

PSIQFT2023

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# Contents

0.1	To * or not to * . . . . .	11
0.2	Convention . . . . .	11
0.3	Overview . . . . .	12
<b>I</b>	<b>When Quantum Mechanics Meets Special Relativity</b>	<b>13</b>
<b>1</b>	<b>A Mathematical Interlude: symmetries and group theory</b>	<b>17</b>
1.1	Review of Quantum Mechanics . . . . .	17
1.2	The operators for symmetry transformation . . . . .	18
1.3	What is a group? . . . . .	20
1.3.1	Finite groups . . . . .	21
1.3.2	Lie groups . . . . .	22
1.3.3	Group trivia . . . . .	24
1.4	Representation of a group . . . . .	25
1.5	The structure constants . . . . .	26
1.6	Specific Representations . . . . .	30
1.6.1	Adjoint Representation . . . . .	30
1.6.2	A general recipe and everything about $SU(2)$ . . . . .	32
1.7	Poincaré group . . . . .	37
1.7.1	*Parity and Time reversal . . . . .	39
<b>2</b>	<b>Scattering amplitude for every theory</b>	<b>41</b>
2.1	Classifying the one particle state . . . . .	41
2.1.1	Massive states . . . . .	45
2.2	Multi: Asymptotic states . . . . .	47
2.3	S-matrix . . . . .	49
2.4	Special relativity comes in . . . . .	50
<b>3</b>	<b>S-matrix computation and the last condition on <math>V</math></b>	<b>53</b>
3.1	Creation and annihilation operator . . . . .	53
3.2	Graphic evaluation . . . . .	57
3.3	The condition on the scattering matrix . . . . .	59
3.3.1	*Interlude: Cluster decomposition principle . . . . .	60
3.4	Condition for $V$ . . . . .	62

3.5	Summary: Requirements for the interaction $V$ . . . . .	62
3.5.1	From Lorentz invariance . . . . .	62
3.5.2	From cluster decomposition principle . . . . .	63
<b>4</b>	<b>Fields! Finally fields!</b>	<b>65</b>
4.1	Why fields instead of operators? . . . . .	65
4.2	*Requirements on the coefficients . . . . .	68
4.2.1	Translations . . . . .	69
4.2.2	Boosts . . . . .	69
4.2.3	Rotations . . . . .	70
4.3	Nod to decomposition cluster principle . . . . .	70
4.4	What about causality? . . . . .	71
4.5	Charged field and the anti-particle . . . . .	71
<b>5</b>	<b>Scalar fields and the Dirac fields</b>	<b>73</b>
5.1	Causal scalar fields . . . . .	73
5.2	The spinor representation . . . . .	75
5.2.1	*Parity in spinor representation . . . . .	79
5.3	*The Dirac field . . . . .	81
<b>6</b>	<b>Concluding remarks</b>	<b>85</b>
<b>II</b>	<b>The Dirac fields and its interactions</b>	<b>87</b>
<b>7</b>	<b>Quantize the Dirac field, again</b>	<b>89</b>
7.1	Step 0: study the classical Dirac field . . . . .	89
7.1.1	Historical Dirac equation derivation . . . . .	89
7.1.2	Relativistic covariance of the Dirac equation . . . . .	94
7.1.3	*Classical solution of the Dirac equation . . . . .	97
7.1.4	Weyl spinors . . . . .	101
7.1.5	Interlude: Majorana spinors and their Lagrangian . . . . .	104
7.2	Step 1: The Dirac Lagrangian . . . . .	107
7.2.1	Weyl Lagrangian . . . . .	110
7.2.2	*Majorana Lagrangian . . . . .	111
7.3	Step 2: The conjugate momentum and the Hamiltonian . . . . .	112
7.3.1	The conjugate momentum and degrees of freedom . . . . .	112
7.3.2	The Hamiltonian . . . . .	112
7.4	Step 3: The commutator/anti-commutator relationship . . . . .	113
7.5	Step 4: Normal ordering . . . . .	115
7.6	Dirac field Quantized . . . . .	116
7.6.1	States . . . . .	116
7.6.2	Anti-particle . . . . .	117
7.6.3	Heisenberg picture . . . . .	118
7.6.4	Causality . . . . .	119
7.7	The propagator . . . . .	120

CONTENTS	5
<b>8 The interaction terms: possibilities and choices</b>	<b>123</b>
8.1 Discrete symmetries and bilinears classification . . . . .	123
8.2 The continuous phase symmetry that leads to QED . . . . .	126
8.3 A case study: QED amuse-bouche . . . . .	127
<b>9 Generic Interactions</b>	<b>131</b>
9.1 Review of the process . . . . .	131
9.2 Fermionic modification . . . . .	132
9.2.1 Initial and final states . . . . .	132
9.2.2 Fermionic LSZ reduction formula . . . . .	133
9.2.3 Dyson's formula . . . . .	133
9.2.4 Wick's theorem . . . . .	134
9.3 Propagator as a Green's function . . . . .	136
9.4 Wick's theorem for multi-fermions . . . . .	138
<b>10 Fermionic Feynman Rules from the Yukawa theory</b>	<b>139</b>
10.1 The Yukawa Lagrangian . . . . .	139
10.2 Computation of a 2-to-2 scattering process . . . . .	140
10.3 Some Feynman rules extracted . . . . .	143
<b>11 *Interlude: Quantization of Maxwell Theory</b>	<b>147</b>
11.0.1 Review of scalar theory . . . . .	147
11.0.2 Action, equation of motion, gauge symmetry . . . . .	148
11.0.3 Quantization . . . . .	151
<b>12 QED at tree level</b>	<b>157</b>
12.0.1 Feynman rules . . . . .	157
12.0.2 Scattering in QED: amplitudes . . . . .	158
12.0.3 *From beginning to end: a cross section in QED . . . . .	160
<b>13 Renormalization</b>	<b>169</b>
13.1 Plan . . . . .	169
13.2 Definition . . . . .	169
13.3 A general scenario: blame the coupling constant . . . . .	169
13.4 Turbulence diffusion . . . . .	170
13.5 Debye-Hückel's theory of the electron gas . . . . .	171
13.5.1 Figuring out the charge density of electrons . . . . .	171
13.5.2 Electric potential from Poisson equation . . . . .	172
13.6 Dimensional Regularization . . . . .	173
13.6.1 An infinite line of charge . . . . .	173
13.6.2 Cutoff regularization . . . . .	174
13.6.3 Dimensional regularization . . . . .	175
13.6.4 Sweeping the infinite away . . . . .	176
13.6.5 Amuse-bouche: Renormalization group . . . . .	176
13.7 Quantum Mechanics . . . . .	177
13.7.1 An Appetizer: two-state system . . . . .	177

13.7.2 *A variation on the quantum harmonic oscillator . . . . .	179
13.8 Renormalization in QFT . . . . .	182
13.8.1 The renormalization procedure: General Remarks . . . . .	182
13.8.2 The renormalization procedure: a recipe and a toy model	183
13.9 Renormalization in QED . . . . .	193
13.9.1 Superficial degree of divergence . . . . .	193
13.9.2 Renormalization in QED . . . . .	195
13.9.3 *Electron propagator/self-energy . . . . .	198
13.9.4 Vacuum polarization . . . . .	200
13.9.5 Vertex . . . . .	200
<b>A How I learned to stop worrying and love group theory: Visualizing Finite Groups</b>	<b>201</b>
A.1 Eat, Play and Sleep . . . . .	201
A.2 Five Families of Finite Groups . . . . .	205
<b>B Quotient group explained with a mattress</b>	<b>213</b>
B.1 Introduction to mattress group . . . . .	213
B.2 Invariant subgroup and the quotient group . . . . .	213
B.3 Why no overlap between cosets? . . . . .	214
<b>C Classification of Simple Lie Algebra</b>	<b>215</b>
C.1 Our good old friend $SU(2)$ is everywhere . . . . .	215
C.2 What does simple geometry tell us? . . . . .	217
<b>D Wigner Theorem</b>	<b>225</b>
D.1 Transformation of complete set gives complete set . . . . .	225
D.2 Consider a special state . . . . .	226
D.3 For states that are made with three states . . . . .	226
D.4 Unitary . . . . .	227
<b>E Why can we not observe boost?</b>	<b>229</b>
E.1 Classically . . . . .	229
E.2 Quantum mechanically . . . . .	229
E.3 From the algebra . . . . .	229
<b>F Massless states</b>	<b>231</b>
<b>G Useful spinor identities</b>	<b>235</b>
G.1 Some calculations . . . . .	236
G.1.1 Properties of the fifth gamma matrix . . . . .	236
G.1.2 Spinor gymnastics . . . . .	237
<b>H <math>\gamma</math> matrices and spinors in higher dimensions</b>	<b>239</b>

<b>I</b>	<b><i>C</i> and <i>P</i>, particle level and field level</b>	<b>241</b>
I.1	Parity . . . . .	241
I.2	Charge conjugation . . . . .	241
<b>J</b>	<b>Wick theorem: can we swap <math>\psi</math> and <math>\bar{\psi}</math> confidently?</b>	<b>243</b>
<b>K</b>	<b>Alternative calculation to extract Feynman rules for fermions</b>	<b>245</b>
<b>L</b>	<b>How many degrees of freedom does photon have?</b>	<b>249</b>
<b>M</b>	<b>Gamma function</b>	<b>251</b>
<b>N</b>	<b>A common way renormalization is introduced</b>	<b>253</b>
N.0.1	QED Lagrangian revisited . . . . .	253
N.0.2	Electron propagator/self-energy . . . . .	255
N.0.3	Vaccum polarization . . . . .	256
N.0.4	Vertex . . . . .	257



# **Disclaimer**

The author is not a native speaker and apologize if sometimes the English is not perfectly understandable. If you find any confusing choice of words or phrases or structures of sentences, please contact the author to help improve the lecture notes together. It will be very appreciated.



# Acknowledgment

We generously consult Carter's "Visual Group Theory" (ch1-ch7), Joglekar's "Mathematical Physics, Advanced Topics" (ch7), Weinberg's "The Quantum Theory of Fields" (ch2-ch5) and Georgi's "Lie Algebras in Particle Physics" (ch2-ch3) for the Part 1 of the book. We draw heavily from the excellent textbooks and lecture notes by David Tong's lecture notes (ch 4-6), Greiner and Reinhardt's Quantum electrodynamics (ch. 5-7), Casalbuoni's quantum field theory, (ch.4) Peskin and Schroeder's Introduction to quantum field theory (ch 3-4). The part of renormalization draws insights from "Renormalization methods" by W.D. McComb (ch 1-3), "Renormalization without infinities-an elementary tutorial" by Arnold Neumaier, "Regularization, renormalization, and dimension analysis: Dimensional regularization meets freshman E&M" by Fredrick Olness and Randall Scalise, "A hint of renormalization" by Bertrand Delamotte.

The author would like to thank her husband and colleague Dan Wohns for many private conversations.

## 0.1 To \* or not to \*

If a part of the material is starred, it is not examined in the interview even if it might be covered during the lectures.

## 0.2 Convention

Throughout the course, we will only incorporate special relativity and we use mostly minus signature for the metric

$$\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1). \quad (1)$$

We always use the natural units

$$c = \hbar = 1. \quad (2)$$

We use Einstein's summation whenever applicable.

We use **1** for identity operator and **1** for identity matrix.

We use upper case Greek letters  $\Psi, \Phi$  for quantum states and reserve the lower case Greek letters  $\psi, \phi, \chi$  for fields. We use lower case Greek letters  $\alpha, \beta$  for quantum multi-states and  $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$  for index run through spacetime.

We use lower case roman letters  $i, j, k = 1, 2, 3$  for index run through space and  $l$  for labeling the dimension of the representation of Poincare group.

We use bold letters to denote three spatial dimensional objects and regular letters to denote four dimensional objects. For example  $\mathbf{p}$  to denote 3-momentum and  $p$  to denote 4-momentum.

We will introduce a few other conventions within the context.

### 0.3 Overview

The goal of this course is to learn about quantum field theory with nonzero spin. In particular, we will learn how the fields interact using an example, Quantum Electrodynamics (QED).

But we hesitate to take the approach of writing down the Lagrangian of the field and canonical quantize it. Here is why.

You may have all heard the saying “QFT is the union of special relativity and quantum mechanics”, but who ordered the fields?

In the first part of the course, we are going to follow Weinberg’s Volume 1 to show that QFT is the only way (we found so far) for quantum mechanics and special relativity to reconcile. It turns out QFT seems to be inevitable.

Then we will build our fields from scratch (first principles like Lorentz covariance). We will then realize the fields we build are the solutions of Klein-Gordon equation, Dirac equation, etc!

Once we have our fields, we will follow the standard approach to study the interaction between them.

But please be patient at the beginning, and enjoy the story of when quantum mechanics meets special relativity.

## Part I

# When Quantum Mechanics Meets Special Relativity



# Plan

In the first part of the course, we will start with quantum mechanics, and study the impact of special relativity.

1. Symmetry: Lorentz transformations and translations. We take a detour to study the mathematical tool for studying symmetries: Lie group and Lie algebra.
2. We classify our states using the symmetry. Then we compute the scattering amplitude between the initial state and final state (the S-matrix) for an arbitrary theory (perturbatively). What does Lorentz invariant of S-matrix require of the interaction?
3. We introduce creation and annihilation operators. Then we further compute the S-matrix. What does cluster decomposition principle require of the interaction?
4. We argue that in order to build Lorentz invariant theory, we need to introduce fields. Fields are linear combination of the creation/annihilation operators. What does Lorentz covariance require of the coefficients?
5. We give two examples of fields: the scalar field and the Dirac field. We build them carefully out of the creation/annihilation operators and choose the correct coefficients. (Important: we have no idea about the Dirac's equation when we build the Dirac field!)



# Chapter 1

## A Mathematical Interlude: symmetries and group theory

The way we go about writing down a relativistic quantum theory is to start with quantum mechanics and see where we get by imposing Lorentz covariance on the theory. In other words, we want our quantum theory to enjoy the symmetry of special relativity: laws of physics in different inertial frames (related by Lorentz transformation and translations) should be the same. Hence the first step of our journey, is to study symmetries, in mathematics, this is done by group theory.

### 1.1 Review of Quantum Mechanics

”Good news, everyone.” QFT is based on the same old quantum mechanics developed in the 1920-30’s. Here we review a bit about quantum mechanics.

1. Physical states are represented by vectors in the Hilbert space<sup>1</sup>. The Hilbert space is a complex vector space. We will use Dirac’s ket-notation. Thus each ket  $|\Phi\rangle$ <sup>2</sup> in the vector space has its corresponding bra  $\langle\Phi|$  living in the adjoint vector space.

The inner product (complex number) defined on the vector space satisfies the following property,

$$\langle\Phi|\Psi\rangle = \langle\Psi|\Phi\rangle^*. \quad (1.1)$$

---

<sup>1</sup>To be precise, it is represented by rays. One ray is made of vectors that are proportional to each other. Rays make things more complicate. We will refer you to Weinberg’s book ch2.7 to read upon rays.

<sup>2</sup>We denote a physical state by uppercase Greek letters  $\Phi, \Psi$  etc and reserve the lowercase ones for the fields when they emerge.

The inner product is linear in the vector argument and anti-linear in the adjoint vector argument.

$$\begin{aligned}\langle \Phi | a\Psi_1 + b\Psi_2 \rangle &= a\langle \Phi | \Psi_1 \rangle + b\langle \Phi | \Psi_2 \rangle \\ \langle a\Psi_1 + b\Psi_2 | \Psi \rangle &= a^* \langle \Psi_1 | \Psi \rangle + b^* \langle \Psi_2 | \Psi \rangle\end{aligned}\quad (1.2)$$

The norm  $\langle \Phi | \Phi \rangle \geq 0$  and is 0 only if  $|\Phi\rangle = 0$ .

2. The observables are represented by Hermitian linear operators. For each operator  $L$ , we can also define its adjoint by

$$\langle \Phi | L^\dagger \Psi \rangle \equiv \langle L\Phi | \Psi \rangle = \langle \Psi | L\Phi \rangle^* \quad (1.3)$$

where  $|L^\dagger \Psi\rangle = L^\dagger |\Psi\rangle$  and  $\langle L\Phi|$  is the adjoint of  $\langle L|\Phi\rangle$ . When  $L^\dagger = L^{T*} = L$ ,  $L$  is a Hermitian operator with real eigenvalues (can possibly be observed). If

$$L|\Psi_n\rangle = \lambda_n |\Psi_n\rangle, \quad (1.4)$$

where we use  $n$  to label different eigenstates, then  $|\Psi_n\rangle$  is the eigenvector: when measured by  $L$ , we will find the observable at value  $\lambda_n$ . Note that eigenvectors corresponding to different eigenvalues are orthogonal.

3. If a system is in  $|\Psi\rangle$  and we measure the observable that corresponds to operator  $L$ , the system will end up in one of the eigenstate  $|\Psi_n\rangle$ . Physical information is coded in probabilities

$$P(|\Psi\rangle \rightarrow |\Psi_n\rangle) = |\langle \Psi | \Psi_n \rangle|^2. \quad (1.5)$$

Note

$$\sum_n P(|\Psi\rangle \rightarrow |\Psi_n\rangle) = 1. \quad (1.6)$$

## 1.2 The operators for symmetry transformation

If observer Alice sees the system in states like  $|\Psi\rangle$ ,  $|\Psi_n\rangle$  etc, another observer Bob (on a train or something, after all we want to put special relativity and quantum mechanics together.) will see the same system in different states like  $|\Psi'\rangle$ ,  $|\Psi'_n\rangle$  etc. Then symmetry means the result of the physical measurement is the same for different observers and we say those observers are connected by a symmetry transformation. In quantum mechanics, the physical information is coded in probabilities, so symmetry means

$$\begin{aligned}P(|\Psi\rangle \rightarrow |\Psi_n\rangle) &= P(|\Psi'\rangle \rightarrow |\Psi'_n\rangle) \\ |\langle \Psi | \Psi_n \rangle|^2 &= |\langle \Psi' | \Psi'_n \rangle|^2.\end{aligned}\quad (1.7)$$

Wigner proved in early 30's the transformation between the corresponding states of the two observers can be described by an operator  $U^{(A)}$ , that is

$$|\Psi'\rangle = U^{(A)}|\Psi\rangle. \quad (1.8)$$

The choice of this funny notation will be clear. Wigner proves that this operator  $U^{(A)}$  should be either unitary and linear  $U$  or anti-unitary and anti-linear  $U^A$ . The unitary and linear operator  $U$  satisfies

$$\begin{aligned} \langle U\Psi|U\Phi\rangle &= \langle\Psi|\Phi\rangle \\ U(a|\Psi\rangle + b|\Phi\rangle) &= aU|\Psi\rangle + bU|\Phi\rangle. \end{aligned} \quad (1.9)$$

And the anti-unitary and anti-linear operator  $U^A$ <sup>3</sup> satisfies

$$\begin{aligned} \langle U^A\Psi|U^A\Phi\rangle &= \langle\Psi|\Phi\rangle^* \\ U^A(a|\Psi\rangle + b|\Phi\rangle) &= a^*U^A|\Psi\rangle + b^*U^A|\Phi\rangle \end{aligned} \quad (1.10)$$

A crude argument<sup>4</sup> is as follows. To preserve the physical information, we demand

$$|\langle\Psi|\Phi\rangle|^2 = |\langle U^{(A)}\Psi|U^{(A)}\Phi\rangle|^2 \quad (1.11)$$

Since the adjoint of an operator is defined by

$$\langle U^{(A)}\Psi|\Phi\rangle = \langle\Psi|U^\dagger\Phi\rangle \text{ or } \langle\Psi|(U^A)^\dagger\Phi\rangle^*. \quad (1.12)$$

Formula (1.11) indicates that we need to have

$$|\langle\Psi|\Phi\rangle|^2 = |\langle U^{(A)}\Psi|U^{(A)}\Phi\rangle|^2 = |\langle\Psi|(U^A)^\dagger U^{(A)}|\Phi\rangle|^2 \quad (1.13)$$

An immediate choice that satisfy our demand is

$$(U^A)^\dagger U^{(A)} = \mathbf{1} \quad (1.14)$$

or equivalently,

$$(U^A)^\dagger = (U^{(A)})^{-1}. \quad (1.15)$$

which describes both unitary and antiunitary operators<sup>5</sup>.

---

<sup>3</sup>Notice that the definition of adjoint operator for anti-linear operators are different  $\langle\Phi|(U^A)^\dagger\Psi\rangle \equiv \langle U^A\Phi|\Psi\rangle^* = \langle\Psi|U^A\Phi\rangle$ . To see this consider  $\langle U^Aa\Phi|\Psi\rangle = a\langle U^A\Phi|\Psi\rangle$ ,  $a$  comes out to be linear because the anti-linear in the bra and anti-linear of the operator. Now let us consider  $\langle a\Phi|(U^A)^\dagger\Psi\rangle$  (assuming the adjoint operator defined the same way as the linear operator), but now  $a$  will come out anti-linear as an  $a^*$ . And we are left with  $a\langle U^A\Phi|\Psi\rangle = a^*\langle\Phi|(U^A)^\dagger\Psi\rangle$ . This is bizarre: the definition of adjoint should not depend the linear constant in front of it! Hence we have the different definition of adjoint for anti-linear operator.

<sup>4</sup>Interested readers are referred to Weinberg's quantum field theory book volume 1 chapter 2 appendix A

<sup>5</sup>Remember the anti-unitary operator's adjoint is defined differently!

Let us make a trivial but very enlightening observation: **1**, the identity, is a unitary and linear operator. Thus symmetries that can be made trivial by continuously varying some parameters must be also described unitary and linear operators. Thus restricted by the scope of this lecture<sup>6</sup>, we will be only interested in unitary and linear operators from this point onward. In other words, we will be only interested in transformations that are continuously connected<sup>7</sup> to identity for this mathematical interlude (We will be very interested in the discrete symmetry parity later).

Now let us think about the composition of the operators. Now we have three observers and for a particular system, the states they observe are  $|\Psi\rangle$ ,  $|\Psi'\rangle$ , and  $|\Psi''\rangle$  respectively. They are related by unitary operators in the following way,

$$\begin{aligned} |\Psi'\rangle &= U_1|\Psi\rangle \\ |\Psi''\rangle &= U_2|\Psi'\rangle \end{aligned} \tag{1.16}$$

Then we immediately have

$$|\Psi''\rangle = U_2U_1|\Psi\rangle \tag{1.17}$$

But on the other hand we can of course relate the states observed by first and the third observers directly,

$$|\Psi''\rangle = U_3|\Psi\rangle \tag{1.18}$$

So we have<sup>8</sup>

$$U_3 = U_2U_1. \tag{1.19}$$

And as we see, the operators that relate different observers seem to form a set in which different elements are related to each other in a particular way.

It turns out this is not a surprise, sets like this have a special name, they are called groups and mathematicians have described symmetries using groups for a long time!

### 1.3 What is a group?

Let us start with the definition of a group. A group is a set  $G$  that with a binary operation  $\circ$ <sup>9</sup> defined and satisfies the following properties.

---

<sup>6</sup>In the future, we will touch on a bit on one of the most famous anti-unitary anti-linear operator: time reversal.

<sup>7</sup>So the transformation can be achieved by a series of infinitesimal transformations

<sup>8</sup>Actually this formula only needs to work up to a phase. For a more rigorous discussion, interested readers are again referred to Weinberg's book.

<sup>9</sup>A binary operator takes two elements of the set to an element of the set. sometimes it is called the multiplication. Be aware that the multiplication in a group might have nothing to do with multiplication we know in arithmetic or linear algebra.

- The group is closed under the binary operation:  $\forall g_1, g_2 \in G$  imply  $g_1 \circ g_2 \in G$ .<sup>10</sup>
- The binary operator is associative:  $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ .
- The group has an identity:  $\exists e$  such that  $\forall g \in G$ ,  $e \circ g = g \circ e = g$ .<sup>11</sup>
- The group has an inverse:  $\forall g \in G \exists g^{-1}$  such that  $g \circ g^{-1} = g^{-1} \circ g = e$ .

Note that we only need to define the right identity and the right inverse, it can be shown that they are also the left identity and left inverse. In other words, the identity of the group and the inverse of a group element is unique.<sup>12</sup>

In particular, we have the cancellation law,

$$\begin{aligned} a \circ b = a \circ c &\text{ implies } b = c \\ b \circ a = c \circ a &\text{ implies } b = c. \end{aligned} \tag{1.20}$$

Basing on the definition of the group, we can see that the transformation operators  $U$  form a group with the identity operator being the identity and inverse operator  $U^{-1}$  being the inverse.

### 1.3.1 Finite groups

Certainly, we can classify groups into classes depending on the number of the element in the group. We will briefly look at finite groups.

Finite groups have finite number of elements. One important example that is very useful in physics is a group with two elements: the parity transformation and identity. This group contains two element:  $e$  and  $P|\mathbf{x}\rangle = |-\mathbf{x}\rangle$ . We immediate see  $P^2 = e$ .

Another example is the *n* – *cyclic* group, for example, all the *n* distinct solutions of the complex equation  $z^n = 1$  form a group  $C_n = \{z_a \equiv e^{i\frac{2\pi}{n}a}, a = 1, 2, \dots, n\}$ <sup>13</sup>. In this group we have  $z_a = z_1^a$ . That is the group that is completely generated by one element  $z_1$ .

There are of course groups that cannot be generated by a single element. Interested readers are referred to first chapter of Georgi's "Lie Algebras in Particle Physics", or chapter seven of Joglekar's "Mathematical Physics: Advanced Topics". Also if you are interested in finite groups, we recommend "Visual Group Theory" by Carter. Some of which is captured in appendix A. As the title indicates, there are lots of pictures!

There are groups that have infinite number of elements. But it is not described by continuous parameters. A favorite example is the addition of integer, with addition being the multiplication and 0 being the identity,  $-n$  being the

<sup>10</sup>Conventionally,  $\forall$  is a short form for "for all".

<sup>11</sup>Conventionally,  $\exists$  is a short form for "there exists".

<sup>12</sup>The proof is in chapter seven of Joglekar's book. Mathematical physics Advanced topics.

<sup>13</sup>A peeled mandarin orange also has a cyclic rotation symmetry described by  $C_n$ .

inverse of  $n$ . On the other hand, integers with multiplication being the multiplication is not a group, since 1 is the identity, but the inverse of an integer is not an integer.

Now let us move to Lie groups which will eventually help us to label our quantum states.

### 1.3.2 Lie groups

Lie groups are the groups that are described by a set of continuous parameters. We will use the following notation to denote a member of a Lie group  $g(\tilde{\varphi})$ , where  $\tilde{\varphi} = (\varphi_1, \varphi_2, \dots)$ . Each of  $\varphi_1, \varphi_2, \dots$  is a continuous parameter. For practical reasons we would like to choose  $g(\tilde{\varphi} = 0) = e$ .

For example, we have the rotational group in 2D-Euclidean space. A typical member is described by a single parameter  $g = g(\theta)$ , the rotational angle between the new  $x$  axis and the old  $x$  axis. The rotational group in 3D-Euclidean space is described by three parameters  $g = g(\theta_1, \theta_2, \theta_3)$ , corresponding to the rotational angle in  $xy$ -plane,  $yz$ -plane and  $zx$ -plane.

Let us take a more careful look at rotations in 3D-Euclidean space.

Any vector in this space can be represented by a ket  $|x\rangle$ , with a certain choice of orthonormal basis, we can find the component of the vector,

$$|x\rangle = \sum_{i=1}^3 x_i |e_i\rangle \equiv x_i |e_i\rangle. \quad (1.21)$$

Here we introduce the Einstein convention and omit the summation sign. From now on, repeated indices are always understood as summed over unless otherwise specified. We define the scalar product as the conventional one<sup>14</sup>,

$$\langle x|y \rangle = x_i y_i. \quad (1.22)$$

And the norm of the vector is

$$\|x\|^2 = x_i x_i. \quad (1.23)$$

The Euclidean metric is defined to be

$$g_{ij} = \langle e_i | e_j \rangle = \delta_{ij}. \quad (1.24)$$

The Kronecker delta is defined as usual

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}.$$

Then we define rotations ( $R$  in  $|x'\rangle = R|x\rangle$ ) to be the operators that preserve the norm of the vectors.  $R$  is a  $3 \times 3$  matrix with real entries.

$$\langle x|x \rangle = \langle x'|x' \rangle. \quad (1.25)$$

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<sup>14</sup>Remember we already start to use the Einstein convention here.

This gives us the condition that defines the rotational group:

$$RR^T = \mathbf{1}. \quad (1.26)$$

It is easy to see that this transformation preserves the Euclidean metric,

$$g'_{ij} = \langle e'_i | e'_j \rangle = \delta_{ij} = g_{ij}. \quad (1.27)$$

This group is known as  $O(3)$ , one of the orthogonal group,  $O(N)$ .  $O(N)$  is just the rotational group in  $N$  dimensional vector space that preserves the Euclidean metric. Equivalently we can define them with condition (1.26). From condition (1.26), we can see,

$$\det(R) \det(R^T) = \det(\mathbf{1}). \quad (1.28)$$

So we have

$$(\det(R))^2 = 1 \Rightarrow \det(R) = \pm 1. \quad (1.29)$$

The operators that correspond to  $\det(R) = +1$  can be continuously connected to the identity and thus are unitary and linear and are thus special. They are known as the special orthogonal group, denoted as  $SO(N)$ .

In particular the  $SO(1, 3)$  group, also known as the Lorentz group contain “rotations” that preserve the  $1 + 3$  Minkowski metric  $\eta_{\mu\nu}$ , where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Before we move to another set of groups, we should count the number of real parameters in each of our group. This is the dimension of our Lie group. For an orthogonal group  $O(N)$ , we can consider them as rotations in 2-planes as we have already done in 2 or 3 dimensions. Thus we can calculate the number of rotational angles by choosing two out of  $N$  axis to form these 2 planes, so the number of real parameters is

$$n = \frac{N(N-1)}{2}. \quad (1.30)$$

We can also work out the number of real parameters as follows. For a general  $N \times N$  matrix, there are  $N^2$  entries. The orthogonal condition (1.26) provide  $N^2$  equations, but since they are symmetric<sup>15</sup>, only  $\frac{N(N+1)}{2}$  equations are independent, thus we are left with  $N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2}$  real parameters. We have the same number of parameters for  $SO(N)$  and  $O(N)$ , as the choice of  $\det R = +1$  is discrete and we do not lose a continuous parameter.

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<sup>15</sup>With indices restored, the condition is  $R_{ij}R_{ik} = \delta_{jk}$ , which is evidently symmetric between indices  $i$  and  $j$

Now let us move on to the next important group of groups, the unitary group  $U(N)$ .  $U(N)$  is defined to be  $N \times N$  matrices with complex entries that leave the metric  $g_{ij} = \delta_{ij}$  invariant. Thus we have the unitary condition,

$$UU^\dagger = \mathbf{1}. \quad (1.31)$$

This condition gives us

$$\det U (\det U)^* = 1. \quad (1.32)$$

In this case,

$$\det U = e^{i\theta}. \quad (1.33)$$

We define the special unitary group  $SU(N)$  in the same way, by demanding the determinant to be  $+1$ , but note that this time we do lose a real parameter  $\theta$ . To count how many real parameters we have for unitary group, we use the entries subtract independent equations strategy. There are  $2N^2$  entries in each  $N \times N$  complex matrix. And we have  $N^2$  complex equations. Because  $UU^\dagger$  is self adjoint, only  $N^2$  real equations are independent. So we have  $n = 2N^2 - N^2 = N^2$  real free parameters for  $U(N)$  group. For  $SU(N)$ , as we just mentioned, there is one less,  $n = N^2 - 1$ .

The Poincaré group (The Lorentz transformations plus translations) contains operators that correspond to both Lorentz transformations and translations and have  $n = \frac{4 \times 3}{2} + 4 = 10$  real parameters. We will look at Poincaré group in great details in the homework and later in the lecture.<sup>16</sup>

### 1.3.3 Group trivia

The followings are a collection of important definitions.

- Abelian group: If  $\forall a \in G$  and  $b \in G$  we have  $a \circ b = b \circ a$ ,  $G$  is Abelian.<sup>17</sup>
- Homomorphic: Two groups  $G$  and  $G'$  are said to be homomorphic, if there exists map  $f$  from  $G$  to  $G'$  that preserves group multiplication. That is we have  $f(g_1) \circ' f(g_2) = f(g_1 \circ g_2)$ , where  $\circ'$  is the multiplication in  $G'$ , which may not be the same as  $\circ$ , the multiplication in  $G$ .
- Isomorphic: Two groups  $G$  and  $G'$  are said to be isomorphic  $G \approx G'$  iff<sup>18</sup> there is a one-to-one correspondence between their elements that is preserved under group multiplication. That is they are homomorphic and the map  $f$  has inverse.

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<sup>16</sup>For other Lie groups, for example like symplectic groups, interested readers are referred to Cornwell's "Group Theory in Physics".

<sup>17</sup>What is purple and commute?An abelian grape.

<sup>18</sup>Iff is short for if and only if

- Direct product of groups: The direct product of the groups is defined to be an ordered pair:  $G \otimes G' \equiv \{(g, g') | \forall g \in G, g' \in G'\}$  and the multiplication is defined to be  $(g_1, g'_1) \circ (g_2, g'_2) \equiv (g_1 \circ g_2, g'_1 \circ g'_2)$ . We can see  $G_1 \equiv \{(g, e') | \forall g \in G\}$  is automatically an invariant subgroup of  $G \otimes G'$  and  $G_1 \approx G$ .
- Subgroup:  $A$  is a subgroup of  $G$  if we have  $A \subset G$ , such that  $A$  is closed under the group multiplication and the inverse of every element exists. Every group has two trivial subgroups that consists only of the identity and the group  $G$  itself.
- \*Invariant subgroup (normal subgroup): For a subgroup  $A$  of  $G$ , if we have  $\forall a \in A, \forall g \in G, g \circ a \circ g^{-1} \in A$ ,  $A$  is an invariant subgroup. All subgroups of an abelian group are invariant subgroups.<sup>19</sup> We can think about the invariant subgroup as a generalization of the concept commute. General commutation is no longer true when we talk about groups. But if we have an invariant subgroup, every element will commute with the invariant subgroup as a collection/set.
- \*Coset and quotient group: If we have an invariant subgroup<sup>20</sup>, we can create the cosets:  $\forall g \in G, x = gA$  is a (left=right) coset. Each coset is an equivalent class and the collection of all cosets  $\{x | x = gA\}$  form the quotient group  $\frac{G}{A}$ . In the appendix, we provide an example using the Klein4 group (also known as the mattress group)
- \*Why there are no non-trivial overlap between cosets? Suppose  $g' \in g_{1N}$  and  $g' \in g_{2N}$  where  $N$  is that normal/invariant subgroup. Then  $g' = g_1 h = g_2 h'$ , where  $h$  and  $h'$  etc belong to the invariant subgroup. According to the definition of the invariant subgroup  $g_2 h' = h'' g_2$ . Then we have  $g_1 h = h'' g_2$ , which means  $g_1 = h^{-1} h'' g_2 = h_{new} g_2$ , this means  $g_1$  is a member of  $g_{2N}$ . And  $g_{1N}$  is  $g_{2N}$ . They completely overlap.

## 1.4 Representation of a group

For  $G = \{a, b, c, \dots\}$ , we can associate a  $M \times M$  matrix<sup>21</sup>  $D(g)$ <sup>22</sup> to each element  $g$ . If these matrices preserve the group multiplication, that is  $\forall a \in G, b \in G, D(a)D(b) = D(a \circ b)$ <sup>23</sup>, then the set of all these matrices  $D(G)$  is a  $M \times M$  matrix representation of the group. The dimension of the representation is then  $M$ . Equivalently, we can also say the dimension of the representation is the dimension of the vector space the matrices act on.

<sup>19</sup>Rotations around  $z$ -axis is a subgroup but not an invariant subgroup of all proper rotations around a point in 3D.

<sup>20</sup>If we don't, we can still create left-cosets and right-cosets but then they are not the same and are less interesting.

<sup>21</sup>For most of the time we will restrict ourselves to matrix representations of a group.

<sup>22</sup>These matrices do not need to be distinct.

<sup>23</sup>Immediately we have  $D(e) = \mathbb{1}$ .

If the matrices associated to all group elements are all distinct, the representation is faithful, otherwise it is unfaithful.

We can always associate identity matrices to every single group element trivially. This trivial representation of the group is certainly unfaithful.

Suppose we have found two representations of the group  $D(G)$  and  $D'(G)$ . But they might not be both interesting! If we can find an  $M \times M$  invertible matrix  $S$ , such that  $\forall g \in G, D'(g) = S^{-1}D(g)S$ . Then the two representation  $D(G)$  and  $D'(G)$  are equivalent because they are related by a similarity transformation. Only inequivalent representations of a group are interesting.

For an  $M \times M$  matrix representation  $D(G)$  of  $G$ , if there exists a similarity transformation  $S$  such that the equivalent representation of the original can be put in block diagonal form,

$$\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_q \end{bmatrix}$$

where  $A_i, i = 1, 2, \dots, q > 1$  are square matrices with dimension  $d_i \times d_i$  such that  $\sum_{i=1}^q d_i = M$ , then the representation is (fully)<sup>24</sup> reducible. If this is not possible for any matrix  $S$ , the representation is irreducible. For a reducible representation, it is possible to find a similarity transformation such that each part of the blocks  $A_i$  is a  $d_i \times d_i$  irreducible representation  $D_i(G)$ .

Now recall transformation operators that map a state observed by one observer to another state of another observer. They furnish a representation<sup>25</sup> of a group!

## 1.5 The structure constants

We will focus on studies of Lie groups. Recall Lie groups are groups that depend on a set of real continuous parameters  $\tilde{\varphi} \equiv \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ . So we can denote group elements  $g^\alpha \equiv g(\tilde{\varphi}^\alpha)$  where  $\tilde{\varphi}^\alpha \equiv \{\varphi_1^\alpha, \varphi_2^\alpha, \dots, \varphi_n^\alpha\}$  and  $g^\beta \equiv g(\tilde{\varphi}^\beta)$ . In other words, we use a Greek upper letter to label the group element. And for their product element  $g^\alpha \circ g^\beta = g^\gamma$ , we must have  $g^\gamma \equiv g(\tilde{\varphi}^\gamma)$  and  $\tilde{\varphi}^\gamma = \tilde{f}(\tilde{\varphi}^\alpha, \tilde{\varphi}^\beta)$ . We use  $\tilde{f}$  to denote that the result will be an  $n$  element array.

Let us make some trivial but very enlightening observations of this function  $\tilde{f}(\tilde{\varphi}^\alpha, \tilde{\varphi}^\beta)$ . If  $\tilde{\varphi}^\alpha = 0$ ,  $g^\alpha = e$ , then  $g^\gamma = e \circ g^\beta = g^\beta$ , we have  $\tilde{f}(0, \tilde{\varphi}^\beta) = \tilde{\varphi}^\beta$ . Similarly, if  $\tilde{\varphi}^\beta = 0$ ,  $g^\beta = e$ , then  $g^\gamma = g^\alpha \circ e = g^\alpha$ , we have  $\tilde{f}(\tilde{\varphi}^\alpha, 0) = \tilde{\varphi}^\alpha$ . And of course we have  $\tilde{f}(0, 0) = 0$ . But what about for general  $\tilde{\varphi}^\alpha, \tilde{\varphi}^\beta$ ?

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<sup>24</sup>All the reducible representations in our course will be fully reducible. Georgi's book discusses more precise definition of reducible in section 1.4 and 1.8. In the more precise definition, representations can be reducible but not fully reducible.

<sup>25</sup>We will restrict ourselves to only ordinary representations here, the ones that differ by a phase are known as projective representations

Let us now make use of the toolbox of matrices and focus our attention to a certain unitary representation  $U$ . That is we associate  $g^\alpha, g^\beta, g^\gamma$  above with  $U(g(\tilde{\varphi}^\alpha)), U(g(\tilde{\varphi}^\beta)), U(g(\tilde{f}(\tilde{\varphi}^\alpha, \tilde{\varphi}^\beta)))$ . Then we have

$$U(\tilde{\varphi}^\alpha)U(\tilde{\varphi}^\beta) = U(\tilde{f}(\tilde{\varphi}^\alpha, \tilde{\varphi}^\beta)). \quad (1.34)$$

Finding a function  $\tilde{f}$  for general arguments seems too ambitious, or mission impossible. Let us restrict our attention to the group elements that are continuously connected to the identity and further only focus on a small vicinity around the identity. So let us consider infinitesimal parameters  $\tilde{\epsilon} \equiv \{\epsilon_1, \dots, \epsilon_n\}$  and a  $M \times M$  representation. We can linear expand the matrix and have<sup>26</sup>

$$U(\tilde{\epsilon}) = \mathbb{1}_{M \times M} + i\epsilon_a (T_a)_{M \times M} \quad (1.35)$$

where  $a = 1, 2, \dots, n$  and  $T_a$ s are  $M \times M$  matrices and are known as generators. Then we can always get the representation matrix of a group element that has finite parameters from the identity by

$$U(\tilde{\varphi}) = \lim_{N \rightarrow \infty} \left( \mathbb{1} + i \frac{\varphi_a}{N} T_a \right)^N \equiv e^{i\varphi_a T_a}. \quad (1.36)$$

As we have mentioned before, Wigner proved that these matrices need to be unitary and linear (or antiunitary and antilinear, but we are more interested in the ones that can be continuously connected to the identity. )

This impose the following condition on  $U(\tilde{\epsilon})$ .

$$\begin{aligned} U^\dagger(\tilde{\epsilon})U(\tilde{\epsilon}) &= \mathbb{1} & (1.37) \\ (\mathbb{1} - i\epsilon_a T_a^\dagger)(\mathbb{1} + i\epsilon_a T_a) &= \mathbb{1} \\ \mathbb{1} - i\epsilon_a T_a^\dagger + i\epsilon_a T_a + O(\epsilon^2) &= \mathbb{1} \\ i\epsilon_a (T_a - T_a^\dagger) &= 0 \end{aligned}$$

This gives the condition on the generators: they need to be Hermitian (so they are observables).

$$T_a = T_a^\dagger. \quad (1.38)$$

Now we are ready to explore the structure of the group by expanding to the second order. This is the MOST important calculation when learning Lie groups and Lie algebra. By following the exciting calculation in the footnote<sup>27</sup>, we obtain a very powerful result in the following simple form

$$[T_a, T_b] = if_{abc} T_c. \quad (1.50)$$

<sup>26</sup>The imaginary number  $i$  is chosen as a convention, such that the generators are Hermitian

<sup>27</sup>To do that, we need to expand to second order:

$$U(\tilde{\epsilon}) = \mathbb{1} + i\epsilon_a T_a + \frac{1}{2} \epsilon_a \epsilon_b l_{ab} + O(\epsilon^3) \quad (1.39)$$

Note that we can certainly choose  $l_{ab}$  to be symmetric, since the antisymmetric part would not contribute. If we apply group multiplication on the infinitesimal representation matrices,

The  $f_{abc}$ s are the structure constants of the group. They remain the same even if we use another representation that comes with a different set of generators, as they are the coefficient of expansion of function  $\tilde{f}$ , which is independent of the

we get

$$U(\tilde{\epsilon}^\alpha)U(\tilde{\epsilon}^\beta) = U(\tilde{f}(\tilde{\epsilon}^\alpha, \tilde{\epsilon}^\beta)) \quad (1.40)$$

We also expand  $\tilde{f}$  to second order in components.

$$f_a(\tilde{\epsilon}^\alpha, \tilde{\epsilon}^\beta) = A_a + B_{ba}\epsilon_b^\alpha + C_{ba}\epsilon_b^\beta + D_{bca}\epsilon_b^\alpha\epsilon_c^\beta + E_{bca}\epsilon_b^\alpha\epsilon_c^\alpha + F_{bca}\epsilon_b^\beta\epsilon_c^\beta + O(\epsilon^3) \quad (1.41)$$

This looks awful, but recall our observations of this expansion when either  $\tilde{\epsilon}^\alpha = 0$  or  $\tilde{\epsilon}^\beta = 0$  or both  $\tilde{\epsilon}^\alpha = 0$  and  $\tilde{\epsilon}^\beta = 0$ .

$$\begin{aligned} f_a(0, 0) &= 0 \\ f_a(0, \tilde{\epsilon}^\beta) &= \epsilon_a^\beta \\ f_a(\tilde{\epsilon}^\alpha, 0) &= \epsilon_a^\alpha. \end{aligned} \quad (1.42)$$

This tells us

$$A_a = 0, B_{ba} = \delta_{ba}, C_{ba} = \delta_{ba}, E_{bca} = 0, F_{bca} = 0. \quad (1.43)$$

Now (13.61) is significantly simplified and reads,

$$f_a(\tilde{\epsilon}^\alpha, \tilde{\epsilon}^\beta) = \epsilon_a^\alpha + \epsilon_a^\beta + D_{bca}\epsilon_b^\alpha\epsilon_c^\beta \quad (1.44)$$

In order for the group multiplication to work out, we expand (1.40) to the second order on both sides.

$$\begin{aligned} (\mathbb{1} + i\epsilon_a^\alpha T_a + \frac{1}{2}\epsilon_a^\alpha\epsilon_b^\alpha l_{ab})(\mathbb{1} + i\epsilon_c^\beta T_c + \frac{1}{2}\epsilon_c^\beta\epsilon_d^\beta l_{cd}) &= \mathbb{1} + i f_a(\tilde{\epsilon}^\alpha, \tilde{\epsilon}^\beta) T_a + \frac{1}{2} f_a(\tilde{\epsilon}^\alpha, \tilde{\epsilon}^\beta) f_b(\tilde{\epsilon}^\alpha, \tilde{\epsilon}^\beta) l_{ab} \\ &= \mathbb{1} + i(\epsilon_a^\alpha + \epsilon_a^\beta + D_{bca}\epsilon_b^\alpha\epsilon_c^\beta) T_a \\ &\quad + \frac{1}{2}(\epsilon_a^\alpha + \epsilon_a^\beta + D_{ef}f_a\epsilon_e^\alpha\epsilon_f^\beta)(\epsilon_b^\alpha + \epsilon_b^\beta + D_{gh}h\epsilon_g^\alpha\epsilon_h^\beta) l_{ab}. \end{aligned} \quad (1.45)$$

To our delight, most terms will just cancel out or be higher order, we are actually only left with one term on the left and two terms on the right at the second order, so the equation becomes

$$\begin{aligned} -\epsilon_a^\alpha\epsilon_b^\beta T_a T_b &= i D_{bca}\epsilon_b^\alpha\epsilon_c^\beta T_a + \epsilon_a^\alpha\epsilon_b^\beta l_{ab} \\ &= i D_{abc}\epsilon_a^\alpha\epsilon_b^\beta T_c + \epsilon_a^\alpha\epsilon_b^\beta l_{ab}. \end{aligned} \quad (1.46)$$

On the last line we cycle our indices so that we have the same indices on the infinitesimal parameters. Since this must work for all infinitesimal parameters, we can drop the  $\epsilon$ s and have

$$\begin{aligned} -T_a T_b &= i D_{abc} T_c + l_{ab} \\ -T_b T_a &= i D_{bac} T_c + l_{ba} \end{aligned} \quad (1.47)$$

In the last line we switch the indices. Use the second equation to subtract the first. Recall  $l_{ab}$  is symmetric, so the unknown  $l_{ab}$ s cancel.

$$T_a T_b - T_b T_a = i(D_{bac} - D_{abc}) T_c \quad (1.48)$$

We recognize the left hand side as the commutator. Let us redefine

$$f_{abc} = D_{bac} - D_{abc} \quad (1.49)$$

generators. By definition, the structure constant is antisymmetric in the first two indices,

$$f_{abc} = -f_{bac} \quad (1.51)$$

What is amazing about structure constants is that this is the only thing we need to know about the group. If (1.40) is expanded to beyond second order, it will not yield new constraints on the structure constants. It will be automatically satisfied at all orders. The commutator is all we need!

So of course, these structure constants deserve more of our attention. First, they are real, we can see this by taking the Adjoint of the commutator,

$$[T_a, T_b]^\dagger = -i f_{abc}^* T_c^\dagger \quad (1.52)$$

Using the fact the generators are Hermitian, it becomes

$$\begin{aligned} [T_b, T_a] &= -i f_{abc}^* T_c \\ &= i f_{bac} T_c \\ &= -i f_{abc} T_c \end{aligned} \quad (1.53)$$

where on the second line we used (1.50) again with indices switched and the third line we used the fact that structure constants are antisymmetric.

The structure constants also satisfies Jacobi identity because of the commutators. We can show that

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0 \quad (1.54)$$

and thus just use the commutator relationship six times, we can show that the structure constants will in turn satisfy Jacobi identity<sup>28</sup> of the following form,

$$f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0. \quad (1.55)$$

The structure of the group is completely determined by the structure constants,

$$[T_a, T_b] = i f_{abc} T_c \quad (1.56)$$

The structure constants  $f_{abc}$ s are real, antisymmetric in the first two indices, and satisfy Jacobi identity.

$f_{abc}$  is the recombination of coefficients in the expansion of  $\tilde{f}$  to second order. And  $\tilde{f}$  is a function of two sets of real parameters. We can expand any function that depends on two sets of parameters in the same way.<sup>29</sup> The expansion has nothing to do with the dimension of the representation. Thus  $f_{abc}$ s are constants are independent of dimension of the representation  $M$ .  $f_{abc}$ s are determined by the Lie group itself<sup>30</sup>.

<sup>28</sup>Also known as Jacobi condition. All numbers satisfy the Jacobi condition are structure constants of some group.

<sup>29</sup>Of course,  $\tilde{f}$  has some nice properties such that the expansion is much simpler.

<sup>30</sup>They are determined up to an overall normalization and a similarity transformation due to change of basis of the generators.

A Lie algebra is a vector space that equips a binary bilinear operator that satisfies the Jacobi identity. If we consider the vector space  $A$  spanned by the generators, the commutator relationship (1.50) is a binary bilinear operator defined on  $A \times A \rightarrow A$  that satisfies the Jacobi identity. So our commutator relationship defines a Lie algebra. The fact that this is the only thing we need to know about function  $\tilde{f}$  implies we only need to study Lie algebra of the Lie group and we will be able to find all the unitary and linear representations.

**Hooray!** Now given a group, we just need to find the generators and the structure constants and focus on the Lie algebra. We will know everything about the group we are interested.

## 1.6 Specific Representations

We learned that in order to find the representations of the Lie group, we only need to find the representations of the Lie algebra and exponentiate them.

Let us start by looking at a representation that always exists for any compact and semi-simple Lie algebra. We will restrict our attention to this subset of the Lie algebras. Compact Lie algebras are Lie algebras that have their free real parameters to be taken within a finite interval. Semi-simple Lie algebras are Lie algebras that are direct sum<sup>31</sup> of simple Lie algebras. do not have any nontrivial abelian invariant algebra.

### 1.6.1 Adjoint Representation

Let us look at our generators in detail  $(T_a)_{ij}$  where  $a = 1, 2, \dots, n$  and  $i, j = 1, 2, \dots, M$ .  $n$  is the number of the real parameters of the group, which is the same as the number of the generators, which is the same as the dimension of the algebra itself and  $M$  is the dimension of the representation. We can take the  $n$  generators  $T_a$  to be  $n$  basis vectors and span to have an  $n$ -dimensional vector space. On the other hand, the representation of  $T_a$ s are  $M \times M$  matrices acting on  $M$ -dimensional space. So when we choose  $M = n$  we can have the matrix representation of  $T_a$  as operators acting the vector space formed by the generators themselves! In this dimension, we have the adjoint representation, and let us denote them as  $T_a^{ad}$  since this representation is very important.

We define the adjoint representation as follows,

$$(T_a^{ad})_{bc} = -if_{abc} \quad (1.57)$$

We can prove this is indeed a representation in a footnote<sup>32</sup>.

<sup>31</sup>For  $x, y \in \mathfrak{g}$  and  $x', y' \in \mathfrak{g}'$ , the direct sum of the two algebras  $\mathfrak{g} \oplus \mathfrak{g}'$  is defined such that  $[(x, x'), (y, y')] = ([x, y], [x', y'])$ , in particular  $[(x, 0), (0, x')] = 0$ .

<sup>32</sup>Starting with Jacobi Identity,

$$f_{bcd}f_{ade} + f_{abd}f_{cde} + f_{cad}f_{bde} = 0 \quad (1.58)$$

I am going to multiply each structure constant with  $-i$ .

$$(-if_{bcd})(-if_{ade}) + (-if_{abd})(-if_{cde}) + (-if_{cad})(-if_{bde}) = 0 \quad (1.59)$$

Now let us define an inner product on the vector space spanned by the generators using Adjoint representation,

$$\begin{aligned}\langle T_a | T_b \rangle &\equiv \text{Tr}[T_a^{ad} T_b^{ad}] \\ &= (T_a^{ad})_{cd} (T_b^{ad})_{dc}.\end{aligned}\quad (1.63)$$

In the last line we write out the indices to indicate the trace of a matrix product. Georgi in his book shows nicely that for compact Lie algebra, we can choose a basis such that

$$\text{Tr}[T_a^{ad} T_b^{ad}] = \lambda \delta_{ab} \quad (1.64)$$

and  $\lambda > 0$ . With this information, we can show that all compact Lie algebras have completely anti-symmetric structure constants  $f_{abc}$ s in all the indices! The proof is in the footnote<sup>33</sup>.

Note that structure constants  $\bar{f}_{abc} \equiv \text{constant} \times f_{abc}$  are still structure constants for the same group, since we can rescale the generators as well to still have  $[\bar{T}_a, \bar{T}_b] = i\bar{f}_{abc}\bar{T}_c$ . Thus we can find a basis such that the  $|\bar{T}_a^{ad}\rangle$ s are orthonormal and  $(\bar{T}_a^{ad})_{bc} = -i\bar{f}_{abc}$ . In particular if

$$\text{Tr}[T_a^{ad} T_b^{ad}] = \lambda \delta_{ab} \quad (1.66)$$

---

Now I am going to use our adjoint representation and realize

$$(-if_{bcd})(-if_{ade}) = (T_b^{ad})_{cd} (T_a^{ad})_{de} = (T_b^{ad} T_a^{ad})_{ce} \quad (1.60)$$

Similarly, I have

$$(-if_{cad})(-if_{bde}) = -(-if_{acd})(T_b^{ad})_{de} = -(T_a^{ad})_{cd} (T_b^{ad})_{de} = -(T_a^{ad} T_b^{ad})_{ce} \quad (1.61)$$

Thus (1.59) becomes

$$\begin{aligned}(T_b^{ad} T_a^{ad})_{ce} - (T_a^{ad} T_b^{ad})_{ce} + (-if_{abd})(-if_{cde}) &= 0 \\ (-if_{abd})(-if_{cde}) &= -(T_b^{ad} T_a^{ad})_{ce} + (T_a^{ad} T_b^{ad})_{ce} \\ -if_{abd}(if_{cde}) &= [T_a^{ad}, T_b^{ad}]_{ce} \\ if_{abd}(T_d^{ad})_{ce} &= [T_a^{ad}, T_b^{ad}]_{ce} \\ [T_a^{ad}, T_b^{ad}] &= if_{abd} T_d^{ad}\end{aligned}\quad (1.62)$$

<sup>33</sup>We will multiply the commutator relationship by  $T_c^{ad}$  on the right on both sides, and take the trace, and use the cyclic property of the trace.

$$\begin{aligned}\text{Tr}[[T_a^{ad}, T_b^{ad}] T_c^{ad}] &= \text{Tr}[if_{abd} T_d^{ad} T_c^{ad}] \\ \text{Tr}[T_a^{ad} T_b^{ad} T_c^{ad} - T_b^{ad} T_a^{ad} T_c^{ad}] &= if_{abd} \delta_{dc} \lambda \\ \text{Tr}[T_b^{ad} T_c^{ad} T_a^{ad} - T_c^{ad} T_b^{ad} T_a^{ad}] &= if_{abc} \lambda \\ \text{Tr}[[T_b^{ad}, T_c^{ad}] T_a^{ad}] &= if_{abc} \lambda \\ \text{Tr}[if_{bcd} T_d^{ad} T_a^{ad}] &= if_{abc} \lambda \\ if_{bcd} \delta_{da} \lambda &= if_{abc} \lambda \\ f_{bca} &= f_{abc} \\ &= -f_{bac}\end{aligned}\quad (1.65)$$

So the  $f_{abc}$  is antisymmetric in the last two indices as well. With this we can easily prove it is completely anti-symmetric.

in one basis, we can choose  $\bar{f}_{abc} = \frac{1}{\sqrt{\lambda}} f_{abc}$ .

In this basis, the answer to the following question, how do the operators  $T_a^{ad}$ 's act on the vector basis  $|T_b\rangle$ 's (note we already dropped all the bars), is particularly simple.

Note that if a Lie algebra is simple, then the Adjoint representation is an irrep.

$$T_a^{ad}|T_b\rangle = |[T_a, T_b]\rangle. \quad (1.67)$$

We prove this in a footnote<sup>34</sup>. This completes our discussion of the adjoint representation.

### 1.6.2 A general recipe and everything about $SU(2)$

$SU(2)$  is very important as we have emphasized.<sup>35</sup> So let us study all the Lie algebras!<sup>36</sup>

We shall study the easiest one first. The one comes with one parameter. Our abelian group  $U(1)$ , but this is kind of boring. We do not even have structure constants...

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<sup>34</sup>To find this out we need to insert an identity,

$$\mathbb{1} = \sum_{c=1}^n |T_c\rangle\langle T_c| \quad (1.68)$$

Then we have

$$T_a^{ad}|T_b\rangle = \sum_{c=1}^n |T_c\rangle\langle T_c|T_a^{ad}|T_b\rangle \quad (1.69)$$

But  $\langle T_c|T_a^{ad}|T_b\rangle$  is the matrix element of  $T_a^{ad}$

$$\langle T_c|T_a|T_b\rangle = (T_a)_{cb} = -if_{acb} \quad (1.70)$$

So now we can complete the calculation to find that the answer is we get the commutator vector.

$$\begin{aligned} T_a^{ad}|T_b\rangle &= \sum_{c=1}^n -if_{acb}|T_c\rangle \\ &= \sum_{c=1}^n if_{abc}|T_c\rangle \\ &= \left| \sum_{c=1}^n if_{abc}T_c \right\rangle \\ &= |[T_a, T_b]\rangle \end{aligned} \quad (1.71)$$

<sup>35</sup>You may have heard  $OB(1)$  is our only hope, but you have been lied to.  $SU(2)$  is our only hope.

<sup>36</sup>To be precise, we will study all the semi-simple Lie algebras. An Lie algebra is considered to be semi-simple if it does not have nontrivial abelian invariant subalgebras. Invariant subalgebras are formed by generators  $X_i$ 's such that we have  $[X_i, T_a]$  is combination of  $X_i$ 's for all  $X_i, T_a$ .

Let us move on to the next one then. We shall always stare at our beautiful structure constants.

$$[T_a, T_b] = i f_{abc} T_c. \quad (1.72)$$

For compact Lie algebra, we have  $f_{abc}$  completely antisymmetric, so the easiest choice is  $\epsilon_{abc}$ . And that is our remarkable  $\mathfrak{su}(2)$  which we have seen in quantum mechanics (angular momentum).

$$[J_a, J_b] = i \epsilon_{abc} J_c \quad (1.73)$$

where the completely antisymmetric quantity  $\epsilon$  is defined

$$\epsilon_{abc} = \begin{cases} 1 & \text{if } a, b, c \text{ is an even permutation of 123;} \\ -1 & \text{if } a, b, c \text{ is an odd permutation of 123;} \\ 0 & \text{otherwise.} \end{cases}$$

Remember this is also our  $\mathfrak{so}(3)$  algebra. This is the first non-trivial algebra, and hence earned its name  $A_1$ . The subscript 1 indicates its rank, which we will explain very soon when we provide recipe for representation hunting.

Before we jump into listing all the representations of  $\mathfrak{su}(2)$ , we shall have a general recipe so that we can apply this strategy to all the other Lie algebras instead of inventing new machinery every time.

- Step 1: Find the maximal set of mutually commuting generators. As we can see for  $\mathfrak{su}(2)$ , we can only choose one generator. In quantum mechanics, it is usually chosen to be  $J_3$ , but  $J_1$  or  $J_2$  would work fine as well. The number of the maximal mutually commuting generators is the rank of the Lie algebra. And these generators form the fabled Cartan-subalgebra.
- Step 2: For the rest of the generators, we need to find linear combinations of them so that  $[J_3, \bullet] = \text{constant} \times \bullet$ . In other words we are looking for the eigenvectors of the commutator operation defined above with our special generator  $J_3$ .
- Step 3: Build a vector space of dimension  $M$  using the highest weight method which we will demonstrate in detail for  $\mathfrak{su}(2)$ . Weight is another name for the eigenvalue of the state of  $J_3$ . The representation is the matrix element of the generator in this vector space we build.

For  $\mathfrak{su}(2)$ , we want  $[J_3, J] = \text{constant} \times J$  for some linear combination of  $J_1$  and  $J_2$ , which we can take to be  $J_1 + aJ_2$ , with  $a$  some constant to be determined. These are the raising and lowering operators well known in quantum mechanics. With a physical convention choice of the overall normalization, they are usually chosen to be,

$$J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2). \quad (1.74)$$

Does not this remind you of the ladder operators in quantum harmonic oscillator? Which generator corresponds to which operator?

And they have their subscript as their eigenvalues (because it looks like an eigenvalue equation except it is a different binary operator, commutator instead of matrix multiplication) or weights (jargon in group theory),

$$[J_3, J_{\pm}] = \pm J_{\pm}. \quad (1.75)$$

We can also work out the commutator relationship between these eigenvectors,

$$[J_+, J_-] = J_3. \quad (1.76)$$

These commutator relationships also describe the  $\mathfrak{su}(2)$  algebra.

So let us now build the vector space using  $J_3$ 's eigenstates. Let us call a typical eigenstate  $|m\rangle$ , then it satisfies

$$J_3|m\rangle = m|m\rangle. \quad (1.77)$$

Using the commutator relationship, we can show

$$J_3(J_{\pm}|m\rangle) = (m \pm 1)(J_{\pm}|m\rangle). \quad (1.78)$$

In other words,

$$J_{\pm}|m\rangle \propto |m \pm 1\rangle. \quad (1.79)$$

So starting from state  $|m\rangle$ , we can go up or down with raising or lowering operator between states. We would like to study the finite representations. So we call the highest weight  $j$ , such that

$$J_+|j\rangle = 0 \quad (1.80)$$

We require our basis vectors are orthonormal, such that

$$\langle m|m' \rangle = \delta_{mm'}. \quad (1.81)$$

Let us define our normalization constant in the following way,

$$J_-|m\rangle = N(m)|m-1\rangle. \quad (1.82)$$

This implies

$$J_+|m-1\rangle = N(m)|m\rangle. \quad (1.83)$$

With the help of the commutator of  $J_+$  and  $J_-$ , we can find a recursion relationship:

$$N^2(m+1) = N^2(m) - m. \quad (1.84)$$

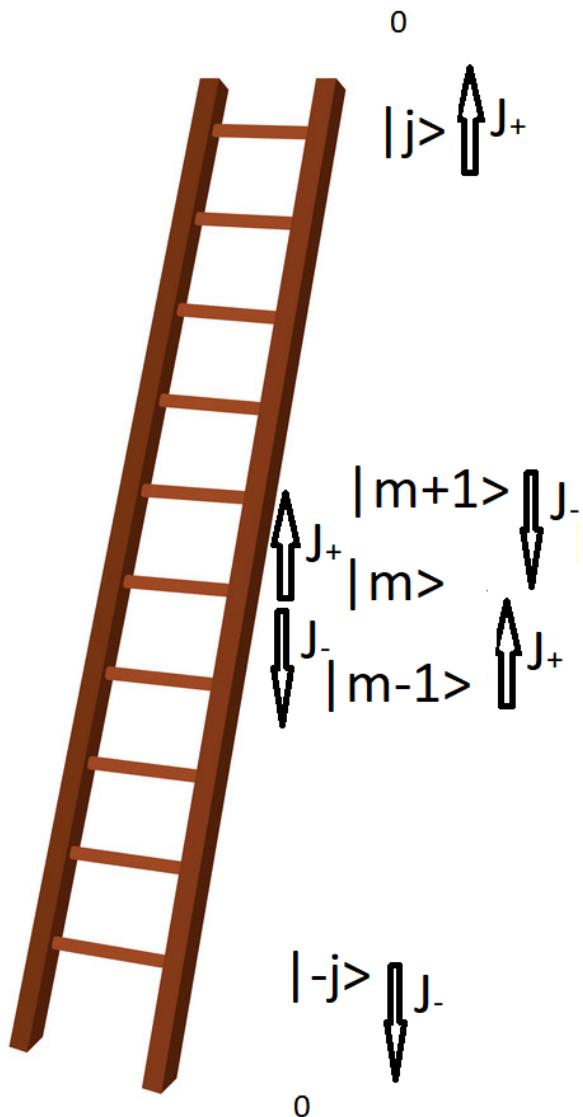


Figure 1.1: Here we show how ladder operators work. Note that an important normalization constant is not shown in the picture.

We know  $N(j+1) = 0$ , this implies  $N(j) = \sqrt{j}$ . The solution to this recursion

problem (this can be done by **Mathematica** using function RSolve.) is

$$N(m) = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}. \quad (1.85)$$

This vanishes at  $m = j + 1$  as  $j$  is chosen to be the highest weight, but we also find  $N(m) = 0$  when  $m = -j$ , that is

$$J_-| -j \rangle = 0. \quad (1.86)$$

This implies that we also have a lowest state  $| -j \rangle$  without assuming that it exists!

From a state  $| j \rangle$  to go down through  $| j-1 \rangle, | j-2 \rangle, \dots$ , to  $| -j \rangle$ , we need  $2j$  steps. This gives us the constraint that  $2j = \text{integer}$ . So the allowed highest weights are

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$$

(1.87)

For each fixed highest weight  $j$ , we have  $2j + 1$  states. We describe each state using the notation  $| j, m \rangle$ , where  $j$  is the highest weight, and  $m$  is the eigenvalue of  $J_3$ , for example  $| \frac{3}{2}, \frac{1}{2} \rangle$ . Now we can write down any representation matrix for any  $j$  we want. For a highest weight  $j$ , we have a  $(2j + 1) \times (2j + 1)$  irreducible representation.  $J_3$  is chosen to be diagonal. For  $J_+$ , we have

$$\begin{aligned} [J_+]_{m'm} &\equiv \langle m' | J_+ | m \rangle \\ &= N(m+1) \langle m' | m \rangle \\ &= N(m+1) \delta_{m'(m+1)}. \end{aligned} \quad (1.88)$$

We can also go back to the  $J_1, J_2$  basis, as  $J_1 = \frac{1}{\sqrt{2}}(J_+ + J_-)$  and so on. Note that now the matrices are labeled by  $m$ , so a typical matrix element would look like  $[J_+]_{\frac{1}{2}, -\frac{1}{2}} = N(\frac{1}{2}) = \frac{1}{\sqrt{2}}$ . For  $j = 0$ , there is only one state, it is often referred to as the singlet. It is trivial as it does not transform under  $\mathfrak{su}(2)$  at all. For  $j = \frac{1}{2}$ , there are two states, so it is referred to as a doublet. Note that the number of the states, in other words, the dimension of the representation is equal to the dimension of the defining matrix of the Lie group.<sup>37</sup> These representations are known as the fundamental representations. So  $SU(3)$  group will have dimension three fundamental representations and so on. For  $j = 1$ , there are three states, so it is referred to as a triplet. Note that the number of the states, in other words, the dimension of the representation is equal to the number of the generators. These are the Adjoint representation we talked about at the beginning of the discussions about representations.<sup>38</sup>

What about all the other groups? Actually using the method we lay out at the beginning, all the Lie groups have been classified. If you are interested, the classification is in Appendix C. There are also many pictures here.

<sup>37</sup>Recall that  $SU(2)$  group is defined to be the special unitary  $2 \times 2$  matrices.

<sup>38</sup>The matrices do not look the same, but are related by a similarity transformation.

## 1.7 Poincaré group

Now we are ready to study the most important group of our interest: the group that describes the symmetry of special relativity, the Poincaré group. Particles living in Minkowski space form irreducible representations of Poincaré group.

The Poincaré group is the symmetry group of 1 + 3 Minkowski space. We will use the following metric.

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then the Poincaré transformation is defined by all the Lorentz transformations together with translations,

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \quad (1.89)$$

where  $\Lambda$  satisfies the following constraint:

$$\eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma} \quad (1.90)$$

that is it preserves the Minkowski metric.

Then  $(\Lambda, a)$  forms the Poincaré group.<sup>39</sup>

Now let us consider infinitesimal transformations, we can use  $\epsilon^\mu$  as infinitesimal translations, for  $\Lambda$ , we can expand it as

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu. \quad (1.91)$$

This infinitesimal element certainly also needs to leave the metric invariant, that is how we will find the constraints of our infinitesimal parameter  $\omega^\mu_\nu$ . We demand

$$\begin{aligned} \eta_{\mu\nu} (\delta^\mu_\rho + \omega^\mu_\rho) (\delta^\nu_\sigma + \omega^\nu_\sigma) &= \eta_{\rho\sigma} \\ \eta_{\rho\sigma} + \eta_{\mu\nu} \omega^\mu_\rho \delta^\nu_\sigma + \eta_{\mu\nu} \delta^\mu_\rho \omega^\nu_\sigma + O(\omega^2) &= \eta_{\rho\sigma} \\ \eta_{\mu\sigma} \omega^\mu_\rho + \eta_{\rho\nu} \omega^\nu_\sigma &= 0 \\ \omega_{\sigma\rho} + \omega_{\rho\sigma} &= 0 \end{aligned} \quad (1.92)$$

So this tells us the  $\omega$  is antisymmetric with two indices down. That is to say that the Lorentz group has  $\frac{4 \times 3}{2} = 6$  real parameters, plus the 4 from the translational group. The Poincaré group has  $6 + 4 = 10$  real parameters. So the dimension of the Poincaré group is 10. It contains an Abelian subgroup  $(1, a)$  and the Lorentz subgroup  $(\Lambda, 0)$ .

Now let us study a group element in a certain representation.

$$U(\delta^\mu_\nu + \omega^\mu_\nu, \epsilon^\mu) = \mathbb{1} + \frac{i}{2} \omega_{\mu\nu} J^{\mu\nu} + i \epsilon_\mu P^\mu \quad (1.93)$$

---

<sup>39</sup>It is also known as the semidirect product of  $\mathcal{R}^{1,3}$  and  $SO(1,3)$  with  $\mathcal{R}^{1,3}$  being the invariant subgroup

So a general group element will be defined by the exponential

$$U = e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\epsilon_\mu P^\mu}. \quad (1.94)$$

where  $J^{\mu\nu}$  and  $P^\mu$  are our ten generators. These generators are Hermitian and are physical observables.  $P^0$  is the Hamiltonian.  $(P^1, P^2, P^3)$  is the momentum operator, while  $(J^{23}, J^{31}, J^{12})$  is the angular momentum operator, and  $(J^{01}, J^{02}, J^{03})$  is the boost.

Under an unitary transformation the states will transform as follows

$$|\Psi'\rangle = U|\Psi\rangle \quad (1.95)$$

The operators should transform in a way so that physical information remains the same,

$$\langle\Phi|O|\Psi\rangle = \langle\Phi'|O'|\Psi'\rangle \quad (1.96)$$

By making use of the property of unitary transformation, we have

$$\begin{aligned} \langle\Phi|O|\Psi\rangle &= \langle\Phi|U^\dagger U O U^\dagger U|\Psi\rangle \\ &= \langle U\Phi|U O U^\dagger|U\Psi\rangle \\ &= \langle\Phi'|O'|\Psi'\rangle \end{aligned} \quad (1.97)$$

Thus

$$O' = U O U^\dagger \quad (1.98)$$

Now let us take

$$O = \mathbb{1} + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\epsilon_\mu P^\mu. \quad (1.99)$$

By performing the following transformation on  $x^\mu$  in sequence, it's not hard to show that<sup>40</sup>

$$U(\Lambda, a)U(\delta + \omega, \epsilon)U^{-1}(\Lambda, a) = U(\delta + \Lambda\omega\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a) \quad (1.100)$$

where in the last equation we adopted the matrix notation to save some index labeling<sup>41</sup>. The matrix multiplication is implied.

