$

By acting with products of powers of $L_n^{\scriptscriptstyle CFT}$, $\overline{L}_n^{\scriptscriptstyle CFT}$ with $n < 0$ on a primary $|\varphi\rangle$, all *descendant states* in its tower can be reached. Descendants $|\varphi'\rangle$ of a primary $|\varphi\rangle$ must have scaling dimension $\Delta_{\varphi'}$ given by

$$\Delta_{\varphi'} = \Delta_\varphi + n, \quad \text{for } n \geq m, \tag{4}$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Furthermore, it follows from (3) that all descendants can be reached from the primary using only $L_{-n}^{\scriptscriptstyle CFT}$, $\overline{L}_{-n}^{\scriptscriptstyle CFT}$ with $n = 1, 2$.

Let Γ_φ be a projector onto all the eigenstates with energy smaller than E_φ,

$$\Gamma_\varphi \equiv \sum_{\varphi_\alpha : E_\alpha < E_\varphi} |\varphi_\alpha\rangle \langle\varphi_\alpha| \,. \tag{5}$$

a) Show we can recast the characterization (3) of a primary state as

$$|\varphi\rangle \text{ primary } \Leftrightarrow \Gamma_\varphi \, O_n|\varphi\rangle = 0, \quad n = 1, 2. \tag{6}$$

To numerically identify primary states, we define $\epsilon^{(n)}$ to be the norm of the matrix elements of O_n that connect an energy eigenstate $|\varphi\rangle$ with states of lower energy:

$$\epsilon_\varphi^{(n)} \equiv |\Gamma_\varphi \, O_n \, |\varphi\rangle| \,, \quad \text{for } n = 1, 2. \tag{7}$$

We then define a primary candidate as a state with small $\epsilon^{(1)}$ and $\epsilon^{(2)}$:

$$|\varphi\rangle \text{ primary candidate} \Leftrightarrow \epsilon_\varphi^{(1)} + \epsilon_\varphi^{(2)} \leq \epsilon_{\max},$$

which is analogous to (6) for $\epsilon_{\max} = 0$.

b) To determine the primary states using this criterion, we will use the sparse ED code that was provided for Tutorials 1 and 2 in this course on Google Colab. Hereafter, we will take a system size $L = 16$ and extract the first 22 lowest energy states. Extract the three primaries $|I\rangle, |\sigma\rangle$ and $|\epsilon\rangle$ (out of the 22 eigenstates) which have energies in increasing order respectively using $\epsilon_{\max} = 10^{-12}$ as a threshold.

c) The Ising Hamiltonians $\hat{H}$ is related to CFT Hamiltonian H^{CFT} through the following transformation:

$$\hat{H} = aH^{CFT} + b,$$

where a and b are constants. Numerically apply the operator $\hat{O}_2$ on the ground state $|I\rangle$ and renormalize it to obtain a descendent state. Use the fact that $\Delta_2^I = 2$ to determine the constant a and to renormalize the Hamiltonian $\hat{H}$ in your code.

Hint: you can use `O2.dot(state)` to perform the operator state multiplication and `np.linalg.norm(state)` to calculate the L_2 norm.

d) After the previous rescaling step, numerically show that $\Delta^\sigma \approx 1/8$ and $\Delta^\epsilon \approx 1$.

e) Use the values you obtained in the previous question to determine the critical exponents $\alpha, \beta, \delta, \eta, \nu$ of the Ising CFT in 1+1d. You can use the formula in `https://en.wikipedia.org/wiki/Ising_critical_exponents` in case you have not seen them before. Compare your estimates with the exact results. How do your critical exponents' estimates improve with increasing the size L in your ED code?

Note: to speed up your code, you can use $k = 3$ instead of $k = 22$ in the sparse diagonalization `sp.linalg.eigsh(H,k, which='SA')` for larger system sizes up to $N = 20$.

f) Numerically verify that you generate descendent states from the three primary states, with scaling dimensions $\Delta_n^I = n$, $\Delta_n^\epsilon = n+1$, and $\Delta_n^\sigma = n+1/8$, using the operators O_1 and O_2 (except for O_1 applied on $|I\rangle$ which has large finite size effects). If you observe some discrepancies (up to errors of about 0.1), you can attribute them to finite-size effects. Compare the energies of the descendent states to the energy spectrum obtained by ED, what do you observe?

g) Using the formula $c \approx 2 \langle I| H_2^\dagger H_2 |I\rangle$ provided in Ref. [1], extract the central charge c and demonstrate that $c = 0.5$ is expected in the thermodynamic limit through a finite-size scaling. Note that you have to rescale H_2 in your code with the constant a obtained in question c to obtain the correct answer.

[1] Ashley Milsted, Guifre Vidal, "Extraction of conformal data in critical quantum spin chains using the Koo-Saleur formula", `https://arxiv.org/pdf/1706.01436.pdf`.