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HOMework 1

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1 The Poincaré Algebra

- (a) The Poincaré is the group of transformation of minkowski space that preserve the spacetime interval between all events. This group contains spacetime translations and Lorentz transformation (boosts and rotations). In a coordinate system where events happenning at x with four-coordinate x^μ , translation by a constant four-vector a with components a^μ reads $x' = x + a$ ($x'^\mu = x^\mu + a^\mu$). The lorentz transformation Λ with components Λ^ν_μ act as $x' = \Lambda x$ ($x'^\mu = \Lambda^\mu_\nu x^\nu$, following the matrix multiplication convention x^ν can be written as a column with ν as a row index and Λ^μ_ν as a square matrix with μ row index and ν column index). We want to find the characteristic of the unitary operator U representing Poincaré transformation near the identity δ (with components δ^μ_ν). To do this, we write the first order Taylor expansions $\Lambda = \delta + \omega + O(\omega^2)$ and $a = \varepsilon$ (exact even for large ε) with respect to an infinitesimal Lorentz shift ω with components $\omega_{\mu\nu}$ (combining infinitesimal rotation angles, and boost angles) and translation ε with components ε^μ . The first order in ω and ε expansion of the unitary is $U(\delta + \omega, \varepsilon) = \mathbf{1} + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\varepsilon_\mu P^\mu + O(\omega^2, \varepsilon^2)$ where $J^{\mu\nu}$, P^μ are the hermitian matrices generating the Poincaré transformation. Since $\Lambda = \delta + \omega$ is a lorentz transformation, we have that it preserves space time intervals. The spacetime interval between events x and y is $(x_\mu - y_\mu)(x^\mu - y^\mu) = x_\mu x^\mu + y_\mu y^\mu - 2y_\mu x^\mu$. Since the first two terms are themselves spacetime intervals between x, y and 0, they are individually preserved by a Poincaré transformation. This forces the invariance of the lorentzian product $y_\mu x^\mu$ for any x, y under Poincaré transformations ($x^\mu y_\mu = x'^\mu y'_\mu$).

In general, we can Taylor expand $x'^\mu y'_\mu$ around $x^\mu y_\mu$ in powers of ω as

$$x'^\mu y'_\mu = (x^\mu + \omega^{\mu\sigma} x_\sigma + O(\omega^2))(y_\mu + \omega_{\mu\nu} y^\nu + O(\omega^2)) = x^\mu y_\mu + \omega_{\mu\nu} x^\mu y^\nu + \omega^{\mu\sigma} x_\sigma y_\mu + O(\omega^2)$$

but we know that length is preserved, and the unique Taylor series stops at $O(1)$ forcing all other orders to vanish. This implies that $0 = \omega_{\mu\nu} x^\mu y^\nu + \omega_{\mu\nu} x^\nu y^\mu = (\omega_{\mu\nu} + \omega_{\nu\mu}) x^\mu y^\nu$. This finally implies that $\omega_{\mu\nu} = -\omega_{\nu\mu}$ since it is true for any x, y which shows $\omega_{\mu\nu}$ is antisymmetric.

- (b) The unitary U is meant to represent a Poincaré symmetry transformation of the quantum states of a module Hilbert space. In quantum mechanics, A symmetry transformation preserves all the statistical properties of observables O . This is equivalent to saying that for any two states $|\phi\rangle$ and $|\psi\rangle$ the quantities $\langle\phi|O|\psi\rangle$ are left unchanged by the symmetry. We have the following transformation of states $|\psi'\rangle = U|\psi\rangle$ and $\langle\phi'| = \langle\phi|U^\dagger$. The transformed O' operator is such that

$$\langle\phi|O|\psi\rangle = \langle\phi'|O'|\psi'\rangle = \langle\phi|U^\dagger O' U|\psi\rangle, \quad \forall |\phi\rangle, |\psi\rangle \iff O = U^\dagger O' U \iff O' = U O U^\dagger.$$

- (c) Following the result of the previous item, we take $O = U(\delta + \omega, \varepsilon)$ and compute the operator O' associated to the general $U(\Lambda, a)$ unitary Poincaré transformation representing the combined lorentz and translation transformation $T(\Lambda, a)$. With the same notation, we write $T(\delta + \omega, \varepsilon)$ to reference the infinitesimal Poincaré transformation. Because the representation is an homomorphism, we have

$$O' = U(\Lambda, a)U(T(\delta + \omega, \varepsilon))U^\dagger(\Lambda, a) = U(T(\Lambda, a)T(\delta + \omega, \varepsilon)T^{-1}(\Lambda, a)).$$