Hence if we expand the above equation to linear order of  $\omega$  and  $\epsilon$ , and take these infinitesimal parameters to be arbitrary, we have the transformation rules for momentum and angular momentum operators,

$$\begin{aligned} U(\Lambda, a)J^{\mu\nu}U^\dagger(\Lambda, a) &= \Lambda_\rho^\mu\Lambda_\sigma^\nu(J^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho) \\ U(\Lambda, a)P^\mu U^\dagger(\Lambda, a) &= \Lambda_\rho^\mu P^\rho \end{aligned} \quad (1.101)$$

<sup>40</sup>Note that this is valid to all orders since we have not expanded around the identity yet.

<sup>41</sup>Here  $\delta$  is short for  $\delta_\nu^\mu$ . I am reluctant to use  $\mathbb{1}$ , as in all other occasions  $\mathbb{1}$  refers to the  $M \times M$  identity matrix in the representation.

Now we expand  $\Lambda$  as well,

$$\Lambda_\nu^\mu = \delta_\nu^\mu + \omega_\nu^\mu \quad (1.102)$$

and take the translation parameter to be small  $a^\mu = \epsilon^\mu$ .

After some calculation (first homework), we arrive at the Poincaré algebra

$$\begin{aligned} i[J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu} \\ i[P^\mu, J^{\rho\sigma}] &= \eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho \\ [P^\mu, P^\nu] &= 0 \end{aligned} \quad (1.103)$$

As we can see  $P^\mu$ 's commute with each other and form an invariant subalgebra because all  $P^\mu$  commutators are equal to some linear combinations of  $P^\mu$ 's.

If we identify  $\mathbf{J} \equiv (J_1, J_2, J_3) = (-J^{23}, -J^{31}, -J^{12})$  as the angular momentum operator<sup>42</sup>, we see they form a subalgebra of the Poincaré algebra themselves,

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad (1.104)$$

where the completely antisymmetric quantity  $\epsilon$  is defined

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is an even permutation of 123;} \\ -1 & \text{if } i, j, k \text{ is an odd permutation of 123;} \\ 0 & \text{otherwise.} \end{cases}$$

Equation (1.104) defines the  $\mathfrak{su}(2)$  that we have studied in detail. It is the Lie algebra of Lie group  $SU(2)$  and Lie group  $SO(3)$ <sup>43</sup>.

For example<sup>44</sup>

$$\begin{aligned} U(\mathbb{1}, a) &= e^{+iP^\mu a_\mu} \\ U(R_\theta, 0) &= e^{i\mathbf{J}\cdot\theta}. \end{aligned} \quad (1.105)$$

### 1.7.1 \*Parity and Time reversal

The following two transformations are Lorentz transformations

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.106)$$

<sup>42</sup>The idea is that the angular momentum in a  $SO(0, 3)$  space is necessarily the opposite of the angular momentum in a  $SO(3, 0)$  space. Thus we use the downstairs angular momentum to include this minus sign so we reproduce the angular momentum algebra. This can be seen from a more carefully written Lie algebra  $[J^i, J^j] = i\epsilon_k^{ij} J^k$ . Notice the downstairs in  $\epsilon$  that gives a minus sign. This is only true in this footnote, in the main text we will not distinguish downstairs and upstairs for Levi-civita and stick with the Euclidean measure.

<sup>43</sup>In fact, we have  $SO(3) \approx SU(2)/Z_2$ .

<sup>44</sup>In the second line, the minus sign we introduced in angular momentum and the minus sign in the metric cancelled. This gives us the familiar result.

And we believed that there are operators corresponding to them

$$P \equiv U(\mathcal{P}, 0), T \equiv U(\mathcal{T}, 0) \quad (1.107)$$

Using the fundamental multiplication law of the Poincare group

$$U(\Lambda', a')U(\Lambda, a) = U(\Lambda'\Lambda, \Lambda'a + a') \quad (1.108)$$

we obtain

$$\begin{aligned} PU(\Lambda, a)P^{-1} &= U(\mathcal{P}\Lambda\mathcal{P}^{-1}, \mathcal{P}a) \\ TU(\Lambda, a)T^{-1} &= U(\mathcal{T}\Lambda\mathcal{T}^{-1}, \mathcal{T}a) \end{aligned} \quad (1.109)$$

This is how we expect them to transform if  $P$  or  $T$  is conserved. Note these symmetries are only an approximate. Now expand  $U(\Lambda, a)$  infinitesimally, we can find the transformation for Poincare generators<sup>45</sup> in particular, the transformation on  $P^0 = H$ . And we have

$$\begin{aligned} PiHP^{-1} &= iH \\ TiHT^{-1} &= -iH. \end{aligned} \quad (1.111)$$

Now we see parity is the linear, unitary operator while time reversal is the anti-linear, anti-unitary operator. We can prove this by contradiction: if we assume parity is anti-linear anti-unitary, then  $PHP^{-1} = -H$ , this tells us for each positive energy state  $\psi$ , there is a negative energy state  $P\psi$ . There are no states that have energy less than the vacuum, our assumption is wrong. And we have a similar argument for time reversal.

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<sup>45</sup>

$$\begin{aligned} PiJ^{\rho\sigma}P^{-1} &= i\mathcal{P}_\mu^\rho\mathcal{P}_\nu^\sigma J^{\mu\nu} \\ PiP^\rho P^{-1} &= i\mathcal{P}_\mu^\rho P^\mu \\ TiJ^{\rho\sigma}T^{-1} &= i\mathcal{T}_\mu^\rho\mathcal{T}_\nu^\sigma J^{\mu\nu} \\ TiP^\rho T^{-1} &= i\mathcal{T}_\mu^\rho P^\mu \end{aligned} \quad (1.110)$$

## Chapter 2

# Scattering amplitude for every theory

Equipped with group theory, now we are ready to write down a relativistic quantum theory. We will be bold: instead of considering a specific theory as an example, we try to consider *every* theory. We consider any quantum theory with a Hamiltonian  $H = H_0 + V$ , where  $H_0$  is the free Hamiltonian and  $V$  is the interaction. We ask the question, if this quantum theory is compatible with special relativity, what kind of constraints we need to impose on the interaction  $V$ . The first step, is to write down the scattering amplitude and see what a Lorentz invariant S-matrix<sup>1</sup> requires of  $V$ .

To do that, first we will use our group theory knowledge to label a one-particle state, then we describe the multi-particle state (asymptotic states) that are useful for interaction and then we compute the  $S$ -matrix for an unspecified theory with interaction  $V$ .

### 2.1 Classifying the one particle state

To label a one-particle state, we usually find a set of mutually commuting operators, and then label the particle state with their eigenvalues. Among the operators (the generators of the Poincaré group), the components of the energy-momentum four vectors all commute with each other<sup>2</sup>. And thus it is natural to use the four momentum  $p^\mu$  to label a one-particle state. As we are clueless about other possible labels (and maybe discrete labels), we will collectively denote them by  $s$  (stands for some) for now. Thus we will consider state  $|p, s\rangle$

---

<sup>1</sup>We use S-matrix and scattering amplitude interchangeably as S-matrix is the matrix element of the scattering amplitude.

<sup>2</sup>On the contrary, the boosts  $K$  do not commute with energy, and hence we do not use boost to label particles.

that satisfies

$$P^\mu |p, s\rangle = p^\mu |p, s\rangle. \quad (2.1)$$

Then for the translation  $U(1, a) = e^{+iP^\mu a_\mu}$ , we know how the state will transform

$$U(1, a)|p, s\rangle = e^{+ip \cdot a}|p, s\rangle, \quad (2.2)$$

But what about for a Lorentz transformation ( $\Lambda$  only, with no translation), how does the corresponding operator  $U(\Lambda)$  act on the state? Well, after Lorentz transformation,  $p^\mu$  becomes  $(\Lambda p)^\mu = \Lambda^\mu_\nu p^\nu$ <sup>3</sup>. So we expect

$$P^\mu U(\Lambda)|p, s\rangle = (\Lambda p)^\mu U(\Lambda)|p, s\rangle. \quad (2.8)$$

Hence  $U(\Lambda)|p, s\rangle$  must be a linear combination of all the states with four-momentum  $\Lambda p$  and some arbitrary  $s'$ :

$$U(\Lambda)|p, s\rangle = \sum_{s'} C_{s's}(\Lambda, p)|\Lambda p, s'\rangle. \quad (2.9)$$

So the ultimate goal is to study these  $C_{s's}(\Lambda, p)$ 's<sup>4</sup>. Maybe we can choose our bases (other linear combinations of  $|p, s\rangle$ 's) cleverly so that the matrix  $C_{s's}(\Lambda, p)$  is block diagonal. In other words, the  $|p, s\rangle$ 's within one block<sup>5</sup> furnish a representation of the Lorentz group. Then it is natural to identify a species of a particle with the corresponding one-particle states within a block (or irreducible representation.). And in that sense, we manage to label the particle, and complete our classification.

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<sup>3</sup>What we want to do is to see the eigenvalue of vector  $U(\Lambda)|p, s\rangle$  under momentum operator, so what we want is,

$$P^\mu U(\Lambda)|p, s\rangle = ? \quad (2.3)$$

As we know the transformation of the momentum operator is

$$U(\Lambda)P^\mu U^\dagger(\Lambda) = \Lambda^\mu_\rho P^\rho \quad (2.4)$$

Apply this to  $\Lambda^{-1}$

$$U(\Lambda^{-1})P^\mu U^{-1}(\Lambda^{-1}) = (\Lambda^{-1})^\mu_\rho P^\rho \quad (2.5)$$

This is really just

$$U^{-1}(\Lambda)P^\mu U(\Lambda) = (\Lambda)^\mu_\rho P^\rho \quad (2.6)$$

So we have

$$\begin{aligned} P^\mu U(\Lambda)|p, s\rangle &= U(\Lambda)(\Lambda)^\mu_\rho P^\rho |p, s\rangle \\ &= U(\Lambda)(\Lambda)^\mu_\rho p^\rho |p, s\rangle \\ &= (\Lambda p)^\mu U(\Lambda)|p, s\rangle \end{aligned} \quad (2.7)$$

Thus we have shown the momentum becomes  $\Lambda p$ .

<sup>4</sup>Note the subscript of  $C$  is ordered like such because in quantum mechanics we have  $A|\Psi\rangle = \sum_j \langle \Phi_j | A | \Psi \rangle |\Phi_j\rangle$ .

<sup>5</sup>assuming not further reducible

The dependence of  $C_{s',s}(\Lambda, p)$  on the entire Lorentz group and the momentum makes it an ominous task. One of the reasons why studying  $C_{s',s}(\Lambda, p)$  is difficult is because the momentum of the state before (the Lorentz transformation)  $p$  is different from the momentum after,  $\Lambda p$ .

Let us consider a significantly simpler case: what if we consider a state  $|k, s\rangle$  with a reference<sup>6</sup> momentum  $k^\mu$ , and only consider the Lorentz transformations that leave this momentum invariant! So now the state before and after will be both have the same momentum  $k^\mu$ .

In other words, consider Lorentz transformations  $W_\nu^\mu$  such that

$$W_\nu^\mu k^\nu = k^\mu. \quad (2.10)$$

This subset of Lorentz transformations form a subgroup of the Lorentz group, called the little group.

Then the Lorentz transformation acting on the state is

$$U(W)|k, s\rangle = \sum_{s'} D_{s's}(W)|k, s'\rangle. \quad (2.11)$$

Then we can show  $D_{s's}(W)$ 's<sup>7</sup> furnish a representation of the little group by showing the group multiplication of the little group  $W_1 W_2$  is preserved.

$$\begin{aligned} U(W_1 W_2)|k, s\rangle &= \sum_{s'} D_{s's}(W_1 W_2)|k, s'\rangle \quad (2.12) \\ &= U(W_1)U(W_2)|k, s\rangle \\ &= U(W_1) \sum_{s''} D_{s''s}(W_2)|k, s''\rangle \\ &= \sum_{s''} D_{s''s}(W_2) \sum_{s'} D_{s's''}(W_1)|k, s'\rangle \\ &= \sum_{s'} \left( \sum_{s''} D_{s's''}(W_1) D_{s''s}(W_2) \right) |k, s'\rangle \end{aligned}$$

Thus

$$D_{s's}(W_1 W_2) = \sum_{s''} D_{s's''}(W_1) D_{s''s}(W_2). \quad (2.13)$$

Now we suppose we have found the little group representation  $D_{s's}(W)$ . How is this going to help us with the original problem

$$U(\Lambda)|p, s\rangle = \sum_{s'} C_{s's}(\Lambda, p)|\Lambda p, s'\rangle? \quad (2.14)$$

---

<sup>6</sup>We are going to refer it back to this momentum later, hence the name.

<sup>7</sup>We dropped the dependence on  $k^\mu$  because all  $D_{s's}(W)$ 's share the same  $k$ . Contrast this with the  $C_{s's}(\Lambda, p)$ 's.

Note the following fun fact: for a single particle of known species with momentum  $p^\mu$  and its reference momentum<sup>8</sup>  $k^\mu$ , we can always find Lorentz transformation  $L_\nu^\mu(p)$ <sup>9</sup> such that

$$p^\mu = L_\nu^\mu(p)k^\nu. \quad (2.15)$$

Then we can define state with momentum<sup>10</sup>  $p^\mu$  using state with reference momentum  $k^\mu$  as

$$|p, s\rangle \equiv N(p)U(L(p))|k, s\rangle \quad (2.16)$$

where  $N(p)$  is a normalization factor. Now let us see what  $U(\Lambda)|p, s\rangle$  becomes, by using the trick “insert the identity” (indicated by the red color),

$$\begin{aligned} U(\Lambda)|p, s\rangle &= N(p)U(\Lambda)U(L(p))|k, s\rangle \\ &= N(p)U(\Lambda L(p))|k, s\rangle \\ &= N(p)\textcolor{red}{U(L(\Lambda p))U(L^{-1}(\Lambda p))}U(\Lambda L(p))|k, s\rangle \\ &= N(p)U(L(\Lambda p))U(L^{-1}(\Lambda p)\Lambda L(p))|k, s\rangle \\ &= N(p)U(L(\Lambda p))U(W(\Lambda, p))|k, s\rangle \end{aligned} \quad (2.17)$$

In the last line we identify  $W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)$ .  $L^{-1}(\Lambda p)\Lambda L(p)$  is indeed a member of the little group because  $L(p)$  brings momentum from  $k$  to  $p$  as defined, and  $\Lambda$  as we show in the first part brings momentum from  $p$  to  $\Lambda p$ , and  $L^{-1}(\Lambda p)$  brings it from  $\Lambda p$  back to  $k$  again.

Now we can use the representations of the little group to continue simplify,

$$\begin{aligned} U(\Lambda)|p, s\rangle &= N(p)U(L(\Lambda p))\sum_{s'}D_{s's}(W(\Lambda, p))|k, s'\rangle \\ &= N(p)\sum_{s'}D_{s's}(W(\Lambda, p))U(L(\Lambda p))|k, s'\rangle \\ &= \frac{N(p)}{N(\Lambda p)}\sum_{s'}D_{s's}(W(\Lambda, p))|\Lambda p, s'\rangle \end{aligned} \quad (2.18)$$

And this shows that in order to find the representation of the Lorentz group, we only need to find the representations of the little group.

Now we compute the normalization factor  $N(p)$ . We show that this normalization factor is 1 if we normalize our states in the same way as you have seen in your scalar QFT course.

$$\langle k', s'|k, s\rangle = (2\pi)^3(2E_k)\delta^3(\mathbf{k}' - \mathbf{k})\delta_{s's} \quad (2.19)$$

Remember you chose this normalization because this combination is Lorentz invariant.

---

<sup>8</sup>This does depend on which species of particle we are talking about, why?

<sup>9</sup>This is why it is not so crazy only studying about the reference momentum: all the possible 4-momenta are all related.

<sup>10</sup>Notice this corresponds to choosing the basis in a way that different states with different  $\sigma$  are not mixed.

Now what about for  $|p, s\rangle$ ?<sup>11</sup>

$$\begin{aligned}\langle p', s' | p, s \rangle &= (2\pi)^3 (2E_p) \delta^3(\mathbf{p}' - \mathbf{p}) \delta_{s's} \\ &= |N(p)|^2 (2\pi)^3 (2E_k) \delta^3(\mathbf{k}' - \mathbf{k}) \delta_{s's}.\end{aligned}\quad (2.21)$$

Now notice the delta function satisfies (that is why we choose this normalization to start with)

$$p^0 \delta^3(\mathbf{p}' - \mathbf{p}) = k^0 \delta^3(\mathbf{k}' - \mathbf{k}). \quad (2.22)$$

We find the normalization factor is

$$N(p) = 1. \quad (2.23)$$

Thus with this normalization, the transformation law under little group is simplified.

$$U(\Lambda) |p, s\rangle = \sum_{s'} D_{s's} (W(\Lambda, p)) |\Lambda p, s'\rangle \quad (2.24)$$

### 2.1.1 Massive states

For a massive particle, the easiest choice of  $k^\mu = (m, 0, 0, 0)$ . The little group is  $SO(3)$ , which is the same as  $SU(2)$ . Since  $SU(2)$ 's representations are called spin- $j$  representations, we realize we can label a massive particle by its mass  $m$  and spin  $j$ . We use  $\sigma$  to label the eigenvalue of  $J_3$  to maintain the same notation as before. The one particle state transform in the following way under a Poincare transformation. Remember that for massive particles, the little group is  $SO(3)$ , which is the same as  $SU(2)$  and we have already studied the  $SU(2)$  representation. For  $D_{s's}^j (1 + \Theta)$  with  $\Theta_{ik} = -\Theta_{ki}$  being infinitesimal,

$$\begin{aligned}D_{s's}^j (1 + \Theta) &= \delta_{s's} + \frac{i}{2} \Theta_{ik} (J_{ik}^j)_{s's} \\ (J_{23}^j \pm i J_{31}^j)_{s's} &= (J_1^j \pm i J_2^j)_{s's} \\ &= \delta_{s's \pm 1} \sqrt{(j \mp s)(j \pm s + 1)} \\ (J_{12}^j)_{s's} &= (J_3^j)_{s's} = s \delta_{s's}.\end{aligned}\quad (2.25)$$

---

<sup>11</sup>A more rigorous derivation:

$$\begin{aligned}\langle p', s' | p, s \rangle &= (2\pi)^3 (2E_p) \delta^3(\mathbf{p}' - \mathbf{p}) \delta_{s's} \\ &= \langle p', s' | N(p) U(L(p)) | k, s \rangle \\ &= N(p) (\langle p', s' | U^{-1}(L(p)) | k, s \rangle) \\ &= N(p) (\langle p', s' | U(L^{-1}(p)) | k, s \rangle) \\ &= N(p) N^*(p') (2\pi)^3 (2E_k) D_{s's}^*(W(L^{-1}(p), p')) \delta^3(\mathbf{k}' - \mathbf{k}) \\ &= |N(p)|^2 (2\pi)^3 (2E_k) \delta^3(\mathbf{k}' - \mathbf{k}) \delta_{s's}\end{aligned}\quad (2.20)$$

where in the second to last line we used the (2.18) and in the last line we took the limit  $p' \rightarrow p$  (only when  $p' = p$  the result is interesting) and also  $W(L^{-1}(p), p) = L^{-1}(k) L^{-1}(p) L(p) = 1$ .

$$U(\Lambda, a)|p, s\rangle = e^{+ia\cdot\Lambda p} \sum_{s'} D_{s's}^j(W(\Lambda, p))|\Lambda p, s'\rangle. \quad (2.26)$$

where the little group element  $W(\Lambda, p)$  are

$$W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p). \quad (2.27)$$

$L(p)$  is a standard boost<sup>12</sup> that boost reference momentum  $k^\mu = (m, 0, 0, 0)$  to  $p^\mu$ .

### \*Parity and one (massive) particle state

What does parity do to one-particle states?<sup>13</sup>

$$\begin{aligned} P\mathbf{J}P^{-1} &= \mathbf{J} \\ P\mathbf{p}P^{-1} &= -\mathbf{p} \\ P\mathbf{H}P^{-1} &= \mathbf{H}. \end{aligned} \quad (2.30)$$

change momentum, keep mass and spin. thus for reference momentum  $k$ , nothing is changed, and we have

$$P|k, s\rangle = \eta_s|k, s\rangle \quad (2.31)$$

From the massive state we just figured out

$$(J_1^j \pm iJ_2^j)_{s's} = \delta_{s's\pm 1} \sqrt{(j \mp s)(j \pm s + 1)} \quad (2.32)$$

we have

$$(J_1^j \pm iJ_2^j)|k, s\rangle = \sqrt{(j \mp s)(j \pm s + 1)}|k, s \pm 1\rangle \quad (2.33)$$

Operating on both sides with  $P$  we find out that

$$\eta_s = \eta_{s\pm 1}. \quad (2.34)$$

---

<sup>12</sup>Can be conveniently chosen as

$$\begin{aligned} L^i{}_k(p) &= \delta^i{}_k + \left( \frac{p^0}{m} - 1 \right) \frac{p^i p^k}{|\mathbf{p}|^2} \\ L^i{}_0(p) &= L^0{}_i(p) = \frac{p^i}{m} \\ L^0{}_0(p) &= \frac{p^0}{m} \end{aligned} \quad (2.28)$$

<sup>13</sup>

$$\begin{aligned} PiJ^{\rho\sigma}P^{-1} &= i\mathcal{P}_\mu^\rho \mathcal{P}_\nu^\sigma J^{\mu\nu} \\ PiP^\rho P^{-1} &= i\mathcal{P}_\mu^\rho P^\mu \end{aligned} \quad (2.29)$$

So  $\eta$  does not depend on  $s$ .

$$P|k, s\rangle = \eta|k, s\rangle. \quad (2.35)$$

Now we boost the state

$$|p, s\rangle = U(L(p))|k, s\rangle. \quad (2.36)$$

Note

$$\begin{aligned} \mathcal{P}L(p)\mathcal{P}^{-1} &= L(\mathcal{P}p) \\ P U(L(p)) P^{-1} &= U(L(\mathcal{P}p)) \end{aligned} \quad (2.37)$$

where first line can be computed explicitly using the standard boost provided in the footnote, and second line we use the fundamental multiplication rule of Poincare algebra. thus

$$\begin{aligned} P|p, s\rangle &= PU(L(p))|k, s\rangle \\ &= PU(L(p))P^{-1}P|k, s\rangle \\ &= U(L(\mathcal{P}p))P|k, s\rangle \\ &= \eta U(L(\mathcal{P}p))|k, s\rangle \\ &= \eta|\mathcal{P}p, s\rangle. \end{aligned} \quad (2.38)$$

We will put the end result in a box, as we are going to use it later when we build the spinor field.

$$\boxed{P|p, s\rangle = \eta|\mathcal{P}p, s\rangle}. \quad (2.39)$$

Massless states are more tricky. We will deal with it in the appendix F.

## 2.2 Multi: Asymptotic states

Now we are finally done with one-particle states, we are ready to move on to multi-particle states.

First, if a state consists of several non-interacting particles, it will transform as a direct product of several one-particle states. We can write down the transformation down according to (2.26).

$$U(\Lambda, a)|p_1, s_1, n_1; \dots\rangle = e^{+ia\cdot\Lambda p_1 + \dots} \sum_{s'} D_{s'_1 s_1}^{j_1}(W(\Lambda, p_1)) \dots |\Lambda p_1, s'_1, n_1; \dots\rangle. \quad (2.40)$$

Note we include a new label: particle species  $n$  because we are talking about multiple particles and they may come from different species.

From now on for simplification we are going to denote the collection of labels  $p_1, s_1, n_1; \dots$  as  $\alpha$

$$|\alpha\rangle \equiv |p_1, s_1, n_1; \dots\rangle. \quad (2.41)$$

Then consider state  $|\alpha\rangle$  and a very special Poincare transformation: where Lorentz transformation is identity  $\Lambda_\nu^\mu = \delta_\nu^\mu$  and there is only translation in time  $a^\mu = (\tau, 0, 0, 0)$ . Since it is a translation, we can immediately write down the transformation according to (1.105)  $U(\mathbf{1}, a) = e^{+iP^\mu a_\mu}$ ,  $U(\mathbf{1}, a) = e^{+iH\tau}$ . On the other hand according to (2.40), the right hand side is  $e^{+i(p_1^0 + \dots)\tau}$  acting on the state. From this we can conclude, this multi-particle state (consists of non-interacting particles)  $|\alpha\rangle$  is the energy eigenstate

$$H|\alpha\rangle = E_\alpha|\alpha\rangle \quad (2.42)$$

with an energy equal to the sum of the one-particle energies

$$E_\alpha = p_1^0 + \dots \quad (2.43)$$

On the other hand, in a typical experiment, we start with particles at time  $t \rightarrow -\infty$  so far apart that they are not yet interacting and end with particles at time  $t \rightarrow +\infty$  so far apart that they have stopped interacting. So it seems we have two sets of states transforming according to (2.40):  $|\alpha\rangle_{in}$  and  $|\alpha\rangle_{out}$ . They both are energy eigenstates of  $H$ . If we make observations at  $t \rightarrow -\infty$ , we will find  $|\alpha\rangle_{in}$  contain particles that are described by label  $\alpha$  and we can make a similar statement for the out state.

Since our focus is to study Poincare transformation, it makes sense that we study in Heisenberg picture where the state vectors do not evolve in time<sup>14</sup>. The state describes the whole spacetime history of a system of particles.

Different observers see equivalent state vectors but not the same ones.

Now consider two observers Alice and Bob, and Bob's clock is set such  $t' = t - \tau$  respect to Alice's clock. Let us say that Alice's clock is tuned such that  $t = 0$  corresponds to when interaction happens. And Bob's clock is set to  $t' = 0$  when  $t = \tau$ . Then if Alice sees state vector  $|\Psi\rangle$ , Bob will see states vector  $U(1, -\tau)|\Psi\rangle = e^{-iH\tau}|\Psi\rangle$ .

We are now going to relate the in/out state to the free states. We define free states  $|\alpha\rangle_0$  to be the eigenstates of  $H_0$  (assume  $H_0$  and  $H = H_0 + V$  has same spectrum and  $V$  is the interaction term.)

$$H_0|\alpha\rangle_0 = E_\alpha|\alpha\rangle_0. \quad (2.44)$$

Now we send in observer Bob to observe both the in/out state and the free state, Bob will see  $e^{-iH\tau}|\alpha\rangle_{in/out}$  for the in/out state and  $e^{-iH_0\tau}|\alpha\rangle_0$  for the free state. Then in the limit  $\tau \rightarrow -\infty$ , Bob should find the in state and the free state the same (similar for out state). This can be summarized in the following equation

$$e^{-iH\tau}|\alpha\rangle_{in/out} \xrightarrow{\tau \rightarrow -\infty} e^{-iH_0\tau}|\alpha\rangle_0. \quad (2.45)$$

where  $\tau \rightarrow -\infty$  is for in state and  $\tau \rightarrow +\infty$  is for out state as usual (we will stop mentioning this every time!).

---

<sup>14</sup>Thus in/out states are **not** the time goes to infinity limit of some state vector  $|\alpha(t)\rangle$ .

From here we can relate the in/out state and the free state via

$$|\alpha\rangle_{in/out} = \Omega(\mp\infty)|\alpha\rangle_0 \quad (2.46)$$

where

$$\Omega(\tau) \equiv e^{+iH\tau} e^{-iH_0\tau}. \quad (2.47)$$

The norm for in/out state is independent of time and thus equals to the norm of the free particle state.

$$_{in/out}\langle\beta|\alpha\rangle_{in/out} =_0 \langle\beta|\alpha\rangle_0 = \delta(\beta - \alpha).$$

For example when  $\alpha = p_1 s_1 n_1 \dots$  and  $\beta = p'_1 s'_1 n'_1 \dots$

$$\delta(\beta - \alpha) = (2\pi)^3 (2E_1) \delta^3(\mathbf{p}_1 - \mathbf{p}'_1) \delta_{s_1 s'_1} \delta_{n_1 n'_1} \dots \pm \text{permutations.} \quad (2.48)$$

The sign comes from boson/fermion statistics.

## 2.3 S-matrix

S-matrix is the probability amplitude (overlap) between in/out state

$$S_{\beta\alpha} =_{out} \langle\beta|\alpha\rangle_{in} \quad (2.49)$$

Now let us emphasize, in and out states live in the same Hilbert space, the S-matrix acts as the expansion coefficient if we want to expand in state as a sum of out state.

S-matrix connects two complete sets of orthonormal states, it must be unitary. we can prove S-matrix is unitary

$$\int d\beta S_{\beta\gamma}^* S_{\beta\alpha} =_{in} \langle\gamma|\alpha\rangle_{in} = \delta(\gamma - \alpha).$$

If there is no interaction, we get  $\delta(\alpha - \beta)$ , thus the rate of reaction is

$$|S_{\beta\alpha} - \delta(\beta - \alpha)|^2. \quad (2.50)$$

Sometimes it is more interesting to work with operator  $S$  (S-matrix is the matrix element of operator  $S$ !)

$$_0\langle\beta|S|\alpha\rangle_0 \equiv S_{\beta\alpha}.$$

Recall

$$|\alpha\rangle_{in/out} = \Omega(\mp\infty)|\alpha\rangle_0 \quad (2.51)$$

thus the in/out state language gives us

$$S = \Omega(\infty)^\dagger \Omega(-\infty) = U(+\infty, -\infty)$$

where  $U$  is time evolution operator.

$$U(\tau, \tau_0) \equiv \Omega(\tau)^\dagger \Omega(\tau_0) = e^{iH_0\tau} e^{-iH(\tau-\tau_0)} e^{-iH_0\tau_0}.$$

where

$$\Omega(\tau) = e^{+iH\tau} e^{-iH_0\tau}. \quad (2.52)$$

By working out the exciting problem in the footnote, we find <sup>15</sup>

$$\begin{aligned} S &= 1 + \sum_n \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 dt_2 \dots dt_n T\{V(t_1) \dots V(t_n)\} \\ &\equiv T e^{-i \int_{-\infty}^{\infty} dt V(t)}. \end{aligned} \quad (2.59)$$

This is known as Dyson formula, which relates the interacting Hamiltonian (what we normally refer as theory) and experimental stuff like S-matrix.

## 2.4 Special relativity comes in

Integral of  $t$  alone is not Lorentz invariant. How to fix? An easy fix is to assume the existence of a Hamiltonian density which itself is a Lorentz scalar (does not transform under Lorentz transformation)

$$V(t) = \int d^3x \mathcal{H}(x, t). \quad (2.60)$$

---

<sup>15</sup>Notice that  $U(t, t_0)$  solves the differential equation

$$i \frac{d}{dt} U(t, t_0) = V(t) U(t, t_0) \quad (2.53)$$

where

$$V(t) = e^{iH_0 t} V e^{-iH_0 t} \quad (2.54)$$

Note that the differential equation shares the same solution as the following integral equation

$$U(t, t_0) = 1 - i \int_{t_0}^t dt V(t) U(t, t_0). \quad (2.55)$$

How can we solve this integral equation? By iteration, using itself again and again

$$U(t, t_0) = 1 - i \int_{t_0}^t dt_1 V(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 V(t_1) V(t_2) + \dots \quad (2.56)$$

Then of course we can put in our upper bound and lower bound and our  $S$  operator is

$$S = U(\infty, -\infty) = 1 - i \int_{-\infty}^{\infty} dt_1 V(t_1) + (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 V(t_1) V(t_2) + \dots \quad (2.57)$$

Realize  $V(t_1) V(t_2)$  is time ordered!

Define time ordering (by the way, time ordering screams for path integral. Why?)

$$\begin{aligned} T\{V(t)\} &= V(t) \\ T\{V(t_1) V(t_2)\} &= \theta(t_1 - t_2) V(t_1) V(t_2) + \theta(t_2 - t_1) V(t_2) V(t_1) \end{aligned} \quad (2.58)$$

Now combine together with the time integral we have a bunch of  $\int d^4x$  which are Lorentz invariant.

The other thing that is not Lorentz invariant is the time ordering. As we know Lorentz transformation can change the order of things happen if they are space separated! Thus we need to require

$$[\mathcal{H}(x), \mathcal{H}(y)] = 0, \text{if spacelike}. \quad (2.61)$$

This is also known as the causality condition, any relativistic theory needs to obey. This makes combining special relativity and quantum mechanics very difficult.



## Chapter 3

# S-matrix computation and the last condition on $V$

Recall S-matrix admits a perturbation expansion

$$\begin{aligned} S &= 1 + \sum_n \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 dt_2 \dots dt_n T\{V(t_1) \dots V(t_n)\} \\ &= T e^{-i \int_{-\infty}^{\infty} dt V(t)}. \end{aligned} \quad (3.1)$$

We are going to compute the matrix element of this  $S$  operator, the S-matrix

$$S_{\beta\alpha} = \delta(\beta - \alpha) + \sum_n \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 dt_2 \dots dt_n T\{{}_0\langle \beta | V(t_1) \dots V(t_n) | \alpha \rangle_0\} \quad (3.2)$$

We will state without proof<sup>1</sup> that if we are interested in the connect piece of S-matrix, we need to compute each connected piece of  ${}_0\langle \beta | V(t_1) \dots V(t_n) | \alpha \rangle_0$ . That is what we will do now. First we convert both the states and the interaction (still, very general interaction) into creation/annihilation operators, then we use graph to help to do the computation.

### 3.1 Creation and annihilation operator

After we transit to the computation of the S-operator, we no longer need the in/out state, so from now on, we will drop the 0 subscript. All our states from now on are understood as the free particle states.

To evaluate  $\langle \beta | V(t_1) \dots V(t_n) | \alpha \rangle$ , we first need to be able to compute  $\langle \beta | \alpha \rangle$ . Again we would like to start with the simplest case, the vacuum state  $|0\rangle$

$$\langle 0 | 0 \rangle = 1. \quad (3.3)$$

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<sup>1</sup>See proof in Weinberg 4.4.

Then one particle state  $|1\rangle$ , where we use the abbreviated notation  $|1\rangle \equiv |\mathbf{p}_1, s_1, n_1\rangle$ . Note here we start to label the one-particle state with only the independent variables. We label the particle by its 3-momentum because we can compute  $p^0 = E = \sqrt{\mathbf{p}^2 + m^2}$ .

$$\begin{aligned}\langle 1' | 1 \rangle &= (2\pi)^3 (2E_1) \delta^3(\mathbf{p}'_1 - \mathbf{p}_1) \delta_{s'_1 s_1} \delta_{n'_1 n_1} \\ &\equiv \delta_{1'1}\end{aligned}\quad (3.4)$$

where we defined abbreviated  $\delta$  function  $\delta_{1'1}$ , note that we include the same normalization factor before.

When we have two particles, things get a bit complicated, because we have to worry about identical particles.

$$\langle 1' 2' | 12 \rangle = \delta_{1'1} \delta_{2'2} \pm \delta_{1'2} \delta_{2'1} \quad (3.5)$$

where top sign + is for bosons and bottom sign - is for fermions. Now we can consider the general states

$$\langle 1' 2' \dots M' | 12 \dots N \rangle = \delta_{MN} \sum_{\mathcal{P}} \delta_{\mathcal{P}} \prod_i \delta_{(\mathcal{P}i)'i} \quad (3.6)$$

Where  $\mathcal{P}$  is a permutation: in the two particle example, the first term corresponds to an identity permutation  $\mathcal{P}1 = 1, \mathcal{P}2 = 2$  and the second term corresponds to  $\mathcal{P}1 = 2, \mathcal{P}2 = 1$ .  $\delta_{\mathcal{P}}$  is a sign factor associated with the permutation: if the permutation involves odd permutation of fermions the sign is minus otherwise it is plus.

The next thing we are going to introduce is a creation operator, which creates a particle,

$$a^\dagger(q) |12 \dots N\rangle \equiv |q12 \dots N\rangle. \quad (3.7)$$

where  $q$  stands for the collection of labels  $\mathbf{p}, s, n$ .<sup>2</sup>

The creation operator is defined as the operator that simply adds a particle's quantum numbers at the front of the list of particles' quantum numbers in the state.

Note this defines all the matrix elements (between any two multiple-particle states) of the creation operator. In particular we can rewrite any multi-particle state as a bunch of creation operators acting on the vacuum.

$$|1 \dots N\rangle = a^\dagger(1) \dots a^\dagger(N) |0\rangle \quad (3.8)$$

Since we have the definition of the inner product we can compute the matrix element of the adjoint of the creation operator, called annihilation operator. Now by doing the exercise in the footnote, we find out that the annihilation

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<sup>2</sup>With the new labeling, it is clear that for a single species for each 3-momentum and spin, there is a corresponding creation operator.

operator is <sup>3</sup>

$$a(q)|1 \dots N\rangle = \sum_{r=1}^N (\pm)^{r-1} \delta_{qr} |1 \dots r-1, r+1 \dots N\rangle. \quad (3.14)$$

As a special case, we have

$$a(q)|0\rangle = 0 \quad (3.15)$$

because there is no particle in the vacuum to annihilate.

With this we can show a very important relationship: the commutation relationship which is essentially assumed in canonical quantization of QFT

$$[a(q), a(q')]_{\mp} = \delta(q' - q) \quad (3.16)$$

where we define  $[\cdot]_+$  to be anti-commutator and  $[\cdot]_-$  to be commutator. Or in the full form, you have seen a similar one in scalar QFT

$$[a_{\mathbf{p}'}^{s'}, (a_{\mathbf{p}}^s)^\dagger]_{\mp} = (2\pi)^3 (2E_p) \delta^3(\mathbf{p}' - \mathbf{p}) \delta_{ss'}. \quad (3.17)$$

Here we introduce the notation for creation/annihilation operators when we write out the 3-momenta and spin label. We use upper index to denote spin and lower index to denote 3-momenta. This is shown in Fig 3.1.

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<sup>3</sup>Now we make this statement explicitly.

$$\langle \beta | (a | \alpha) = (\langle \beta | a) | \alpha \rangle \quad (3.9)$$

where we use  $(\langle \beta | a)$  to denote the bra that corresponds to  $a^\dagger | \beta \rangle$

$$(\langle \beta | a) \equiv (a^\dagger | \beta)^\dagger \quad (3.10)$$

Now let us explicitly compute with  $|\alpha\rangle = |1 \dots N\rangle$  and  $\langle \beta| = \langle 1' \dots M'|$ . Thus

$$\begin{aligned} \langle 1' \dots M' | a(q) &= (a(q)^\dagger | 1' \dots M')^\dagger \\ &= |q 1' \dots M' \rangle^\dagger \\ &= \langle q 1' \dots M' | \end{aligned} \quad (3.11)$$

According to our normalization (3.6), considering each permutation as two parts: first permute one of them, say  $r$ , to the very front and then permute the rest  $N-1=M$  of them.  $\bar{\mathcal{P}}$  maps  $1 \dots M$  to  $1 \dots r-1, r+1 \dots N$ .

$$\begin{aligned} \langle 1' \dots M' | (a(q) | 1 \dots N) &= \langle q 1' \dots M' | 1 \dots N \rangle \\ &= \delta_{M+1, N} \sum_{r=1}^N \sum_{\bar{\mathcal{P}}} (\pm)^{r-1} \delta_{\bar{\mathcal{P}}} \delta_{qr} \prod_{i=1}^M \delta_{i' \bar{\mathcal{P}}_i} \\ &= \delta_{M+1, N} \sum_{r=1}^N (\pm)^{r-1} \delta_{qr} \sum_{\bar{\mathcal{P}}} \prod_{i=1}^M \delta_{i' \bar{\mathcal{P}}_i} \\ &= \sum_{r=1}^N (\pm)^{r-1} \delta_{qr} \langle 1' \dots M' | 1 \dots r-1, r+1 \dots N \rangle. \end{aligned} \quad (3.12)$$

Now compare left side and right side, we find out that the annihilation operator is

$$a(q) |1 \dots N\rangle = \sum_{r=1}^N (\pm)^{r-1} \delta_{qr} |1 \dots r-1, r+1 \dots N\rangle. \quad (3.13)$$

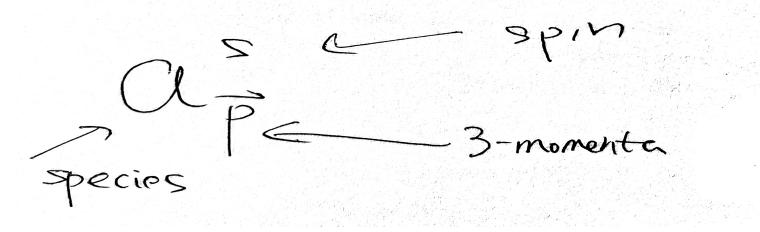


Figure 3.1: I am a notation box.

And we use the convention the top sign is for bosons and the bottom sign is for fermions.

In the footnote we provide the proof.<sup>4</sup>

In addition, we have

$$[a(q'), a(q)]_{\mp} = [a(q')^{\dagger}, a(q)^{\dagger}]_{\mp} = 0. \quad (3.23)$$

As we can see, we have not assumed any pre-existing field theories, but we end up with the same (anti-)commutator relationship.

To study any operator, we express it as an expansion of creation and anni-

---

<sup>4</sup>on one hand,

$$\begin{aligned} a(q')a(q)^{\dagger}|1\dots N\rangle &= a(q')|q1\dots N\rangle \\ &= \delta(q' - q)|1\dots N\rangle + \sum_{r=1}^N (\pm)^r \delta_{q'r}|q1\dots r-1, r+1\dots N\rangle \end{aligned} \quad (3.18)$$

On the other hand

$$\begin{aligned} a(q)^{\dagger}a(q')|1\dots N\rangle &= a(q)^{\dagger} \sum_{r=1}^N (\pm)^{r-1} \delta_{q'r}|1\dots r-1, r+1\dots N\rangle \\ &= \sum_{r=1}^N (\pm)^{r-1} \delta_{q'r}|q1\dots r-1, r+1\dots N\rangle \end{aligned} \quad (3.19)$$

Suppose we are talking about bosons, then  $(+)^r$  is the same as  $(+)^{r-1}$ , we subtract the above two equations and get

$$\begin{aligned} (a(q')a(q)^{\dagger} - a(q)^{\dagger}a(q'))|1\dots N\rangle &= \delta(q' - q)|1\dots N\rangle \\ &= [a(q'), a(q)]_{-}|1\dots N\rangle \end{aligned} \quad (3.20)$$

Suppose we are talking about fermions, then  $(-)^r$  is negative of  $(-)^{r-1}$ , we add the above two equations and get

$$\begin{aligned} (a(q')a(q)^{\dagger} + a(q)^{\dagger}a(q'))|1\dots N\rangle &= \delta(q' - q)|1\dots N\rangle \\ &= [a(q'), a(q)]_{+}|1\dots N\rangle \end{aligned} \quad (3.21)$$

And together we proved

$$[a(q'), a(q)^{\dagger}]_{\mp} = \delta(q' - q). \quad (3.22)$$

hilation operators<sup>5</sup>

$$\begin{aligned}\mathcal{O} = & \sum_N \sum_M \int dq'_1 \dots dq'_N dq_1 \dots dq_M \\ & \times a^\dagger(q'_1) \dots a^\dagger(q'_N) a(q_1) \dots a(q_M) \\ & \times C_{NM}(q'_1 \dots q'_N q_1 \dots q_M)\end{aligned}\quad (3.24)$$

where we grossly use the notation  $\int dq_1 \equiv \sum_{s_1} \int \frac{d^3 p_1}{(2\pi)^3 (2E_1)}$ . (Again we choose a normalization so that the measure is Lorentz invariant.)

For example: the free particle Hamiltonian is always<sup>6</sup>

$$H_0 = \int dqa^\dagger(q)a(q)E(q) \quad (3.25)$$

where  $E(\mathbf{p}, s, n) = \sqrt{\mathbf{p}^2 + m_n^2}$ .

## 3.2 Graphic evaluation

Now recall our S operator,

$$\begin{aligned}S = & 1 + \sum_n \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 dt_2 \dots dt_n T\{V(t_1) \dots V(t_n)\} \\ = & Te^{-i \int_{-\infty}^{\infty} dt V(t)}\end{aligned}\quad (3.26)$$

and in order to compute S-matrix, we need to be able to evaluate  $\langle \beta | V(t_1) \dots V(t_n) | \alpha \rangle$ . Now state  $|\alpha\rangle$  and  $|\beta\rangle$  can be expressed as products of creation operators acting on the vacuum.  $V(t)$  is an operator, so it is a sum of products of creation and annihilation operators. In the end, what we need to compute is a bunch of creation and annihilation operators sandwiched by the vacuum.

Thus we need to evaluate

$$\begin{aligned}\langle \beta | V(t_1) \dots | \alpha \rangle & \quad (3.27) \\ = & \sum \int \text{coefficients} \\ & \times \langle 0 | a_{1'} \dots (a^\dagger \dots a \dots) \dots a_1^\dagger \dots | 0 \rangle.\end{aligned}$$

Let us break it down a bit

$$\underbrace{\langle 0 |}_{\text{left vacuum}} \underbrace{a_{1'} \dots}_{\text{acting left}} \underbrace{(a^\dagger \dots a \dots)}_{\text{one factor of } V} \underbrace{\dots}_{\text{other } V\text{'s}} \underbrace{a_1^\dagger \dots}_{\text{acting right}} \underbrace{| 0 \rangle}_{\text{right vacuum}}. \quad (3.28)$$

The left bunch of annihilation operators comes from creation operators from final state  $\langle \beta |$ , but we can always use the adjoint property to move the creation

<sup>5</sup>For a proof of existence, see Weinberg 4.2.

<sup>6</sup>If you substitute  $dq$  in, you should find the formula familiar from scalar QFT.

operators acting left to be annihilation operators acting on the right. The right bunch comes from creation operators from initial state  $|\alpha\rangle$ , and the operators in the middle are from the expansion of  $V$  in terms of the creation/annihilation operators. We write it out explicitly for one of them and denote the rest by ....

Note that each of the creation/annihilation operator comes from the expansion of  $V$  will also come with a momentum integral and sum over spins, but we can deal with them later. The interaction also comes with some coefficients, since we are keeping everything general, we don't worry about them at this point.

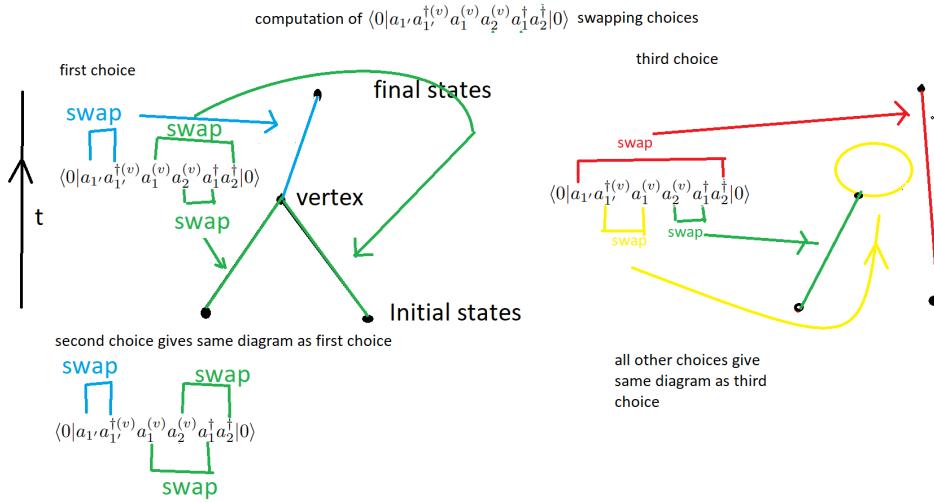


Figure 3.2: In this figure, we show how to draw a graph for the corresponding calculation of creation and annihilation operators. In this figure, green indicates swapping between initial state and the interaction, blue indicates swapping between final state and the interaction, yellow indicates swapping within the interactions and red indicates swapping between initial and final states.

Now let us think about how to evaluate this thing. This seems crazy, but we have all the tools: we just swap them using the commutators, once we swap an annihilation operator to the right end, we are done because  $a|0\rangle = 0$ . Let us organize our swappings by graphs : let us draw a few points in the middle representing how many  $V$ 's we have, and draw a few points on the bottom to represent the initial state and a few more points on the top to represent the final state. Now if we swapped an annihilation operator in the middle with the creation operators from the initial states, we are going to draw a line coming from the diagram below (choose an initial particle) to the vertex. Similarly if we swapped an creation operator in the middle with the annihilation operators from the final states, we are going to draw a line from the vertex to the diagram above (choose a final particle). If we swapped a creation operator from one  $V$  to an annihilation operator from another  $V$  (or even the same one), then we

draw a line between the vertex (or a loop on the same vertex). If we moved the annihilation operator from final states to the creation operator to the initial states, we draw a line directly between the initial particle and final particle without passing through any vertices. Then we can convince ourselves each of this diagram represent a non-zero result.

So graphically speaking, the connected part of the S-matrix is the sum of the contribution to the S-matrix that are connected, in the sense that we drop all terms where any initial or final particles or any operator  $V$  is not connected to all the others by a sequence of particle creation and annihilation.

As you have seen scalar QFT, you probably already recognized something: hey, aren't these Feynman diagrams? Sure, we just need some Feynman rules and the vertex rule is going to depend on the specific choice of theory/interaction.

Isn't it amazing that without a QFT, Feynman diagrams still emerge?

Now we know the strategy to compute the scattering matrix (via graphs), let us focus on the question we have: what kind of interactions ( $V$ ) can we have? But before that, what kind of S-matrix can we have?

### 3.3 The condition on the scattering matrix

The connected S-matrix in the momentum space needs to include a part due to the overall momentum conservation (Lorentz invariance)

$$S_{1' \dots M', 1 \dots N}^C = \delta^4(p_1 + \dots + p_N - p_{1'} - \dots - p_{M'}) \times \text{stuff} \quad (3.29)$$

where superscript C is for connected.

But not anything proportional to the overall momentum conservation delta can be a connected S-matrix. This is actually as singular as connected S-matrix can be. Any more singular will be disallowed. For example, *stuff* is not allowed to contain one or more factors of

$$\delta\left(\sum_{\text{some subset}} p - \sum_{\text{some subset}} p'\right) \quad (3.30)$$

Why is that so?

Suppose such a factor also exists in  $S^C$ . A momentum conservation indicates a translational symmetry due to Noether theorem. The total of a subset of momenta is conserved means the translation  $x \rightarrow x+b$  that acts on that subset of particles is a symmetry transformation. In other words, the S-matrix (now let us in position space) for some original configuration should be the same (invariant) as the S-matrix for some new configuration where a subset of particles have been simultaneously shifted by the same amount.

Why is that a problem? Well, when we take the shift to be large, the subset of particles will be well separated from the original bunch. And there is a principle that says in this case the S-matrix should vanish (instead of being the same due to partial translation symmetry). This principle is the cluster decomposition principle. And the argument we just made is depicted in figure 3.3.

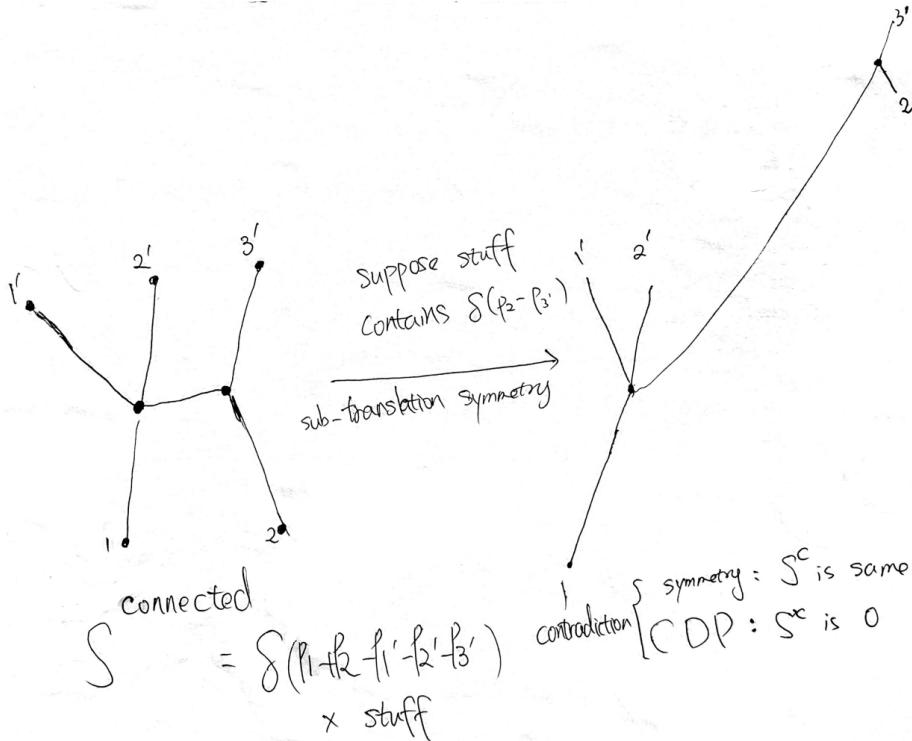


Figure 3.3: In this figure, we show why cluster decomposition principle indicates the lack of other momentum delta.

### 3.3.1 \*Interlude: Cluster decomposition principle

This principle has a daunting name, but it is a very realistic principle. It says if there are several experiments going on at different labs, then the results should not affect each other. This is of course true otherwise science cannot advance.

We hope it is reasonable to say that our S-matrix is then factorized according to the clusters (labs).

$$S_{\beta\alpha} = S_{\beta_1\alpha_1} S_{\beta_2\alpha_2} \dots \quad (3.31)$$

where  $\beta = \beta_1 + \beta_2 + \dots$  and similar for  $\alpha$ .  $\beta_1, \alpha_1$  are in one cluster etc. Here the discussion is in position space because we are talking about particles being close to each other physically.

Now we are going to introduce the connected part of S-matrix (without referring to graphs)

$$S_{\beta\alpha} \equiv \sum_{\text{all partition}} S_{\beta_a\alpha_a}^C S_{\beta_b\alpha_b}^C \dots \quad (3.32)$$

Note this time we no longer require  $\beta_a, \alpha_a$  to be in a cluster, this is over any

partition. This is a definition that we will use recursively to define all sorts of  $S^C$ 's.

Claim: in terms of the connected part of the S-matrix, the decomposition principle states that  $S_{\beta_a \alpha_a}^C$  vanishes if  $\beta_a, \alpha_a$  are not all in the same cluster.

Why the above statement about connected part of S-matrix is the same as S-matrix factorizes? We will show by an example.

Let us start from the beginning. Suppose we have 1 particle in the initial state and 1 particle in the final state. Then there is only one partition and

$$S_{1',1}^C \equiv S_{1',1} = \delta_{1'1} \quad (3.33)$$

The right hand side follows from conservation laws.

Now let us move to the case with two particles.

$$\begin{aligned} S_{1'2',12} &= S_{1'2',12}^C + S_{1',1}^C S_{2',2}^C + S_{1',2}^C S_{2',1}^C \\ &= S_{1'2',12}^C + \delta_{1'1} \delta_{2'2} + \delta_{1'2} \delta_{2'1}, \end{aligned} \quad (3.34)$$

where we use the wisdom from one particle to one particle S-matrix and write them all out as  $\delta$  function.

In the exciting exercise in the footnote, you will continue the journey to 4 particles. Then we can show that the claim is true: factorization in S-matrix is equivalent to connected S-matrix being 0 if all constituents are not in the same cluster.<sup>7</sup>

---

<sup>7</sup>We are going to skip 3 particles for now, and look at 4 particles. This is going to be a bit complicated

$$\begin{aligned} S_{1'2'3'4',1234} &= S_{1'2'3'4',1234}^C \\ &+ S_{1',1}^C S_{2'3'4',234}^C + \text{perm} \\ &+ S_{1',1}^C S_{2',2}^C S_{3'4',34}^C + \text{perm} \\ &+ S_{1',1}^C S_{2',2}^C S_{3',3}^C S_{4',4}^C + \text{perm} \\ &+ S_{1'2',12}^C S_{3'4',34}^C + \text{perm} \end{aligned} \quad (3.35)$$

Now we are going to assume  $S^C$  is zero if the particles are not in the same cluster. Let us say the cluster are  $1'2', 12$  and  $3'4', 34$ .

The above computation simplifies dramatically. for example the first two lines vanish. Last line one term remain. The result is

$$\begin{aligned} S_{1'2'3'4',1234} &= S_{1'2',12}^C S_{3'4',34}^C \\ &+ (\delta_{1'1} \delta_{2'2} + \delta_{1'2} \delta_{2'1}) S_{3'4',34}^C \\ &+ (\delta_{3'3} \delta_{4'4} + \delta_{3'4} \delta_{4'3}) S_{1'2',12}^C \\ &+ (\delta_{1'1} \delta_{2'2} + \delta_{1'2} \delta_{2'1})(\delta_{3'3} \delta_{4'4} + \delta_{3'4} \delta_{4'3}) \end{aligned} \quad (3.36)$$

Now let us look at the two particle to two particle scattering matrix and we find that the above is exactly

$$\begin{aligned} S_{1'2'3'4',1234} &= (S_{1'2',12}^C + (\delta_{1'1} \delta_{2'2} + \delta_{1'2} \delta_{2'1})) \\ &\times (S_{3'4',34}^C + (\delta_{3'3} \delta_{4'4} + \delta_{3'4} \delta_{4'3})) \\ &= S_{1'2',12} S_{3'4',34} \end{aligned} \quad (3.37)$$

We showed that the scattering matrix indeed factorize!

### 3.4 Condition for $V$

In the last part, we conclude that the connected part of S-matrix includes exactly one overall momentum conservation delta.

Now let us look at a connected part of the diagram. And ask ourselves the question again what constraints can we put on the theory with interaction  $V$ , so that the connected part of the momentum S-matrix has only one overall momentum conservation.

Let us assume  $V$  comes with  $x$  momentum  $\delta$  function, for each internal line we need to use a  $\delta$  function to fixed the momentum, but for each loop there is one momentum not fixed, if overall we have one left, we get this equation to solve for  $x$ .

$$1 = xV - I + L \quad (3.38)$$

Now a little bit of graph theory: a graph has a single vertex has  $V = 1, I = 0, L = 0$  plug in above we seem to find  $x = 1!$  In general<sup>8</sup> we have

$$C = V - (I - L) \quad (3.39)$$

where  $C$  stands for number of connected parts.

So it is determined we need interaction that contains a single momentum  $\delta$  so that decomposition principle is satisfied. This graph theory fun fact is the reason why in the Feynman rules, we dont have to write the vertex rule with a momentum  $\delta$ : they will all go into the final 4-momentum conservation in the S-matrix and we often define the scattering amplitude by extracting that 4-momentum  $\delta$  out.

### 3.5 Summary: Requirements for the interaction $V$

#### 3.5.1 From Lorentz invariance

We require the interaction can be written as a 3-volume integral of a Hamitonian density,

$$V(t) = \int d^3x \mathcal{H}(\mathbf{x}, t) \quad (3.40)$$

such that the Hamiltonian density is Lorentz invariant

$$U_0(\Lambda, a) \mathcal{H}(x) U_0^{-1}(\Lambda, a) = \mathcal{H}(\Lambda x + a) \quad (3.41)$$

---

<sup>8</sup>If we add  $V - 1$  vertices and add lines to just keep the graph connected, we need to add  $V - 1$  internal lines and 0 loops. So we have  $I = V - 1$  and  $L = 0$ . each new lines added produce a loop, so  $I = V - 1 + L$  which is  $1 = V - (I - L)$ .

where  $U_0(\Lambda, a)$  are the Lorentz transformation operators with the free Hamiltonian<sup>9</sup>.

Further more we require the Hamiltonian density commutes at space-separation,

$$[\mathcal{H}(x), \mathcal{H}(y)] = 0 \text{ for spacelike.} \quad (3.42)$$

### 3.5.2 From cluster decomposition principle

We require  $V(t)$  contains one and only one momentum conservation delta.

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<sup>9</sup>See Weinberg 3.3 for an explanation why the asymptotic states transform the way as the free particle state. Thus the operator also transforms with the free particle Lorentz transformation operators.



## Chapter 4

# Fields! Finally fields!

In our point of view, we define fields as linear combinations of creation/annihilation operators with some coefficients. The fields are operators that only depend on spacetime (not on momentum and spin).

It turns out fields are very useful to build Lorentz invariant Hamiltonian densities.

In this chapter, we will first make an argument about why fields are useful to build Lorentz invariant quantities (or why do we not build directly from creation/annihilation operators?).

Then we will find in order for Lorentz invariance to be implemented, what requirements we need to impose on the coefficients of the operators.

And then we will comment how this will satisfy the constraint from cluster decomposition principle. And comment on how causality condition could be satisfied.

Finally we will comment on why anti-particles exist.

In the next chapter we will solve the requirements for a couple cases, and to our delight, we will find the same scalar field that you learned during your journey with QFT for scalar theory.

### 4.1 Why fields instead of operators?

In order to make the Hamiltonian density Lorentz invariant, let us first take a look at its building blocks: the creation/annihilation operators. A creation operator acts on vacuum produce a one particle state. Here we briefly write out the label of  $a^\dagger$  and omit the particle species for now.

$$(a_p^s)^\dagger |0\rangle = |\mathbf{p}, s\rangle \quad (4.1)$$

We know the vacuum is Lorentz invariant and one particle state transforms as follows under Lorentz transformation

$$U_0(\Lambda, b)|\mathbf{p}, s\rangle = e^{+ib\cdot\Lambda p} \sum_{s'} D_{s's}^j(W(\Lambda, p))|\mathbf{p}_\Lambda, s'\rangle. \quad (4.2)$$

where we use  $\mathbf{p}_\Lambda$  to denote the 3-momentum part of  $\Lambda p$ .

This gives us the way creation operator transforms

$$U_0(\Lambda, b)(a_{\mathbf{p}}^s)^\dagger U_0^{-1}(\Lambda, b) = e^{+ib \cdot \Lambda p} \sum_{s'} D_{s's}^j(W(\Lambda, p))(a_{\mathbf{p}_\Lambda}^{s'})^\dagger \quad (4.3)$$

We can prove this by acting the equation above (4.3) on vacuum on both sides.

Similarly, the annihilation operator transforms as

$$U_0(\Lambda, b)a_{\mathbf{p}}^s U_0^{-1}(\Lambda, b) = e^{-ib \cdot \Lambda p} \sum_{s'} D_{s's}^{j*}(W(\Lambda, p))a_{\mathbf{p}_\Lambda}^{s'}. \quad (4.4)$$

Now consider Hamiltonian density  $\mathcal{H}$  as an expansion of these creation and annihilation operators with coefficients. It will be challenging to choose the coefficients for a Hamiltonian density to be a Lorentz scalar as the transformation of the coefficient needs to cancel various transformations from numerous of creation/annihilation operators. Especially all these transformations of the operators are momentum dependent!

The solution is to first make some other building blocks out of the creation/annihilation operators. Ideally these building blocks will transform nicely under Lorentz group. And here our star, the fields, makes an entrance. (Applause please!)

We can make two kinds of fields, the creation field and the annihilation field. Fields depend on spacetime, so we will introduce some coefficients that depend on  $x; \mathbf{p}, s$  to integrate the momentum and sum the spin with the creation/annihilation operator

$$\begin{aligned} \chi_l^-(x) &= \sum_s \int \frac{d^3 p}{(2\pi)^3 (2E_p)} u_l^s(x, \mathbf{p}) a_{\mathbf{p}}^s \\ \chi_l^+(x) &= \sum_s \int \frac{d^3 p}{(2\pi)^3 (2E_p)} v_l^s(x, \mathbf{p}) (a_{\mathbf{p}}^s)^\dagger, \end{aligned} \quad (4.5)$$

where we denote the creation field  $\chi^+$  and the annihilation field  $\chi^-$ . And we use the convention to write the discrete label  $s$  as an upper index. We keep the  $\mathbf{p}$  dependence in the parentheses to indicate this is not a quantized quantity. The subscript  $l$  labels the possible different components of the field. Since we don't know what fields we are dealing with yet, we give it a mysterious name  $\chi$  (or X).

From now on, we will see a lot of this normalized volume integral  $\frac{d^3 p}{(2\pi)^3 (2E_p)}$  and it is a huge headache to tex with them, from now on, we will use the following convention to absorb all the normalization factor

$$dV_p \equiv \frac{d^3 p}{(2\pi)^3 (2E_p)}. \quad (4.6)$$

In our new notation, our creation and annihilation fields become

$$\begin{aligned} \chi_l^-(x) &= \oint_s dV_p u_l^s(x, \mathbf{p}) a_{\mathbf{p}}^s \\ \chi_l^+(x) &= \oint_s dV_p v_l^s(x, \mathbf{p}) (a_{\mathbf{p}}^s)^\dagger, \end{aligned} \quad (4.7)$$

where we introduced  $\oint \equiv \sum \int$  as well.

We hope, by carefully choosing the coefficients, under Lorentz transformation, each field is multiplied with a position-independent (constant) matrix<sup>1</sup> (compared to the operators that transform with  $p$  dependent coefficients and  $p$  dependent matrix).

$$\begin{aligned} U_0(\Lambda, a)\chi_l^+(x)U_0^{-1}(\Lambda, a) &= \sum_{\bar{l}} D_{l\bar{l}}(\Lambda^{-1})\chi_{\bar{l}}^+(\Lambda x + a) \\ U_0(\Lambda, a)\chi_{\bar{l}}^-(x)U_0^{-1}(\Lambda, a) &= \sum_{\bar{l}} D_{l\bar{l}}(\Lambda^{-1})\chi_{\bar{l}}^-(\Lambda x + a) \end{aligned} \quad (4.8)$$

In the footnote<sup>2</sup> we show the constant matrices furnishes a representation of Lorentz group!

Now we can write our Hamiltonian density as an expansion of creation field and annihilation field,

$$\mathcal{H}(x) = \sum_{N,M} \sum_{l',l} g_{l',l} \chi_{l'}^+(x) \dots \chi_{l'_N}^+(x) \chi_{l_1}^-(x) \dots \chi_{l_M}^-(x) \quad (4.11)$$

where the summed term has  $N$  annihilation field and  $M$  creation field.  $l'$  and  $l$  collectively label  $l'_1 \dots l'_N$  and  $l_1 \dots l_M$ .

If we know how each field transform, we should be able to figure out what kind of coefficient we need so that each term of the Hamiltonian density transforms as a Lorentz scalar. This is in principle similar to using Clebsch-Gordan coefficients to couple various reps of the three dimension rotation group to form rotational scalars. In the footnote<sup>3</sup> we elaborate with equations.

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<sup>1</sup>Can always choose the creating field and the annihilating field such that the matrices are the same.

<sup>2</sup>Now take a second Lorentz transformation, paying attention to only creation field,

$$\begin{aligned} &U_0(\bar{\Lambda}, b)U_0(\Lambda, a)\chi_l^+(x)U_0^{-1}(\Lambda, a)U_0^{-1}(\bar{\Lambda}, b) \\ &= U_0(\bar{\Lambda}, b) \sum_{l'} D_{ll'}(\Lambda^{-1})\chi_{l'}^+(\Lambda x + a)U_0^{-1}(\bar{\Lambda}, b) \\ &= \sum_{l'} D_{ll'}(\Lambda^{-1}) \sum_{l''} D_{ll''}(\bar{\Lambda}^{-1})\chi_{l''}^+(\bar{\Lambda}\Lambda x + \bar{\Lambda}a + b) \\ &= \sum_{l'} D_{ll'}((\bar{\Lambda}\Lambda)^{-1})\chi_{l'}^+(\bar{\Lambda}\Lambda x + \bar{\Lambda}a + b) \end{aligned} \quad (4.9)$$

This shows us

$$\begin{aligned} D(\Lambda^{-1})D(\bar{\Lambda}^{-1}) &= D(\bar{\Lambda}\Lambda)^{-1} \\ &= D(\Lambda^{-1}\bar{\Lambda}^{-1}) \end{aligned} \quad (4.10)$$

<sup>3</sup>For each fixed  $N$  and  $M$ ,  $\mathcal{H}(x)$  needs to be a scalar separately.

$$\begin{aligned} U_0(\Lambda, a)\mathcal{H}(x)_{N,M}U_0^{-1}(\Lambda, a) &= \mathcal{H}_{N,M}(\Lambda x + a) \\ &= \sum_{\bar{l}', \bar{l}} g_{\bar{l}', \bar{l}} \chi_{\bar{l}'}^+(\Lambda x + a) \dots \psi_{\bar{l}'}^+(\Lambda x + a) \chi_{\bar{l}_1}^-(\Lambda x + a) \dots \chi_{\bar{l}_M}^-(\Lambda x + a) \\ &= \sum_{l', l} g_{l', l} \sum_{\bar{l}_1} D_{l'_1 \bar{l}_1}(\Lambda^{-1})\chi_{\bar{l}_1}^+(\Lambda x + a) \dots D_{l'_N \bar{l}_N}(\Lambda^{-1})\chi_{\bar{l}_N}^+(\Lambda x + a) \\ &\quad \times \sum_{\bar{l}_1} D_{l_1 \bar{l}_1}(\Lambda^{-1})\chi_{\bar{l}_1}^-(\Lambda x + a) \dots D_{l_M \bar{l}_M}(\Lambda^{-1})\chi_{\bar{l}_M}^-(\Lambda x + a) \end{aligned} \quad (4.12)$$

## 4.2 \*Requirements on the coefficients

So now we need to decide the coefficient  $u_l^s(x, \mathbf{p})$  and  $v_l^s(x, \mathbf{p})$ . Since we know the transformation for both the field and the creation/annihilation operator, we should be able to figure this out. Let us focus on the annihilation field first.

