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Homework 2 : Linearized gravity

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1 Linearized field equations

Weak gravitational effects can be modeled as a perturbation of the flat Minkowski metric η. On the level of manifolds, this perturbation can be seen as a diffeomorphism $\phi : M \to M'$ mapping flat spacetime M into a weakly curved manifold M'. A global coordinate chart $\psi : M \to \mathbb{R}^4$ on the flat spacetime can be converted to a coordinate chart ψ' on the disformed manifold as $\psi' = \psi \circ \phi^{-1} : M' \to \mathbb{R}^4$. Taking the coordinates on M to be cartesian, we work with the inherited coordinates on M' as a starting point. In these coordinates, the full metric $g_{\mu\nu}$ can be Taylor expanded in a small parameter λ as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(\lambda^2)$ where $h_{\mu\nu}$ is the perturbation depending linearly on λ. For all the following calculations, we drop the $O(\lambda^2)$ but keep in mind that everything represents a first-order expansion in λ.

To write the first-order contribution to the Einstein equations arising from this perturbation, we first compute the inverse metric. Expanding it in λ around the inverse Minkowski metric, we have $g_{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu}$ and

$$\delta^{\nu}_{\rho} = g_{\rho\mu}g^{\mu\nu} = \eta_{\rho\mu}\eta^{\mu\nu} + h_{\rho\mu}\eta^{\mu\nu} + \eta_{\rho\mu}f^{\mu\nu} \iff f_{\rho}{}^{\nu} = -h_{\rho}{}^{\nu} \iff f^{\rho\nu} = -h^{\rho\nu}.$$

Then the expansion of the Christoffel symbols read

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(g_{\mu\rho,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) = \frac{1}{2}(\eta^{\sigma\rho} - h^{\sigma\rho})(h_{\mu\rho,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho}) = \frac{1}{2}\eta^{\sigma\rho}(h_{\mu\rho,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho})$$

because $\eta_{\mu\nu,\rho} = 0$ in cartesian coordinates. The Riemann tensor can now be expressed as

$$\begin{aligned}
R^{\rho}{}_{\sigma\mu\nu} &= \Gamma^{\rho}{}_{\nu\sigma,\mu} - \Gamma^{\rho}{}_{\mu\sigma,\nu} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma} \\
&= \Gamma^{\rho}{}_{\nu\sigma,\mu} - \Gamma^{\rho}{}_{\mu\sigma,\nu} = \frac{1}{2}\eta^{\rho\lambda}(h_{\nu\lambda,\sigma\mu} + h_{\lambda\sigma,\nu\mu} - h_{\nu\sigma,\lambda\mu}) - \frac{1}{2}\eta^{\rho\lambda}(h_{\mu\lambda,\sigma\nu} + h_{\lambda\sigma,\mu\nu} - h_{\mu\sigma,\lambda\nu}) \\
&= \frac{1}{2}\eta^{\rho\lambda}(h_{\nu\lambda,\sigma\mu} - h_{\nu\sigma,\lambda\mu} - h_{\mu\lambda,\sigma\nu} + h_{\mu\sigma,\lambda\nu}).
\end{aligned}$$

Contracting the ρ and μ indices, we get the following Ricci tensor:

$$\begin{aligned}
R_{\sigma\nu} &= \frac{1}{2}\eta^{\mu\lambda}(h_{\nu\lambda,\sigma\mu} - h_{\nu\sigma,\lambda\mu} - h_{\mu\lambda,\sigma\nu} + h_{\mu\sigma,\lambda\nu}) = \frac{1}{2}(h_{\nu}{}^{\mu}{}_{,\sigma\mu} - h_{\nu\sigma,\lambda}{}^{\lambda} - h^{\lambda}{}_{\lambda,\sigma\nu} + h^{\mu}{}_{\sigma,\mu\nu}) \\
&= \frac{1}{2}(h_{\nu}{}^{\mu}{}_{,\sigma\mu} + h^{\mu}{}_{\sigma,\mu\nu} - \Box h_{\nu\sigma} - h_{,\sigma\nu}), \quad h = h^{\lambda}{}_{\lambda}
\end{aligned}$$

where we used the fact raising indices with $g^{\mu\nu}$ for tensors proportionnal to λ reduces to contracting them with $\eta^{\mu\nu}$ at first order in λ (the $-h^{\mu\nu}$ term only contributes to second order). Contracting the remaining indices (with the Minkowski) metric yields the Ricci scalar

$$R = \eta^{\sigma\nu}R_{\sigma\nu} = \frac{1}{2}(h^{\sigma\mu}{}_{,\sigma\mu} + h^{\sigma\mu}{}_{,\mu\sigma} - \Box h^{\nu}{}_{\nu} - h_{,\nu}{}^{\nu}) = h^{\sigma\mu}{}_{,\sigma\mu} - \Box h.$$

