Homework 1: Entanglement measures and POVMs

Deadline I: March 22, 2024, Dropbox link
Deadline II: April 04, 2024 Dropbox link

Kindly acknowledge the individuals or references that contributed to your successful completion of the assignment.

To help us structure the course in a better way, please submit anonymous feedback for the course here.

1 Concurrence and negativity

In this question we will study several entanglement measures and how to work with them.

(a) The *concurrence* is a *faithful* entanglement monotone for arbitrary mixed states of *bipartite qubits* $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathcal{H})$, with $\mathcal{H} \cong \mathbb{C}^2$. It is defined by

$$C[\rho] \coloneqq \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}\,, \tag{1}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the eigenvalues of the Hermitian operator

$$R \coloneqq \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}\,, \tag{2}$$

$$\tilde{\rho} \coloneqq (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\,. \tag{3}$$

The complex conjugation is taken with respect to eigenbasis of σ_z, i.e., write ρ as a 4×4 matrix in the $|0\rangle, |1\rangle$ basis and then take complex conjugate of the matrix elements.

Calculate the concurrence for the Bell state $|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and product state $|00\rangle$.

Hint: if you like, you can use *Mathematica* to calculate this, but show your working anyway (as snapshot, or written out).

Remark: Note that this definition looks very *ad hoc* as a trade-off for being easier to compute; the original definition is an optimization over all ensembles of pure states: that is, if we write $\rho = \sum_j p_j |\psi^j_{AB}\rangle\langle\psi^j_{AB}|$, then the concurrence is given by

$$C[\rho] = \inf_{\{p_j, |\psi^j_{AB}\rangle\}} \sum_j p_j C(|\psi^j_{AB}\rangle\langle\psi^j_{AB}|) \tag{4}$$

where $C(|\psi_{AB}\rangle\langle\psi_{AB}|) = \sqrt{2(1 - \mathrm{tr}(\rho_A^2)}$ with $\rho_A = \mathrm{tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$.

(b) If you tried to calculate concurrence by hand, hopefully it is clear that it is not very easy to calculate. An easier entanglement monotone for bipartite system is *negativity*, defined by

$$\mathcal{N}[\rho] = \frac{||\rho^\Gamma||_1 - 1}{2}, \tag{5}$$

where ρ^Γ is the *partial transpose* of ρ. It can be checked that it does not matter if you take partial transpose with respect to system A or system B. Here the trace norm is defined by

$$||X||_1 = \mathrm{tr}|X| = \mathrm{tr}\sqrt{X^\dagger X}, \tag{6}$$

that is, it is the sum of absolute values of the *singular values* of X (if X is positive-semidefinite operators, then these will be the eigenvalues).

Calculate the negativity for the Bell state $|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and product state $|00\rangle$.

(c) Consider the bipartite state

$$\rho_{AB} = \frac{1}{4}\mathbb{1}_A \otimes \mathbb{1}_B. \tag{7}$$

Calculate the negativity and show that this entanglement measure *does* say it is separable, unlike the von Neumann entropy of the reduced state (which is only a measure of entanglement if ρ_{AB} is pure).

2 POVM accounts for errors

In reality, our measurements are never perfect. For example, a photodetector may have "dark counts": it clicks even when no photons hit the detector. The POVM framework can account for such measurements, as you will demonstrate below.

(a) Suppose that a source emits spin-1/2 particles (let's call them qubits) in some state given by a density operator ρ. Any qubit state can be written in terms of Bloch vector $\boldsymbol{r} = (r_x, r_y, r_z)$ by

$$\rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma}) \tag{8}$$

where $\boldsymbol{r} \cdot \boldsymbol{\sigma} = r_x\sigma_x + r_y\sigma_y + r_z\sigma_z$ and $0 \leq |\boldsymbol{r}| \leq 1$. Clearly, the state is fully determined if you measure $\boldsymbol{r}$.

Show that

$$r_j = \mathrm{tr}(\rho\sigma_j)\,, \quad j = x, y, z\,. \tag{9}$$

That is, to measure r_j you just need to do (in principle) projective measurement in the eigenbasis of σ_j and repeat many times (to take expectation values).

(b) Now suppose that your σ_z measurement is faulty with some small error ϵ: instead of performing a perfect measurement along the eigenbasis of σ_z, namely $|0\rangle, |1\rangle$, **50% of the time** you are measuring in the eigenbasis of $\sigma'_z := |1'\rangle\langle 1'| - |0'\rangle\langle 0'|$, where

$$|0'\rangle = \sqrt{1-\epsilon}\,|0\rangle + \sqrt{\epsilon}\,|1\rangle\,,$$
$$|1'\rangle = -\sqrt{1-\epsilon}\,|1\rangle + \sqrt{\epsilon}\,|0\rangle\,,$$

and $0 \leq \epsilon \ll 1$. If we did not have this error, we would simply do a projective measurement with projectors $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$.