Because we know the field expansion

$$\chi_l^-(x) = \sum_s \int dV_p u_l^s(x, \mathbf{p}) a_{\mathbf{p}}^s \quad (4.14)$$

and how fields transform

$$U_0(\Lambda, b) \chi_l^-(x) U_0^{-1}(\Lambda, b) = \sum_{\bar{l}} D_{l\bar{l}}(\Lambda^{-1}) \chi_{\bar{l}}^-(\Lambda x + b) \quad (4.15)$$

and how the operators transform

$$U_0(\Lambda, b) a_{\mathbf{p}}^s U_0^{-1}(\Lambda, b) = e^{-ib \cdot \Lambda p} \sum_{s'} D_{s's}^{j*}(W(\Lambda, p)) a_{\mathbf{p}_\Lambda}^{s'}, \quad (4.16)$$

we must be able to find how the coefficients transform. This is derived in the footnote.<sup>4</sup> The result is

$$\sum_{s'} u_{l'}^{s'}(\Lambda x + b; \mathbf{p}_\Lambda) D_{s's}^j(W(\Lambda, p)) = \sum_{\bar{l}} D_{l\bar{l}}(\Lambda) u_{\bar{l}}^s(x, \mathbf{p}) e^{-ib \cdot \Lambda p}. \quad (4.20)$$

Compare the last two results, we need to impose on the constants  $g_{l',l}$  that

$$g_{l',\bar{l}} = \sum_{l',l} g_{l',l} D_{l_1 \bar{l}_1}(\Lambda^{-1}) \dots D_{l_N \bar{l}_N}(\Lambda^{-1}) D_{l_1 \bar{l}_1}(\Lambda^{-1}) \dots D_{l_M \bar{l}_M}(\Lambda^{-1}). \quad (4.13)$$

<sup>4</sup>

$$\begin{aligned} U_0(\Lambda, b) \chi_l^-(x) U_0^{-1}(\Lambda, b) &= \sum_s \int dV_p u_l^s(x, \mathbf{p}) e^{-ib \cdot \Lambda p} \sum_{s'} D_{s's}^{j*}(W(\Lambda, p)) a_{\mathbf{p}_\Lambda}^{s'} \\ &= \sum_s \int dV_{\Lambda p} u_l^s(x, \mathbf{p}) e^{-ib \cdot \Lambda p} \sum_{s'} D_{s's}^{j*}(W(\Lambda, p)) (a_{\mathbf{p}_\Lambda}^{s'}) \\ &= \sum_{s'} \int dV_{\Lambda p} u_{l'}^{s'}(x, \mathbf{p}) e^{-ib \cdot \Lambda p} \sum_s D_{ss'}^{j*}(W(\Lambda, p)) a_{\mathbf{p}_\Lambda}^s \\ &= \sum_{l'} D_{l\bar{l}}(\Lambda^{-1}) \chi_{\bar{l}}^-(\Lambda x + b) \\ &= \sum_{l'} D_{l\bar{l}}(\Lambda^{-1}) \sum_s \int dV_{\Lambda p} u_{l'}^s(\Lambda x + b; \mathbf{p}_\Lambda) a_{\mathbf{p}_\Lambda}^s. \end{aligned} \quad (4.17)$$

Compare line 3 and line 5,

$$\sum_{l'} D_{l\bar{l}}(\Lambda^{-1}) u_{l'}^s(\Lambda x + b; \mathbf{p}_\Lambda) = e^{-ib \cdot \Lambda p} \sum_{s'} D_{ss'}^{j*}(W(\Lambda, p)) u_{l'}^{s'}(x, \mathbf{p}) \quad (4.18)$$

or we can multiply  $D(\Lambda)$  and  $D(W^{-1})$  on both sides, and write it as

$$\begin{aligned} \sum_{s'} D_{ss'}^{j*}(W^{-1}(\Lambda, p)) u_{l'}^{s'}(\Lambda x + b; \mathbf{p}_\Lambda) &= \sum_l D_{l\bar{l}}(\Lambda) u_l^s(x, \mathbf{p}) e^{-ib \cdot \Lambda p} \\ &= \sum_{s'} u_{l'}^{s'}(\Lambda x + b; \mathbf{p}_\Lambda) D_{s's}^j(W(\Lambda, p)) \end{aligned} \quad (4.19)$$

In the last line, we used the fact the presentation is unitary so  $D^\dagger(W^{-1}) = D(W)$ , we also switch the position of the two factors so if somehow we can interpret  $u$  as a matrix, it will be the proper order of matrix multiplication.

We are going to simplify this formula a bit by considering some simple iconic (not generic) Poincare transformations.

### 4.2.1 Translations

First we take  $\Lambda = \delta$  and  $b$  arbitrary, formula (4.20) becomes (all the  $D$ 's becomes identity now.)

$$u_l^s(x + b; \mathbf{p}) = u_l^s(x, \mathbf{p})e^{-ib \cdot p}. \quad (4.21)$$

Thus we can choose

$$u_l^s(x, \mathbf{p}) = u_l^s(\mathbf{p})e^{-ix \cdot p}. \quad (4.22)$$

where we extract the  $x$  dependence. Let us take a look at our fields again,

$$\chi^-(x) = \sum_s \int dV_p u_l^s(\mathbf{p}) e^{-ix \cdot p} a_{\mathbf{p}}^s \quad (4.23)$$

We realize fields are Fourier transforms of the creation/annihilation operators! So now we use our insight (4.22) in our general formula (4.20). The formula becomes

$$\sum_{s'} u_{l'}^{s'}(\mathbf{p}_\Lambda) D_{s's}^j(W(\Lambda, p)) = \sum_l D_{l'l}(\Lambda) u_l^s(\mathbf{p}). \quad (4.24)$$

### 4.2.2 Boosts

Now take  $\mathbf{p} = 0$ , so we have  $p^\mu = (m, 0, 0, 0)$ . Let  $\Lambda$  be the standard boost  $L(q)$ <sup>5</sup> that takes a particle from mass with momentum  $(m, 0, 0, 0)$  to some four momentum  $q^\mu$ .

Then  $L(p) = \mathbb{1}$  and  $\Lambda p = L(q)p = q$ .

$$W(\Lambda, p) \equiv L^{-1}(\Lambda p) \Lambda L(p) = L^{-1}(q) L(q) \mathbb{1} = \mathbb{1}. \quad (4.25)$$

In this special case,  $D(W)$  becomes trivial (identity matrix) and formula (4.24) becomes

$$u_{l'}^s(\mathbf{q}) = \sum_l D_{l'l}(\Lambda) u_l^s(0). \quad (4.26)$$

In other words, if we know  $u_l$  at 0 momentum, then for a given representation  $D_{ll'}$ , we can compute  $u_l$  for all momentum.

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<sup>5</sup>Recall we have provided the standard boost before in a footnote.

### 4.2.3 Rotations

Now let us take  $\mathbf{p} = 0$  and only consider Lorentz transformation that give  $\mathbf{p}_\Lambda = 0$ , that is we consider a pure rotation  $R$ . In this case

$$W(\Lambda, p) \equiv L^{-1}(\Lambda p) \Lambda L(p) = \Lambda = R \quad (4.27)$$

because there is no boost,  $L^{-1}(\Lambda p) = L(p) = \mathbb{1}$ ! Our formula (4.24) simplifies

$$\sum_{s'} u_{l'}^{s'}(0) D_{s's}^j(R) = \sum_l D_{l'l}(R) u_l^s(0). \quad (4.28)$$

Or we can express using the angular momentum matrix (remember  $D = 1 + i/2\Theta_{jk} \cdot J_{jk}$ )

$$\boxed{\sum_{s'} u_{l'}^{s'}(0) \mathbf{J}_{s's}^j(R) = \sum_l \mathcal{J}_{l'l}(R) u_l^s(0)}. \quad (4.29)$$

where we use  $\mathbf{J}$  to denote the angular momentum matrix for representation  $D^j(R)$  for the rotation. And we use  $\mathcal{J}$  to denote the angular momentum matrix for  $D(R)$  for the  $l$  representation.

We can perform the similar calculation for the creation operator, then we will find the condition on the  $v$ 's

$$\boxed{\sum_{s'} v_{l'}^{s'}(0) \mathbf{J}_{s's}^{j*}(R) = - \sum_l \mathcal{J}_{l'l}(R) v_l^s(0)}. \quad (4.30)$$

where the minus sign comes from the  $D^{j*}(R)$  but no  $*$  on the other side  $D(R)$ .

## 4.3 Nod to decomposition cluster principle

Now consider a Hamiltonian density (one term) made of fields

$$\mathcal{H}_{N,M}(x) = \sum_{l',l} g_{l',l} \chi_{l'_1}^+(x) \dots \chi_{l'_N}^+(x) \chi_{l_1}^-(x) \dots \chi_{l_M}^-(x) \quad (4.31)$$

Let us substitute the field with its expansion and integrate over the space to give  $V$ ,

$$\begin{aligned} V_{N,M} &= \sum_{l',l} g_{l',l} \int d^3x \\ &\times \sum_{\text{all } s'} \int dV_{l'} v_{l'_1}^{s'_1}(\mathbf{p}'_1) a_{\mathbf{p}'_1}^{s'_1\dagger} e^{+i\mathbf{x}\cdot\mathbf{p}'_1} \dots \\ &\times \sum_{\text{all } s} \int dV_1 u_{l_1}^{s_1}(\mathbf{p}_1) a_{\mathbf{p}_1}^{s_1} e^{-i\mathbf{x}\cdot\mathbf{p}_1} \dots \\ &= \delta(p'_1 + \dots + p'_N - p_1 - \dots - p_M) \times \text{stuff}. \end{aligned} \quad (4.32)$$

As we can see, the momentum conservation delta is guaranteed to show up in the interaction and there is only one delta! Cluster decomposition principle is guaranteed to be satisfied.

## 4.4 What about causality?

If all we wanted is a Lorentz scalar Hamiltonian density, we are done. But then we recall the second requirement from special relativity,

$$[\mathcal{H}(x), \mathcal{H}(y)] = 0 \text{ for spacelike.} \quad (4.33)$$

Let us quickly compute the commutation relationship of our creation and annihilation field

$$\begin{aligned} \chi_l^-(x) &= \sum_s \int dV_{\mathbf{p}} u_l^s(x, \mathbf{p}) a_{\mathbf{p}}^s \\ \chi_l^+(x) &= \sum_s \int dV_{\mathbf{p}} v_l^s(x, \mathbf{p}) (a_{\mathbf{p}}^s)^\dagger, \end{aligned} \quad (4.34)$$

the commutator is

$$[\chi_l^-(x), \chi_{l'}^+(y)]_{\mp} = \sum_s \int dV_{\mathbf{p}} u_l^s(\mathbf{p}) v_{l'}^s(\mathbf{p}) e^{-ip \cdot (x-y)} \quad (4.35)$$

where we used our insight (4.22). This in general does not vanish for spacelike separations.

The commutator of the same type operators vanish. But we cannot use all creation or all annihilation fields to make our Hamiltonian density, because that is not hermitian.

The only way to get out of this difficulty is to use the linear combination of the creation/annihilation fields,

$$\chi_l(x) = \kappa \chi_l^-(x) + \lambda \chi_l^+(x) \quad (4.36)$$

and hopefully for some good choice of coefficients, we have

$$[\chi_l(x), \chi_{l'}(y)]_{\mp} = 0 \text{ for spacelike.} \quad (4.37)$$

We will discuss this method for the specific fields we will introduce.

## 4.5 Charged field and the anti-particle

Suppose there is some conserved charge  $Q$  with  $[Q, \mathcal{H}] = 0$ . Suppose a particle of species  $n$  carries a value  $q_n$  for this charge,

$$Q a^\dagger |0\rangle = q_n a^\dagger |0\rangle. \quad (4.38)$$

What if it is a multi particle state  $|\alpha\rangle \equiv |\mathbf{p}_1, s_1, n_1; \dots\rangle$  that already carries some charge  $q_\alpha$ ?

$$\begin{aligned} Q |\alpha\rangle &= q_\alpha |\alpha\rangle \\ Q a^\dagger |\alpha\rangle &= (q_n + q_\alpha) a^\dagger |\alpha\rangle \end{aligned} \quad (4.39)$$

Acting  $a^\dagger$  on the first line and use the first line to subtract second line we get commutator relationship

$$[Q, a^\dagger] = q_n a^\dagger. \quad (4.40)$$

and take a hermitian conjugate of this equation, we have

$$[Q, a] = -q_n a. \quad (4.41)$$

If we want to have  $[Q, \mathcal{H}] = 0$ ,  $\mathcal{H}$  should be made of some nice fields that satisfy

$$[Q, \chi_l(x)] = q \chi_l(x) \quad (4.42)$$

But this is not possible for our current situation, because  $\chi_l(x)$  is made of both creation and annihilation operators and they carry different charges!

For this to work, for the particle carries charge  $q$ , there must exist another particle carries charge  $-q$ . This way we can combine the creation operator that creates charge  $q$  and the annihilation operator that destroys charge  $-q$  together to make a field operator that has a definite charge. This is the reason for anti-particles.

## Chapter 5

# Scalar fields and the Dirac fields

Finally we are going to use the condition on the coefficients

$$\begin{aligned}\sum_{s'} u_{l'}^{s'}(0) \mathbf{J}_{s's}^j(R) &= \sum_l \mathcal{J}_{l'l}(R) u_l^s(0) \\ \sum_{s'} v_{l'}^{s'}(0) \mathbf{J}_{s's}^{j*}(R) &= - \sum_l \mathcal{J}_{l'l}(R) v_l^s(0)\end{aligned}\tag{5.1}$$

to find the fields corresponding to specific representation  $l$ . We are going to demonstrate for two cases: the scalar field and the Dirac field (the one that describes an electron (and other fermions)!).

### 5.1 Causal scalar fields

For scalar, we have trivial representation all around,  $\mathcal{J} = 0$ . The only way to satisfy this is to choose  $\mathbf{J} = 0$  too, that is we have a spin  $j = 0$  field. There is no spin to sum for spin 0 field. We denote this field as  $\phi$  and its expansion is very simple compared to the general case.

$$\begin{aligned}\phi^-(x) &= \int dV_p u(x, \mathbf{p}) a_{\mathbf{p}} \\ \phi^+(x) &= \int dV_p v(x, \mathbf{p}) a_{\mathbf{p}}^\dagger,\end{aligned}\tag{5.2}$$

with

$$\begin{aligned}u(x, \mathbf{p}) &= u(\mathbf{p}) e^{-ix \cdot p} = u(0) e^{-ix \cdot p} \\ v(x, \mathbf{p}) &= v(\mathbf{p}) e^{+ix \cdot p} = v(0) e^{+ix \cdot p}\end{aligned}\tag{5.3}$$

In principle we can choose  $u(0)$  and  $v(0)$  to be any number, so it is convenient to choose them to be 1. Thus we have in the scalar case,

$$\begin{aligned}\phi^-(x) &= \int dV_p a_{\mathbf{p}} e^{-i\mathbf{p}\cdot x} \\ \phi^+(x) &= \int dV_p a_{\mathbf{p}}^\dagger e^{+i\mathbf{p}\cdot x} = (\phi^-)^\dagger(x).\end{aligned}\quad (5.4)$$

Now we will compute the commutator and anticommutator (because we don't know it is boson or fermion yet) attempting to satisfy the causality constraint.

We consider a linear combination (which is very arbitrary)

$$\begin{aligned}\phi(x) &\equiv \kappa\phi^-(x) + \lambda\phi^+(x) \\ \phi^\dagger(x) &= \lambda^*\phi^-(x) + \kappa^*\phi^+(x).\end{aligned}\quad (5.5)$$

We will need both so that Hamiltonian is hermitian.

Let us first compute for the annihilation and creation fields,

$$\begin{aligned}D(x-y) &\equiv [\phi^-(x), \phi^+(y)]_\mp \\ &= \int dV_p dV_{p'} e^{-i\mathbf{p}\cdot x + i\mathbf{p}'\cdot y} [a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger]_\mp \\ &= \int dV_p e^{-i\mathbf{p}\cdot(x-y)}\end{aligned}\quad (5.6)$$

At spacelike separation  $D(x-y)$  is even. Because at spacelike we can choose  $x^0 = y^0$ , then

$$\begin{aligned}\int dV_p e^{-i\mathbf{p}\cdot(x-y)} &= \int dV_p e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \int dV_p e^{-i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})} \\ &= \int dV_p e^{+i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})} \\ &= \int dV_p e^{-i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}.\end{aligned}\quad (5.7)$$

Now we compute the commutator relationship between  $\phi(x)$  and  $\phi^\dagger(x)$ ,

$$\begin{aligned}[\phi(x), \phi^\dagger(y)] &= |\kappa|^2 [\phi^-(x), \phi^+(y)]_\mp + |\lambda|^2 [\phi^+(x), \phi^-(y)]_\mp \\ &= (|\kappa|^2 \mp |\lambda|^2) D(x-y)\end{aligned}\quad (5.8)$$

where we used the fact that  $D(x-y)$  is even.

$$\begin{aligned}[\phi(x), \phi(y)]_\mp &= \kappa\lambda ([\phi^-(x), \phi^+(y)]_\mp + [\phi^+(x), \phi^-(y)]_\mp) \\ &= \kappa\lambda(1 \mp 1) D(x-y).\end{aligned}\quad (5.9)$$

From the second contribution, it is clear that we need to choose minus sign, that is commutator, our calculation shows that spin 0 leads to a boson field! (spin-statistics theorem anyone?)

From the first calculation we deduce  $|\kappa| = |\lambda|$ , the phase can be absorbed by redefining our field to be

$$\tilde{\phi}(x) = e^{i\alpha} \phi^+(x) + e^{-i\alpha} \phi^-(x). \quad (5.10)$$

In the end we can choose

$$\phi(x) = \phi^+(x) + \phi^-(x) = \phi^\dagger(x). \quad (5.11)$$

The Hamiltonian density will commute at spacelike separation if it is made of  $\phi(x)$ , the well known scalar field.

Note that of course this scalar field satisfies Klein Gordon equation (because it looks exactly the same as the mode expansion you saw in scalar QFT.), but we don't need Klein Gordon equation to find it!

## 5.2 The spinor representation

The next field we are interested are the Dirac fields. Recall that to determine the coefficients in the field expansion, we need to start with a specific representation of Lorentz group. In the case of scalar field, we used the trivial representation.

The Dirac field is the result of choosing the spinor representation of Lorentz group.

Recall that to find a representation of the Lorentz group, equivalently we can find a representation of the Lorentz algebra, which you found in your first homework

$$i[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = \eta^{\nu\rho} \mathcal{J}^{\mu\sigma} - \eta^{\mu\rho} \mathcal{J}^{\nu\sigma} - \eta^{\sigma\mu} \mathcal{J}^{\rho\nu} + \eta^{\sigma\nu} \mathcal{J}^{\rho\mu}. \quad (5.12)$$

where  $\mathcal{J}^{\mu\nu}$ 's are the generators of the Lorentz group defined as follows (to first order)<sup>1</sup>

$$D(\Lambda) = 1 + \frac{i}{2} \omega_{\mu\nu} \mathcal{J}^{\mu\nu}. \quad (5.13)$$

The spinor representation is constructed by making use of gamma matrices defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}. \quad (5.14)$$

Then the above Lorentz algebra is solved by

$$\mathcal{J}^{\mu\nu} = -\frac{i}{4} [\gamma^\mu, \gamma^\nu]. \quad (5.15)$$

---

<sup>1</sup>The  $i$  is a convention in hope to find Hermitian generators to be observable, the factor of  $\frac{1}{2}$  cancels the factor 2 comes from, say  $\omega_{12} \mathcal{J}^{12}$  repeated by  $\omega_{21} \mathcal{J}^{21}$ .

We will show this in a footnote<sup>2</sup>. In order to solve the constraints for the coefficients, it is easier to work with specific representation. It is even easier if we work with an easy representation (for example maybe we can hope it is reducible, i.e. block diagonal?).

To do that, let us make some specific choice of gamma matrices. Since they cannot be just numbers, let us try  $2 \times 2$  matrices. **what are special  $2 \times 2$  matrices that we are familiar with and are related to spin?** The famous Pauli matrices are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (5.19)$$

and they satisfy the following anti-commutator relationship

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}. \quad (5.20)$$

Unfortunately we cannot find a fourth  $2 \times 2$  matrix to anti-commute with them. We need to go for more dimensions. Three dimension is no good<sup>3</sup>. It turns out at 4 dimensions, we can find the following representation, known as the chiral or Weyl representation,

$$\gamma_w^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_w^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \quad (5.21)$$

---

<sup>2</sup>First we show

$$\begin{aligned} [\mathcal{J}^{\mu\nu}, \gamma^\rho] &= -\frac{i}{4} [[\gamma^\mu, \gamma^\nu], \gamma^\rho] \\ &= -\frac{i}{4} (\{\gamma^\mu, \{\gamma^\nu, \gamma^\rho\}\} - \{\gamma^\nu, \{\gamma^\mu, \gamma^\rho\}\}) \\ &= -\frac{i}{2} (\{\gamma^\mu, \eta^{\nu\rho}\} - \{\gamma^\nu, \eta^{\mu\rho}\}) \\ &= -i\gamma^\mu \eta^{\nu\rho} + i\gamma^\nu \eta^{\mu\rho}. \end{aligned} \quad (5.16)$$

where we use commutator identity

$$[[A, B], C] = \{A, \{B, C\}\} - \{B, \{A, C\}\}. \quad (5.17)$$

Then we commute the commutator of  $\mathcal{J}$ 's.

$$\begin{aligned} i[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] &= -\frac{i}{4} i[\mathcal{J}^{\mu\nu}, [\gamma^\rho, \gamma^\sigma]] \\ &= +\frac{1}{4} ([[[\mathcal{J}^{\mu\nu}, \gamma^\rho], \gamma^\sigma] + [[\gamma^\sigma, \mathcal{J}^{\mu\nu}], \gamma^\rho]]) \\ &= +\frac{1}{4} ([-i\gamma^\mu \eta^{\nu\rho} + i\gamma^\nu \eta^{\mu\rho}, \gamma^\sigma] - [-i\gamma^\mu \eta^{\nu\sigma} + i\gamma^\nu \eta^{\mu\sigma}, \gamma^\rho]) \\ &= \eta^{\nu\rho} \mathcal{J}^{\mu\sigma} - \eta^{\mu\rho} \mathcal{J}^{\nu\sigma} - \eta^{\sigma\mu} \mathcal{J}^{\rho\nu} + \eta^{\sigma\nu} \mathcal{J}^{\rho\mu} \end{aligned} \quad (5.18)$$

where in the second line we used Jacobi identity for commutators  $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$  and in the last line we used definition of  $\mathcal{J}$  and remember they are anti-symmetric.

<sup>3</sup>For a discussion about dimension of spinor with general spacetime dimension, see Appendix H.

It is easy to show that any set of matrices related to  $\gamma_w$  by a similar transformation also satisfy the conditions satisfied by  $\gamma$  matrices<sup>4</sup>.

Thus for any  $\gamma$  matrix that satisfies the Dirac algebra and relates to the Weyl representation<sup>5</sup> by a unitary similar transformation, it automatically satisfies the Hermitian condition! But for the ones related by a non-unitary similar transformation, there is no reason for the hermitian condition to be true. And counter examples can easily be found. So we can now say that the Hermitian condition that comes from the Hamiltonian  $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$  puts a strong constraint on the  $\gamma$  matrix<sup>6</sup>.

With this specific representation, we can immediately compute the Lorentz generators, the rotations and the boosts.

Now we ask an interesting question: what would happen to the Dirac field (that we are going to find) if we rotate  $2\pi$  in the say  $xy$ -plane? Would it come back to itself? To answer this question, we are going to compute for the rotation in  $xy$ -plane,  $\mathcal{J}_{12}$ <sup>7</sup>,

$$\begin{aligned}\mathcal{J}_{12} &= -\frac{i}{4} [\gamma_{w,1}, \gamma_{w,2}] \\ &= -\frac{i}{2} \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \\ &= -\frac{i}{2} \begin{pmatrix} -\sigma^1 \sigma^2 & 0 \\ 0 & -\sigma^1 \sigma^2 \end{pmatrix} \\ &= -\frac{1}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}.\end{aligned}\tag{5.24}$$

We define angular momentum vector<sup>8</sup>

$$\mathcal{J}^3 \equiv -\mathcal{J}_{12} = \frac{1}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}.\tag{5.25}$$

---

<sup>4</sup>Let

$$\gamma^\mu = U \gamma_w^\mu U^{-1}\tag{5.22}$$

where  $U$  is an arbitrary unitary matrix. Then we have

$$\{\gamma^\mu, \gamma^\nu\} = U \{\gamma_w^\mu, \gamma_w^\nu\} U^{-1} = U 2\eta^{\mu\nu} \mathbb{1} U^{-1} = 2\eta^{\mu\nu} \mathbb{1}\tag{5.23}$$

The converse of the above statement is also true: all representations of the Dirac algebra are equivalent. Such  $U$  matrix can always be constructed. Interested reader can see the construction in “An introduction to relativistic quantum field theory” by S.S. Schweber in ch 4b.

<sup>5</sup>There are other famous representations, such as Majorana basis that we will encounter later. There is also a basis known as Dirac basis which are the ones Dirac discovered when he wrote down his equation.

<sup>6</sup>We thank Ivan Mauricio Burbano Aldana points this out to us.

<sup>7</sup>Note with our choice of signature of the metric, superscripts/subscripts don't matter.

<sup>8</sup>This minus sign comes from the fact with the most minus sign signature, the Lorentz algebra indicates  $[J^{23}, J^{31}] = -iJ^{12}$ . In order for it to agree with angular momentum algebra. We should define our  $J^3 = -J_{12}$ .

We can then compute the rotational matrix

$$\begin{aligned} D(\Lambda_{\text{rot by } 2\pi}) &= e^{\frac{i\theta}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}} \\ &= e^{\text{diag}(i\pi, -i\pi, i\pi, -i\pi)} \\ &= -\mathbb{1}. \end{aligned} \quad (5.26)$$

We conclude whatever field that is transform under this spinor representation probably have spin  $\frac{1}{2}$  because we believe spin  $x$  object will come back to itself in  $\frac{2\pi}{x}$  radians of rotation. This argument is not really a solid one, because with this argument, a windmill should be spin 4. We will provide a solid argument why Dirac field is spin  $\frac{1}{2}$  next section.

Now we find all the rotation generators

$$\begin{aligned} \mathcal{J}^{ij} &= -\frac{i}{4} [\gamma_w^i, \gamma_w^j] = -\frac{i}{2} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \\ &= -\frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \end{aligned} \quad (5.27)$$

where  $\varepsilon^{ijk}$  is the Levi-Civita symbol,

$$\varepsilon^{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 123} \\ -1 & \text{if } ijk \text{ is an odd permutation of 123} \\ 0 & \text{otherwise} \end{cases}. \quad (5.28)$$

or in vector notation<sup>9</sup>

$$\mathcal{J} = \frac{1}{2} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \quad (5.29)$$

Then the rotational matrix becomes,

$$D(\Lambda_{\text{rot}}) = \begin{pmatrix} e^{\frac{i}{2}\theta \cdot \sigma} & 0 \\ 0 & e^{\frac{i}{2}\theta \cdot \sigma} \end{pmatrix}. \quad (5.30)$$

We can also compute the boost generators

$$\begin{aligned} \mathcal{J}_{0i} &= -\frac{i}{4} [\gamma_{w,0}, \gamma_{w,i}] \\ &= -\frac{i}{2} (-) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \\ &= -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}. \end{aligned} \quad (5.31)$$

---

<sup>9</sup>As we discussed before, in order to agree with regular angular momentum, we need to take the downstairs angular momentum, which differs by a minus sign.

Notice the boost generators has a factor of  $i$  in front compared to the rotation generators. The boost matrix is then

$$D(\Lambda_{\text{boost}}) = \begin{pmatrix} e^{\frac{1}{2}\mathbf{x}\cdot\boldsymbol{\sigma}} & 0 \\ 0 & e^{-\frac{1}{2}\mathbf{x}\cdot\boldsymbol{\sigma}} \end{pmatrix}. \quad (5.32)$$

Notice that boosts are not unitary.

### 5.2.1 \*Parity in spinor representation

A while ago we have studied how parity transforms a one-particle state,

$$P|p, s\rangle = \eta|\mathcal{P}p, s\rangle. \quad (5.33)$$

Now that we learned about the creation/annihilation operators. We express the above equation in terms of the operators and

$$\begin{aligned} P(b_p^s)^\dagger|0\rangle &= \eta(b_{-p}^s)^\dagger|0\rangle \\ P(c_p^s)^\dagger|0\rangle &= \eta^c(c_{-p}^s)^\dagger|0\rangle \end{aligned} \quad (5.34)$$

where  $b^\dagger$  is the creation operator for the particle and  $c^\dagger$  is the creation operator for the anti-particle.

By inserting an identity  $P^{-1}P = 1$  and notice that  $P|0\rangle = |0\rangle$ , We find for operators <sup>1011</sup>

$$\begin{aligned} Pb_p^s P^{-1} &= \eta^* b_{-p}^s \\ P(c_p^s)^\dagger P^{-1} &= \eta^c(c_{-p}^s)^\dagger. \end{aligned} \quad (5.35)$$

It is interesting to know how parity works in spinor representation. It will hopefully be clear later that we need parity to build the spinor field.

Now we have a specific representation, we can even try to find  $U(\mathcal{P}, 0)$ .

First we need to find a vector: if we have a vector we know exactly what parity does, it flips the spatial components and leave the time component unchanged.

Without a field, building a vector will be difficult, but nevertheless, we can try to define something that "in some sense" it is a vector. But it is not a vector.

Vectors are related to Lorentz transformation, so we are going to take a look at the Lorentz transformation of an object  $X$

$$D^{-1}(\Lambda)XD(\Lambda) = ? \quad (5.36)$$

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<sup>10</sup>Some of them and for the annihilation operator, you need to find the ones for creation operator and then take a dagger. We thank Jacqueline Caminiti for discussion.

<sup>11</sup>We have also computed for not acting on vacuum but acting on one-particle state. It does not work unless we assume the relationship is an operator level relationship. This is how we gain confidence the relationship must work on the operator level. We thank Jessica Yeh for the discussion.

and define the object is an vector if the right hand side is  $\Lambda X$ . Similarly we could define tensors of various kind.

First let us look at a trivial example.

$$D^{-1}(\Lambda) \mathbb{1} D(\Lambda) = \mathbb{1}. \quad (5.37)$$

The identity matrix transforms as a scalar. The next example we are going to look at is the  $\gamma$  matrices, they are nicely labeled as  $\gamma^\mu$  after all.

It is shown in a footnote<sup>12</sup> that

$$D(\Lambda^{-1}) \gamma^\mu D(\Lambda) = \Lambda^\mu_\rho \gamma^\rho \quad (5.39)$$

In this sense  $\gamma^\mu$  is a vector (in reality,  $\gamma^\mu$  is a bunch of numbers that do **not** change under Lorentz transformation.). More precisely the fermion bilinear that has  $\gamma^\mu$  in the middle is a vector. We will come back to this point after we have Dirac field.

It is in this sense, we look for the Parity operator  $\beta \equiv U(\mathcal{P}, 0)$ . Under parity, we should have

$$\begin{aligned} \beta \gamma^0 \beta^{-1} &= +\gamma^0 \\ \beta \gamma^i \beta^{-1} &= -\gamma^i \end{aligned} \quad (5.40)$$

Move  $\beta$  to the other side, we have

$$\begin{aligned} \beta \gamma^0 &= \gamma^0 \beta \\ \beta \gamma^i &= -\gamma^i \beta \end{aligned} \quad (5.41)$$

Clearly  $\beta \propto \gamma^0$  is a solution, conveniently we can take  $\beta \equiv \gamma^0$ , so that it looks nice in Chiral/Weyl representation,

$$\beta_w = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \quad (5.42)$$

It is the same matrix, but we will use the notation  $\beta$  to emphasize we are talking about parity here.

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<sup>12</sup>To first order

$$\begin{aligned} \left(1 - \frac{i}{2} \omega_{\rho\sigma} \mathcal{J}^{\rho\sigma}\right) \gamma^\mu \left(1 + \frac{i}{2} \omega_{\alpha\beta} \mathcal{J}^{\alpha\beta}\right) &= \gamma^\mu - \frac{i}{2} \omega_{\rho\sigma} \mathcal{J}^{\rho\sigma} \gamma^\mu + \gamma^\mu \frac{i}{2} \omega_{\rho\sigma} \mathcal{J}^{\rho\sigma} \\ &= \gamma^\mu - \frac{i}{2} \omega_{\rho\sigma} [\mathcal{J}^{\rho\sigma}, \gamma^\mu] \\ &= \gamma^\mu - \frac{i}{2} \omega_{\rho\sigma} (-i\gamma^\rho \eta^{\sigma\mu} + i\gamma^\sigma \eta^{\rho\mu}) \\ &= \gamma^\mu + \omega^\mu_\rho \gamma^\rho \\ &= \Lambda^\mu_\rho \gamma^\rho. \end{aligned} \quad (5.38)$$

### 5.3 \*The Dirac field

Recall we need to find coefficients via

$$\begin{aligned}\sum_{s'} u_{l'}^{s'}(0) \mathbf{J}_{s's}^j(R) &= \sum_l \mathcal{J}_{l'l}(R) u_l^s(0) \\ \sum_{s'} v_{l'}^{s'}(0) \mathbf{J}_{s's}^{j*}(R) &= - \sum_l \mathcal{J}_{l'l}(R) v_l^s(0)\end{aligned}\quad (5.43)$$

for different representations  $l$  of Lorentz group.

In general this field takes the form of

$$\begin{aligned}\psi_l^-(x) &= \oint_s dV_{\mathbf{p}} u_l^s(\mathbf{p}) b_{\mathbf{p}}^s e^{-ip \cdot x} \\ \psi_l^+(x) &= \oint_s dV_{\mathbf{p}} v_l^s(\mathbf{p}) (c_{\mathbf{p}}^s)^\dagger e^{+ip \cdot x}.\end{aligned}\quad (5.44)$$

Here we assume there is a charge symmetry and  $b^\dagger$  creates particle and  $c^\dagger$  creates anti-particle.

Let us again focus on  $u$

$$\sum_{s'} u_{l'}^{s'}(0) \mathbf{J}_{s's}^j(R) = \sum_l \mathcal{J}_{l'l}(R) u_l^s(0). \quad (5.45)$$

For spinor representation, the angular momentum matrix looks like

$$\mathcal{J} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \quad (5.46)$$

Thus we are going to choose our index  $l$  to become index  $m\pm$ , where  $\pm$  denotes upper or lower block and  $m$  denotes within the  $2 \times 2$  block. Then the equation becomes

$$\sum_{s'} u_{m',\pm}^{s'}(0) \mathbf{J}_{s's}^j(R) = \sum_m \frac{1}{2} \sigma_{m'm} u_{m,\pm}^s(0) \quad (5.47)$$

This is because  $\mathcal{J}_{m+,\bar{m}-} = 0$  because it is block diagonal.

The equation above can be written a matrix if we consider  $u_{\bar{m}\pm}^s(0)$  the  $m, s$  component of the matrix  $U_\pm$ . Then the above equation in matrix form is

$$U_\pm \mathbf{J}^j = \frac{1}{2} \boldsymbol{\sigma} U_\pm. \quad (5.48)$$

Now  $\mathbf{J}^j$  and  $\boldsymbol{\sigma}$  are both irreducible representations of the rotation group. Shur's lemma tells us matrix like  $U_\pm$  that link representations together must either vanish or be square and non-singular. Vanish does not help us. Thus  $U_\pm$  are square matrices, in other words the range of  $m$  is the same as range of  $s$  but  $m$  is the label of Pauli matrices so the range is 2, thus  $s$ 's range is also 2.  $2 = 2j + 1$ , this tells us the Dirac field can only describe spin 1/2 fields! This is the rigorous way to show that the spinor representation corresponds to a particle with spin  $j = \frac{1}{2}$ .

We have  $\mathbf{J}^{(\frac{1}{2})} = \frac{1}{2}\boldsymbol{\sigma}$ , hence  $U_{\pm}$  must commute with  $\boldsymbol{\sigma}$  and hence must be unit matrix.

$$u_{m,\pm}^s(0) = c_{\pm}\delta_{ms}. \quad (5.49)$$

With similar argument we find<sup>13</sup>

$$v_{m,\pm}^s(0) = -id_{\pm}(\sigma_2)_{ms}. \quad (5.50)$$

From here we need parity to further constraint  $c$ 's and  $d$ 's. Recall for the operators,

$$\begin{aligned} P(c_{\mathbf{p}}^s)^{\dagger}P^{-1} &= \eta^c(c_{-\mathbf{p}}^s)^{\dagger} \\ Pb_{\mathbf{p}}^sP^{-1} &= \eta^*b_{-\mathbf{p}}^s \end{aligned} \quad (5.51)$$

Now look at Dirac field,

$$\begin{aligned} \psi_l^-(x) &= \oint dV_p u_l^s(\mathbf{p}) e^{-ip \cdot x} b_{\mathbf{p}}^s \\ \psi_l^+(x) &= \oint dV_p v_l^s(\mathbf{p}) e^{+ip \cdot x} (c_{\mathbf{p}}^s)^{\dagger} \end{aligned} \quad (5.52)$$

In terms of field

$$\begin{aligned} P\psi_l^-(x)P^{-1} &= \eta^* \oint dV_p u_l^s(\mathbf{p}) e^{-ip \cdot x} b_{-\mathbf{p}}^s \\ &= \eta^* \oint dV_p u_l^s(-\mathbf{p}) e^{-i\mathcal{P} \cdot p \cdot x} b_{\mathbf{p}}^s \\ &= \eta^* \oint dV_p u_l^s(-\mathbf{p}) e^{-ip \cdot \mathcal{P} \cdot x} b_{\mathbf{p}}^s \end{aligned} \quad (5.53)$$

where we change integral over  $\mathbf{p}$  to integrate over  $-\mathbf{p}$ . We want

$$P\psi_l^-(x)P^{-1} = \text{constant matrix} \times \psi_l^-(\mathcal{P}x). \quad (5.54)$$

This impose some constraints on the coefficients as well.

Boosting in the reverse direction is given in terms of parity operator

$$U(L(\mathcal{P}p)) = PU(L(p))P. \quad (5.55)$$

For spinor representation it is in terms of  $\beta$

$$\begin{aligned} u^s(\mathbf{p}) &= D(L(p))u^s(0) \\ u^s(-\mathbf{p}) &= \beta D(L(p))\beta u^s(0) \end{aligned} \quad (5.56)$$

where we used  $\beta^{-1} = \beta$ .

Then it is necessarily to have

$$\beta u^s(0) = b_u u^s(0) \quad (5.57)$$

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<sup>13</sup> $\sigma_2$  shows up because  $-\boldsymbol{\sigma}^* = \sigma_2 \boldsymbol{\sigma} \sigma_2$ . The  $-i$  factor is chosen so  $-i\sigma_2$  is real.

where  $b_u^2 = 1$  are sign factors. Then, indeed we have<sup>14</sup>

$$\begin{aligned} P\psi_l^-(x)P^{-1} &= \eta^* b_u \beta \psi_l^-(\mathcal{P}x) \\ P\psi_l^+(x)P^{-1} &= \eta^c b_v \beta \psi_l^+(\mathcal{P}x) \end{aligned} \quad (5.59)$$

Then recall our solution

$$\begin{aligned} u_{m,\pm}^s(0) &= c_\pm \delta_{ms} \\ v_{m,\pm}^s(0) &= -id_\pm(\sigma_2)_{ms}. \end{aligned} \quad (5.60)$$

We will choose  $c_+ = 1, c_- = b_u c_+$  and  $d_+ = 1, d_- = b_v d_+$ . To write it out, we have

$$u^{\frac{1}{2}}(0) = \begin{pmatrix} 1 \\ 0 \\ b_u \\ 0 \end{pmatrix}, u^{-\frac{1}{2}}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ b_u \end{pmatrix} \quad (5.61)$$

and

$$v^{\frac{1}{2}}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ b_v \end{pmatrix}, v^{-\frac{1}{2}}(0) = \begin{pmatrix} 1 \\ 0 \\ b_v \\ 0 \end{pmatrix}. \quad (5.62)$$

Again in order to satisfy causality condition, we make linear combination of the fields in the second homework. we obtain our final Dirac field<sup>15</sup>

$$\psi_a(x) = \oint_s dV_{\mathbf{p}} (u(\mathbf{p})_a^s b_{\mathbf{p}}^s e^{-ipx} + v(\mathbf{p})_a^s c_{\mathbf{p}}^{s,\dagger} e^{+ipx}) \quad (5.63)$$

where we choose  $b_u = 1$  and  $b_v = -1$  and we list our final  $u(0), v(0)$  here.

$$u^{\frac{1}{2}}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, u^{-\frac{1}{2}}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad (5.64)$$

and

$$v^{\frac{1}{2}}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, v^{-\frac{1}{2}}(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \quad (5.65)$$

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<sup>14</sup>One more later comment about parity after we figure out  $b_u = -b_v$ : in order for the field  $\psi(x)$  transforms like constant matrix  $\times \psi(\mathcal{P}x)$  under parity. The creation and annihilation fields must transform the same, this gives  $\eta^c = -\eta^*$ . And we have

$$P\psi(x)P^{-1} = \eta^* \beta \psi(\mathcal{P}x). \quad (5.58)$$

<sup>15</sup>Now that we realize this is the Dirac field everybody talks about, we will use spinor index  $a, b, c, d$  instead of the generic  $l, l'$  to be consistent with the other textbooks.

And it satisfies the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \quad (5.66)$$

This is the celebrated Dirac equation for a spin  $\frac{1}{2}$  charged fermionic particle.  
How did we know it is spin  $\frac{1}{2}$ ? fermionic? charged?

Note in our point of view, the Dirac equation is not fundamental.

# Chapter 6

## Concluding remarks

In the first part of the course, we quantized Dirac field, but not in the usual sense. Our line of logic is:

1. First, we build our one-particle states according to Lorentz symmetry.
2. We then introduce creation operator (and its adjoint annihilation operator.).
3. We realize the (anti-)commutator of the creation and annihilation operator is simple and enlightening.
4. We then assemble the operators in a way that is compatible with special relativity. In other words, we assemble the field such that it transforms in some representation of the Lorentz group. Remember in this process we figure out that it has to be anti-commutator!

In some sense, we already have the quanta of the theory, we simply realize the field is a very convenient mathematical tool to organize things.



## Part II

# The Dirac fields and its interactions



## Chapter 7

# Quantize the Dirac field, again

So how do people usually quantize the Dirac field?[ask question](#)

1. First start with the Dirac equation and study its solutions.
2. Then write down the Dirac Lagrangian according to Lorentz invariance and the Dirac equation.
3. Compute the Hamiltonian.
4. Promote the coefficients to creation/annihilation operator and impose (anti)-commutator relationship.
5. Impose normal ordering.
6. Study the states.

We will now follow this order to quantize the Dirac field, again. And we will start by looking at how historically Dirac equation<sup>1</sup> is found.

### 7.1 Step 0: study the classical Dirac field

#### 7.1.1 Historical Dirac equation derivation

In the last couple of weeks when you were learning about the Klein-Gordon theory, you took the approach of writing down the Lagrangian and then happily applying Lagrangian mechanics to write down its e.o.m and solve the e.o.m to get the classical solution to quantize. This is very nice, the Klein Gordon Lagrangian looks like harmonic oscillator (actually infinite of them) and looks

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<sup>1</sup>As Dirac admitted: The equation is smarter than its creator.

believable. The Dirac's Lagrangian, is however a different story. Writing down Dirac's Lagrangian needs some insight from the Dirac's equation.

In the last couple of weeks when you were learning about the Klein-Gordon theory, you took the approach of writing down the Lagrangian and then happily applying Lagrangian mechanics to write down its e.o.m and solve the e.o.m to get the classical solution to quantize. This is very nice, the Klein Gordon Lagrangian looks like harmonic oscillator (actually infinite of them) and looks believable. But if I just write down the Dirac Lagrangian and carry on from there, there is way too much to be explained: what are all these new symbols? where is this come from?

Thus we will take a historical approach, to learn a bit of history of the Dirac equation, or the birth of quantum mechanics: we will take a look at Schrödinger equation first. Starting from the non-relativistic energy momentum relationship

$$E = \frac{\mathbf{p}^2}{2m}, \quad (7.1)$$

and making the substitution<sup>2</sup>

$$E \rightarrow i\frac{\partial}{\partial t}, \quad \mathbf{p} \rightarrow -i\nabla. \quad (7.2)$$

We arrive at the Schrödinger's equation for a free particle

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2m}\nabla^2\Psi, \quad (7.3)$$

and the probability density is given by

$$\rho_S = \Psi^*\Psi, \quad (7.4)$$

which is positive definite.

Now if we want to include special relativity to quantum mechanics<sup>3</sup>, following the same procedure, we shall start with the relativistic energy momentum relationship

$$p^2 \equiv p^\mu p_\mu = E^2 - |\mathbf{p}|^2 = m^2, \quad (7.5)$$

where  $p^\mu = (E, \mathbf{p})$ <sup>4</sup>, and make the substitution

$$p_\mu \rightarrow i\frac{\partial}{\partial x^\mu} = (i\partial_t, i\nabla), \quad (7.6)$$

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<sup>2</sup>This substitution is valid because the differential operator is acting on a wavefunction that can be thought of as a sum of planewaves. We thank Caroline de Lima Vargas Simoes and Danaan Cordoni-Jordan for the discussion.

<sup>3</sup>Of course, we also want to include general relativity, but that will, hopefully be done by the interested readers.

<sup>4</sup>With our choice of metric,  $p_\mu = (E, -\mathbf{p})$ . And the substitution is consistent with the one done in the non-relativistic case.

where we use notation  $\partial_t \equiv \frac{\partial}{\partial t}$ , we arrive at the Klein-Gordon equation,

$$(\partial_\mu \partial^\mu + m^2)\Phi = 0. \quad (7.7)$$

The density<sup>5</sup> defined following the continuity equation<sup>6</sup> is given by

$$\rho_{\text{KG}} = i((\partial_t \Phi^*)\Phi - \Phi^* \partial_t \Phi). \quad (7.8)$$

Because the appearance of second time derivative in the Klein-Gordon equation, the “probability density”<sup>7</sup> includes first time derivative, and is no longer positive definite. For example, for a plane wave  $\Phi = e^{-ik \cdot x}$ , where  $k \cdot x \equiv k^\mu x_\mu$ ,

$$\rho_{\text{KG}} = i((ik^0)e^{ik \cdot x}e^{-ik \cdot x} - e^{ik \cdot x}(-ik^0)e^{-ik \cdot x}) = -2k^0, \quad (7.9)$$

since there can be both positive and negative frequencies for the wave  $k^0 = \pm\sqrt{|\mathbf{k}|^2 + m^2}$ ,  $\rho_{\text{KG}}$  is no longer positive definite.

Dirac considered this a problem and he started to look for another equation, according to some apparently true legend, while staring into a fire. What does Dirac want?

1. The equation needs to be first order in time derivative, so that a positive definite probability density can be defined.
2. Since Dirac was looking for something that unites special relativity and quantum mechanics, it needs to treat time and space in the same footing, the equation also needs to be first order in space derivative.
3. The solution of the equation should still satisfy Klein-Gordon equation, so that the relativistic energy momentum condition is satisfied.
4. Ultimately, Dirac equation should be Lorentz covariant.

So we are basically looking for the “square root” of Klein-Gordon equation. Based on the first two requirements, it is easy to write down something like

$$(\gamma^t(i\partial_t) + \gamma \cdot (i\nabla) - m)\psi = 0, \quad (7.10)$$

where  $\gamma$ ’s are linear coefficients<sup>8</sup> to be determined. Or in fancy notation,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \quad (7.11)$$

---

<sup>5</sup> $|\Phi|^2$  is invariant and cannot serve the purpose of a density, as relativistic density transforms as the time component of a 4-current.

<sup>6</sup>As you have seen in the first part of the course, you arrive at this by considering the global symmetry of  $\Phi \rightarrow \Phi e^{i\alpha}, \Phi^* \rightarrow \Phi^* e^{-i\alpha}$  and the current follows from the standard procedure using Noether theorem.

<sup>7</sup>As you have learned last two weeks, this interpretation is quickly abandoned. Rather it is the particle number density from Noether’s theorem, see Dan’s notes on Noether’s theorem.

<sup>8</sup>Dirac made the simplest choice: he simply demand  $\gamma$ ’s are numbers that do not transform under Lorentz transformation, just like the number  $2\pi$  does not transform under Lorentz transformation. There is a catch, as we will see,  $\gamma$ ’s are actually matrices that acts on spinor space. So naturally there is a choice of the basis in the spinor space, hence there is a choice of the basis for  $\gamma$  matrices. This is similar to the basis in group representation. We thank class PSI2011-2012 for discussion.

Now we need to check the third condition is satisfied<sup>9</sup>. That is we need to find an operator  $\mathcal{O}$  such that

$$\mathcal{O}(i\gamma^\nu \partial_\nu - m) = \partial_\mu \partial^\mu + m^2. \quad (7.12)$$

Inspired by the following identity in complex number

$$(-ia - b)(ia - b) = a^2 + b^2, \quad (7.13)$$

we propose the operator is  $-i\gamma^\mu \partial_\mu - m$  and demand

$$(-i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m) = \partial_\mu \partial^\mu + m^2, \quad (7.14)$$

thus we need

$$\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2 = \partial_\mu \partial^\mu + m^2. \quad (7.15)$$

we can symmetrize the left hand side as  $\partial_\mu \partial_\nu$  is symmetric,

$$\frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu = \partial^\mu \partial_\mu. \quad (7.16)$$

Now as long as

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}. \quad (7.17)$$

We indeed satisfy the third requirement, that the solution is also a solution of Klein-Gordon equation. As the number of equations is equal to  $\frac{4(4+1)}{2} = 10$ , which is bigger than number of variables 4, these linear coefficients  $\gamma$ 's are certainly not ordinary numbers. These are matrices. We will use the following notation to denote the anti-commutator as it will show up frequently throughout the lectures.

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu. \quad (7.18)$$

Thus we arrive at the Dirac equation<sup>10</sup>

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}, \quad (7.19)$$

provided the  $\gamma$  matrices satisfy the Dirac algebra<sup>11</sup>,

$$\boxed{\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \mathbb{1}.} \quad (7.20)$$

<sup>9</sup>Although you have only learned that Klein-Gordon equation and Klein-Gordon fields (scalars), because  $p^2 = m^2$ , Klein-Gordon equation is way more general than you think: any special relativity compatible field has to satisfy KG equation. We thank the discussion with Hanne Silverans.

<sup>10</sup>The Dirac equation indeed admits a positive definite density  $\rho_D = \psi^\dagger \psi$  as desired. We will have to wait until we can write down the Lagrangian to be able to show, this density is the time component of the Noether current following from the global symmetry similar to the scalar case  $\psi \rightarrow \psi e^{i\alpha}$ .

<sup>11</sup>To see a connection with the Clifford algebra in mathematics, see Clifford Algebras and Spinors by Ivan Todorov at arXiv: 1106.3197.

Note that after we realize that  $\gamma$ 's have to be matrices, we also make it clear that the right hand side should be a matrix. Also note that we have used the convention that we can use metric to bring the indices of  $\gamma$  matrices upstairs or downstairs,

$$\gamma_\mu \equiv \eta_{\mu\nu} \gamma^\nu. \quad (7.21)$$

But this does not mean they transform as a vector! They are constant matrices that remain constant under Lorentz transformations.

The combination  $\gamma^\mu \partial_\mu$  shows up way too often, hence we introduce Feynman's famous slash notation

$$\not{A} \equiv \gamma^\mu \partial_\mu. \quad (7.22)$$

The Dirac equation is then often written as

$$(i\not{A} - m)\psi = 0. \quad (7.23)$$

Of course this notation applies to any 4-vector that is contracted with  $\gamma$  matrices  $\not{A} \equiv A_\mu \gamma^\mu$ .

Before we moved on, I must confess that I have lied. This was actually not how historically the Dirac equation was written down. Dirac wanted to make the analogy with Schrödinger equation and write his equation in the form of

$$i \frac{\partial}{\partial t} \psi = H\psi, \quad (7.24)$$

where  $H$  is the Hamiltonian of the system. So he actually wrote his equation in a slightly different form. Define  $\partial_i \equiv \frac{\partial}{\partial x^i}$ <sup>12</sup>,

$$\begin{aligned} i \frac{\partial}{\partial t} \gamma^0 \psi &= (-i\partial_i \gamma^i + m)\psi \\ i \frac{\partial}{\partial t} \psi &= (-i\partial_i \gamma^0 \gamma^i + m\gamma^0)\psi, \end{aligned} \quad (7.25)$$

and identified the Hamiltonian operator as

$$H = -i\gamma^0 \gamma^i \partial_i + m\gamma^0. \quad (7.26)$$

The Hamiltonian operator describes an observable, and thus must be Hermitian. Demanding this, we immediately have

$$\begin{aligned} (\gamma^0)^\dagger &= \gamma^0 \\ (-i\gamma^0 \gamma^i \partial_i)^\dagger &= i(\gamma^i)^\dagger (\gamma^0)^\dagger (-\partial_i) = -i\gamma^0 \gamma^i \partial_i \\ (\gamma^i)^\dagger \gamma^0 &= \gamma^0 \gamma^i \\ (\gamma^i)^\dagger &= \gamma^0 \gamma^i \gamma^0, \end{aligned} \quad (7.27)$$

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<sup>12</sup>The author has largely revised the notations in the notes. We thank Cynthia Arias Pruna for discussion.

where in the above derivation, the second line we used  $\partial_i$  is anti-Hermitian because momentum  $i\partial_i$  is an observable, thus Hermitian, the third line we used the result from first line  $\gamma^0$  is Hermitian, and the last line we used the fact that  $(\gamma^0)^2 = 1$  from the Dirac algebra.

The above condition on the  $\gamma$  matrices can be summarized covariantly,

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0. \quad (7.28)$$

This is an important property on  $\gamma$  matrices that we will make use of frequently.

In Dirac's point of view, he wanted to identify

$$i \frac{\partial}{\partial t} \psi = E \psi. \quad (7.29)$$

Assuming the energy eigenstates are still some sort of plane wave<sup>13</sup>,  $\psi \propto e^{\pm ik \cdot x}$ <sup>14</sup>, we are in trouble: from the relativistic energy momentum relationship, we still have  $k^0 = \pm \sqrt{|\mathbf{k}|^2 + m^2}$ , and thus we have negative energy eigenstates with arbitrary negativeness in our spectrum.

This is very bad news: the spectrum is unbounded below, we can just keep producing these states to gain energy. Dirac had a genius/ingenious solution. He claimed he wrote down the equation for electrons, and they obey Pauli exclusion principle. And hence he proposed in the true vacuum of universe, all the negative energy states are already filled up to form the Dirac sea.

Only positive energy states are accessible. Then Dirac made one of the most triumph prediction of theoretical physics in the history: if an electron is knocked out of his sea to become one that with positive energy, it will leave a hole behind. The hole will behave just like the electron except it has negative charge. Hence Dirac predicted the anti-particle positron must exist. Since the electron can fall back to fill in the hole, the electron and positron must be able to annihilate. It only took two years for positron to be seen in the cloud chambers in 1932.

From now on, we shall consider  $\psi$  as a field, and the story of anti-particles from the modern point of view shall return once we quantize the Dirac field.

### 7.1.2 Relativistic covariance of the Dirac equation

This discussion is very parallel to the spinor representation discussion in the first part! In this section, we ask the question what the transformation needs to be, so that the Dirac field is compatible with Lorentz symmetry (special relativity).

But there is still the fourth requirement: How does  $\psi$  transform under Lorentz transformation so that the Dirac equation is Lorentz covariant? We will be able to conclude that  $\psi$  is a spinor and under Lorentz transformation, it transforms like a spinor<sup>15</sup>.

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<sup>13</sup>Which is not a wild guess, considering we take care to make sure they are also solutions of Klein-Gordon equation.

<sup>14</sup>We will study very carefully what is hidden by the proportional sign soon in the course.

<sup>15</sup>Once upon a time, a dictionary tells me ooze is the thing that oozes or is oozing. A Zee says in his book, tensors are objects that transform as tensors.

Let us consider two frame related by Lorentz transformation,

$$x'^\mu = \Lambda^\mu_\nu x^\nu. \quad (7.30)$$

Then  $\psi(x)$  and  $\psi'(x')$  should both satisfy Dirac equation<sup>16</sup>, What do we mean Dirac equation is covariant?

$$\begin{aligned} (i\cancel{D} - m)\psi(x) &= 0 \\ (i\cancel{D}' - m)\psi'(x') &= 0. \end{aligned} \quad (7.31)$$

Since both Dirac equation and Lorentz transformation is linear, we required that  $\psi(x)$  and  $\psi'(x')$  are linearly related, by an invertible matrix  $D(\Lambda)$  that depends on the Lorentz transformation  $\Lambda$

$$\begin{aligned} \psi'(x') &= D(\Lambda)\psi(x) \\ \psi(x) &= D^{-1}(\Lambda)\psi'(x'). \end{aligned} \quad (7.32)$$

Thus we can rewrite the Dirac equation in frame  $x$  as

$$\begin{aligned} \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right) D^{-1}(\Lambda)\psi'(x') &= 0 \\ D(\Lambda) \left(i\gamma^\mu \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} - m\right) D^{-1}(\Lambda)\psi'(x') &= 0 \\ D(\Lambda) \left(i\gamma^\mu \Lambda^\nu_\mu \frac{\partial}{\partial x'^\nu} - m\right) D^{-1}(\Lambda)\psi'(x') &= 0. \end{aligned} \quad (7.33)$$

Now we compare with the Dirac equation in frame  $x'$ ,

$$\left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m\right) \psi'(x') = 0 \quad (7.34)$$

and conclude that  $D(\Lambda)$  must satisfy

$$D(\Lambda)\gamma^\nu \Lambda^\mu_\nu D^{-1}(\Lambda) = \gamma^\mu, \quad (7.35)$$

or

$$D(\Lambda)\gamma^\nu \Lambda_{\mu\nu} D^{-1}(\Lambda) = \gamma_\mu. \quad (7.36)$$

Obviously  $D(\Lambda) = \mathbb{1}$  is not a solution. So  $D(\Lambda)$  probably has something to do with  $\gamma$  matrices. In order to solve the above equation, we are going to take our favorite approach: we will consider the infinitesimal case, and then exponentiate the result.

<sup>16</sup>Because  $\partial_\mu$  transforms under Lorentz transformation and  $\gamma^\mu$  are just a bunch constant matrices,  $\psi$  has to transform: the hope that  $\psi$  can also just be a scalar simply will not work. In fact, from the fact  $\gamma$  has to be at least 4-dimensional, we realize  $\psi$  is a column vector with 4 components. This screams for linear algebra and hence a spinor rep to transform between them.

First we linearly expand the Lorentz transformation

$$x'_\mu = \Lambda_{\mu\nu} x^\nu = x_\mu + \epsilon_{\mu\nu} x^\nu, \quad (7.37)$$

and demand the length is invariant  $x'^2 = x^2$  to first order, this gives us

$$x^\mu \epsilon_{\mu\nu} x^\nu = 0, \quad (7.38)$$

which is to say  $\epsilon_{\mu\nu}$  is anti-symmetric

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}. \quad (7.39)$$

This is not completely unexpected. In electrodynamics, we combine three electric fields and three magnetic fields into field strength tensor which is antisymmetric and has six components. Now we have three rotations and three boosts and we should be able to also combine them into some antisymmetric 4d thing, which comes with an antisymmetric  $\epsilon$  as the associated infinitesimal parameter.

Now we expand  $D(\Lambda)$  to first order<sup>17</sup> in terms of the anti-symmetric infinitesimal parameters  $\epsilon_{\mu\nu}$

$$D(\Lambda) = 1 + \frac{i}{2} \epsilon_{\mu\nu} \mathcal{J}^{\mu\nu}, \quad (7.40)$$

where  $\mathcal{J}^{\mu\nu}$  are the generators, which are of course also anti-symmetric. Now the condition on  $D(\Lambda)$  in (7.36) is given by

$$\begin{aligned} \left(1 + \frac{i}{2} \epsilon_{\alpha\beta} \mathcal{J}^{\alpha\beta}\right) \gamma^\nu (\eta_{\mu\nu} + \epsilon_{\mu\nu}) \left(1 - \frac{i}{2} \epsilon_{\rho\sigma} \mathcal{J}^{\rho\sigma}\right) &= \gamma_\mu \\ \frac{i}{2} \epsilon_{\alpha\beta} \mathcal{J}^{\alpha\beta} \gamma_\mu + \gamma^\nu \epsilon_{\mu\nu} - \frac{i}{2} \gamma_\mu \epsilon_{\alpha\beta} \mathcal{J}^{\alpha\beta} &= 0 \\ \frac{i}{2} \mathcal{J}^{\alpha\beta} \gamma_\mu - \frac{i}{2} \gamma_\mu \mathcal{J}^{\alpha\beta} + \gamma^\beta \delta_\mu^\alpha &= 0. \end{aligned} \quad (7.41)$$

From here, we have

$$\frac{i}{2} [\mathcal{J}_{\alpha\beta}, \gamma_\mu] = -\eta_{\alpha\mu} \gamma_\beta = -\frac{1}{2} (\eta_{\alpha\mu} \gamma_\beta - \eta_{\beta\mu} \gamma_\alpha). \quad (7.42)$$

Thus we conclude

$$[\mathcal{J}_{\alpha\beta}, \gamma_\mu] = -i(\gamma_\alpha \eta_{\beta\mu} - \gamma_\beta \eta_{\alpha\mu}). \quad (7.43)$$

Since  $\mathcal{J}_{\alpha\beta}$  is antisymmetric, and has something to do with  $\gamma$  matrices, and has two indices, a reasonable **guess** will be

$$\mathcal{J}_{\alpha\beta} \propto [\gamma_\alpha, \gamma_\beta]. \quad (7.44)$$

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<sup>17</sup>Unfortunately different people use different convention, but the combination of  $\frac{i}{2}$  and the  $-\frac{i}{4}$  in the generators later should be the same.

With a few lines of calculation. It turns out the proportional constant is  $-\frac{i}{4}$ .

$$\mathcal{J}_{\alpha\beta} = -\frac{i}{4} [\gamma_\alpha, \gamma_\beta]. \quad (7.45)$$

Now to prove it, we can go through a slightly tedious calculation,

$$\begin{aligned} [\mathcal{J}_{\alpha\beta}, \gamma_\mu] &= -\frac{i}{4} [\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha, \gamma_\mu] \\ &= -\frac{i}{4} (\gamma_\alpha \gamma_\beta \gamma_\mu - \gamma_\beta \gamma_\alpha \gamma_\mu - \gamma_\mu \gamma_\alpha \gamma_\beta + \gamma_\mu \gamma_\beta \gamma_\alpha) \\ &= -\frac{i}{4} (\gamma_\alpha (2\eta_{\beta\mu} - \gamma_\mu \gamma_\beta) - \gamma_\beta (2\eta_{\alpha\mu} - \gamma_\mu \gamma_\alpha) - (2\eta_{\mu\alpha} - \gamma_\alpha \gamma_\mu) \gamma_\beta + (2\eta_{\mu\beta} - \gamma_\beta \gamma_\mu) \gamma_\alpha) \\ &= -i(\gamma_\alpha \eta_{\beta\mu} - \gamma_\beta \eta_{\alpha\mu}). \end{aligned} \quad (7.46)$$

A finite transformation is then given by

$$D(\Lambda) = e^{\frac{i}{2}\epsilon^{\mu\nu}\mathcal{J}_{\mu\nu}}, \quad (7.47)$$

where

$$\mathcal{J}_{\mu\nu} = -\frac{i}{4} [\gamma_\mu, \gamma_\nu]. \quad (7.48)$$

So this is how Dirac's  $\psi$  transforms, let us label the indices of  $\gamma$  matrices by  $l$  as we did before. restoring the indices we have

$$\psi'^a(x') = D(\Lambda)_b^a \psi^b(x), \quad (7.49)$$

where

$$D(\Lambda)_b^a = e^{\frac{i}{2}(\mathcal{J}_{\mu\nu})_b^a \epsilon^{\mu\nu}}, \quad (7.50)$$

where  $\mu\nu$  is the generator index, there are  $\frac{4 \times 3}{2} = 6$  generators, and each generator is a matrix, where  $a$  and  $b$  are the indices for the matrices, the spinor indices. Recall we decide to use the common  $a$  and  $b$  indices for spinors specially.

### 7.1.3 \*Classical solution of the Dirac equation

This is parallel to the discussion of finding the coefficients in part 1! This time around we start with the equation and simply look for the classical solution..

Now we have made sure the Dirac equation is compatible with Lorentz transformation, and explored a bit about the Lorentz group, we shall solve the Dirac equation.

$$(i\cancel{D} - m)\psi = 0. \quad (7.51)$$

We are going to use the chiral/Weyl representation of the  $\gamma$  matrices,

$$\gamma_w^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma_w^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (7.52)$$

and will introduce the new notation of  $\sigma^\mu$  and  $\bar{\sigma}^\mu$  to unify them,

$$\sigma^\mu = (\mathbb{1}, \sigma^i), \quad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i). \quad (7.53)$$

Then we can write  $\gamma$  matrices as,

$$\gamma_w^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \quad (7.54)$$

We want to find the plane wave solution of the Dirac equation and propose the following ansatz with positive frequency<sup>18</sup>  $p^0 > 0$ ,

$$\psi = u(\mathbf{p}) e^{-ip \cdot x}, \quad (7.55)$$

where  $u(\mathbf{p})$  is a four component spinor, independent of the spacetime  $x$ , but can depend on 3-momentum  $\mathbf{p}$ . Plug our ansatz into the Dirac equation, we have

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m)u(\mathbf{p})e^{-ip \cdot x} &= 0 \\ (i\gamma^\mu (-ip_\mu) - m)u(\mathbf{p})e^{-ip \cdot x} &= 0 \\ (\gamma^\mu p_\mu - m)u(\mathbf{p})e^{-ip \cdot x} &= 0 \\ \begin{pmatrix} -m & \sigma \cdot p \\ \bar{\sigma} \cdot p & -m \end{pmatrix}u(\mathbf{p}) &= 0 \\ \begin{pmatrix} -m & \sigma \cdot p \\ \bar{\sigma} \cdot p & -m \end{pmatrix} \begin{pmatrix} u_-(\mathbf{p}) \\ u_+(\mathbf{p}) \end{pmatrix} &= 0, \end{aligned} \quad (7.56)$$

where in the last line we split the four component spinor  $u(\mathbf{p})$  to two two-component spinors  $u(\mathbf{p}) = \begin{pmatrix} u_-(\mathbf{p}) \\ u_+(\mathbf{p}) \end{pmatrix}$ <sup>19</sup>. Then the coupled equations for  $u_-(\mathbf{p})$  and  $u_+(\mathbf{p})$  are given by

$$\begin{aligned} (\sigma \cdot p)u_+(\mathbf{p}) &= mu_-(\mathbf{p}) \\ (\bar{\sigma} \cdot p)u_-(\mathbf{p}) &= mu_+(\mathbf{p}). \end{aligned} \quad (7.57)$$

Notice the following identity<sup>20</sup>,

$$\begin{aligned} (\sigma \cdot p)(\bar{\sigma} \cdot p) &= (p_0 - p^i \sigma^i)(p_0 + p^j \sigma^j) \\ &= p_0^2 - p^i p^j \sigma^i \sigma^j \\ &= p_0^2 - p^i p^j \delta^{ij} = p^\mu p_\mu = m^2, \end{aligned} \quad (7.58)$$

<sup>18</sup>This is completely in analogous to the complex scalar cases that you have studied. We will study the negative frequency solutions later.

<sup>19</sup>In Chinese,  $-$  is 1 and  $=$  is 2. We use Chinese subscripts here so we can reserve the Arabian numbers for more commonly used occasion (also easier to compare to literature.). We thank Anna Knorr for discussion.

<sup>20</sup>For the first line, the minus sign in the first factor comes from the metric. For the spatial components, we only use upper index to avoid the confusion which metric we should use to pull the index down. This will be the convention we always use. And we extend the Einstein convention to omit a pair of indice (if they are both up, assuming Euclidean metric)

where we use the fact that  $p^i p^j$  is symmetric, only symmetric part of  $\sigma^i \sigma^j$  will matter. This identity means that the above equations imply each other. Now we can look at the first equation, and propose,

$$u_-(\mathbf{p}) = (\sigma \cdot \mathbf{p}) \xi', \quad (7.59)$$

where  $\xi'$  is any two-component spinor. Then the second equation and the identity give

$$u_+(\mathbf{p}) = m \xi', \quad (7.60)$$

which also solves the first equation. Thus we conclude, as the equations are linear, the solution is

$$u(\mathbf{p}) = c_0 \begin{pmatrix} (\sigma \cdot \mathbf{p}) \xi' \\ m \xi' \end{pmatrix}, \quad (7.61)$$

where  $c_0$  is any normalization constant and  $\xi'$  is any spinor. We will choose  $c_0$  and  $\xi'$  so that our solution looks more symmetric by rewriting it as<sup>21</sup>

$$u(\mathbf{p}) = c_0 \begin{pmatrix} \sqrt{\sigma \cdot \mathbf{p}} \sqrt{\sigma \cdot \mathbf{p}} \xi' \\ m \xi' \end{pmatrix}. \quad (7.62)$$

Then we pick

$$\begin{aligned} \sqrt{\sigma \cdot \mathbf{p}} \xi' &= \xi \\ m \xi' &= \sqrt{\sigma \cdot \mathbf{p}} \xi, \end{aligned} \quad (7.63)$$

and our solution turns into

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{\sigma \cdot \mathbf{p}} \xi \\ \sqrt{\sigma \cdot \mathbf{p}} \xi \end{pmatrix}, \quad (7.64)$$

where we take normalization constant to be  $c_0 = 1$  and  $\xi$  is any two-component spinor that normalized to  $\xi^\dagger \xi = 1$ , where  $\xi^\dagger = (\xi^*)^T$  the adjoint is defined in the usual way, the transpose of the complex conjugate. Now let us follow our footstep and solve Dirac equation again for the negative frequency with the following ansatz

$$\psi = v(\mathbf{p}) e^{i \mathbf{p} \cdot \mathbf{x}}. \quad (7.65)$$

The Dirac equation gives,

$$(\gamma^\mu p_\mu + m)v(\mathbf{p}) = \begin{pmatrix} m & \sigma \cdot \mathbf{p} \\ \bar{\sigma} \cdot \mathbf{p} & m \end{pmatrix} v(\mathbf{p}) = 0. \quad (7.66)$$

---

<sup>21</sup>where when we take the square root of the matrix, we always pick the positive roots of the eigenvalues. We can that both  $\sigma \cdot \mathbf{p}$  and  $\bar{\sigma} \cdot \mathbf{p}$  are semi-positive definite if  $p^0 > 0$

We split  $v(\mathbf{p})$  into two two-component spinors  $v_-(\mathbf{p})$  and  $v_+(\mathbf{p})$  as before, we have equations for them

$$\begin{aligned} (\sigma \cdot \mathbf{p}) v_+(\mathbf{p}) &= -m v_-(\mathbf{p}) \\ (\bar{\sigma} \cdot \mathbf{p}) v_-(\mathbf{p}) &= -m v_+(\mathbf{p}). \end{aligned} \quad (7.67)$$

We can easily check this is solved by

$$v(\mathbf{p}) = \begin{pmatrix} \sqrt{\sigma \cdot p} \eta \\ -\sqrt{\bar{\sigma} \cdot p} \eta \end{pmatrix}, \quad (7.68)$$

where  $\eta$  is an arbitrary two-component spinor that we normalized to  $\eta^\dagger \eta = 1$ .

Let us introduce orthonormal basis for the two component spinors  $\xi^s$  and  $\eta^s$ , with  $s = 1, 2$  such that (from now we will also use  $r$  for spin index. ).

$$\begin{aligned} (\xi^r)^\dagger \xi^s &= \delta^{rs} \\ (\eta^r)^\dagger \eta^s &= \delta^{rs}. \end{aligned} \quad (7.69)$$

A common and good choice is  $\xi^1 = \eta^1 = (1, 0)^T$  and  $\xi^2 = \eta^2 = (0, 1)^T$ , because with those choices we have physical meanings for the different component of the solution, we can call them spin  $\pm \frac{1}{2}$  electron or positron. That will always be the choice we make from now on. Then the two independent positive frequency solutions can be expressed as

$$u^s(\mathbf{p}) = \begin{pmatrix} \sqrt{\sigma \cdot p} \xi^s \\ \sqrt{\bar{\sigma} \cdot p} \xi^s \end{pmatrix}, \quad (7.70)$$

similarly for the two independent negative frequency solutions

$$v^s(\mathbf{p}) = \begin{pmatrix} \sqrt{\sigma \cdot p} \eta^s \\ -\sqrt{\bar{\sigma} \cdot p} \eta^s \end{pmatrix}. \quad (7.71)$$

But we have also computed in the other way, trying to satisfy the constraint of the coefficient, and found them to be

$$u^{\frac{1}{2}}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, u^{-\frac{1}{2}}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad (7.72)$$

and

$$v^{\frac{1}{2}}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, v^{-\frac{1}{2}}(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \quad (7.73)$$

We realize that indeed when we take  $\mathbf{p} = 0$ , the solutions agree when we pick  $(1, 0)$  and  $(0, 1)$  to be our solution basis. Since spin has physical meaning, from

now on we will refer the solutions as the spin up (spin  $\frac{1}{2}$ ) and spin down (spin  $-\frac{1}{2}$ ) solutions. And from now on  $r, s$  takes on physical value  $\frac{1}{2}, -\frac{1}{2}$ . And by  $\sum_s$  we meant sum over those two values.