Combining all the previous results, the linearised Einstein tensor can be written as

$$G_{\sigma\nu} = R_{\sigma\nu} - \frac{1}{2}\eta_{\sigma\nu}R = \frac{1}{2}(h_{\nu}{}^{\mu}{}_{,\sigma\mu} + h^{\mu}{}_{\sigma,\mu\nu} - \Box h_{\nu\sigma} - h_{,\sigma\nu} - \eta_{\sigma\nu}h^{\rho\mu}{}_{,\rho\mu} + \eta_{\sigma\nu}\Box h).$$

We define $\bar{h}_{\sigma\nu} = h_{\sigma\nu} - \frac{1}{2}\eta_{\sigma\nu}h$ with trace $\bar{h} = \eta^{\sigma\nu}\bar{h}_{\sigma\nu} = h - \frac{4}{2}h = -h$. With this in mind, the perturbation can be written as $h_{\sigma\nu} = \bar{h}_{\sigma\nu} + \frac{1}{2}\eta_{\sigma\nu}(-\bar{h})$. Substitution of this form in the Einstein tensor leads to

$$\begin{aligned}
G_{\sigma\nu} &= \frac{1}{2}(h_{\nu}{}^{\mu}{}_{,\sigma\mu} + h^{\mu}{}_{\sigma,\mu\nu} - \Box h_{\nu\sigma} - h_{,\sigma\nu} - \eta_{\sigma\nu}h^{\rho\mu}{}_{,\rho\mu} + \eta_{\sigma\nu}\Box h) \\
&= \frac{1}{2}(\bar{h}_{\nu}{}^{\mu}{}_{,\sigma\mu} - \frac{1}{2}\bar{h}_{,\sigma\nu} + \bar{h}^{\mu}{}_{\sigma,\mu\nu} - \frac{1}{2}\bar{h}_{,\sigma\nu} - \Box\bar{h}_{\sigma\nu} + \frac{1}{2}\eta_{\sigma\nu}\Box\bar{h} + \bar{h}_{,\sigma\nu} - \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu} + \frac{1}{2}\eta_{\sigma\nu}\Box\bar{h} - \eta_{\sigma\nu}\Box\bar{h}) \\
&= \frac{1}{2}(\bar{h}_{\nu}{}^{\mu}{}_{,\sigma\mu} + \bar{h}^{\mu}{}_{\sigma,\mu\nu} - \Box\bar{h}_{\sigma\nu} - \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu})
\end{aligned}$$

with

$$\begin{aligned}
&h_{\nu}{}^{\mu}{}_{,\sigma\mu} = \bar{h}_{\nu}{}^{\mu}{}_{,\sigma\mu} - \frac{1}{2}\eta_{\nu}{}^{\mu}\bar{h}_{,\sigma\mu} = \bar{h}_{\nu}{}^{\mu}{}_{,\sigma\mu} - \frac{1}{2}\bar{h}_{,\sigma\nu}, \quad h^{\mu}{}_{\sigma,\mu\nu} = \bar{h}^{\mu}{}_{\sigma,\mu\nu} - \frac{1}{2}\eta^{\mu}{}_{\sigma}\bar{h}_{,\mu\nu} = \bar{h}^{\mu}{}_{\sigma,\mu\nu} - \frac{1}{2}\bar{h}_{,\sigma\nu} \\
&\Box h_{\sigma\nu} = \bar{h}_{\sigma\nu} - \frac{1}{2}\eta_{\sigma\nu}\Box\bar{h}, \quad h_{,\sigma\nu} = -\bar{h}_{,\sigma\nu}, \quad \eta_{\sigma\nu}h^{\rho\mu}{}_{,\rho\mu} = \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu} - \frac{1}{2}\eta_{\sigma\nu}\eta^{\rho\mu}\bar{h}_{,\rho\mu} = \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu} - \frac{1}{2}\eta_{\sigma\nu}\Box\bar{h}.
\end{aligned}$$

