Perimeter Institute for Theoretical Physics

PSI Study Text

Introduction to Relativity

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Abstract

This is a PSI study text for the PSI 2021/22 *Relativity* course. It provides an introductory treatment of general relativity and its basic applications. The text is based on a number of sources stated below, as well as builds on the Relativity courses taught by Neil Turok and myself in the previous PSI years.

- <u>Introductory books on general relativity</u>
  - S. Carroll, *Spacetime and geometry. An introduction to general relativity* [1]. (Easy to read good introduction to the subject.)
  - J.B. Hartle, *Gravity: An introduction to Einstein's general relativity* [2]. (The book is more "experimentally oriented".)
  - L.D. Landau and E.M. Lifshitz: *Classical Field Theory* [3]. (Concise introduction to gravitational physics, with an excellent selection of topics.)
  - A. Zee *Einstein gravity in a nutshell* [4].
- <u>More advanced books</u>:
  - E. Poisson, *A relativist's toolkit: the mathematics of black-hole mechanics* [5]. (Excellent slightly advanced treatment of the subject.)
  - T. Ortin, *Gravity and strings* [6]. (Nice book regarding the advanced topics in general relativity, stringy part is too sketchy.)
  - R.M. Wald, *General Relativity* [7]. (Advanced book on general relativity.)

Contents

Chapter 1: First Look at General Relativity

1.1 Conceptual path to the theory

Following the exposure in [8] let us briefly discuss two key ideas that Einstein had when he first approached a problem of re-formulating the theory of gravitational force.

- Motivation 1: Field theory for the Newtonian interaction. It was obvious to Einstein that he wanted to find a field theory for gravitation that is consistent with special relativity. In particular, he wanted to eliminate the "instantaneous influence from far away", similar to how Maxwell theory replaced the theory of electrostatics. Of course, action at a distance does not make any sense (there are no 3d simultaneity surfaces in Minkowski space).

  We recall that the Newton's law has an apparent similarity with the Coulomb law of electrostatics:

  $$F_{\text{Coulomb}} = k\frac{q_1 q_2}{r^2} \quad \leftrightarrow \quad F_{\text{Newton}} = G\frac{m_1 m_2}{r^2}\,, \tag{1.1}$$

  where G stands for the Newton constant. This is even more obvious, if one writes the Newton's theory in a field theory language, in terms of the gravitational potential ϕ:

  $$\begin{aligned} \text{field equations:} \qquad & \Delta\phi = 4\pi G\rho_m\,, \\ \text{equations of motion:} \qquad & \frac{dp}{dt} = -m\nabla\phi\,, \end{aligned} \tag{1.2}$$

  where ρ_m stands for the matter density.[1]

  What Einstein wanted was the action at a distance to be replaced by a field theory whose static limit yields Newton's law, similar to how the Maxwell theory yields the Coulomb force. In other words he wanted to generalize (1.2).

[1] This is to be compared with the Maxwell theory in the electrostatic limit. Obviously, in the static limit the vector potential $A^\mu = (\phi, \vec{A}) = (\phi, 0)$, where $\phi \neq \phi(t)$, and the 4-current reads $J_\mu = (-\rho, 0)$. The Lorenz gauge condition, $\partial_\mu A^\mu = 0$, is now trivially satisfied, and the wave equation together with the Lorentz force law reduce to

$$\Delta\phi = -4\pi\rho\,, \quad \frac{dp}{dt} = eE = -e\nabla\phi\,. \tag{1.3}$$

- Motivation 2: Relativity of motion. However, can we really just generalize Newton's gravity to comply with special relativity in the way this is done for the electromagnetism? Or is gravity somehow special? To answer this question let us return back to the fundamental understanding of space and the motion in it.

  - Throughout the history of science there were two proposals as to how one may understand the space and the motion.

    - Space as
      - entity (absolute) ... Newton
      - relation ≈ adjacency relation between objects Aristotel, Descartes..

    - motion
      - absolute: "going from one part of space to another"
      - relative: "going from contiguity of one object to contiguity of another object"

  - Of course Einstein did not like the absolute notions. He was also very much influenced and inspired by a famous philosopher of the time Mach. One of the key problems was to understand the very essence of what is called the Newton's bucket experiment:

    As the bucket rotates, the water surface gets curved and takes a concave shape. This concavity is presumably caused by fluid rotation, but with respect to what?

    * *Descartes:* surrounding objects... perhaps the bucket?
    * *Newton:* space itself, not the surrounding bodies. The curved surface is thence a proof of the existence of an absolute space. (Interestingly, Newton's argument could not be defeated for 300 years.)
    * *Mach:* the full matter content of the Universe (inertia caused by "distance stars")

∗ *Einstein:* local physical quantity, the gravitational field (interaction between matter and gravitational field causes the shaped surface).
However, this brought Einstein to an important question: Why should the Newtonian acceleration be anyhow related to the gravitational field? Are gravitational field and Newtonian accelerations similar? What if gravity determines what inertial frame means?
These foundational questions then lead Einstein to a brilliant idea of the *principle of equivalence.* (This principle is in the heart of general relativity and allowed Einstein to eventually formulate its laws.)

1.2 Principle of equivalence

- Principle of equivalence. Perhaps the simplest formulation of this principle is the following observation (supported by experiment): *All bodies in a given gravitational field experience the same acceleration.* We have the following equality:

$$F = m_I a = m_g g\,. \tag{1.4}$$

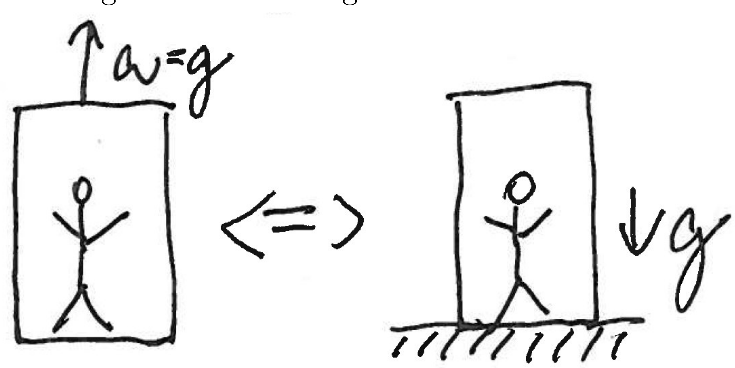
Here, m_I represents the *inertial mass*, that is a quantity preventing bodies from being accelerated, and m_g is the *gravitational mass*, that is how the body responds to the gravitational field. The principle of equivalence than states

$$m_I = m_g \quad \Rightarrow \quad \boxed{a = g\,.} \tag{1.5}$$

(At the moment there are not many theories explaining as to why $m_I = m_g$. Mach's theory of inertia is one of them.)