Now we can write down the general solution for the Dirac equation

$$\psi(x) = \int dV_{\mathbf{p}} \sum_s (u^s(\mathbf{p}) b^s(\mathbf{p}) e^{-ip \cdot x} + v^s(\mathbf{p}) c^{s*}(\mathbf{p}) e^{+ip \cdot x}), \quad (7.74)$$

where  $b^s(\mathbf{p}), c^{s*}(\mathbf{p})$  and similarly  $c^s(\mathbf{p}), b^{s*}(\mathbf{p})$  for  $\psi^*(x)$  for now are just expansion coefficients, but as you probably would expect, at quantization, we will promote them to operators.

Note that there is a spinor label omitted in the above expansion, with all indices restored, we have

$$\psi_a(x) = \int_s dV_{\mathbf{p}} (u_a^s(\mathbf{p}) b^s(\mathbf{p}) e^{-ip \cdot x} + v_a^s(\mathbf{p}) c^{s*}(\mathbf{p}) e^{+ip \cdot x}). \quad (7.75)$$

where  $a = 1, 2, 3, 4$  goes through the four different component of the solutions  $u^{\frac{1}{2}}$  (or  $u^{-\frac{1}{2}}$  or  $v^{\frac{1}{2}}$  or  $v^{-\frac{1}{2}}$ ). Note that we have a total number of 4 independent solutions for the Dirac equation. Upon quantization, they are known to describe the spin-up and spin-down particles and spin-up and spin-down anti-particles.

In the appendix, we gave a list of spinor identities that will be useful in calculations. We also provide the proof.

### 7.1.4 Weyl spinors

As we have seen, the rotational matrices are

$$D(\Lambda_{\text{rot}}) = \begin{pmatrix} e^{\frac{i}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma}} & 0 \\ 0 & e^{\frac{i}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma}} \end{pmatrix}. \quad (7.76)$$

And the boost matrices are,

$$D(\Lambda_{\text{boost}}) = \begin{pmatrix} e^{\frac{1}{2}\boldsymbol{\chi} \cdot \boldsymbol{\sigma}} & 0 \\ 0 & e^{-\frac{1}{2}\boldsymbol{\chi} \cdot \boldsymbol{\sigma}} \end{pmatrix}. \quad (7.77)$$

As we have seen, all the three rotations and three boosts are block diagonal in the Weyl representation. This indicates that the Dirac spinor we found is a reducible representation of Lorentz group. By the form of the spinor rotations and boosts, the Dirac spinor representation of the Lorentz group decomposes into two irreducible inequivalent representations<sup>22</sup>, acting only on two-component spinors  $w_{\pm}$  which, in the chiral representation, are defined by

$$\psi = \begin{pmatrix} w_+ \\ w_- \end{pmatrix}. \quad (7.78)$$

<sup>22</sup>To show they are inequivalent, note the following identity  $\sigma^2(\sigma^i)^* \sigma^2 = -\sigma^i$ . Consider  $\sigma^2 w_+^*$ , under an infinitesimal Lorentz transformation, it should transform  $\sigma^2 w_+^* \rightarrow \sigma^2 (\mathbb{1} - \frac{i}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma}^* + \frac{1}{2}\boldsymbol{\chi} \cdot \boldsymbol{\sigma}^*) w_+^* = (\mathbb{1} + \frac{i}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma} - \frac{1}{2}\boldsymbol{\chi} \cdot \boldsymbol{\sigma}) \sigma^2 w_+^*$ , but this is how  $w_-$  transforms.  $w_+$  and  $w_+^*$  are not related by a unitary transformation, thus the two representations are not related by similar transformations, and hence are inequivalent.

The two component objects  $w_{\pm}$  are called Weyl spinors or chiral spinors. They transform in the same way under rotations,

$$w_{\pm} \rightarrow e^{\frac{i}{2}\theta \cdot \sigma} w_{\pm}, \quad (7.79)$$

and transform in the opposite way under boosts,

$$w_{\pm} \rightarrow e^{\pm \frac{1}{2}\chi \cdot \sigma} w_{\pm}, \quad (7.80)$$

You may question the way we do things, as we did pick a particular nice representation, the Weyl representation. In other representations that are related by similar transformations, the spinor will not split up this nicely. But we know that the Dirac representation is reducible from the fact that all its generators are block diagonal in the Weyl representation, so how to reduce the reducible 4-d Dirac spinor into two 2-d spinors in general? Inspired by quantum mechanics, we should use two projection operators, let us name it  $P_-$  and  $P_+$ <sup>23</sup>. As good projection operators, they should be complete, project to themselves when applying more than once, and orthogonal to each other. So we demand

$$P_- + P_+ = \mathbb{1} \quad (7.81)$$

$$\begin{aligned} P_-^2 &= P_-, & P_+^2 &= P_+ \\ P_- P_+ &= P_+ P_- = 0. \end{aligned}$$

Further, most importantly, we need to respect the Lorentz symmetry: it is kind of nonsense if the irreducible representation of the Lorentz group we find changes in a different Lorentz frame. This means we need the projection operators to commute<sup>24</sup> with the generators of Lorentz symmetry. We are bound to succeed: according to Schur's Lemma<sup>25</sup>, if we cannot find a matrix other than the multiple of the identity matrix to commute with all the generators of Lorentz symmetry of the Dirac representation, it will be irreducible, but we know it is reducible. Since the generators are the commutators of the  $\gamma$  matrices, we will be in business if we can find something anti-commutes with all the existing  $\gamma$  matrices. And this matrix indeed can be found, it is known as the fantastic fifth  $\gamma$  matrix:

$$\boxed{\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3.} \quad (7.82)$$

By using the completely anti-symmetry  $\epsilon$  tensor, we can also write  $\gamma^5$  as

$$\gamma^5 = -i\epsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho. \quad (7.83)$$

In addition to the fact that it anti-commutes with all the rest of the  $\gamma$  matrices and hence commutes with all the Lorentz generators  $\mathcal{J}^{\mu\nu}$

$$\{\gamma^5, \gamma^\mu\} = 0, \quad [\gamma^5, \mathcal{J}^{\mu\nu}] = 0. \quad (7.84)$$

<sup>23</sup>The subscript for now is just a label. Some people call it left and right. We will explore the left and right thing in great detail in the tutorial.

<sup>24</sup>Of course something proportional to identity matrix commutes with everything, but we cannot build projection operators out of them.

<sup>25</sup>Schur's lemma: If a matrix commutes with all the group matrices of an irreducible representation, then this matrix must be proportional to the identity.

it is also Hermitian and square to identity

$$(\gamma^5)^\dagger = \gamma^5, \quad (\gamma^5)^2 = 1 \quad (7.85)$$

The proof of these statements is in the appendix G.1.1. This means  $\gamma^5$  is a Hermitian matrix that squares to 1 and it is a Lorentz scalar. Hence we can form Lorentz invariant projection operators

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5). \quad (7.86)$$

They are projection operators because

$$P_+^2 = \frac{1}{4}(1 + \gamma^5)(1 + \gamma^5) = \frac{1}{4}(2 + 2\gamma^5) = \frac{1}{2}(1 + \gamma^5) = P_+, \quad (7.87)$$

and similarly

$$P_-^2 = P_-, \quad (7.88)$$

and

$$P_+P_- = \frac{1}{4}(1 + \gamma^5)(1 - \gamma^5) = 0 = \frac{1}{4}(1 - \gamma^5)(1 + \gamma^5) = P_-P_+. \quad (7.89)$$

Hence for an arbitrary representation of the Dirac algebra, we can always define the chiral or Weyl spinors to be

$$\psi_\pm = P_\pm \psi. \quad (7.90)$$

We can calculate the specific form of  $\gamma^5$  in the chiral representation we have been using

$$\begin{aligned} \gamma_w^5 &= -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \\ &= -i \begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (7.91)$$

where we used the product of the Pauli matrices is given by

$$\sigma^i \sigma^j = \delta^{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma^k. \quad (7.92)$$

Then in this presentation,  $P_\pm$  are given by

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (7.93)$$

and

$$P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (7.94)$$

which will indeed project onto the Weyl spinors<sup>26</sup>  $w_{\pm}$ :

$$\psi_+ = \begin{pmatrix} w_+ \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ w_- \end{pmatrix}. \quad (7.95)$$

$\psi_+$  is often called a “left-handed” spinor, while  $\psi_-$  is often called “right-handed” spinor. The handedness is measured by the eigenvalue of  $\gamma^5$ , known as chirality. As we can see, the “left-chiral” spinor has chirality +1 and the “right-chiral” spinor has chirality -1. We will see in the tutorial in great detail why the names “left-handed” and “right-handed” are misnomers and how they are actually connected to our hands.

Of course, at this point, you should have asked, why do I care? Well, irreducible representation is the simplest building block, and simple<sup>27</sup> is good. Also it turns out that our world is chiral. The “left-chiral” part of our Dirac electron behaves very differently from the “right-chiral” part. Only the “left”-electron participates in weak interactions and hence why in  $\beta$  decays we will only see “right” anti-neutrinos. We will leave it to the standard model course to explore our chiral world.

### 7.1.5 Interlude: Majorana spinors and their Lagrangian

Along the lines that simple things are good. Is there any other way we can make things simpler?

We have been studying the Dirac field as a complex field. This makes sense, as so far our representation  $D(\Lambda)$  and its generators  $\mathcal{J}^{\mu\nu}$  are complex. So even if we start with a real 4-component spinor, under Lorentz transformation, it will become complex any way. But is it possible to have a real spinor field, in analogy of the real scalar field we studied a while ago? Certainly it is not in the chiral representation, where Lorentz transformation will change the realness of the spinor. Hence we need a new basis, it has the property that the Lorentz representation has pure imaginary generators.

$$\mathcal{J}^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]. \quad (7.96)$$

One way to achieve this is to have all of the  $\gamma$  matrices to be pure imaginary<sup>28</sup>, satisfying  $(\gamma^\mu)^* = -\gamma^\mu$ .

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<sup>26</sup>Weyl spinors only exist in even dimensions. As the analogous  $\gamma^5$  is the identity matrix in odd dimensions.

<sup>27</sup>You might want to ask, can we just write down the equation of motion for the simpler Weyl spinors and study them since they are simple? Yes we can, but there is a subtle issue of Weyl spinor cannot have mass and we will come back to this once we write down the Lagrangian of the Dirac theory. So we need to wait a while for the Weyl theory.

<sup>28</sup>For a general discussion of how to achieve things like this, see “Notes on Clifford Algebra and Spin(N) Representations” by Hitoshi Murayama.

One basis that satisfies this condition is the Majorana basis

$$\gamma_m^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma_m^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma_m^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma_m^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}. \quad (7.97)$$

One can check indeed they satisfy the Dirac algebra. In this base, we can impose the reality condition

$$\psi^* = \psi, \quad (7.98)$$

and it will remain real under Lorentz transformation. The spinor that satisfies the reality condition is called the Majorana spinor. What about the other bases? Can we define some “reality” condition that is basis independent and Lorentz invariant such that we can reduce the complexity of the Dirac spinor by a factor of 2?

The problem of imposing  $\psi^* = \psi$  is that, in a general basis,

$$\begin{aligned} \psi &\rightarrow D(\Lambda)\psi \\ \psi^* &\rightarrow (D(\Lambda))^*\psi^* \neq D(\Lambda)\psi^*. \end{aligned} \quad (7.99)$$

The condition imposed will no longer hold. Thus what we are looking for is a c-conjugation,

$$\psi^{(c)} \equiv C\psi^*. \quad (7.100)$$

and hope for some matrix  $C$ ,  $\psi^{(c)}$  transforms in the same way as  $\psi$ , and then we can impose the condition  $\psi^{(c)} = \psi$  with no problem with Lorentz symmetry. On the other hand, the condition indicates that  $\psi^{(c)}$  should also satisfy the Dirac equation. Another good thing to have for a conjugate, is that  $(\psi^{(c)})^{(c)} = \psi$ , which gives  $CC^* = 1$ . Let us deal with the following requirements now.

- $\psi^{(c)}$  transforms like a Dirac spinor.
- $\psi^{(c)}$  satisfies Dirac equation.

Under Lorentz transformation, this new spinor transforms in the following way,

$$\psi'^{(c)} = C\psi'^* = CD^*(\Lambda)\psi^* \text{ expected to be } D(\Lambda)C\psi^*. \quad (7.101)$$

This gives the condition

$$CD^*(\Lambda) = D(\Lambda)C \quad (7.102)$$

$$D^*(\Lambda) = C^{-1}D(\Lambda)C$$

$$\mathcal{J}^{*,\mu\nu} = C^{-1}\mathcal{J}^{\mu\nu}C = \frac{i}{4}C^{-1}[\gamma^\mu, \gamma^\nu]C = \frac{i}{4}[C^{-1}\gamma^\mu C, C^{-1}\gamma^\nu C]$$

which gives

$$C^{-1}\gamma^\mu C = \eta_1(\gamma^\mu)^*, \quad (7.103)$$

where  $\eta_1$  is a plus minus sign we are going to fix now. Let us look at the Dirac equation and take a complex conjugate of it,

$$\begin{aligned} (i\partial_\mu\gamma^\mu - m)\psi &= 0 \\ (-i\partial_\mu(\gamma^\mu)^* - m)\psi^* &= 0 \\ (-i\partial_\mu\eta_1 C^{-1}\gamma^\mu C - m)\psi^* &= 0 \\ C^{-1}(-i\eta_1\partial_\mu\gamma^\mu - m)C\psi^* &= 0 \\ C^{-1}(-i\eta_1\cancel{D} - m)\psi^{(c)} &= 0. \end{aligned} \tag{7.104}$$

This fix the sign ambiguity  $\eta_1 = -1$  if Dirac equation  $(i\cancel{D} - m)\psi^{(c)} = 0$  needs to be obeyed.

To summarize, in a general basis, the way to define Majorana spinor is

$$\boxed{\psi^{(c)} \equiv C\psi^* = \psi,} \tag{7.105}$$

where  $C$  satisfies,

$$\boxed{CC^* = 1, \quad C^{-1}\gamma^\mu C = -(\gamma^\mu)^*.} \tag{7.106}$$

In the Majorana basis, because  $\gamma$  matrices are pure imaginary, the matrix  $C$  is simply the identity and the Majorana condition is simply the reality condition as we discussed at first.

For the chiral/Weyl basis we have been fond of, we can define

$$C_w = i\gamma_w^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}. \tag{7.107}$$

Since we are in chiral/Weyl basis, a Dirac spinor is simply decomposed into two Weyl spinors,

$$\psi = \begin{pmatrix} w_+ \\ w_- \end{pmatrix}. \tag{7.108}$$

The Majorana condition is given by

$$\begin{pmatrix} w_+ \\ w_- \end{pmatrix} = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} w_+^* \\ w_-^* \end{pmatrix}. \tag{7.109}$$

This is  $w_- = -i\sigma^2 w_+^*$ <sup>29</sup>. So Majorana spinor can be written in terms of two component spinor,

$$\psi = \begin{pmatrix} w_+ \\ -i\sigma^2 w_+^* \end{pmatrix}. \tag{7.110}$$

<sup>29</sup>This is not surprising, as previously when discussing irreducible representation of Lorentz group, we have shown  $\sigma^2 w_+^*$  transforms like  $w_-$ .

Majorana condition imposes a constraint on the lower half of the spinor to be related to the upper half. Weyl condition projects half the spinor out. They cannot be both imposed<sup>30</sup>.

We are going to explore the Dirac, Weyl and Majorana spinors in more details in the tutorial.

## 7.2 Step 1: The Dirac Lagrangian

Now that we know how the Dirac field transforms under Lorentz transformation, we can write down the Dirac Lagrangian and do some perturbative calculations, after all, if we want to have a relativistic field theory, the Lagrangian needs to be Lorentz invariant.

First we will try the mass term, enlightened by the case of complex scalar field theory, we make a first guess:  $\psi^\dagger \psi$ <sup>31</sup>, where  $\psi^\dagger = (\psi^*)^T$  is the adjoint defined the usual way, the transpose of the complex conjugate.

Let us check if  $\psi^\dagger \psi$  is a Lorentz scalar. As we have learned, the Dirac field transforms as

$$\psi'(x') = D(\Lambda)\psi(x), \quad (7.111)$$

where

$$D(\Lambda) = e^{\frac{i}{2}\mathcal{J}_{\mu\nu}\epsilon^{\mu\nu}} \quad (7.112)$$

and

$$\mathcal{J}_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu]. \quad (7.113)$$

So its adjoint (hermitian conjugate) transforms as

$$\psi'^\dagger(x') = \psi(x)^\dagger D^\dagger(\Lambda). \quad (7.114)$$

Thus we have

$$\psi'^\dagger(x')\psi'(x') = \psi(x)^\dagger D^\dagger(\Lambda)D(\Lambda)\psi(x). \quad (7.115)$$

If  $D^\dagger(\Lambda)D(\Lambda) = \mathbb{1}$  in other words, if  $D(\Lambda)$  is unitary, we will be all set. So let us take a look at our representation  $D(\Lambda)$

$$D(\Lambda) = e^{\frac{i}{2}\mathcal{J}_{\mu\nu}\epsilon^{\mu\nu}} \quad (7.116)$$

again, using our favorite method of linear expansion, we have

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<sup>30</sup>Majorana and Weyl conditions can be imposed simultaneously in  $8k+2$  dimensions. See the discussion in “Notes on Clifford Algebra and Spin(N) Representations” by Hitoshi Murayama.

<sup>31</sup>We will ignore the mass coupling constant for now, since mass is a Lorentz scalar.

$$D^\dagger(\Lambda)D(\Lambda) = \left(1 - \frac{i}{2}\mathcal{J}_{\mu\nu}^\dagger\epsilon^{\mu\nu}\right)\left(1 + \frac{i}{2}\mathcal{J}_{\alpha\beta}\epsilon^{\alpha\beta}\right) \quad (7.117)$$

$$= 1 - \frac{i}{2}\epsilon^{\mu\nu}(\mathcal{J}_{\mu\nu}^\dagger - \mathcal{J}_{\mu\nu}), \quad (7.118)$$

which gives

$$\mathcal{J}_{\mu\nu}^\dagger - \mathcal{J}_{\mu\nu} = 0. \quad (7.119)$$

So requiring unitarity is the same as requiring the generators  $\mathcal{J}_{\mu\nu}$  to be Hermitian. Let us work out  $\mathcal{J}_{\mu\nu}^\dagger$ ,

$$\begin{aligned} \mathcal{J}_{\mu\nu}^\dagger &= \frac{i}{4}[\gamma_\mu, \gamma_\nu]^\dagger \\ &= \frac{i}{4}(\gamma_\nu^\dagger \gamma_\mu^\dagger - \gamma_\mu^\dagger \gamma_\nu^\dagger) \\ &= \frac{i}{4}[\gamma_\nu^\dagger, \gamma_\mu^\dagger]. \end{aligned} \quad (7.120)$$

But recall the property of  $\gamma$  matrices

$$\gamma_\mu^\dagger = \gamma^0 \gamma_\mu \gamma^0. \quad (7.121)$$

So

$$\begin{aligned} \mathcal{J}_{\mu\nu}^\dagger &= \frac{i}{4}[\gamma_\nu^\dagger, \gamma_\mu^\dagger] \\ &= \frac{i}{4}[\gamma^0 \gamma_\nu \gamma^0, \gamma^0 \gamma_\mu \gamma^0] \\ &= \gamma^0 \mathcal{J}_{\mu\nu} \gamma^0. \end{aligned} \quad (7.122)$$

So this will work for  $\mathcal{J}_{ij}$ , but not going to work for  $\mathcal{J}_{0i}$ . In other words, the rotations are unitary, but the boosts are not, as we can see in the chiral/Weyl representation.

$$\begin{aligned} D(\Lambda_{\text{rot}}) &= \begin{pmatrix} e^{\frac{i}{2}\theta \cdot \sigma} & 0 \\ 0 & e^{\frac{i}{2}\theta \cdot \sigma} \end{pmatrix} \\ D(\Lambda_{\text{boost}}) &= \begin{pmatrix} e^{\frac{1}{2}\mathbf{x} \cdot \sigma} & 0 \\ 0 & e^{-\frac{1}{2}\mathbf{x} \cdot \sigma} \end{pmatrix}. \end{aligned} \quad (7.123)$$

Actually, there is a mathematical theorem in group theory against our wish: Lorentz group is a non-compact semi-simple group and thus cannot have finite dimensional unitary representation.

But nevertheless we are going to proceed and try to see what we are missing to make an invariant scalar out of  $\psi^\dagger$  and  $\psi$ 's. Now that we have  $\mathcal{J}_{\mu\nu}^\dagger$

$$\mathcal{J}_{\mu\nu}^\dagger = \gamma^0 \mathcal{J}_{\mu\nu} \gamma^0. \quad (7.124)$$

We can calculate  $D(\Lambda)^\dagger$

$$\begin{aligned} D(\Lambda)^\dagger &= e^{-\frac{i}{2}\mathcal{J}_{\mu\nu}\epsilon^{\mu\nu}} \\ &= e^{-\frac{i}{2}\gamma^0\mathcal{J}_{\mu\nu}\epsilon^{\mu\nu}\gamma^0} \\ &= \gamma^0 e^{-\frac{i}{2}\mathcal{J}_{\mu\nu}\epsilon^{\mu\nu}}\gamma^0 \\ &= \gamma^0 D(\Lambda)^{-1}\gamma^0. \end{aligned} \quad (7.125)$$

Now we can look at how  $\psi^\dagger\psi$  transforms again, suppressing the  $x$  dependence

$$\begin{aligned} \psi'^\dagger\psi' &= \psi^\dagger D(\Lambda)^\dagger D(\Lambda)\psi \\ &= \psi^\dagger\gamma^0 D(\Lambda)^{-1}\gamma^0 D(\Lambda)\psi. \end{aligned} \quad (7.126)$$

So what is the remedy? It seems we need some  $\gamma^0$  to cancel out the  $\gamma^0$  between  $D(\Lambda)$  and  $D^{-1}(\Lambda)$ . Thus we propose  $\psi^\dagger\gamma^0\psi$ , let us check how this works

$$\begin{aligned} \psi'^\dagger\gamma^0\psi' &= \psi^\dagger\gamma^0 D(\Lambda)^{-1}\gamma^0\gamma^0 D(\Lambda)\psi \\ &= \psi^\dagger\gamma^0\psi. \end{aligned} \quad (7.127)$$

Yeah! We find a Lorentz scalar. This object  $\psi^\dagger\gamma^0$  turns out to be extremely useful in making Lorentz covariant objects, so it has a special name and special notation, it is called a Dirac adjoint or Dirac conjugate, and it is denoted by

$$\boxed{\bar{\psi} \equiv \psi^\dagger\gamma^0.} \quad (7.128)$$

So our first Lorentz scalar is given by  $\bar{\psi}\psi$ . Encouraged by this, we charge forward and start building more Dirac bilinears, for example, we decide to take a look at  $\bar{\psi}\gamma^\mu\psi$ , and hope it transforms as a vector. If it would transform as a vector, we will have,

$$\bar{\psi}'\gamma^\mu\psi' = \Lambda^\mu_\nu \bar{\psi}\gamma^\nu\psi. \quad (7.129)$$

On the other hand, according to the transformation rule, we have

$$\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}D(\Lambda)^{-1}\gamma^\mu D(\Lambda)\psi. \quad (7.130)$$

Thus we need to show

$$D(\Lambda)^{-1}\gamma^\mu D(\Lambda) = \Lambda^\mu_\nu \gamma^\nu. \quad (7.131)$$

But this is the property of  $\gamma$  matrices we used to show  $\gamma^\mu$  is in some sense a vector.

Thus we have proved that  $\bar{\psi}\gamma^\mu\psi$  is indeed a Lorentz vector.

We will encounter more Dirac bilinears in the future, we will delay the discussion of them until when we learn about interactions. But these two are enough to help us build the free Dirac Lagrangian, recall Dirac equation is,

$$(i\cancel{\partial} - m)\psi = 0. \quad (7.132)$$

And Euler-Lagrangian equation is

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right) = \frac{\partial \mathcal{L}}{\partial \psi} \quad (7.133)$$

and the one for the Dirac adjoint  $\bar{\psi}$  is

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} \right) = \frac{\partial \mathcal{L}}{\partial \bar{\psi}}. \quad (7.134)$$

We expect the two equations we obtain should be Dirac equation and its conjugate. The easiest one is that the Lagrangian does not depend on  $\partial_\mu \bar{\psi}$  and the Lagrangian density is simply

$$\boxed{\mathcal{L} = \bar{\psi} (i\cancel{D} - m) \psi.} \quad (7.135)$$

Now let us vary respect to  $\psi$ , we have

$$i\partial_\mu \bar{\psi} \gamma^\mu = -m\bar{\psi} \quad (7.136)$$

which is indeed the conjugate of the Dirac equation.

After we write down the action, we find the dimension of a spinor is  $\frac{3}{2}$ , or in  $d$  dimensions, it will be  $\frac{d-1}{2}$ , because the mass dimension of the Lagrangian density needs to be  $d$ , so after integrate over the space time, the action is dimensionless. By looking at either the kinetic term or the mass term, we conclude  $1 + 2\dim(\psi) = d$ , hence  $\boxed{\dim(\psi) = \frac{d-1}{2}}.$

### 7.2.1 Weyl Lagrangian

As we have seen in the tutorial, Weyl spinor by definition has a certain chirality, so it can only be massless. We also saw at the end of the tutorial, we can make a Dirac spinor out of two Weyl spinors, but a Dirac spinor can have mass, as manifested in the equation and the Lagrangian. So how? Now that we have the Dirac Lagrangian, we can answer this question. Let us look at the mass term of the Dirac Lagrangian,

$$m\bar{\psi}\psi = m((P_+ + P_-)\psi)^\dagger \gamma^0 (P_+ + P_-)\psi = m\psi^\dagger (P_+ + P_-) \gamma^0 (P_+ + P_-)\psi. \quad (7.137)$$

Note that the projection operators, which have  $\gamma^5$  anti-commute with  $\gamma^0$ ,

$$\begin{aligned} P_+ \gamma^0 &= \gamma^0 P_- \\ P_- \gamma^0 &= \gamma^0 P_+. \end{aligned} \quad (7.138)$$

Recall

$$P_+ P_- = 0. \quad (7.139)$$

This means the surviving terms are

$$m\bar{\psi}\psi = m\psi^\dagger P_+ \gamma^0 P_- \psi + m\psi^\dagger P_- \gamma^0 P_+ \psi = m(\bar{\psi}_+ \psi_- + \bar{\psi}_- \psi_+), \quad (7.140)$$

where we used

$$\begin{aligned} \psi_+ &= P_+ \psi \\ \bar{\psi}_+ &= (P_+ \psi)^\dagger \gamma^0 = \psi^\dagger P_+ \gamma^0. \end{aligned} \quad (7.141)$$

As we can see, if we only have one Weyl fermion, it is impossible to write down mass term for them. The mass term for the Dirac field is actually the interaction between two Weyl fermions of different chirality.

On the other hand, we can also look at the kinetic term of the Dirac Lagrangian,

$$\bar{\psi}(i\cancel{D})\psi = ((P_+ + P_-)\psi)^\dagger \gamma^0 (i\gamma^\mu \partial_\mu) (P_+ + P_-)\psi = \psi^\dagger (P_+ + P_-) \gamma^0 (i\gamma^\mu \partial_\mu) (P_+ + P_-)\psi, \quad (7.142)$$

This time the projection operator, which has  $\gamma^5$  commute with  $\gamma^0 \gamma^\mu$ ,

$$\begin{aligned} P_+ \gamma^0 \gamma^\mu &= \gamma^0 \gamma^\mu P_+ \\ P_- \gamma^0 \gamma^\mu &= \gamma^0 \gamma^\mu P_-. \end{aligned} \quad (7.143)$$

Thus the surviving terms are

$$\bar{\psi}(i\cancel{D})\psi = \psi^\dagger P_+ \gamma^0 (i\gamma^\mu \partial_\mu) P_+ \psi + \psi^\dagger P_- \gamma^0 (i\gamma^\mu \partial_\mu) P_- \psi = \bar{\psi}_+ (i\cancel{D})\psi_+ + \bar{\psi}_- (i\cancel{D})\psi_-. \quad (7.144)$$

Each Weyl fermion has its own kinetic term. As you probably would expect, in the chiral/Weyl representation, where

$$\psi = \begin{pmatrix} w_+ \\ w_- \end{pmatrix}, \quad \gamma_w^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad (7.145)$$

the Lagrangian is simpler and obviously Weyl fermion has two degrees of freedom

$$\mathcal{L}_{\text{Weyl}} = i w_+^\dagger \bar{\sigma}^\mu \partial_\mu w_+ + i w_-^\dagger \sigma^\mu \partial_\mu w_-. \quad (7.146)$$

### 7.2.2 \*Majorana Lagrangian

Majorana fermion, is not restricted to be massless, and we can write down the Lagrangian. The Majorana Lagrangian in Weyl basis, written in terms of a Weyl spinor  $w_+$ <sup>32</sup> is given by

$$\mathcal{L}_{\text{Majorana}} = i w_+^\dagger \bar{\sigma}^\mu \partial_\mu w_+ + \frac{im}{2} (w_+^T \sigma_2 w_+ - w_+^\dagger \sigma_2 w_+^*). \quad (7.147)$$

The mass term comes from plugging Majorana fermion in Weyl basis in the Dirac Lagrangian mass term. For kinetic term, first obtain the equation of motion for the Majorana fermion from the Dirac equation and then write down the kinetic term that together with the mass term will reproduce the equation of motion.

<sup>32</sup>as Majorana spinor can be considered to be a Weyl spinor plus its c-conjugate spinor, so this is ok.

## 7.3 Step 2: The conjugate momentum and the Hamiltonian

### 7.3.1 The conjugate momentum and degrees of freedom

First we calculate the conjugate momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0 = i \psi^\dagger. \quad (7.148)$$

How many degrees of freedom does Dirac spinor have? If you have remembered the solutions we found recently, you would know the answer is 4, namely  $u^{\frac{1}{2}}(\mathbf{p})$ ,  $u^{-\frac{1}{2}}(\mathbf{p})$ ,  $v^{\frac{1}{2}}(\mathbf{p})$ , and  $v^{-\frac{1}{2}}(\mathbf{p})$ . But on the other hand the Dirac field is a four component complex field, which naively should indicate that there are  $4 \times 2 = 8$  degrees of freedom. After all, we are all familiar with scalar field theory, and it is well known that the real scalar field has one degrees of freedom and complex scalar field has two degrees of freedom. So why the Dirac field is different? How should we actually count degrees of freedom? In classical mechanics, a particle moving in one dimension has one degrees of freedom, and its phase space has 2 dimensions, the position and momentum. So we will use this definition: the number of degrees of freedom is half the dimension of the phase space. This works for both the real and complex scalar field, for which the phase space includes  $\phi$  and  $\dot{\phi}$ . Since Dirac equation is a first order differential equation, the conjugate momentum does not involve derivative, so in order to specify the system, we need to specify  $\psi$  and  $\psi^\dagger$  at  $t = 0$ , but no derivative information is needed. Put it in other words, Although  $\psi$  is a four component complex spinor, since the phase space dimension is eight, the number of degrees of freedom for Dirac spinor is actually four instead of eight. The four including a particle spin up and down and an antiparticle spin up and down.

### 7.3.2 The Hamiltonian

The Hamiltonian is

$$\mathcal{H} = \psi^\dagger (-i \gamma^0 \gamma^i \partial_i + m \gamma^0) \psi, \quad (7.149)$$

where the operator in between is no surprise the same as the Hamiltonian found by Dirac.

Now we are going to utilize the solution with mode expansion, and express the Hamiltonian in terms of the modes. What do we expect? We expect the density should be something like energy multiply by something that is going to be promoted to be the number operator.

Since Hamiltonian is conserved, we can set  $t = 0$  in the mode expansion<sup>33</sup>:

$$\begin{aligned}\psi(\mathbf{x}) &= \oint_s dV_{\mathbf{p}} (u^s(\mathbf{p})b^s(\mathbf{p})e^{+i\mathbf{p}\cdot\mathbf{x}} + v^s(\mathbf{p})c^{s*}(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}) \\ \psi^\dagger(\mathbf{x}) &= \oint_s dV_{\mathbf{p}} (u^{\dagger s}(\mathbf{p})b^{s*}(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + v^{\dagger s}(\mathbf{p})c^s(\mathbf{p})e^{+i\mathbf{p}\cdot\mathbf{x}}).\end{aligned}\quad (7.150)$$

This simplifies the Hamiltonian to be

$$H = \int dV_{\mathbf{p}} E_{\mathbf{p}} (b^{s*}(\mathbf{p})b^s(\mathbf{p}) - c^s(\mathbf{p})c^{s*}(\mathbf{p})). \quad (7.151)$$

The derivation is so important that it is left as an exciting exercise.

## 7.4 Step 3: The commutator/anti-commutator relationship

The next step is to impose commutator relationship between the field and the conjugate momentum, which means the field will become a field operator. The natural way to accomplish that is to promote the  $b$ 's and  $c$ 's as annihilation operators and promote the  $b^*$ 's and  $c^*$ 's creation operators. Note that we label  $b, c, b^*, c^*$ 's with 3-momentum, because the mass shell condition  $p^2 = m^2$  is imposed so only three out of four components of the momentum are independent.

$$H = \int dV_{\mathbf{p}} E_{\mathbf{p}} (b_{\mathbf{p}}^{s\dagger}b_{\mathbf{p}}^s - c_{\mathbf{p}}^s c_{\mathbf{p}}^{s\dagger}). \quad (7.152)$$

If we impose the commutator relationship,  $[c_{\mathbf{p}}^r, c_{\mathbf{q}}^{s\dagger}] = -\delta^{rs}\delta^3(\mathbf{p} - \mathbf{q})(2\pi)^3 2E_{\mathbf{p}}$ <sup>34</sup>, and demand normal ordering, we encounter a disaster: it seems that we have a Hamiltonian that is unbounded from below: the more  $c$  type particles we produce, the lower the energy will be. One way to fix this, is to assume an anti-commutator relationship,

$$\{c_{\mathbf{p}}^r, c_{\mathbf{q}}^{s\dagger}\} = \delta^{rs}\delta^3(\mathbf{p} - \mathbf{q})(2\pi)^3 2E_{\mathbf{p}}. \quad (7.153)$$

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<sup>33</sup>To avoid confusion of whether the 1 in  $p_1$  is a four-index or three-index, we will avoid using  $p_i$  as much as possible. On the other hand, everybody agrees what is  $\mathbf{p} \cdot \mathbf{x}$ . We thank Cynthia Arias Pruna for discussion.

<sup>34</sup>The extra minus sign is needed if we want to properly try to quantize Dirac field with commutators. See Peskin and Schroeder 3.5 and Tong 5.1 for a discussion on how not to quantize Dirac field. We thank Nika Sokolova to point this out.

But what does this suppose to mean? <sup>35</sup> It means that anti-commutator relationship leads to fermions<sup>37</sup>.

What we have seen is an example of spin-statistics theorem: spin  $\frac{1}{2}$  particles are fermions.

So we will impose the anti-commutator relationship, with all others vanishing,

$$\begin{aligned}\{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} &= \delta^{rs} \delta^3(\mathbf{p} - \mathbf{q}) 2E_{\mathbf{p}} (2\pi)^3 \\ \{c_{\mathbf{p}}^r, c_{\mathbf{q}}^{s\dagger}\} &= \delta^{rs} \delta^3(\mathbf{p} - \mathbf{q}) 2E_{\mathbf{p}} (2\pi)^3.\end{aligned}\quad (7.158)$$

Now the field operators become

$$\psi(\mathbf{x}) = \oint_s dV_{\mathbf{p}} (u^s(\mathbf{p}) b_{\mathbf{p}}^s e^{i\mathbf{p}\cdot\mathbf{x}} + v^s(\mathbf{p}) c_{\mathbf{p}}^{s\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}}) \quad (7.159)$$

and

$$\psi^\dagger(\mathbf{x}) = \oint_s dV_{\mathbf{p}} (u^{\dagger s}(\mathbf{p}) b_{\mathbf{p}}^{s\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} + v^{\dagger s}(\mathbf{p}) c_{\mathbf{p}}^s e^{+i\mathbf{p}\cdot\mathbf{x}}). \quad (7.160)$$

What does this tell us the equal time relationship between the field and its conjugate momentum? This time we will write the spinor index explicitly, as otherwise it does not make sense in terms of matrix multiplication,

$$\begin{aligned}\{\psi_a(\mathbf{x}), i\psi_b^\dagger(\mathbf{y})\} &= i\delta^3(\mathbf{x} - \mathbf{y})\delta_{ab} \\ \{\psi_a(\mathbf{x}), \psi_b(\mathbf{y})\} &= 0, \quad \{\psi_a^\dagger(\mathbf{x}), \psi_b^\dagger(\mathbf{y})\} = 0.\end{aligned}\quad (7.161)$$

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<sup>35</sup>Let us look at an example in quantum mechanics<sup>36</sup>, so no momentum dependence, with just one type of particle,

$$\{a, a^\dagger\} = 1, \quad \{a, a\} = 0, \quad \{a^\dagger, a^\dagger\} = 0. \quad (7.154)$$

Now in analogy to quantum harmonic oscillator, we define number operator  $N = a^\dagger a$ . What are the possible eigenvalues of the number operator? Note that

$$N^2 = a^\dagger a a^\dagger a = a^\dagger (1 - a^\dagger a) a = a^\dagger a = N. \quad (7.155)$$

Solve this algebraic equation:  $N = 0$  or  $N = 1$ . We can label the eigenstates with  $|0\rangle$  and  $|1\rangle$ . There cannot be more than one particle! This can also be seen by  $(a^\dagger)^2 |n\rangle = 0$  from the anti-commutator relationship. This is the Pauli exclusion principle.

<sup>37</sup>Still in quantum mechanics: To proceed, in analogous of quantum harmonic oscillator, we compute the following commutator,

$$\begin{aligned}[N, a] &= a^\dagger a a - a a^\dagger a = -a a^\dagger a = -a(1 - a a^\dagger) = -a \\ [N, a^\dagger] &= a^\dagger a a^\dagger - a^\dagger a^\dagger a = a^\dagger a a^\dagger = (1 - a a^\dagger) a^\dagger = a^\dagger.\end{aligned}\quad (7.156)$$

Then we can show,

$$\begin{aligned}Na|n\rangle &= ([N, a] + aN)|n\rangle = (-1 + n)a|n\rangle \\ Na^\dagger|n\rangle &= ([N, a^\dagger] + a^\dagger N)|n\rangle = (1 + n)a^\dagger|n\rangle.\end{aligned}\quad (7.157)$$

$a$  and  $a^\dagger$  are indeed the annihilation and creation operator as we expected.

Let us show this in a footnote. Note that we stop writing the sums just like in Einstein convention. We also stopped writing the spinor index because nobody writes it in textbooks or literature.<sup>38</sup>.

It is easy to see the other anti-commutators vanish, as all the anti-commutators of the operators vanish in the derivation.

## 7.5 Step 4: Normal ordering

To see the ordering prescription, we take a look at the Hamiltonian again.

$$\begin{aligned} H &= \int dV_{\mathbf{p}} E_{\mathbf{p}} (b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s - c_{\mathbf{p}}^s c_{\mathbf{p}}^{s\dagger}) \\ &= \int dV_{\mathbf{p}} E_{\mathbf{p}} (b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s - \{c_{\mathbf{p}}^s, c_{\mathbf{p}}^{s\dagger}\} + c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s) \\ &= \int dV_{\mathbf{p}} E_{\mathbf{p}} (b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s + c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s) - \int d^3 p E_{\mathbf{p}} \delta^3(0). \end{aligned} \quad (7.165)$$

We want to define

$$:H := \int dV_{\mathbf{p}} E_{\mathbf{p}} (b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s + c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s). \quad (7.166)$$

Thus normal ordering in the case of fermions, means putting all the annihilation operators to the right without using the anti-commutation relations but each switch of two fermionic operators will generate a minus sign because of their fermioness. For example<sup>39</sup>,

$$:cc^\dagger := -:c^\dagger c := -c^\dagger c. \quad (7.167)$$

<sup>38</sup>

$$\begin{aligned} \{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} &= \int dV_{\mathbf{p}} dV_{\mathbf{q}} (u^r(\mathbf{p}) u^{\dagger s}(\mathbf{q}) \{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} e^{+i\mathbf{p}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{y}} + v^r(\mathbf{p}) v^{\dagger s}(\mathbf{q}) \{c_{\mathbf{p}}^{r\dagger}, c_{\mathbf{q}}^s\} e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{q}\cdot\mathbf{y}}) \\ &= \int dV_{\mathbf{p}} (u^r(\mathbf{p}) u^{\dagger r}(\mathbf{p}) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + v^r(\mathbf{p}) v^{\dagger r}(\mathbf{p}) e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}) \\ &= \int dV_{\mathbf{p}} (u^r(\mathbf{p}) \bar{u}^r(\mathbf{p}) \gamma^0 e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + v^r(\mathbf{p}) \bar{v}^r(\mathbf{p}) \gamma^0 e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}). \end{aligned} \quad (7.162)$$

Now we use spinor outer sums

$$\begin{aligned} u^s(\mathbf{p}) \bar{u}^s(\mathbf{p}) &= \not{p} + m \\ v^s(\mathbf{p}) \bar{v}^s(\mathbf{p}) &= \not{p} - m. \end{aligned} \quad (7.163)$$

And then we have

$$\begin{aligned} \{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} &= \int dV_{\mathbf{p}} (\not{p} + m) \gamma^0 e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + (\not{p} - m) \gamma^0 e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \int dV_{\mathbf{p}} (p_0 \gamma^0 - \mathbf{p} \cdot \boldsymbol{\gamma} + m) \gamma^0 e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + (p_0 \gamma^0 - \mathbf{p} \cdot \boldsymbol{\gamma} - m) \gamma^0 e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \int dV_{\mathbf{p}} 2p_0 e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \delta^3(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (7.164)$$

Note that in the second to last step, we used the fact that integral of integrand that is linear in the integration variable is 0, because we can change the integration variable to be  $-\mathbf{p}$ .

<sup>39</sup>where the spin labels and the momentum dependence are omitted.

On the other hand, notice the negative infinity zero point energy we get in this case as oppose to the positive infinite zero point energy in the case of the bosons. Supersymmetry(SUSY) propose for each particle that exists in the universe, it comes with a partner that is every way the same except the boson/fermion label is switched. In a supersymmetric vacuum, the cosmological constant will be zero. In our world, it is clear that SUSY is broken and despite the effort, no SUSY partner has been discovered yet. Oh well, it is still pretty successful, as fifty percent of the particles are already discovered.

Today we have seen an explicitly example of spin-statistics theorem: Particles with integer spin are bosons. Particles with half-integer spin are fermions.

For deeper connection between spin and statistics, we will refer interested readers to Pauli's paper<sup>40</sup>. In which he showed the following: by assuming that the energy must be a positive quantity, he concludes that particles with half-integer spins cannot be bosons<sup>41</sup>; by assuming microcausality, he concludes that particles with integer spin cannot be fermions<sup>42</sup>.

## 7.6 Dirac field Quantized

### 7.6.1 States

Just as in the bosonic case, we define the vacuum  $|0\rangle$  to be

$$b_{\mathbf{p}}^s |0\rangle = 0, \quad c_{\mathbf{p}}^s |0\rangle = 0 \quad (7.168)$$

with the normalization,

$$\langle 0|0\rangle = 1. \quad (7.169)$$

It is easy to see that we have

$$H|0\rangle = 0 \quad (7.170)$$

for our normal ordered Hamiltonian

$$:H := \int dV_{\mathbf{p}} E_{\mathbf{p}} (b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s + c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s). \quad (7.171)$$

Again we can construct energy eigenstates using creation operators on vacuum, we can define one-particle state and one anti-particle state as

$$\begin{aligned} |\mathbf{p}, s\rangle &= b_{\mathbf{p}}^{s\dagger} |0\rangle \\ |\overline{\mathbf{p}, s}\rangle &= c_{\mathbf{p}}^{s\dagger} |0\rangle. \end{aligned} \quad (7.172)$$

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<sup>40</sup>The Connection Between Spin and Statistics, by W Pauli. Phys. Rev. 58, 716.

<sup>41</sup>This we have seen explicitly for the Dirac theory.

<sup>42</sup>This can be shown explicitly in the Klein-Gordon theory: microcausality crucially depends on the fact that the theory was quantized using the commutator relationship. The observable would not commute at spacelike separations if the theory was quantized using the anti-commutators, in particular the field itself at spacelike separation would not anti-commute even if the creation and annihilation operators satisfy the anti-commutators. See an explicitly example in “Quantum field theory in a nutshell.” 11.4 A. Zee.

We show that they are energy eigenstates in a footnote<sup>43</sup>.

The one particle states are not unit normed. Based on our normalization, they are normalized such that

$$\langle \mathbf{q}, r | \mathbf{p}, s \rangle = (2\pi)^3 2E_{\mathbf{p}} \delta^3(\mathbf{p} - \mathbf{q}) \delta^{rs}. \quad (7.177)$$

This is a direct result of the normalization of the vacuum and the anti-commutator relationship between the creation operator and the annihilation operator. We can continue to define multi particle states,

$$|(\mathbf{p}_1, s_1), (\mathbf{p}_2, s_2)\rangle = b_{\mathbf{p}_1}^{s_1\dagger} b_{\mathbf{p}_2}^{s_2\dagger} |0\rangle = -b_{\mathbf{p}_2}^{s_2\dagger} b_{\mathbf{p}_1}^{s_1\dagger} |0\rangle = -|(\mathbf{p}_2, s_2), (\mathbf{p}_1, s_1)\rangle \quad (7.178)$$

Indeed the particles obey the Fermi-Dirac statistics. In particular, we have Pauli exclusion principle, as we have shown before the number operator can only have eigenvalues 1 and 0, also it is clear from above that  $|\mathbf{p}, s; \mathbf{p}, s\rangle = 0$ .

### 7.6.2 Anti-particle

So what are these  $b$  particles and  $c$  particles? It seems we quantize one Dirac field and get two types of particles. And you might have heard of the terms particles and anti-particles. But how do we see that? One way, is to see through the electric charge. Recall from global phase symmetry  $\psi \rightarrow e^{-iQ\alpha}\psi$ , we obtain a conserved current,

$$J^\mu = Q \bar{\psi} \gamma^\mu \psi. \quad (7.179)$$

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<sup>43</sup>First we show

$$\begin{aligned} [H, b_{\mathbf{p}}^s] &= \int dV_{\mathbf{q}} E_{\mathbf{q}} [b_{\mathbf{q}}^{s\dagger} b_{\mathbf{q}}^s, b_{\mathbf{p}}^s] \\ &= \int dV_{\mathbf{q}} E_{\mathbf{q}} (-\{b_{\mathbf{q}}^{s\dagger}, b_{\mathbf{p}}^s\}) b_{\mathbf{q}}^s \\ &= -E_{\mathbf{p}} b_{\mathbf{p}}^s, \end{aligned} \quad (7.173)$$

where we make use the commutator properties

$$[AB, C] = A\{B, C\} - \{A, C\}B. \quad (7.174)$$

Similarly, we have

$$[H, b_{\mathbf{p}}^{s\dagger}] = E_{\mathbf{p}} b_{\mathbf{p}}^{s\dagger} \quad (7.175)$$

and same with  $c$ , the anti-particles. Then we have, where we omit the normal ordering “::”,

$$\begin{aligned} H|\mathbf{p}, s\rangle &= H b_{\mathbf{p}}^{s\dagger} |0\rangle = [H, b_{\mathbf{p}}^{s\dagger}] |0\rangle + b_{\mathbf{p}}^{s\dagger} H |0\rangle = E_{\mathbf{p}} b_{\mathbf{p}}^{s\dagger} |0\rangle \\ &= E_{\mathbf{p}} |\mathbf{p}, s\rangle. \end{aligned} \quad (7.176)$$

After coupling to electromagnetism, the current is still conserved, and the conserved charge is the electric charge,

$$\begin{aligned}
Q_e &= Q \int d^3x \bar{\psi} \gamma^0 \psi = Q \int d^3x \psi^\dagger \psi \\
&= Q \int d^3x dV_{\mathbf{p}} dV_{\mathbf{q}} (u^{\dagger s}(\mathbf{p}) b_{\mathbf{p}}^{s\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} + v^{\dagger s}(\mathbf{p}) c_{\mathbf{p}}^s e^{+i\mathbf{p}\cdot\mathbf{x}}) (u^r(\mathbf{q}) b_{\mathbf{q}}^r e^{+i\mathbf{q}\cdot\mathbf{x}} + v^r(\mathbf{q}) c_{\mathbf{q}}^{r\dagger} e^{-i\mathbf{q}\cdot\mathbf{x}}) \\
&= Q \int dV_{\mathbf{p}} \frac{1}{2E_{\mathbf{p}}} (u^{\dagger s}(\mathbf{p}) u^r(\mathbf{p}) b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^r + v^{\dagger s}(\mathbf{p}) v^r(\mathbf{p}) c_{\mathbf{p}}^s c_{\mathbf{p}}^{r\dagger}) \\
&= Q \int dV_{\mathbf{p}} (b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s + c_{\mathbf{p}}^s c_{\mathbf{p}}^{s\dagger}).
\end{aligned} \tag{7.180}$$

According to our normal ordering prescription, the normal ordered charge operator, will be,

$$:Q_e := Q \int dV_{\mathbf{p}} (b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s - c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s). \tag{7.181}$$

Thus we see the  $c$  particles contribute to the charge with a minus sign, it indeed makes sense to call them the anti-particles of  $b$  particles.

### 7.6.3 Heisenberg picture

So far we have been taken  $t = 0$  for our operators. It will be nice to study the operator that depends on the spacetime, not just the space. The Heisenberg picture operator  $\mathcal{O}(t)$  is related to the time independent operator in the usual way

$$\mathcal{O}(t) = e^{iHt} \mathcal{O} e^{-iHt}. \tag{7.182}$$

The Heisenberg picture annihilation operator can be computed using the Baker–Campbell–Hausdorff (BCH) formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2} [A, [A, B]] + \dots \tag{7.183}$$

and

$$[H, b_{\mathbf{p}}^s] = -E_{\mathbf{p}} b_{\mathbf{p}}^s. \tag{7.184}$$

The result is:

$$\begin{aligned}
b_{\mathbf{p}}^s(t) &= e^{iHt} b_{\mathbf{p}}^s e^{-iHt} \\
&= e^{-iE_{\mathbf{p}} t} b_{\mathbf{p}}^s.
\end{aligned} \tag{7.185}$$

Similarly by using

$$[H, b_{\mathbf{p}}^{s\dagger}] = E_{\mathbf{p}} b_{\mathbf{p}}^{s\dagger}, \tag{7.186}$$

the Heisenberg picture creation operator is:

$$b_p^{s\dagger}(t) = e^{+iE_p t} b_p^{s\dagger}. \quad (7.187)$$

Similar things hold for  $c$  operators. Then the Heisenberg field operator is given by,

$$\begin{aligned} \psi(x) = \psi(\mathbf{x}, t) &= \int dV_p (u^s(\mathbf{p}) b_p^s e^{-i(E_p - \mathbf{p} \cdot \mathbf{x})} + v^s(\mathbf{p}) c_p^{s\dagger} e^{+i(E_p t - \mathbf{p} \cdot \mathbf{x})}) \quad (7.188) \\ &= \int dV_p (u^s(\mathbf{p}) b_p^s e^{-ip \cdot x} + v^s(\mathbf{p}) c_p^{s\dagger} e^{+ip \cdot x}) \\ \psi^\dagger(x) = \psi^\dagger(\mathbf{x}, t) &= \int dV_p (u^{\dagger s}(\mathbf{p}) b_p^{s\dagger} e^{+ip \cdot x} + v^{\dagger s}(\mathbf{p}) c_p^s e^{-ip \cdot x}). \end{aligned}$$

Indeed they obey Heisenberg equation, by using the commutator of the Hamiltonian and the annihilation and creation operator.

$$\frac{\partial \psi(x)}{\partial t} = i[H, \psi(x)]. \quad (7.189)$$

To prove it, we do simple calculations,

$$\frac{\partial \psi(x)}{\partial t} = \int dV_p (-iE_p u^s(\mathbf{p}) b_p^s e^{-ip \cdot x} + iE_p v^s(\mathbf{p}) c_p^{s\dagger} e^{+ip \cdot x}) \quad (7.190)$$

$$i[H, \psi(x)] = i \int dV_p (-E_p u^s(\mathbf{p}) b_p^s e^{-ip \cdot x} + E_p v^s(\mathbf{p}) c_p^{s\dagger} e^{+ip \cdot x}) = \frac{\partial \psi(x)}{\partial t}. \quad (7.191)$$

### 7.6.4 Causality

Recall that in the bosonic case, the amplitude for a particle to propagate from  $y$  to  $x$  is given by

$$D(x - y) = \int dV_p e^{-ip \cdot (x - y)} \quad (7.192)$$

Causality requires that the commutator of two local observables to vanish at space-like separations

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \text{ for spacelike.} \quad (7.193)$$

This is simple in scalar field theory, which satisfies

$$[\varphi(x), \varphi(y)] = 0 \text{ for spacelike.} \quad (7.194)$$

Let us show this,

$$\begin{aligned} \varphi_-(x) &= \int dV_p a_p e^{-ip \cdot x} \\ \varphi_+(x) &= \int dV_p a_p^\dagger e^{+ip \cdot x}. \end{aligned} \quad (7.195)$$

The commutator is given by

$$[\varphi(x), \varphi(y)] = [\varphi_-(x), \varphi_+(y)] + [\varphi_+(x), \varphi_-(y)] = D(x - y) - D(y - x). \quad (7.196)$$

Outside the lightcone, we can choose  $x^0 = y^0$ ,  $D(x - y) = \int dV_{\mathbf{p}} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} = \int dV_{\mathbf{p}} e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} = \int dV_{\mathbf{p}} e^{i\mathbf{p} \cdot (\mathbf{y} - \mathbf{x})} = \int dV_{\mathbf{p}} e^{-i\mathbf{p} \cdot (y - x)} = D(y - x)$ , where for the second equal sign, we perform the integral with  $-\mathbf{p}$  instead of  $\mathbf{p}$ .

This is not true for Dirac theory. Fermion fields do not commute outside the light cone.

But this is not as worrisome. All fermion observables come in pairs, odd number of fermions are not Lorentz invariant<sup>44</sup>. Fermionic operators are not observable: nobody has seen an apparatus that rotates  $2\pi$  and pick up a minus sign. Consider a generic operator of the following form

$$\mathcal{O} = \bar{\psi} \Gamma \psi, \quad (7.197)$$

where  $\Gamma$  is some combinations of  $\gamma$  matrices. What we demand is that

$$[\bar{\psi} \Gamma_1 \psi, \bar{\psi} \Gamma_2 \psi] = 0 \text{ for spacelike..} \quad (7.198)$$

We show this demand can be met as long as the ant-commutation relationship between fermions are as we described before. This is done in a footnote<sup>45</sup>. If object  $\{\psi_a, \bar{\psi}_b\}$  always vanish outside the light cone, causality will be saved.

## 7.7 The propagator

This motivates us to study this object  $iS(x - y) \equiv \{\psi(x), \bar{\psi}(y)\}$ , which is known as the fermion propagator. We often omit the spinor indices, but we should

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<sup>44</sup>In next chapter we will show this.

<sup>45</sup>Let us restore the spinor indices:

$$[\bar{\psi}_a(\Gamma_1)_{ab}\psi_b, \bar{\psi}_c(\Gamma_2)_{cd}\psi_d] = (\Gamma_1)_{ab}(\Gamma_2)_{cd}[\bar{\psi}_a\psi_b, \bar{\psi}_c\psi_d] = 0 \text{ for spacelike.} \quad (7.199)$$

We want causality to hold for all operators, so we need to look at the commutator  $[\bar{\psi}_a\psi_b, \bar{\psi}_c\psi_d]$  outside the light cone. Now we use the identities for commutators

$$\begin{aligned} [A, BC] &= [A, B]C + B[A, C] \\ [A, BC] &= \{A, B\}C - B\{A, C\}, \end{aligned} \quad (7.200)$$

to get

$$\begin{aligned} [\bar{\psi}_a\psi_b, \bar{\psi}_c\psi_d] &= [\bar{\psi}_a\psi_b, \bar{\psi}_c]\psi_d + \bar{\psi}_c[\bar{\psi}_a\psi_b, \psi_d] \\ &= (\bar{\psi}_a\{\psi_b, \bar{\psi}_c\} - \{\bar{\psi}_a, \bar{\psi}_c\}\psi_b)\psi_d + \bar{\psi}_c(\{\bar{\psi}_a, \psi_d\}\psi_b - \bar{\psi}_a\{\psi_b, \psi_d\}). \end{aligned} \quad (7.201)$$

We always have

$$\begin{aligned} \{\psi_a, \psi_b\} &= 0 \\ \{\bar{\psi}_a, \bar{\psi}_b\} &= 0, \end{aligned} \quad (7.202)$$

just because the “wrong” combinations of creation and annihilation operators are involved. Then  $[\bar{\psi}_a\psi_b, \bar{\psi}_c\psi_d]$  is simplified to be

$$[\bar{\psi}_a\psi_b, \bar{\psi}_c\psi_d] = \bar{\psi}_a\{\psi_b, \bar{\psi}_c\}\psi_d - \bar{\psi}_c\{\bar{\psi}_a, \psi_d\}\psi_b. \quad (7.203)$$

remember that the fermion propagator is a  $4 \times 4$  matrix: it tells us the amplitude for a certain component of a fermion at position  $x$  to propagate to position  $y$  with the same component or a different component. This makes sense from the analogy of complex scalar. We use Dirac conjugate so that the propagator is Lorentz covariant. So let us calculate the propagator by using our mode expansion

$$\begin{aligned}\psi(x) &= \int_s dV_{\mathbf{p}} (u^s(\mathbf{p}) b_{\mathbf{p}}^s e^{-ip \cdot x} + v^s(\mathbf{p}) c_{\mathbf{p}}^{s\dagger} e^{+ip \cdot x}) \\ \psi^\dagger(x) &= \int_s dV_{\mathbf{p}} (u^{\dagger s}(\mathbf{p}) b_{\mathbf{p}}^{s\dagger} e^{+ip \cdot x} + v^{\dagger s}(\mathbf{p}) c_{\mathbf{p}}^s e^{-ip \cdot x}).\end{aligned}\quad (7.204)$$

$$\begin{aligned}iS(x-y) &= \{\psi(x), \bar{\psi}(y)\} \\ &= \int dV_{\mathbf{p}} dV_{\mathbf{q}} (u^s(\mathbf{p}) \bar{u}^r(\mathbf{q}) \{b_{\mathbf{p}}^s, b_{\mathbf{q}}^{r\dagger}\} e^{-i(p \cdot x - q \cdot y)} + v^s(\mathbf{p}) \bar{v}^r(\mathbf{q}) \{c_{\mathbf{p}}^{s\dagger}, c_{\mathbf{q}}^r\} e^{+i(p \cdot x - q \cdot y)}) \\ &= \int dV_{\mathbf{p}} (u^s(\mathbf{p}) \bar{u}^s(\mathbf{p}) e^{-ip \cdot (x-y)} + v^s(\mathbf{p}) \bar{v}^r(\mathbf{p}) e^{+ip \cdot (x-y)}) \\ &= \int dV_{\mathbf{p}} ((\not{p} + m) e^{-ip \cdot (x-y)} + (\not{p} - m) e^{+ip \cdot (x-y)}) \\ &= \int dV_{\mathbf{p}} ((i\not{\partial}_x + m) e^{-ip \cdot (x-y)} + (-i\not{\partial}_x - m) e^{+ip \cdot (x-y)}) \\ &= (i\not{\partial}_x + m) \int dV_{\mathbf{p}} (e^{-ip \cdot (x-y)} - e^{+ip \cdot (x-y)}) \\ &= (i\not{\partial}_x + m) (D(x-y) - D(y-x)),\end{aligned}\quad (7.205)$$

where we used that in bosonic case, the amplitude for a particle to propagate from  $y$  to  $x$  is

$$D(x-y) = \int dV_{\mathbf{p}} e^{-ip \cdot (x-y)}. \quad (7.206)$$

For spacelike separated points, we have already seen in the bosonic case that  $D(x-y) - D(y-x) = 0$  everywhere. Thus the fermion propagator<sup>46</sup> also vanishes everywhere outside the light cone.

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<sup>46</sup>The propagator satisfies the following differential equation aside from singular points,

$$(i\not{\partial}_x - m) S(x-y) = 0. \quad (7.207)$$



## Chapter 8

# The interaction terms: possibilities and choices

We have already seen Parity as one of the discrete transformations emerge from the full Poincare group.

But is Parity a symmetry of the nature? What does our universe choose?

In this chapter we will write down all possible interaction terms (the fermionic parts), classify them with parity and point out a few ones clearly chosen by nature.

Before doing all that, we want to know how parity acts on spinors. And it turns out it is easier to study how parity acts on "half" of a Dirac spinor, the Weyl spinors.

This chapter is organized as follows,

1. What about parity (in class activity)?
2. List of all possible fermionic interactions. What has nature chosen?

### 8.1 Discrete symmetries and bilinears classification

Weyl spinors are simpler and we can use them to build other spinors. So it is worth exploring a little more. We have already seen one way to relate the Weyl spinors of different chirality: using the c-conjugation.  $\psi^{(c)} = C\psi^*$  is of a different chirality from  $\psi$ . Now we are going to see another way to relate them. Use discrete symmetry! So far we have steer clear from the discrete symmetries, as they are not continuously connected to identity, so our favorite method of expansion with small parameters fails miserably. But the importance of them cannot be underestimated: they certainly lead to the understanding of the particle physics, and ultimately the standard model.

And surprise, surprise, it is all related to our beloved Lorentz group. The rotations and boosts make up the Lorentz transformations that are continuously connected to the identity, which we study by looking at the infinitesimal form. However, in Lorentz group, there are also two discrete symmetries that are part of Lorentz group, time reversal and parity

$$\begin{aligned} \text{Time Reversal } T : x^0 &\rightarrow -x^0, x^i \rightarrow x^i \\ \text{Parity } P : x^0 &\rightarrow x^0, x^i \rightarrow -x^i. \end{aligned} \quad (8.1)$$

We will leave the readers who are interested with time reversal to read in Peskin ch 3 and A Zee ch 11. Parity, on the other hand, has a very important role to play in standard model – you might have heard the saying that the standard model is a chiral theory – and is related to Nobel prizes<sup>1</sup>, so we will discuss with care here.

First, notice that in three spacial dimensions, rotating around  $z$  axis by 180 degrees is equivalent to flipping the signs of  $x$  and  $y$  axis. Then parity which flip the signs of all spacial dimensions, is equivalent to flipping only one axis plus rotation. Flipping one axis is also known as reflection through a mirror.

How would a Dirac spinor transform under parity? Maybe it is time to make use of our building block, the Weyl spinor. Both Weyl spinor transform the same way under rotation. Now consider the right-handed Weyl spinor we considered that has a definite helicity in a mirror. There are two cases:

- The momentum could be pointing into (or out of) the mirror.
- The momentum could be parallel to the mirror.

In the first case, the momentum is changed, while the spin is not. In the second case, the momentum is the same, but the spin changes direction. Either way: the mirror changed a right-handed spinor into a left-handed spinor and vice versa.

$$P : \psi_{\pm} \rightarrow \psi_{\mp}. \quad (8.2)$$

On the other hand, we must have  $P^2 = 1$ , then this suggest that the Dirac spinor transforms in the following way<sup>2</sup>

$$P\psi(t, \mathbf{x})P^{-1} = \beta\psi(t, -\mathbf{x}). \quad (8.3)$$

Now we will check if the new spinor transformed under parity will also satisfy Dirac equation.

$$\begin{aligned} (i\partial_{\mu'}\gamma^{\mu} - m)\psi(t, -\mathbf{x}) &= (i\partial_0\gamma^0 + i\partial_i\gamma^i - m)\beta P\psi(t, \mathbf{x})P^{-1} \\ &= (i\partial_0\gamma^0 - i\partial_i\gamma^i - m)\gamma^0 P\psi(t, \mathbf{x})P^{-1} \\ &= \gamma^0(i\partial_0\gamma^0 + i\partial_i\gamma^i - m)P\psi(t, \mathbf{x})P^{-1} \\ &= \beta(i\cancel{\partial} - m)P\psi(t, \mathbf{x})P^{-1} = 0, \end{aligned} \quad (8.4)$$

<sup>1</sup>Time reversal is also related to Nobel prizes. We will leave it for the standard model to explain those.

<sup>2</sup>See an explicit calculation to derive this result in Peskin and Schroeder 3.6 parity.

where we use  $\partial_{\mu'}$  to denote taking derivative respect to  $x^{\mu'} = (x^0, -x^i)$  and we used the fact the Dirac operator is parity even: both derivative and  $\gamma$  will flip under parity but together they will not.

Now it is time to explore other ways to write down spinor bilinears and classify them with our new symmetry, parity. This is important thing to do as this gives guidelines to write down Lagrangian terms for fermions. We have already seen two of them, so we can just keep the list going

$$\begin{aligned} & \bar{\psi}\psi \\ & \bar{\psi}\gamma^\mu\psi \\ & \bar{\psi}\gamma^\mu\gamma^\nu\psi \\ & \bar{\psi}\gamma^\mu\gamma^\nu\gamma^\rho\psi \\ & \bar{\psi}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\psi. \end{aligned} \tag{8.5}$$

The list terminates at 4  $\gamma$  matrices, as they all square to 1 or  $-1$ , so we can have at most one of each. Using the same logic, we can actually rewrite the above list. The last one which has  $\gamma$  matrix each appearing once is precisely equivalent to some multiple of  $\gamma^5$ . The one that have three  $\gamma$  matrix can use Levi-Civita symbol in 4-dimension to contract to give  $\gamma^5\gamma^{\mu 3}$ . As for  $\gamma^\mu\gamma^\nu$ , only the commutator matters, as the anti-commutator gives  $\eta^{\mu\nu}$ . The simple list of all bilinears are given by

$$\begin{aligned} & \bar{\psi}\psi \\ & \bar{\psi}\gamma^5\psi \\ & \bar{\psi}\gamma^\mu\psi \\ & \bar{\psi}\gamma^\mu\gamma^5\psi \\ & \bar{\psi}\mathcal{J}^{\mu\nu}\psi. \end{aligned} \tag{8.6}$$

There are  $1 + 1 + 4 + 4 + \frac{4 \times 3}{2} = 16 = 4 \times 4$  of them, which is expected from making bilinears out of two 4-component spinors.

Now we can study how our bilinears transform under Parity in a footnote<sup>4</sup>

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<sup>3</sup>Consider  $\frac{i}{6}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho$ , it is clearly something has one free index. But what is it? First take this free index to be 0, and multiply with  $(\gamma^0)^2$  which is identity, now take the first  $\gamma^0$  and combine with  $\gamma^\mu\gamma^\nu\gamma^\rho$ , because the  $\epsilon$  tensor, they have to choose permutation of 123 and hence there will be six of them. Take one permutation  $\gamma^1\gamma^2\gamma^3$  and realize it will combine with  $\gamma^0$  to give  $\gamma^5$ . Together with the other  $\gamma^0$  left from the square we inserted, this gives  $\gamma^5\gamma^0$ . You can check all the rest of the terms worked out, they will give  $\gamma^5\gamma^1$  etc. We thank Wirawat Kokaew for the question.

<sup>4</sup>

$$\begin{aligned} P\bar{\psi}(t, \mathbf{x})P &= P\psi^\dagger(t, \mathbf{x})P\gamma^0 \\ &= (P\psi(t, \mathbf{x})P)^\dagger\gamma^0 \\ &= \psi^\dagger(t, -\mathbf{x})\beta\gamma^0 \\ &= \bar{\psi}(t, -\mathbf{x})\gamma^0 \end{aligned} \tag{8.7}$$

To summarize

$\bar{\psi}\psi$	scalar	mass term, Yukawa
$\bar{\psi}\gamma^5\psi$	pseudoscalar	effective theory of nuclear interaction
$\bar{\psi}\gamma^\mu\psi$	vector	QED and QCD
$\bar{\psi}\gamma^\mu\gamma^5\psi$	axial vector	weak
$\bar{\psi}\mathcal{J}^{\mu\nu}\psi$	tensor	consist with none, will give electric dipole

(8.12)

In nature, parity is broken by observation. We need a chiral theory: in which the two Weyl spinors behave differently. Thus the weak theory of the standard model uses axial vector type of interaction.

## 8.2 The continuous phase symmetry that leads to QED

Review of Noether theorem: Recall that for infinitesimal transformation of field

$$\delta\psi = Y(\psi). \quad (8.13)$$

If we have change of Lagrangian

$$\delta\mathcal{L} = \partial_\mu F^\mu(\psi). \quad (8.14)$$

where we used  $P = P^{-1}$ . For scalar, we have

$$\begin{aligned} P\bar{\psi}(t, \mathbf{x})\psi(t, \mathbf{x})P &= \bar{\psi}(t, -\mathbf{x})\gamma^0\beta\psi(t, -\mathbf{x}) \\ &= \bar{\psi}(t, -\mathbf{x})\psi(t, -\mathbf{x}) \end{aligned} \quad (8.8)$$

Similarly For the vector, we have

$$\begin{aligned} P\bar{\psi}(t, \mathbf{x})\gamma^\mu\psi(t, \mathbf{x})P &= \bar{\psi}(t, -\mathbf{x})\gamma^0\gamma^\mu\beta\psi(t, -\mathbf{x}) \\ &= \begin{cases} \bar{\psi}(t, -\mathbf{x})\gamma^0\psi(t, -\mathbf{x}) & \text{time component} \\ -\bar{\psi}(t, -\mathbf{x})\gamma^i\psi(t, -\mathbf{x}) & \text{space component} \end{cases} \end{aligned} \quad (8.9)$$

For time component, it does not change, for spacial components, it picks up a minus sign, this is preciously how vector will transform. For the pseudoscalar, we have

$$\begin{aligned} P\bar{\psi}(t, \mathbf{x})\gamma^5\psi(t, \mathbf{x})P &= \bar{\psi}(t, -\mathbf{x})\gamma^0\gamma^5\beta\psi(t, -\mathbf{x}) \\ &= -\bar{\psi}(t, -\mathbf{x})\gamma^5\psi(t, -\mathbf{x}). \end{aligned} \quad (8.10)$$

And for axial vector, we have

$$\begin{aligned} P\bar{\psi}(t, \mathbf{x})\gamma^\mu\gamma^5\psi(t, \mathbf{x})P &= \bar{\psi}(t, -\mathbf{x})\gamma^0\gamma^\mu\gamma^5\beta\psi(t, -\mathbf{x}) \\ &= \begin{cases} -\bar{\psi}(t, -\mathbf{x})\gamma^0\gamma^5\psi(t, -\mathbf{x}) & \text{time component} \\ +\bar{\psi}(t, -\mathbf{x})\gamma^i\gamma^5\psi(t, -\mathbf{x}) & \text{space component} \end{cases} \end{aligned} \quad (8.11)$$

This behaves the opposite of a vector, the time component picks up a minus sign, while the spacial component does not change. It can be checked that  $\bar{\psi}\mathcal{J}^{\mu\nu}\psi$  transforms as a suitable tensor.

The current is given by

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} Y(\psi) - F^\mu(\psi). \quad (8.15)$$

Not surprisingly, the Dirac theory is invariant under spacetime translation and is Lorentz invariant. Interested readers are referred to David Tong's notes ch4.6 to read upon the corresponding conserved current: the energy-momentum tensor, and the current which gives angular momentum and boosts.

