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Homework 1

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Quantum Field Theory III

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Contents

1 Conformal invariance of the Maxwell action for $D = 4$

(a) Consider a classical abelian gauge field A_μ on $D = 4$ dimensionnal Minkowski spacetime. Under an infinitesimal conformal transformation, spacetime undergoes the transformation $\tilde{x}^\mu = f(x) = x^\mu + \xi^\mu(x)$ where $\xi^\mu(x)$ is a smal deformation. We want to calculate the effect of this transformation on the gauge field A_μ. The starting point is that we expect A_μ to transform as a tensor under the lorenz transformation subgroup of the conformal group. This implies that A_μ is a primary operator and we denote its scaling dimension Δ. The transformed field $\tilde{A}_\mu$ at $\tilde{x}$ is related to the original field A_μ at x by an internal rotation, scaling, and special conformal transformation. The rotation operation acts on the components A_μ through its spin 1 representation which is the defnining representation of rotations. The scaling and special conformal transformation act together through the multiplication of A_μ by the jacobian factor $|\partial x/\partial\tilde{x}|_x^{\Delta/D}$. Finally, translations act trivially internally. This can be summarized with the relation $\tilde{A}_\mu(\tilde{x}) = |\partial x/\partial\tilde{x}|_x^{\Delta/D} R_\mu^\nu A_\nu(x)$ where R_μ^ν is the matrix associated with the part of $\xi^\mu(x)$ that does not change the metric components (after the Weyl and diffeomorpshim transformations). With this in mind, we calculate the jacobian of the infinitesimal transformation to be

$$\left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|_x = \left|\frac{\partial\tilde{x}^\mu}{\partial x^\nu}\right|_x^{-1} = |\delta_\nu^\mu + \partial_\nu\xi^\mu|_x^{-1} \approx |e^{-\partial_\nu\xi^\mu}|_x = e^{-\mathrm{Tr}\,\partial_\nu\xi^\mu(x)} = 1 - \partial_\mu\xi^\mu(x) + O(\xi^2).$$

The matrix $R_\mu^\nu(x)$ can be extracted by dividing the matrix $(\partial x/\partial\tilde{x})_x$ by a factor $\Omega(x)$ such that we extract the "metric component preserving" operation. To find this factor we consider the effect on the metric of $\Omega^{-1}(x)(\partial x/\partial\tilde{x})_x$. We can write the "metric component preserving" property as

$$\Omega^{-2}(x)\left(\frac{\partial x^\mu}{\partial\tilde{x}^\sigma}\right)_x\left(\frac{\partial x^\nu}{\partial\tilde{x}^\rho}\right)_x\eta_{\mu\nu} = \eta_{\sigma\rho}.$$

Since $\Omega(x)$ is a factor, we can extract it by taking the determinant on both sides of the previous relation to get

$$\det(\eta)\Omega(x)^{-2D}\left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|_x^2 = \det(\eta) \iff \Omega(x) = \left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|_x^{-\frac{1}{D}}.$$

This result can be intuitively undertsood from the fact the jacobian measures D-volume rescaling. Since we want metric components (associated to distances) to be preserved by the rescaled trasnformation, we need to divide by the D-root of the jacobian. The matrix $R_\mu^\nu(x)$ provided by the rescaling is given by

$$\begin{aligned}R_\mu^\nu(x) = \frac{1}{(1-\partial_\sigma\xi^\sigma(x)+O(\xi^2))^{1/D}}\left(\frac{\partial x^\nu}{\partial\tilde{x}^\mu}\right)_x &= (1+\partial_\sigma\xi^\sigma(x)/D + O(\xi^2))(\delta_\nu^\mu + \partial_\mu\xi^\nu(x)+O(\xi^2))^{-1}\\ &= \delta_\nu^\mu(1+\partial_\sigma\xi^\sigma(x)/D) - \partial_\mu\xi^\nu(x) + O(\xi^2).\end{aligned}$$