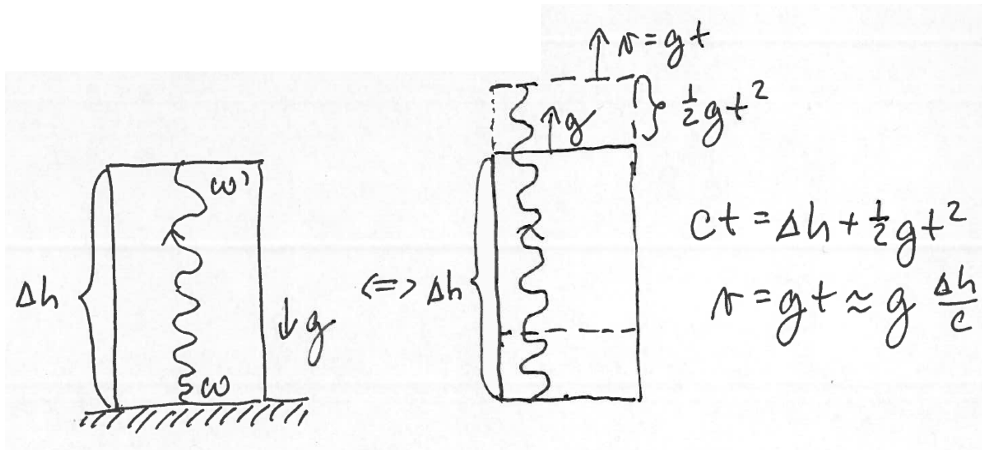
Based on this simple observation Einstein devised the following ingenious idea:

- *Einstein's elevator:* An observed confined to an elevator (without a window) cannot distinguish the following two cases:



That is, we can mimic the (at least uniform) gravitational field by going to the accelerated frame. Using this simple idea we can derive many physical consequences that are key predictions of general relativity.

- *Gravitational redshift.*



The two observers describe the light by the following wave vectors:

$$\begin{aligned} \text{bottom:} \qquad & k^\mu = \frac{\omega}{c}(1,0,0,1)\,, \\ \text{up:} \qquad & k'^\mu = \frac{\omega'}{c}(1,0,0,1) = \gamma\frac{\omega}{c}(1-\beta)(1,0,0,1)\,. \end{aligned} \tag{1.6}$$

The latter formula is related to the Doppler shift and will be derived in your tutorial. So we have

$$\omega' = \gamma\omega(1-\beta) \approx \omega\Big(1 - \frac{v}{c}\Big) \approx \omega\Big(1 - \frac{g\Delta h}{c^2}\Big)\,. \tag{1.7}$$

So we derived the following result for the gravitational redshift:

$$\boxed{z = \frac{\omega' - \omega}{\omega} = -\frac{g\Delta h}{c^2}\,.} \tag{1.8}$$

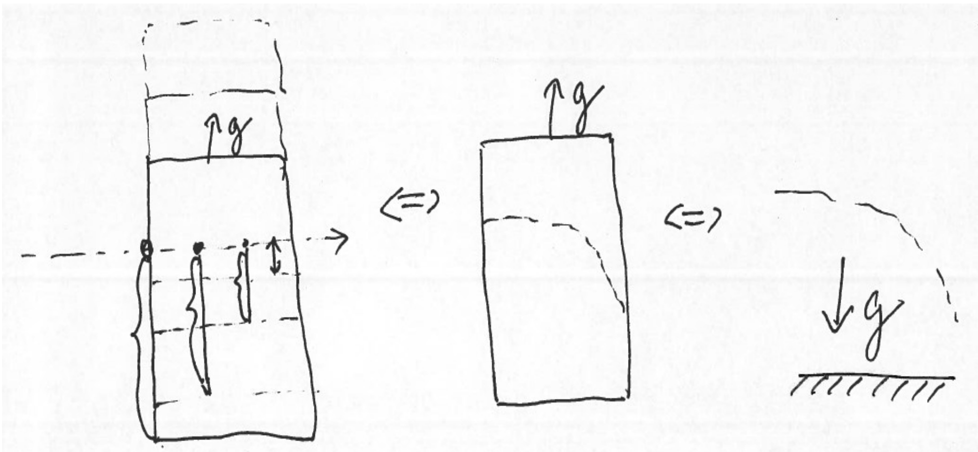
Photons are "redder" up there.

This prediction has been experimentally measured by Paund & Rebka (1960) who used a flash light and a tower of $\Delta h \approx 22.6m$. They observed

$$z \approx \frac{2.3\times 10^2}{(3\times 10^8)^2} \approx 2.5\times 10^{-15}\,. \tag{1.9}$$

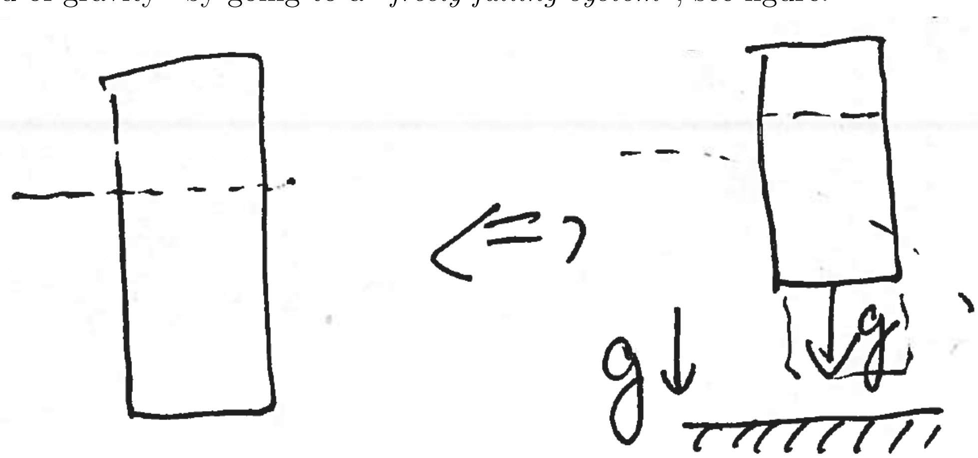
Note also that since $\Delta\tau \propto 1/\omega$, in strong gravitational field the time clicks more slowly. Hence come the advise: move fast and sleep in the library (close to heavy books) to stay young.