What we will study here is a global symmetry that you have seen for the complex scalar: the phase symmetry. Just like the complex scalar, Dirac Lagrangian is also invariant under rotating the phase of the spinor  $\psi' = e^{-i\alpha}\psi$ . Applying Noether's theorem, we have

$$\begin{aligned} \delta\psi &= -i\alpha\psi \\ \delta\bar{\psi} &= i\alpha\bar{\psi}. \end{aligned} \quad (8.16)$$

The Lagrangian is invariant.  $F^\mu = 0$ . Then we have the conserved current

$$J^\mu = i\bar{\psi}\gamma^\mu(-i)\psi = \bar{\psi}\gamma^\mu\psi. \quad (8.17)$$

The conserved charge is

$$N_f = \int d^3x \bar{\psi}\gamma^0\psi = \int d^3x \psi^\dagger\psi. \quad (8.18)$$

This can be interpreted as the fermion number. Note that the density is positive definite. Dirac's original goal was achieved.

### 8.3 A case study: QED amuse-bouche

We can start to write down interaction term. For reasons that will be clear in QFT II course, we are mostly interested in Lagrangian terms that have a mass dimensions equal or smaller than four in four dimensions. Since Dirac field has dimension  $\frac{3}{2}$ , and it has to come in pairs because otherwise Lorentz symmetry is broken, so we could either couple Dirac field to a scalar or a vector. The interaction term we can write down could either be  $\varphi\bar{\psi}\psi$ , which is known as the Yukawa theory<sup>5</sup>, which standard model use it to give mass to fermions.<sup>6</sup> The other one we can write down is  $\bar{\psi}\gamma^\mu A_\mu\psi$ . But we should be cautious: a photon field always have a gauge symmetry, and if  $A_\mu \rightarrow A_\mu + \partial_\mu\alpha$  is all we have, the term we wrote down is not going to be gauge invariant. The fermion field has to transform. Luckily we just talked about the fermion field has a phase symmetry, we can try to employ that and gauge that global symmetry,

<sup>5</sup>We will study Yukawa theory very carefully in a few lectures to derive Feynman rules for fermions.

<sup>6</sup>We have a fermion mass term, unfortunately that does not obey the full gauge symmetry of the standard model.

we propose  $\psi \rightarrow e^{-iQ\alpha(x)}\psi$ , now the parameter depends on  $x$ . The mass term is still invariant, but because of the derivative acting on  $\alpha(x)$ , the kinetic term is no longer invariant. We compute the change in the kinetic term and interaction term

$$\delta\mathcal{L}_{\text{kinetic}} = e^{iQ\alpha}\bar{\psi}i\gamma^\mu(-iQ\partial_\mu\alpha)e^{-iQ\alpha}\psi = (Q\partial_\mu\alpha)\bar{\psi}\gamma^\mu\psi, \quad (8.19)$$

and

$$\delta\mathcal{L}_{\text{interaction}} = e^{iQ\alpha}\bar{\psi}\gamma^\mu(\partial_\mu\alpha)e^{-iQ\alpha}\psi = (\partial_\mu\alpha)\bar{\psi}\gamma^\mu\psi. \quad (8.20)$$

And we realize they cancel only if the interaction term comes with a coefficient of  $-Q$ . The Lagrangian looks like

$$\begin{aligned} \mathcal{L}_{\text{QED}} &= \bar{\psi}(i\gamma^\mu\partial_\mu - Q\gamma^\mu A_\mu - m)\psi \\ &= \bar{\psi}(i\gamma^\mu(\partial_\mu + iQA_\mu) - m)\psi \end{aligned} \quad (8.21)$$

Now we defined the covariant derivative

$$D_\mu \equiv \partial_\mu + iQA_\mu. \quad (8.22)$$

In terms of the covariant derivative,

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(iD\!\!\!/ - m)\psi. \quad (8.23)$$

And the equation of motion for the Dirac spinor is simply

$$(iD\!\!\!/ - m)\psi = 0. \quad (8.24)$$

The equation of motion for EM field, is the expected Maxwell equation

$$\partial_\mu F^{\mu\nu} = -J^\nu. \quad (8.25)$$

where  $J^\nu$  is the current

$$J^\mu = Q\bar{\psi}\gamma^\mu\psi, \quad (8.26)$$

which is proportional to the fermion number current, with the proportional coefficient equal to the charge of the fermion.

Now we can understand the c-conjugation we introduced in a new level. Let us take a look at the Dirac equation of motion including QED interaction

$$(iD\!\!\!/ - m)\psi = 0, \quad (8.27)$$

and again take a complex conjugate of it

$$\begin{aligned} (-i\partial_\mu(\gamma^\mu)^* - QA_\mu(\gamma^\mu)^* - m)\psi^* &= 0 \\ C^{-1}(iD\!\!\!/ + QA_\mu\gamma^\mu - m)\psi^{(c)} &= 0, \end{aligned} \quad (8.28)$$

where we use

$$(\gamma^\mu)^* = -C^{-1}\gamma^\mu C. \quad (8.29)$$

The interaction term changed sign: it is telling us  $\psi^{(c)}$  should be considered to be a field that has opposite charge of  $\psi$ . That is why we usually call  $\psi^{(c)}$  the charge conjugation spinor, as it couples to the EM field with an opposite charge. Now we also have a new level of understanding Majorana spinor: it is a particle that is its own anti-particle

$$\psi^{(c)} = \psi. \quad (8.30)$$

The trick of simply replacing partial derivative with covariant derivative works for other cases as well. It is known as the minimal coupling between gauge theory and fermions. Standard model has only minimal couplings.



# Chapter 9

## Generic Interactions

Yeah, interaction theory!

### 9.1 Review of the process

We are interested in  $2 \rightarrow 2$  scattering processes, and we want to compute,

$$\langle f|S|i\rangle_H = \langle f; t = \infty | i; t = -\infty \rangle_S. \quad (9.1)$$

Let us first review how things are done in the scalar case:

- Step 1: we choose the initial and final states to be

$$\begin{aligned} S|i\rangle_H &= a_1^\dagger(-\infty) a_2^\dagger(-\infty) |\Omega\rangle \\ |f\rangle_H &= a_3^\dagger(+\infty) a_4^\dagger(+\infty) |\Omega\rangle, \end{aligned} \quad (9.2)$$

where  $a_1^\dagger(t) \equiv a_{\mathbf{k}_1}^\dagger(t)$ . So we have

$$\langle f|S|i\rangle_H = \langle \Omega | T a_4(+\infty) a_3(+\infty) a_1^\dagger(-\infty) a_2^\dagger(-\infty) | \Omega \rangle. \quad (9.3)$$

- Step 2: Now we invoke the following trick

$$a_4(+\infty) = a_4(-\infty) + a_4(+\infty) - a_4(-\infty) = a_4(-\infty) + I_4, \quad (9.4)$$

and similarly we can define  $I_3$ ,  $I_1^\dagger$  and  $I_2^\dagger$ , the time ordering will bring the annihilation operator to the right and the creation operator to the left, so all terms except the following vanish

$$\langle f|S|i\rangle_H = \langle \Omega | T I_4 I_3 I_1^\dagger I_2^\dagger | \Omega \rangle. \quad (9.5)$$

In the scalar theory, we worked out

$$I_1^\dagger = -i \int d^4x e^{-ik_1 \cdot x} (\partial^2 + m^2) \varphi(x). \quad (9.6)$$

Note the appearance of the Klein-Gordon operator, since the theory is interacting,  $(\partial^2 + m^2)\varphi(x)$  is no longer zero. Hence we arrive at the LSZ reduction formula, and relate  $\langle f|S|i\rangle_H$  to compute vacuum expectation values of time-ordered products of fields<sup>1</sup>, also known as Green's functions

$$\langle f|S|i\rangle_H = i^{2+2} \int \left( \prod_{j=1}^{2+2} d^4 x_j e^{-i\lambda_j k_j \cdot x_j} (\partial_j^2 + m^2) \right) \langle \Omega | T\varphi_1 \varphi_2 \varphi_3 \varphi_4 | \Omega \rangle, \quad (9.7)$$

where  $\lambda_j$ 's are 1 for initial states and  $-1$  for final states.

- Step 3: By going to the interacting picture, we use Dyson's formula, to relate interacting Green's functions to free Green's functions

$$\langle \Omega | T\varphi(x_1)\varphi(x_2)\dots\varphi(x_n) | \Omega \rangle = \frac{\langle 0 | T\varphi_0(x_1)\varphi_0(x_2)\dots\varphi_0(x_n) \exp(i \int d^4x \mathcal{L}_{int}(\varphi_0)) | 0 \rangle}{\langle 0 | T \exp(i \int d^4x \mathcal{L}_{int}(\varphi_0)) | 0 \rangle}. \quad (9.8)$$

- Step 4: We use Wick's theorem to compute free Green's functions from propagators

$$\begin{aligned} T\varphi_1\dots\varphi_n &= : \varphi_1\dots\varphi_n : + : \text{all possible contractions} : \\ \overline{\varphi(x)}\varphi(y) &= \Delta_F(x-y). \end{aligned} \quad (9.9)$$

## 9.2 Fermionic modification

We are going to study Yukawa theory, a theory describing interaction between scalars and fermions. But before then, we don't need the specific interaction term yet. Today like the scalar case, we will study the process of two fermions with momentum  $\mathbf{k}_1$  polarization  $s_1$  and  $\mathbf{k}_2, s_2$  scattering to two fermions with momentum  $\mathbf{k}_3, s_3$  and  $\mathbf{k}_4, s_4$ . We will study the fermionic LSZ reduction formula and the fermionic Wick's theorem.

The generalization to  $n$  to  $n'$  kind of process should be straight forward. The generalization to processes like from one scalar and one fermion to one scalar and one fermion should also be straight forward.

### 9.2.1 Initial and final states

We swap out the bosonic creation operator for the fermionic ones. Since we only have fermions, we only involve type  $b$  operators, we use a similar short handed notation,  $b_1^\dagger(t) \equiv (b_{\mathbf{k}_1}^{s_1})^\dagger(t)$ .

$$\begin{aligned} S|i\rangle_{f,H} &= b_1^\dagger(-\infty) b_2^\dagger(-\infty) |\Omega\rangle \\ |f\rangle_{f,H} &= b_3^\dagger(+\infty) b_4^\dagger(+\infty) |\Omega\rangle \end{aligned} \quad (9.10)$$

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<sup>1</sup>where we used the shorthanded notation  $\varphi_1 \equiv \varphi(x_1)$ .

### 9.2.2 Fermionic LSZ reduction formula

If we look into the derivation of LSZ reduction formula, it seems the most important thing is to figure out what are these analogous  $I_1$ 's for fermions. As soon as we find

$$I_{f,1} \equiv b_1(+\infty) - b_1(-\infty), \quad (9.11)$$

we can just plug in

$$\langle f|S|i\rangle_{f,H} = \langle \Omega|TI_{f,4}I_{f,3}I_{f,1}^\dagger I_{f,2}^\dagger|\Omega\rangle, \quad (9.12)$$

and we will arrive at the fermionic LSZ reduction formula.

The following formula is so important that it is left as an exciting exercise.

$$I_{f,1} = (-i) \int d^4x e^{ik_1 \cdot x} \bar{u}_1(i\cancel{\partial} - m)\psi(x). \quad (9.13)$$

Note the appearance of the Dirac operator in  $I_{f,1}$ : we know we are on the right track.

Now let us take complex conjugate of  $I_{f,1}$  carefully,

$$\begin{aligned} I_{f,1}^\dagger &= i \int d^4x e^{-ik_1 \cdot x} (-i\partial_\mu \psi^\dagger (\gamma^\mu)^\dagger - m\psi^\dagger) \gamma^0 u_1 \\ &= i \int d^4x e^{-ik_1 \cdot x} (-i\partial_\mu \bar{\psi} \gamma^\mu - m\bar{\psi}) u_1 \\ &= -i \int d^4x e^{-ik_1 \cdot x} (i\partial_\mu \bar{\psi} \gamma^\mu + m\bar{\psi}) u_1 \\ &\equiv -i \int d^4x \bar{\psi} (i\cancel{\partial} + m) u_1 e^{-ik_1 \cdot x} \end{aligned} \quad (9.14)$$

where the operator is the Dirac operator for  $\bar{\psi}$ .

Now let us put everything together and assemble the LSZ reduction formula,

$$\langle f|S|i\rangle_H = i^{2+2} \int \left( \prod_{j=3}^4 d^4x_j e^{ik_j \cdot x_j} \bar{u}_j(i\cancel{\partial}_j - m) \right) \langle \Omega|T\psi_4\psi_3\bar{\psi}_1\bar{\psi}_2|\Omega\rangle \left( \prod_{j=1}^2 d^4x_j (i\cancel{\partial}_j + m) u_j e^{-ik_j \cdot x_j} \right).$$

$$(9.15)$$

### 9.2.3 Dyson's formula

To relate interacting Green's function to free Green's function, we use equal numbers of  $\psi$  and  $\bar{\psi}$  and relate the correlation functions the same way

$$\langle \Omega|T\psi(x_1) \dots \psi(x_n) \bar{\psi}(x_1) \dots \bar{\psi}(x_n)|\Omega\rangle = \frac{\langle 0|T\psi_0(x_1) \dots \psi_0(x_n) \bar{\psi}_0(x_1) \dots \bar{\psi}_0(x_n) \exp(i \int d^4x \mathcal{L}_{int}(\psi_0, \bar{\psi}_0))|0\rangle}{\langle 0|T\exp(i \int d^4x \mathcal{L}_{int}(\psi_0, \bar{\psi}_0))|0\rangle}. \quad (9.16)$$

### 9.2.4 Wick's theorem

We need Wick's theorem to compute correlation functions. As we have already seen in the scalar theory, Wick's theorem will allow us to compute objects like  $\langle 0|T\psi_0(x_1)\dots\psi_0(x_n)\bar{\psi}_0(x_1)\dots\bar{\psi}_0(x_n)\varphi_0(z_1)\dots\varphi_0(z_m)|0\rangle$ . As we will see, we should have matching number of  $\psi$  and  $\bar{\psi}$ 's. Analogous to the scalar's case, the Wick's theorem for one pair of  $\psi$  and  $\bar{\psi}$  should look something like

$$T[\psi(x)\bar{\psi}(y)] =: \psi(x)\bar{\psi}(y) : + \overline{\psi(x)\bar{\psi}(y)}. \quad (9.17)$$

And of course, the important question will be how is the contraction defined? We use the same trick. We separate the fermion field into a part that contains the anti-particle creation operator  $\psi_+$  and a part contains the particle annihilation operator  $\psi_-$ , and similarly the Dirac conjugate of the fermion field is separated to a part that contains the particle creation operator  $\bar{\psi}_+$  and a part that contains the anti-particle annihilation operator  $\bar{\psi}_-$ . Normal ordering says that we put the annihilation operator to the right and if we switch fermionic operators, we pick up a minus sign.

First we should define time ordering. In the scalar case, it is defined in a very intuitive way,

$$T(\varphi(x)\varphi(y)) = \begin{cases} \varphi(x)\varphi(y), & \text{if } x^0 > y^0 \\ \varphi(y)\varphi(x), & \text{if } y^0 > x^0 \end{cases} \quad (9.18)$$

The fermion fields anti-commute,

$$\psi_a(x)\psi_b(y) = -\psi_b(y)\psi_a(x). \quad (9.19)$$

So if we adopt the same definition of time ordering that we had for scalars, assuming  $x^0 > y^0$

$$\begin{aligned} T(\psi_a(x)\psi_b(y)) &= -T(\psi_b(y)\psi_a(x)) \\ \psi_a(x)\psi_b(y) &= -\psi_a(x)\psi_b(y) \\ \psi_a(x)\psi_b(y) &= 0. \end{aligned} \quad (9.20)$$

Well, that certainly cannot be always true. We should have defined our time ordering differently, to include a minus sign. So for fermions, we should have defined

$$T(\psi(x)\bar{\psi}(y)) \equiv \begin{cases} \psi(x)\bar{\psi}(y), & \text{if } x^0 > y^0 \\ -\bar{\psi}(y)\psi(x), & \text{if } y^0 > x^0 \end{cases}. \quad (9.21)$$

The above time-ordering definition applies if any  $\psi$  is replaced by  $\bar{\psi}$  and vice versa. Now Take the case  $x^0 > y^0$ ,

$$\begin{aligned} T\psi(x)\bar{\psi}(y) &= \psi(x)\bar{\psi}(y) \\ &= (\psi(x)_+ + \psi(x)_-)(\bar{\psi}(y)_+ + \bar{\psi}(y)_-) \\ &= \psi(x)_+\bar{\psi}(y)_+ + \psi(x)_+\bar{\psi}(y)_- + \psi(x)_-\bar{\psi}(y)_+ + \psi(x)_-\bar{\psi}(y)_- \\ &= \psi(x)_+\bar{\psi}(y)_+ + \psi(x)_+\bar{\psi}(y)_- + \{\psi(x)_-, \bar{\psi}(y)_+\} - \bar{\psi}(y)_+\psi(x)_- + \psi(x)_-\bar{\psi}(y)_- \\ &=: \psi(x)\bar{\psi}(y) : + \{\psi(x)_-, \bar{\psi}(y)_+\}. \end{aligned} \quad (9.22)$$

It is tricky with minus signs of fermions, so let us do it carefully with  $y^0 > x^0$ ,

$$\begin{aligned} T\psi(x)\bar{\psi}(y) &= -\bar{\psi}(y)\psi(x) \\ &= -(\bar{\psi}(y)_+ + \bar{\psi}(y)_-)(\psi(x)_+ + \psi(x)_-) \\ &= -(:\bar{\psi}(y)\psi(x): + \{\bar{\psi}(y)_-, \psi(x)_+\}) \\ &=: \psi(x)\bar{\psi}(y) : - \{\bar{\psi}(y)_-, \psi(x)_+\}. \end{aligned} \quad (9.23)$$

This implies we have

$$\overline{\psi(x)\bar{\psi}(y)} = \begin{cases} \{\psi(x)_-, \bar{\psi}(y)_+\} & \text{if } x^0 > y^0 \\ -\{\bar{\psi}(y)_-, \psi(x)_+\} & \text{if } y^0 > x^0 \end{cases}. \quad (9.24)$$

None of the above calculation used the fact one of them is  $\psi$  and the other is  $\bar{\psi}$ , so the contraction definition certainly works for two  $\psi$  fields or two  $\bar{\psi}$  fields, but since we know that the anticommutator of two  $\psi$  fields or two  $\bar{\psi}$  fields vanish because they have the “wrong” type of annihilation/creation operators, we immediately have,

$$\overline{\psi(x)\bar{\psi}(y)} = \overline{\bar{\psi}(x)\bar{\psi}(y)} = 0. \quad (9.25)$$

For the one that contains both  $\psi$  and  $\bar{\psi}$ , we will need to look at the precise definition of these  $\psi_+$  things, from the mode expansion

$$\begin{aligned} \psi(x) &= \oint dV_{\mathbf{p}} (u^s(\mathbf{p})b_{\mathbf{p}}^s e^{-ip\cdot x} + v^s(\mathbf{p})c_{\mathbf{p}}^{s\dagger} e^{+ip\cdot x}) \\ \psi^{\dagger}(x) &= \oint dV_{\mathbf{p}} (u^{\dagger s}(\mathbf{p})b_{\mathbf{p}}^{s\dagger} e^{+ip\cdot x} + v^{\dagger s}(\mathbf{p})c_{\mathbf{p}}^s e^{-ip\cdot x}), \end{aligned} \quad (9.26)$$

we have

$$\begin{aligned} \psi(x)_+ &= \int dV_{\mathbf{p}} v^s(\mathbf{p})c_{\mathbf{p}}^{s\dagger} e^{+ip\cdot x} \\ \psi(x)_- &= \int dV_{\mathbf{p}} u^s(\mathbf{p})b_{\mathbf{p}}^s e^{-ip\cdot x} \\ \bar{\psi}(x)_+ &= \int dV_{\mathbf{p}} \bar{u}^s(\mathbf{p})b_{\mathbf{p}}^{s\dagger} e^{+ip\cdot x} \\ \bar{\psi}(x)_- &= \int dV_{\mathbf{p}} \bar{v}^s(\mathbf{p})c_{\mathbf{p}}^s e^{-ip\cdot x}. \end{aligned} \quad (9.27)$$

Now we can evaluate the anti-commutator in the contraction. We have done this type of calculation a few times now, the result is

$$\begin{aligned} \{\psi(x)_-, \bar{\psi}(y)_+\} &= \int dV_{\mathbf{p}} (\not{p} + m) e^{-ip\cdot(x-y)} \\ -\{\bar{\psi}(y)_-, \psi(x)_+\} &= - \int dV_{\mathbf{p}} (\not{p} - m) e^{-ip\cdot(y-x)}. \end{aligned} \quad (9.28)$$

We can combine them neatly using the step function. This is the Feynman propagator.

$$\begin{aligned} \overline{\psi(x)\bar{\psi}(y)} &= \Theta(x^0 - y^0) \{\psi(x)_-, \bar{\psi}(y)_+\} + \Theta(y^0 - x^0) (-\{\bar{\psi}(y)_-, \psi(x)_+\}) \\ &\equiv S_F(x - y) \equiv \langle 0 | T(\psi(x)\bar{\psi}(y)) | 0 \rangle, \end{aligned} \quad (9.29)$$

where in the last step we use the fact that it is a  $c$ -function and it does not contain creation or annihilation operators. It is equivalent to the vacuum expectation value of itself. Notice that the Feynman propagator is a matrix, it describes a fermion at  $y$  with spinor index  $l'$  propagating to a fermion at  $x$  with spinor index  $l$

$$S_F(x - y)_{ll'} \equiv \langle 0 | T(\psi_l(x)\bar{\psi}_{l'}(y)) | 0 \rangle. \quad (9.30)$$

So the Feynman propagator is

$$S_F(x - y) = (i\cancel{\partial}_x + m) \left( \Theta(x^0 - y^0) \int dV_p e^{-ip \cdot (x-y)} + \Theta(y^0 - x^0) \int dV_p e^{-ip \cdot (y-x)} \right). \quad (9.31)$$

And we recognize the expression in the parenthesis is nothing but the bosonic Feynman propagator  $\Delta_F(x - y)$ , recall that

$$\Delta_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}. \quad (9.32)$$

The fermionic Feynman propagator<sup>2</sup> is then given by

$$S_F(x - y) = (i\cancel{\partial}_x + m)\Delta_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\cancel{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}. \quad (9.33)$$

Sometimes it is worrisome to use normal ordering because in its definition infinities are thrown away. In the case of Wick theorem, it is actually ok. We have done a brute force calculation to show this in appendix J.

### 9.3 Propagator as a Green's function

Since the bosonic Feynman propagator is a green function of the Klein-Gordon operator

$$(\partial^2 + m^2)\Delta_F(x - y) = -i\delta(x - y), \quad (9.34)$$

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<sup>2</sup>You might worry about anti-fermion propagator, but you can always use the same fermion propagator without worrying if it is a fermion or anti-fermion. The idea is that the anti-fermion propagator is given by  $\langle 0 | T\bar{\psi}(x)\psi(y) | 0 \rangle$ , which is nothing but the fermion propagator with  $x$  and  $y$  switched, then in momentum space, it means the momentum is flipped. But an anti-fermion's momentum is exactly the opposite of fermion, so then anti-fermion propagates at momentum  $-p^\mu$  is nothing but a fermion propagate at momentum  $p^\mu$  and we do not need to deal with separately.

we conclude Feynman propagator is a green function of the Dirac operator<sup>3</sup>.

$$(i\cancel{\partial}_x - m)S_F(x - y) = (i\cancel{\partial}_x - m)(i\cancel{\partial}_x + m)\Delta_F(x - y) = -(\partial_x^2 + m^2)\Delta_F(x - y)$$

$$(9.37)$$

$(i\cancel{\partial}_x - m)S_F(x - y) = i\delta(x - y).$

The momentum space Feynman propagator is

$$S_F(p) = \frac{i(\cancel{p} + m)}{p^2 - m^2} \equiv \frac{i}{\cancel{p} - m}. \quad (9.38)$$

Note that the last expression  $\frac{i}{\cancel{p} - m}$  should be only understood as a short-handed notation: it does not make much sense and cannot be used for calculations.

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<sup>3</sup>We can also show that the Feynman propagator is a green function of the Dirac operator for  $\bar{\psi}$ . The equation of motion for  $\bar{\psi}$  is

$$(i\partial_\mu \bar{\psi} \gamma^\mu) + m\bar{\psi} = 0 \quad (9.35)$$

$$\bar{\psi}(\overset{\leftarrow}{i\cancel{\partial}} + m) = 0.$$

Now we calculate

$$\begin{aligned} S_F(x - y)(\overset{\leftarrow}{i\cancel{\partial}}_y + m) &= -i\partial_{x,\mu}S_F(x - y)\gamma^\mu + mS_F(x - y) \quad (9.36) \\ &= -i\partial_{x,\mu}(i\cancel{\partial}_x + m)\gamma^\mu\Delta_F(x - y) + m(i\cancel{\partial}_x + m)\Delta_F(x - y) \\ &= \partial_{x,\mu}\partial_{x,\nu}\gamma^\nu\gamma^\mu\Delta_F(x - y) + m^2\Delta_F(x - y) \\ &= (\partial_x^2 + m^2)\Delta_F(x - y) = -i\delta(x - y). \end{aligned}$$

## 9.4 Wick's theorem for multi-fermions

Now we come back to look at the fermion propagator, which is just a matrix with ordinary numbers, so we have

$$\{\psi(x), \bar{\psi}(y)\} = \langle 0 | \{\psi(x), \bar{\psi}(y)\} | 0 \rangle = \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle + \langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle. \quad (9.39)$$

Since  $\bar{\psi}$  contains creation operator for particles and  $\psi$  contains creation operator for anti-particles, for spacelike separations, for  $x^0 > y^0$ , we can describe the first part as a particle propagating from space-time point  $y$  to  $x$  and the second part as an anti-particle propagating from  $x$  to  $y$  backward in time. As we have seen from the calculation, for spacelike separations, this two amplitude cancel each other and so does the fermion propagator. From the argument we made earlier about observables, this means causality is preserved.

One can extend the Wick's theorem to multi-fermion fields to obtain

$$T\psi_1 \bar{\psi}_2 \psi_3 \dots = : \psi_1 \bar{\psi}_2 \psi_3 \dots : + \text{all possible contractions} \times : \text{remaining fields} :. \quad (9.40)$$

The minus sign problem again! Let us see an example how minus signs arise:

$: \psi_1 \bar{\psi}_2 \bar{\psi}_3 \dots := - : \bar{\psi}_1 \bar{\psi}_3 \psi_2 \dots := - \bar{\psi}_1 \bar{\psi}_3 : \psi_2 \dots :.$

(9.41)

If the field and the conjugate field are not adjacent to each other, one needs to permute the field inside the normal ordering with appropriate number of minus signs until they are next to each other. Then we can bring the contraction out of the normal ordering. The contraction is a  $c$ -number function. It is the Feynman propagator. It is trivial, but we will point out that Wick contraction of different species of particles say  $\varphi$  and  $\psi$  will give 0, as they contain “wrong” type of annihilation/creation operators.

# Chapter 10

## Fermionic Feynman Rules from the Yukawa theory

We use Yukawa theory (simple, interaction between scalars and spinors) as an example to show where various kinds of Feynman rules come from.

### 10.1 The Yukawa Lagrangian

In 1934, Yukawa theory was written down as a proposal for nucleon interaction. It is a general renormalizable theory that has wide use, such as describing fermion Higgs interaction. We will study a simple version of Yukawa theory, which contains one fermion and its anti-fermion partner and one real scalar field.<sup>1</sup> The Lagrangian<sup>2</sup><sup>3</sup> of the theory can be separated in three terms

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_\varphi + \mathcal{L}_{\text{Yuk}}, \quad (10.3)$$

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<sup>1</sup>The Yukawa theory that originally proposed including a pseudo-scalar field, a massive meson, which will complicate our calculation with some  $\gamma^5$ s.

<sup>2</sup>Yukawa wants to get a theory for nucleons and mesons, hence he wants an attractive potential. To see a derivation of the Yukawa potential, the reader is referred to end of ch5 of Tong's notes.

<sup>3</sup>Notice that the Lagrangian is invariant under the global  $U(1)$  phase rotation:

$$\psi(x) \rightarrow e^{i\alpha} \psi(x) \quad (10.1)$$

and consequently

$$\bar{\psi}(x) \rightarrow e^{-i\alpha} \bar{\psi}(x) \quad (10.2)$$

The conserved charge associated with this global symmetry is known as the baryon or lepton number.

where

$$\begin{aligned}\mathcal{L}_f &= \bar{\psi}(i\cancel{\partial} - m)\psi \\ \mathcal{L}_\varphi &= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 \\ \boxed{\mathcal{L}_{\text{Yuk}} &= -\lambda\varphi\bar{\psi}\psi.}\end{aligned}\quad (10.4)$$

What is the dimension of the coupling?  $[\lambda] = 0$ . As you will see in QFT II, determining the dimension of a coupling constant is very important for determining the renormalizability of the theory.

## 10.2 Computation of a 2-to-2 scattering process

Finally we are ready to extract Feynman rules for Yukawa theory by computing a scattering amplitude explicitly using the steps we just lay out<sup>4</sup>. Let us first calculate the relevant correlation function and then stick it into LSZ reduction formula to see what happens.

The process we want to calculate is the two fermions to two fermions scattering. So we need to compute

$$\langle 0|T\psi_4\psi_3\bar{\psi}_1\bar{\psi}_2\exp(-i\lambda\int d^4y\phi_y\bar{\psi}_y\psi_y)|0\rangle, \quad (10.5)$$

and remove the vacuum diagrams as we go because they are going to cancel with the denominator anyway. At zeroth order, we are going to get two fermions passing each other without interacting, which is not very interesting, so we should start our journey at first order. At first order after contracting all the fermion fields, we will end up with one scalar with no partner to contract with, the part of the scalar field with annihilation operator would annihilate the right vacuum, and the other part contains creation operator that annihilate the left vacuum. So this order also does not contribute.

The real fun begins at the second order, where we need to compute

$$\begin{aligned}\frac{1}{2}\langle 0|T\psi_4\psi_3\bar{\psi}_1\bar{\psi}_2(-i\lambda)^2\int d^4y d^4z\phi_y\bar{\psi}_y\psi_y\phi_z\bar{\psi}_z\psi_z|0\rangle \\ = \frac{1}{2}(-\lambda^2)\int d^4y d^4z\Delta_F(y-z)\langle 0|T\psi_4\psi_3\bar{\psi}_1\bar{\psi}_2\bar{\psi}_y\psi_y\bar{\psi}_z\psi_z|0\rangle,\end{aligned}\quad (10.6)$$

where we already contract the scalars and write it in terms of the propagator.

At first contracting 8 fermion fields seems daunting, but then we realize if we contract any of the external field with another external field, we are going to get disconnected diagrams which are not interesting. So all external fields will each contract with an internal field, more specifically, the two external  $\psi$  fields will contract with the two internal  $\bar{\psi}$  fields and the two external  $\bar{\psi}$  fields

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<sup>4</sup>See an alternate calculation in Appendix K.

will contract with the two internal  $\psi$  fields. A typical contraction looks like

$$\langle 0 | T \psi_4 \psi_3 \bar{\psi}_1 \bar{\psi}_2 \bar{\psi}_y \psi_y \bar{\psi}_z \psi_z | 0 \rangle = (-1)^{3+3+2+1} S_F(x_4 - y) S_F(x_3 - z) S_F(y - x_1) S_F(z - x_2), \quad (10.7)$$

where the exponent of the negative sign is how many switches to bring each contraction together<sup>5</sup>. Since  $y$  and  $z$  are internal points, the contraction like the above except  $y$  and  $z$  are switched will give exactly the same contribution, this is expected and will cancel the  $\frac{1}{2}$  from the Taylor expansion of the exponential.

The other distinct contraction<sup>6</sup> looks like

$$\langle 0 | T \psi_4 \psi_3 \bar{\psi}_1 \bar{\psi}_2 \bar{\psi}_y \psi_y \bar{\psi}_z \psi_z | 0 \rangle = (-1)^{3+3+3+1} S_F(x_4 - y) S_F(x_3 - z) S_F(z - x_1) S_F(y - x_2). \quad (10.8)$$

The crucial difference is we need one more switch<sup>7</sup>, so this has a relative negative sign compared to the first contribution. Now let us bring back the LSZ reduction formula

$$\langle f | S | i \rangle_H = i^{2+2} \int \left( \prod_{j=3}^4 d^4 x_j e^{ik_j \cdot x_j} \bar{u}_j(i\cancel{\partial}_j - m) \right) \langle \Omega | T \psi_4 \psi_3 \bar{\psi}_1 \bar{\psi}_2 | \Omega \rangle \left( \prod_{j=1}^2 d^4 x_j (-i\cancel{\partial}_j - m) e^{-ik_j \cdot x_j} u_j \right), \quad (10.9)$$

and the relevant contribution at second order comes from

$$\int \left( \prod_{j=3}^4 d^4 x_j e^{ik_j \cdot x_j} \bar{u}_j(i\cancel{\partial}_j - m) \right) \langle 0 | T \psi_4 \psi_3 \bar{\psi}_1 \bar{\psi}_2 \varphi_y \bar{\psi}_y \psi_y \varphi_z \bar{\psi}_z \psi_z | 0 \rangle \left( \prod_{j=1}^2 d^4 x_j (-i\cancel{\partial}_j - m) e^{-ik_j \cdot x_j} u_j \right) d^4 y d^4 z, \quad (10.10)$$

where we omitted an overall numeric factor of  $\frac{(-i\lambda)^2}{2}$ .

Now we restore all the spinor indices to the full wrath<sup>8</sup> so we will really know what's going on,

$$\begin{aligned} & \int \prod_{j=1}^4 d^4 x_j d^4 y d^4 z e^{ik_3 \cdot x_3} e^{ik_4 \cdot x_4} \bar{u}_{3a}(i\cancel{\partial} - m)_{3a,3b} \bar{u}_{4a}(i\cancel{\partial} - m)_{4a,4b} \times \quad (10.11) \\ & \quad \langle 0 | T \psi_{4b} \psi_{3b} \bar{\psi}_{1a} \bar{\psi}_{2a} \varphi_y \bar{\psi}_{ya} \psi_{ya} \varphi_z \bar{\psi}_{za} \psi_{za} | 0 \rangle \times \\ & \quad (-i\cancel{\partial} - m)_{1a,1b} e^{-ik_1 \cdot x_1} u_{1b} (-i\cancel{\partial} - m)_{2a,2b} e^{-ik_2 \cdot x_2} u_{2b}, \end{aligned}$$

<sup>5</sup>Of course this will vary depending on in which order you decide to contract them. The important thing to notice is that this contraction and the one a few lines below has a relative negative sign.

<sup>6</sup>there is another one only differ by switching  $y$  and  $z$  which contributes the same.

<sup>7</sup>or in general, an odd number of switches more or fewer.

<sup>8</sup>We thank Haoxing Du's very insightful question in class for this to form.

Note that all the spinor indices  $1a, 2a$  etc all run from 1 to 4 just like indices  $a$  and  $b$ . There are way too many of them and I decide to name them systematically.<sup>9</sup>

Notice that all the spinor indices are contracted. The whole expression is Lorentz invariant, as a scattering amplitude should be.

Let us now contract all the Wick pairs<sup>10</sup> and focus on the first contribution,

$$\begin{aligned} & - \int \prod_{j=1}^4 d^4 x_j d^4 y d^4 z e^{ik_3 \cdot x_3} e^{ik_4 \cdot x_4} \bar{u}_{3a} (i\cancel{\partial} - m)_{3a,3b} \bar{u}_{4a} (i\cancel{\partial} - m)_{4a,4b} \times \quad (10.12) \\ & S_{4b,ya}(x_4 - y) S_{3b,za}(x_3 - z) S_{ya,1a}(y - x_1) S_{za,2a}(z - x_2) \Delta_F(y - z) \times \\ & \quad (-i \cancel{\partial} - m)_{1a,1b} e^{-ik_1 \cdot x_1} u_{1b} (-i \cancel{\partial} - m)_{2a,2b} e^{-ik_2 \cdot x_2} u_{2b}, \end{aligned}$$

where we have carefully kept track of where the spinor indices come from. Now we use the fact that the propagator is green's function of the Dirac operator<sup>11</sup> and arrives at

$$\begin{aligned} & - \int \prod_{j=1}^4 d^4 x_j d^4 y d^4 z e^{ik_3 \cdot x_3} e^{ik_4 \cdot x_4} \bar{u}_{3a} \bar{u}_{4a} \times \quad (10.14) \\ & \delta_{4a,ya}(x_4 - y) \delta_{3a,za}(x_3 - z) \delta_{ya,1b}(y - x_1) \delta_{za,2b}(z - x_2) \Delta_F(y - z) \times \\ & \quad e^{-ik_1 \cdot x_1} u_{1b} e^{-ik_2 \cdot x_2} u_{2b}. \end{aligned}$$

Now if we carefully trace the remaining spinor indices, we will find that  $\bar{u}_4$  is contracted to  $u_1$  and  $\bar{u}_3$  is contracted to  $u_2$ . Now we will use the inner product notation and omit the spinor indices and integrated over  $x_1, x_2, x_3, x_4$  using  $\delta$  distribution.

We see the first contribution is

$$\begin{aligned} & - \int d^4 y d^4 z \Delta_F(y - z) e^{i(k_3 - k_2) \cdot y} e^{i(k_4 - k_1) \cdot z} \bar{u}_3 \cdot u_2 \bar{u}_4 \cdot u_1 \quad (10.15) \\ & = - \int d^4 y d^4 z \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (y - z)} e^{i(k_3 - k_2) \cdot y} e^{i(k_4 - k_1) \cdot z} \bar{u}_3 \cdot u_2 \bar{u}_4 \cdot u_1 \\ & = - \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} (2\pi)^8 \delta(k_3 - k_2 - p) \delta(k_4 - k_1 + p) \bar{u}_3 \cdot u_2 \bar{u}_4 \cdot u_1 \\ & = -(2\pi)^4 \delta(k_1 + k_2 - k_3 - k_4) \frac{i}{(k_1 - k_4)^2 - m^2 + i\epsilon} \bar{u}_3 \cdot u_2 \bar{u}_4 \cdot u_1. \end{aligned}$$

<sup>9</sup>We thank Yile Yang to point out an error in the original clarification.

<sup>10</sup>Notice that if we keep track of all spinor indices, we can write the propagators in any order we want.

<sup>11</sup>And recall the property of the Feynman propagator

$$\begin{aligned} (i\cancel{\partial}_x - m) S_F(x - y) &= i\delta(x - y) \quad (10.13) \\ S_F(x - y) (i\cancel{\partial}_y + m) &= -i\delta(x - y). \end{aligned}$$

Similarly, the other contribution is

$$\begin{aligned}
& + \int d^4y d^4z \Delta_F(y - z) e^{i(k_3 - k_1)y} e^{i(k_4 - k_2)z} \bar{u}_3 \cdot u_1 \bar{u}_4 \cdot u_2 \\
& = + \int d^4y d^4z \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (y-z)} e^{i(k_3 - k_1)y} e^{i(k_4 - k_2)z} \bar{u}_3 \cdot u_1 \bar{u}_4 \cdot u_2 \\
& = + \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} (2\pi)^8 \delta(k_3 - k_1 - p) \delta(k_4 - k_2 + p) \bar{u}_3 \cdot u_1 \bar{u}_4 \cdot u_2 \\
& = + (2\pi)^4 \delta(k_1 + k_2 - k_3 - k_4) \frac{i}{(k_1 - k_3)^2 - m^2 + i\epsilon} \bar{u}_3 \cdot u_1 \bar{u}_4 \cdot u_2.
\end{aligned} \tag{10.16}$$

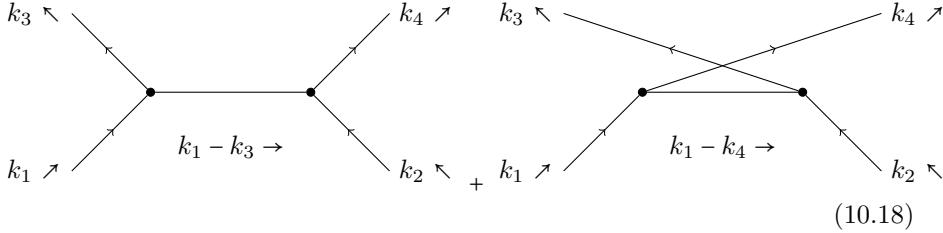
So the amplitude is given by

$$i\mathcal{M} = i(-i\lambda)^2 \left( \frac{\bar{u}_3 \cdot u_1 \bar{u}_4 \cdot u_2}{(k_1 - k_3)^2 - m^2 + i\epsilon} - \frac{\bar{u}_3 \cdot u_2 \bar{u}_4 \cdot u_1}{(k_1 - k_4)^2 - m^2 + i\epsilon} \right). \tag{10.17}$$

Now it is time to move onward and calculate the contribution from the next order. By the same argument, the  $\mathcal{O}(\lambda^3)$  contribution vanish. Onward,  $\mathcal{O}(\lambda^4)$  contribution, we need to calculate  $\frac{(-i\lambda)^4}{4!} \langle 0 | T \psi_4 \psi_3 \bar{\psi}_1 \bar{\psi}_2 \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 (\varphi \bar{\psi} \psi)_1 (\varphi \bar{\psi} \psi)_2 (\varphi \bar{\psi} \psi)_3 (\varphi \bar{\psi} \psi)_4 \rangle$ . We can calculate this with brute force, but that will take a while.

### 10.3 Some Feynman rules extracted

Instead we could stare at the result and use the powerful technique of Feynman diagrams you have seen in the scalar theory. First let us draw the diagrams, or the doodling form, without worrying too much about the details: we need to draw a diagram with two incoming particles and two outgoing particles, and the interaction is a 3-vertex, and to the order of  $\mathcal{O}(\lambda^2)$ , we only have two vertices. Taking into account the fact that identical particles are identical, we have the following two sketches



and now we want to match them with our amplitude. The denominator should look familiar to you, as that is the scalar propagator. For the numerator, the  $u$ 's and  $\bar{u}$ 's will be associated with external legs. Notice that  $\bar{u}$ 's are always with the outgoing momenta, and  $u$ 's are always with the incoming momenta, we have the following momentum space Feynman rules, consider all Feynman rules are boxed:

- For each incoming fermion of momentum  $\mathbf{p}$  and spin polarization  $r$ , we associate  $u^r(\mathbf{p})$  and draw the following diagram

$$\overrightarrow{\text{---}} \mathbf{p} = u^r(\mathbf{p}). \quad (10.19)$$

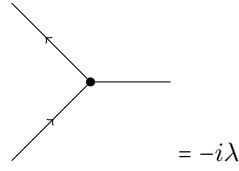
We draw an arrow on the fermion line to indicate the flow of fermions (as opposed to anti-fermions), we often draw a small arrow next to it to indicate the momentum of the particle.

- For each outgoing fermion of momentum  $\mathbf{p}$  and spin polarization  $r$ , we associate  $\bar{u}^r(\mathbf{p})$  and draw the following diagram

$$\overleftarrow{\text{---}} \mathbf{p} = \bar{u}^r(\mathbf{p}). \quad (10.20)$$

The factor in the numerator of the amplitude is then the spin factors you wrote down in the order of reversing the arrow of the fermion line.

- We associate with each vertex a factor  $(-i\lambda)$ , the diagram looks like



$$= -i\lambda. \quad (10.21)$$

- We impose 4-momentum conservation at each vertex.
- For fermions in the final states with indistinguishable particles, we introduce a relative minus sign if a diagram can be converted into another by an odd number of permutation of the momenta and spin labels of the final fermionic particles.

Here are a list of rules we cannot get by studying just one scattering experiment, or gedanken experiment. We will explore these rules in the tutorial,

- For each incoming anti-fermion of momentum  $\mathbf{p}$  and spin polarization  $r$ , we associate  $\bar{v}^r(\mathbf{p})$  and the diagram

$$\overleftarrow{\text{---}} \mathbf{p} = \bar{v}^r(\mathbf{p}). \quad (10.22)$$

Again, we draw an arrow on the fermion line to indicate the flow of fermions, so in opposite direction of where anti-fermion go.

- For each outgoing anti-fermion of momentum  $\mathbf{p}$  and spin polarization  $r$ , we associate  $v^r(\mathbf{p})$  and the diagram.

$$\overrightarrow{\text{---}} \mathbf{p} = v^r(\mathbf{p}). \quad (10.23)$$

- The fermion propagator is of course given by the Feynman propagator we calculated before, but to confirm that, we need to study a fermion-scalar scattering.

$$\overline{p \rightarrow} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \quad (10.24)$$

- For final fermion particles that are distinguishable we can still have a relative sign. This comes from how operators in the normal ordered term for the Dyson series expansion with the external states. We will study a fermion-anti-fermion scattering in the tutorial to uncover this mysterious minus sign.
- The way to figure out the minus sign<sup>12</sup> in Feynman diagrams is to use Wick's contraction on the external legs following the Feynman diagrams. This is valid because fermion bilinears from the interaction vertex will not change this minus sign because bilinears commute with any operator. It is possible because any fermion external leg eventually has to be connected to some other external leg: this is required by Lorentz invariance, some  $\bar{u}$  or  $\bar{v}$  from one leg has to be eventually connect to some  $u$  or  $v$  from some other leg.
- When there is an internal loop and there is an undetermined momentum in the loop, then we need to integrate over the loop momentum

$$\int \frac{d^4q}{(2\pi)^4}. \quad (10.25)$$

We introduce a  $-1$  for each fermion loop and trace over the fermionic loop spinor indices. Again, we will uncover this minus sign in the tutorial.

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<sup>12</sup>We thank Haoxing Du for her insightful questions on this matter.



## Chapter 11

# \*Interlude: Quantization of Maxwell Theory

*Let there be light.* Einstein's photoelectric effect has earned him the Nobel prize. The most important part of the explanation is that the light behaves like particles and one light particle is called a photon. Luckily in this case, we already have a field for it, the Maxwell's theory for electromagnetism. So how do we quantize the field to get the particle in this case? (Hint: the gauge symmetry complicated things quite a bit here.)

### 11.0.1 Review of scalar theory

At the beginning of this course, we have studied free scalar theory with Klein-Gordon equation of motion

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0. \quad (11.1)$$

In order to quantize this theory, we solve the equation classically

$$\varphi(t, \mathbf{x}) = \int dV_{\mathbf{k}} [a(\mathbf{k}) e^{-ik \cdot x} + a^*(\mathbf{k}) e^{ik \cdot x}]. \quad (11.2)$$

Then we impose the equal-time commutator relationship

$$\begin{aligned} [\varphi(\mathbf{x}), \pi_\varphi(\mathbf{y})] &= i\delta^3(\mathbf{x} - \mathbf{y}) \\ [\varphi(\mathbf{x}), \varphi(\mathbf{y})] &= 0 = [\pi_\varphi(\mathbf{x}), \pi_\varphi(\mathbf{y})]. \end{aligned} \quad (11.3)$$

Now we promote the coefficients in the classical solution to operators, and obtain (setting  $t = 0$ )

$$\varphi(\mathbf{x}) = \int dV_{\mathbf{k}} [a_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} + a_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}}]. \quad (11.4)$$

The commutator relationship for the operators deduced by the equal-time commutator relationship is

$$\begin{aligned} [a_{\mathbf{k}}, a_{\mathbf{p}}^\dagger] &= (2\pi)^3 2E_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{p}) \\ [a_{\mathbf{k}}, a_{\mathbf{p}}] &= 0 = [a_{\mathbf{k}}^\dagger, a_{\mathbf{p}}^\dagger], \end{aligned} \quad (11.5)$$

and the normal ordered Hamiltonian is

$$:H_\varphi := \int dV_{\mathbf{k}} E_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}. \quad (11.6)$$

The Feynman propagator of a scalar field is

$$\Delta_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip \cdot (x-y)}. \quad (11.7)$$

### 11.0.2 Action, equation of motion, gauge symmetry

As we have seen before, the Lagrangian for Maxwell's free theory is

$$\mathcal{L}_{\text{Max}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (11.8)$$

In terms of the field potential  $A^\mu$ , which has mass dimension 1, the field strength  $F_{\mu\nu}$  is defined to be

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (11.9)$$

The field strength defined this way automatically satisfies the Bianchi identity:

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \quad (11.10)$$

The equations of motion from Euler-Lagrangian equation from this action are

$$\partial_\mu \left( \frac{\partial \mathcal{L}_{\text{Max}}}{\partial (\partial_\mu A_\nu)} \right) = -\partial_\mu F^{\mu\nu} = 0. \quad (11.11)$$

Now we just need to solve the equation of motion, and then expand  $A^\mu(x)$  in modes, and quantize the theory, basically repeat the things we have done for scalars and fermions. Third time is a charm. Except of course, there are some caveats.

For one thing, the photon has only two degrees of freedom<sup>1</sup> even if it is described by the vector field  $A^\mu(x)$ .

On a related note, Maxwell theory enjoys gauge symmetry<sup>2</sup>. The gauge transformation is

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x), \quad (11.12)$$

---

<sup>1</sup>We will explain why in the Appendix L.

<sup>2</sup>The Maxwell action enjoys some other important symmetries

- Poincaré symmetry: when coordinate transforms like  $x'^\mu = \Lambda_\nu^\mu x^\nu + a^\mu$ , where we have line element is unchanged  $\eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\mu\nu} dx^\mu dx^\nu$ ,  $A^\mu$  transforms like a vector  $A'^\mu = \Lambda_\nu^\mu A^\nu$ , and the Lagrangian is already Lorentz invariant. Poincaré symmetry is global

where  $\alpha(x)$  is an arbitrary function that dies suitably fast at spacial infinity. The field strength  $F_{\mu\nu}$  is invariant under this transformation:

$$F_{\mu\nu} \rightarrow \partial_\mu(A_\nu(x) - \partial_\nu\alpha(x)) - \partial_\nu(A_\mu(x) - \partial_\mu\alpha(x)) = F_{\mu\nu}. \quad (11.13)$$

Hence the Lagrangian is invariant. The gauge symmetry is parametrized by not a constant but a function, does that mean we have infinite number of conserved currents? The answer is **NO!** A gauge symmetry is actually a misnomer: it is not a symmetry of the system, but rather should be viewed as a redundancy<sup>3</sup>. In our description: the states related by gauge symmetry should be identified as the same physical state.

One way to see this is to look at Maxwell's equations. In terms of  $A_\mu$ , we have

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = (\delta^\nu_\mu \partial_\rho \partial^\rho - \partial_\mu \partial^\nu) A^\mu. \quad (11.14)$$

Note the following is an identity for any arbitrary function  $\alpha(x)$ :

$$(\delta^\nu_\mu \partial_\rho \partial^\rho - \partial_\mu \partial^\nu) \partial^\mu \alpha(x) = 0. \quad (11.15)$$

This means we cannot distinguish  $A^\mu(x)$  from  $A^\mu(x) + \partial^\mu \alpha(x)$  given any initial data. In other words, the operator  $\delta^\nu_\mu \partial_\rho \partial^\rho - \partial_\mu \partial^\nu$  is not invertible. So we should understand the phase space of Maxwell theory as an enlarged phase space foliated by gauge orbits. All states along the same gauge orbits are equivalent, and to proceed, we need to pick a representative from each gauge orbit. This process is called gauge fixing, and is often described graphically (The figure is on the next page.).

In this course we will use the Lorentz<sup>4</sup> gauge

$$\partial_\mu A^\mu = 0. \quad (11.16)$$

As the name suggested, this gauge is Lorentz invariant. Let us first examine the Lorentz gauge: first this is always possible, suppose we started out with

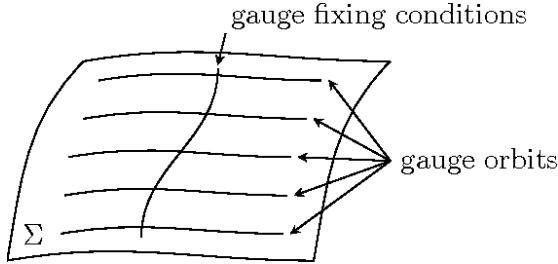
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symmetry, as  $\Lambda_\mu^\nu$  and  $a^\mu$  are constants. The Poincaré symmetry leads to conserved currents.

- Electromagnetic duality symmetry: the equation of motions of the electric field strength  $\mathbf{E}$  and the magnetic field strength  $\mathbf{B}$  are unchanged under  $\mathbf{E} \rightarrow \mathbf{B}$  and  $\mathbf{B} \rightarrow -\mathbf{E}$ .
- Conformal symmetry: maximal extension of the Poincaré symmetry, the largest space-time symmetry a theory can have without including supersymmetry: the line element is changed by a conformal factor  $\eta_{\mu\nu} dx'^\mu dx'^\nu = \Omega^2(x) \eta_{\mu\nu} dx^\mu dx^\nu$ . Maxwell symmetry is conformal invariant as you will see in QFTIII.

<sup>3</sup>So why do we use  $A_\mu$  which cannot be measured instead of  $\mathbf{E}$  and  $\mathbf{B}$ ? The Aharonov-Bohm effect actually can measure  $A_\mu$  where both  $\mathbf{E}$  and  $\mathbf{B}$  are zero. Consider the well known double slit experiment, if we have a solenoid in front of the slits, the interference pattern of the particles will change because the particle will pick up a phase due to the non-zero  $A_\mu$  around the solenoid even if the field strength is zero.

<sup>4</sup>But really should be called the Lorenz gauge, but poor Lorenz has the misfortune to have his name one letter away from the great Lorentz



some  $\partial_\mu(A')^\mu = \alpha'(x)$ , we can always choose  $A^\mu = (A')^\mu + \partial^\mu \alpha(x)$  to satisfy the Lorentz gauge condition

$$\partial_\mu A^\mu = \alpha'(x) + \partial^\mu \partial_\mu \alpha(x) = 0. \quad (11.17)$$

The above equation always has a solution for  $\alpha(x)$ . Notice that Lorentz gauge does not pick a unique representative from the gauge orbit. We can make further gauge transformation as long as  $\partial^\mu \partial_\mu \alpha(x) = 0$ , which has non-trivial solutions.

For example, the Coulomb gauge uses this residue gauge transformations to pick  $\nabla \cdot \mathbf{A} = 0$ . As we mentioned in the appendix,  $A_0$  is determined by (L.3) which gives  $A_0 = 0$ . In this gauge, it is manifest that photons have two degrees of freedom, but we have broken the Lorentz invariance. You have seen how the quantization is done in Coulomb's gauge. In this course, we like the Lorentz invariance, and we will stick with Lorentz gauge.

With this gauge choice, the equation of motion becomes

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu \partial^\mu A^\nu = 0. \quad (11.18)$$

But this is nothing but four massless Klein Gordon equations. If we forget about the fact that actually only two degrees of freedom are independent, we can write down its general solution as a sum of four planewave solutions with polarization vector  $\epsilon^\nu$

$$A^\nu = \sum_{\lambda=0}^3 a^\lambda(\mathbf{k}) (\epsilon^\nu)^\lambda(\mathbf{k}) e^{-ik \cdot x} + c.c., \quad (11.19)$$

where  $(\epsilon^\nu)^\lambda(\mathbf{k})$ , with  $\lambda = 0, 1, 2, 3$  are 4 polarization vectors with normalization

$$\eta^{\mu\nu} (\epsilon_\mu)^\lambda (\epsilon_\nu)^{\lambda'} = \eta^{\lambda\lambda'}, \quad (11.20)$$

which also means

$$(\epsilon_\mu)^\lambda (\epsilon_\nu)^{\lambda'} \eta_{\lambda\lambda'} = \eta_{\mu\nu}. \quad (11.21)$$

The polarization vectors depend on the photon 4-momentum  $k^\mu = (|\mathbf{k}|, \mathbf{k})$ . Of the two spacelike polarization, we choose  $\epsilon^1$  and  $\epsilon^2$  to be transverse to the momentum,

$$\epsilon^1 \cdot k = \epsilon^2 \cdot k = 0 \quad (11.22)$$

The third polarization vector is longitudinal.

For example, if the photon is moving in the  $x^3$  direction, then we have momentum  $p^\mu \propto (1, 0, 0, 1)$ . We can then choose polarization vectors to be

$$\epsilon^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (11.23)$$

With this choice of polarization vector, the transversality condition in the third and time direction can be written as

$$\epsilon^0 \cdot k = -\epsilon^3 \cdot k. \quad (11.24)$$

For other 4-momenta, the polarization vectors will be the Lorentz transformation of these vectors. Note that both  $\epsilon^1 \cdot k = \epsilon^2 \cdot k = 0$  and  $\epsilon^0 \cdot k = -\epsilon^3 \cdot k$  are invariant under Lorentz transformations and has nothing to do with gauge condition<sup>5</sup>. Note that so far we have completely ignored the fact the theory is gauge invariant. And we are going to upgrade all the coefficients  $a^\lambda(\mathbf{k})$  and their Hermitian conjugates to be operators.<sup>6</sup>

### 11.0.3 Quantization

Again we recall our recipe to quantize a field theory:

1. Pick a Lagrangian.
2. Find the canonical momentum and Hamiltonian.
3. Impose equal time commutator relationship.
4. Define normal ordering.

Using the original Lagrangian  $\mathcal{L}_{\text{Max}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ , we quickly run into a problem: since the Lagrangian is independent of  $\partial_0 A^0$ , the canonical momentum for  $A^0$ ,  $\pi^0 = 0$ . This is not very good for quantization. It is very hard to impose the equal time commutator relationship if  $\pi^0 = 0$  all the time!

Remember when we get the equation of motion  $\partial^\mu \partial_\mu A^\nu = 0$ , it is the Euler-Lagrangian equation from the Maxwell's Lagrangian with gauge condition imposed after the eom is derived<sup>7</sup>. We propose to change the Lagrangian such that the above equation of motion follows directly from the new Lagrangian<sup>8</sup>

$$\mathcal{L}_{\text{new}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2. \quad (11.25)$$

---

<sup>5</sup>It only has to do with the fact that in some frame we can choose  $p^\mu \propto (1, 0, 0, 1)$ .

<sup>6</sup>We thank Shuwei Liu for insightful discussions on this point.

<sup>7</sup>See Reinhardt 7.2 that we only need to impose Lorentz condition at one point.

<sup>8</sup>The way to change the Lagrangian is not unique, interested readers can read in Reinhardt 7.2.

The equation of motion is,

$$\partial_\mu \left( \frac{\partial \mathcal{L}_{\text{new}}}{\partial (\partial_\mu A_\nu)} \right) = -\partial_\mu F^{\mu\nu} - \partial^\nu (\partial_\mu A^\mu) = -\partial_\mu \partial^\mu A^\nu = 0. \quad (11.26)$$

We like this equation of motion! This is just four copies of massless free Klein-Gordon theory. We know how to quantize this, but only if the canonical momenta also look nice. Thus we use the technique of integrate by parts to further massage this Lagrangian,

$$\begin{aligned} \mathcal{L}_{\text{new}} &= -\frac{1}{2} \partial_\mu A_\nu F^{\mu\nu} - \frac{1}{2} \partial_\mu A^\mu \partial_\nu A^\nu \\ &= -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2} \partial_\mu A^\mu \partial_\nu A^\nu \\ &= -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial_\mu (A_\nu \partial^\nu A^\mu - A^\mu \partial_\nu A^\nu). \end{aligned} \quad (11.27)$$

The last term is a four divergence that does not influence field equations. We will use the simpler Lagrangian

$$\boxed{\mathcal{L}_q = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu} \quad (11.28)$$

to quantize the theory. This Lagrangian would in particular have a nice form of canonical momentum  $\pi_A^\mu$  that treats time and space on equal footing. So let us proceed to quantize this theory with this new Lagrangian, and will later impose the gauge condition on the Hilbert space, as we know there is some residue gauge symmetry also needs to be fixed.

Now we find the canonical momentum  $\pi_A^\mu$  and impose the usual equal-time commutation relations.

We have

$$\pi_{\mu, A} = \frac{\partial \mathcal{L}_{\text{new}}}{\partial (\partial_0 A^\mu)} = -\partial^0 A_\mu. \quad (11.29)$$

Before we rush to quantize this theory. As we can see, after adding the gauge fixing term,  $\pi_{0,A}$  is no longer 0, and the canonical momenta look just like in the scalar theory!

Now we can profit from the massage we gave to the Lagrangian and the hard work you put in learning about the scalar theory. We will eventually have to pay a price, but let us profit first. Let us follow the standard procedure you are familiar with from the first part of the course, the canonical quantization: we promote the classical fields  $A^\mu$  and  $\pi_A^\mu$  to be field operators that satisfy the equal-time commutation relations,

$$[A^\mu(\mathbf{x}, t), A^\nu(\mathbf{y}, t)] = [\pi_A^\mu(\mathbf{x}, t), \pi_A^\nu(\mathbf{y}, t)] = 0, \quad (11.30)$$

and

$$[A^\mu(\mathbf{x}, t), \pi_A^\nu(\mathbf{y}, t)] = i\eta^{\mu\nu}\delta^3(\mathbf{x} - \mathbf{y}). \quad (11.31)$$

Note that the spatial component of the equation above is, substituting in the form of  $\pi_A^i$  from (11.29)

$$[A^i(\mathbf{x}, t), -\partial^0 A^j(\mathbf{y}, t)] = i\eta^{ij}\delta^3(\mathbf{x} - \mathbf{y}), \quad (11.32)$$

which is equivalent to,

$$[A^i(\mathbf{x}, t), \dot{A}^j(\mathbf{y}, t)] = i\delta^{ij}\delta^3(\mathbf{x} - \mathbf{y}), \quad (11.33)$$

which mimics the commutation relations of scalars. On the other hand, the 0th component satisfies the equal-time commutator relationship with a wrong sign, as the metric tensor shows up on the right hand side. This is unavoidable and maybe later we will have to pay some price. Now we can write down the mode expansion of the potential  $A^\mu$  and its conjugate momentum in terms of creation and annihilation operators

$$\begin{aligned} A^\mu(\mathbf{x}) &= \int dV_{\mathbf{k}} \sum_{\lambda=0}^3 (\epsilon^\mu)^\lambda(\mathbf{k}) \left( a_{\mathbf{k}}^\lambda e^{-ik^i \cdot x_i} + (a_{\mathbf{k}}^\lambda)^\dagger e^{ik^i \cdot x_i} \right) \\ \pi_A^\mu(\mathbf{x}) &= \int dV_{\mathbf{k}} (+iE_{\mathbf{k}}) \sum_{\lambda=0}^3 (\epsilon^\mu)^\lambda(\mathbf{k}) \left( a_{\mathbf{k}}^\lambda e^{-ik^i \cdot x_i} - (a_{\mathbf{k}}^\lambda)^\dagger e^{ik^i \cdot x_i} \right). \end{aligned} \quad (11.34)$$

The equal-time commutator relationship leads to the following commutator relationship between the annihilation and creation operator,

$$[a_{\mathbf{k}}^\lambda, a_{\mathbf{k}'}^{\lambda'}] = [(a_{\mathbf{k}}^\lambda)^\dagger, (a_{\mathbf{k}'}^{\lambda'})^\dagger] = 0, \quad (11.35)$$

and

$$[a_{\mathbf{k}}^\lambda, (a_{\mathbf{k}'}^{\lambda'})^\dagger] = -\eta^{\lambda\lambda'}(2\pi)^3 2E_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}'). \quad (11.36)$$

For spacelike components  $\lambda = 1, 2, 3$ , it just looks like our relationship for scalars as we could have expected. For the timelike annihilation and creation operators, we have

$$[a_{\mathbf{k}}^0, (a_{\mathbf{k}'}^0)^\dagger] = -(2\pi)^3 2E_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}'). \quad (11.37)$$

The wrong sign of the commutator relationship of field and its momentum has carried over to the annihilation and creation operators as expected. From this, we can calculate the Hamiltonian by using the usual method

$$\begin{aligned} :H_A: &= \int d^3x : \pi_A^\mu \dot{A}_\mu - \mathcal{L}_q : \\ &= - \int dV_k E_k \sum_{\lambda=0}^3 (a_{\mathbf{k}}^\lambda)^\dagger (a_{\mathbf{k}}^\lambda) \eta^{\lambda\lambda}. \end{aligned} \quad (11.38)$$

Analogously, we can write down the Feynman propagator of  $A^\mu$

$$\begin{aligned} D_F^{\mu\nu} &= \langle 0 | T(A^\mu(x) A^\nu(y)) | 0 \rangle \\ &= \lim_{m \rightarrow 0} (-\eta^{\mu\nu} \Delta_F(x - y)) \\ &= \int \frac{d^4k}{(2\pi)^4} \frac{-i\eta^{\mu\nu} e^{-ik \cdot x}}{k^2 + i\varepsilon}. \end{aligned} \quad (11.39)$$

Let us define our Lorentz invariant vacuum and one-particle states as usual. The vacuum is defined to be

$$a_{\mathbf{k}}^{\lambda}|0\rangle = 0. \quad (11.40)$$

Now we can define one particle state

$$|\mathbf{k}, \lambda\rangle = (a_{\mathbf{k}}^{\lambda})^{\dagger}|0\rangle. \quad (11.41)$$

There is nothing unusual in the spacelike direction, but let us calculate the norm of the state  $|\mathbf{k}, 0\rangle$

$$\langle \mathbf{k}, 0 | \mathbf{k}', 0 \rangle = \langle 0 | a_{\mathbf{k}}^0 (a_{\mathbf{k}'}^0)^{\dagger} | 0 \rangle = -(2\pi)^3 2E_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}'). \quad (11.42)$$

This is very strange, but it is expected, it comes from the wrong sign in the commutator of annihilation and creation operators, which comes from the wrong sign in the time component of equal time commutator relationship. We can actually trace this wrong sign all the way back to the simple Lagrangian, written in a way really mimic scalar fields,

$$\mathcal{L}_q = -\frac{1}{2} \partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu} = -\frac{1}{2} \partial_{\mu} A^0 \partial^{\mu} A^0 + \frac{1}{2} \partial_{\mu} A^i \partial^{\mu} A^i \quad (11.43)$$

where field  $A^0$  has a kinetic term with a wrong sign.

We can fix the problem, by imposing our long forgotten gauge condition.

$$\partial_{\mu} A^{\mu} = 0 \quad (11.44)$$

It does not make sense to impose  $\partial_{\mu} A^{\mu} = 0$  as an operator equation because the condition is too strong. We can apply  $\partial_{\mu}$  to our propagator and have

$$\begin{aligned} & \langle 0 | T(\partial_{\mu} A^{\mu}(x) A^{\nu}(y)) | 0 \rangle \\ &= \partial_{\mu} \int \frac{d^4 k}{(2\pi)^4} \frac{-i\eta^{\mu\nu} e^{-ik\cdot x}}{k^2 + i\varepsilon}. \end{aligned} \quad (11.45)$$

The right hand side is certainly not zero, so it does not make sense to impose the left hand side to be zero.

Instead, we are going to attempt to impose the condition on the Hilbert space and try to throw away the negative norm states. We could try to impose the condition in a way that the Hilbert space is split into good states and bad states, and hopefully the bad states will decouple from the system and will include those troublesome negative norm states. For example, we can propose for all good, physical states,  $|\Psi\rangle$ , we have

$$\partial_{\mu} A^{\mu} |\Psi\rangle = 0 \quad (11.46)$$

This does not work! If we decompose  $A_{\mu}(x) = A_{\mu}^-(x) + A_{\mu}^+(x)$ , with

$$\begin{aligned} A_{\mu}^-(x) &= \int dV_{\mathbf{k}} \sum_{\lambda=0}^3 (\epsilon_{\mu})^{\lambda} a_{\mathbf{k}}^{\lambda} e^{-ik\cdot x} \\ A_{\mu}^+(x) &= \int dV_{\mathbf{k}} \sum_{\lambda=0}^3 (\epsilon_{\mu})^{\lambda} (a_{\mathbf{k}}^{\lambda})^{\dagger} e^{+ik\cdot x}. \end{aligned} \quad (11.47)$$

Then on the vacuum, we have  $\partial^\mu A_\mu^- |0\rangle = 0$  automatically, but  $\partial^\mu A_\mu^+ |0\rangle \neq 0$ . Our vacuum is not a physical state if we impose our gauge condition like this!

In order to keep the vacuum as a good state, we will instead impose the gauge condition as, for any physical state  $|\Psi\rangle$

$$\partial^\mu A_\mu^- |\Psi\rangle = 0. \quad (11.48)$$

So for the bra, we have

$$\langle \Psi | \partial^\mu A_\mu^+ = 0. \quad (11.49)$$

This will ensure that for any physical states  $|\Psi\rangle$  and  $|\Psi'\rangle$ , we have

$$\boxed{\langle \Psi' | \partial_\mu A^\mu | \Psi \rangle = 0.} \quad (11.50)$$

So the operator  $\partial_\mu A^\mu$  always have vanishing matrix elements between physical states. This way of fixing the gauge (11.48) is known as the Gupta-Bleuler condition. The condition is linear in physical states. So the physical states do span a physical Hilbert space.

Substitute the mode expansion of  $A_\mu^-$  (11.47) into the Gupta-Bleuler condition (11.48), we have

$$\begin{aligned} \partial^\mu A_\mu^- |\Psi\rangle &= \int dV_k \sum_{\lambda=0}^3 (\epsilon \cdot (-ik))^\lambda a_k^\lambda e^{-ik \cdot x} |\Psi\rangle \\ &= \int dV_k (\epsilon \cdot (-ik))^0 (a_k^0 - a_k^3) e^{-ik \cdot x} |\Psi\rangle. \end{aligned} \quad (11.51)$$

where we use that  $\epsilon_1$  and  $\epsilon_2$  are transverse to momentum and  $\epsilon_3 \cdot k = -\epsilon_0 \cdot k$ . For the above equation to be held for all momenta, the constraint becomes

$$(a_k^0 - a_k^3) |\Psi\rangle = 0 \quad (11.52)$$

for all momenta. This means physical states must contain particular combinations of timelike and longitudinal photons.

Write the above relation in terms of  $a_k^0 |\Psi\rangle = a_k^3 |\Psi\rangle$  and take the norm on both sides, we discover the number operator for timelike photons and the number operator for longitudinal photons always yield the same result when evaluated on physical states

$$\langle \Psi | (a_k^0)^\dagger a_k^0 |\Psi\rangle = \langle \Psi | (a_k^3)^\dagger a_k^3 |\Psi\rangle. \quad (11.53)$$

Recall the form of our Hamiltonian,

$$\begin{aligned} :H_A: &= \int d^3x : \pi_A^\mu \dot{A}_\mu - \mathcal{L}_q : \\ &= - \int dV_k E_k \sum_{\lambda=0}^3 (a_k^\lambda)^\dagger (a_k^\lambda) \eta^{\lambda\lambda} \\ &= \int dV_k E_k \sum_{\lambda=1}^2 (a_k^\lambda)^\dagger (a_k^\lambda) \end{aligned} \quad (11.54)$$

The contribution of the timelike photon and the longitudinal photon to the energy always cancel.