We note that $R_\mu^\nu(x)$ will represent a rotation if $\partial_\sigma\xi^\sigma(x) = 0$ (bring the conformal Killing equation to the normal Killing equation with a rotation isometry as its solution). If $\partial_\sigma\xi^\sigma(x) \neq 0$, the rescaled transformation contains a special conformal trasformation. The special conformal transformation as a Weyl transformation does not preserve distances, but can be combined with a diffeomorpshim to preserve the initial components of the metric. With these results, we can write the effect of the infinitesimal transformation as

$$\begin{aligned}\tilde{A}_\mu(\tilde{x}) &= (1-\partial_\rho\xi^\rho(f^{-1}(\tilde{x}))+O(\xi^2))^{\Delta/D}(A_\mu(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2))\\ &= \left(1-\frac{\Delta}{D}\partial_\rho\xi^\rho(f^{-1}(\tilde{x}))+O(\xi^2)\right)(A_\mu(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2))\\ &= A_\mu(f^{-1}(\tilde{x})) - A_\mu(f^{-1}(\tilde{x}))\frac{\Delta}{D}\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x})) + A_\mu(f^{-1}(\tilde{x}))\partial_\sigma\xi^\sigma(f^{-1}(\tilde{x}))\frac{1}{D} - A_\nu(f^{-1}(\tilde{x}))\partial_\mu\xi^\nu(f^{-1}(\tilde{x})) + O(\xi^2).\end{aligned}$$

Since $\xi(f^{-1}(\tilde{x}))$ is already first order in ξ, the only term contribution to its expansion around $\xi = 0$ at $O(\xi)$ is $\xi(\tilde{x})$. To go further, we expand $f^{-1}(\tilde{x})$ at first order in $\xi(\tilde{x})$ with the ansatz $f^{-1}(\tilde{x})^\nu = \tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x})$ (the first term of this ansatz is justified by noticing the transformation reduces to identity at $\xi = 0$). From $f(f^{-1}(\tilde{x})) = \tilde{x}$, we find

$$\tilde{x}^\nu = \tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x}) + \xi(\tilde{x}^\nu + B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x})) + O(\xi^2) \implies B_\mu^\nu(\tilde{x})\xi^\mu(\tilde{x}) + \xi^\nu(\tilde{x}) = 0, \quad \forall\xi(\tilde{x}) \implies B_\mu^\nu(\tilde{x}) = -\delta_\mu^\nu.$$

Using this result, we can expand $A_\mu(f^{-1}(\tilde{x}))$ as

$$A_\mu(f^{-1}(\tilde{x})) = A_\mu(\tilde{x}^\nu - \xi^\nu(\tilde{x}) + O(\xi^2)) = A_\mu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2)$$

Combining this expression with the internal transformation at first order in ξ, we get

$$\begin{aligned}\tilde{A}_\mu(\tilde{x}) &= \left(1-\frac{\Delta}{D}\partial_\sigma\xi^\sigma(\tilde{x}) + \partial_\sigma\xi^\sigma(\tilde{x}) - \partial_\mu\xi^\nu(\tilde{x})\right)(A_\mu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x})) + O(\xi^2)\\ &= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta-1}{D}\partial_\sigma\xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\partial_\mu\xi^\nu(\tilde{x}) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2)\end{aligned}$$

with $\xi(f^{-1}(\tilde{x})) = \xi(\tilde{x}) + O(\xi^2)$. This result can be simplified by using the conformal killing equation $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2\eta_{\mu\nu}\partial_\sigma \xi^\sigma / D$ as follows:

$$
\begin{aligned}
\tilde{A}_\mu(\tilde{x}) &= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta - 1}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\left(\frac{1}{2}\partial_\mu \xi^\nu(\tilde{x}) + \frac{1}{2}\partial_\mu \xi^\nu(\tilde{x})\right) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2) \\
&= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta - 1}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\left(\frac{1}{2}\partial_\mu \xi^\nu(\tilde{x}) - \frac{1}{2}\partial_\nu \xi^\mu(\tilde{x}) + \delta^\nu_\mu \partial_\sigma \xi^\sigma(\tilde{x})\frac{1}{D}\right) - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2) \\
&= A_\mu(\tilde{x}) - A_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\nu(\tilde{x})\underbrace{\left(\frac{1}{2}\partial_\mu \xi^\nu(\tilde{x}) - \frac{1}{2}\partial^\nu \xi_\mu(\tilde{x})\right)}_{M_\mu{}^\nu} - \xi^\nu(\tilde{x})\partial_\nu A_\mu(\tilde{x}) + O(\xi^2).
\end{aligned}
$$