- *Bending of light.* The light bending is clearly predicted by the equivalence principle, see the following figure:



The immediate consequences of this is: "gravity changes what it means for a path to be straight".

- Local inertial frame. Without gravity, I see a straight line. So perhaps I can "get rid of gravity" by going to a "*freely falling system*", see figure.



As we shall see, for a general gravitational field, this can only be done locally, in a *local inertial frame.*

- *Geodesic equation.* (From now on we set $c = 1$.)
  - Let ξ^μ be the local coordinates in the (local) inertial frame. In these coordinates a free particle (or light) moves on a straight line,

$$\boxed{\frac{d^2\xi^\mu}{d\tau^2} = 0\,.} \tag{1.10}$$

  - Let $x^\alpha = x^\alpha(\xi^\mu)$ be any other (global) coordinates. We want to know how the equation of motion for a free particle looks like in these coordinates. We have

$$\frac{d^2\xi^\mu}{d\tau^2} = \frac{d}{d\tau}\Big(\frac{\partial\xi^\mu}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\Big) = \frac{\partial\xi^\mu}{\partial x^\alpha}\frac{d^2x^\alpha}{d\tau^2} + \frac{\partial^2\xi^\mu}{\partial x^\alpha\partial x^\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} = 0\,. \tag{1.11}$$

Let us next use the fact that

$$\frac{\partial x^\nu}{\partial x^\alpha} = \delta^\nu_\alpha = \frac{\partial x^\nu}{\partial \xi^\mu}\frac{\partial \xi^\mu}{\partial x^\alpha}\,. \tag{1.12}$$

Therefore, upon multiplying the previous equation by $\frac{\partial x^\nu}{\partial \xi^\mu}$, we have

$$\underbrace{\frac{d^2x^\nu}{d\tau^2}}_{\frac{du^\nu}{d\tau}} + \underbrace{\frac{\partial x^\nu}{\partial \xi^\mu}\frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\beta}}_{\Gamma^\nu{}_{\alpha\beta}}\underbrace{\frac{dx^\beta}{d\tau}}_{u^\beta}\underbrace{\frac{dx^\alpha}{d\tau}}_{u^\alpha} = 0\,. \tag{1.13}$$

Therefore we have derived the <u>geodesic equation:</u>

$$\boxed{\frac{du^\nu}{d\tau} + \Gamma^\nu{}_{\alpha\beta}u^\alpha u^\beta = 0\,, \quad u^\alpha = \frac{dx^\alpha}{d\tau}\,.} \tag{1.14}$$

– Let us further introduce a "<u>curved metric</u>" $g_{\alpha\beta}$ as follows:

$$ds^2 = -d\tau^2 = \eta_{\mu\nu}d\xi^\mu d\xi^\nu = \underbrace{\eta_{\mu\nu}\frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial \xi^\nu}{\partial x^\beta}}_{g_{\alpha\beta}}dx^\alpha dx^\beta\,. \tag{1.15}$$

It is symmetric ($g_{\alpha\beta} = g_{\beta\alpha} = g_{(\alpha\beta)}$). In general curved metric will have 10 components. Note also

$$u^2 = g_{\alpha\beta}u^\alpha u^\beta = -1\,. \tag{1.16}$$

– Obviously $\Gamma \propto \partial g$. Let us introduce two types of <u>Christoffel symbols</u>:

$$\underbrace{\Gamma^\nu{}_{\alpha\beta}}_{\text{2nd kind}} = g^{\nu\mu}\underbrace{\Gamma_{\mu\alpha\beta}}_{\text{1st kind}}\,. \tag{1.17}$$

Then we have the following statement:

$$\boxed{\Gamma_{\mu\alpha\beta} = \frac{1}{2}\big(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\big)\,.} \tag{1.18}$$

To prove this statement, we (painfully) calculate

$$\begin{aligned} g_{\mu\alpha,\beta} &= \frac{\partial}{\partial x^\beta}\left(\eta_{\kappa\sigma}\frac{\partial \xi^\kappa}{\partial x^\mu}\frac{\partial \xi^\sigma}{\partial x^\alpha}\right) = \eta_{\kappa\sigma}\left(\underbrace{\frac{\partial^2 \xi^\kappa}{\partial x^\beta \partial x^\mu}}_{\frac{\partial \xi^\kappa}{\partial x^\iota}\Gamma^\iota{}_{\beta\mu} = \frac{\partial \xi^\kappa}{x^\iota}\frac{\partial x^\iota}{\partial \xi^\delta}\frac{\partial^2 \xi^\delta}{\partial x^\beta \partial x^\mu}}\frac{\partial \xi^\sigma}{\partial x^\alpha} + \frac{\partial \xi^\kappa}{\partial x^\mu}\frac{\partial^2 \xi^\sigma}{\partial x^\beta \partial x^\alpha}\right) \\ &= g_{\iota\alpha}\Gamma^\iota{}_{\beta\mu} + g_{\mu\iota}\Gamma^\iota{}_{\alpha\beta}\,, \end{aligned} \tag{1.19}$$

and similarly for the other 2 terms.

1.3 Fake gravity: Rindler frame

In the previous section we have mimicked the gravitational field by performing the coordinate transformation that took us outside of the inertial frame. A prototype of such transformation is a transformation to the accelerated frame of a Rindler observer.