Finally, the relation between the Einstein tensor and the stress-energy tensor $T_{\mu\nu}$ is provided by Einstein equations. We take $T_{\mu\nu}$ to be of the order of λ consistently with the weak field on almost flat space ($T_{\mu\nu}$ has no zeroth order contribution) assumptions. The perturbation satisfies the equation

$$\frac{1}{2}(\bar{h}^{\mu}{}_{\nu,\sigma\mu} + \bar{h}^{\mu}{}_{\sigma,\nu\mu} - \Box\bar{h}_{\sigma\nu} - \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu}) = \frac{1}{2}(2\bar{h}^{\mu}{}_{(\sigma,\nu)\mu} - \Box\bar{h}_{\sigma\nu} - \eta_{\sigma\nu}\bar{h}^{\rho\mu}{}_{,\rho\mu}) = 8\pi G T_{\sigma\nu}$$

with gravitational coupling strength G.

2 Let's simplify our lives

(a) Since coordinate transformations locally transform the metric components without changing the spacetime it describes, we can interpret them as gauge transformations on a tensor component field $g_{\mu\nu}$. To preserve the validity of our linearized expansion, we consider the effect of infinitesimal coordinate transformations $x^{\mu\prime}(x) = x^\mu - \xi^\mu(x)$ with ξ at order in λ. This ensures that a coordinate change preserves $\eta_{\mu\nu}$ at zeroth order and sends $h_{\mu\nu}$ to a perturbation in the range satisfying the linearized Einstein equations. The transformed components $h'_{\mu\nu}(x')$ will satisfy the equation and we recover a notion of linearized covariance. Relating the $g_{\mu\nu}(x)$ components and the gauge transformed components $g'_{\mu\nu}(x')$ at first order, we have

$$\begin{aligned}\eta_{\mu\nu} + h_{\mu\nu}(x) = g_{\mu\nu}(x) &= x^{\mu\prime}{}_{,\mu} x^{\nu\prime}{}_{,\nu} g'_{\mu\nu}(x'(x)) \\ &= (\delta^\sigma_\mu - \xi^\sigma{}_{,\mu}(x))(\delta^\rho_\nu - \xi^\rho{}_{,\nu}(x))(\eta_{\sigma\rho} + h'_{\sigma\rho}(x'(x))) \\ &= \eta_{\mu\nu} + h'_{\mu\nu}(x'(x)) - \delta^\sigma_\mu \eta_{\sigma\rho}\xi^\rho{}_{,\nu}(x) - \delta^\rho_\nu \eta_{\sigma\rho}\xi^\sigma{}_{,\mu}(x) \\ &= \eta_{\mu\nu} + h'_{\mu\nu}(x'(x)) - \xi_{\mu,\nu} - \xi_{\nu,\mu}\end{aligned}$$

Comparing the right and left-hand sides of this expression yields $h'_{\mu\nu}(x'(x)) = h_{\mu\nu}(x) + \xi_{\mu,\nu}(x) + \xi_{\nu,\mu}(x)$. To bring the dependency of $h'_{\mu\nu}$ to x explicitly, we write the expansion $h'_{\mu\nu}(x'(x)) = h'_{\mu\nu}(x) + \xi^\sigma(x) h'_{\mu\nu,\sigma}(x)$ where the second term is second order in λ and does not contribute so $h'_{\mu\nu}(x'(x)) = h'_{\mu\nu}(x)$.

(b) Using the previous result, the gauge transformation of $\bar{h}_{\sigma\nu}$ to $\bar{h}'_{\sigma\nu}$ reads

$$\begin{aligned}\bar{h}'_{\mu\nu}(x) &= h_{\mu\nu}(x) + \xi_{\mu,\nu}(x) + \xi_{\nu,\mu}(x) - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}(h_{\sigma\rho}(x) + \xi_{\sigma,\rho}(x) + \xi_{\sigma,\rho}(x)) \\ &= \bar{h}_{\mu\nu}(x) + \xi_{\mu,\nu}(x) + \xi_{\nu,\mu}(x) - \eta_{\mu\nu}\xi_{\sigma,}{}^{\sigma}(x).\end{aligned}$$

