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HOMEWORK 1

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Contents

1 Cartan in a FLRW universe

(a) The Friedmann-Lemaitre-Robinson-Walker (FLRW) metric two-form describes a spacetime with spacelike foliation in homogeneous and isotropic hypersurfaces. In a coordinate chart with coordinates $x^\mu = \{t,\theta,\phi,r\}$ making the isotropy and foliation manifest, this metric reads
$$g_{\mu\nu}\underline{d}x^\mu \otimes \underline{d}x^\nu \equiv \underline{d}t\otimes\underline{d}t - a^2(t)\left(\frac{\underline{d}r\otimes\underline{d}r}{1-kr^2} + r^2\left(\underline{d}\theta\otimes\underline{d}\theta + \sin^2\theta\underline{d}\phi\otimes\underline{d}\phi\right)\right)$$
where $\{\underline{d}x^\mu\}_{\mu=0}^3 = \{\underline{d}t,\underline{d}\theta,\underline{d}\phi,\underline{d}r\}$ are the coordinate on-forms dual to the vector basis $\underline{e}_a = \{\partial_t,\partial_\theta,\partial_\phi,\partial_r\}$, $a(t)>0$ is the scale factor and $k=0,-1,1$ gives the sign of the curvature of the spacelike hypersurfaces (respectively flat, Anti-de Sitter, de Sitter). In what follows, the tensor products are implicit. At every point in our chart, we define an orthonormal basis of one-forms $\underline{\omega}^a = c^a_\mu\underline{d}x^\mu$ such that $g_{\mu\nu}\underline{d}x^\mu\underline{d}x^\nu = \eta_{ab}\underline{\omega}^a\underline{\omega}^b$ where η_{ab} is the Minkowski metric components with signature $(+,-,-,-)$. We can write
$$\begin{aligned} g_{\mu\nu}\underline{d}x^\mu\underline{d}x^\nu &= \underline{d}t\underline{d}t - \left(\frac{a(t)\underline{d}r}{\sqrt{1-kr^2}}\right)\left(\frac{a(t)\underline{d}r}{\sqrt{1-kr^2}}\right) - \left(a(t)r\underline{d}\theta\right)\left(a(t)r\underline{d}\theta\right) - (a(t)r\sin\theta\underline{d}\phi)(a(t)r\sin\theta\underline{d}\phi) \\ &= \underline{\omega}^0\underline{\omega}^0 - \underline{\omega}^1\underline{\omega}^1 - \underline{\omega}^2\underline{\omega}^2 - \underline{\omega}^3\underline{\omega}^3 \end{aligned}$$
where $\{\underline{\omega}^a\}_{a=0}^3 = \{\underline{d}t,\ a(t)r\underline{d}\theta,\ a(t)r\sin\theta\underline{d}\phi,\ \frac{a(t)}{\sqrt{1-kr^2}}\underline{d}r\}$ is shown to satisfy the orthonormality condition. We note that the resulting choice of basis is unique up to a local lorentz transformation (which preserves orthonormality).