From this transformed gauge field, we calculate the transformation of gauge field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ to $\tilde{F}_{\mu\nu}$. We start by writting the transformation law of the derivatives used to construct $F_{\mu\nu}$. The chain rule yields

$$
\tilde{\partial}_\mu \equiv \frac{\partial}{\partial \tilde{x}^\mu} = \left(\frac{\partial f^{-1}(\tilde{x})^\nu}{\partial \tilde{x}^\mu}\right)_{\tilde{x}} \left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} = \left(\frac{\partial \tilde{x}^\nu - \xi^\nu(\tilde{x})}{\partial \tilde{x}^\mu}\right)_{\tilde{x}} \left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} = \left(-\frac{\partial \xi^\nu(\tilde{x})}{\partial \tilde{x}^\mu}\right)_{\tilde{x}} \left(\frac{\partial}{\partial x^\nu}\right)_{\tilde{x}} + \left(\frac{\partial}{\partial x^\mu}\right)_{\tilde{x}} \equiv -\partial_\mu \xi^\nu(\tilde{x})\partial_\nu + \partial_\mu.
$$

where the subscripts indicate thet a partial derivative with respect to x^μ should be precomposed with $x = f^{-1}(x^\mu)$ to yield a function dependant on the left hand side variable $\tilde{x}$. Now we can calculate the transformed field strength at first order in ξ to be

$$
\begin{aligned}
\tilde{F}_{\mu\nu} &= \tilde{\partial}_\mu \tilde{A}_\nu - (\mu \leftrightarrow \nu) \\
&= \left(-\partial_\mu \xi^\rho(\tilde{x})\partial_\rho + \partial_\mu\right)\left(A_\nu(\tilde{x}) - A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - A_\lambda(\tilde{x})M_\nu{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x})\right) - (\mu \leftrightarrow \nu) \\
&= \partial_\mu A_\nu(\tilde{x}) - (\partial_\mu \xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x})\right) - \partial_\mu\left(A_\lambda(\tilde{x})M_\nu{}^\lambda\right) - \partial_\mu\left(\xi^\lambda(\tilde{x})\partial_\lambda A_\nu(\tilde{x})\right) - (\mu \leftrightarrow \nu) \\
&= \partial_\mu A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x})\right) - \partial_\mu A_\lambda(\tilde{x})\partial_\nu \xi^\lambda(\tilde{x}) - A_\lambda(\tilde{x})\partial_\mu\partial_\nu \xi^\lambda(\tilde{x}) - \xi^\lambda(\tilde{x})\partial_\lambda\partial_\mu A_\nu(\tilde{x}) - 2(\partial_\mu \xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - (\mu \leftrightarrow \nu) \\
&= \partial_\mu A_\nu(\tilde{x}) - \partial_\mu\left(A_\nu(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x})\right) - (\partial_\mu A_\lambda(\tilde{x}))M_\nu{}^\lambda - A_\lambda(\tilde{x})\partial_\mu M_\nu{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda\partial_\mu A_\nu(\tilde{x}) - 2(\partial_\mu \xi^\lambda(\tilde{x}))\partial_\lambda A_\nu(\tilde{x}) - (\mu \leftrightarrow \nu) \\
&= F_{\mu\nu}(\tilde{x}) - F_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x}) - A_{(\nu}(\tilde{x})\frac{\Delta}{D}\partial_{\mu)}\partial_\lambda \xi^\lambda(\tilde{x}) - (\partial_{(\mu} A_\lambda(\tilde{x}))M_{\nu)}{}^\lambda - A_\lambda(\tilde{x})\partial_{(\mu} M_{\nu)}{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda F_{\mu\nu}(\tilde{x}) - 2(\partial_{(\mu} \xi^\lambda(\tilde{x}))\partial_\lambda A_{\nu)}(\tilde{x}) \\
&= F_{\mu\nu}(\tilde{x}) - F_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\lambda \xi^\lambda(\tilde{x}) - A_{(\nu}(\tilde{x})\frac{\Delta}{D}\partial_{\mu)}\partial_\lambda \xi^\lambda(\tilde{x}) - (\partial_{(\mu} A_\lambda(\tilde{x}))M_{\nu)}{}^\lambda - \xi^\lambda(\tilde{x})\partial_\lambda F_{\mu\nu}(\tilde{x}) - 2(\partial_{(\mu} \xi^\lambda(\tilde{x}))\partial_\lambda A_{\nu)}(\tilde{x})
\end{aligned}
$$