Now we contract the μ index of $\bar{h}_{\mu\nu}$ with a derivative and get

$$\begin{aligned}\bar{h}'{}_{\mu\nu,}{}^{\mu}(x) &= \bar{h}_{\mu\nu,}{}^{\mu}(x) + \xi_{\mu,\nu}{}^{\mu}(x) + \xi_{\nu,\mu}{}^{\mu}(x) - \eta_{\mu\nu}{}^{\mu}\xi_{\sigma,}{}^{\sigma}(x) \\ &= \bar{h}_{\mu\nu,}{}^{\mu}(x) + \xi_{\mu,\nu}{}^{\mu}(x) + \xi_{\nu,\mu}{}^{\mu}(x) - \xi_{\sigma,\nu}{}^{\sigma}(x) \\ &= \bar{h}_{\mu\nu,}{}^{\mu}(x) + \Box\xi_\nu(x).\end{aligned}$$

Choosing ξ_ν to make $\bar{h}'{}_{\mu\nu,}{}^{\mu}(x)$ vanish constitutes a choice of gauge called the *De Donder gauge*. The coordinate transforms leading to this gauge are constrained by

$$\Box\xi_\nu(x) = -\bar{h}_{\mu\nu,}{}^{\mu}(x)$$

which is a wave equation with $-\bar{h}_{\mu\nu,}{}^{\mu}(x)$ sources for each ν. Given any starting $\bar{h}_{\mu\nu}$, we can compute the associated source and solve the wave equation to go to the De Donder gauge. In this gauge, the Einstein equations derived above become

$$8\pi G T_{\sigma\nu} = \frac{1}{2}(\bar{h}'_{\mu\sigma,}{}^{\mu}{}_{\nu} + \bar{h}'_{\mu\nu,}{}^{\mu}{}_{\sigma} - \Box\bar{h}'_{\sigma\nu} - \eta_{\sigma\nu}(\bar{h}'_{\rho\mu,}{}^{\mu})^{\rho}) = -\frac{1}{2}\Box\bar{h}'_{\sigma\nu} \iff \Box\bar{h}'_{\sigma\nu} = -16\pi G T_{\sigma\nu}.$$

In the following steps, we work in De Donder gauge and drop $'$ to simplify notation.

3 Gravitomagnetism

(a) The linearization of gravity works for $T_{\sigma\nu}$ of the order of λ which is realised far from sources. Going further, the Newtonian limit is taken by approximating that the only significant $T_{\sigma\nu}$ component is mass density $\rho = T_{00}$. Then, all other components of Einstein equations have no significant sources at all times and vanish in the Newtonian limit. We can identify the Newtonian gravitationnal potential ϕ with $-\frac{1}{4}\bar{h}_{00}$ (or $h_{00} = \bar{h}_{00} - \frac{1}{2}\bar{h}_{00} = -2\phi$, $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -\bar{h}_{00}$ because only one diagonal element is non-zero). The Einstein equation associated with this component reads

$$\Box\bar{h}_{00} = -16\pi G T_{00} \iff 4\pi G\rho = -\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\phi$$

and in the quasi-static field limit (slowly changing ϕ, not enough to emit considerable gravitational radiation, of the order of the field variations in celestial mechanics) we recover $\nabla^2\phi = 4\pi G\rho$. The solution for $h_{\mu\nu}$ consistent with $\bar{h}_{\mu\nu}$ is

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}(-\bar{h}) = \begin{cases} \frac{1}{2}\bar{h}_{00}\delta_{ij}, & (\mu,\nu) = (i,j) \\ 0, & (\mu,\nu) = (i,0) \\ \frac{1}{2}\bar{h}_{00}, & (\mu,\nu) = (0,0). \end{cases}$$

which leads to the linearized metric

$$ds^2 = \left(-1 + \frac{1}{2}\bar{h}_{00}\right)dt^2 + \left(1 + \frac{1}{2}\bar{h}_{00}\right)(dx^2 + dy^2 + dz^2).$$

We make sense of the g_{ij} elements by comparing them to the weak field limit of the Schwarzschild for spherically symmetric energy density ρ.