(b) To calculate the connection one-forms $\underline{\theta}^a{}_b$, we use the orthonormal basis found in (a) and Cartan's first structure equation for vanishing torsion to get
$$\underline{\theta}^a{}_b\wedge\underline{\omega}^b = -\underline{d}\underline{\omega}^a = \begin{cases} -\partial_\mu(1)\ \underline{d}x^\mu\wedge\underline{d}t \\ -\partial_\mu(a(t)r)\ \underline{d}x^\mu\wedge\underline{d}\theta \\ -\partial_\mu(a(t)r\sin\theta)\ \underline{d}x^\mu\wedge\underline{d}\phi \\ -\partial_\mu\left(\frac{a(t)}{\sqrt{1-kr^2}}\right)\ \underline{d}x^\mu\wedge\underline{d}r \end{cases} = \begin{cases} 0 \\ -a'(t)r\underline{d}t\wedge\underline{d}\theta - a(t)\underline{d}r\wedge\underline{d}\theta \\ -a'(t)r\sin\theta\underline{d}t\wedge\underline{d}\phi - a(t)\sin\theta\underline{d}r\wedge\underline{d}\phi - a(t)r\cos\theta\underline{d}\theta\wedge\underline{d}\phi \\ -\frac{a'(t)}{\sqrt{1-kr^2}}\underline{d}t\wedge\underline{d}r - [\cdots]\underline{d}r\wedge\underline{d}r \end{cases}$$
$$= \begin{cases} 0 \\ \frac{a'(t)}{a(t)}\underline{\omega}^1\wedge\underline{\omega}^0 + \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1\wedge\underline{\omega}^3 \\ \frac{a'(t)}{a(t)}\underline{\omega}^2\wedge\underline{\omega}^0 + \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2\wedge\underline{\omega}^3 + \frac{1}{a(t)r}\cot\theta\underline{\omega}^2\wedge\underline{\omega}^1 \\ \frac{a'(t)}{a(t)}\underline{\omega}^3\wedge\underline{\omega}^0 \end{cases} = \begin{cases} \underline{\theta}^0{}_b\wedge\underline{\omega}^b \\ \underline{\theta}^1{}_b\wedge\underline{\omega}^b \\ \underline{\theta}^2{}_b\wedge\underline{\omega}^b \\ \underline{\theta}^3{}_b\wedge\underline{\omega}^b \end{cases}$$
Since the $\wedge$ product with $\underline{\omega}^b$ maps $\underline{\omega}^{c\neq b}$ to linearly independant two-forms, we can read the coefficients of $\underline{\omega}^{c\neq b}$ preceeding the $\wedge$ product in the previous expressions. We have
$$\begin{cases} \underline{\theta}^0{}_1 = [\cdots]\underline{\omega}^1,\quad \underline{\theta}^0{}_2 = [\cdots]\underline{\omega}^2,\quad \underline{\theta}^0{}_3 = [\cdots]\underline{\omega}^3 \\ \underline{\theta}^1{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^1 + [\cdots]\underline{\omega}^0,\quad \underline{\theta}^1{}_2 = [\cdots]\underline{\omega}^2,\quad \underline{\theta}^1{}_3 = \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1 + [\cdots]\underline{\omega}^3 \\ \underline{\theta}^2{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^2 + [\cdots]\underline{\omega}^0,\quad \underline{\theta}^2{}_3 = \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2 + [\cdots]\underline{\omega}^3,\quad \underline{\theta}^2{}_1 = \frac{1}{a(t)r}\cot\theta\underline{\omega}^2 + [\cdots]\underline{\omega}^1 \\ \underline{\theta}^3{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^3 + [\cdots]\underline{\omega}^0,\quad \underline{\theta}^3{}_1 = [\cdots]\underline{\omega}^1,\quad \underline{\theta}^3{}_2 = [\cdots]\underline{\omega}^2 \end{cases}$$
where $[\cdots]$ terms represent the terms mapped to 0 by the $\wedge$ product from which information about $\underline{\theta}^a{}_b$ was read. From the first line we can also read $\underline{\theta}^0{}_{1,2,3} = [\cdots]\underline{\omega}^{1,2,3}$