A uniformly accelerated observer with an acceleration $a = \sqrt{a_\mu a^\mu} =$ const. has a trajectory described in Minkowski coordinates (t, x) by

$$t = \frac{1}{a}\sinh(a\tau)\,, \quad x = \frac{1}{a}\cosh(a\tau)\,, \tag{1.20}$$

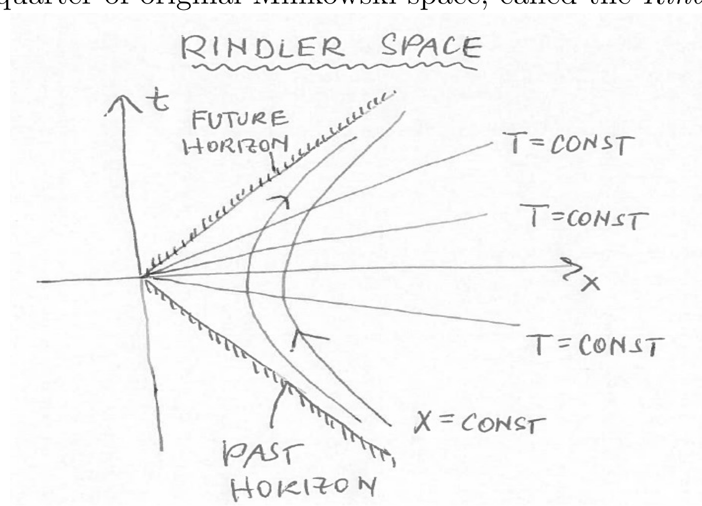
where τ is the observer's proper time. Introducing *Rindler coordinates* (T, X) by

$$t = \Bigl(\frac{1}{a} + X\Bigr)\sinh(aT)\,, \quad x = \Bigl(\frac{1}{a} + X\Bigr)\cosh(aT)\,, \tag{1.21}$$

the metric reads

$$\boxed{ds^2 = -dt^2 + dx^2 = -(1 + aX)^2 dT^2 + dX^2\,,} \tag{1.22}$$

and the trajectory is described by $X = 0$ and $T = \tau$. The observer has 2-velocity given by $u = \partial_T$. Note that the range $X \in (-1/a, \infty)$ and $T \in (-\infty, \infty)$ covers only one quarter of original Minkowski space, called the *Rindler space*, see picture



The boundary of the 'wedge' is called the *Rindler horizon*. As we shall see later, the situation is in many respects similar to what happens in black hole spacetimes.

At the quantum level an accelerated observer sees the Minkowski vacuum as Unruh radiation bath, which is a black-body like radiation with the following temperature:

$$\boxed{T = \frac{a}{2\pi}}\,. \tag{1.23}$$

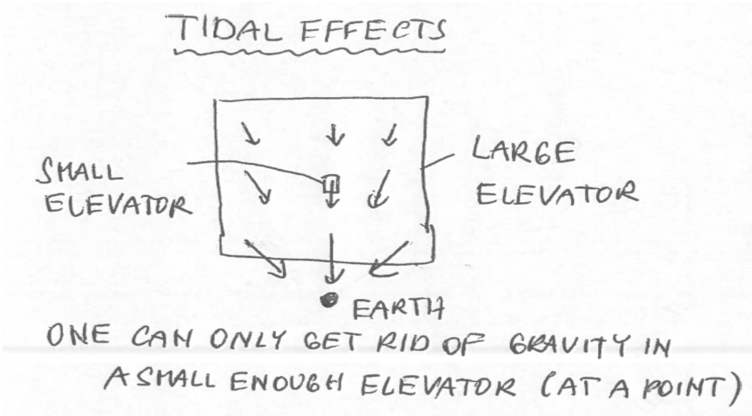
In the black hole case, this then leads to the Hawking radiation, with

$$T = \frac{\kappa}{2\pi}\,, \tag{1.24}$$

where κ is the surface gravity (acceleration at infinity needed to hold a test particle on the horizon of a black hole).

1.4 Real gravitational field

General (inhomogeneous, time-varying) gravitational field cannot be fully mimicked by acceleration. One cannot get rid of gravity except at a point.



The gravitational field is described by a (general) metric $g_{\mu\nu}$:

$$\boxed{ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,.} \tag{1.25}$$

In other words:

$$\boxed{\text{gravity} = \text{geometry}} \tag{1.26}$$

However, beware! Einstein's gravity is a gauge theory: metric is not physical. It is the *Riemann tensor* that governs physical effects such as tidal forces. (We shall need to introduce Riemannian geometry, manifolds, tensors,...)

1.5 Newtonian limit

Let us now study the Newtonian limit of the geodesic equation. Thus we consider the following approximations:

- *Slow motion.* $\left|\frac{dx^i}{d\tau}\right| \ll \frac{dt}{d\tau}$ or in other words $|v^i| \ll 1$. This implies that the geodesic equation (1.14) reduces to

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{tt}\left(\frac{dt}{d\tau}\right)^2 = 0\,. \tag{1.27}$$

- *Stationary field:* $g_{\mu\nu,t} = 0$. Then we have

$$\Gamma^\mu{}_{tt} = \frac{1}{2} g^{\mu\sigma}(g_{\sigma t,t} + g_{t\sigma,t} - g_{tt,\sigma}) = -\frac{1}{2} g^{\mu\sigma} g_{tt,\sigma}\,. \tag{1.28}$$

- *Weak field:*

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)\,. \tag{1.29}$$

We then have

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \qquad [(\eta - h)(\eta + h) = \delta + h - h + O(h^2)]\,, \tag{1.30}$$

and also

$$\Gamma^{\mu}{}_{tt} = -\frac{1}{2}\eta^{\mu\sigma} h_{tt,\sigma}\,. \tag{1.31}$$

- Writing the equation (1.27) then yields for $\mu = t$:

$$\frac{d^2 t}{d\tau^2} = 0 \quad \Rightarrow \quad \frac{dt}{d\tau} = c = \text{const.} \tag{1.32}$$

Next, for $\mu = i$ we have

$$\underbrace{\frac{d^2 x^i}{d\tau^2}}_{c^2 \frac{d^2 x^i}{dt^2}} + \underbrace{\Gamma^i{}_{tt}}_{-\frac{1}{2}h_{tt,}{}^{i}}\; c^2 = 0\,, \tag{1.33}$$

that is,

$$\boxed{\frac{d^2 x^i}{dt^2} = \frac{1}{2} h_{tt,}{}^{i}\,.} \tag{1.34}$$

- We want to compare the last equation with Newton:

$$\boxed{\frac{d^2 \vec{x}}{dt^2} = -\vec{\nabla}\phi\,,} \tag{1.35}$$

where ϕ stands for the <u>*Newtonian gravitational potential*</u>. Thus we find $\phi = -\frac{1}{2}h_{tt}$, or,

$$\boxed{g_{tt} = -\left(1 + \frac{2\phi}{c^2}\right),} \tag{1.36}$$

recovering the velocity of light c for the moment again.