where we simplified further by expliciting

$$
2\partial_{(\mu} M_{\nu)}{}^\lambda = \partial_\mu\partial_\nu \xi^\lambda(\tilde{x}) - \partial_\mu\partial^\lambda \xi_\nu(\tilde{x}) - \partial_\nu\partial_\mu \xi^\lambda(\tilde{x}) - \partial_\nu\partial^\lambda \xi_\mu(\tilde{x}) = 0.
$$

We note that the transformation law of $F_{\mu\nu}$ involves A_μ homogeneously which is an example of mixing of CFT fields under the transformation of a descendant.

(b) For a D-dimensionnal spacetime, the Maxwell action reads

$$
S = \int \mathrm{d}^D x \sqrt{|g|}\,\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \int \mathrm{d}^D x \sqrt{|g|}\; g^{\mu\sigma} g^{\nu\rho}\,\frac{1}{4}F_{\mu\nu}F_{\sigma\rho}.
$$

where g is the metric (which we suppose conformaly flat). We aim to apply the results found in (a) to determine when this action gains conformal symmetry. Under a conformal transformation given by the killing vector $\xi^\mu(x)$ and the scaling $\Omega(x) = 1 + \partial_\mu \xi^\mu(x)/D + O(\xi^2)$ of the metric components, we have

$$
\begin{aligned}
&g_{\nu\rho}(x) = \Omega(f(x))^{-2}\tilde{g}_{\nu\rho}(f(x)) = \Omega(\tilde{x})^{-2}\tilde{g}_{\nu\rho}(\tilde{x}) \quad \text{Defining property of a conformal transformation} \\
&|g|(x) = \Omega(f(x))^{-2D}|\tilde{g}|(f(x)), \quad g^{\nu\rho}(x) = \Omega(f(x))^{+2}\tilde{g}^{\nu\rho}(f(x)) = \Omega(\tilde{x})^2\tilde{g}^{\nu\rho}(\tilde{x}), \quad \mathrm{d}^D x\sqrt{|g|} = \mathrm{d}^D \tilde{x}\;\Omega(\tilde{x})^{-D}\sqrt{|\tilde{g}|(\tilde{x})}
\end{aligned}
$$

Without loss of generality, we take the traget metric $\tilde{g}$ to be the Minkowski metric. Inverting the result found in (a) for the transformation of the gauge field, we write

$$
\begin{aligned}
A_\mu(x) = |\partial x/\partial\tilde{x}|_{\tilde{x}}^{-\Delta/D}(R^{-1})^\nu_\mu \tilde{A}_\nu(\tilde{x}) &= \tilde{A}_\mu(\tilde{x}) + \tilde{A}_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) - \tilde{A}_\mu(\tilde{x})\partial_\sigma \xi^\sigma(\tilde{x})\frac{1}{D} + \tilde{A}_\nu(\tilde{x})\partial_\mu \xi^\nu(\tilde{x}) + O(\xi^2) \\
&= \tilde{A}_\mu(\tilde{x}) + \tilde{A}_\mu(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) + \frac{1}{2}\tilde{A}_\nu(\tilde{x})\left(\partial_\mu \xi^\nu(\tilde{x}) - \partial^\nu \xi_\mu(\tilde{x})\right) + O(\xi^2).
\end{aligned}
$$