To make this expression more precise, we look for Λ' and a' such that $T^{-1}(\Lambda, a) = T(\Lambda', a')$. Acting with the identity on an arbitrary four-vector x leads to

$$\begin{aligned} x &= T^{-1}(\Lambda, a)T(\Lambda, a)x = T(\Lambda', a')T(\Lambda, a)x = \Lambda'(\Lambda x + a) + a' \\ &= \Lambda' \Lambda x + \Lambda' a + a', \quad \forall x \iff \Lambda' \Lambda = 1 \text{ \& } \Lambda' a + a' = 0 \end{aligned}$$

With these relations in hand, the action on x of the product Poincaré transformation represented by O' is expanded as follows:

$$\begin{aligned} T(\Lambda, a)T(\delta + \omega, \varepsilon)T^{-1}(\Lambda, a)x &= T(\Lambda, a)T(\delta + \omega, \varepsilon)(\Lambda^{-1}x - \Lambda^{-1}a) \\ &= T(\Lambda, a)((\Lambda^{-1}x - \Lambda^{-1}a) + \omega(\Lambda^{-1}x - \Lambda^{-1}a) + \varepsilon) \\ &= T(\Lambda, a)((\Lambda^{-1} + \omega\Lambda^{-1})x - \omega\Lambda^{-1}a - \Lambda^{-1}a + \varepsilon) \\ &= ((\delta + \Lambda\omega\Lambda^{-1})x - \Lambda\omega\Lambda^{-1}a - a + \Lambda\varepsilon + a) = T(\delta + \Lambda\omega\Lambda^{-1}, -\Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon)x. \end{aligned}$$

This relation holds for all x and we can finally write $O' = U(\delta + \Lambda\omega\Lambda^{-1}, -\Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon)$.

- (d) Combining the hermitian generator expansion of $U(\delta + \omega, a)$ (we omit Landau order notation in the next calculations) given in item (a) to the expression for the transformed operator O' of item (c), we find

$$\begin{aligned}
O' &= U(\Lambda, a)U(\delta + \omega, \varepsilon)U^\dagger(\Lambda, a) \tag{*} \\
&= 1 + \frac{i}{2}\omega_{\mu\nu}U(\Lambda, a)J^{\mu\nu}U^\dagger(\Lambda, a) + i\varepsilon_\mu U(\Lambda, a)P^\mu U^\dagger(\Lambda, a) \\
&= 1 + \frac{i}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu}J^{\mu\nu} + i(-\Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon)_\mu P^\mu \\
&= 1 + \frac{i}{2}(\Lambda_\mu{}^\rho\omega_{\rho\sigma}\Lambda_\nu{}^\sigma)J^{\mu\nu} + i(-\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma a^\sigma\omega_{\rho\sigma} + \Lambda_\mu{}^\nu\varepsilon_\nu)P^\mu \\
&= 1 + \frac{i}{2}(\Lambda_\rho{}^\mu\omega_{\mu\nu}\Lambda_\sigma{}^\nu)J^{\rho\sigma} + i(-\Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu a^\sigma\omega_{\mu\nu}P^\rho + \varepsilon_\mu\Lambda_\nu{}^\mu P^\nu) \\
&= 1 + \frac{i}{2}(\Lambda_\rho{}^\mu\omega_{\mu\nu}\Lambda_\sigma{}^\nu)J^{\rho\sigma} + i\varepsilon_\mu\Lambda_\nu{}^\mu P^\nu + \frac{i}{2}(-\Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu a^\sigma\omega_{\mu\nu}P^\rho + \Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu a^\sigma\omega_{\nu\mu}P^\rho) \\
&= 1 + \frac{i}{2}(\Lambda_\rho{}^\mu\omega_{\mu\nu}\Lambda_\sigma{}^\nu)J^{\rho\sigma} + i\varepsilon_\mu\Lambda_\nu{}^\mu P^\nu + \frac{i}{2}(-\Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu a^\sigma\omega_{\mu\nu}P^\rho + \Lambda_\sigma{}^\nu\Lambda_\rho{}^\mu a^\rho\omega_{\mu\nu}P^\sigma)
\end{aligned}$$