Write down the POVM that describes this imperfect measurement: that is, construct two POVM elements E_0, E_1 so that $E_0 + E_1 = \mathbb{1}$ and $E_0, E_1 \geq 0$ (i.e., they are positive semidefinite operators). If the prepared state is $\rho = |0\rangle\langle 0|$, what is the probability of getting outcome "1" with this faulty measurement?

Hint: your POVM should reflect the fact that this error only occurs 50% of the time.

(c) As before suppose the prepared state is $\rho = |0\rangle\langle 0|$, i.e., $\boldsymbol{r} = (0, 0, 1)$. Suppose that in your state tomography, you are performing an imperfect measurement of σ_z as in part (b), while the other two measurements σ_x and σ_y are still perfect. On other words, instead of performing a perfect projective measurement in the z-basis, you are instead measuring the POVM you found in part b). This procedure will lead you to infer a "wrong" z-component for the $\boldsymbol{r}$ vector. Use the result in part (a) to calculate the z-component of the "wrong" Bloch vector r'_z.

3 Entanglement-breaking channel

Recall that a quantum channel is a completely-positive trace-preserving (CPTP) linear map between density matrices, i.e., $\Phi : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}')$, that gives us a very general way of describing physically allowed operations on quantum systems. For example, in Tutorial 2 we studied some examples of quantum channels and also analyzed how LOCC cannot be used to create entanglement (but it can create classical correlations).

Suppose we have a channel $\Phi : \mathcal{D}(\mathcal{H}_A) \to \mathcal{D}(\mathcal{H}_{A'})$. We say that Φ is *entanglement-breaking* if for any auxiliary environment B with Hilbert space $\mathcal{H}_B$, we have that

$$\hat{\rho}_{A'B} := (\Phi \otimes \mathbb{1})(\hat{\rho}_{AB}) \text{ is separable.} \tag{10}$$

The identity operator acting on B is the *identity channel*[1] that does nothing: $\mathbb{1}(\hat{\rho}) = \hat{\rho}$ for all $\hat{\rho}_B \in \mathcal{D}(\mathcal{H}_B)$. Eq. (10) even if we prepare a state that is entangled across the system and the environment, acting on the system A with channel Φ always breaks entanglement with any degrees of freedom external to the output system A'.

[1] In quantum information literature, sometimes this is written as $\mathbb{1}_{\mathcal{D}(\mathcal{H}_B)}$ to denote that it is a identity channel rather than just the identity matrix for conceptual clarity, since quantum channels do not in general act by "multiplication".

(a) Suppose we have a system S and an environment E which are both qubits and they undergo a joint unitary operation of the form

$$\hat{U} = e^{i\theta \hat{\sigma}_z \otimes \hat{\sigma}_z} \,, \tag{11}$$

We are interested in what happens to the *environment state* after the interaction. Let the initial state of the system and the environment be $\hat{\rho}_S$ and $\hat{\rho}_E$ respectively. Compute the channel Φ, defined by

$$\hat{\rho}'_E \coloneqq \Phi(\hat{\rho}_S) = \mathrm{tr}_S(\hat{U}\hat{\rho}_S \otimes \hat{\rho}_E \hat{U}^\dagger) \,. \tag{12}$$

(b) Suppose the environment is in the ground state $\hat{\rho}_E = |0\rangle\langle 0|$ (for simplicity). Show that if we have another system S' such that the joint system $\hat{\rho}_{SS'}$ is prepared in the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ then the final state

$$\hat{\rho}'_{ES'} \coloneqq (\Phi \otimes \mathbb{1})(\hat{\rho}_{SS'}) \tag{13}$$

is separable.

(c) A *classical-quantum* (QC) channel is defined by

$$\Phi_{qc}(\hat{\rho}) \coloneqq \sum_j \mathrm{tr}(\hat{E}_j \hat{\rho})\hat{\sigma}_j \,, \tag{14}$$

where $\hat{\sigma}_j$ are density operators and $\{\hat{E}_j\}$ defines a POVM. Roughly speaking, the channel takes in the input state and outputs a statistical mixture of different states whose probability distribution is determined by the input state with respect to some measurement (in this case, the POVM elements). Ruskai proved that *all* CQ channels are entanglement-breaking.

Now consider more general unitary of the form

$$\hat{U} = e^{i\theta \hat{A} \otimes \hat{B}} \tag{15}$$

where $\hat{A}$, $\hat{B}$ are arbitrary Hermitian operators acting on the system and environment respectively. Show that the channel

$$\Phi(\hat{\rho}_S) = \mathrm{tr}_S(\hat{U}\hat{\rho}_S \otimes \rho_E \hat{U}^\dagger) \tag{16}$$

can be recast into the QC form, hence the joint unitary between the system and the environment defines a subclass of entanglement-breaking channels.

Hint: use the spectral decomposition of $\hat{A}$ and use the fact that

$$e^{\sum_j \hat{P}_j \otimes c_j \hat{X}} = \sum_j \hat{P}_j \otimes e^{c_j \hat{X}} \,. \tag{17}$$

if $\hat{P}_j$ are orthogonal projectors.