Gross-Neveu Model

22 November 2023

Deadline: Friday, 1 December 2023, 4 pm - submit here
Late deadline: Friday, 15 December 2023, 4 pm - submit here
Pass/fail deadline: Friday, 12 January 2024, 4 pm - submit here

You are encouraged to discuss the homework problems with your classmates.

Academic integrity requires that any solutions you submit are either your own or properly cited. If a classmate explains to you how to solve part of a problem, you should indicate this on your submission.

1 Gross-Neveu Model

In this problem we will apply the rules of functional integration to show the equivalence (in the sense that the functional integrals defining the theories are equal) between a theory of fermions and a theory of scalars.

a) Consider the Gross-Neveu model of interacting Dirac fermions in 2 dimensions:

$$\mathcal{L}_{\mathrm{GN}} = \bar{\psi}_a i \partial\!\!\!/ \psi^a + \frac{g^2}{2N} \left(\bar{\psi}_a \psi^a\right)^2 \tag{1}$$

where the "color" index a runs from 1 to N and the suppressed Dirac indices run from 1 to 2. What is the dimension of g?

b) In two dimensions we can choose the gamma matrices to be

$$\begin{aligned} \gamma^0 &= \sigma^2 \\ \gamma^1 &= i\sigma^1 \\ \gamma^5 &= \gamma^0\gamma^1 = \sigma^3 \end{aligned} \tag{2}$$

where the σ^i are the Pauli matrices. Show that the Gross-Neveu Lagrangian is invariant under $\psi_a \to \gamma^5\psi_a$ and that this symmetry forbids a mass term.

c) Draw the Feynman diagram(s) that contribute to the two-point function at one-loop and write down the associated analytic expressions.

In general the one-loop contributions to the two-point function provide the leading contribution to the renormalization of the mass. It can be shown that the one-loop diagram(s) you found vanish.

d) Show that the Gross-Neveu theory is equivalent (in the sense that the functional integrals defining the theories are equal) to the following theory

$$\mathcal{L}_{\psi,\sigma} = \bar{\psi}_a(i\partial\!\!\!/ - \sigma)\psi^a - \frac{N}{2g^2}\sigma^2 \tag{3}$$

where $\sigma(x)$ is a scalar field. *Hint*: Perform a functional integral over σ.

e) By performing the functional integral over ψ, show that the Gross-Neveu model is equivalent (in the sense that the functional integrals defining the theories are equal) to the following scalar field theory

$$\int d^2x\mathcal{L}_\sigma = \int d^2x \left[-\frac{N}{2g^2}\sigma^2 + N\mathrm{tr}\log(i\partial\!\!\!/ - \sigma)\right] . \tag{4}$$

f) In the $N \to \infty$ limit, we can use the saddle-point approximation to evaluate the functional integral with action given by (4). We will make two assumptions. First we will assume that the field is constant at the saddle-point $\sigma(x) = \sigma_c$. Secondly, we will assume[1] that after we introduce a UV regulator Λ, the derivative of the trace becomes

$$\frac{d}{d\sigma}\mathrm{tr}\log(i\partial\!\!\!/ - \sigma) = \frac{\sigma}{4\pi}\log\left(\frac{\Lambda^2}{\sigma^2}\right) . \tag{5}$$

Using these assumptions, find the saddle-point field configuration σ_c.

[1] We will learn how to derive similar formulae later in the course.

g) What does your result tell you about the mass of the fermion? How is this result consistent with parts (b) and (c)?

h) Who did you collaborate with on this homework assignment?