where we expanded the result of item (c) at $O(\omega, \varepsilon)$ in the third line and used antisymmetry of $\omega_{\mu\nu}$ in the second last line. To obtain the component representation of the previous result, the vector/matrix multiplication was written and then converted to appropriated index structure:

$$\begin{aligned}
(-\Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon)^\mu &= -\Lambda^\mu{}_\rho\omega^\rho{}_\sigma(\Lambda^{-1})^\sigma{}_\nu a^\nu + \Lambda^\mu{}_\nu\varepsilon^\nu \\
\iff (-\Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon)_\mu &= -\Lambda_\mu{}^\rho\omega_{\rho\sigma}(\Lambda^{-1})^\sigma{}_\nu a^\nu + \Lambda_\mu{}^\nu\varepsilon_\nu = -\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma a_\nu\omega_{\rho\sigma} + \Lambda_\mu{}^\nu\varepsilon_\nu \\
(\Lambda\omega\Lambda^{-1})^\mu{}_\nu &= \Lambda^\mu{}_\rho\omega^\rho{}_\sigma(\Lambda^{-1})^\sigma{}_\nu \\
\iff (\Lambda\omega\Lambda^{-1})_{\mu\nu} &= \eta_{\mu\lambda}(\Lambda\omega\Lambda^{-1})^\lambda{}_\nu = \Lambda_\mu{}^\rho\omega_{\rho\sigma}(\Lambda^{-1})^\sigma{}_\nu = \Lambda_\mu{}^\rho\omega_{\rho\sigma}\Lambda_\nu{}^\sigma.
\end{aligned}$$

The inverse transformation components could be related to the direct components because the Lorentz matrices preserve the Lorentzian product of arbitrary x, y . Indeed, this property implies

$$\begin{aligned}
\eta_{\rho\sigma}x^\rho y^\sigma &= \eta_{\mu\nu}(\Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma y^\sigma), \quad \forall x, y \iff \eta_{\rho\sigma} = \eta_{\mu\nu}(\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma) \\
\iff \eta_{\rho\sigma}(\Lambda^{-1})^\sigma{}_\lambda &= \eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma (\Lambda^{-1})^\sigma{}_\lambda = \eta_{\mu\lambda}\Lambda^\mu{}_\rho \iff (\Lambda^{-1})^\nu{}_\lambda = \eta^{\nu\rho}\eta_{\rho\sigma}(\Lambda^{-1})^\sigma{}_\lambda = \eta_{\mu\lambda}\eta^{\nu\rho}\Lambda^\mu{}_\rho = \Lambda_\lambda{}^\nu.
\end{aligned}$$

Equality of the second and last lines of (*) for all ω, ε ensures that the tensors contracted with ω and ε are equal. Regrouping terms proportionnal to $\omega_{\mu\nu}$ and ε_μ We have

$$\begin{aligned}
U(\Lambda, a)J^{\mu\nu}U^\dagger(\Lambda, a) &= \Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu(J^{\rho\sigma} + a^\rho P^\sigma - a^\sigma P^\rho), \\
U(\Lambda, a)P^\mu U^\dagger(\Lambda, a) &= \Lambda_\rho{}^\mu P^\rho.
\end{aligned}$$

- (e) To extract the commutation relations of the generators $J^{\mu\nu}, P^\mu$, we start by setting $\Lambda = \delta + \omega, a = \varepsilon$ in the final result of item (d). At first order in ω, ε , we get

$$\begin{aligned}
P^\mu + \eta^{\mu\sigma}\omega_{\rho\sigma}P^\rho &= \Lambda_\rho{}^\mu P^\rho = U(\Lambda, a)P^\mu U^\dagger(\Lambda, a) = \left(1 + \frac{i}{2}\omega_{\lambda\varepsilon}J^{\lambda\varepsilon} + i\varepsilon_\nu P^\nu\right)P^\mu \left(1 - \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} - i\varepsilon_\sigma P^\sigma\right) \\
&= P^\mu + \frac{i}{2}(\omega_{\lambda\varepsilon}J^{\lambda\varepsilon}P^\mu - P^\mu\omega_{\rho\sigma}J^{\rho\sigma}) + i(\varepsilon_\nu P^\nu P^\mu - P^\mu\varepsilon_\nu P^\nu) \\
&= P^\mu + \frac{i}{2}\omega_{\rho\sigma}[J^{\rho\sigma}, P^\mu] + i\varepsilon_\nu[P^\nu, P^\mu].
\end{aligned}$$