We consider a point mass moving on a curve $\gamma : \mathbb{R} \to M'$. Its points are represented in the coordinate chart inherited from cartesian coordinates on M by $x^\mu(\tau)$ parametrized by proper time τ (Lorentzian arc length for timelike velocities). In the Newtonian limit, an infinitesimal proper time change on γ reads

$$-d\tau^2 = -(1-2\phi)dt^2 + (1+2\phi)\left(dx^2 + dy^2 + dz^2\right) \implies -1 = -(1-2\phi)\left(\frac{dt}{d\tau}\right)^2 + (1+2\phi)\left(\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right).$$

At leading order in λ ($\sim$ neglecting gravitational time dilation and associated space effect) the proper time parametrization behaves the same way it does in Minkowski space. To go from Minkowsk-like proper time to Galilean-like absolute time we take $\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}$ to be small (neglecting special relativistic time dilation). This means that the Minkowski-like coordinate system is such that the three-velocity of the mass stays close to the time direction for all τ where the Newtonian limit applies. This three-velocity constraint reduces the previous equation to $1 = \frac{dt}{d\tau}$ implying parametrizing by proper time is equivalent to parametrizing by coordinate time t. A schematic way to write this conclusion is $x^\mu(\tau(t)) = x^\mu(t + O(v^2) + O(\lambda)) = x_0^\mu(t) + O(v^2) + O(\lambda)$ and $\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = (1 + O(\lambda) + O(v^2))\frac{d}{dt}$ where v represents three-velocity (the dependency starts at $O(v^2)$ because of the Minkowski lorentz factor expansion). With this conclusion in mind, we can write the geodesic equation describing the free trajectory in M' as follows

$$\begin{aligned} 0 &= \frac{d^2x^\mu(\tau)}{d\tau^2} + \Gamma^\mu{}_{\alpha\tau}\frac{dx^\alpha(\tau)}{d\tau}\frac{dx^\beta(\tau)}{d\tau} \\ &= (1 + O(\lambda) + O(v^2))^2\frac{d^2x_0^\mu(t) + O(v^2) + O(\lambda)}{dt^2} + \Gamma^\mu{}_{\alpha\beta}(1 + O(\lambda) + O(v^2))^2\frac{dx_0^\alpha(t) + O(v^2) + O(\lambda)}{dt}\frac{dx_0^\beta(t) + O(\lambda) + O(v^2)}{dt}. \end{aligned}$$

Since $\Gamma^\mu{}_{\alpha\tau}$ is first order in λ (see linearized expression given above), the leading order in v and λ of the geodesic equation is $0 = \frac{d^2x^\mu(\tau)}{dt^2}$ (Newton's principle of inertia). To retrieve the dominant gravitationnal effets we go to first order in λ and define $x^\mu(\tau(t)) = x_1^\mu(t) + O(v^2) + O(\lambda^2)$ to get

$$0 = (1 + O(\lambda) + O(v^2))^2\frac{d^2x_1^\mu(t) + O(v^2) + O(\lambda^2)}{dt^2} + \Gamma^\mu{}_{\alpha\beta}(1 + O(\lambda) + O(v^2))^2\frac{dx_1^\alpha(t) + O(v^2) + O(\lambda^2)}{dt}\frac{dx_1^\beta(t) + O(\lambda^2) + O(v^2)}{dt}$$

Gravitationnal dilation effects vanish in the first term at $O(\lambda)$ because x_1^μ is already at $O(\lambda)$ and all other $O(\lambda)$ factors are neglected

$$= \frac{d^2x_1^\mu(t)}{dt^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx_0^\alpha(t)}{dt}\frac{dx_0^\beta(t)}{dt}, \quad \text{For the } \Gamma^\mu{}_{\alpha\beta} = O(\lambda) \text{ term, only the zeroth order contributions } x_0^\mu \text{ are preserved}$$

$$= \frac{d^2x_1^\mu(t)}{dt^2} + \Gamma^\mu{}_{00}\frac{dx_0^0(t)}{dt}\frac{dx_0^0(t)}{dt}, \quad dx^i/d\tau = (1 + O(\lambda) + O(v^2))dx^i/dt = O(v) + O(\lambda) : \text{spacial velocities factors vanish at } O(v^0)$$

$$= \frac{d^2x_1^\mu(t)}{dt^2} + \frac{1}{2}\eta^{\mu\rho}(h_{0\rho,0} + h_{\rho0,0} - h_{00,\rho}), \quad \text{principle of inertia at } O(\lambda^0) \impliedby x_0^0 = t$$

$$\implies 0 = \frac{d^2x_{1,i}(t)}{dt^2} - \frac{1}{2}h_{00,i} = \frac{d^2x_{1,i}(t)}{dt^2} + \phi_{,i}, \quad \text{lowering the index to get a gradient, } h_{00} = \bar{h}_{00} - \frac{1}{2}\bar{h}_{00} = -2\phi$$

