Homework 2: Renormalization of ϕ^4 Theory

4 December 2023

Deadline: Monday, 18 December 2023, noon - submit here
Late deadline: Monday, 15 January 2023, noon - submit here
Pass/fail deadline: Monday, 29 January 2024, noon - submit here

You are encouraged to discuss the homework problems with your classmates.

Academic integrity requires that any solutions you submit are either your own or properly cited. If a classmate explains to you how to solve part of a problem, you should indicate this on your submission.

1 Sharp Cutoff Regularization

In this question we will work in the Euclidian framework.

In lectures we saw how to deal with coupling constant renormalization in massless ϕ^4 theory (at one loop order). In this question we will derive some of the results that are presented in the lecture notes without proof in the massive case.

The place to start is to define the renormalized action (at one loop)

$$S_R[\phi] = S[\phi] + \hbar\Delta_1 S[\phi] + O(\hbar^2) \tag{1}$$

which is made up of the original action (written in terms of renormalized couplings/masses)

$$S[\phi] = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 + \frac{m_R^2}{2}\phi^2 + \frac{g_R}{4!}\phi^4\right] \tag{2}$$

and the counterterms[1]

$$\Delta_1 S[\phi] = \int d^4x \left[\frac{B_1}{2}\phi^2 + \frac{C_1}{4!}\phi^4\right] \tag{3}$$

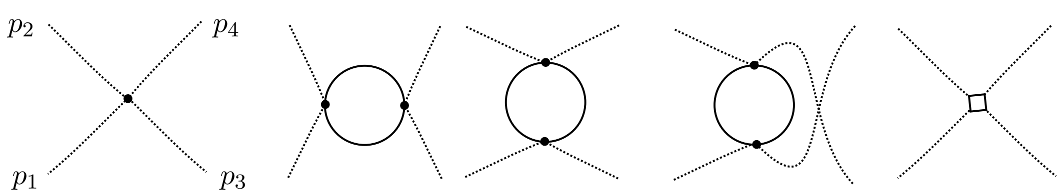
whose coefficient functions B_1 and C_1 are chosen such that physical quantities (such as scattering amplitudes) calculated with S_R, and expressed in terms of the renormalized masses and couplings, do not contain ultraviolet divergences. To recap the story: The UV divergences found when computing simply with S (and not with S_R) must be regulated to make all expressions finite. Counterterms are introduced, which depend on the UV regulator Λ in a special way that makes physical quantities finite in the $\Lambda \to \infty$ limit.

We'll first deal with the divergence in the irreducible four-point diagrams – this leads to coupling constant renormalization.

The irreducible four-point function calculated with the renormalized action S_R is

$$\begin{aligned}\hat{\Gamma}_R^{(4)}(p_i) =& g_R - \frac{\hbar g_R^2}{2}\Big[I(p_1+p_2, m_R) + I(p_1+p_3, m_R) + I(p_1+p_4, m_R)\Big] \\ &+ \hbar C_1 + O(\hbar^2)\end{aligned} \tag{4}$$

where $I(p, m_R)$ is the integral associated to the one loop diagram. If we introduce a special vertex to represent the quartic counterterm, (4) corresponds to the following Feynman diagrams:

$\hat{\Gamma}_R^{(4)}(p_i) =$ 

[1]In principle there should be a counterterm $\int \frac{A_1}{2}(\partial\phi)^2$ but it turns out that $A_1 = 0$ at order $\mathcal{O}(\hbar)$.

a) Working in 4-dimensional Euclidean spacetime, write down the integral corresponding to

$$I(p_1 + p_2, m_R; \Lambda)$$

where Λ is a large momentum cutoff, introduced to regularize the UV divergence.

b) Suppose for simplicity that $m_R^2 = 0$. Calculate the dominant large Λ behavior of $I(p, 0; \Lambda)$.

c) By differentiating under the integral, show that

$$\frac{\partial}{\partial(m_R^2)} I(p, m_R; \Lambda)$$

is finite in the limit $\Lambda \to \infty$. This tells us that the only large Λ divergences in $I(p_i, m_R; \Lambda)$ in the massive theory are those that we identified in part b) for the massless theory. (These finite differences are not necessary for the purposes of this question.)

For parts d)–h), you should work in *massless* ϕ^4 theory.

d) What is the counterterm C_1 that is required in the massless theory to ensure $\hat{\Gamma}_R^{(4)}(p_i)$ is free from large Λ divergences at the renormalization scale μ? More precisely, determine the choice of counterterm that allows us to impose the renormalization condition

$$\hat{\Gamma}_R^{(4)}(p_i) = g_R \quad for \quad (p_1 + p_2)^2 = (p_1 + p_3)^2 = (p_1 + p_4)^2 = \mu^2 \,. \tag{5}$$

You need only give an explicit expression for the parts of the counterterm that diverge in the $\Lambda \to \infty$ limit.

e) What is $\hat{\Gamma}_R^{(4)}(p_i, g_R, \mu)$ in terms of g_R and μ for general momenta?

As the renormalization scale is arbitrary physics as determined by the $\hat{\Gamma}_R^{(4)}$ should not depend on what choice of μ we make. This leads to the requirement that

$$\hat{\Gamma}_R^{(4)}(p_i, g_R, \mu) = \hat{\Gamma}_R^{(4)}(p_i, g'_R, \mu') \tag{6}$$

where the primed quantities on the right-hand side depend on a different renormalization scale μ'. For this equation to be a consistent, one might expect the couplings to have some dependence on μ.

f) By considering (6) for a particular choice of external momenta, derive an equation that expresses $g'_R(\mu')$ in terms of $g_R(\mu)$, μ and μ'.

g) Compute the 1-loop β-function for the effective coupling $g_R(\mu)$ and plot its behavior as a function of $g_R(\mu)$.

h) What does this β-function tell you about the strength of the effective coupling at high energies and low energies?