where $U^\dagger(\Lambda, a) = U((\delta + \omega)^{-1}, -(\delta + \omega)^{-1}a) = U(\delta - \omega, -\varepsilon + \omega\varepsilon) = U(\delta - \omega, -\varepsilon)$ was used. Invoquing the validity of the last set of equalities for all ω, ε , we find:

$$[P^\nu, P^\mu] = 0 \quad \& \quad \frac{i}{2}[J^{\rho\sigma}, P^\mu] = \frac{1}{2}(\eta^{\mu\sigma}P^\rho - \eta^{\mu\rho}P^\sigma).$$

with the antisymmetry of ω used to write

$$2\eta^{\mu\sigma}\omega_{\rho\sigma}P^\rho = \eta^{\mu\sigma}\omega_{\rho\sigma}P^\rho - \eta^{\mu\sigma}\omega_{\sigma\rho}P^\rho = \eta^{\mu\sigma}\omega_{\rho\sigma}P^\rho - \eta^{\mu\rho}\omega_{\rho\sigma}P^\sigma.$$

The expanded transformation of $J^{\mu\nu}$ reads

$$\begin{aligned}
J^{\mu\nu} + \frac{i}{2}\omega_{\rho\sigma}[J^{\rho\sigma}, J^{\mu\nu}] + O(\varepsilon) &= \left(1 + \frac{i}{2}\omega_{\lambda\varepsilon}J^{\lambda\varepsilon} + i\varepsilon_\alpha P^\alpha\right)J^{\mu\nu}\left(1 - \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} - i\varepsilon_\sigma P^\sigma\right) \\
&= U(\Lambda, a)J^{\mu\nu}U^\dagger(\Lambda, a) \\
&= \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu (J^{\rho\sigma} + \varepsilon^\rho P^\sigma - \varepsilon^\sigma P^\rho) \\
&= (\delta_\rho{}^\mu + \omega_\rho{}^\mu)(\delta_\sigma{}^\nu + \omega_\sigma{}^\nu)J^{\rho\sigma} + O(\varepsilon) \\
&= J^{\mu\nu} + J^{\mu\sigma}\omega_\sigma{}^\nu + J^{\rho\nu}\omega_\rho{}^\mu + O(\varepsilon) \\
&= J^{\mu\nu} + \frac{1}{2}\omega_{\rho\sigma}(\eta^{\nu\sigma}J^{\mu\rho}\omega_{\rho\sigma} - \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\sigma}J^{\rho\nu} - \eta^{\mu\rho}J^{\sigma\nu}) + O(\varepsilon)
\end{aligned}$$

The expansion in ε was not explicited because it only serves to determine $[J^{\rho\sigma}, P^\mu]$ which is already known at that point. Finally, using the fact η is symmetric and ω arbitrary, we get

$$\frac{i}{2}[J^{\rho\sigma}, J^{\mu\nu}] = \frac{1}{2}(\eta^{\nu\sigma}J^{\mu\rho} - \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\sigma\mu}J^{\rho\nu} - \eta^{\rho\mu}J^{\sigma\nu}).$$

- (f) We now define the angular momentum vector $\mathbf{J} = (J^{23}, J^{31}, J^{12}) \equiv (J^1, J^2, J^3)$. We can also write $\varepsilon_{ij}{}^k J^{ij} = J^k$ where $-\varepsilon_{ij}{}^k = \varepsilon_{ijk}$ is the levi-civita tensor with one up index in our mostly minus signature. The commutation relation of these operators is given by

$$[J^k, J^n] = \varepsilon_{ij}{}^k \varepsilon_{lm}{}^n [J^{ij}, J^{lm}] = -i\varepsilon_{ij}{}^k \varepsilon_{lm}{}^n (\eta^{vj}J^{\mu i} - \eta^{vi}J^{\mu j} + \eta^{j\mu}J^{i\nu} - \eta^{i\mu}J^{j\nu})$$

(g)

2 Acknowledgement