# Chapter 12

## QED at tree level

### 12.0.1 Feynman rules

We have already seen, the QED Lagrangian is

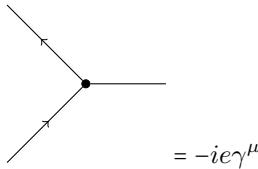
$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\cancel{D} - m)\psi, \quad (12.1)$$

where

$$D_\mu \equiv \partial_\mu + ieA_\mu. \quad (12.2)$$

Right away, we can write down the Feynman rules for QED. For vertices and internal lines, we have

- For each interaction between fermions and the photon we associate  $-iQ\gamma^\mu$ ,



$$= -ie\gamma^\mu. \quad (12.3)$$

- The photon propagator is

$$\mu \xrightarrow[p \rightarrow]{} \nu = \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}. \quad (12.4)$$

- For incoming and outgoing photons, we have

$$\xrightarrow{\rightarrow p} = \epsilon^{\text{in}}(\mathbf{p}) \quad (12.5)$$

$$\xleftarrow{\leftarrow p} = \epsilon^{\text{out}}(\mathbf{p}). \quad (12.6)$$

### 12.0.2 Scattering in QED: amplitudes

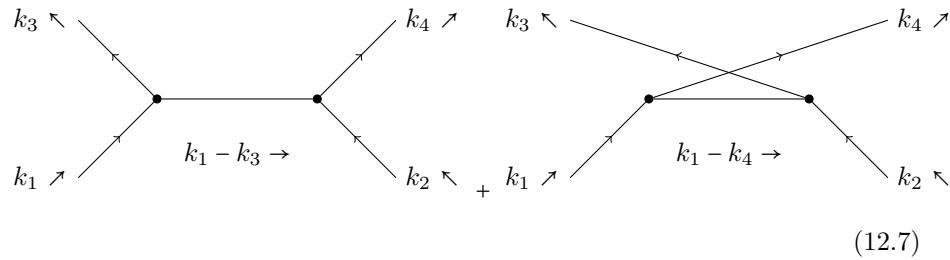
Now we have Feynman rules. We can draw Feynman diagrams and study a lot of processes. Here are some tips for drawing Feynman rules,

- Pay attention to your conventions: label your time direction and make a little legend for the different fields.
- Make sure that you are drawing diagrams for the same process. Process is determined by the experimentalists, pay attention to your external momenta.
- Make sure that you are drawing different diagrams. Some diagrams could look different but they are actually the same. Pay attention to your interaction vertices. Interaction determines the diagram.
- If it is a 2 to 2 scattering, think about the three possible channels: 1 and 2 could interact; 1 and 3 could interact; 1 and 4 could interact.

Now we can study all sorts of process in QED:

#### Electron electron scattering

$e^-e^- \rightarrow e^-e^-$  scattering is captured in the following Feynman diagram



$$= (-i)(-ie)^2 \left( \frac{\bar{u}_3 \gamma^\mu u_1 \bar{u}_4 \gamma_\mu u_2}{(k_1 - k_3)^2 + i\epsilon} - \frac{\bar{u}_3 \gamma^\mu u_2 \bar{u}_4 \gamma_\mu u_1}{(k_1 - k_4)^2 + i\epsilon} \right) \quad (12.8)$$

## Electron positron scattering

$$\begin{aligned}
 & k_3 \uparrow \\
 & k_1 \nearrow \quad \quad \quad k_4 \nearrow \\
 & \quad \quad \quad k_1 - k_3 \rightarrow \\
 & \quad \quad \quad k_2 \nwarrow \quad + \quad k_1 \nearrow \\
 & \quad \quad \quad k_2 \nwarrow \quad \quad \quad k_4 \nearrow \\
 & \quad \quad \quad k_1 + k_2 \uparrow
 \end{aligned} \tag{12.9}$$

$$= (-i)(-ie)^2 \left( \frac{\bar{v}_1 \gamma^\mu v_3 \bar{u}_4 \gamma_\mu u_2}{(k_1 - k_3)^2 + i\epsilon} - \frac{\bar{u}_4 \gamma^\mu v_3 \bar{v}_1 \gamma_\mu u_2}{(k_1 + k_2)^2 + i\epsilon} \right)$$

## Electron positron annihilation

$$\begin{aligned}
 & k_3 \uparrow \\
 & k_1 \nearrow \quad \quad \quad k_4 \nearrow \\
 & \quad \quad \quad k_1 - k_3 \rightarrow \\
 & \quad \quad \quad k_2 \nwarrow \quad + \quad k_1 \nearrow \\
 & \quad \quad \quad k_1 - k_4 \rightarrow \\
 & \quad \quad \quad k_2 \nwarrow \quad \quad \quad k_4 \nearrow
 \end{aligned} \tag{12.10}$$

$$= i(-ie)^2 \bar{v}_2 \left( \frac{\gamma^\nu (\not{k}_1 - \not{k}_3 + m) \gamma^\mu}{(k_1 - k_3)^2 + i\epsilon} + \frac{\gamma^\mu (\not{k}_1 - \not{k}_4 + m) \gamma^\nu}{(k_1 - k_4)^2 + i\epsilon} \right) u_1 \epsilon_\mu^{3,\text{out}} \epsilon_\nu^{4,\text{out}}$$

## Compton scattering

The scattering amplitude of compton scattering can be written as  $\epsilon_\mu^{4,out} \mathcal{M}^\mu$ . Now we ask an interesting question, what is the polarization of the outgoing photon  $\epsilon^{4,out}$  is parallel to the outgoing photon momentum  $k_4$ ? This physically is asking for the scattering amplitude when the outgoing photon is longitudinal, but longitudinal photons are unphysical. The only way to reconcile that is the scattering amplitude vanish for longitudinal photons  $k_4^\mu \mathcal{M}^\mu = 0^1$ . This is known as Ward identity, which is closely related to the gauge invariance.

### 12.0.3 \*From beginning to end: a cross section in QED

In the end, we want to compute a cross section to hand over to our experimentalists friends to compare with their data. As we have seen in the scalar case, the cross section is

$$d\sigma = d\Phi_2 \frac{|\mathcal{M}_{i \rightarrow f}|^2}{2E_{\mathbf{k}} 2E_p |\mathbf{v}_1 - \mathbf{v}_2|}. \quad (12.12)$$

Scalars do not have spins. When we are doing real experiments with real particles, we do not know the specific spin of the electrons in the initial beams in the collider and our detectors are not sensitive to the spin of the final states. So in order to relate to cross section, we need to average the initial spin and sum the final spin, that is to compute,

$$d\sigma = d\Phi_2 \frac{\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_{i \rightarrow f}|^2}{2E_k 2E_p |\mathbf{v}_1 - \mathbf{v}_2|}, \quad (12.13)$$

<sup>1</sup>David Tong's notes carried out the calculation in details and showed it is indeed zero

for electrons which are spin  $\frac{1}{2}$  massive<sup>2</sup> particles and hence have two different spin states. We have seen the two-body phase space is

$$d\Phi_2 = \frac{d\Omega}{32\pi^2} \frac{2|\mathbf{p}|_{\text{cm}}}{E_{\text{cm}}}. \quad (12.14)$$

We will take care all the kinematics later, let us concentrate on calculating the “initially average, finally summed” matrix element square.

It turns out it's much easier—a still somewhat long calculation—to introduce a new fermion species muon which does exist in nature, and study the process of electron positron scatter to produce muon-anti-muon pair. This is the matrix element square we will compute, and in your homework, you will compute a slightly more complicated one.

The Lagrangian for electron and muon is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_e (i\cancel{D} - m_e) \psi_e + \bar{\psi}_{\text{muon}} (i\cancel{D} - m_{\text{muon}}) \psi_{\text{muon}}, \quad (12.15)$$

and the only<sup>3</sup> tree diagram is

$$= \frac{\bar{v}_r(\mathbf{k})(-ie\gamma^\mu)u_s(\mathbf{p})(-i\eta_{\mu\nu})\bar{u}_{s'}(\mathbf{p}')(-ie\gamma^\nu)v_{r'}(\mathbf{k}')}{(p+k)^2 + i\epsilon}, \quad (12.16)$$

Comparing to the electron positron scattering, the other diagram disappears because there is no interaction vertex that converts an electron to a muon in QED.

If you follow the fermion lines when writing down the matrix element, it should be clear that different species of fermions would never contract each other.<sup>4</sup>

Now a long calculation begins

$$\frac{1}{4} \sum_{s,r,s',r'} |\mathcal{M}|^2 = \frac{1}{4} \sum_{s,r,s',r'} \frac{e^4}{(p+k)^4} (\bar{v}_r(\mathbf{k})\gamma^\mu u_s(\mathbf{p})\bar{u}_{s'}(\mathbf{p}')\gamma_\mu v_{r'}(\mathbf{k}'))^* (\bar{v}_r(\mathbf{k})\gamma^\nu u_s(\mathbf{p})\bar{u}_{s'}(\mathbf{p}')\gamma_\nu v_{r'}(\mathbf{k}')). \quad (12.17)$$

<sup>2</sup>You will learn in the standard model course that the standard model neutrino has spin  $\frac{1}{2}$  and is considered massless and hence possible to have only one state.

<sup>3</sup>which is why it is the simplest.

<sup>4</sup>Of course this changes in standard model, where the flavor changing interaction will change the species of the fermions.

How to take complex conjugate of the matrix element? It is a number and we have  $\#^* = \#^\dagger$ , and we are pretty good at taking Hermitian conjugate of spinors by now, so we have

$$\begin{aligned} (\bar{v}_r(\mathbf{k})\gamma^\mu u_s(\mathbf{p})\bar{u}_{s'}(\mathbf{p}')\gamma_\mu v_{r'}(\mathbf{k}'))^\dagger &= v_{r'}^\dagger(\mathbf{k}')\gamma_\mu^\dagger\gamma_0^\dagger u_{s'}(\mathbf{p}')u_s^\dagger(\mathbf{p})\gamma^{\mu\dagger}\gamma_0^\dagger v_r(\mathbf{k}) \\ &= \bar{v}_{r'}(\mathbf{k}')\gamma_\mu u_{s'}(\mathbf{p}')\bar{u}_s(\mathbf{p})\gamma^\mu v_r(\mathbf{k}), \end{aligned} \quad (12.18)$$

where we used the properties of  $\gamma$  matrices,  $\gamma_0^\dagger = \gamma_0$ ,  $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$ , hence what we have is

$$\begin{aligned} \frac{1}{4} \sum_{s,r,s',r'} |\mathcal{M}|^2 &= \frac{1}{4} \sum_{s,r,s',r'} \frac{e^4}{(p+k)^4} (\bar{v}_{r'}(\mathbf{k}')\gamma_\mu u_{s'}(\mathbf{p}')\bar{u}_s(\mathbf{p})\gamma^\mu v_r(\mathbf{k}))(\bar{v}_r(\mathbf{k})\gamma^\nu u_s(\mathbf{p})\bar{u}_{s'}(\mathbf{p}')\gamma_\nu v_{r'}(\mathbf{k}')) \\ &= \frac{1}{4} \sum_{s,r,s',r'} \frac{e^4}{(p+k)^4} (\bar{v}_{r'}(\mathbf{k}')\gamma_\mu u_{s'}(\mathbf{p}')\bar{u}_s(\mathbf{p})\gamma_\nu v_{r'}(\mathbf{k}'))(\bar{u}_s(\mathbf{p})\gamma^\mu v_r(\mathbf{k})\bar{v}_r(\mathbf{k})\gamma^\nu u_s(\mathbf{p})), \end{aligned} \quad (12.19)$$

where we group spinors with same momenta and spinor labels together so we can perform spin sums.

Now we are going to restore the spinor index so we can freely move things around, otherwise they are matrices that do not commute,

$$\begin{aligned} \sum_{r',s'} \bar{v}_{r'}(\mathbf{k}')_a(\gamma_\mu)_{ab}u_{s'}(\mathbf{p}')_b\bar{u}_{s'}(\mathbf{p}')_c(\gamma_\nu)_{cd}v_{r'}(\mathbf{k})_d &= \sum_{r',s'} v_{r'}(\mathbf{k}')_d\bar{v}_{r'}(\mathbf{k}')_a(\gamma_\mu)_{ab}u_{s'}(\mathbf{p}')_b\bar{u}_{s'}(\mathbf{p}')_c(\gamma_\nu)_{cd} \\ &= \text{Tr}[(\not{k}' - m_{\text{muon}})\gamma_\mu(\not{p}' + m_{\text{muon}})\gamma_\nu]. \end{aligned} \quad (12.20)$$

Similarly, we have

$$\bar{u}_s(\mathbf{p})\gamma^\mu v_r(\mathbf{k})\bar{v}_r(\mathbf{k})\gamma^\nu u_s(\mathbf{p}) = \text{Tr}[(\not{p} + m_e)\gamma^\mu(\not{k} - m_e)\gamma^\nu]. \quad (12.21)$$

Experimentally, a muon is about 200 times as heavy as an electron, we can safely ignore electron's mass, and proceed with,

$$\frac{1}{4} \sum_{s,r,s',r'} |\mathcal{M}|^2 = \frac{1}{4} \frac{e^4}{(p+k)^4} \text{Tr}[(\not{k}' - m)\gamma_\mu(\not{p}' + m)\gamma_\nu] \text{Tr}[\not{p}\gamma^\mu\not{k}\gamma^\nu], \quad (12.22)$$

where we omit the subscript as the electron is taken to be massless from now on.

The trace identities of  $\gamma$  matrices will be very handy now, notice if we have odd number of  $\gamma$  matrices the trace will be zero, and the other two that are useful for this calculation are

$$\begin{aligned} \text{Tr}[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu} \\ \text{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\rho\nu}). \end{aligned} \quad (12.23)$$

Charging ahead, we have

$$\begin{aligned}\text{Tr}[(\not{k}' - m)\gamma_\mu(\not{p}' + m)\gamma_\nu] &= -4m^2g_{\mu\nu} + 4k'^\rho p'^\sigma(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\sigma\mu}) \\ &= -4m^2g_{\mu\nu} + 4(k'_\mu p'_\nu - k'\cdot p' g_{\mu\nu} + k'_\nu p'_\mu),\end{aligned}\quad (12.24)$$

and

$$\text{Tr}[\not{p}\gamma^\mu\not{k}\gamma^\nu] = 4(p^\mu k^\nu - p\cdot k g^{\mu\nu} + p^\nu k^\mu). \quad (12.25)$$

Now we multiply them together

$$\begin{aligned}\text{Tr}[(\not{k}' - m)\gamma_\mu(\not{p}' + m)\gamma_\nu]\text{Tr}[\not{p}\gamma^\mu\not{k}\gamma^\nu] &= (-4m^2g_{\mu\nu} + 4(k'_\mu p'_\nu - k'\cdot p' g_{\mu\nu} + k'_\nu p'_\mu))4(p^\mu k^\nu - p\cdot k g^{\mu\nu} + p^\nu k^\mu) \\ &= 16(-m^2g_{\mu\nu} + k'_\mu p'_\nu - k'\cdot p' g_{\mu\nu} + k'_\nu p'_\mu)(p^\mu k^\nu - p\cdot k g^{\mu\nu} + p^\nu k^\mu) \\ &= 16(-m^2p\cdot k + 4m^2p\cdot k - m^2p\cdot k \\ &\quad + k'\cdot pp'\cdot k - p\cdot kk'\cdot p' + k'\cdot kp'\cdot p \\ &\quad - k'\cdot p'k\cdot p + 4k'\cdot p'p\cdot k - k'\cdot p'p\cdot k \\ &\quad + p'\cdot pk'\cdot k - k'\cdot p'p\cdot k + k\cdot p'p\cdot k') \\ &= 16(2m^2p\cdot k + 2k'\cdot pp'\cdot k + 2k'\cdot kp'\cdot p).\end{aligned}\quad (12.26)$$

So finally we have

$$\frac{1}{4}\sum_{s,r,s',r'}|\mathcal{M}|^2 = \frac{4e^4}{(p+k)^4}(2m^2p\cdot k + 2k'\cdot pp'\cdot k + 2k'\cdot kp'\cdot p). \quad (12.27)$$

Remember the three channels that I keep talking about when dealing with 2 to 2 scattering, they are so useful that they actually have names, or rather the variable associated with them have names. In terms of the external momenta we have in this case, the Mandelstam variables are defined as

$$\begin{aligned}s &\equiv (p+k)^2 = (p' + k')^2 \\ t &\equiv (p-p')^2 = (k-k')^2 \\ u &\equiv (p-k')^2 = (p' - k)^2\end{aligned}\quad (12.28)$$

they corresponding to the momentum square of the internal line for particle 1 and 2, particle 1 and 3, particle 1 and 4 meeting together respectively. Because we neglect electron mass  $p^2 = k^2 = 0$  and  $p'^2 = k'^2 = m^2$ , thus  $s = 2p\cdot k$ ,  $t = m^2 - 2p\cdot p' = m^2 - 2k\cdot k'$  and  $u = m^2 - 2p\cdot k' = m^2 - 2p'\cdot k$ . Note that  $s + t + u = 2m^2$  (this is in general true  $s + t + u = \sum_{\text{external}} m_{\text{ex}}^2$  only two of the Mandelstam variables are independent.).

Thus the above quarter amplitude square can be written as

$$\begin{aligned}\frac{1}{4}\sum_{s,r,s',r'}|\mathcal{M}|^2 &= \frac{4e^4}{s^2}\left(m^2s + \frac{1}{2}(m^2-u)(m^2-u) + \frac{1}{2}(m^2-t)(m^2-t)\right) \\ &= \frac{4e^4}{s^2}\left(\frac{1}{2}u^2 + \frac{1}{2}t^2\right)\end{aligned}\quad (12.29)$$

where in the last line we have taken the massless limit, so we are talking about high energy scattering. Now we ask the question what is the quarter squared amplitude for electron  $\mu$  scattering to electron  $\mu$  scattering  $e^- \mu^- \rightarrow e^- \mu^-$ . Of course we can follow the same steps and do it all over again, but there is actually simple way to get to the answer. This trick is called crossing symmetry.

Crossing symmetry in general is a relationship between scattering amplitudes: the scattering amplitude for any process involving a particle with momentum  $p$  in the initial state is equal to the amplitude for an otherwise identical process but with antiparticle of momentum  $p' = -p$  in the final state.<sup>5</sup>

$$\mathcal{M}(\phi(p) + \dots \rightarrow \dots) = \mathcal{M}(\dots \rightarrow \dots + \bar{\phi}(-p)) \quad (12.30)$$

where  $\bar{\phi}$  is anti-particle of  $\phi$ . So if we know process  $A + B \rightarrow C + D$  occurs, then so does  $A \rightarrow \bar{B} + C + D$  (provided  $A$  is heavy enough) and  $A + \bar{C} \rightarrow \bar{B} + D$  and their scattering amplitudes are related by crossing symmetry.

So let us first apply the crossing symmetry: we know the amplitude of  $e^- e^+ \rightarrow \mu^- \mu^+$  and we want to know the amplitude of  $e^- \mu^- \rightarrow e^- \mu^-$ . we have

$$\mathcal{M}(e^-(p)\mu^-(k) \rightarrow e^-(p')\mu^-(k')) = \mathcal{M}(e^-(p) \rightarrow \mu^-(k')\mu^+(-k)e^-(p')) \quad (12.31)$$

$$= \mathcal{M}(e^-(p)e^+(-p') \rightarrow \mu^-(k')\mu^+(-k)) \quad (12.32)$$

Let us compare the resulting scattering amplitude to the one we have already calculated. The new Mandelstam variables are

$$\begin{aligned} s' &\equiv (p - p')^2 = t \\ t' &\equiv (p - k')^2 = u \\ u' &\equiv (p - (-k))^2 = s \end{aligned} \quad (12.33)$$

Now we can apply crossing symmetry, and immediately have

$$\frac{1}{4} \sum_{s,r,s',r'} |\mathcal{M}|^2 = \frac{4e^4}{s'^2} \left( \frac{1}{2} u'^2 + \frac{1}{2} t'^2 \right) = \frac{4e^4}{t^2} \left( \frac{1}{2} s^2 + \frac{1}{2} u^2 \right) \quad (12.34)$$

Since the result should not depend on which particle we label as 3rd particle and which we label as 4th particle when we perform the crossing symmetry, in which case this symmetry is also called  $s \leftrightarrow t$  crossing symmetry, we will of course get the same result.<sup>6</sup>

Now let us see why crossing symmetry works. Without loss of generality, we still consider 2 particles to 2 particles scattering which is related to the process of 1 particle decays to 2 particles and 1 anti-particle<sup>7</sup> by crossing symmetry. Let us consider the simplest case, where we study a pure scalar theory like in the QFT1 course (actually complex scalar theory, so that there is anti-particle), then we can represent the crossing symmetry in the following picture

<sup>5</sup>Note the second process is unphysical: a physical  $p$  will mean  $-p$  is with negative energy.

<sup>6</sup>We thank Dan Wohns to point this out.

<sup>7</sup>An incoming particle provides the same amount of charge as an outgoing anti-particle, so the appearance of anti-particle is because of charge conservation.

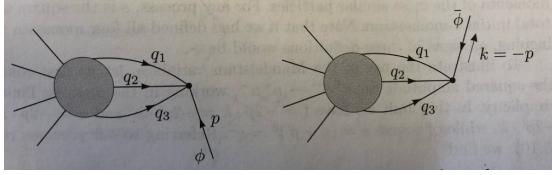


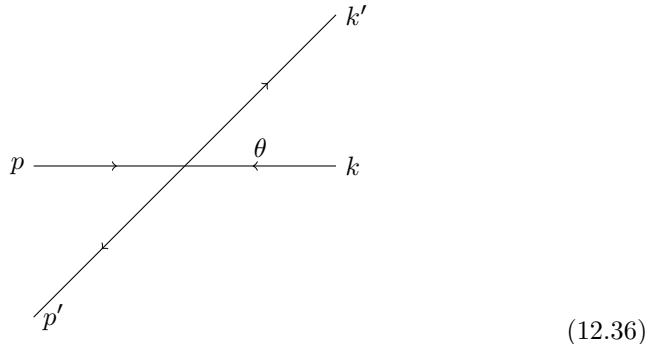
Figure 12.1: This figure is from Peskin and Schroeder.

Note that from the overall momentum conservation, we immediately require the antiparticle's momentum is negative of that of the particle. (And this result of course holds when we have more external legs). The scattering amplitude, according to the Feynman rules, receives contribution from external legs, propagators, vertices, loop integrals. The two processes related by crossing symmetry are different on the external leg but are otherwise identical. Then crossing symmetry works trivially in the scalar case because the Feynman rule for scalar external leg is 1. Now let us consider QED, with external legs being fermions. The fermion case is subtle, but we can see from the spin averaged case

$$\sum u(p)\bar{u}(p) = \not{p} + m = -(-\not{p} - m) = -(\sum v(-p)\bar{v}(-p)) \quad (12.35)$$

So when we state the crossing symmetry, there is one more thing we need to pay attention: for each fermion crossed, the amplitude pick up an overall minus sign.

Now let us come back to evaluate the cross section: we need to go to center of mass frame (or any frame, com is more common), note the figure below is not a Feynman diagram, the arrow on the line indicate the direction of the momentum.



Remember that electron is massless

$$\begin{aligned} p^\mu &= (E, 0, 0, E) & (12.37) \\ k^\mu &= (E, 0, 0, -E) \\ k^{\mu'} &= (E, \mathbf{k}') \\ p^{\mu'} &= (E, -\mathbf{k}'), \end{aligned}$$

where  $|\mathbf{k}'| = \sqrt{E^2 - m^2}$ .

$$(p+k)^2 = 4E^2 \quad (12.38)$$

Now we can calculate

$$\begin{aligned} p \cdot k &= 2E^2 \\ k' \cdot p &= E^2 - |\mathbf{k}'|E \cos \theta \\ p' \cdot k &= E^2 - |\mathbf{k}'|E \cos \theta \\ k' \cdot k &= E^2 + |\mathbf{k}'|E \cos \theta \\ p' \cdot p &= E^2 + |\mathbf{k}'|E \cos \theta. \end{aligned} \quad (12.39)$$

Thus we have

$$\begin{aligned} \frac{1}{4} \sum_{s,r,s',r'} |\mathcal{M}|^2 &= \frac{4e^4}{16E^4} (2m^2 2E^2 + 2(E^2 - |\mathbf{k}'|E \cos \theta)^2 + 2(E^2 + |\mathbf{k}'|E \cos \theta)^2) \\ &= \frac{e^4}{E^4} (m^2 E^2 + E^4 + |\mathbf{k}'|^2 E^2 \cos^2 \theta) \\ &= \frac{e^4}{E^4} (m^2 E^2 + E^4 + (E^2 - m^2) E^2 \cos^2 \theta) \\ &= e^4 \left( \left( 1 + \frac{m^2}{E^2} \right) + \left( 1 - \frac{m^2}{E^2} \right) \cos^2 \theta \right). \end{aligned} \quad (12.40)$$

Now use the cross section formula,

$$\begin{aligned} d\sigma &= \frac{d\Omega}{32\pi^2} \frac{2|\mathbf{p}|_{\text{cm}}}{E_{\text{cm}}} \frac{\frac{1}{4} \sum_{\text{spins}} |M_{i \rightarrow f}|^2}{2E_{\mathbf{k}} 2E_{\mathbf{p}} |\mathbf{v}_1 - \mathbf{v}_2|} \\ &= \frac{d\Omega}{32\pi^2} \times \frac{2\sqrt{E^2 - m^2}}{2E} \times \frac{1}{2E} \frac{1}{2E} \frac{1}{2} e^4 \left( \left( 1 + \frac{m^2}{E^2} \right) + \left( 1 - \frac{m^2}{E^2} \right) \cos^2 \theta \right) \\ &= \frac{\alpha^2}{4(2E)^2} \sqrt{1 - \frac{m^2}{E^2}} \left( \left( 1 + \frac{m^2}{E^2} \right) + \left( 1 - \frac{m^2}{E^2} \right) \cos^2 \theta \right) d\Omega, \end{aligned} \quad (12.41)$$

where we calculate

$$|\mathbf{v}_1 - \mathbf{v}_2| = \frac{|p|}{E} - \left( -\frac{|k|}{E} \right) = 2, \quad (12.42)$$

where again remember electron is treated as massless. And we use the commonly known fine structure constant

$$\alpha_e \equiv \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137.04}. \quad (12.43)$$

Integrate over the angle, we have

$$\begin{aligned}\sigma &= 2\pi \frac{\alpha^2}{4(2E)^2} \sqrt{1 - \frac{m^2}{E^2}} \left( 2 \left( 1 + \frac{m^2}{E^2} \right) + \frac{2}{3} \left( 1 - \frac{m^2}{E^2} \right) \right) \\ &= \frac{4\pi\alpha^2}{3(2E)^2} \sqrt{1 - \frac{m^2}{E^2}} \left( 1 + \frac{1}{2} \frac{m^2}{E^2} \right).\end{aligned}\quad (12.44)$$

In the limit that  $E \gg m$ , this reduced to

$$\sigma \approx \frac{4\pi\alpha^2}{3(2E)^2}. \quad (12.45)$$

This of course agree with our expectation from dimensional analysis.  $\sigma \propto \frac{\alpha^2}{E_{\text{com}}^2}$ . If the spin average and summed scattering amplitude square does not depend on energy, then the only energy dependence comes from the phase space. Thus experimental deviation from the phase space only prediction is important for a test of QED.



# Chapter 13

## Renormalization

### 13.1 Plan

Our plan is to demystify.

The objective is to understand the reasons for renormalization in a practical theory, such as Quantum Electro-Dynamics. Normally, that includes complicated calculation and fancy techniques that makes our vision misty and blurry and asking "why are we doing renormalization?"

In the following, I thought it would be fun to show that we need renormalization in Q:quantum mechanics, E: electric statics and D: Dynamical system like fluids. Then we must also need it in QED.

Thus, we will first make a case for renormalization in simpler scenarios, and in the end make some comments on the beta function of QED.

### 13.2 Definition

A physical problem often consists of many scales and some "bare" parameters<sup>1</sup>. During a renormalization process, we replace the "bare" parameters with "renormalized" quantities (that typically depends on some relevant "scale").

### 13.3 A general scenario: blame the coupling constant

The idea of renormalization does not have to be associated with quantum field theory, it does not have to be associated with any specific theory at all.

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<sup>1</sup>("bare parameters" are usually non-physical but easily available, like the "coupling constant"  $g$ , or in a dynamical system, some statistical randomness as a result of some microscopic interactions)

Here we consider two observables  $F$  and  $G$  and a perturbative coupling  $g$ , some perturbative computation hands us the result

$$\begin{aligned} F &= g + g^2(S+1) + g^3(S+1)^2 + \dots \\ G &= g + g^2(S-1) + g^3(S-1)^2 \end{aligned} \quad (13.1)$$

where  $S$  is some divergent integral or some that renders our answer useless.

What do we do? One thing renormalization taught us is the coupling constant is suspicious, we shall expand results in terms of physical quantities. Hence we expand the coupling in terms of  $G$

$$g = \alpha_1 G + \alpha_2 G^2 + \dots \quad (13.2)$$

And to find out the coefficient, we only have to plug it back to the expression for  $G$  and demand it is consistent

$$G = \alpha_1 G + \alpha_2 G^2 + (\alpha_1 G + \alpha_2 G^2)^2(S-1) \quad (13.3)$$

The first order gives  $\alpha_1 = 1$  and the second order gives  $\alpha_2 + \alpha_1^2(S-1) = 0$  so  $\alpha_2 = -(S-1)$ . Hence

$$g = G - (S-1)G^2 + \dots \quad (13.4)$$

Now we can substitute this in expression of  $F$

$$F = (G - (S-1)G^2) + (G - (S-1)G^2)^2(S+1) + \dots = G + 2G^2 + \dots \quad (13.5)$$

And we get rid of the undesired coupling  $g$  and expanding observable in terms of observable.

But in this case we can actually perform the infinite sum and find

$$F - G = 2FG \quad (13.6)$$

which of course agrees with our perturbative result.

### 13.4 Turbulence diffusion

The idea of renormalization initially comes from statistical physics (which you will see in the statistical physics course). But originally, it comes from an experiment with diffusion of particles in a tank of water. If the water is still, the particles (let's say colored dot) will remain relatively static for a few hours before observable molecular diffusion occurs. This process is described by a molecular diffusion coefficient  $D$  (the "bare" parameter determined by thermal quantities like temperature, pressure, volume). However, if the water is stirred, the colors spread out rapidly. Boussinesq understood this phenomenon by using the same diffusion equation

$$\frac{\partial \rho}{\partial t} = D \nabla^2 \rho, \quad (13.7)$$

but replacing the molecular diffusion coefficient  $D$  with the turbulence diffusion coefficient  $D(r)$  (the renormalized parameter), where  $r$  represents the scale of stirring (i.e, the size of the eddy).

## 13.5 Debye-Hückel's theory of the electron gas

We aim to compute the electrostatic potential at a distance  $r$  from the origin, considering the effect of all the other electrons.

The Coulomb potential for an electron is famously (we have set some proportional constant irrelevant to our discussion to be equal to one)

$$\phi(r) = \frac{e}{r}. \quad (13.8)$$

where  $e$  is the "bare" coupling, which corresponds to the (negative) electric charge of an electron. In the D-H theory, the influence of the electron gas is depicted through a renormalized coupling  $e \rightarrow e(r)$ . Here, we shall try to derive the explicit expression for  $e(r)$ .

We will study an ideal case. There are total of  $N$  free-moving electrons in the system. Additionally, there exists a uniformly distributed positively charge density, ensuring the system is neutral. If the electrons were uniformly distributed, the number density would be  $n_\infty$ . As a result, the average charge density of the positive and negative charges is  $\pm en_\infty$ . It is important to note that the system is in thermal equilibrium.

It is clear how the system is going to behave in the limiting case. When the location we are interested in is very close to one of the electron, the renormalized electric charge  $e(r \rightarrow 0) = e$  as it cannot see the rest of the electric charges. And when the electron is very far away, the renormalized electric charge  $e(r \rightarrow \infty) = 0$  as it will see a neutral cloud.

It is reasonable to assume  $e(r)$  is monotonic; therefore, we have a pretty good idea of the function we are looking for. If it is a simple function, it is probably either a power law or an exponential.

So now let us figure out what it is. The strategy we use is to smear the individual electrons and describe what we have using a charge density field  $\rho(r)$ . We will then use the Poisson's equation to solve for the electric potential

$$\nabla^2 \phi(r) = -4\pi\rho(r). \quad (13.9)$$

Since we still assume the potential takes the Coulomb form, we can read off the renormalized charge. Now, we take the following steps to finish the computation.

### 13.5.1 Figuring out the charge density of electrons

The total charge density (remember the uniform positive charge background and  $e$  is negative) is

$$\rho(r) = en(r) - en_\infty. \quad (13.10)$$

Now we will figure out  $n(r)$  from  $p(r)$  the probability density.

We choose coordinate such that there is an electron at  $r = 0$ . Then, we are interested to know the probability to find a second electron at  $r$ . This is given

by the partition function  $\mathcal{Z}$  from statistical physics.

$$p(r) = \frac{e^{-\frac{E(r)}{kT}}}{\mathcal{Z}}. \quad (13.11)$$

where  $E(r) = \frac{ee(r)}{r}$  represents the energy of the electron in a potential  $\phi(r) = \frac{e(r)}{r}$ , where  $e(r)$  is the renormalized charge. As mentioned before, this energy should vanish as  $r \rightarrow \infty$ . This approach is also known as a mean-field approximation: we replace the effect of all the electrons by a field  $e(r)$ .

Then we can compute the number density in the shell  $n(r)$  as

$$\begin{aligned} n(r) &\equiv Np(r) \\ &= Np(r) \\ &= N \frac{e^{-\frac{E(r)}{kT}}}{\mathcal{Z}}. \end{aligned} \quad (13.12)$$

To highlight the dependence on  $r$ , we compute the number density ratio of  $n(r)$  and  $n(R)$ .

$$\frac{n(r)}{n(R)} = \frac{e^{-\frac{E(r)}{kT}}}{e^{-\frac{E(R)}{kT}}}. \quad (13.13)$$

Now let us take  $R \rightarrow \infty$ ; as we have argued,  $E(r \rightarrow \infty) \rightarrow 0$ , thus

$$\frac{n(r)}{n(\infty)} = e^{-\frac{E(r)}{kT}}. \quad (13.14)$$

But for an electron  $\infty$  away, the number density should be the same as the uniform density  $n(\infty) = n_\infty$ . Hence we find out the number density is

$$n(r) = n_\infty e^{-\frac{E(r)}{kT}}. \quad (13.15)$$

### 13.5.2 Electric potential from Poisson equation

Then we can use Poisson's equation to solve for the potential in the continuous limit

$$\begin{aligned} \nabla^2 \phi &= -4\pi\rho(r) \\ &= -4\pi e n_\infty \left( e^{-\frac{e\phi(r)}{kT}} - 1 \right). \end{aligned} \quad (13.16)$$

Let us restrict to the high temperature case and expand the exponential

$$\begin{aligned} \nabla^2 \phi &= -4\pi e n_\infty \left( -\frac{e\phi}{kT} \right) \\ &= \frac{4\pi e^2 n_\infty \phi}{kT}. \end{aligned} \quad (13.17)$$

We assume the electron gas is isotropic, and use the spherical Laplacian

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\phi) = \phi \times \frac{4\pi e^2 n_\infty}{kT} \quad (13.18)$$

This clearly indicates that  $r\phi$  is an exponential. We choose the decaying solution due to our boundary condition, and express the solution in a way that highlight the characteristic scale of the system.

$$\phi = \frac{e}{r} e^{-\frac{r}{l_D}} \quad (13.19)$$

and the Debye length  $l_D$  is

$$l_D = \left( \frac{4\pi e^2 n_\infty}{kT} \right)^{-\frac{1}{2}}. \quad (13.20)$$

As you can see, this can be interpreted as the Coulomb potential while the "bare" charge is replaced by the renormalized charge that depends on the scale  $r$

$$e \rightarrow e(r) = e \times e^{-\frac{r}{l_D}}. \quad (13.21)$$

The physical interpretation of this result is that each electron has a screening cloud with a radius of  $l_D$  around it. And the continuous limit implies that there are many other electrons in this cloud.

## 13.6 Dimensional Regularization

People often confuse regularization with renormalization. Regularization is a mathematical technique to make sense of infinities. As we have shown in the previous case and some later cases, renormalization does not necessarily involve infinities. However, of course, QFT is plagued with many infinities without being properly renormalized (or dressed)<sup>2</sup>. Many regularization technique emerge, among which introducing a momentum cutoff and then taking the cutoff back to infinity is a popular one that you have seen.

Another popular one is named dimensional regularization because we take the dimension of spacetime to be close to 4:  $d = 4 - \epsilon$ , and hope we can take  $\epsilon$  to zero at the end of the day. This sounds nuts since we don't live in, say, 3.999 dimensions. However, it is a very useful mathematical tool, as we will demonstrate in the following familiar electrostatics example.

### 13.6.1 An infinite line of charge

Consider an infinite line of charge with a constant charge density. Our goal is to determine the electric potential  $V(x)$  at a distance  $x$  away from the line

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<sup>2</sup>According to Arnold Neumaier: it is socially unacceptable for a not-dressed-bare human; and it is also unacceptable for a not-dressed/renormalized theory.

assuming the line extends in the  $y$  direction. Since  $V(x)$  is clearly proportional to the charge density, for simplicity, we will take the density to be one.

Infinitesimally (we have taken some irrelevant proportional constant with a unit of potential to be one

$$dV = \frac{dQ}{\sqrt{x^2 + y^2}}. \quad (13.22)$$

The reasonable thing to do is, of course, integrate along the line, which gives a nonsense result:  $\infty$ .

$$V(x) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{x^2 + y^2}} = \infty. \quad (13.23)$$

Because the line is infinite long, the integration range is from  $-\infty$  to  $\infty$ , we can actually show that the potential (integral) is scale invariant

$$V(x) = V(\lambda x) \rightarrow V(x_1) = V(x_2). \quad (13.24)$$

To prove, we scale  $y$  in the same way so the integrand will be the same and the integration interval, of course stay the same at infinity.

This is a disaster: as the difference of the potential for a test charge tells us how the test charge will move. In this case it is zero, but we know the test charge will certainly move if put next to an infinite line of charge.

This is actually okay. Because the above argument assumes the potential is finite. Infinity has this property that  $\infty - \infty = c \neq 0$  is totally possible. But to extract this  $c$  correctly, we need some work; hence, we need regularization.

### 13.6.2 Cutoff regularization

Let us first review the cutoff regularization. The plan is simple: instead of integrating from  $-\infty$  to  $\infty$ , we integrate from  $-L$  to  $L$ . The potential is then

$$\begin{aligned} V(x) &= \int_{-L}^L \frac{dy}{\sqrt{x^2 + y^2}} \\ &= \ln \left( \frac{+L + \sqrt{L^2 + x^2}}{-L + \sqrt{L^2 + x^2}} \right) \end{aligned} \quad (13.25)$$

Of course, now the result is finite. The bad news is that our electric potential now depends on the cutoff (aka regulator)  $L$ . But it is okey; electric-potential is not physical— the difference is. So we compute the difference and then take  $L \rightarrow \infty$ . After a few lines of algebra/calculus, we arrive at

$$V(x_1) - V(x_2) \rightarrow \ln \frac{x_2^2}{x_1^2}. \quad (13.26)$$

We actually get a sensible result that is independent of the regulator.

So everything is ok, physical observables remain finite. Why the fuss about changing the dimension that kind of crazy stuff? While, the system used to have a translational symmetry (only location in the  $x$  direction matter in our result), in particular moving in parallel to the infinite line does not change the result. This symmetry is broken by the cutoff technique. The integral we perform to get  $V(x)$  is no longer the same if we change variable to  $y' = y + c$ . As we have seen in QFT, the Lorentz symmetry plays an important role. Now we attempt to regularize in a symmetry- preserving way and the price we pay is to change the spacetime dimension a little bit.

### 13.6.3 Dimensional regularization

The essence of dimensional regularization is that the integral that results in the troublesome infinity becomes finite if we live in non-integer dimension.

Instead of performing a one dimensional integral over  $y$ , we generalize to  $d$  dimensional (no integrand here; we will consider specific integrand later)

$$\int dy \rightarrow \int dy^d = \int d\Omega_d \int_0^\infty y^{d-1} dy \quad (13.27)$$

where we separate into the angular direction integral and the radial part (which we will still call it  $y$ ).

The result of the angular integral is a known result: the solid-angle in  $d$  dimensions (it is easy to check it works for low dimensions:  $\Omega_2 = 2\pi$  the circle's circumference for example)

$$\Omega_d = \int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \quad (13.28)$$

Thus, instead of a one dimensional integral we had before, we insist integrating

$$V(x) = \Omega_d \int_0^\infty \left(\frac{y}{L}\right)^{d-1} \frac{dy}{\sqrt{x^2 + y^2}} \quad (13.29)$$

where we are forced to introduce some scale  $L$  so that the dimension of  $V(x)$  does not change.

We will now take  $d = 1 - \epsilon$  as we want to expand around  $d = 1$  and perform the integral

$$\begin{aligned} V(x) &= \Omega_d \int_0^\infty \left(\frac{y}{L}\right)^{-\epsilon} \frac{dy}{\sqrt{x^2 + y^2}} \\ &= \frac{2\pi^{\frac{1-\epsilon}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{x}{L}\right)^{-\epsilon} \frac{\Gamma\left(\frac{\epsilon}{2}\right) \Gamma\left(\frac{d}{2}\right)}{2\sqrt{\pi}} \\ &= \left(\frac{L}{x}\right)^\epsilon \frac{\Gamma\left(\frac{\epsilon}{2}\right)}{\pi^{\frac{\epsilon}{2}}} \end{aligned} \quad (13.30)$$

where we used a trig trick and **Mathematica** to perform the integral.

In order to preserve the translational symmetry, we introduce a dimensional regulator,  $\epsilon$  but we are also forced to introduce a auxiliary scale  $L$ . In the cutoff case,  $L$  serves both as the regulator and the auxiliary scale.

To make sense of this result, let's compute the potential difference

$$\lim_{\epsilon \rightarrow 0} V(x_1) - V(x_2) = \lim_{\epsilon \rightarrow 0} L^\epsilon \frac{\Gamma\left(\frac{\epsilon}{2}\right)}{\pi^{\frac{\epsilon}{2}}} \left( \frac{1}{x_1^\epsilon} - \frac{1}{x_2^\epsilon} \right) \xrightarrow{\text{Mathematica}} \ln \frac{x_2^2}{x_1^2}. \quad (13.31)$$

And we arrive at the same result as before! Interested readers could also take a derivative for both regularization schemes and verify that the electric field strength also agrees (and agrees with the known result from Gauss's law).

### 13.6.4 Sweeping the infinite away

Now we turn to discuss the famous renormalization prescriptions to sweep the infinity away and produce something finite that we use to easily find physical quantities. (As you can see, the taking the limit for regulator style is not easy to extract results.)

The idea is to expand in  $\epsilon$  and drop the infinite part, as we know they will cancel when we look for either the potential difference or the electric field strength (the derivative).

The expansion of our result above is to-be-famously (it is everywhere in QFT dimensional regularization)

$$V(x) = \frac{2}{\epsilon} + \text{finite constant} + \ln\left(\frac{L^2}{x^2}\right) + O(\epsilon) \quad (13.32)$$

<sup>3</sup>

Typically, the minimal subtraction (MS) prescription propose to drop the infinite  $\frac{2}{\epsilon}$  part and the modified minimal subtraction ( $\overline{MS}$ ) prescription propose to drop the finite part as well. As you can see from the process of taking the limit, both work equally well and will give the correct potential difference, the electric field strength.

But stick with the same prescription!

### 13.6.5 Amuse-bouche: Renormalization group

The potential difference  $\delta V$  should not depend on the auxilary scale  $L$

$$\frac{d\delta V}{dL} = 0. \quad (13.33)$$

Since the idea of something invariant is related to symmetry, and symmetry is related to group. This is so called renormaliztion group equation. Often log derivative is used. We will come back to discuss this topic a bit more and QFT 2 will also discuss about this point.

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<sup>3</sup>The finite constant part comes from expanding  $\Gamma(\epsilon) \sim -\gamma_E + \frac{1}{\epsilon}$ , where  $\gamma_E$  is the Euler constant.

## 13.7 Quantum Mechanics

There is another way to talk about these "bare" and "dressed" (renormalized) quantities. As you probably have heard in your General Relativity course: the coordinates are suspicious; the physical quantities do not depend on the coordinates you choose.

"Bare" quantities are kind of like that. We should not be too frustrated with their divergence since they are parameters chosen by some theoretical physicists to write down their model, and maybe they just choose very bad parameters and we get some bad results analogous to some coordinate singularities. And renormalization, in some sense, is the process to look for the "dressed"/physical quantities.

Another myth I mentioned at the beginning is about perturbation theory. So when we are working out simple quantum mechanics problem, we can find exact systems we can solve but yet still prefer to renormalize them because the result could be suspiciously sensitive to the "bare" parameters we choose.

### 13.7.1 An Appetizer: two-state system

Let us consider a two-state system and the Hamiltonian is

$$H = H_0 + H_I. \quad (13.34)$$

where

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix} \quad (13.35)$$

is a system has two energy states at 0 and  $\omega$  and

$$H_I = g \begin{pmatrix} -1 & \Lambda \\ \Lambda & 0 \end{pmatrix}, \quad (13.36)$$

where  $g$  is a dimensionful (energy) coupling and  $\Lambda \gg 1$ .  $g\Lambda$  is often called a cutoff in a QFT. As we can see this interacting Hamiltonian is problematic when we take the limit  $\Lambda \rightarrow \infty$ .

Nevertheless we know how to solve the problem: diagonalize the Hamiltonian to find the energy eigenvalues. Let's do it right now

$$H = \begin{pmatrix} -g & g\Lambda \\ g\Lambda & \omega \end{pmatrix}. \quad (13.37)$$

The characteristic equation for this matrix is

$$\begin{aligned} (E + g)(E - \omega) - g^2\Lambda^2 &= 0 \\ E^2 + (g - \omega)E - g\omega - g^2\Lambda^2 &= 0 \end{aligned} \quad (13.38)$$

This matrix has two eigenvalues at

$$E_{\pm} = \frac{1}{2}(\omega - g \pm \sqrt{(w + g)^2 + 4g^2\Lambda^2}). \quad (13.39)$$

As we can see if  $\Lambda$  is very large, then the formula is extremely sensitive to changes in  $g$ . The extreme sensitivity is a hint that we have very poor choice of parametrization of our model, because in reality we cannot measure a physical parameter that well in the lab! Also the physical quantity energy splitting will be completely dominated by  $g\Lambda$ , also something not desirable.

Maybe the model is not described by the "bare" parameters  $\omega$  and  $g$ . Intuitively, it should be described by the true energy eigenvalues  $E_{\pm}$ .

Now we solve for  $\omega$  and  $g$  in terms of the good parameters.  $E_{\pm}$  satisfy

$$\begin{aligned} E_+ + E_- &= \omega - g \\ E_+ E_- &= -gw - g^2\Lambda^2. \end{aligned} \quad (13.40)$$

We can see that

$$\omega = E_+ + E_- + g \quad (13.41)$$

and  $g$  is the solution of

$$E_+ E_- = -g(E_+ + E_- + g) - g^2\Lambda^2. \quad (13.42)$$

As we can see,  $g$  and hence  $\omega$  depends on  $\Lambda$ .

If the physical energy  $E_{\pm}$  is reasonable (not too large), it is clear  $g = O(\Lambda^{-1})$  in that case. Thus, we propose to use a different parameter (the renormalized parameter)  $g_{ren} \equiv g\Lambda$ . To proceed, we replace the "bare" coupling  $g$  by the renormalized coupling  $g_{ren}$ . And obtain the renormalized Hamiltonian,

$$H_{ren} = \begin{pmatrix} -g_{ren}\Lambda^{-1} & g_{ren} \\ g_{ren} & \omega \end{pmatrix}. \quad (13.43)$$

This is of course completely equivalent to the Hamiltonian we were working with.

This Hamiltonian, on the other hand, is perfectly reasonable to take  $\Lambda \rightarrow \infty$ .

$$H_{ren,\infty} = \begin{pmatrix} 0 & g_{ren} \\ g_{ren} & \omega \end{pmatrix}. \quad (13.44)$$

This is a simple system to solve using the same technique: the eigenvalues are

$$E_{\pm} = \frac{1}{2} \left( \omega \pm \sqrt{\omega^2 + 4g_{ren}^2} \right). \quad (13.45)$$

These are perfectly reasonable physical energies. No more infinities!

So here is another way to understand renormalization: suppose we have a family of Hamiltonians that depends on some huge cutoff and other "bare" parameters, and the physical results (that can be compared with experiments) are extremely sensitive to changes in the "bare" coupling. This is an indication that the extreme sensitivity is an artifact of the parametrization we chose. And we can express the "bare" coupling in terms of the cutoff  $\Lambda$  and some "dressed" (renormalized) coupling, so that the model characterized by the renormalized coupling gives sensible results when we take  $\Lambda \rightarrow \infty$ .

The idea of effective field theory (like all of the Standard Model) is built on the principle of separation: the low energy physics we are interested in is not so significantly impacted by the high energy (large cutoff) physics that we don't know. That is how we are making progress without a proper theory of quantum gravity. In this sense, it is never good to live with a model whose prediction depends sensitively on the cutoff. After all, we do not know at which energy scale new physics starts.

### 13.7.2 \*A variation on the quantum harmonic oscillator

After it is written up, the author found an error to show the sensitivity of the parameter from the original paper. And this part is no longer covered.

#### The eigenvalue problem

Let us assume our Hamiltonian has two parts

$$H = H_0 + H_I. \quad (13.46)$$

The free Hamiltonian is our familiar one of the quantum harmonic oscillator (we have shifted the ground energy to 0 for convenience.)

$$H_0 = \omega a^\dagger a \quad (13.47)$$

where  $a$  and  $a^\dagger$  are the creation and annihilation operator in the quantum harmonic oscillator problem and they enjoy the familiar commutator relationship

$$[a, a^\dagger] = 1. \quad (13.48)$$

This Hamiltonian has a well known set of eigenstates  $|i\rangle, i = 0, 1, 2, \dots$  with eigenvalue  $E_i = i\omega$ .

Now  $|e\rangle \equiv e_i|i\rangle$  is a normalized state in the Hilbert space with  $e_i$ 's being real numbers. In this problem we consider a finite case

$$e_i = 0 \quad \forall i > \text{some finite number.} \quad (13.49)$$

Then the matrix element of the interacting Hamiltonian is

$$\langle i | H_I | j \rangle = -g \langle i | e \rangle \langle e | j \rangle. \quad (13.50)$$

where  $g$  is again a dimensionful ( $[g]=[Energy]$ ) coupling. In this theory we consider eigenvalue equation

$$H|\psi\rangle = E|\psi\rangle \quad (13.51)$$

where  $|\psi\rangle = \sum \psi_k |k\rangle$ . The goal is to solve for all the possible  $E$ 's and their corresponding  $\psi_k$ 's. First substitute  $|\psi\rangle$ 's expansion in the eigenvalue equation gives us (no summing over  $k$ )

$$(E_k - E)\psi_k = \gamma_E e_k \quad (13.52)$$

where  $\gamma_E = g\langle e|\psi\rangle$ . Now we have two situations

- Case one: For some  $l$ ,  $e_l = 0$ . Then we conclude  $E = E_l$ , and  $|\psi\rangle = |l\rangle$ . This is the same as the original eigenvalue and eigenstate. Now let us show the converse is true, assume we find an eigenvalue  $E = E_l$ , then we have  $\gamma_E e_l = 0$ . If  $e_l \neq 0$ , we have  $\gamma_E = 0$ , but that will change the eigenvalue equation to  $(E_k - E)\psi_k = 0 \forall k$ . Then  $\psi_k$  can only be nonzero when  $E_k - E = 0$  which means  $k = l$  and  $\psi_l$  is the only nonzero entry. That means  $|\psi\rangle$  is parallel to  $|l\rangle$ . But then

$$\gamma_E = g\langle e|\psi\rangle = g\psi_l\langle e|l\rangle = ge_l\psi_l = 0 \quad (13.53)$$

implies  $\psi_l = 0$ . This indicates  $|\psi\rangle = 0$ , which contradicts the assumption it is an eigenvector. Thus

$$E = E_l \Leftrightarrow e_l = 0 \quad (13.54)$$

From this we learned that the eigenvalue structure of  $H$  agrees with the original Hamiltonian whenever  $e_l = 0$ .

- Case two: Now let us look at the other case when  $E_k \neq E$ . In this case, we can immediately solve for  $\psi_k$  and we get

$$\psi_k = \frac{\gamma_E e_k}{E_k - E}. \quad (13.55)$$

Now let us use the normalization condition of  $|\psi\rangle$ , which allows us to find

$$\gamma_E = \left( \sum_k \frac{e_k^2}{(E_k - E)^2} \right)^{-\frac{1}{2}}. \quad (13.56)$$

Note that we also have

$$\gamma_E = g e_l \psi_l. \quad (13.57)$$

Combine the last equation and the equation for  $\psi_k$  and eliminate  $\gamma_E$ , we obtain the characteristic equation <sup>4</sup>.

$$g \sum_k \frac{e_k^2}{E_k - E} = 1. \quad (13.58)$$

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<sup>4</sup>note that because we consider a finite situation, the sum is finite.

Now let us denote the lowest index  $k = k_{min}$  such that  $e_k = 0 \forall k < k_{min}$ . In other words we are not so interested if the eigenvalues and eigenvectors below this energy level are exactly the same as the original QHO, and  $E_{k_{min}}$  is the first eigenvalue of the full problem that is different from the original QHO.

Now Consider  $E < E_{k_{min}}$ . The left hand side of the characteristic equation (note that the sum starts at  $k_{min}$ ) increases monotonically (from zero to  $\infty$ ) as we increase  $E$  (but still keep it lower than  $E_{k_{min}}$ ). That means we are guaranteed to find a unique eigenvalue smaller than  $E_{k_{min}}$ .

Now for the subset of  $\{\tilde{k} | e_{\tilde{k}} \neq 0\}$ , consider  $E_{\tilde{k}} < E < E_{\tilde{k}+1}$ . The left hand side of the characteristic equation in this range also increases monotonically (this time from  $-\infty$  to  $\infty$ ) as we increase  $E$ . This tells us there is always a unique eigenvalue between a pair of the original eigenvalues for the non changing ones.

So what is the effect of the interaction? The interaction moves the eigenvalue to slightly smaller values. The eigenvalues slide. This makes sense because it is an attractive potential.

### Renormalization

There is a problem when we consider our characteristic equation at tiny coupling constant  $g$  and large  $\sum_{E_k \neq 0} \frac{e_k^2}{E_k}$ . For  $E < E_{k_{min}}$  (the first eigenvalue we found), when the characteristic equation is approximately satisfied (when we are around the first true eigenvalue  $E$ ), the left hand side changes extremely slowly<sup>5</sup>.

This means we have to acquire knowledge of  $g$  to a ridiculously high accuracy in order to solve the problem to some acceptable level. Or this is an indicator that  $g$ , the "bare" coupling is a problematic parameter to start with.

Thus we should change our variable to something the solution does not so sensitively depend on

$$g = g(\Delta) \equiv \left( \sum_k \frac{e_k^2}{E_k + \Delta} \right)^{-1}. \quad (13.59)$$

where  $\Delta > 0$ . Note that this function ranges from 0 to infinity and it is monotonic increasing with  $\Delta$ . Thus our problem can be described by  $\Delta$  instead of  $g$ .

Put this back in the characteristic equation. An obvious eigenvalue is  $E = -\Delta$  so  $\Delta$  can be interpreted physically as the shift of the ground state eigenvalue. This is certainly an observable that we can measure.

Now taking a difference between  $g^{-1}(\Delta)$  and  $g^{-1}$  solved from the characteristic equation, we obtain the new renormalized characteristic equation for nontrivial eigenvalue, which is  $g$  independent.

$$\sum_k \frac{e_k^2(E + \Delta)}{(E_k + \Delta)(E_k - \Delta)} = 0 \quad (13.60)$$

Because it is  $g$  independent, the previous problem of sensitivity respect to  $g$  is solved. We can make the same argument about how to solve for the sliding eigenvalues as before and they are not very sensitive to  $\Delta$ .

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<sup>5</sup>This is true for all the sliding eigenvalues but it is harder to see

## 13.8 Renormalization in QFT

### 13.8.1 The renormalization procedure: General Remarks

As we have discussed, the idea of renormalization comes from all aspects of physics. The common idea is to replace a non-physical "bare" parameter by a physical "dressed"/renormalized parameter.

However, this is a QFT class after all, so from now on, we will use the common mythical definition: renormalization is an algorithm to get rid of infinities that appear at each order of perturbation theory in almost all QFT. But deep in our hearts, we know we are simply replacing the parameters chosen arbitrarily by the theoretical physicists with the physical parameters that we can measure in the lab, which is a very sensible thing to do.

Instead of studying a particular QFT, let us first look at some general remarks that apply to a class of them. In this class, the quantity we are interested is something very general, say function  $F(x)$  and we only have one "bare" parameter  $g_0$ . Then perturbation theory via Feynman diagrams tells us that a general form of  $F(x)$  looks like

$$F(x) = g_0 + g_0^2 F_1(x) + g_0^3 F_2(x) + \dots \quad (13.61)$$

where  $x$  in reality is something like the incoming 4-momentum. Notice the first term is  $g_0$  itself, that is because in this toy example, we consider  $F(x)$  is the coupling computed via Feynman diagrams and it should agree with the "bare" coupling in the tree level.

Note that since we have only one free "bare" parameter, we only need one measurement to fix it. Knowing  $F(x = \mu)$  is enough to fix  $g_0$ . This means we have the following procedure:

1. Write down a sensible theory (Lorentz invariance, gauge invariance etc).
2. Compute the quantity using perturbation theory up to a chosen order (like 1-loop).
3. Fix the parameter(s) such that our computation (at 1-loop) above agree with the experiment data.

In the end, only the experiment data is the reality. Thus, it should not matter if we parameterize our theory with the "bare" parameter or the "dressed"/renormalized parameter  $F(\mu)$ . Then the renormalization procedure is again replace the "bare" parameter by the "dressed" parameter. The expansion of  $F(x)$  in terms of the "dressed" parameter will no longer be ill-defined. And we get rid of the infinities as a by-product. This is the renormalization hypothesis: the problem is the choice of the parameter  $g_0$ , not the perturbation expansion itself.

Then it makes sense to define the renormalized coupling

$$g_R \equiv F(\mu). \quad (13.62)$$

This renaming is often called a renormalization prescription (a crazy name).

But unfortunately, we cannot directly go back and write up a well-defined (no infinities) expansion of  $F(x)$ , because the only expansion we have is ill-defined.

This is where (cutoff or dimensional) regularization comes in. We need to

1. Employ a regulator (say cutoff  $\Lambda$ ) such that the divergent terms of the expansion  $F_{i,\Lambda}(x)$  (c.f. eq (13.61)) are regular before taking the limit (commonly  $\Lambda \rightarrow \infty$ ).
2. Then perform renormalization: use  $g_R$  to replace  $g_0$ . We hope we can now take the limit (commonly  $\Lambda \rightarrow \infty$ ) and recover the original form of the expansion (but infinity no more, yeah).

These procedure can be implemented as follows

1. Introduce a regulator  $\Lambda$  (e.g. the momentum cutoff) and write the regulator dependent expansion as

$$F_\Lambda(x) \equiv F(x, g_0, \Lambda) = g_0 + g_0^2 F_{1,\Lambda}(x) + g_0^3 F_{2,\Lambda}(x) + \dots \quad (13.63)$$

2. Since everything is finite, we will trade  $g_0$  for  $g_R = F(\mu)$ ,

$$F_\Lambda(x, g_0, \Lambda) \rightarrow F_\Lambda(x, g_R, \mu). \quad (13.64)$$

3. Now take the limit  $\Lambda \rightarrow \infty$  to get rid of the regulator. The renormalization hypothesis states that

$$F(x) = F(x, g_R, \mu) = \lim_{\Lambda \rightarrow \infty} F_\Lambda(x, g_R, \mu). \quad (13.65)$$

Where did the infinity go? Infinities are conserved, we can only move them around but never get rid of them (it is like a squishy). The renormalization procedure has moved them to the relationship between  $g_0$  and  $g_R$ , in other words the "bare" coupling we started with were actually infinite after taking  $\Lambda$  to  $\infty$ .

But this is ok!  $g_0$  is not a physical quantity: we cannot measure it.

In reality, this procedure is implemented order by order. So let us practice it with this toy model!

### 13.8.2 The renormalization procedure: a recipe and a toy model

Recall the general expansion of  $F(x)$  is

$$F(x) = g_0 + g_0^2 F_1(x) + g_0^3 F_2(x) + \dots \quad (13.66)$$

To take a concrete example and to pay respect to the fact that QFT is plagued with infinities, we can take

$$\begin{aligned} F_1(x) &= \alpha \int_0^\infty \frac{dt}{t+x} \\ F_{1,\Lambda}(x) &= \alpha \int_0^\Lambda \frac{dt}{t+x}. \end{aligned} \quad (13.67)$$

where  $\alpha$  denotes some numeric factor (2's and  $\pi$ 's etc). The integral represents (at the 1-loop level) a loop integral over some propagator. As you have seen in Dan's examples, integrating four momenta over some internal propagator is usually where these infinities come from.

1. Order  $g_0$ . At order  $g_0$ , the original expansion (13.61) becomes

$$F(x) = g_0 + O(g_0^2). \quad (13.68)$$

This is independent of  $x$ ! Thus  $g_R = F(\mu) = g_0$ . There is nothing we need to do!

2. Order  $g_0^2$ . Remember, according to the renormalization hypothesis, all we need to do is to replace  $g_0$  by  $g_R$  (c.f. eq(13.64)). However, we don't know this relationship yet. As good perturbative physicists, we expand (the first term is guaranteed from our trivial computation at order  $g_0$ ).

$$g_0 = g_R + \delta_2 g_R^2 + \delta_3 g_R^3 + \dots \quad (13.69)$$

And we already discussed that  $\delta_i$  is where infinities hide, and they diverge. Thus we need a regulator.

Hence, we substitute this into the regulator dependent expansion

$$F_\Lambda(x) \equiv F(x, g_0, \Lambda) = g_0 + g_0^2 F_{1,\Lambda}(x) + g_0^3 F_{2,\Lambda}(x) + \dots, \quad (13.70)$$

and keep it to the second order

$$F_\Lambda(x) = g_R + \delta_2 g_R^2 + g_R^2 F_{1,\Lambda}(x) + \dots \quad (13.71)$$

Apply our renormalization prescription  $F(\mu) = g_R$ , we find explicitly

$$\begin{aligned} \delta_2 &= -F_{1,\Lambda}(\mu) \\ &= -\alpha \int_0^\Lambda \frac{dt}{t+\mu} \\ &= -\alpha \ln \frac{\mu+\Lambda}{\mu}. \end{aligned} \quad (13.72)$$

where the first line applies to any theory; in the second line, we computed explicitly for the toy model. This indeed diverges in the limit  $\Lambda \rightarrow \infty$ .

Now substitute the expression of  $\delta_2$  into the second order expansion eq(13.71)

$$F_\Lambda(x) = g_R + g_R^2 (F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)) + \dots \quad (13.73)$$

We hope  $F(x)$  is well defined as the limit of  $F_\Lambda(x)$  as  $\Lambda \rightarrow \infty$ . Then the divergent part of  $F_{1,\Lambda}(x)$  and  $F_{1,\Lambda}(\mu)$  must cancel each other. However, because  $\mu$  represents a fixed scale at which the experiment is performed, the divergent part of  $F_{1,\Lambda}(x)$  must be a constant, independent of  $x$ . If this happens, the regulator-free  $F(x)$  will be well defined and finite.

Let us take a look at the toy example again. Previously we computed that

$$\delta_2 = -F_{1,\Lambda}(\mu) = -\alpha \int_0^\Lambda \frac{dt}{t+\mu} \quad (13.74)$$

$$\begin{aligned} F(x) &= \lim_{\Lambda \rightarrow \infty} F_\Lambda(x) \\ &= g_R + g_R^2 \lim_{\Lambda \rightarrow \infty} (F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)) + \dots \\ &= g_R + \alpha g_R^2 \lim_{\Lambda \rightarrow \infty} \int_0^\Lambda \left( \frac{dt}{t+x} - \frac{dt}{t+\mu} \right) + \dots \\ &= g_R + \alpha g_R^2 \lim_{\Lambda \rightarrow \infty} \ln \frac{x+\Lambda}{x} - \ln \frac{\mu+\Lambda}{\mu} + \dots \\ &= g_R + \alpha g_R^2 \ln \frac{\mu}{x} + \dots \end{aligned} \quad (13.75)$$

The integral is finite and the answer is well behaved. We have successfully renormalized the theory. The renormalization hypothesis is confirmed to hold in this case at this order.

Now let us make two comments.

- Notice that the way we cancel the infinity is to introduce a divergent term at order 2 to the expansion of  $g_0$  in terms of  $g_R$ , and this divergence cancels the one coming from  $F_1(x)$ . This is a general phenomenon: a divergence coming from  $n$ th order in the  $g_0, g_R$  relationship will cancel the divergence in  $F_{n-1}(x)$ .
  - For the case the divergence of  $F_{1,\Lambda}(x)$  is not merely a constant, it is an indication that our assumption there is only one parameter in the theory is wrong, and the theory is not renormalizable.
3. Let us carry it to two-loop. The reason we need one more loop is that we will be able to see some enlightening/cool structures: we will see that the divergence at two-loop separate into two kinds— one inherited from the one-loop and the other novel. This structure is maintained in higher orders. We can partially sum the expansion and give some closer-to-exact results. It will also be very important when we discuss about renormalization group.

Also it is the first two-loop calculation we can perform within reasonable amount of time! So let us do it!

We expand  $g_0$  to the third order

$$g_0 = g_R + \delta_2 g_R^2 + \delta_3 g_R^3 + \dots \quad (13.76)$$

At the third order, the expansion becomes

$$\begin{aligned} F_\Lambda(x) &= g_0 + g_0^2 F_{1,\Lambda}(x) + g_0^3 F_{2,\Lambda}(x) + \dots \\ &= g_R + \delta_2 g_R^2 + \delta_3 g_R^3 + (g_R^2 + 2\delta_2 g_R^3) F_{1,\Lambda}(x) + g_R^3 F_{2,\Lambda}(x) + \dots \\ &= g_R + g_R^2 (\delta_2 + F_{1,\Lambda}(x)) + g_R^3 (\delta_3 + 2\delta_2 F_{1,\Lambda}(x) + F_{2,\Lambda}(x)). \end{aligned} \quad (13.77)$$

Use the renormalization prescription  $F_\Lambda(\mu) = g_R$ , we recover our one-loop result

$$\delta_2 = -F_{1,\Lambda}(\mu) \quad (13.78)$$

and obtain the new result

$$\delta_3 = 2(F_{1,\Lambda}(\mu))^2 - F_{2,\Lambda}(\mu). \quad (13.79)$$

Now we substitute  $\delta$ 's back, the expansion becomes

$$\begin{aligned} F_\Lambda(x) &= g_R + g_R^2 (F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)) \\ &\quad + g_R^3 (-2F_{1,\Lambda}(\mu)(F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)) + F_{2,\Lambda}(x) - F_{2,\Lambda}(\mu)). \end{aligned} \quad (13.80)$$

For the theory to be renormalizable, we need the  $g_R^3$  term to be finite after taking the limit  $\Lambda \rightarrow \infty$ . In other words, we need the divergent part of that term with  $x$  cancel the term with  $\mu$  to be a constant (independent of  $x$ ). The first half is already a constant due to the success of the previous order. We can even pick this constant to be 0 at one-loop<sup>6</sup>. Of course, we wish the divergent part to be a constant (independent of  $x$ ) at the second order with  $F_{2,\Lambda}$ . Although that is not the case, we now show that the  $x$ -dependent term is completely determined by the previous order.

Let us separate each  $F_{i,\Lambda}$  into a regular part and a divergent part. As we discussed, at first order, we can have the divergent part independent of  $x$ <sup>7</sup>.

$$F_{1,\Lambda}^s(x) = f_1(\Lambda). \quad (13.81)$$

Now let us look at the coefficient of  $g_R^3$ , and with the same argument at the previous footnote, we argue that the divergent part can be taken to be zero by tuning the regular part,

$$\begin{aligned} &-2f_1(\Lambda)(F_{1,\Lambda}^r(x) - F_{1,\Lambda}^r(\mu)) + F_{2,\Lambda}^s(x) - F_{2,\Lambda}^s(\mu) \\ &= F_{2,\Lambda}^s(x) - 2f_1(\Lambda)F_{1,\Lambda}^r(x) - (F_{2,\Lambda}^s(\mu) - 2f_1(\Lambda)F_{1,\Lambda}^r(\mu)) \\ &\xrightarrow{\Lambda \rightarrow \infty} 0 \end{aligned} \quad (13.82)$$

<sup>6</sup>Here is a proof. Suppose  $F_{1,\Lambda}^s(x) - F_{1,\Lambda}^s(\mu) \xrightarrow{\Lambda \rightarrow \infty} f_1^s(x, \mu) \neq 0$ . Note that  $f_1^s(x, \mu)$  is anti-symmetric in the two arguments. Now  $f_1^s(x, \mu) = F_{1,\Lambda}^s(x) - F_{1,\Lambda}^s(\mu) = f_1^s(1, \mu) + f_1^s(x, 1) = f_1^s(x, 1) - f_1^s(\mu, 1)$ . Then the difference in divergence is the difference of the same function evaluated at  $x$  and  $\mu$ , call it  $f'(x)$ . Then redefine  $F_{1,\Lambda}^s(x) \rightarrow F_{1,\Lambda}^s(x) - f'(x)$  will make the divergent part difference to be 0.

<sup>7</sup>In our toy model,  $f_1(\Lambda) = \alpha \log \Lambda$ .

Thus we conclude

$$F_{2,\Lambda}^s(x) - 2f_1(\Lambda)F_{1,\Lambda}^r(x) = f_2(\Lambda) \quad (13.83)$$

where  $f_2(\Lambda)$  is again  $x$ -independent just as before. As you can see, there is an  $x$  dependence, but it is completely determined at the previous order, and ultimately due to the fact that  $\delta_2$  also participate in producing a divergent term at the  $g_R^3$  order.

In a renormalizable theory, this always happens: the divergence of the  $n$ th order contains some part with a precise structure from the previous  $(n-1)$ th order. At order  $n$ , the singular part of  $F_{n,\Lambda}$  contains some completely determined  $x$ -dependent terms and one new term independent of  $x$ .

To summarize, because we started with a "bare" coupling that is non-physical, when we compute observables, we are likely to encounter infinities. To regularize these infinities, we introduce a regulator like  $\Lambda$  and compute the renormalized coupling using perturbation theory. Then we eliminate the "bare" coupling. The expansion now contains no divergences and we can take the limit and get rid of the regulator. Happy ending.

We have also explored a bit of higher-loop and discovered the divergences in each order have a precise structure. We will come back to discuss this when we start the topic of the renormalization group.

The lesson here is, if we start with a "bare" coupling that is infinite, we will have to introduce something else that is infinite to extract the physical/renormalized coupling.

### Renormalization group

As we briefly mentioned before, there is a group associated with renormalization because in this process something is invariant (due to some natural symmetry), thus related to a group.

This idea is realized beautifully through multiple examples in McComb's book, using Ising model on a lattice. Here we will explain it on the field side following "A Hint of Renormalization".

Remember, a crucial step of renormalization is to compare with experiment and carry out a renormalization prescription

$$g_R = F(\mu). \quad (13.84)$$

But why at  $x = \mu$ ? What about  $x = \mu' \rightarrow g'_R \equiv F(\mu')$ ? Or  $x = \mu'' \rightarrow g''_R \equiv F(\mu'')$ ? They should be equally good choices. In other words, for our ultimate goal  $F(x)$ , we must have

$$F(x) = F(x, \mu, g_R) = F(x, \mu', g'_R) = F(x, \mu'', g''_R) = \dots \quad (13.85)$$

In other words, there is an equivalent class of parametrization of the theory and it does not matter which element in this equivalent class is chosen. The physical quantity should be independent of how we parametrize the theory.

Thus we should be able to freely change from one parametrization to another, and this "change" (due to the invariance of physics) should follow a group law: Going from  $(\mu, g_R)$  parametrization to  $(\mu', g'_R)$  and then to  $(\mu'', g''_R)$  is the same as going directly from  $(\mu, R)$  to  $(\mu'', g''_R)$ .