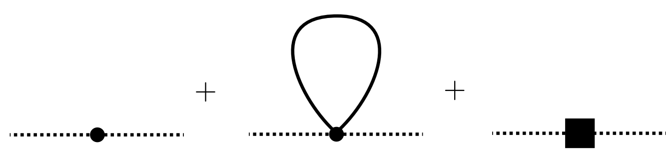
As the 4-point diagrams do not acquire any new divergences, the above calculations can also be done for the massive theory; the end result is very similar, but the algebra is more involved.

We will now look at how to renormalize the mass, which requires us to consider the irreducible two-point function to 1-loop order

$$\hat{\Gamma}_R^{(2)}(p) = p^2 + m_R^2 + \hbar g_R \frac{1}{2} T(m_R) + \hbar B + O(\hbar^2)\,. \tag{7}$$

Introducing a new (square) vertex to represent the counterterm B_1 the irreducible two-point function is equal to sum of three diagrams

$\hat{\Gamma}_R^{(2)}(p) =$



i) Write down the integral corresponding to $T(m_R;\Lambda)$ where Λ is a large momentum cutoff, introduced to regularize the UV divergences.

j) You can compute this integral by going to hyperspherical coordinates. First compute the angular integral. Let $r = \sqrt{\sum_{i=1}^{d} x_i^2}$ be the length of a D-dimensional vector, and change variables from x_i to r and a set of angles denoted by Ω_D, which parameterize the surface of a unit sphere in D-dimensions. Show that

$$\int_{-\infty}^{\infty} dx_1 \dots dx_D \, \exp\left(-\sum_{i=1}^{D} x_i^2\right) = \int d\Omega_D \int_0^{\infty} dr\, r^{D-1} e^{-r^2}\,. \tag{8}$$

Use this result to show that the area of the unit sphere in D-dimensions is

$$\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,. \tag{9}$$

Hint: Recall the integral representation of the gamma function:

$$\Gamma(n) = \int_0^\infty dx\, e^{-x} x^{n-1} \,.$$

k) Calculate the large Λ behavior of $T(m_R;\Lambda)$ by computing the integral. You should identify both quadratic and logarithmic divergences in Λ, and a term that is finite in the large Λ limit.

The *physical mass* of the field is the location of the pole in the two-point correlation function (see lecture 7 from QT). Since the two-point irreducible function is the inverse of the two-point connected function (see tutorial 2), the two-point irreducible function vanishes at $p^2 = -m^2_{phys}$

$$\hat{\Gamma}^{(2)}_R(p_i^{\text{ref}}, m_R, g_R, \mu) = 0 \qquad\qquad (p_i^{\text{ref}})^2 = -m^2_{\text{phys}} \,. \tag{10}$$

This should be true for *any* μ.

Rather than compute everything in terms of this physical mass, to ensure the two-point function is finite, it turns out to be easier to impose the renormalization condition

$$\hat{\Gamma}^{(2)}_R(0, m_R, g_R, \mu) = m_R^2 \qquad\qquad \mu = m_R \tag{11}$$

while assuming the necessary counterterm B_1 takes the following special form (linear in m_R^2)

$$B_1(m_R, g_R, \mu, \Lambda) = B_{1,0}(g_R, \mu, \Lambda) + m_R^2 \times B_{1,1}(g_R, \mu, \Lambda) \,. \tag{12}$$

This choice makes clear which divergences are present due to the non-zero mass of the theory, and which divergences were already present in the massless case.

l) What are the counterterms $B_{1,0}(g_R, \mu, \Lambda)$ and $B_{1,1}(g_R, \mu, \Lambda)$ that are required to ensure the renormalization condition (11) holds?

m) Check that the counterterms (12) make $\hat{\Gamma}^{(2)}_R(p_i, m_R, g_R, \mu)$ finite for any value of m_R and the momentum p.

n) By requiring that $\hat{\Gamma}^{(2)}_R(p_i)$ is independent of the renormalization scale (just like equation (6)), derive an equation relating the renormalized square-masses

$\left[m'_R(\mu')\right]^2$ and $\left[m_R(\mu)\right]^2$ found at the renormalization scales μ' and μ respectively.

Hint: Your answer should depend on g_R and be accurate only up to order $\hbar$.

o) The requirements (11) and (12) may be natural/convenient from a renormalization point of view, but you may worry about what the m_R we have defined actually means physically. Show, that when $\mu^2 = m_R^2$, the renormalized mass squared is actually equal to the physical mass squared: $m_R^2 = m_{\text{phys}}^2$.

p) Calculate the anomalous dimension γ_{m^2} of the mass-squared operator in terms of g_R

$$\gamma_{m^2}(g_R) \equiv \mu \frac{\partial}{\partial\mu} \ln m_R^2(\mu)\,. \tag{13}$$

This tells us how quantum effects modify the scaling properties of the operator.

q) Who did you collaborate with on this assignment?

Note in this question we dealt with the four-point interactions first because of the similarity with the massless case. In general one should worry about wavefunction renormalization (which shows up as the residue of the propagator when it goes on-shell), so one should look at the irreducible two-point diagrams first, or at least at the same time as the four-point ones. We were lucky here because (it turns out) there is no need for wavefunction renormalization at one-loop order in ϕ^4 theory.