To fully determine the one-forms components from these relations, we invoke the relation $\underline{\theta}_{ab} + \underline{\theta}_{ba} = \underline{d}g_{ab}$ where $\underline{\theta}_{ba} = g_{bc}\underline{\theta}^c{}_a$. Recalling that in our orthonormal basis $g_{ab} = \eta_{ab}$, we get the antisymmetry relation $\underline{\theta}_{ab} + \underline{\theta}_{ba}$. It follows that $\underline{\theta}^a{}_a = 0,\ \forall a$ and we can use it to determine [...]. Expliciting the relation between $\underline{\theta}^b{}_a$ and $\underline{\theta}^a{}_b$ yields
$$\begin{cases} b \text{ spacelike} \implies \underline{\theta}^b{}_a = \eta^{bc}\underline{\theta}_{ca} = (-1)\underline{\theta}_{ba} = \underline{\theta}_{ab} \implies \begin{cases} a \text{ spacelike} \implies \underline{\theta}^b{}_a = -\underline{\theta}^a{}_b \\ a \text{ timelike} \implies \underline{\theta}^b{}_a = \underline{\theta}^a{}_b \end{cases} \\ b \text{ timelike} \implies \underline{\theta}^b{}_a = \eta^{bc}\underline{\theta}_{ca} = \underline{\theta}_{ba} = -\underline{\theta}_{ab} \implies \begin{cases} a \text{ spacelike} \implies \underline{\theta}^b{}_a = \underline{\theta}^a{}_b \\ a \text{ timelike} \implies \underline{\theta}^b{}_a = -\underline{\theta}^a{}_b \quad \text{never happens } (a\neq b) \end{cases} \end{cases}$$
Comparing $\underline{\theta}^a{}_b$ with $\underline{\theta}^b{}_a$, we finally see
$$[\cdots]\underline{\omega}^1 = \underline{\theta}^0{}_1 = \underline{\theta}^1{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^1 + [\cdots]\underline{\omega}^0 \iff \underline{\theta}^1{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^1,\quad \underline{\theta}^0{}_1 = \frac{a'(t)}{a(t)}\underline{\omega}^1$$
$$[\cdots]\underline{\omega}^2 = \underline{\theta}^0{}_2 = \underline{\theta}^2{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^2 + [\cdots]\underline{\omega}^0 \iff \underline{\theta}^2{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^2,\quad \underline{\theta}^0{}_2 = \frac{a'(t)}{a(t)}\underline{\omega}^2$$
$$[\cdots]\underline{\omega}^3 = \underline{\theta}^0{}_3 = \underline{\theta}^3{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^3 + [\cdots]\underline{\omega}^0 \iff \underline{\theta}^3{}_0 = \frac{a'(t)}{a(t)}\underline{\omega}^3,\quad \underline{\theta}^0{}_3 = \frac{a'(t)}{a(t)}\underline{\omega}^3$$
$$[\cdots]\underline{\omega}^2 = \underline{\theta}^1{}_2 = -\underline{\theta}^2{}_1 = -\frac{1}{a(t)r}\cot\theta\underline{\omega}^2 - [\cdots]\underline{\omega}^1 \iff \underline{\theta}^1{}_2 = -\frac{1}{a(t)r}\cot\theta\underline{\omega}^2,\quad \underline{\theta}^2{}_1 = \frac{1}{a(t)r}\cot\theta\underline{\omega}^2$$
$$[\cdots]\underline{\omega}^2 = \underline{\theta}^3{}_2 = -\underline{\theta}^2{}_3 = -\frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2 - [\cdots]\underline{\omega}^3 \iff \underline{\theta}^3{}_2 = -\frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2,\quad \underline{\theta}^3{}_2 = \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^2$$
$$[\cdots]\underline{\omega}^1 = \underline{\theta}^3{}_1 = -\underline{\theta}^1{}_3 = -\frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1 - [\cdots]\underline{\omega}^3 \iff \underline{\theta}^3{}_1 = -\frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1,\quad \underline{\theta}^1{}_3 = \frac{1}{a(t)r}\sqrt{1-kr^2}\underline{\omega}^1$$

(c) The curvature two-forms are obtained from the connection one-forms calculated above with the relation $\underline{R}^a{}_b = \underline{d\theta}^a{}_b + \underline{\theta}^a{}_c \wedge \underline{\theta}^c{}_b$. Using $H = a'(t)/a(t)$, $A = \frac{1}{a(t)r}\sqrt{1-kr^2}$ and $B = \frac{1}{a(t)r}\cot\theta$ the connection one form can be organised as

$$[\underline{\theta}^a{}_b] == \begin{pmatrix} 0 & H\underline{\omega}^1 & H\underline{\omega}^2 & H\underline{\omega}^3 \\ H\underline{\omega}^1 & 0 & B\underline{\omega}^2 & A\underline{\omega}^1 \\ H\underline{\omega}^2 & -B\underline{\omega}^2 & 0 & A\underline{\omega}^2 \\ H\underline{\omega}^3 & -A\underline{\omega}^1 & -A\underline{\omega}^2 & 0 \end{pmatrix}$$