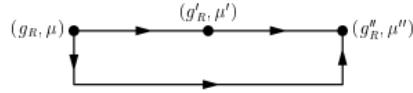


FIG. 1: An illustration of the renormalization group: the two equivalent ways to compose changes of parametrizations.

This group law contains no physical information. It simply states that the theory is reparametrization invariant. It is not a global symmetry that will lead to a physical conservation law. It is more like a pineapple symmetry: if you don't have this invariance, you can no longer talk about a well defined theory that is independent of the scale at which we choose to renormalize.

We can compare this to solving a differential equation with initial condition. If we change the initial condition to be provided at an earlier or later time, we will still arrive at the same solution.

Let us look at a simple problem

$$\dot{y}(t) = \epsilon y(t), \quad y(t_0) = r_0. \quad (13.86)$$

The solution is an exponential

$$y(t) = f(t - t_0, r_0) = r_0 e^{\epsilon(t-t_0)}. \quad (13.87)$$

Then the group law is

$$y(t) = f(t - t_0, r_0) = f(t - \tau, y(\tau) = f(\tau - t_0, r_0)) \forall \tau. \quad (13.88)$$

The last entry signifies the fact that if we choose the initial condition at  $\tau$  to be the same as what the solution implies, then we can use the new initial condition to find the same solution.

This group law can be verified as

$$r_0 e^{\epsilon(t-t_0)} = \underbrace{r_0 e^{\epsilon(\tau-t_0)}}_{\text{new initial condition}} e^{\epsilon(t-\tau)}. \quad (13.89)$$

This group law is the property of the whole theory, and it is actually violated at any order. Expand the two sides we obtain

$$r_0(1 + \epsilon(t - t_0)) = r_0(1 + \epsilon(t - t_0)) + \epsilon^2 r_0(t - \tau)(\tau - t_0) \quad (13.90)$$

And the  $\epsilon^2$  term can be enhanced by an arbitrarily large  $t - t_0$ . In other words, we can consider  $t_0$  to play a role as a cutoff and  $t_0 \rightarrow -\infty$  divergences occur. And the relic of these divergences when  $t_0$  is finite is the source of why the group law is violated at every order.

Now let's do an interesting exercise: we will pretend we don't know the exact solution and we use the perturbation theory to solve it to order one with the result given above. Could we manage to use the information that a group law is obeyed and somehow find the next term?

Thus our strategy is to guess the result of the next order is

$$f(t - t_0, r_0) = r_0(1 + \epsilon(t - t_0) + \epsilon^2 G(t - t_0)) + \dots \quad (13.91)$$

Let us impose the group law is obeyed to the second order

$$r_0(1 + \epsilon(t - t_0) + \epsilon^2 G(t - t_0)) = \underbrace{r_0(1 + \epsilon(\tau - t_0) + \epsilon^2 G(\tau - t_0))}_{\text{new initial condition}} r_0(1 + \epsilon(t - \tau) + \epsilon^2 G(t - \tau)) \quad (13.92)$$

If we expand this and extract the second order, we obtain an equation for this term we guessed:

$$G(t - t_0) = G(t - \tau) + G(\tau - t_0) + (t - \tau)(\tau - t_0). \quad (13.93)$$

To solve this equation, we take a derivative on both side respect to  $t_0$ ,

$$-G'(t - t_0) = -G'(\tau - t_0) - (t - \tau) \quad (13.94)$$

and take  $t_0 = \tau$  and set  $x = t - \tau$

$$G'(x) = G'(0) + x \quad (13.95)$$

And because  $G(0) = 0$ , we integrate and find

$$G(x) = \frac{1}{2}x^2 + ax \quad (13.96)$$

where  $a$  is arbitrary. But when  $a = 0$ , this is exactly the second order perturbative result for our exact solution (exponential functions)!

Thus, we have shown the first order combined with the group law determines the second order. In fact, we can extend this to all orders: the first-order result, together with the group law will allow us to determine all the expansion of the exponential function  $\frac{1}{n!}\epsilon^n(t - t_0)^n$ . The perturbation calculations generate the subdominant terms,  $\frac{1}{n!}\epsilon^n(t - t_0)^p$  with  $p < n$ .

This ends our general discussion. Next, we will come back to our toy model to implement the renormalization group technique.

### Toy model and renormalization group

Our toy model is chosen carefully: the coupling is dimensionless, similar to  $\lambda\phi^4$ , QED, and all the gauge theories. We aim to demonstrate the following crucial result: by dimension analysis and renormalizability constraint, we can determine almost the entire structure of divergences.

Because we are considering a dimensionless coupling,  $F(x)$  is also dimensionless as  $F(\mu) = g_R$ . From the general discussion, we can deduce that  $F(x)$  (where  $x$  has the unity of energy) can only depend on the ratio of  $x$  and some other momentum, which is the cutoff  $\Lambda$ .  $F_\Lambda(x)$  or  $F_{i,\Lambda}^s(x)$  depends solely on  $\frac{x}{\Lambda}$ . It turns out we can make stronger statement:  $\frac{x}{\Lambda}$  is a sum of powers of logarithms with mostly prescribed factors.

Let's begin with  $F_{1,\Lambda}^s(x)$ . We have seen that we can adjust the regular part such that the singular part  $f_1(\Lambda)$  is independent of  $x$ . We should also be able to readjust the regular part such that the regular part  $r(x)$  only depends on  $x$  (because it is regular, we can take the  $\Lambda \rightarrow \infty$  limit.)

$$F_{1,\Lambda}^s(x) = f\left(\frac{x}{\Lambda}\right) = f_1(\Lambda) + r(x) \quad (13.97)$$

$F_{1,\Lambda}^s(x)$  can be separated into a function of  $x$  only and a function of  $\Lambda$  only" for correct phrasing. This separation turns out to be in the form of logarithms only<sup>8</sup>. And

$$F_{1,\Lambda}^s(x) = \alpha \log \frac{\Lambda}{x} \quad (13.98)$$

Now let us press on to show that we can also determine the structure of  $F_{2,\Lambda}^s(x)$  by only dimension analysis and the renormalizability.

Recall that from the success of renormalization we conclude that

$$F_{2,\Lambda}^s(x) = -2\alpha^2 \log \Lambda \log x + f_2(\Lambda). \quad (13.99)$$

But because of dimensionless coupling, this can be a function of  $\frac{\Lambda}{x}$ . As indicated by the first term of  $F_{2,\Lambda}^s(x)$ , we need to include

$$\log^2\left(\frac{\Lambda}{x}\right) = (\log \Lambda)^2 - 2 \log \Lambda \log x + (\log x)^2 \quad (13.100)$$

Then

$$\begin{aligned} F_{2,\Lambda}^s(x) &= -2\alpha^2 \log \Lambda \log x + f_2(\Lambda) \\ &\xrightarrow{\text{add regular term}} -2\alpha^2 \log \Lambda \log x + \alpha^2(\log \Lambda)^2 + f_2(\Lambda) - \alpha^2(\log \Lambda)^2 + \alpha^2(\log x)^2 \\ &= \alpha^2 \log^2\left(\frac{\Lambda}{x}\right) + f_2(\Lambda) - \alpha^2(\log \Lambda)^2. \end{aligned} \quad (13.101)$$

Now for  $f_2(\Lambda) - \alpha^2(\log \Lambda)^2$ , this is a function of  $\Lambda$  and needs to have a function of  $x$  added to become function of  $\frac{\Lambda}{x}$  (sounds familiar?), it is a logarithm!

Thus final result

$$F_{2,\Lambda}^s(x) = \alpha^2 \log^2\left(\frac{\Lambda}{x}\right) + \beta \log \frac{\Lambda}{x}. \quad (13.102)$$

<sup>8</sup> $f\left(\frac{x}{\Lambda}\right) = f_1(\Lambda) + r(x)$ . Take a derivative respect to  $x$ .  $f'\left(\frac{x}{\Lambda}\right) \frac{1}{\Lambda} = r'(x)$  and then take  $x = 1$  and  $\Lambda = \frac{1}{y}$ .  $yf'(y) = r'(1)$ . Integrate gives  $f(y) = -\alpha \log y$ .

The first term is determined; only the second subleading term is new. This process can be repeated. At every order, all but the newest logarithm term is determined by dimensional analysis and renormalizability.

In appendix to be written can show up to order 4 using  $g_0$  and  $g_R$  give similar series.

We can write

$$F_\Lambda(x, g_0, \Lambda) = g_0 + F_\Lambda^s(x, g_0, \Lambda) + F_\Lambda^r(x, g_0, \Lambda). \quad (13.103)$$

Then

$$F_\Lambda(x = \Lambda) = g_0 \quad (13.104)$$

because the rest of the terms are logarithms of  $\frac{x}{\Lambda}$ .

Now we want to study the renormalization group flow of this theory and define

$$\begin{aligned} F_\Lambda(\mu) &= g_R \\ F_\Lambda(\mu') &= g'_R \end{aligned} \quad (13.105)$$

And study different descriptions with different parameterizations of the theory.

For example we found

$$F_\Lambda(x, g_0, \Lambda) = g_0 + \alpha g_0^2 \log \frac{\Lambda}{x} + \dots \quad (13.106)$$

Thus

$$\begin{aligned} g_R &= g_0 + \alpha g_0^2 \log \frac{\Lambda}{\mu} + \dots \\ g'_R &= g_0 + \alpha g_0^2 \log \frac{\Lambda}{\mu'} + \dots \end{aligned} \quad (13.107)$$

We can get rid of  $g_0$  and relate the two renormalized coupling by

$$g'_R = g_R + \alpha g_R^2 \log \frac{\mu}{\mu'} + \dots, \quad (13.108)$$

according to the renormalization group law. Even better: now we get rid of the cutoff  $\Lambda$ . This elimination occurs at each order: in a renormalizable theory, we can always consistently exchange  $(g_0, \Lambda)$  for the renormalized pair  $(g_R, \mu)$ . Additionally, we can introduce an improvement term according to the renormalization group law, and that will determine the next order's leading divergence of the next order as we've previously demonstrated using renormalizability.

In summary, at each order, certain divergences are already indicated by the leading order divergence. Through renormalization constraint/group law we can deduce these higher order divergences.

### The beta function

$$f(g_R, \frac{\mu}{\mu''}) = f(g'_R = f(g_R, \frac{\mu}{\mu'}), \frac{\mu'}{\mu''}). \quad (13.109)$$

Because the regulator  $\Lambda$  is large, the large logarithm will invalid the use of the perturbation theory in function  $f$ . Thus we are interested in the construction of  $f(\Lambda/\mu)$  by performing a series of little steps. Thus we need to know the tangent, the beta function.

$$\beta(g_R) = \frac{\partial g_R}{\partial \log \mu}|_{g_0, \Lambda}. \quad (13.110)$$

For example, for our current case,

$$\beta(g_R) = -\alpha g_0^2 = -\alpha g_R^2 \quad (13.111)$$

Now we can integrate and find

$$g'_R = \frac{g_R}{1 - \alpha g_R \log \frac{\mu}{\mu'}}. \quad (13.112)$$

Several comments about this calculation to follow

- If expanded to the second order, the perturbative result is recovered. This is not surprising: we computed to this order.
- When expand to all orders, we have a whole series of logarithms: These are the logarithms determined by the first order, and we have successfully resummed all of them!
- This expression precisely follows the renormalization group law (check!).
- Moreover, including the already summed logarithm in our perturbative result will not alter the beta function result (check!)

$$g_R = g_0 + \alpha g_0^2 \log \left( \frac{\Lambda}{\mu} \right) + \beta g_0^3 \log^2 \left( \frac{\Lambda}{\mu} \right) + \dots \quad (13.113)$$

The  $\beta$  function is independent of these higher order logarithms!

Now, let's delve into more comments about the beta function.

We use the logarithm derivative to define the beta function and this allows the beta function to remain dimensionless. The beta function cannot depend on the cutoff because the theory is perturbatively renormalizable, the perturbative relation between  $g_R(\mu)$  and  $g'_R(\mu')$  depends only on  $\mu$  and  $\mu'$  but not on  $\Lambda$ . Since beta function is dimensionless, it cannot depend on  $\mu$  as it is the only scale now. Thus, beta function can only depend on  $g_R$  the coupling: In the parameter space: beta function is local. Consequently, the beta function is expanded in terms of  $g_R$  not  $g_R \log \frac{\Lambda}{\mu}$ .

## 13.9 Renormalization in QED

### 13.9.1 Superficial degree of divergence

Any Feynman diagram in QED contains the following components:

$N_e$	number of external electron lines	(13.114)
$N_\gamma$	number of external photon lines	
$P_e$	number of electron propagators	
$P_\gamma$	number of photon propagators	
$V$	number of vertices	
$L$	number of loops	

We define the superficial degree of divergence  $D$  as

$$D \equiv \text{power of momentum in numerator} - \text{power of momentum in denominator} \\ = 4L - P_e - 2P_\gamma \quad (13.115)$$

But it is going to be hard to count the propagators in a diagram like this. Thus

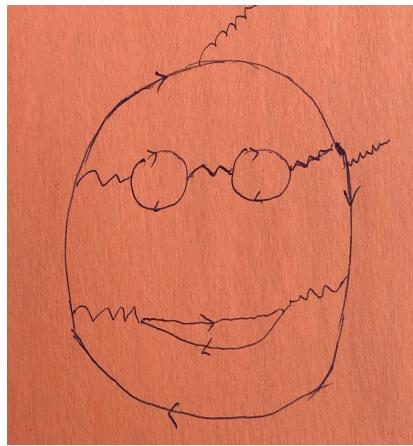


Figure 13.1: A complicated Feynman diagram that remotely resemble a ninja turtle or a mandarin orange.

we use the following relation

$$2V = N_e + 2P_e. \quad (13.116)$$

Each vertex has two electron lines coming out and internal electron lines (propagators) are counted twice.

A similar relation holds for photons

$$V = N_\gamma + 2P_\gamma. \quad (13.117)$$

Finally we use the relation

$$L = P_e + P_\gamma - V + 1 \quad (13.118)$$

This comes from applying Feynman rules without applying the momentum conservation at the vertex, then each propagator corresponds to a momentum integral, each vertex provides one momentum conservation, but there is the over-counting because of the overall momentum conservation. Now let us substitute these relations in  $D$ ,

$$\begin{aligned} D &= 4L - P_e - 2P_\gamma & (13.119) \\ &= 4(P_e + P_\gamma - V + 1) - P_e - 2P_\gamma \\ &= 3P_e + 2P_\gamma - 4V + 4 \\ &= 3P_e + 2P_\gamma - \frac{3}{2}N_e - 3P_e - N_\gamma - 2P_\gamma + 4 \\ &= 4 - \frac{3}{2}N_e - N_\gamma \end{aligned}$$

Turns out the superficial degrees of divergent only has to do with the number of external legs<sup>9</sup>, and there cannot be too many of them!

There are so few of them actually we can list them all: has zero one, two, three or four photon legs, two electron legs, and finally two electron legs and a photon legs.

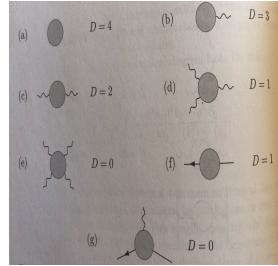


Figure 13.2: Taken from Peskin and Schroeder figure 10.2.

If there are no external legs, it is called the vacuum diagram, which is not very useful for scattering amplitudes. As you will see in the quiz, the diagram

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<sup>9</sup>if you start with a diagram like the T-rex: you can only add an internal photon line (there is no vertex supporting adding a fermion line). If you add an internal photon line you increase the loop by 1, photon line by 1 and fermion line by 2 (because you broke two fermion lines) Then based on  $4L-2N\text{photon}-N\text{fermion}$  you have  $4 - 2 \times 1 - 2 = 0$ . So internal lines and loops don't affect  $D$ . We thank for Ali Saraer Toosi to ask the insightful question.

with only odd number of photon legs (1 or 3) vanishes due to symmetry. Now the diagram with 4-photon legs. We will use the Ward Identity we have learned from compton scattering. The Ward identity states  $k^\mu \mathcal{M}_\mu = 0$ <sup>10</sup>. There are four copies of it because there are four photon legs. This is only possible if the amplitude contains a factor of  $g^{\mu\nu}k^\sigma - g^{\mu\sigma}k^\nu$ , with a similar factor for each other leg. By dimensional analysis, for each momentum factor we introduce, we should introduce it as  $\frac{k}{\Lambda}$ , so each of these momentum factors will lower the superficial degree of divergence by 1. Since this diagram starts with  $D = 0$ , with the symmetry considered it lowers to  $D = 0 - 4 = -4$ . So this diagram is actually finite.

So in the end, there are only three “primitive” divergent amplitudes in QED. They are called primitive because all the other divergent diagrams contain these primitive ones as components. The next few subsections are devoted to computing these primitive divergent diagrams at 1-loop.

### 13.9.2 Renormalization in QED

Just we have seen in all other cases -- as in Quantum mechanics, Electric statics, Dynamical system: when theoretical physicists write down a ”naive” theory, it is likely the hand-picked ”naive” constant-parameters are bad. They are not the (experimentally) good parameters and we should replace them by some physical parameters that we can measure and compare with experimentalists. Often there will be divergences involved, but regardless we shall renormalize/dress our parameters..

So let us write down our ”naive” theory:

$$\mathcal{L}_{naive} = i\bar{\psi}\not{\partial}\psi - m_0\bar{\psi}\psi - e_0\bar{\psi}\gamma_\mu\psi A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \text{gauge terms.} \quad (13.120)$$

There are two naive parameters here:  $m_0$ , the ”bare” mass and  $e_0$ , the ”bare” coupling between the electron and photon. Both are pure number (have no energy dependence). What about mass of photon? The massless of photon is given by the  $U(1)$  gauge symmetry. Renormalization maintains the zero mass of the photon! (don’t turn an apple to a pineapple!)

Ok, let us calculate something with our ”naive” theory then.

”Not so fast!” says L, S, and Z. Let us briefly put on our scalar field hat and review the discussion there. If we are going to compute this interaction theory, presumably we are going to use LSZ reduction formula. We have made some important assumption there, including  $\langle k|\varphi(x)|\Omega = e^{ik\cdot x}$ . On the other hand, we can use translation to move the field to be at  $x = 0$

$$\langle k|\varphi(x)|\Omega = e^{ik\cdot x}\langle k|\varphi(0)|\Omega\rangle. \quad (13.121)$$

There is no reason why  $\langle k|\varphi(0)|\Omega\rangle = 1$ .

---

<sup>10</sup>where we define  $\mathcal{M}^\mu$  as follows: the scattering amplitude with four external photon legs is  $\mathcal{M} = \epsilon^\mu\epsilon^\nu\epsilon^\rho\epsilon^\sigma\mathcal{M}_{\mu\nu\rho\sigma}$ , then we define  $\mathcal{M} = \epsilon^\mu\mathcal{M}_\mu$ .

To agree with other textbooks

$$|\langle k|\varphi(0)|\Omega\rangle|^2 \equiv Z_\varphi \quad (13.122)$$

Similar argument works for a fermion or photon field too! We are forced to rescale our "naive field" to "normalized field"<sup>11</sup>

$$\begin{aligned} \psi_r &\equiv \psi \frac{1}{Z_\psi^{\frac{1}{2}}} & (13.123) \\ A_r^\mu &\equiv A^\mu \frac{1}{Z_A^{\frac{1}{4}}} \end{aligned}$$

Since normalized fields are what used in LSZ formula, it makes sense to write our "naive" theory with them. (From now on we ignore the gauge fixing term<sup>12</sup>)

$$\mathcal{L} = Z_\psi i\bar{\psi}_r \not{\partial} \psi_r - m_0 Z_\psi \bar{\psi}_r \psi_r - e_0 Z_\psi Z_A^{\frac{1}{2}} \bar{\psi}_r \gamma_\mu \psi_r A_r^\mu - \frac{Z_A}{4} F_{r,\mu\nu} F_r^{\mu\nu} \quad (13.124)$$

This is our consistent "naive" Lagrangian we can compute with. As we can see the complication of doing renormalization in field theory is mostly because of the normalization of the field. But "this is the way"<sup>13</sup>.

Now we will follow standard perturbative QFT. From now on, we will omit the indices  $r$  for easy computation. But we should remember, everything field from on is normalized. Free Lagrangian is respect to the normalized field etc.

We separate the Lagrangian into the free part (respect to  $\psi_r$  and  $A_r$  with fermion physical mass being  $m$  (or  $m_R$ )

$$\mathcal{L}_{free} = i\bar{\psi} \not{\partial} \psi - m\bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (13.125)$$

and the interaction part which including a term looked like our QED interaction and the rest called the counter terms<sup>14</sup>.

$$\mathcal{L}_{int} = -e Z_e \bar{\psi} \gamma_\mu \psi A^\mu + \mathcal{L}_{ct}. \quad (13.126)$$

where for book keeping reason we define  $e Z_e \equiv e_0 Z_\psi Z_A^{\frac{1}{2}}$ <sup>15</sup> and  $e$  (or  $e_R$ ) is the physical coupling. We have introduced the counter term Lagrangian

$$\mathcal{L}_{ct} = (Z_\psi - 1) i\bar{\psi} \not{\partial} \psi - (Z_m - 1) m\bar{\psi} \psi - \frac{Z_A - 1}{4} F_{\mu\nu} F^{\mu\nu} \quad (13.127)$$

<sup>11</sup>Note it is normalize, nothing was normalized before, so not renormalize.

<sup>12</sup>The shift in gauge-fixing part of the Lagrangian turns out not to change physics. Interested reader can see a discussion in Casalbuoni chapter 10 (10.147)-(10.156).

<sup>13</sup>If you insist on the simple Lagrangian, you will need a new LSZ and the mode expansion interpretation of the field in terms of creation/annihilation operators will be different too.

<sup>14</sup>They are named so probably because together with the free part, their form looks like the free part if we define some "bare" fields.

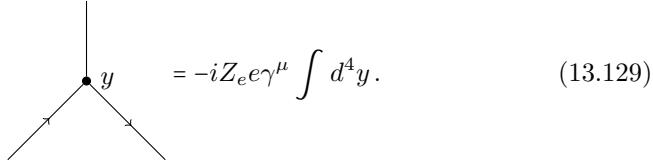
<sup>15</sup>This is the  $g_R$  and  $g_0$  relationship we were discussing about!

where similarly we define  $mZ_m \equiv m_0 Z_\psi$ <sup>16</sup>. We should recover the free theory in the limit  $e \rightarrow 0$  so at the leading order all the  $Z_i$  are unity.

$$Z_i = 1 + \mathcal{O}(e^2) \quad (13.128)$$

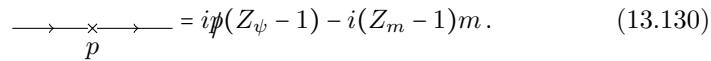
The first order corrections to the  $Z_i$ 's vanish for the same reason as in  $\phi^3$  theory: the corrections come from the Feynman diagrams with two extra vertices.

Compared to our naive theory, the Feynman rule for the vertex changes



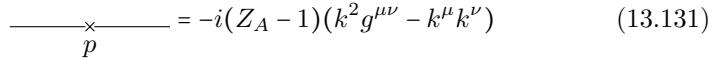
$$= -iZ_e e \gamma^\mu \int d^4 y. \quad (13.129)$$

There is a new vertex where two lines meet that is easier in momentum space



$$= i p (Z_\psi - 1) - i (Z_m - 1) m. \quad (13.130)$$

There is also another new vertex where two photon lines meet.



$$= -i (Z_A - 1) (k^2 g^{\mu\nu} - k^\mu k^\nu) \quad (13.131)$$

In the following subsections we are going to determine  $Z_\psi$ ,  $Z_A$ ,  $Z_m$  and  $Z_e$  at next to leading order. Naively it seems we need four conditions to determine these four parameters. But gauge invariance makes our life simpler: the new Lagrangian is only gauge invariant if  $Z_\psi = Z_e$ . Thus we need only three conditions to determine three parameters. Two of the conditions relate the two physical parameters (electron mass and the coupling  $e_{phys}$ ) of the theory to the parameters that appear in Lagrangian. The other condition has to do with photon propagator: the photon remains massless after renormalization, in other words, the gauge symmetry (which is not a symmetry) is preserved.

The general procedure is clear as before in other cases of renormalization we have seen before:

1. Use perturbation theory to evaluate something physical.
2. Apply the renormalization prescription.
3. Find perturbative expansion for  $Z$ 's. This is where the infinities hide: in the relationship between the "bare" parameter and the physical parameter.
4. Combine to get something nice like the  $\beta$  function. (HW3)

<sup>16</sup>This is the  $g_R$  and  $g_0$  relationship we were discussing about! Again!

The propagator of fermion will provide us two Zs<sup>17</sup> and the propagator of photon will give us another one. The vertex, on the other hand will be fixed by gauge symmetry.

The rest is about details how to compute the Feynman diagram and how to regularize the infinity.

### 13.9.3 \*Electron propagator/self-energy

This part is on calculation intensive details. And it is not much different from the bosonic case aside from  $\gamma$  matrices manipulations. Uninterested readers may skip.

Here we will start to calculate the 1PI (one-particle irreducible diagram, amputated) with two external electron legs, denoted by  $-i\Sigma(p^2)$ <sup>18</sup>.

$$\begin{aligned} -i\Sigma(p^2) &= \text{---} \overset{\curvearrowright}{\text{---}} \text{---} + \text{---} \overset{\curvearrowright}{\text{---}} \text{---} + \mathcal{O}(e^4) \\ &= i(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma^\mu}{((p-k)^2 - m^2)k^2} + i\not{p}(Z_\psi - 1) - i(Z_m - 1)m \end{aligned} \quad (13.132)$$

We are going to compute this integral using Feynman parameters

$$\frac{1}{ab} = \int_0^1 \frac{dz}{(az + b(1-z))^2}. \quad (13.133)$$

The denominator of our integral becomes

$$\begin{aligned} \frac{1}{((p-k)^2 - m^2)k^2} &= \int_0^1 dz \frac{1}{[((p-k)^2 - m^2)z + k^2(1-z)]^2} \\ &= \int_0^1 dz \frac{1}{k^2 + p^2z - 2pkz - m^2z} \\ &= \int_0^1 dz \frac{1}{(k - pz)^2 + p^2z - p^2z^2 - m^2z} \end{aligned} \quad (13.134)$$

Let  $k' = k - pz$ , making use of the fact that when the numerator is linear in  $k'$ , the integral goes to 0, our integral becomes

$$\int_0^1 dz (\gamma_\mu(\not{p}(1-z) + m)\gamma^\mu) \int \frac{d^4k'}{(2\pi)^4} \frac{1}{(k'^2 - \Delta)^2} \quad (13.135)$$

where  $\Delta = m^2z - p^2z(1-z)$ .

It seems useful to know how to do integral of the type  $\int d^4k \frac{1}{(k'^2 - \Delta)^2}$ , except as you have learned in QFT1, it is divergent.

We are going to introduce a new way to regulate this integral: dimensional regularization. We are going analytically continue our space time dimension 4

<sup>17</sup>The Kallen-Lehmann formula determines the pole and the residue, and thus provide two conditions.

<sup>18</sup>The constants are chosen so it contribute to mass  $\Sigma(p^2)$ .

to  $d = 4 - \epsilon$  dimensions. This might seem bizarre, but many divergent integrals in  $d = 4$  become well behaved in  $d = 4 - \epsilon$  dimensions, so we are going to go ahead to do it. You can find in (say Peskin and Schroeder 7.5) the derivation of all sorts of integrals useful in dimension regularization. We are just going to quote the ones that are useful for us (from Peskin and Schroeder equation (A.44))

$$\int \frac{d^d k}{(2\pi)^4} \frac{1}{(k^2 - \Delta)^2} = \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{1}{\Delta}\right)^{2-d/2} \quad (13.136)$$

with the following expansion (if needed) to extract the behavior near  $d = 4$ :

$$\left(\frac{1}{\Delta}\right)^{2-d/2} = 1 - \left(2 - \frac{d}{2}\right) \log \Delta \quad (13.137)$$

and near  $x = 0$ ,

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x)^{19}. \quad (13.138)$$

One more thing about putting dimension to  $d$ : if we have dimension  $d$ , then the dimension for fermion is  $\frac{d-1}{2}$  and dimension for vector boson is  $\frac{d-2}{2}$ , then the dimension for the interaction operator  $\bar{\psi} \mathcal{A} \psi$  is  $\frac{3d}{2} - 2$ , the dimension of the coupling then is  $2 - \frac{d}{2}$ , so in order for  $e$  to remain dimensionless, we need  $e \rightarrow e\mu^{2-\frac{d}{2}}$ , where  $\mu$  is the renormalization scale.

Then the first term in the self-energy 1PI contributed by a loop diagram becomes

$$\begin{aligned} & -ie^2 \mu^{4-d} \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dz (\gamma_\mu (\not{p}(1-z) + m) \gamma^\mu) \Delta^{d/2-2} \\ &= -ie^2 \mu^{4-d} \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dz ((2-d)\not{p}(1-z) + dm) \Delta^{d/2-2} \\ &= -ie^2 \mu^{4-d} \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{2-d}{2}\right) \not{p} + dm + \text{finite} \\ &= -ie^2 \frac{1}{\epsilon/2} \frac{1}{(4\pi)^2} (-\not{p} + 4m) + \text{finite} = -i \frac{e^2}{8\pi^2 \epsilon} (-\not{p} + 4m) + \text{finite} \end{aligned} \quad (13.139)$$

where in the second line we dealt with the  $\gamma$  matrices using the following  $d$  dimension upgraded identities

$$\gamma^\mu \gamma_\mu = d, \quad \gamma^\mu \gamma^\nu \gamma_\mu = (2-d)\gamma^\nu. \quad (13.140)$$

In the third line we have taken the leading contribution of  $\Delta^{d/2-2}$  which is 1 (the rest will eventually contribute to some finite part of the result of the integral) and perform the  $z$  integral.

In last line we use  $\Gamma(x)$  expansion and put  $d = 4 - \epsilon$  and discard all finite contribution. Now the full calculation together with the counter terms is

$$-i\Sigma(p^2) = -i \frac{e^2}{8\pi^2 \epsilon} (-\not{p} + 4m) + \text{finite} + i\not{p}(Z_\psi - 1) - i(Z_m - 1)m \quad (13.141)$$

Compare and use our renormalization condition  $\Sigma(p^2 = m^2) = \Sigma'(p^2 = m^2) = 0$  we have our results

$$\begin{aligned} Z_\psi &= 1 - \frac{e^2}{8\pi^2\epsilon} \\ Z_m &= 1 + \frac{e^2}{2\pi^2\epsilon}. \end{aligned} \tag{13.142}$$

### 13.9.4 Vacuum polarization

We will leave this important study to the homework 3 of beta function of QED.

### 13.9.5 Vertex

The vertex diagram is much more complicated to evaluate. But we dont have to do it:  $Z_e = Z_\psi$  due to gauge symmetry. And we already evaluate  $Z_\psi$  in the electron self energy part.

## Appendix A

# How I learned to stop worrying and love group theory: Visualizing Finite Groups

The first lecture introduces the basic concepts of group theory, such as subgroup, direct product and quotient group with the visualization aid of Cayley diagrams.

### A.1 Eat, Play and Sleep

<sup>1</sup>

Group theory is the study of symmetry. We say an object is symmetrical if some transformations leave the object unchanged. Take an orange as an example<sup>2</sup>, we can turn an orange around its axis by any angle and it looks essentially the same. Once we peel the orange, the continuous rotational symmetry is broken down to a discrete one. And of course, if we eat it, that transforms the orange into something completely different. Many other foods, like pancakes and donuts are also symmetrical.

After devouring the orange, now let us turn our eyes to the famous toy, Rubik's cube. To play the game, we can perform the following moves: take one of the six faces, front, back, up, down, left, right and rotate it clockwise or counter clockwise for a quarter of a turn, then we bring the cube into a new configuration. As these moves are enough, actually, more than enough, to generate all possible configurations, we call them the generators. If we ever regret, no problem, just take the same side and rotate the other way will undo

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<sup>1</sup>a.k.a, learning group theory as a baby

<sup>2</sup>This example is taken from Mirman's " Group Theory: An Intuitive Approach".

the deed. If we combine two or more moves in a certain order indicated by some online algorithm, we can bring the cube to some desirable configuration. The collection of all the possible moves form a group, the Rubik's cube group.

Let us now introduce the formal definition of a group:

**Definition 1:**

A group is a set  $G$  that with a binary operation  $\circ$ <sup>3</sup> defined and satisfies the following properties.

- The group is closed under the binary operation:  $\forall g_1, g_2 \in G$  imply  $g_1 \circ g_2 \in G$ .<sup>4</sup>
- The operation is associative:  $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ .
- The group has an identity:  $\exists e$  such that  $\forall g \in G$ ,  $e \circ g = g \circ e = g$ .<sup>5</sup>
- The group has an inverse:  $\forall g \in G \exists g^{-1}$  such that  $g \circ g^{-1} = g^{-1} \circ g = e$

For the Rubik's cube group, the set is the ensemble of all moves that lead to all possible configurations and the binary operator is to perform two moves in sequence. Since the set contains all possible configuration, the group is clearly closed. Performing moves in sequence implies it is associative. With doing nothing being the identity, each move clearly has an inverse and we also just demonstrated the inverse of a generator move few minutes ago.

To solve this puzzle, we could either explore the group in a random-walk fashion, or follow some online algorithms, but neither is optimal. What would be the best is a Big Book: each page contains one possible configuration, and the locations of all the possible configurations that can be reached by a generator, and a comment on if that move will lead you towards or away from the final destination. Illustrating a group's structure in this fashion is known as the Cayley diagram. Producing the dream Big Book is actually a daunting task for anybody, as it will include more than 43 billion billion pages. So let us put the Rubik's cube down and take a rest.

Although some people believe sleep is for the weak, many are troubled by the question how to flip mattresses<sup>6</sup> to prolong their lifetime. As a theoretical physicist, my first reaction is to consider a spherical mattress in the vacuum. But this problem in reality turns out not to be so hard after all, let us try to solve it together.

First, let us label our mattress to differentiate between different statuses. We realize there are four different statuses, each of which represented by the side on top next to the headboard of the bed. We can therefore label them as "A,B,C,D".

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<sup>3</sup>Be aware sometimes it is referred to as the group multiplication, but the multiplication in a group might have nothing to do with multiplication we know in arithmetic or linear algebra.

<sup>4</sup>Conventionally,  $\forall$  is a short form for "for all".

<sup>5</sup>Conventionally,  $\exists$  is a short form for "there exists".

<sup>6</sup>This example is taken from Brian Hayes' "Group Theory in the Bedroom" published in American Scientist.

For such a mattress, we can perform the following two basic operations: horizontal flip and vertical flip. As we can see, mattress A horizontal flips to C and then back to A, vertical flips to D and then back to A again. If we perform a horizontal flip and then a vertical flip, we arrive at the status B. We can summarize these observations in figure A.1.

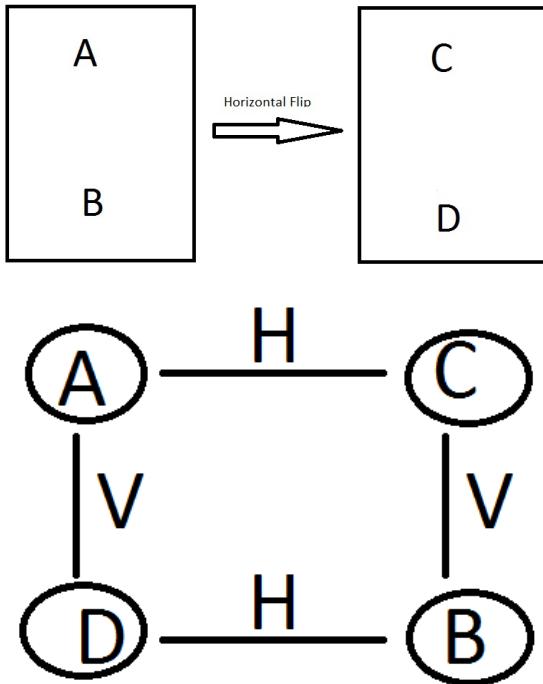


Figure A.1: The left panel showed the way to label the mattress and the right panel shows the relationship between different statuses.

But we realize that this diagram could describe something completely different. If we have a pair of light switches, and the move is to either change the first switch or the second switch, then the resulting 4 statuses are also related in the same way as our mattresses if we label them carefully. Thus we can totally abstract our diagram, and use different colors to represent different operations and obtain the Cayley diagram of the mattress group<sup>7</sup>, also known as the Klein-4 group ( $V_4$ )<sup>8</sup> for mathematicians.

Feynman was refused by army psychiatrist as he was diagonalized with mental illness. During the psychiatric exam, he was asked to extend his two hands.

<sup>7</sup>Shown on the left in A.1

<sup>8</sup>Klein-4 group can also be represented algebraically:

$$\langle a, b | a^2 = e, b^2 = e, ab = ba \rangle \quad (\text{A.1})$$

So he put out his hands with one palm up and the other palm down. The psychiatrist said, “No, the other way.” Thus he flipped both hands, still with one palm up and the other palm down. In this situation, Feynman made a move that is the combination of the two basic moves we have chosen, flipping both hands, which is equivalent to rotate the matrix by 180 degrees. We can also draw this move in the Cayley diagram as well, after rearranging, it looks like the one on the right of figure A.1.

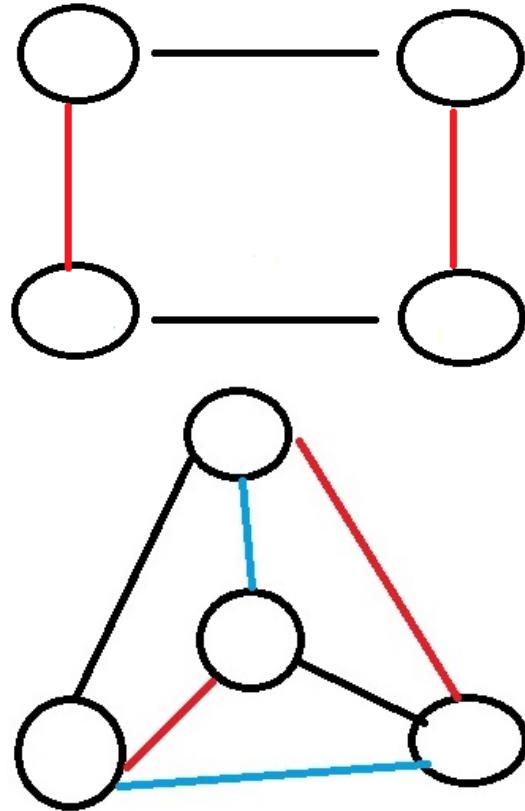


Figure A.2: The left panel shows the Cayley diagram of Klein-4 group derived from the horizontal and vertical flips of a mattress. The right panel is an alternative Cayley diagram for the same group using one more generator, the 180-degree rotation of the mattress.

They are both correct Cayley diagrams as Cayley diagrams do not have to have the minimal sets of generators.

To solve the mattress problem, we need to sample all different status as even as possible. So  $H \rightarrow V \rightarrow H \rightarrow V \dots$  would work, but flipping a mattress

vertically is hard work. Thus the golden rule of the mattress company is to spin in the spring and flip in the fall.

Let us summarize how to draw a Cayley diagram for a group:

1. Pick a set of generators that can generate all the elements of the group, note that they don't have to be minimal.
2. Pick our favorite configuration and draw a starting labeled node to represent it.
3. Pick a node<sup>9</sup> to explore by using one of the generator, draw an arrow that represents the generator, and draw another labeled node if we arrive at a new configuration.
4. Repeat step 3 until we have used all the generators on this node.
5. Repeat step 2-3 until we have been through all nodes.

## A.2 Five Families of Finite Groups

Now let us explore the five families of groups. The first family of groups are the cyclic groups  $C_n$ . They represent the symmetry of the propellers with  $n$  leaves. Algebraically  $C_n$  is represented by  $\langle a | a^n = e \rangle$ . All cyclic groups have the Cayley diagram-representation like the one on the very left in figure A.2.

Nevertheless, we can look at  $C_6, \langle a | a^6 = e \rangle$  more carefully. We have picked  $a$  as our generator previously. Suppose we pick  $a^2$  instead, but this does not generate the entire group, so we need to also pick  $a^3$ , together they generate our group  $C_6$ . So the diagram on the right of figure A.2 also represent  $C_6$ , and we can arrange it to look very nicely. In this rearranged form, we can easily identify parts of the Cayley diagrams that resemble groups that we already know. Thus Cayley diagrams show us the subgroups in a very visual way. In the group  $C_6$ , we find subgroups  $C_2$  and  $C_3$ . To be complete, let us introduce the official definition of subgroups as well:

### Definition 2:

**Subgroup:**  $A$  is a subgroup of  $G$  if we have  $A \subset G$ , such that  $A$  is closed under the group multiplication and the inverse of every element exists. Every group has a trivial subgroup that consists only of the identity.

This is also a good time to introduce the official definition of isomorphic:

### Definition 3:

**Isomorphic:** Two groups  $G$  and  $G'$  are said to be isomorphic  $G \approx G'$  iff<sup>10</sup> there is a one-to-one correspondence between their elements that is preserved under group multiplication.

Since the group structure does not change as we pick different sets of generators to draw Cayley diagrams, the groups that correspond to any version of the  $C_6$  Cayley diagrams are all isomorphic.

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<sup>9</sup>The first time there will be only one choice, the starting node.

<sup>10</sup>Iff is short for if and only if

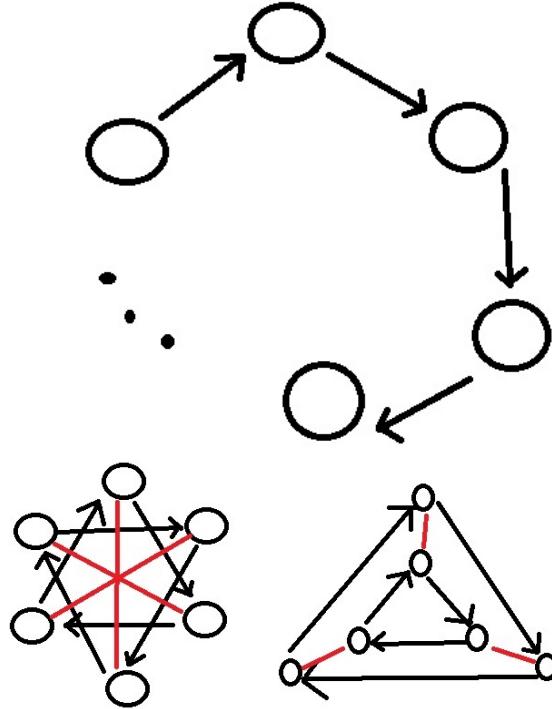


Figure A.3: The left panel shows the Cayley diagrams for all cyclic groups. The right panel shows Cayley diagrams for  $C_6$  with different choice of generators.

Let us move on to the next family of groups, the Abelian groups<sup>11</sup>. An Abelian group has the property that the binary operator of the group is commutative. This is a very strong constraint and makes the Abelian groups simple to study. Actually the fundamental theorem of Abelian groups tell us that all finite Abelian groups are isomorphic to the direct product of several cyclic groups.

In terms of Cayley diagrams, direct product is very simple. Suppose we are interested in studying what is the Cayley diagram for  $C_2 \times C_3$ , we draw Cayley diagram for  $C_2$  first, blow the two nodes up and insert a  $C_3$  Cayley diagram in each of them, and then connect the corresponding nodes with the  $C_2$  generators. This works also the other way, we can blow up  $C_3$  and insert  $C_2$ s. As we can see,  $C_2 \times C_3$  is just a rearrangement of  $C_6$ . This is shown in figure A.2. So in our new fancy language, we say  $C_2 \times C_3$  is isomorphic to  $C_6$ .<sup>12</sup>

Now we introduce the official definition.

**Definition 4:**

<sup>11</sup>What is purple and commute? An Abelian grape.

<sup>12</sup>Note that for our mattress group, we have  $V_4 = C_2 \times C_2$ .

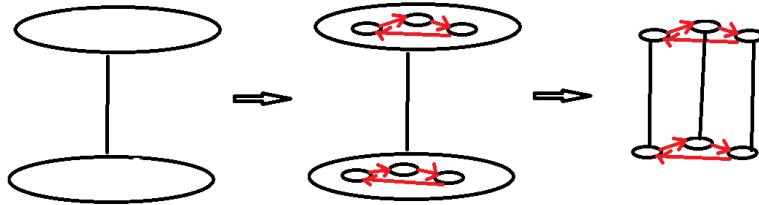


Figure A.4: This graph shows how to use Cayley diagrams to perform direct products of groups.

**Direct product of groups:** The direct product of the groups is defined to be an ordered pair:  $G \oplus G' \equiv \{(g, g') | \forall g \in G, g' \in G'\}$  and the multiplication is defined to be  $(g_1, g'_1) \circ (g_2, g'_2) \equiv (g_1 \circ g_2, g'_1 \circ' g'_2)$ . <sup>13</sup>

As we see, direct product is not very complicated, it introduces a pair of pointers that moves independently in two groups and each different location of the pair of pointers gives a different group element in the direct product group.

Let us move on to the next family, the dihedral groups, also known as the polygon group. The dihedral group is generated by the reflection, which is  $C_2$ , and rotations  $C_n$  within the plane. Let us start with the easiest case, a unigon<sup>14</sup>, also known as a point. A point is more symmetrical than  $D_1$ , but according to the definition of dihedral group, it has the reflection, which is  $C_2$ . A bi-gon, also known as a line, has the symmetry of reflection, and 180 degree rotation. This group is equivalent to the mattress group  $V_4$ . The first non-trivial dihedral group is  $D_3$ , which describes the symmetry of a equilateral triangle. Let us try to draw the Cayley diagram of  $D_3$ . We start with the configuration 123, rotated counterclockwise a third of way, we get 312, and rotate again we get 231, and again we get back to the original configuration. With reflection, we reaches the configuration 132, rotate and get 213, notice that this is the reflection of 231, rotate again and get 321, notice this is the reflection of 312. Thus we obtain figureA.2. Note that we actually went through all the permutations of 123. So  $D_3$  is actually isomorphic to  $S_3$ , where  $S_n$  is the symmetry group of all the  $n$ -permutations. Actually all dihedral groups have similar Cayley diagram, shown in figure A.2

The Cayley diagram looks very similar to the  $C_6$  we encountered before. So let us look at them together again, this time with labels of the nodes in figure A.2

The first thing we immediately notice, is that  $D_3$  is not Abelian, while  $C_6$  is. Thus although  $D_3$  also has  $C_2$  and  $C_3$  as its subgroups, it cannot be the direct product of  $C_2$  and  $C_3$ . So what is the structure of  $D_3$  then? To investigate further, we need to learn a couple more definitions related to subgroups.

**Definition 5:**

<sup>13</sup>We can see  $G_1 \equiv \{(g, e') | \forall g \in G\}$  is automatically an invariant subgroup (see definition 6 next page) of  $G \oplus G'$  and  $G_1 \approx G$ .

<sup>14</sup>Unigon also means the hybrid of unicorn and dragon, which has mystical powers

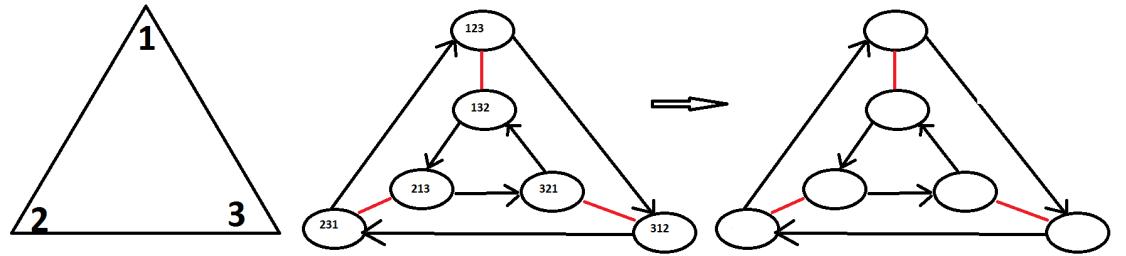
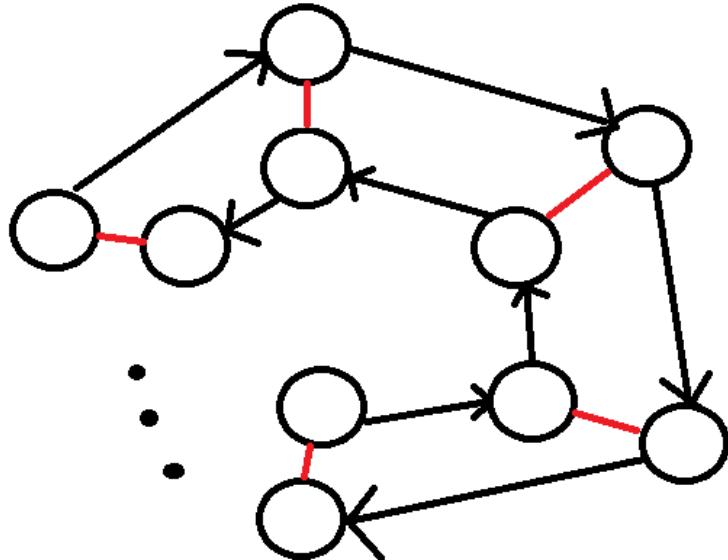
Figure A.5: The Cayley diagram for  $D_3$ .

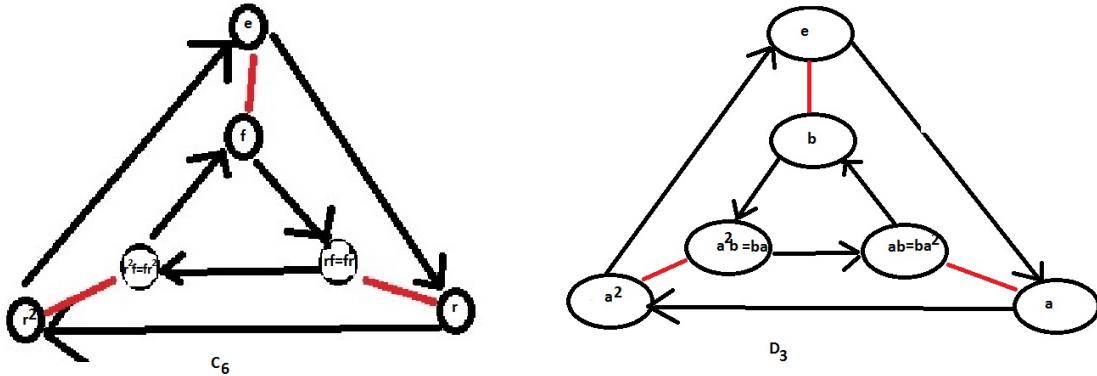
Figure A.6: Cayley diagrams for all dihedral groups.

Cosets: If group  $H$  is group  $G$ 's subgroup, then  $aH$  ( $\forall a \in G$ ) are the left cosets,  $Ha$  are the right cosets.

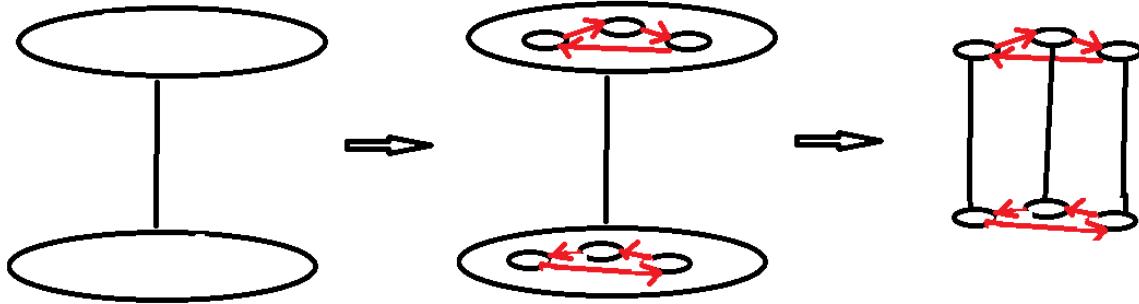
**Definition 6:** Invariant (normal) subgroup: If  $\forall a \in G$ ,  $aH = Ha$ .  $H$  is the invariant subgroup of  $G$ .

As  $C_6$  is Abelian, both  $C_2$  and  $C_3$  are the invariant subgroups. But for  $D_3$  the situation is complicated, the  $C_3$  is still invariant, but the  $C_2$  is not. Note that  $a(e, b) = (a, ab)$  and  $(e, b)a = (a, ba) = (a, a^2b) \neq (a, ab)$ . So if we want to generate  $D_3$  from  $C_2$  and  $C_3$ , instead of two copies of  $C_3$ s, we put a copy of  $C_3$  and a copy of the rewiring of  $C_3$ <sup>15</sup> in the blown up two nodes of  $C_2$ . In other

<sup>15</sup>A rewiring of a Cayley diagram has the same nodes as the original Cayley diagram, but

Figure A.7: A comparison of the Cayley diagram of  $C_6$  and  $D_3(S_3)$ .

words, we say  $D_3$  is the semi-direct product of  $C_2$  and  $C_3$ , with  $C_3$  being the invariant subgroup.  $D_3 = C_3 \rtimes C_2$ . This is shown in figure A.2.

Figure A.8: This graph shows that  $D_3$  is the semi-direct product of  $C_2$  and  $C_3$ .

Now that we have the concept of invariant subgroup, we can learn one more operation with groups. We can find the quotient group  $\frac{G}{H}$ , which is the group formed by all the cosets<sup>16</sup>. In terms of the Cayley diagram, finding quotient group is rather simple, first we find an invariant subgroup and its cosets, and then arrange the Cayley diagram such that each coset is clustered together and cosets are besides each other. We collapse the cosets into single nodes and obtain the quotient group. This is like the reverse process of direct product and semi-direct product<sup>17</sup>. Thus we have  $\frac{C_6}{C_2} = C_3$ ,  $\frac{C_6}{C_3} = C_2$ ,  $\frac{D_3}{C_3} = C_2$ , but we cannot divide  $D_3$  by  $C_2$  as  $C_2$  is not an invariant subgroup.

has arrows arranged differently

<sup>16</sup>Since we are considering invariant subgroup, the left cosets are the same as the right cosets

<sup>17</sup>But not quite, for example, we have  $\frac{C_4}{C_2} = C_2$ , but  $C_4 \neq C_2 \times C_2$

Let us continue our journey with the five families of finite groups, the next family is something you have already encountered, the symmetric or permutation group. The symmetric group is actually very hard to study just by looking at Cayley's theorem.

**Cayley's theorem:**

Any finite group is a collection of permutations.

Thus we will skip ahead to the alternating groups. Alternating groups are basically half of the symmetric group. We take every permutation and square them, and only half of the elements will remain, they will form the alternating groups. Let us look at the platonic solids. The symmetry of the tetrahedron is described by  $A_4$ . The symmetry of the cube and the octahedron are both described by  $S_4$ . The symmetry of dodecahedron and icosahedron are both described by  $A_5$ . Let us look at the simplest example, the tetrahedron. As we can see, there are twelve different configurations. Each of four vertex can be on top, and we can rotate the bottom three vertices. There are the following two kinds of symmetrical operations we can perform on the tetrahedron, rotate a third of a cycle around each vertex and rotate 180 degrees around the axis formed by joining opposite sides of the tetrahedron.

We can label each configuration carefully as before, and produce the Cayley diagram. After reorganizing to be pretty, the Cayley Diagram for  $A_4$  looks like a truncated tetrahedron, shown in figure A.2

Unfortunately, this pretty Diagram does not tell us the structure of  $A_4$ , for that we need to organize the Cayley Diagram according to its invariant subgroup Klein-4 group  $V_4$ . This is shown in figure A.2

Using this diagram, it is easy to see that if we collapse all the cosets, we have  $\frac{A_4}{V_4} = C_3$ . Indeed, we can consider  $A_4$  as a group formed by some rewirings of  $V_4$ , in other words we have  $A_4 = V_4 \rtimes C_3$ .

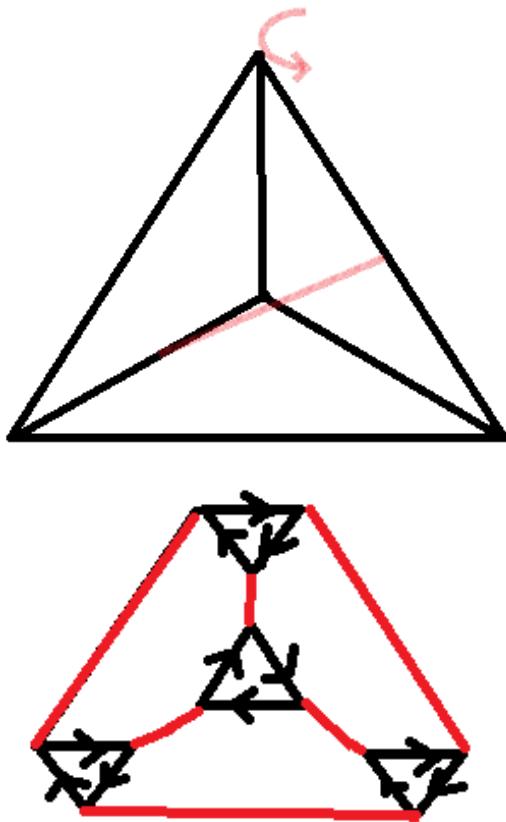


Figure A.9: The left Panel: The symmetries of tetrahedron. The right Panel: One form of Cayley diagram for  $A_4$ .

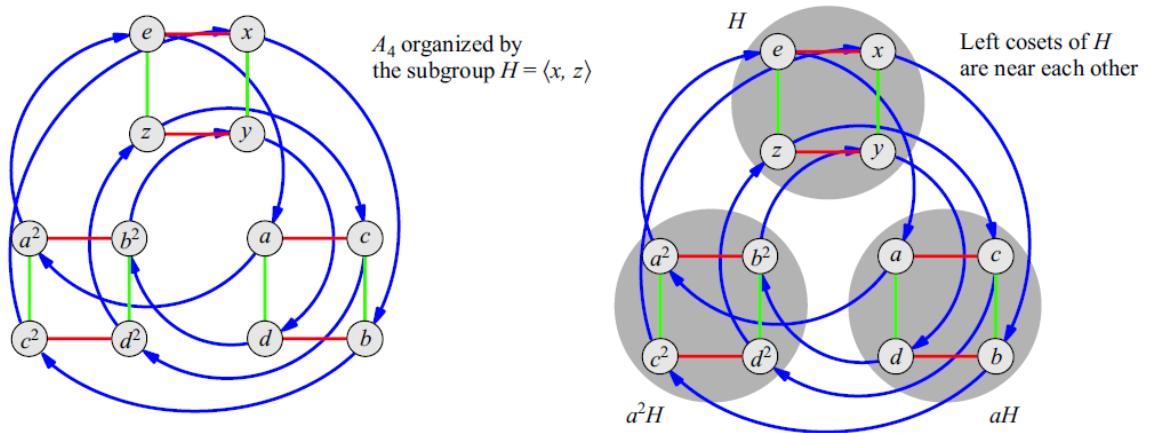


Figure A.10: Cayley diagram taken from Nathan Carter's "Visual Group Theory", for  $A_4$  organized by the  $V_4$ . In this diagram  $x, y, z$ , represents the 180 degree rotation along the axis between two opposite ends. And the  $a, b, c, d$  represent the 120 degree rotation around the vertex.

## Appendix B

# Quotient group explained with a mattress

### B.1 Introduction to mattress group

Mattress group, also known as the Klein-4 group consists 4 elements

$$\{e, H, V, HV\} \quad (\text{B.1})$$

H stands for horizontal flip and V stands for vertical flip. So  $H^2 = V^2 = 1$ .  $HV = VH$  stands for do one flip after the other (the order does not matter). This is an Abelian group.

### B.2 Invariant subgroup and the quotient group

$\{e, H\}$  is clearly subgroup. Since the mattress group is abelian, this subgroup is an invariant subgroup. Now let us create cosets

$$\begin{aligned} \{e, H\}HV &= \{HV, V\} \\ \{e, H\}V &= \{V, HV\} \end{aligned} \quad (\text{B.2})$$

These two cosets are the same. So there are two coset, the subgroup itself and this new one. Now we make a set of the cosets,

$$\{\{e, H\}, \{HV, V\}\}. \quad (\text{B.3})$$

Now we define binary product: the multiplication of the two cosets is defined to be the set of all possible multiplication of the individual elements. Thus you can show

$$\begin{aligned} \{e, H\} \times \{HV, V\} &= \{HV, V\} \\ \{HV, V\} \times \{HV, V\} &= \{e, H\} \end{aligned} \quad (\text{B.4})$$

So the cosets form a group! And this is the quotient group.

### B.3 Why no overlap between cosets?

suppose  $g' \in g_1N$  and  $g' \in g_2N$  but  $g_1N \neq g_2N$ , then

$$g' = g_1h = g_2h' = h''g_2 \quad (\text{B.5})$$

now take a look at  $g_1h = h''g_2$ . The invariant algebra is defined as for any  $h \in N$ ,  $g_1hg_2^{-1} = h''$ , where  $h'' \in N$ . But  $g_1N \neq g_2N$ , so for some  $h$ ,  $g_1h \neq g_2h'$  for any  $h' \in N$ , but this means  $N$  is not invariant. Thus  $g_1N = g_2N$ .

## Appendix C

# Classification of Simple Lie Algebra

### C.1 Our good old friend $SU(2)$ is everywhere

So now let us attempt to classify the simple Lie Algebra by following the same recipe. We will focus on rank 2.<sup>1</sup>

Let us follow our general recipe:

- Step 1: Find maximal sets of commuting generators:  $\{H_1, H_2, \dots, H_r\} \equiv \tilde{H}$  such that  $\forall H_i, H_j$ , we have  $[H_i, H_j] = 0$ . Here  $r$  is the rank of the Lie algebra. We will focus on  $r = 2$ .
- Step 2: Find “eigenvectors” that is linear combinations of other generators  $E_{\vec{\alpha}}$  such that  $[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$ , so  $\vec{\alpha} = \alpha_1, \alpha_2, \dots, \alpha_r \in R^n$ . We again call the  $\vec{\alpha}$  weights. In particular if it is an Adjoint representation, we call them roots, which will be our focus today as well. So all the following  $\vec{\alpha}$ s are roots.
- Step 3: We will be able to use simple geometry on 2D Euclidean space to draw all the  $r = 2$  Lie Algebras!

First we shall notice  $\forall E_{\vec{\alpha}}$ , we have  $E_{\vec{\alpha}}^\dagger = E_{-\vec{\alpha}}$ , because

$$\begin{aligned}[H_i, E_{\vec{\alpha}}^\dagger] &= [E_{\vec{\alpha}}, H_i]^\dagger \\ &= -\alpha_i E_{\vec{\alpha}}^\dagger\end{aligned}\tag{C.1}$$

The above equation shows that  $E_{\vec{\alpha}}^\dagger$  has eigenvalue  $-\alpha_i$  under  $H_i$ , so it is by definition  $E_{-\vec{\alpha}}$ . Let us figure out what  $E_{\vec{\alpha}}|\vec{\mu}\rangle$ , where  $|\vec{\mu}\rangle$  is the eigenstate of  $H_i$  with eigenvalue  $\mu_i$ :

$$H_i|\vec{\mu}\rangle = \mu_i|\vec{\mu}\rangle\tag{C.2}$$

---

<sup>1</sup>The interested reader is referred to chapter 8 of Georgi for the complete classification

This is analogue to  $J_3|m\rangle = m|m\rangle$  Now let us calculate the eigenvalue of the state after acted on by  $E_{\vec{\alpha}}$

$$\begin{aligned} H_i E_{\vec{\alpha}} |\vec{\mu}\rangle &= ([H_i, E_{\vec{\alpha}}] + E_{\vec{\alpha}} H_i) |\vec{\mu}\rangle \\ &= (\alpha_i E_{\vec{\alpha}} + \mu_i E_{\vec{\alpha}}) |\vec{\mu}\rangle \end{aligned} \quad (\text{C.3})$$

This means  $E_{\vec{\alpha}}$  brings us from state  $|\vec{\mu}\rangle$  to  $|\vec{\mu} + \vec{\alpha}\rangle$ . This tells us  $E_{\vec{\alpha}}$  brings us from state  $|- \vec{\alpha}\rangle$  to  $|\vec{\alpha} - \vec{\alpha}\rangle = |0\rangle$  Recall in adjoint representation, we have  $T_a |T_b\rangle = |[T_a, T_b]\rangle$  from last lecture, so

$$E_{\vec{\alpha}} |E_{-\vec{\alpha}}\rangle = [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] \quad (\text{C.4})$$

but this has eigenvalue 0 under all  $H_i$ s, as we just found out. In other words, we have

$$[H_i, [E_{\vec{\alpha}}, E_{-\vec{\alpha}}]] = 0 [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = 0 \quad (\text{C.5})$$

which is to say the commutator commutes with all  $H_i$ . The only way this is going to work, is that  $[E_{\vec{\alpha}}, E_{-\vec{\alpha}}]$  is actually a linear combination of  $H_i$ s, say  $a_i H_i$ .

$$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = a_i H_i \quad (\text{C.6})$$

In adjoint representation, this can be written as

$$E_{\vec{\alpha}} |E_{-\vec{\alpha}}\rangle = a_i |H_i\rangle \quad (\text{C.7})$$

remember every  $|H_i\rangle$  is  $|0\rangle$  as all  $H_i$ s commute. Now we realize  $a_i$  is a matrix element and we have,

$$\begin{aligned} a_i &= \langle H_i | E_{\vec{\alpha}} | E_{-\vec{\alpha}} \rangle \\ &= \langle H_i | [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] \rangle \\ &= \text{Tr}[H_i [E_{\vec{\alpha}}, E_{-\vec{\alpha}}]] \\ &= \text{Tr}[H_i E_{\vec{\alpha}} E_{-\vec{\alpha}} - H_i E_{-\vec{\alpha}} E_{\vec{\alpha}}] \\ &= \text{Tr}[E_{\vec{\alpha}} E_{-\vec{\alpha}} H_i - E_{\vec{\alpha}} H_i E_{-\vec{\alpha}}] \\ &= \text{Tr}[E_{\vec{\alpha}} [E_{-\vec{\alpha}}, H_i]] \\ &= \text{Tr}[E_{\vec{\alpha}} \alpha_i E_{-\vec{\alpha}}] \\ &= \alpha_i \text{Tr}[E_{\vec{\alpha}} \alpha_i E_{\vec{\alpha}}^\dagger] \\ &= \alpha_i \end{aligned} \quad (\text{C.8})$$

This should not be surprising, as we only one  $r$ -vector,  $\vec{\alpha}$  in game.

$$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha} \cdot \vec{H} \quad (\text{C.9})$$

As we can see this really reminds us of our  $\mathfrak{su}(2)$  algebra, which is

$$\begin{aligned} [J_3, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= J_3 \end{aligned} \quad (\text{C.10})$$

Let us rescale our generators,

$$\begin{aligned} E_+ &= aE_{\vec{\alpha}} \\ E_- &= aE_{-\vec{\alpha}} \\ E_3 &= b\vec{\alpha} \cdot \vec{H} \end{aligned} \tag{C.11}$$

and demand they form  $\mathfrak{su}(2)$ ,

$$\begin{aligned} [E_3, E_{\pm}] &= \pm E_{\pm} \\ [E_+, E_-] &= E_3 \end{aligned} \tag{C.12}$$

It is not hard to find that the constants we are looking for are

$$\begin{aligned} b &= \frac{1}{|\vec{\alpha}|^2} \\ a &= \frac{1}{|\vec{\alpha}|} \end{aligned} \tag{C.13}$$

So we have

$$E_3 = \frac{\vec{\alpha} \cdot \vec{H}}{|\vec{\alpha}|^2} \tag{C.14}$$

Suppose we have another  $E_{\vec{\alpha}'}$  with the same eigenvalue  $\vec{\alpha}$ . Then we have

$$[E_{-\vec{\alpha}}, E_{\vec{\alpha}'}] = 0$$

as we have shown that  $\vec{H}E_{\vec{\alpha}}|E_{\vec{\beta}}\rangle = (\vec{\alpha} + \vec{\beta})E_{\vec{\alpha}}|E_{\vec{\beta}}\rangle = (\vec{\alpha} + \vec{\beta})|E_{\vec{\alpha} + \vec{\beta}}\rangle$ . Thus we must have  $|E_{\vec{\alpha}'}\rangle$  as the lowest state, as it is annihilated by the lowering operator. On the other hand it is supposed to have eigenvalue 1 under  $E_3$ , because  $E_3|E_{\vec{\alpha}'}\rangle = |E_{\vec{\alpha}'}\rangle$ . This is inconsistent, as the lowest weight cannot be positive. Thus  $E_{\vec{\alpha}}$  is unique for given root vector  $\alpha$ .

## C.2 What does simple geometry tell us?

Assume that we can find another pair  $E_{\vec{\beta}}, E_{-\vec{\beta}}$ , that satisfies  $H_i|E_{\vec{\beta}}\rangle = \beta_i|E_{\vec{\beta}}\rangle$ . We can ask what is  $E_3|E_{\beta}\rangle$ ?

$$\begin{aligned} E_3|E_{\beta}\rangle &= \frac{\vec{\alpha} \cdot \vec{H}}{|\vec{\alpha}|^2}|E_{\beta}\rangle \\ &= \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}|^2}|E_{\beta}\rangle \end{aligned} \tag{C.15}$$

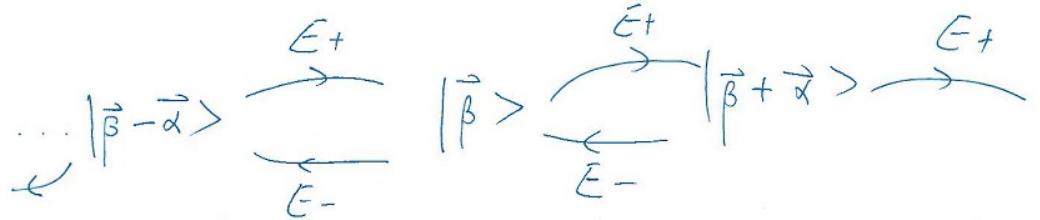
But this is an eigenvalue of our old friend  $J_3$ , and we already know the eigenvalues can only be half integers, so we have

$$2 \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}|^2} = \text{integer} \tag{C.16}$$

Now let us see what our raising operator  $E_+$  will do to  $|E_{\vec{\beta}}\rangle$ ,

$$\begin{aligned} E_3 E_+ |E_{\vec{\beta}}\rangle &= (E_+ E_3 + E_+) |E_{\vec{\beta}}\rangle \\ &= \left( \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}|^2} + 1 \right) E_+ |E_{\vec{\beta}}\rangle \\ &= \frac{\vec{\alpha} \cdot (\vec{\beta} + \vec{\alpha})}{|\vec{\alpha}|^2} E_+ |E_{\vec{\beta}}\rangle \end{aligned} \quad (\text{C.17})$$

This is very interesting, as it tells us this is what our raising and lowering operator do. The raising operator takes us from state  $|\vec{\beta}\rangle$  to  $|\vec{\beta} + \vec{\alpha}\rangle$ . So again, if



we focus on the finite representations, there is highest weight  $j$ . And suppose that takes our  $p$  steps of  $E_+$  from  $|\beta\rangle$  to get up to  $|j\rangle$

$$\frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}|^2} + p = j \quad (\text{C.18})$$

and it takes  $q$  steps of  $E_-$  from  $|\beta\rangle$  to get down to the lowest weight state  $|-j\rangle$

$$\frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}|^2} - q = -j \quad (\text{C.19})$$

Let us get rid of the highest weight dependence, we have

$$2 \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}|^2} + p - q = 0 \quad (\text{C.20})$$

which tells us  $2 \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}|^2}$  must be an integer, as  $p$  and  $q$  are integers, but we already know that... so what is so amazing about that? The amazing thing is that we can do the same calculation switching the roles of  $\vec{\alpha}$  and  $\vec{\beta}$ , and of course this formula still holds, with  $\vec{\alpha}$  and  $\vec{\beta}$  switched.

$$2 \frac{\vec{\beta} \cdot \vec{\alpha}}{|\vec{\beta}|^2} + p' - q' = 0 \quad (\text{C.21})$$

And now we find something truly amazing,

$$4 \frac{(\vec{\beta} \cdot \vec{\alpha})^2}{|\vec{\alpha}|^2 |\vec{\beta}|^2} = (p - q)(p' - q') \quad (\text{C.22})$$

As the left hand side the equation is nothing but  $4\cos^2(\theta_{\vec{\alpha}\vec{\beta}})^2$  and this is an integer!