Then, with the derivative $(\partial_\mu)_{\tilde{x}} = \tilde{\partial}_\mu \xi^\nu(\tilde{x})\tilde{\partial}_\nu + \tilde{\partial}_\mu$, the field strength transforms as

$$
\begin{aligned}
F_{\mu\nu} = \partial_\mu A_\nu(x) - (\mu \leftrightarrow \nu) &= \left(\tilde{\partial}_\mu \xi^\lambda(\tilde{x})\tilde{\partial}_\lambda + \tilde{\partial}_\mu\right)\left(\tilde{A}_\nu(\tilde{x}) + \tilde{A}_\nu(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_\lambda(\tilde{x})M_\nu{}^\lambda\right) - (\mu \leftrightarrow \nu) \\
&= \tilde{\partial}_\mu\left(\tilde{A}_\nu(\tilde{x}) + \tilde{A}_\nu(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_\lambda(\tilde{x})M_\nu{}^\lambda\right) + \tilde{\partial}_\mu \xi^\lambda(\tilde{x})\tilde{\partial}_\lambda \tilde{A}_\nu(\tilde{x}) - (\mu \leftrightarrow \nu) \\
&= \tilde{F}_{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x})\frac{\Delta}{D}\partial_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_{(\nu}(\tilde{x})\frac{\Delta}{D}\tilde{\partial}_{\mu)}\tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{\partial}_{(\mu}(\tilde{A}_\lambda(\tilde{x}))M_{\nu)}{}^\lambda + \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x})\tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x})
\end{aligned}
$$

The contravariant equivalent of this result is given by

$$\begin{aligned}
F^{\mu\nu} = g^{\mu\sigma} g^{\nu\rho} F_{\sigma\rho} &= \Omega(\tilde{x})^4 \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} F_{\sigma\rho} \\
&= \Omega(\tilde{x})^4 \left(\tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}^{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} (\tilde{A}_\lambda(\tilde{x})) M_{\rho)}{}^\lambda + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\rho)}(\tilde{x}) \right)
\end{aligned}$$

Next, we calculate

$$\begin{aligned}
F_{\mu\nu} F^{\mu\nu} = &\left(\tilde{F}_{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + O(\xi^2) \right) \\
&\times \Omega(\tilde{x})^4 \left(\tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}^{\mu\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} (\tilde{A}_\lambda(\tilde{x})) M_{\rho)}{}^\lambda + \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho} \tilde{\partial}_{(\sigma} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\rho)}(\tilde{x}) \right) \\
&= \Omega(\tilde{x})^4 \left(\tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) \frac{2\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{A}_{(\nu}(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_{\mu)} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) \right) \\
&= \Omega(\tilde{x})^4 \left(\tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) + \tilde{F}_{\mu\nu}(\tilde{x}) \tilde{F}^{\mu\nu}(\tilde{x}) \frac{2\Delta}{D} \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} (\tilde{A}_\lambda(\tilde{x})) M_{\nu)}{}^\lambda + 2\tilde{F}^{\mu\nu} \tilde{\partial}_{(\mu} \xi^\lambda(\tilde{x}) \tilde{\partial}_\lambda \tilde{A}_{\nu)}(\tilde{x}) + 4\tilde{F}^{\mu\nu} \tilde{A}_\nu(\tilde{x}) \frac{\Delta}{D} \tilde{\partial}_\mu \tilde{\partial}_\sigma \xi^\sigma(\tilde{x}) \right)
\end{aligned}$$

We want to show that the last line vanishes or is a total derivative.

2 Axial anomaly

(a)

(b)

(c)

(d)

3 OPE coefficients from three point functions

(a)

(b)

(c)

(d)

4 Acknowledgement

References

[1] *Frobenius inner product*. https://en.wikipedia.org/wiki/Frobenius_inner_product. Accessed: 2024-03-14.

[2] *Schmidt decomposition*. https://en.wikipedia.org/wiki/Schmidt_decomposition. Accessed: 2024-03-14.