and the second term in the curvature two-forms can be expressed as a matrix multiplication where the elementwise multiplication is a $\wedge$. We have

$$\begin{aligned}
&[\underline{\theta}^a{}_c \wedge \underline{\theta}^c{}_b] \\
&= \begin{pmatrix} 0 & H\underline{\omega}^1 & H\underline{\omega}^2 & H\underline{\omega}^3 \\ H\underline{\omega}^1 & 0 & B\underline{\omega}^2 & A\underline{\omega}^1 \\ H\underline{\omega}^2 & -B\underline{\omega}^2 & 0 & A\underline{\omega}^2 \\ H\underline{\omega}^3 & -A\underline{\omega}^1 & -A\underline{\omega}^2 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & H\underline{\omega}^1 & H\underline{\omega}^2 & H\underline{\omega}^3 \\ H\underline{\omega}^1 & 0 & B\underline{\omega}^2 & A\underline{\omega}^1 \\ H\underline{\omega}^2 & -B\underline{\omega}^2 & 0 & A\underline{\omega}^2 \\ H\underline{\omega}^3 & -A\underline{\omega}^1 & -A\underline{\omega}^2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & H\underline{\omega}^2 \wedge (-B\underline{\omega}^2) + H\underline{\omega}^3 \wedge (-A\underline{\omega}^1) & H\underline{\omega}^1 \wedge (B\underline{\omega}^2) + H\underline{\omega}^3 \wedge (-A\underline{\omega}^2) & 0 \\ A\underline{\omega}^1 \wedge (H\underline{\omega}^3) & 0 & H\underline{\omega}^1 \wedge (H\underline{\omega}^2) + A\underline{\omega}^1 \wedge (-A\underline{\omega}^2) & H\underline{\omega}^1 \wedge (H\underline{\omega}^3) \\ -B\underline{\omega}^2 \wedge (H\underline{\omega}^1) + A\underline{\omega}^2 \wedge (H\underline{\omega}^3) & H\underline{\omega}^2 \wedge (H\underline{\omega}^1) + A\underline{\omega}^2 \wedge (-A\underline{\omega}^1) & 0 & H\underline{\omega}^2 \wedge (H\underline{\omega}^3) - B\underline{\omega}^2 \wedge (A\underline{\omega}^1) \\ 0 & H\underline{\omega}^3 \wedge (H\underline{\omega}^1) & H\underline{\omega}^3 \wedge (H\underline{\omega}^2) - A\underline{\omega}^1 \wedge (B\underline{\omega}^2) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (HA)\underline{\omega}^1 \wedge \underline{\omega}^3 & (HB)\underline{\omega}^1 \wedge \underline{\omega}^2 - (HA)\underline{\omega}^3 \wedge \underline{\omega}^2 & 0 \\ (HA)\underline{\omega}^1 \wedge \underline{\omega}^3 & 0 & (H^2 - A^2)\underline{\omega}^1 \wedge \underline{\omega}^2 & (H^2)\underline{\omega}^1 \wedge \underline{\omega}^3 \\ (HB)\underline{\omega}^1 \wedge \underline{\omega}^2 - (HA)\underline{\omega}^3 \wedge \underline{\omega}^2 & -(H^2 - A^2)\underline{\omega}^1 \wedge \underline{\omega}^2 & 0 & (H^2)\underline{\omega}^2 \wedge \underline{\omega}^3 - (AB)\underline{\omega}^2 \wedge \underline{\omega}^1 \\ 0 & -(H^2)\underline{\omega}^1 \wedge \underline{\omega}^3 & -((H^2)\underline{\omega}^2 \wedge \underline{\omega}^3 - (AB)\underline{\omega}^2 \wedge \underline{\omega}^1) & 0 \end{pmatrix}
\end{aligned}$$

(d)

2 Acknowledgement

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