(b) If we allow significant energy flux ($T_{0i} = T_{i0}$ components) while keeping the stress (T_{ij} components) negligible, the non-vanishing components of $\bar{h}_{\mu\nu}$ becomes $\bar{h}_{\mu0} = \bar{h}_{0\mu}$. These components can be associated with a four-potential $A_\mu = -\bar{h}_{\mu0}/4$ sourced by the four-courent $J_\mu = -T_{0\mu}$. Writing Einstein's equations and the De Donder gauge condition for the nonzero components gives

$$-4\Box A_\mu = \Box\bar{h}_{0\mu} = -16\pi G T_{0\mu} = 16\pi G J_\mu \iff \Box A_\mu = -4\pi G J_\mu, \quad \bar{h}_{0\mu,}{}^\mu = -4A_{\mu,}{}^\mu = 0 \iff A_{\mu,}{}^\mu = 0$$

which is analogous to Maxwell's equations for electromagnetism (an important difference remains through energy conditions that forbid negative charge densities T_{00}). As before we work in the quasi-static field limit where $A_{\mu,0}$ is taken negligible. Again, the Newtonian limit is used to write the geodesic equation for a point mass up to $O(\lambda)$ and $O(v)$ (we go further than to extract leading order gravitomagnetic effects) as

$$\begin{aligned}
0 &= (1+O(\lambda)+O(v^2))^2 \frac{d^2x_1^\mu(t)+O(v^2)+O(\lambda^2)}{dt^2} + \Gamma^\mu{}_{\alpha\beta}(1+O(\lambda)+O(v^2))^2 \frac{dx_1^\alpha(t)+O(v^2)+O(\lambda^2)}{dt}\frac{dx_1^\beta(t)+O(\lambda^2)+O(v^2)}{dt} \\
&= \frac{d^2x_1^\mu(t)}{dt^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx_0^\alpha(t)}{dt}\frac{dx_0^\beta(t)}{dt} = \frac{d^2x_1^\mu(t)}{dt^2} + \Gamma^\mu{}_{00}\frac{dx_0^0(t)}{dt}\frac{dx_0^0(t)}{dt} + \Gamma^\mu{}_{0i}\frac{dx_0^0(t)}{dt}v^i(t), \quad \text{with } v^i = \frac{dx_0^i(t)}{dt} \\
&= \frac{d^2x_1^\mu(t)}{dt^2} + \frac{1}{2}\eta^{\mu\rho}(h_{0\rho,0}+h_{\rho0,0}-h_{00,\rho}) + \eta^{\mu\rho}(h_{\rho i,0}+h_{\rho 0,i}-h_{0i,\rho})v^i = \frac{d^2x_1^\mu(t)}{dt^2} - \frac{1}{2}h_{00,}{}^\mu - 4(A^\mu{}_{,i} - A_{i,}{}^\mu)v^i,\ h_{0i} = \bar{h}_{0i} - \frac{1}{2}\eta_{0i}\bar{h} = \bar{h}_{0i} \\
\implies 0 &= \frac{d^2x_{1,j}(t)}{dt^2} - E_j - 4(\varepsilon_{ij}{}^k B_k)v^i = \frac{d^2x_{1,j}(t)}{dt^2} - E_j + 4\varepsilon_i{}^k{}_j v^i B_k.
\end{aligned}$$

where we identified $\varepsilon_{ij}{}^k B_k = A_{j,i} - A_{i,j}$ and $E_i = -\phi_{,i} - A_{i,0} = -\phi_{,i}$ in analogy with the electromagnetic field extracted from the four-potential. The equivalent of Lorentz force was recovered. Its associated *electric* charge q equals the inertial mass m of the particle and cancels with it. The coefficient of the magnetic term is 4 times as big as the coefficient in electromagnetism and has a reversed sign.

4 Acknowledgement

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References

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