The linearization is valid for

$$\frac{\phi}{c^2} \approx \frac{GM}{c^2 r} \ll 1\,. \tag{1.37}$$

We have the following values of ϕ/c^2 on the surface of various objects:

PROTON	10^{-39}	NEUTRON STAR	$10^{-2} - 10^{-1}$
EARTH	10^{-9}	BH	$10^{-1} - 1$
SUN	10^{-6}	$\left(r = r_g = \frac{2GM}{c^2}\right)$	

- In fact, as we shall see later, in a weak field (slow motion) approximation, the full metric can be written as

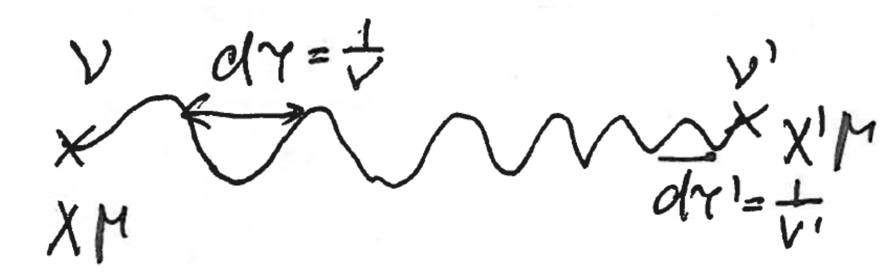
$$\boxed{ds^2 = -\Big(1+\frac{2\phi}{c^2}\Big)dt^2 + \Big(1-\frac{2\phi}{c^2}\Big)\delta_{ij}dx^i dx^j\,.} \tag{1.38}$$

1.6 Gravitational redshift again

In strong gravitational fields time ticks more slowly. We have

$$d\tau = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}dt\,, \tag{1.39}$$

where t is the coordinate time. Let us consider 2 observer's in the spacetime, each using a different coordinate system and carrying her watch measuring the proper time, as displayed in the following figure:



We have

$$\frac{\nu'}{\nu} = \frac{d\tau}{d\tau'} = \frac{\sqrt{}|_x dt}{\sqrt{}|_{x'} dt'} = \frac{\sqrt{-g_{tt}(x)}}{\sqrt{g_{tt}(x')}}\frac{dt}{dt'}\,, \tag{1.40}$$

where the last equality is valid for static (not moving) observers. In stationary spacetimes, we can choose 'global' coordinate time $dt = dt'$. Hence we have derived the following simple formula for the gravitational redshift:

$$\boxed{\frac{\nu'}{\nu} = \frac{\sqrt{-g_{tt}(x)}}{\sqrt{-g_{tt}(x')}}\,.} \tag{1.41}$$

Note that in the Newtonian limit

$$\frac{\nu'}{\nu} = \sqrt{\frac{1+2\phi}{1+2\phi'}} \approx (1+\phi)(1-\phi') \approx 1+\phi-\phi'\,, \tag{1.42}$$

which gives the following redshift:

$$z = \frac{\Delta\nu}{\nu} = \frac{\nu'-\nu}{\nu} \approx \phi - \phi'\,, \tag{1.43}$$

and specifically, for the homogeneous gravitational field

$$z = -\frac{g\Delta h}{c^2}\,, \tag{1.44}$$

as derived previously using the idea of Einstein's elevator.

1.7 Field theory for gravity

- The classical field theory has 3 basic ingredients: field, field equations, and equations of motion for the matter. Let us see what we already know about the 3 most famous classical field theories:

  CLASSICAL FIELD THEORIES

	MAXWELL	NEWTON	EINSTEIN
FIELD	A^μ	Φ	$g_{\mu\nu}$
FIELD EQ	$\Box A^\mu = -4\pi J^\mu$ $\partial_\mu A^\mu = 0$	$\nabla^2 \Phi = 4\pi G \rho$	?
EQ. MOTION	$\frac{dp^\mu}{d\tau} = e F^{\mu\nu} u_\nu$	$\frac{d\vec{p}}{dt} = -m \vec{\nabla} \phi$	$\frac{dp^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} p^\alpha u^\beta = 0$

  We see, that all it remains is to find (following Einstein and Hilbert) the field equations for gravity.

- Spoiler. To write down the field equations is actually not that difficult. We already argued that gravity is described by a metric and in the Newtonian approximation we have $g_{tt} = -1 - 2\phi$. We also know the Poisson equation and the fact that the energy density is a zero-zero component of the energy momentum tensor, $\rho = T_{tt}$:

  $$\nabla^2 \phi = 4\pi G \rho \quad \Leftrightarrow \quad -\nabla^2 g_{tt} = 8\pi G T_{tt} \,. \tag{1.45}$$

  We want to make this a tensor equation, and expect that the Einstein equation writes as

  $$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu} \,,} \tag{1.46}$$

  where $G_{\mu\nu} = G_{\mu\nu}(\partial^2 g, \partial g, g)$ is such that in the Newtonian limit it reduces to $-\nabla^2 g_{tt}$. To further limit our freedom, we realize that the energy momentum tensor has to be 'conserved', which as we shall see later, means $\nabla_\mu T^{\mu\nu} = 0$. For consistency we thus require that also

  $$\nabla_\mu G^{\mu\nu} = 0 \,. \tag{1.47}$$

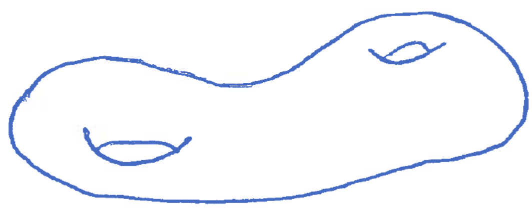
  Now we can ask our mathematician friend (Grosmann in case of Einstein) if such $G_{\mu\nu}$ exists, and he will tell us that it does and is more or less unique—given by the Einstein tensor. We can thus claim the victory and be happy :)

  To make these steps more precise, we will need to study a bit of geometry now. (This took Einstein a couple of years. We shall do it in about 3 lectures...)

Chapter 2: Geometric Tools

2.1 Manifolds

- Manifolds.