As we all know the square of the cosine function is bounded by 0 and 1, and with the new constraint, we obtain the following table,

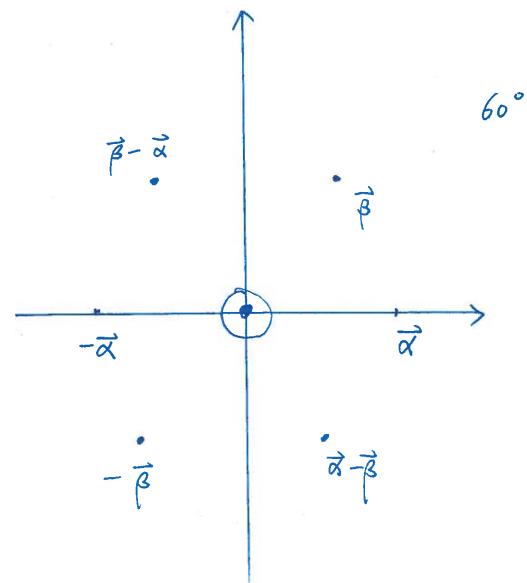
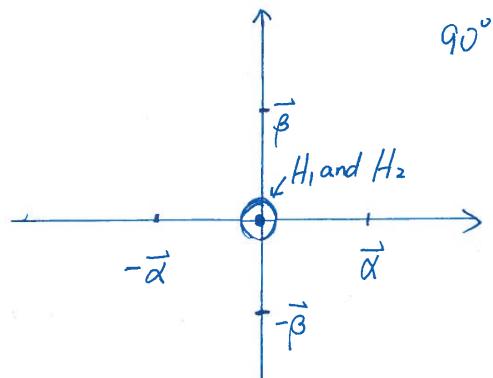
$\cos^2(\theta_{\vec{\alpha}\vec{\beta}})$	$\theta_{\vec{\alpha}\vec{\beta}}$
$\frac{0}{4}$	$90^\circ$
$\frac{1}{4}$	$60^\circ, 120^\circ$
$\frac{2}{4}$	$45^\circ, 135^\circ$
$\frac{3}{4}$	$30^\circ, 150^\circ$
$\frac{4}{4}$	$0^\circ, 180^\circ$

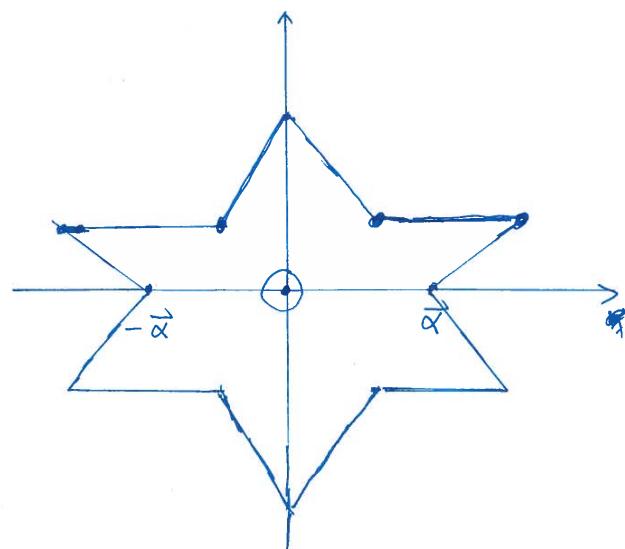
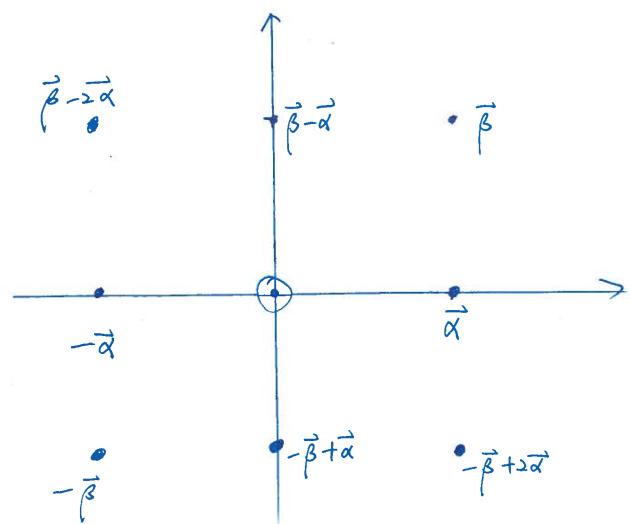
That is to say, the angle between *any* pair of roots of a Lie algebra can only be chosen from the above list! This turns out to be a strong enough constraint for us to sort out all the possible Lie algebras. So now we are ready to classify all the rank 2 Lie algebra just by drawing roots. As we have already picked the two mutually commuting generators  $H_1$  and  $H_2$ , a rank 2 Lie algebras always has two roots at the origin. So let us start by trying with the first angle,  $90^\circ$ . We end up with two  $\mathfrak{su}(2)$  subalgebras that are orthogonal. So they are invariant subalgebra,  $|\vec{\beta}\rangle$  transforms as a singlet of the  $\vec{\alpha}$   $\mathfrak{su}(2)$ . This is  $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ . When we try with  $60^\circ$ , we get an algebra with 8 roots, this is our  $\mathfrak{su}(3)$ , or  $A_2$ .  $\vec{\beta}$  and  $\vec{\beta} - \vec{\alpha}$  transform as a doublet under the  $\mathfrak{su}(2)$  subalgebra. When we try with  $45^\circ$ , we get 10 roots, this is our  $\mathfrak{so}(5)$ , also known as  $B_2$ . When we try with  $30^\circ$ , we get 14 roots, this is a bit weird, and is called  $G_2$ .

If we continue to higher ranks, we will be able to find all the possible simple Lie algebra using this root method summarized in Dynkin diagrams. Interested reader is referred to Georgi's chapter 8 for more details. In Dynkin's system, we only draw the simple roots. We define the simple roots in the following way: First we define positive roots: on the root diagram, draw an arbitrary  $r - 1$  plane that pass through origin. We call the ones on one side positive roots. For the positive roots, we find the  $r$  linear independent ones that we can express all other roots as linear combinations of with positive coefficients. These  $r$  roots are called simple roots.

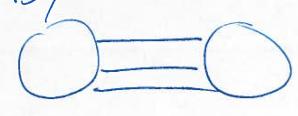
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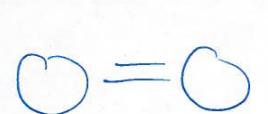
<sup>2</sup> $\theta_{\vec{\alpha}\vec{\beta}}$  denotes the angle between  $\vec{\alpha}$  and  $\vec{\beta}$  in usual sense.

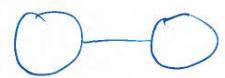




Dynkin

 if angle is  $150^\circ$

 if angle is  $135^\circ$

 if angle is  $120^\circ$

 if angle is  $90^\circ$   
filled means short vector

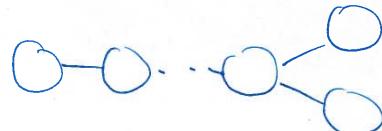
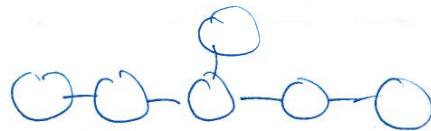
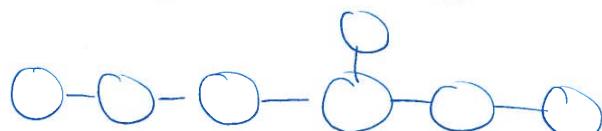
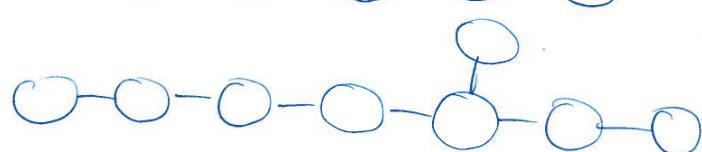
$SU(2)$  

$SU(3)$  

$SO(4)$  

$SO(5)$  

$G_2$  

$A_n$  $B_n$  $C_n$  $D_n$  $G_2$  $F_4$  $E_6$  $E_7$  $E_8$ 



# Appendix D

## Wigner Theorem

This is a shortened version of Weinberg Appendix 2.A.

### D.1 Transformation of complete set gives complete set

The probability (physics) invariance gives us

$$|\langle \psi'_1 | \psi'_2 \rangle|^2 = |\langle \psi_1 | \psi_2 \rangle|^2. \quad (\text{D.1})$$

Assume we start with a complete set

$$\langle k | l \rangle = \delta_{kl}. \quad (\text{D.2})$$

Apply the above probability invariance

$$|\langle k' | l' \rangle|^2 = |\langle k | l \rangle|^2 = \delta_{kl}. \quad (\text{D.3})$$

This immediately leads to

$$\langle k' | k' \rangle = 1 \quad (\text{D.4})$$

As it is an self-inner product it is positive and real.

Thus we have to pick

$$|\langle k' | l' \rangle| = \delta_{kl} \quad (\text{D.5})$$

for consistency. And hence we conclude  $|k'\rangle$  also form a complete set.

Suppose it is not complete, then we can find some state  $|\Psi'\rangle \perp |k'\rangle$  for any  $|k'\rangle$ . Then the inverse transformation makes  $|\Psi\rangle \perp |k\rangle$  for any  $k$ , but this is not possible as the original set is complete.

## D.2 Consider a special state

The special state is

$$|\Gamma_k\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |k\rangle). \quad (\text{D.6})$$

Now we transform this state, knowing the complete set we can expand

$$|\Gamma_k\rangle = \sum_l c_{kl} |l'\rangle \quad (\text{D.7})$$

Clearly from the probability preservation  $|c_{k1}| = |c_{kk}| = \frac{1}{\sqrt{2}}$  and the others equal to 0. We can adjust phase between  $|\Gamma'_k\rangle$  and  $|k'\rangle$ , we can fix the phase to be 1 and arrive at

$$c_{kk} = c_{k1} = \frac{1}{\sqrt{2}}. \quad (\text{D.8})$$

Then

$$U|\Gamma_k\rangle = \frac{1}{\sqrt{2}}(U|k\rangle + U|1\rangle) \quad (\text{D.9})$$

But for any genera state  $|\Psi\rangle = \sum_k c_k |k\rangle$  that is transformed to  $|\Psi'\rangle = \sum_k c_{k'} U|k\rangle$  where  $U|k\rangle$  is also a complete set (proved last section). Then the physics tells me

$$|\langle k|\Psi\rangle|^2 = |\langle k|U|\Psi'\rangle|^2 \quad (\text{D.10})$$

This leads to  $|c_k|^2 = |c'_{k'}|^2$  including  $k = 1$ . Now let us compute with our special state

$$|\langle \Gamma_k|\Psi\rangle|^2 = |\langle \Gamma_k|U|\Psi'\rangle|^2 \quad (\text{D.11})$$

This gives

$$\begin{aligned} |c_k + c_1|^2 &= |c'_k + c'_1|^2 \\ \left| \frac{c_1}{c_k} \right|^2 &= \left| \frac{c'_1}{c'_k} \right|^2. \end{aligned} \quad (\text{D.12})$$

This is only true if

$$\frac{c_k}{c_1} = \frac{c'_k}{c'_1} \text{ or } \frac{c_k}{c_1} = \frac{c'_k}{c'_1}^* \quad (\text{D.13})$$

this is need to be true for all  $k$  and for the same choice. So linear or anti-linear.

## D.3 For states that are made with three states

Weinberg uses same argument shows that the choice cannot differ.

## D.4 Unitary

Then we have (only linear  $\rightarrow$  unity shown here)

$$U\left(\sum_k c_k |k\rangle\right) = \sum_k c_k U|k\rangle \quad (\text{D.14})$$

compute  $(U\Psi, U\Phi) = (\Psi, \Phi) = \sum_k A_k^* B_k$ .



## Appendix E

# Why can we not observe boost?

### E.1 Classically

Recall angular momentum in third direction is

$$L^3 = x^1 p^2 - x^2 p^1. \quad (\text{E.1})$$

Then analogously, the boost in third direction should be

$$\begin{aligned} K^3 &= x^3 p^0 - x^0 p^3 \\ &= p^0 \left( x^3 - \frac{x^0 p^3}{p^0} \right) \\ &= p^0 (x^3 - t v^3) \end{aligned} \quad (\text{E.2})$$

where we used  $v^3 \equiv \frac{p^3}{p^0}$ . Then we can interpret boost classically as the center of mass position times the energy.

### E.2 Quantum mechanically

The expression  $\frac{x^0 p^3}{p^0}$  includes an uncertainty principle pair  $x_0$  and  $p_0$ , they cannot be simultaneously measured!

Also boost does not commute with  $H = P^0$ , thus it is not conserved. It is weird to label a state of particle with something not conserved. (Think about a free particle!)

### E.3 From the algebra

Note that boost corresponds to the non-compact part of the algebra (as the range of boost parameter, rapidity can take infinity). There is a theorem says

non-compact algebra does not omit finite unitary (read: observable) representation.

To connect to the fine observable angular momentum: The boost algebra is not complete, it will always involve angular momentum which admits finite representation. Thus any finite rep of the angular momentum will force the boost rep to be finite, which is then non-unitary because it is not compact algebra. And hence boost is not an observable. This is different from the momentum subalgebra, which is not mixed with the Lorentz algebra and hence can take the infinite unitary representation.

## Appendix F

# Massless states

Just as in the massive case, we only need to consider little group. For the massless case, the reference momentum is chosen to be  $(k, 0, 0, k)$ . The little group is  $ISO(2)$ , also known as the Euclidean group. The  $ISO(2)$  group consists of a translation (the  $I$  part,  $I$  stands for inhomogeneous. Note Poincare group is often called the inhomogeneous Lorentz group. ) of a two vector  $S(\alpha, \beta)$  and a rotation in the two plane  $R(\theta)$ . Both translation and rotation are abelian. Further when  $\theta = 0$ , the subgroup  $S(\alpha, \beta)$  is invariant, so it is the normal subgroup. (This also makes sense as the 4-translation group is also the invariant/normal subgroup of Poincare.).

Now let us actually gives an example element of this little group  $ISO(2)$ .

$$W(\theta, \alpha, \beta) = T(\alpha, \beta)R(\theta) \quad (\text{F.1})$$

where

$$T_\nu^\mu(\alpha, \beta) = \begin{pmatrix} 1 + \xi & \alpha & \beta & -\xi \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \xi & \alpha & \beta & 1 - \xi \end{pmatrix} \quad (\text{F.2})$$

where  $\xi = \frac{\alpha^2 + \beta^2}{2}$ . And  $R(\theta)$  is just the rotational matrix around the third axis. Then we can expand the Lorentz transformation acting on the state infinitesimally as usual,

$$S(W(\theta, \alpha, \beta)) = 1 + i\alpha A + i\beta B + i\theta J_3 \quad (\text{F.3})$$

it turns out that the three generators<sup>1</sup> are

$$\begin{aligned} A &= -S^{13} + S^{10} \equiv S^{1-} \\ B &= -S^{23} + S^{20} \equiv S^{2-} \\ J_3 &= S^{12} \end{aligned} \quad (\text{F.4})$$

---

<sup>1</sup>we also used the lightcone coordinates and labeled A and B as 1- and 2-

and they have commutators,

$$\begin{aligned} [J_3, A] &= +iB \\ [J_3, B] &= -iA \\ [A, B] &= 0 \end{aligned} \tag{F.5}$$

The first idea is to simultaneously diagonalize  $A$  and  $B$ ,

$$\begin{aligned} A\Psi_{k,a,b} &= a\Psi_{k,a,b} \\ B\Psi_{k,a,b} &= b\Psi_{k,a,b} \end{aligned} \tag{F.6}$$

But it turns out we can find a continuum of states

$$\Psi_{k,a,b}^\theta = S^{-1}(R(\theta))\Psi_{k,a,b} \tag{F.7}$$

such that

$$\begin{aligned} A\Psi_{k,a,b}^\theta &= (a \cos \theta - b \sin \theta)\Psi_{k,a,b}^\theta \\ B\Psi_{k,a,b}^\theta &= (a \sin \theta + b \cos \theta)\Psi_{k,a,b}^\theta \end{aligned} \tag{F.8}$$

We do not observe such continuous states, so we must impose physical states satisfy

$$\begin{aligned} A\Psi_{k,\sigma} &= 0 \\ B\Psi_{k,\sigma} &= 0 \end{aligned} \tag{F.9}$$

and label them with the third operator  $S^{12}$ , which is the spin  $J_3$  in the third direction. Remember we have chosen a reference momentum to be in the third direction? this tells us for massive particle, the momentum and spin are always in the same or opposite direction. There is no angle between them! This spin in the third direction is the helicity. Hence we classify the particles under Lorentz transformation by their helicity, which is to say helicity is Lorentz invariant.

There are a couple comments we want to make,

- Notice that the  $A, B, J_3$  algebra (contains mixture of boost and rotation) of  $ISO(2)$  is very different from  $SU(2)$  (which does not include boost operators at all). Thus we cannot say a massless particle has spin. But interestingly, assuming a massless particle in the helicity-1 representation and try to build a field will fail as shown in Weinberg 5.7. So photo, although massless, is considered a spin-1 particle with gauge symmetry (which makes it massless).
- Once we imposed the physical state condition:  $A$  and  $B$  are effectively 0 acting on the physical state. Then the algebra is really only formed by  $J^3$  alone, and the group element is  $e^{iJ_3\theta}$ , which is really  $U(1)$  and the eigenvalue of  $J_3$  is the helicity.

- Now let us look at  $W^\mu$ , Pauli–Lubanski pseudovector (this monster is going to give us a label!)

$$W_\mu \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma. \quad (\text{F.10})$$

We can also define helicity as  $W^\mu = hP^\mu$ . With  $P^\mu = (k, 0, 0, k)$ , what is  $W^0$ ? It is  $W^0 = J_3 P_3 = hP_0$ , so clearly  $h$  is the eigenvalue of  $J^3$ . Our two definitions of helicity agree!

- Why helicity are integers? Well, it can be  $1/2$ , then we will have Weyl spinor (has to be massless), it does not exist in nature, but is very good building blocks. We use them to make our electrons for example. We will come back to this point in the next appendix.

When the particle is not massless, we define helicity as the projection of spin along the momentum (it of course agrees with our massless case).

$$h \equiv \frac{\mathbf{J} \cdot \mathbf{P}}{|\mathbf{P}|}. \quad (\text{F.11})$$

In this case we can show that the helicity is invariant under rotation (it looks like a dot product of two three vector) but it is not invariant under boost.

To show that, we need some variations of the Lorentz algebra,

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad (\text{F.12})$$

$$(\text{F.13})$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \quad (\text{F.14})$$

$$[J_i, P_j] = i\epsilon_{ijk} P_k \quad (\text{F.15})$$

$$[K_i, P_j] = iH\delta_{ij}$$

Then we can do a straight forward computation,

$$[J_i P_i, J_k] = [J_i, J_k] P_i + J_i [P_i, J_k] \quad (\text{F.16})$$

$$(\text{F.17})$$

$$= i\epsilon_{ikj} J_j P_i - J_i \epsilon_{kij} P_j \quad (\text{F.18})$$

$$= i\epsilon_{ikj} J_j P_i - J_j \epsilon_{kji} P_i = 0$$

The other one is not so lucky (as expected),

$$[J_i P_i, K_k] = [J_i, K_k] P_i + J_i [P_i, K_k] \quad (\text{F.19})$$

$$(\text{F.20})$$

$$= i\epsilon_{ikj} K_j P_i - iJ_k H$$

We have no reason to believe this vanish.



## Appendix G

# Useful spinor identities

Before we move on to quantize the Dirac field, we list the inner and outer product properties of the spinors  $u(\mathbf{p})$ 's and  $v(\mathbf{p})$ 's which are useful for quantization.

The spinors satisfy

$$\begin{aligned} (-\not{p} + m)u^s(\mathbf{p}) &= 0 \\ (\not{p} + m)v^s(\mathbf{p}) &= 0 \end{aligned} \tag{G.1}$$

Recall our solutions are given by,

$$\begin{aligned} u^s(\mathbf{p}) &= \begin{pmatrix} \sqrt{\sigma \cdot \mathbf{p}}\xi^s \\ \sqrt{\bar{\sigma} \cdot \mathbf{p}}\xi^s \end{pmatrix} \\ v^s(\mathbf{p}) &= \begin{pmatrix} \sqrt{\sigma \cdot \mathbf{p}}\eta^s \\ -\sqrt{\bar{\sigma} \cdot \mathbf{p}}\eta^s \end{pmatrix} \end{aligned} \tag{G.2}$$

where

$$\begin{aligned} (\xi^r)^\dagger \xi^s &= \delta^{rs} \\ (\eta^r)^\dagger \eta^s &= \delta^{rs} \end{aligned} \tag{G.3}$$

We will list all the useful results for future reference, the derivation can be found at G.1.2,

- The inner product<sup>1</sup> of the spinors,

$$\begin{aligned} (u^r)^\dagger(\mathbf{p}) \cdot u^s(\mathbf{p}) &= 2p_0\delta^{rs} \\ (v^r)^\dagger(\mathbf{p}) \cdot v^s(\mathbf{p}) &= 2p_0\delta^{rs} \\ (u^r)^\dagger(\mathbf{p}) \cdot v^s(-\mathbf{p}) &= 0 \\ (v^r)^\dagger(\mathbf{p}) \cdot u^s(-\mathbf{p}) &= 0 \end{aligned} \tag{G.4}$$

---

<sup>1</sup>The inner product refers to the contraction of the spinor indices that are omitted here.

- The Lorentz invariant, that is with Dirac conjugate, inner product of the spinors,

$$\begin{aligned}\bar{u}^r(\mathbf{p}) \cdot u^s(\mathbf{p}) &= 2m\delta^{rs} \\ \bar{v}^r(\mathbf{p}) \cdot v^s(\mathbf{p}) &= -2m\delta^{rs} \\ \bar{u}^r(\mathbf{p}) \cdot v^s(\mathbf{p}) &= 0 \\ \bar{v}^r(\mathbf{p}) \cdot u^s(\mathbf{p}) &= 0\end{aligned}\tag{G.5}$$

- The outer product<sup>2</sup> of spinors,

$$\sum_{s=1}^2 u^s(\mathbf{p}) \bar{u}^s(\mathbf{p}) = \not{p} + m \sum_{s=1}^2 v^s(\mathbf{p}) \bar{v}^s(\mathbf{p}) = \not{p} - m\tag{G.6}$$

## G.1 Some calculations

### G.1.1 Properties of the fifth gamma matrix

Now let us prove these statements, recall that from Dirac algebra  $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$  and  $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$ , which written for specific components.

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i\tag{G.7}$$

$$\begin{aligned}(\gamma^5)^\dagger &= i(\gamma^3)^\dagger (\gamma^2)^\dagger (\gamma^1)^\dagger (\gamma^0)^\dagger \\ &= i(-\gamma^3)(-\gamma^2)(-\gamma^1)(\gamma^0) \\ &= -i\gamma^3 \gamma^2 \gamma^1 \gamma^0 \\ &= -i\gamma^0 \gamma^1 \gamma^2 \gamma^3\end{aligned}\tag{G.8}$$

The last line is because they all anti-commute, and it takes  $\gamma^3$  three switches to get to the end, and then two switches for  $\gamma^2$  to get into position, and one last switch of  $\gamma^1$  and  $\gamma^0$ , a total of six switches which is even, so there is no sign change.

$$\begin{aligned}(\gamma^5)^2 &= -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^1 \gamma^2 \gamma^3 \\ &= -(\gamma^0)^2 (\gamma^1)^2 (\gamma^2)^2 (\gamma^3)^2 \\ &= 1\end{aligned}\tag{G.9}$$

Again the first line to second line takes six switches so there is no sign change. For the next statement about  $\gamma^5$  anti-commuting all the others, we will prove it for  $\gamma^0$  and  $\gamma^i$  separately

$$\begin{aligned}\{\gamma^5, \gamma^0\} &= -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 - i\gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 \\ &= +i\gamma^1 \gamma^2 \gamma^3 - i\gamma^1 \gamma^2 \gamma^3 = 0\end{aligned}\tag{G.10}$$

---

<sup>2</sup>Here we are multiplying a column vector by a row vector, hence the result will be a  $4 \times 4$  matrix.

$$\{\gamma^5, \gamma^i\} = -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^i - i\gamma^i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (\text{G.11})$$

No matter what is  $i$ , to make the two terms look the same, we need to make a total number of three switches, for example, for  $i = 1$ , the first term needs two switches and the second term needs one switches. Thus one term will pick up a minus sign from the switches while the other one will not. This relative sign will guarantee  $\gamma^5$  anti-commutes with the other  $\gamma$  matrices.

Now finally,

$$[\gamma^5, \mathcal{J}^{\mu\nu}] \propto [\gamma^5, [\gamma^\mu, \gamma^\nu]] \quad (\text{G.12})$$

$$\begin{aligned} &= \gamma^5\gamma^\mu\gamma^\nu - \gamma^5\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\gamma^5 + \gamma^\nu\gamma^\mu\gamma^5 \\ &= 0 \end{aligned} \quad (\text{G.13})$$

where the first and the third term cancels after anti-commuting  $\gamma^5$  to the end, and the second and the fourth term cancels for the same reason.

### G.1.2 Spinor gymnastics

We can calculate the inner product of the spinors

$$\begin{aligned} (u^r)^\dagger(\mathbf{p}) \cdot u^s(\mathbf{p}) &= ((\xi^r)^\dagger\sqrt{\sigma \cdot p}, (\xi^r)^\dagger\sqrt{\bar{\sigma} \cdot p}) \begin{pmatrix} \sqrt{\sigma \cdot p}\xi^s \\ \sqrt{\bar{\sigma} \cdot p}\xi^s \end{pmatrix} \quad (\text{G.14}) \\ &= (\xi^r)^\dagger\sigma \cdot p\xi^s + (\xi^r)^\dagger\bar{\sigma} \cdot p\xi^s \\ &= (\xi^r)^\dagger((\sigma + \bar{\sigma}) \cdot p)\xi^s \\ &= 2p_0\delta^{rs} \end{aligned}$$

The Lorentz invariant version is calculated with the Dirac conjugate

$$\begin{aligned} \bar{u}^r(\mathbf{p}) \cdot u^s(\mathbf{p}) &= ((\xi^r)^\dagger\sqrt{\sigma \cdot p}, (\xi^r)^\dagger\sqrt{\bar{\sigma} \cdot p}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\sigma \cdot p}\xi^s \\ \sqrt{\bar{\sigma} \cdot p}\xi^s \end{pmatrix} \quad (\text{G.15}) \\ &= ((\xi^r)^\dagger\sqrt{\sigma \cdot p}, (\xi^r)^\dagger\sqrt{\bar{\sigma} \cdot p}) \begin{pmatrix} \sqrt{\bar{\sigma} \cdot p}\xi^s \\ \sqrt{\sigma \cdot p}\xi^s \end{pmatrix} \\ &= (\xi^r)^\dagger m\xi^s + (\xi^r)^\dagger m\xi^s \\ &= 2m\delta^{rs} \end{aligned}$$

Analogously, for the negative frequency solution, we have the following result

$$\begin{aligned} (v^r)^\dagger(\mathbf{p}) \cdot v^s(\mathbf{p}) &= 2p_0\delta^{rs} \quad (\text{G.16}) \\ \bar{v}^r(\mathbf{p}) \cdot v^s(\mathbf{p}) &= -2m\delta^{rs} \end{aligned}$$

We can also take inner product between positive and negative frequency solution

$$\begin{aligned}
\bar{u}^r(\mathbf{p}) \cdot v^s(\mathbf{p}) &= ((\xi^r)^\dagger \sqrt{\sigma \cdot p}, (\xi^r)^\dagger \sqrt{\bar{\sigma} \cdot p}) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\sigma \cdot p} \eta^s \\ -\sqrt{\bar{\sigma} \cdot p} \eta^s \end{pmatrix} \quad (\text{G.17}) \\
&= ((\xi^r)^\dagger \sqrt{\sigma \cdot p}, (\xi^r)^\dagger \sqrt{\bar{\sigma} \cdot p}) \begin{pmatrix} -\sqrt{\bar{\sigma} \cdot p} \eta^s \\ \sqrt{\sigma \cdot p} \eta^s \end{pmatrix} \\
&= (\xi^r)^\dagger (-m) \xi^s + (\xi^r)^\dagger m \xi^s \\
&= 0
\end{aligned}$$

and similarly,

$$\bar{v}^r(\mathbf{p}) \cdot u^s(\mathbf{p}) = 0 \quad (\text{G.18})$$

It is also useful to take the inner product without the Dirac conjugate, it turns out that it is only interesting if we take the three momentum to be equal and opposite, so we take  $p'^\mu = (p^0, -\mathbf{p})$

$$\begin{aligned}
(u^r)^\dagger(\mathbf{p}) \cdot v^s(-\mathbf{p}) &= ((\xi^r)^\dagger \sqrt{\sigma \cdot p}, (\xi^r)^\dagger \sqrt{\bar{\sigma} \cdot p}) \begin{pmatrix} \sqrt{\sigma \cdot p'} \eta^s \\ -\sqrt{\bar{\sigma} \cdot p'} \eta^s \end{pmatrix} \quad (\text{G.19}) \\
&= (\xi^r)^\dagger \sqrt{\sigma \cdot p} \sqrt{\sigma \cdot p'} \eta^s - (\xi^r)^\dagger \sqrt{\bar{\sigma} \cdot p} \sqrt{\bar{\sigma} \cdot p'} \eta^s \\
&= 0
\end{aligned}$$

where we use the identity

$$(\sigma \cdot p)(\sigma \cdot p') = (p_0 - p^i \sigma^i)(p_0 + p^j \sigma^j) = (\bar{\sigma} \cdot p)(\bar{\sigma} \cdot p') \quad (\text{G.20})$$

The last thing we want to calculate is something called an outer product, instead of contracting the two spinors, we place them back to back to give a  $4 \times 4$  matrix,

$$\begin{aligned}
\sum_{s=1}^2 u^s(\mathbf{p}) \bar{u}^s(\mathbf{p}) &= \sum_{s=1}^2 \begin{pmatrix} \sqrt{\sigma \cdot p} \xi^s \\ \sqrt{\bar{\sigma} \cdot p} \xi^s \end{pmatrix} ((\xi^s)^\dagger \sqrt{\sigma \cdot p}, (\xi^s)^\dagger \sqrt{\bar{\sigma} \cdot p}) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad (\text{G.21}) \\
&= \sum_{s=1}^2 \begin{pmatrix} \sqrt{\sigma \cdot p} \xi^s \\ \sqrt{\bar{\sigma} \cdot p} \xi^s \end{pmatrix} ((\xi^s)^\dagger \sqrt{\bar{\sigma} \cdot p}, (\xi^s)^\dagger \sqrt{\sigma \cdot p}) \\
&= \begin{pmatrix} m & \sigma \cdot p \\ \bar{\sigma} \cdot p & m \end{pmatrix} = p_\mu \gamma^\mu + m = \not{p} + m
\end{aligned}$$

where we used <sup>3</sup>

$$\sum_s \xi^s (\xi^s)^\dagger = \mathbb{1} \quad (\text{G.22})$$

Similarly we can show that

$$\sum_{s=1}^2 v^s(\mathbf{p}) \bar{v}^s(\mathbf{p}) = p_\mu \gamma^\mu - m = \not{p} - m \quad (\text{G.23})$$

---

<sup>3</sup>if we take first spinor to be  $(\cos \theta, \sin \theta)$ , the other one has to be  $(-\sin \theta, \cos \theta)$ , then it is easy to show the statement is true.

## Appendix H

# $\gamma$ matrices and spinors in higher dimensions

In higher dimensions, we need to figure out the dimensionality of the vector space the  $\gamma$  matrices act on. Let us work in an even dimension of  $2k + 2$  and define

$$\begin{aligned}\Gamma^{0\pm} &= \frac{1}{2}(\gamma^0 \pm \gamma^1) \\ \Gamma^{i\pm} &= \frac{1}{2}(i\gamma^{2i} \pm \gamma^{2i+1}), \quad i = 1 \dots k\end{aligned}\tag{H.1}$$

It is easy to show that  
rac algebra gives

$$\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab}\tag{H.2}$$

with all other anti-commutators vanish. Thus each pair of  $\Gamma^{a+}, \Gamma^{a-}$  is analogous to the fermionic annihilation/creation operators. Thus the base of the vector space the  $\gamma$  matrices act on can be built by a “lowest weight” manner. Starting with a state that is annihilate by all the  $\Gamma^{a-}$ , then each creation operator  $\Gamma^{a+}$  has a choice of acting on it or not. Thus we conclude the total number of states must be  $2^{k+1}$ , which is the dimension of the vector spaces the  $\gamma$  matrices act on. Thus  $\gamma$  matrices in  $2k + 2$  dimensions are represented by  $2^{k+1} \times 2^{k+1}$  matrices. It is completely a coincidence that we live in 4 dimensions, where  $k = 1$ , and the  $\gamma$  matrices also act on  $2^{1+1} = 4$  dimensional vector space.

For the  $\gamma$  matrices in higher even dimensions, we can also construct the analogous matrix  $\gamma^5$  in 4 dimensions, let us name it  $\gamma^{2k+3}$  and call it chiral matrix,

$$\gamma^{2k+3} \equiv (-i)^k \gamma^0 \gamma^1 \dots \gamma^{2k+1}\tag{H.3}$$

This matrix will also enable us to define analogous Weyl spinors.

In the odd dimension  $2k + 3$ , the  $\gamma$  matrices are in the same dimension,  $2^{k+1} \times 2^{k+1}$ , the extra  $\gamma$  matrix is nothing but the chiral matrix in one lower dimension,  $2k + 2$ . It can be shown that the Dirac algebra will be satisfied. We can try to construct the analogous chiral matrix in the odd dimension, but it is nothing but the identity matrix so there are no Weyl spinors in odd dimensions.

Weyl condition and Majorana condition can be simultaneously imposed in 2, 10,  $8k + 2$  dimensions. For interested reader, we encourage you to further read upon the lecture notes by Hitoshi Murayama on the representation on Clifford Algebra and  $\text{Spin}(N)$ .

$$(\Gamma^{a\pm})^\dagger = \Gamma^{a\mp} \quad (\text{H.4})$$

where  $a = 0, i$ .

# Appendix I

## $C$ and $P$ , particle level and field level

### I.1 Parity

For the parity operator

$$\begin{aligned} Pb_{\mathbf{p}}^s P^{-1} &= \eta^* b_{-\mathbf{p}}^s \\ P c_{\mathbf{p}}^{\dagger, s} P^{-1} &= \eta^* c_{-\mathbf{p}}^{s, \dagger}. \end{aligned} \quad (\text{I.1})$$

From here and the mode expansion, we can compute

$$\begin{aligned} P\psi(x)P^{-1} &= P \oint_s dV_{\mathbf{p}} (u^s(\mathbf{p})b_{\mathbf{p}}^s e^{+i\mathbf{p}\cdot\mathbf{x}} + v^s(\mathbf{p})c_{\mathbf{p}}^{s, \dagger} e^{-i\mathbf{p}\cdot\mathbf{x}}) P^{-1} \\ &= \oint_s dV_{\mathbf{p}} (u^s(\mathbf{p})\eta^* b_{-\mathbf{p}}^s e^{+i\mathbf{p}\cdot\mathbf{x}} + v^s(\mathbf{p})\eta^* c_{-\mathbf{p}}^{s, \dagger} e^{-i\mathbf{p}\cdot\mathbf{x}}) \end{aligned} \quad (\text{I.2})$$

On the other hand

$$\begin{aligned} \psi(t, -\mathbf{x}) &= \oint_s dV_{\mathbf{p}} (u^s(\mathbf{p})b_{\mathbf{p}}^s e^{-i\mathbf{p}\cdot\mathbf{x}} + v^s(\mathbf{p})c_{\mathbf{p}}^{s, \dagger} e^{+i\mathbf{p}\cdot\mathbf{x}}) \\ &= \oint_s dV_{\mathbf{p}} (u^s(-\mathbf{p})b_{-\mathbf{p}}^s e^{-i\mathbf{p}\cdot\mathbf{x}} + v^s(-\mathbf{p})c_{-\mathbf{p}}^{s, \dagger} e^{+i\mathbf{p}\cdot\mathbf{x}}) \end{aligned} \quad (\text{I.3})$$

Notice that  $u(-\mathbf{p}) = \beta u(\mathbf{p})$  and  $v(-\mathbf{p}) = -\beta v(\mathbf{p})$  there is a sign cancelled out from  $\eta$ . This shows  $P\psi(x)P^{-1} = \beta\psi(t, -\mathbf{x})$ .

### I.2 Charge conjugation

For the charge conjugation operator

$$\begin{aligned} Cb_{\mathbf{p}}^s C^{-1} &= c_{\mathbf{p}}^s \\ C c_{\mathbf{p}}^{s, \dagger} C^{-1} &= b_{\mathbf{p}}^{s, \dagger}. \end{aligned} \quad (\text{I.4})$$

From here and the mode expansion, we can compute

$$\begin{aligned} C\psi(x)C^{-1} &= C \oint_s dV_{\mathbf{p}} (u^s(\mathbf{p})b_{\mathbf{p}}^s e^{+i\mathbf{p}\cdot\mathbf{x}} + v^s(\mathbf{p})c_{\mathbf{p}}^{s,\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}})C^{-1} \\ &= \oint_s dV_{\mathbf{p}} (u^s(\mathbf{p})c_{\mathbf{p}}^s e^{+i\mathbf{p}\cdot\mathbf{x}} + v^s(\mathbf{p})b_{\mathbf{p}}^{s,\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}}) \end{aligned} \quad (\text{I.5})$$

On the other hand, we need to compute  $C\psi^*$

$$\psi^* = \oint_s dV_{\mathbf{p}} (u^{s,*}(\mathbf{p})b_{\mathbf{p}}^{s,\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} + v^{s,*}(\mathbf{p})c_{\mathbf{p}}^s e^{+i\mathbf{p}\cdot\mathbf{x}}) \quad (\text{I.6})$$

It is nice to observe that  $C = i\gamma_2$  turns  $u^*$  into  $v$ 's and  $v^*$  into  $u$ 's. From here we can show

$$C\psi(x)C^{-1} = C\psi^*. \quad (\text{I.7})$$

## Appendix J

# Wick theorem: can we swap $\psi$ and $\bar{\psi}$ confidently?

The short answer is yes we can. What about the  $\delta$  infinities in the anti-commutator? The short answer is normal ordering will take care of it. But is it ok to use normal ordering to throw away so many infinities? Turns out the normal ordering is legit. To show that we decided to compute the Feynman propagator using brute force instead of the Wick theorem to unveil what is going on behind the curtain<sup>1</sup>.

The important computation is to figure out what is the time ordered two point correlation function

$$\langle 0 | T\psi(x)\bar{\psi}(y) | 0 \rangle. \quad (\text{J.1})$$

We will still split the fields into creation and annihilation parts because it makes it easy to discuss each part.

Let us choose a time ordering  $x^0 > y^0$ :

So there will be four terms  $\psi(x)^+\bar{\psi}(y)^+$ ,  $\psi(x)^+\bar{\psi}(y)^-$ ,  $\psi(x)^-\bar{\psi}(y)^+$  and  $\psi(x)^-\bar{\psi}(y)^-$ . Clearly if there are two creation or two annihilation operators one of the vacuum will be annihilated and result is 0. and even the  $+-$  is not going to survive for the same reason. There is really one term contributing which is  $\psi(x)^-\bar{\psi}(y)^+$ . Let us compute this term honestly using the mode expansion and swapping operators using the anti-commutator relationship, completely old school way, (as it should be clear only the anti-commutator remains, the other

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<sup>1</sup>We thank Raquel Izquierdo Garcia and Vincent He for asking insightful questions.

term vanishes after acting on the vacuum.)

$$\langle 0 | \psi(x)^- \bar{\psi}(y)^+ | 0 \rangle = \langle 0 | \int dV_{\mathbf{p}} u(\mathbf{p}) b_{\mathbf{p}} e^{-ipx} \int dV_{\mathbf{q}} \bar{u}(\mathbf{q}) b_{\mathbf{q}}^\dagger e^{iqy} \bar{\psi}(y)^+ | 0 \rangle \quad (\text{J.2})$$

$$\quad \quad \quad (\text{J.3})$$

$$= \langle 0 | \int dV_{\mathbf{p}} \int dV_{\mathbf{q}} u(\mathbf{p}) \bar{u}(\mathbf{q}) \delta(\mathbf{p} - \mathbf{q}) N_D e^{-ipx + iqy} | 0 \rangle$$

$$\quad \quad \quad (\text{J.4})$$

$$= \int dV_{\mathbf{p}} (\psi + m) e^{-ip(x-y)}$$

where we conveniently used Dan's normalization. In the last line we took everything out of the vacuum as there are no more operators it is just a function of  $(x - y)$ .

As you can see the result is exactly what we expected, the half of the Feynman propagator with this time ordering. It is hard to believe the other half will be any different but gives us exactly the right thing.

As you can see from the calculation, we did not throw away any infinities by using normal ordering, in the contrast, normal ordering helped us to organize things in two category: acting on vacuum gives 0 result and the ones give non zero result (also known as Wick contractions). There is no infinity issue: it comes from anti-commutator and was integrated over and we all know  $\delta$  distribution integrated to something finite.

## Appendix K

# Alternative calculation to extract Feynman rules for fermions

Let us consider a fermion fermion scattering process  $ff \rightarrow ff$ . Let us consider a fermion with momentum  $\mathbf{k}$  and polarization  $s$  and a fermion with momentum  $\mathbf{p}$  and polarization  $r$  scatter to a fermion with momentum  $\mathbf{k}'$  and polarization  $s'$  and a fermion with momentum  $\mathbf{p}'$  and polarization  $r'$ . The in and out state are then given by

$$\begin{aligned} |\text{in}\rangle &= |(\mathbf{k}, s), (\mathbf{p}, r)\rangle = b_{\mathbf{k}}^{s\dagger} b_{\mathbf{p}}^{r\dagger} |0\rangle \\ |\text{out}\rangle &= |(\mathbf{k}', s'), (\mathbf{p}', r')\rangle = b_{\mathbf{k}'}^{s'\dagger} b_{\mathbf{p}'}^{r'\dagger} |0\rangle \end{aligned} \quad (\text{K.1})$$

Then  $\langle \text{out}|$  is given by, being careful taking the Hermitian conjugate, keeping everything in order ( the order of the operators did change because Hermitian conjugate has property  $(AB)^\dagger = B^\dagger A^\dagger$ .)

$$\langle \text{out}| = \langle 0| b_{\mathbf{p}'}^{r'} b_{\mathbf{k}'}^{s'} \quad (\text{K.2})$$

Thus we are interested to compute

$$\langle \text{out}| T \exp[-i \int_T^T dt H_I(t)] |\text{in}\rangle \quad (\text{K.3})$$

where the interaction Hamiltonian is

$$H_I = \lambda \int d^3x \varphi(x) \bar{\psi}(x) \psi(x) \quad (\text{K.4})$$

So we are interested to compute

$$\langle \text{out}| T \exp[i \int d^4x \mathcal{L}_{\text{int}}[\varphi_0, \psi_0, \bar{\psi}_0]] |\text{in}\rangle \quad (\text{K.5})$$

We separate the  $\varphi(x)$  field into parts that contain annihilation and creation operators as before,

$$\varphi(x) = \varphi_+(x) + \varphi_-(x) \quad (\text{K.6})$$

Since there are no  $\varphi$  particles in the initial or final states, so we have

$$\begin{aligned} \varphi_-(x)|\text{in}\rangle &= 0 \\ \langle \text{out}|\varphi_+(x) &= 0 \end{aligned} \quad (\text{K.7})$$

This immediately gives us the  $\mathcal{O}(\lambda)$  contribution vanishes,

$$\langle \text{out}|Ti \int d^4x (-\lambda) \varphi(x) \bar{\psi}(x) \psi(x) |\text{in}\rangle = +i\lambda \langle \text{out}| \int d^4x : \varphi(x) : \bar{\psi}(x) \psi(x) |\text{in}\rangle \quad (\text{K.8})$$

Onward to  $\mathcal{O}(\lambda^2)$  contribution, where we adopt the notation  $\psi_1 \equiv \psi(x_1)$

$$\begin{aligned} i\mathcal{T} &= \frac{(-i\lambda)^2}{2!} \langle \text{out}|T \int d^4x_1 d^4x_2 (\varphi \bar{\psi} \psi)_1 (\varphi \bar{\psi} \psi)_2 |\text{in}\rangle \\ &= \frac{(-i\lambda)^2}{2!} \langle \text{out}|T \int d^4x_1 d^4x_2 (\bar{\psi} \psi)_1 (\bar{\psi} \psi)_2 |\text{in}\rangle \varphi_1 \varphi_2 \\ &= \frac{(-i\lambda)^2}{2!} \langle \text{out}|T \int d^4x_1 d^4x_2 (\bar{\psi} \psi)_1 (\bar{\psi} \psi)_2 |\text{in}\rangle \Delta_F(x_1 - x_2) \end{aligned} \quad (\text{K.9})$$

where by the same argument we made before, we contracted  $\varphi_1 \varphi_2$  using Wick theorem, otherwise they will either annihilate the in or out state.

The term that contribute to the scattering process is when the fermion field actually act on the in and out state. Let us leave  $\bar{\psi}$  alone for now, and expand  $\psi$ . Note that at this order, we need two fields to annihilate the two particles in the in state, and the other two fields to annihilate the two particles in the in state to have a non-zero amplitude, and we have no left over fields to do anything with the anti-particles. Hence we will only keep the expansion involving  $b$  and discard the  $c$  part. Based on the mode expansion,

$$\begin{aligned} \psi(x)_+ &= \int dV_{\mathbf{p}} v^s(\mathbf{p}) c_{\mathbf{p}}^{s\dagger} e^{+ip \cdot x} \\ \psi(x)_- &= \int dV_{\mathbf{p}} u^s(\mathbf{p}) b_{\mathbf{p}}^s e^{-ip \cdot x} \\ \bar{\psi}(x)_+ &= \int dV_{\mathbf{p}} \bar{u}^s(\mathbf{p}) b_{\mathbf{p}}^{s\dagger} e^{+ip \cdot x} \\ \bar{\psi}(x)_- &= \int dV_{\mathbf{p}} \bar{v}^s(\mathbf{p}) c_{\mathbf{p}}^s e^{-ip \cdot x} \end{aligned} \quad (\text{K.10})$$

We have,

$$\begin{aligned}
i\mathcal{T} &= \frac{(-i\lambda)^2}{2!} \langle \text{out} | T \int d^4x_1 d^4x_2 \bar{\psi}_{1+} \psi_{1-} \bar{\psi}_{1+} \psi_{1-} \Delta_F(x_1 - x_2) | \text{in} \rangle \\
&= \frac{(-i\lambda)^2}{2!} \langle \text{out} | T \int d^4x_1 d^4dV_{\mathbf{p}_1} dV_{\mathbf{p}_2} x_2 \bar{\psi}_{1+} \cdot u^{s_1}(\mathbf{p}_1) b_{\mathbf{p}_1}^{s_1} e^{-ip_1 \cdot x_1} \bar{\psi}_{2+} \cdot u^{s_2}(\mathbf{p}_2) b_{\mathbf{p}_2}^{s_2} e^{-ip_2 \cdot x_2} \Delta_F(x_1 - x_2) | \text{in} \rangle \\
&= -\frac{(-i\lambda)^2}{2!} \langle \text{out} | T \int d^4x_1 d^4dV_{\mathbf{p}_1} dV_{\mathbf{p}_2} \bar{\psi}_{1+} \cdot u^{s_1}(\mathbf{p}_1) \bar{\psi}_{2+} \cdot u^{s_2}(\mathbf{p}_2) e^{-ip_1 \cdot x_1 - ip_2 \cdot x_2} b_{\mathbf{p}_1}^{s_1} b_{\mathbf{p}_2}^{s_2} b_{\mathbf{k}}^{s_1 \dagger} b_{\mathbf{p}}^{r \dagger} \Delta_F(x_1 - x_2) | 0 \rangle
\end{aligned} \tag{K.11}$$

Let us calculate the string of annihilation/creation operators acting on vacuum by using anti-commutator relationship,

$$\begin{aligned}
b_{\mathbf{p}_1}^{s_1} b_{\mathbf{p}_2}^{s_2} b_{\mathbf{k}}^{s_1 \dagger} b_{\mathbf{p}}^{r \dagger} | 0 \rangle &= b_{\mathbf{p}_1}^{s_1} (\delta^{s_2 s} \delta^3(\mathbf{p}_2 - \mathbf{k}) (2\pi)^3 2E_{\mathbf{k}} - b_{\mathbf{k}}^{s_1 \dagger} b_{\mathbf{p}_2}^{s_2}) b_{\mathbf{p}}^{r \dagger} | 0 \rangle \\
&= (\delta^{s_2 s} \delta^3(\mathbf{p}_2 - \mathbf{k}) (2\pi)^3 2E_{\mathbf{p}_2} \delta^{s_1 r} \delta^3(\mathbf{p}_1 - \mathbf{p}) (2\pi)^3 2E_{\mathbf{p}_1} \\
&\quad - \delta^{s_1 s} \delta^3(\mathbf{p}_1 - \mathbf{k}) (2\pi)^3 2E_{\mathbf{p}_1} \delta^{s_2 r} \delta^3(\mathbf{p}_2 - \mathbf{p}) (2\pi)^3 2E_{\mathbf{p}_2}) | 0 \rangle
\end{aligned} \tag{K.12}$$

Plug it back and perform the momentum integrals, we have

$$\begin{aligned}
i\mathcal{T} &= -\frac{(-i\lambda)^2}{2!} \langle \text{out} | T \int d^4x_1 d^4x_2 (\bar{\psi}_{1+} \cdot u^r(\mathbf{p}) \bar{\psi}_{2+} \cdot u^s(\mathbf{k}) e^{-ip \cdot x_1 - ik \cdot x_2} \\
&\quad - \bar{\psi}_{1+} \cdot u^s(\mathbf{k}) \bar{\psi}_{2+} \cdot u^r(\mathbf{p}) e^{-ik \cdot x_1 - ip \cdot x_2}) \Delta_F(x_1 - x_2) | 0 \rangle \\
&= -2 \frac{(-i\lambda)^2}{2!} \langle \text{out} | T \int d^4x_1 d^4x_2 \bar{\psi}_{1+} \cdot u^r(\mathbf{p}) \bar{\psi}_{2+} \cdot u^s(\mathbf{k}) e^{-ip \cdot x_1 - ik \cdot x_2} \Delta_F(x_1 - x_2) | 0 \rangle
\end{aligned} \tag{K.13}$$

The last line is due to the fact that the second term is nothing but the first term relabeling 1 to be 2 and 2 to be 1, which are then integrated over except the fermion fields are switched. Hence the two terms contribute the same. Now we are going to do the same with the  $\bar{\psi}$  terms, we have

$$\begin{aligned}
i\mathcal{T} &= -(-i\lambda)^2 \langle 0 | b_{\mathbf{p}'}^{r'} b_{\mathbf{k}'}^{s'} T \int d^4x_1 d^4dV_{\mathbf{p}_1} dV_{\mathbf{p}_2} \bar{u}^{s_1}(\mathbf{p}_1) \cdot u^r(\mathbf{p}) b_{\mathbf{p}_1}^{s_1 \dagger} e^{+ip_1 \cdot x_1} \\
&\quad \times \bar{u}^{s_2}(\mathbf{p}_2) \cdot u^s(\mathbf{k}) b_{\mathbf{p}_2}^{s_2 \dagger} e^{+ip_2 \cdot x_2} e^{-ip \cdot x_1 - ik \cdot x_2} \Delta_F(x_1 - x_2) | 0 \rangle \\
&= -(-i\lambda)^2 \langle 0 | b_{\mathbf{p}'}^{r'} b_{\mathbf{k}'}^{s'} T \int d^4x_1 d^4dV_{\mathbf{p}_1} dV_{\mathbf{p}_2} b_{\mathbf{p}_1}^{s_1 \dagger} b_{\mathbf{p}_2}^{s_2 \dagger} \\
&\quad \times \bar{u}^{s_1}(\mathbf{p}_1) \cdot u^r(\mathbf{p}) \bar{u}^{s_2}(\mathbf{p}_2) \cdot u^s(\mathbf{k}) e^{i(p_1 \cdot x_1 + p_2 \cdot x_2) - ip \cdot x_1 - ik \cdot x_2} \Delta_F(x_1 - x_2) | 0 \rangle
\end{aligned} \tag{K.14}$$

Again, we need to calculate

$$\langle 0 | b_{\mathbf{p}'}^{r'} b_{\mathbf{k}'}^{s'} b_{\mathbf{p}_1}^{s_1 \dagger} b_{\mathbf{p}_2}^{s_2 \dagger} \tag{K.15}$$

But this is exactly the adjoint of what we calculated with the other half except

with 1 and 2 switched and  $k \rightarrow k'$  etc

$$\begin{aligned}
\langle 0 | b_{\mathbf{p}'}^{r'} b_{\mathbf{k}'}^{s'} b_{\mathbf{p}_1}^{s_1 \dagger} b_{\mathbf{p}_2}^{s_2 \dagger} &= -\langle 0 | b_{\mathbf{p}'}^{r'} b_{\mathbf{k}'}^{s'} b_{\mathbf{p}_2}^{s_2 \dagger} b_{\mathbf{p}_1}^{s_1 \dagger} \\
&= -(b_{\mathbf{p}_1}^{s_1} b_{\mathbf{p}_2}^{s_2} b_{\mathbf{k}'}^{s' \dagger} b_{\mathbf{p}'}^{r' \dagger} | 0 \rangle)^\dagger \\
&= -\langle 0 | (\delta^{s_2 s'} \delta^3(\mathbf{p}_2 - \mathbf{k}')) (2\pi)^3 2E_{\mathbf{p}_2} \delta^{s_1 r'} \delta^3(\mathbf{p}_1 - \mathbf{p}') (2\pi)^3 2E_{\mathbf{p}_1} \\
&\quad - \delta^{s_1 s'} \delta^3(\mathbf{p}_1 - \mathbf{k}')) (2\pi)^3 2E_{\mathbf{p}_1} \delta^{s_2 r'} \delta^3(\mathbf{p}_2 - \mathbf{p}') (2\pi)^3 2E_{\mathbf{p}_2})
\end{aligned} \tag{K.16}$$

Now we plug this back, and perform the momentum integrals, we have

$$\begin{aligned}
i\mathcal{T} &= (-i\lambda)^2 \langle 0 | T \int d^4 x_1 d^4 x_2 (\bar{u}^{r'}(\mathbf{p}') \cdot u^r(\mathbf{p}) \bar{u}^{s'}(\mathbf{k}') \cdot u^s(\mathbf{k}) e^{i(p' \cdot x_1 + k' \cdot x_2) - ip \cdot x_1 - ik \cdot x_2} \\
&\quad - \bar{u}^{s'}(\mathbf{k}') \cdot u^r(\mathbf{p}) \bar{u}^{r'}(\mathbf{p}') \cdot u^s(\mathbf{k}) e^{i(k' \cdot x_1 + p' \cdot x_2) - ip \cdot x_1 - ik \cdot x_2}) \Delta_F(x_1 - x_2) | 0 \rangle \\
&= (-i\lambda)^2 \langle 0 | T \int d^4 x_1 d^4 x_2 (\bar{u}^{r'}(\mathbf{p}') \cdot u^r(\mathbf{p}) \bar{u}^{s'}(\mathbf{k}') \cdot u^s(\mathbf{k}) e^{i(p' - p) x_1 + (k' - k) \cdot x_2} \\
&\quad - \bar{u}^{s'}(\mathbf{k}') \cdot u^r(\mathbf{p}) \bar{u}^{r'}(\mathbf{p}') \cdot u^s(\mathbf{k}) e^{i(k' - p) \cdot x_1 + (p' - k) \cdot x_2}) \Delta_F(x_1 - x_2) | 0 \rangle
\end{aligned} \tag{K.17}$$

Now we are going to use the scalar Feynman propagator

$$\Delta_F(x_1 - x_2) = i \int \frac{d^4 q}{(2\pi)^4} \frac{e^{iq \cdot (x_1 - x_2)}}{q^2 - M^2 + i\epsilon} \tag{K.18}$$

With this, we can perform the  $x_1$  and  $x_2$  integral,

$$\begin{aligned}
\int d^4 x_1 d^4 x_2 e^{i((p' - p + q) x_1 + (k' - k - q) \cdot x_2)} &= (2\pi)^8 \delta^4(p' - p + q) \delta^4(k' - k - q) \tag{K.19} \\
\int d^4 x_1 d^4 x_2 e^{i((k' - p + q) \cdot x_1 + (p' - k - q) \cdot x_2)} &= (2\pi)^8 \delta^4(k' - p + q) \delta^4(p' - k - q)
\end{aligned}$$

Plug this back and perform the  $q$  integral, we have finally our result,

$$\begin{aligned}
i\mathcal{T} &= i(-i\lambda)^2 \langle 0 | T \int d^4 q \frac{(2\pi)^4}{q^2 - M^2 + i\epsilon} (\bar{u}^{r'}(\mathbf{p}') \cdot u^r(\mathbf{p}) \bar{u}^{s'}(\mathbf{k}') \cdot u^s(\mathbf{k}) \delta^4(p' - p + q) \delta^4(k' - k - q) \\
&\quad - \bar{u}^{s'}(\mathbf{k}') \cdot u^r(\mathbf{p}) \bar{u}^{r'}(\mathbf{p}') \cdot u^s(\mathbf{k}) \delta^4(k' - p + q) \delta^4(p' - k - q)) | 0 \rangle \\
&= i(-i\lambda)^2 (2\pi)^4 \delta^4(p' - p + k' - k) \left( \frac{\bar{u}^{r'}(\mathbf{p}') \cdot u^r(\mathbf{p}) \bar{u}^{s'}(\mathbf{k}') \cdot u^s(\mathbf{k})}{(k' - k)^2 - M^2 + i\epsilon} - \frac{\bar{u}^{s'}(\mathbf{k}') \cdot u^r(\mathbf{p}) \bar{u}^{r'}(\mathbf{p}') \cdot u^s(\mathbf{k})}{(p' - k)^2 - M^2 + i\epsilon} \right)
\end{aligned} \tag{K.20}$$

## Appendix L

# How many degrees of freedom does photon have?

Why does photon have two degrees of freedom?

First we notice that since the field strength is anti-symmetric, the time derivative of  $A_0$ ,  $\partial_0 A_0$  does not show up in the equation of motion, so  $A_0$  is not dynamical. This can also be seen by looking at the canonical momentum  $\pi_{\text{Max}}^0 = 0$ , which is a problem when we need to promote the fields to be operators and impose the commutation relationship. The equation involves  $A_0$  is the 0 component:

$$\partial_i F^{i0} = 0 \quad (\text{L.1})$$

Spell this out, we have

$$\partial_i \partial^i A^0 - \partial_i \partial^0 A^i = \nabla^2 A^0 - \partial^0 (\nabla \cdot \mathbf{A}) = 0. \quad (\text{L.2})$$

Here  $\partial^i \equiv \delta^{ij} \partial_j$ .

Thus if we are given some initial data  $A_i$  and  $\dot{A}_i$  at a time  $t_0$ ,  $A_0$  is fully determined, as the above differential equation has the solution,

$$A_0(\mathbf{x}) = \int d^3 x' \frac{\partial^0 (\nabla \cdot \mathbf{A})(\mathbf{x}')}{4\pi |\mathbf{x} - \mathbf{x}'|} \quad (\text{L.3})$$

And we lose another degree of freedom in the process of gauge fixing described in the main text.



# Appendix M

## Gamma function

The Gamma function<sup>1</sup> is the most commonly used extension of the factorial function to complex numbers. For any positive integer  $n$ ,  $\Gamma$  function is defined to be the same as factorial function.

$$\Gamma(n) = (n-1)! \quad (\text{M.1})$$

The gamma function is defined via the following integral for any complex number with a positive real part,

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \quad (\text{M.2})$$

This definition is then extended to the entire complex plane aside from 0 and negative integers, where the function has simple poles. Near 0, the expansion of  $\Gamma(x)$  is

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x). \quad (\text{M.3})$$

Near  $x = -n$ , the expansion is

$$\Gamma(x) = \frac{(-1)^n}{n!} \left( \frac{1}{x+n} - \gamma + 1 + \dots + \frac{1}{n} + \mathcal{O}(x+n) \right). \quad (\text{M.4})$$

$\Gamma$  function has no zeros.

---

<sup>1</sup>For more information, see wikipedia Gamma function entry.



## Appendix N

# A common way renormalization is introduced

### N.0.1 QED Lagrangian revisited

Just like what you have seen in the scalar theory, we must renormalize our theory (QED) if we want to make predictions using perturbation theory beyond leading order. We made similar assumptions using LSZ reduction formula, to acknowledge which we need to shift and rescale our fields. Our naive Lagrangian is

$$\mathcal{L}_{\text{naive}} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi - e\bar{\psi}\gamma_\mu\psi A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2. \quad (\text{N.1})$$

After shifting and rescaling, it becomes

$$\mathcal{L} = Z_\psi i\bar{\psi}\not{\partial}\psi - Z_m m\bar{\psi}\psi - eZ_e\bar{\psi}\gamma_\mu\psi A^\mu - \frac{Z_A}{4}F_{\mu\nu}F^{\mu\nu} + \text{gauge fixing}. \quad (\text{N.2})$$

This Lagrangian (from now on we ignore the gauge fixing term<sup>1</sup>) is often written in terms of bare field, so that it resumes the form of our naive Lagrangian

$$\mathcal{L}_B = i\bar{\psi}_B\not{\partial}\psi_B - m_B\bar{\psi}_B\psi_B - e_B\bar{\psi}_B\gamma_\mu\psi_B A_B^\mu - \frac{1}{4}F_{B,\mu\nu}F^{B,\mu\nu} \quad (\text{N.3})$$

where  $\psi_B = Z_\psi^{1/2}\psi$ ,  $m_B = m\frac{Z_m}{Z_\psi}$ ,  $A_B^\mu = A^\mu Z_A^{1/2}$ ,  $e_B = e\frac{Z_e}{Z_\psi Z_A^{1/2}}$ .

The Lagrangian can be separated into a free part and an interaction part, where

$$\mathcal{L}_{\text{free}} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (\text{N.4})$$

---

<sup>1</sup>The shift in gauge-fixing part of the Lagrangian turns out not to change physics. Interested reader can see a discussion in Casalbuoni chapter 10 (10.147)-(10.156).

and

$$\mathcal{L}_{int} = -eZ_e\bar{\psi}\gamma_\mu\psi A^\mu + \mathcal{L}_{ct}. \quad (\text{N.5})$$

We have introduced the counter term Lagrangian

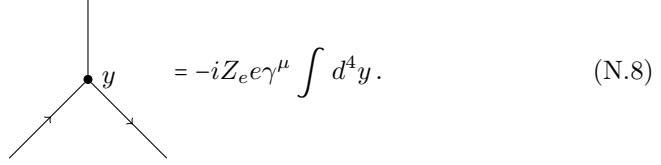
$$\mathcal{L}_{ct} = (Z_\psi - 1)i\bar{\psi}\not{\partial}\psi - (Z_m - 1)m\bar{\psi}\psi - \frac{Z_A - 1}{4}F_{\mu\nu}F^{\mu\nu} \quad (\text{N.6})$$

Counterterms are the terms we are forced to add. We should recover the free theory in the limit  $e \rightarrow 0$  so at the leading order all the  $Z_i$  are unity.

$$Z_i = 1 + \mathcal{O}(e^2) \quad (\text{N.7})$$

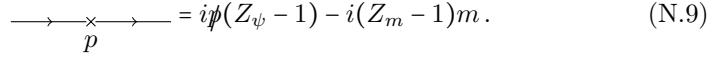
The first order corrections to the  $Z_i$ 's vanish for the same reason as in  $\phi^3$  theory: the corrections come from the Feynman diagrams with two extra vertices.

Compared to our naive theory, the Feynman rule for the vertex changes



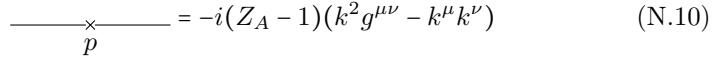
$$= -iZ_e e \gamma^\mu \int d^4 y. \quad (\text{N.8})$$

There is a new vertex where two lines meet that is easier in momentum space



$$= i\not{p}(Z_\psi - 1) - i(Z_m - 1)m. \quad (\text{N.9})$$

There is also another new vertex where two photon lines meet.



$$= -i(Z_A - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu) \quad (\text{N.10})$$

In the following subsections we are going to determine  $Z_\psi$ ,  $Z_A$ ,  $Z_m$  and  $Z_e$  at next to leading order. Naively it seems we need four conditions to determine these four parameters. But gauge invariance makes our life simpler: the new Lagrangian is only gauge invariant if  $Z_\psi = Z_e$ . Thus we need only three conditions to determine three parameters. Two of the conditions relate the two physical parameters (electron mass and the coupling  $e_{phys}$ ) of the theory to the parameters that appear in Lagrangian. The other condition has to do with photon propagator: the photon remains massless after renormalization, in other words, the gauge symmetry (which is not a symmetry) is preserved.

### N.0.2 Electron propagator/self-energy

Here we will start to calculate the 1PI (one-particle irreducible diagram, amputated) with two external electron legs, denoted by  $-i\Sigma(p^2)$ <sup>2</sup>.

$$\begin{aligned} -i\Sigma(p^2) &= \text{---} \circlearrowright \text{---} + \text{---} \bowtie \text{---} + \mathcal{O}(e^4) \\ &= i(-ie)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma^\mu}{((p - k)^2 - m^2)k^2} + i\not{p}(Z_\psi - 1) - i(Z_m - 1)m \end{aligned} \quad (\text{N.11})$$

We are going to compute this integral using Feynman parameters

$$\frac{1}{ab} = \int_0^1 \frac{dz}{(az + b(1-z))^2}. \quad (\text{N.12})$$

The denominator of our integral becomes

$$\begin{aligned} \frac{1}{((p - k)^2 - m^2)k^2} &= \int_0^1 dz \frac{1}{[((p - k)^2 - m^2)z + k^2(1-z)]^2} \\ &= \int_0^1 dz \frac{1}{k^2 + p^2 z - 2pkz - m^2 z} \\ &= \int_0^1 dz \frac{1}{(k - pz)^2 + p^2 z - p^2 z^2 - m^2 z} \end{aligned} \quad (\text{N.13})$$

Let  $k' = k - pz$ , making use of the fact that when the numerator is linear in  $k'$ , the integral goes to 0, our integral becomes

$$\int_0^1 dz (\gamma_\mu(\not{p}(1-z) + m)\gamma^\mu) \int \frac{d^4 k'}{(2\pi)^4} \frac{1}{(k'^2 - \Delta)^2} \quad (\text{N.14})$$

where  $\Delta = m^2 z - p^2 z(1-z)$ .

It seems useful to know how to do integral of the type  $\int d^4 k \frac{1}{(k'^2 - \Delta)^2}$ , except as you have learned in QFT1, it is divergent.

We are going to introduce a new way to regulate this integral: dimensional regularization. We are going analytically continue our space time dimension 4 to  $d = 4 - \epsilon$  dimensions. This might seem bizarre, but many divergent integrals in  $d = 4$  become well behaved in  $d = 4 - \epsilon$  dimensions, so we are going to go ahead to do it. You can find in (say Peskin and Schroeder 7.5) the derivation of all sorts of integrals useful in dimension regularization. We are just going to quote the ones that are useful for us (from Peskin and Schroeder equation (A.44))

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^2} = \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{1}{\Delta}\right)^{2-d/2} \quad (\text{N.15})$$

with the following expansion (if needed) to extract the behavior near  $d = 4$ :

$$\left(\frac{1}{\Delta}\right)^{2-d/2} = 1 - \left(2 - \frac{d}{2}\right) \log \Delta \quad (\text{N.16})$$

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<sup>2</sup>The constants are chosen so it contribute to mass  $\Sigma(p^2)$ .

and near  $x = 0$ ,

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x)^3. \quad (\text{N.17})$$

One more thing about putting dimension to  $d$ : if we have dimension  $d$ , then the dimension for fermion is  $\frac{d-1}{2}$  and dimension for vector boson is  $\frac{d-2}{2}$ , then the dimension for the interaction operator  $\bar{\psi}A\psi$  is  $\frac{3d}{2} - 2$ , the dimension of the coupling then is  $2 - \frac{d}{2}$ , so in order for  $e$  to remain dimensionless, we need  $e \rightarrow e\mu^{2-\frac{d}{2}}$ , where  $\mu$  is the renormalization scale.

Then the first term in the self-energy 1PI contributed by a loop diagram becomes

$$\begin{aligned} & -ie^2\mu^{4-d}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dz (\gamma_\mu(\not{p}(1-z) + m)\gamma^\mu)\Delta^{d/2-2} \\ &= -ie^2\mu^{4-d}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dz ((2-d)\not{p}(1-z) + dm)\Delta^{d/2-2} \\ &= -ie^2\mu^{4-d}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{2-d}{2}\not{p} + dm\right) + \text{finite} \\ &= -ie^2\frac{1}{\epsilon/2}\frac{1}{(4\pi)^2}(-\not{p} + 4m) + \text{finite} = -i\frac{e^2}{8\pi^2\epsilon}(-\not{p} + 4m) + \text{finite} \end{aligned} \quad (\text{N.18})$$

where in the second line we dealt with the  $\gamma$  matrices using the following  $d$  dimension upgraded identities

$$\gamma^\mu\gamma_\mu = d, \quad \gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu. \quad (\text{N.19})$$

In the third line we have taken the leading contribution of  $\Delta^{d/2-2}$  which is 1 (the rest will eventually contribute to some finite part of the result of the integral) and perform the  $z$  integral.

In last line we use  $\Gamma(x)$  expansion and put  $d = 4 - \epsilon$  and discard all finite contribution. Now the full calculation together with the counter terms is

$$-i\Sigma(p^2) = -i\frac{e^2}{8\pi^2\epsilon}(-\not{p} + 4m) + \text{finite} + i\not{p}(Z_\psi - 1) - i(Z_m - 1)m \quad (\text{N.20})$$

Compare and use our renormalization condition  $\Sigma(p^2 = m^2) = \Sigma'(p^2 = m^2) = 0$  we have our results

$$\begin{aligned} Z_\psi &= 1 - \frac{e^2}{8\pi^2\epsilon} \\ Z_m &= 1 + \frac{e^2}{2\pi^2\epsilon}. \end{aligned} \quad (\text{N.21})$$

### N.0.3 Vacuum polarization

We will leave this important study to the tutorial of beta function of QED.

#### N.0.4 Vertex

The vertex diagram is much more complicated to evaluate. But we dont have to do it:  $Z_e = Z_\psi$  due to gauge symmetry. And we already evaluate  $Z_\psi$  in the electron self energy part.