An n-dimensional manifold has a local differential structure of $\mathbb{R}^n$, but not necessarily its global properties (not necessarily embedded in higher-dimensional Euclidean space). Slightly more precisely:

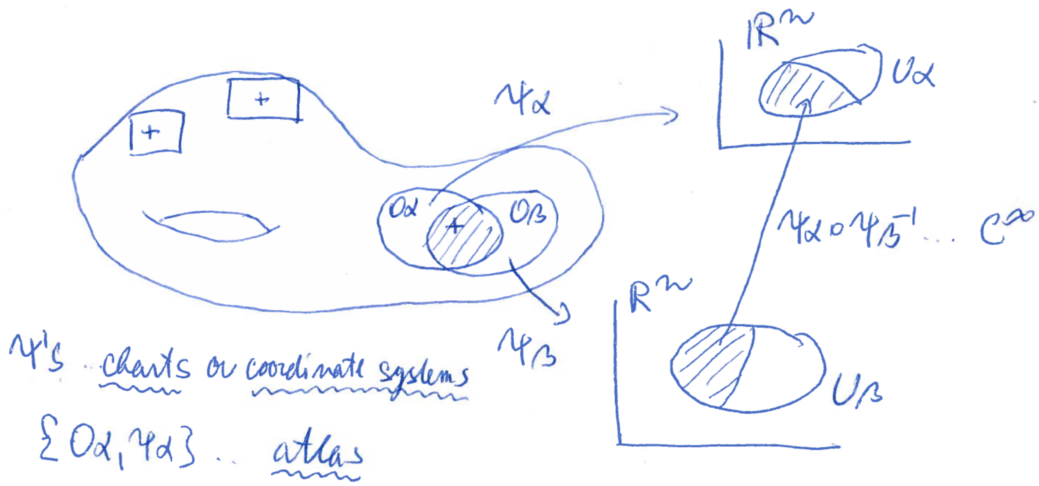
Definition. An n-dimensional *manifold* M is a 'set of points' together with a collection of subsets $\{O_\alpha\}$ satisfying:

i) Each $p \in M$ lies in at least one O_α, i.e., $\{O_\alpha\}$ cover M.

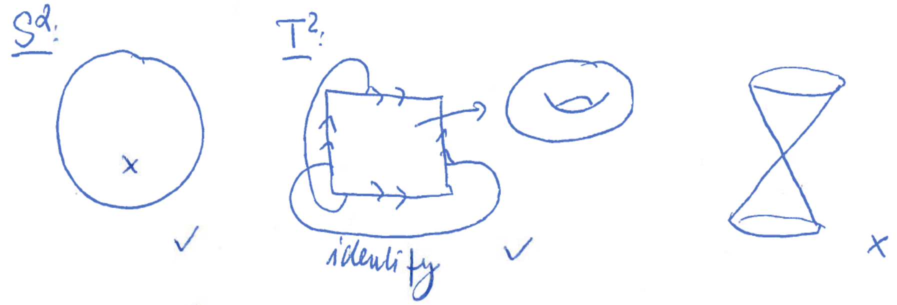
ii) For each α, there is 1-1, onto, map $\psi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of $\mathbb{R}^n$ (a union of open balls).

iii) If any two sets of O_α and O_β overlap, $O_\alpha \cap O_\beta \neq 0$, the maps $\psi_\beta \circ \psi_\alpha^{-1}$ is C^∞.

In other words: "a manifold is made of pieces that look like open subsets of $\mathbb{R}^n$ which are sewn together smoothly".

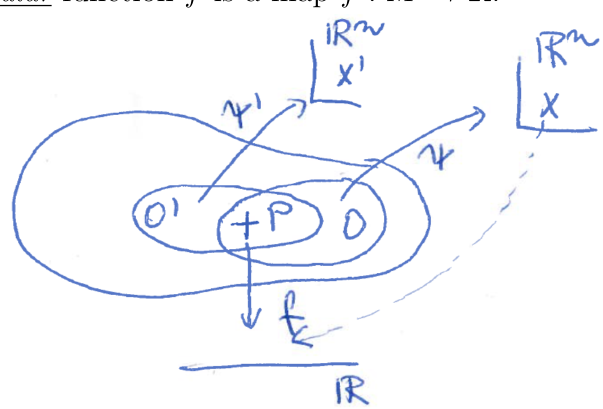


Examples:



Note. GR (particles) need manifolds. String theory works fine on orbifolds (e.g. $x \sim -x$, i.e. $\mathbb{R}^1/\mathbb{Z}_2$: fundamental domain has boundary…singular in GR.)

- A *scalar* function f is a map $f : M \to \mathbb{R}$.

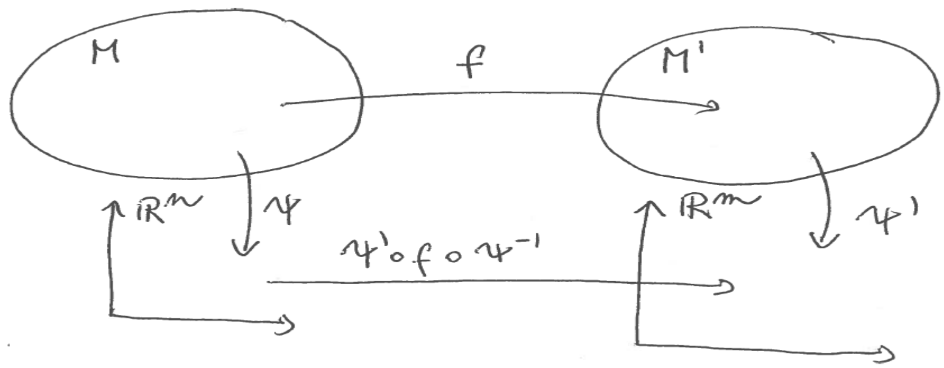


In this definition we exploit the coordinate map ψ associated with the manifold. That is, the function is defined by first going to $\mathbb{R}^n$ and then by defining a function from $\mathbb{R}^n$ to $\mathbb{R}$. If this definition is to be any good, under a change of coordinates we must have: $f(p) = f(x(p)) = f'(x'(p))$, giving

$$\boxed{f'(x') = f(x)} \tag{2.1}$$

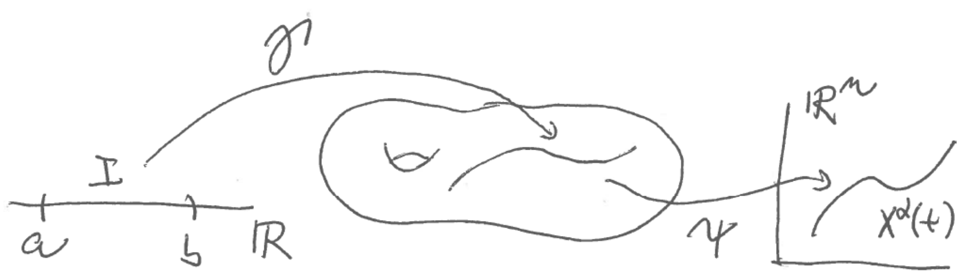
as a transformation rule for scalar functions. An example of a scalar function on a manifold is intrinsic curvature scalar, or "temperature on Earth".

- More generally, a map between manifolds $f : M \to M'$ is defined as displayed in the picture:



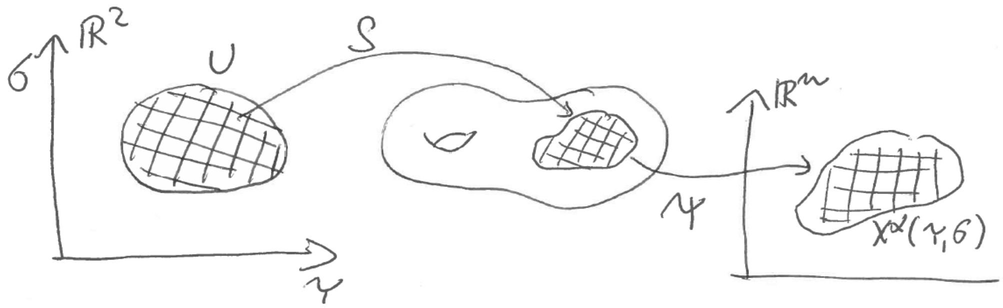
If $f : M \to M'$ is C^∞, one-on-one, onto, and has C^∞ inverse, it is called a diffeomorphism. The manifolds M and M' are then called diffeomorphic and have an identical manifold structure. Roughly speaking one can think of them as one manifold and of diffeomorphism as a map on this manifold to itself. (See Appendix A for more details on maps in between manifolds.)

- A *curve* γ on M is a map $\gamma : I \subset \mathbb{R}^1 \to M$, s.t., $(\psi_\alpha \circ \gamma)(t) = [x^1(t), \dots, x^n(t)]$ are smooth.



E.g. river, trajectory (worldline) of a particle.

Similarly a *surface* S is a map $S : U \subset \mathbb{R}^2 \to M$ such that $(\psi_\alpha \circ S)(\tau, \sigma) = [x^1(\tau, \sigma), \dots, x^n(\tau, \sigma)]$ are smooth.

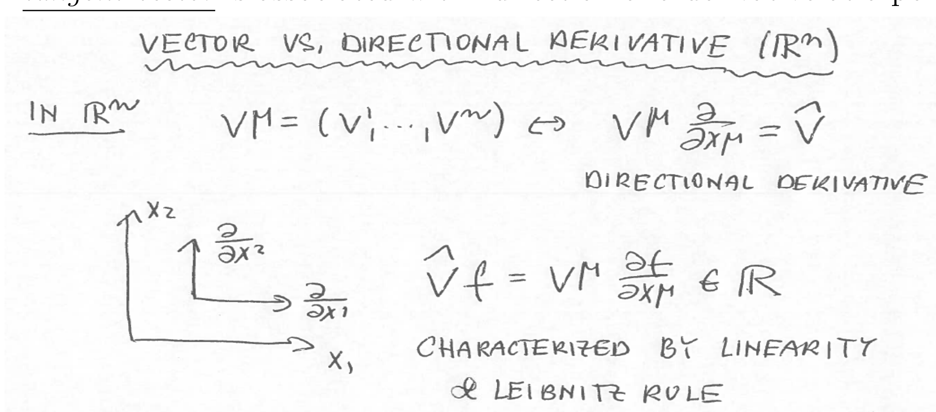


E.g. lake surface, worldsheet of a string.

2.2 Tensors

Tensors are invariant objects that can "live on a manifold". This means that they are independent of coordinates. We have already seen a simplest example of a tensor: a scalar function. (Remind yourself why its definition is independent of coordinates.) Let us now proceed to a more complicated example of a tensor: that of a tangent vector.

- A *tangent vector* is associated with "direction of a derivative at a point".



Definition. Let $\mathfrak{F}$ be a collection of C^∞ scalar functions. A *tangent vector* V at point $p \in M$ is a map $V : \mathfrak{F} \to \mathbb{R}$ that is

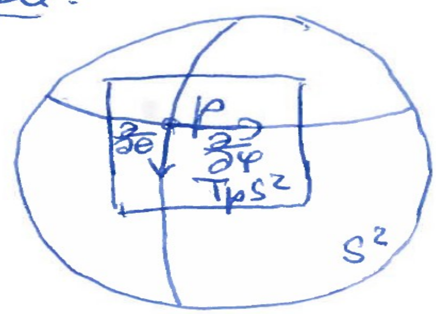
linear:	$V(af + bg) = aV(f) + bV(g)$ for all $a, b \in \mathbb{R}$.
obeys Leibnitz rule:	$V(fg) = f(p)V(g) + g(p)V(f)$.

Theorem. The set of tangent vectors at p forms a *tangent vector space* T_pM which has the same dimensionality as M, with coordinate basis $\frac{\partial}{\partial x^\mu}$. Any vector V can be expressed in the form

$$\boxed{V = V^\mu \frac{\partial}{\partial x^\mu}\,,} \tag{2.2}$$

where V^μ are vector components.

• Example:



Under a transformation of coordinates (using the chain rule) we have

$$V = V^\mu(x)\frac{\partial}{\partial x^\mu} = V^\mu(x)\frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = V'^\nu(x')\frac{\partial}{\partial x'^\nu}\,,$$

and hence:

$$\boxed{V'^\nu(x') = \frac{\partial x'^\nu}{\partial x^\mu}V^\mu(x)\,.} \tag{2.3}$$

This is how components of a vector transform under change of coordinates. (Note that vector itself is a geometric object and remains invariant under such a change.)

- We now want to extend these notions to the 'whole manifold'.

> **Definition.** A *tangent vector field* V is defined as $\{V|_p \in T_pM$ for all $p \in M$: $V(f)$ is smooth$\}$. A *tangent bundle* $TM = \bigcup_p T_pM$.

Note that TM has local coordinates (x^μ, V^ν).

- **Definition.** A *cotangent vector (1-form)* ω at a point $p \in M$ is a map $\omega : T_pM \to \mathbb{R}$. 1-forms form a *cotangent vector space* T_p^*M with coordinate basis dx^μ defined by

$$\boxed{dx^\mu\Big(\frac{\partial}{\partial x^\nu}\Big) = \delta^\mu_\nu\,.} \tag{2.4}$$

One can write

$$\boxed{\omega = \omega_\mu dx^\mu\,.} \tag{2.5}$$

Changing coordinates we have

$$\omega = \omega_\mu(x)dx^\mu = \omega_\mu(x)\frac{\partial x^\mu}{\partial x'^\nu}dx'^\nu = \omega'_\nu(x')dx'^\nu\,, \tag{2.6}$$

that is

$$\boxed{\omega'_\nu(x') = \frac{\partial x^\mu}{\partial x'^\nu}\omega_\mu(x)\,.} \tag{2.7}$$

This is how components of a covector transform under a change of coordinates.[1]

- A *cotangent bundle* $T^*M = \bigcup_p T_p^*M$. It has local coordinates (x^μ, ω_ν). Tangent and cotangent bundles are specific examples of *fibre bundles*.

- Canonical projection. Note that $V \in TM$ uniquely defines: i) $p \in M$ and $v \in T_pM$. This allows one to define *canonical projection* π:

$$\pi : TM \to M : \pi(V) = p\,. \tag{2.8}$$

  In local coordinates $V \in TM$ has components $(x^1, \dots, x^n, v^1, \dots, v^n)$. We then have

$$\pi(x^1, \dots, x^n, v^1, \dots v^n) = (x^1, \dots, x^n)\,. \tag{2.9}$$

  Similarly for T^*M.

- Tensor = "product of vectors and forms, has components $T^{\alpha\beta\dots}{}_{\gamma\delta\dots}$."
Definition. A *tensor of type (k,l) of rank $(k+l)$* is a multilinear map

$$T : \underbrace{T_p^* \times \dots \times T_p^*}_{k\text{-times}} \times \underbrace{T_p \times \dots \times T_p}_{l\text{-times}} \to \mathbb{R}\,. \tag{2.10}$$

  So the tensor "eats vectors and forms." Tensor field extends this notion to the whole manifold. Its components transform as

$$T'^{\alpha\dots}{}_{\mu\dots}(x') = \underbrace{\frac{\partial x'^\alpha}{\partial x^\delta}\dots}_{k}\underbrace{\frac{\partial x^\kappa}{\partial x'^\mu}\dots}_{l} T^{\delta\dots}{}_{\kappa\dots}(x)\,. \tag{2.11}$$

  For example: Tensor of type $(2,1)$ is of rank 3 and writes as

$$T = T^{\alpha\beta}{}_\gamma \partial_{x^\alpha} \otimes \partial_{x^\beta} \otimes dx^\gamma\,. \tag{2.12}$$

  Here $T^{\alpha\beta}{}_\gamma$ are the components of T and $\partial_{x^\alpha} \otimes \partial_{x^\beta} \otimes dx^\gamma$ is its basis. The components are nothing else then the tensor evaluated on the corresponding basis vectors:

$$T^{\kappa\delta}{}_\iota = T(dx^\kappa, dx^\delta, \partial_{x^\iota})\,. \tag{2.13}$$

[1] Note that the 'transformation matrix' for a covector $A^\alpha_\beta = \frac{\partial x^\alpha}{\partial x'^\beta}$ is an inverse of the transformation matrix for the vector $B^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x^\beta}$: $B^\alpha_\beta A^\beta_\gamma = \delta^\alpha_\gamma$.

Indeed, we have

$$T(dx^\kappa, dx^\delta, \partial_{x^\iota}) = T^{\alpha\beta}{}_\gamma dx^\kappa(\partial_{x^\alpha}) dx^\delta(\partial_{x^\beta}) dx^\gamma(\partial_{x^\iota}) = T^{\alpha\beta}{}_\gamma \delta^\kappa_\alpha \delta^\delta_\beta \delta^\gamma_\iota = T^{\kappa\delta}{}_\iota \,. \tag{2.14}$$

By linearity we then have

$$T(\omega, \nu, W) = T(\omega_\alpha dx^\alpha, \nu_\beta dx^\beta, W^\gamma \partial_{x^\gamma}) = \omega_\alpha \nu_\beta W^\gamma T(dx^\alpha, dx^\beta, \partial_{x^\gamma}) = T^{\alpha\beta}{}_\gamma \omega_\alpha \nu_\beta W^\gamma \,, \tag{2.15}$$

which is a scalar function.

- Cook book (tensor algebra). Let T, S be two tensors of rank t and s, then

  i) $T + S$ is a tensor (if they are the same type).

  ii) *tensor product* $\otimes$ "creates bigger tensors", namely, $T \otimes S$ is a tensor of rank $(t+s)$. For example, considering the above tensor T, together with a 1-form $S = S_\delta dx^\delta$, we have

  $$T \otimes S = \underbrace{T^{\alpha\beta}{}_\gamma S_\delta}_{(T\otimes S)^{\alpha\beta}{}_{\gamma\delta}} \partial_{x^\alpha} \otimes \partial_{x^\beta} \otimes dx^\gamma \otimes dx^\delta \,. \tag{2.16}$$

  iii) *contraction* $\cdot$ "creates smaller tensors". Each contraction 'connects' the corresponding vector and covector basis, reducing the rank of a tensor by two. For example, considering the above (2,1) tensor T, we can contract the first and the last index, obtaining a $(1,0)$ tensor T_{contr}, given by

  $$T_{contr} = T^{\alpha\beta}{}_\gamma \partial_{x^\beta} \underbrace{dx^\gamma(\partial_{x^\alpha})}_{\delta^\gamma_\alpha} = T^{\alpha\beta}{}_\alpha \partial_{x^\beta} = T^\beta_{contr} \partial_{x^\beta} \,. \tag{2.17}$$

  One can also contract indices of two tensors, creating a tensor $T \cdot S$ of rank $(t + s - 2 \times \#_{\text{contractions}})$. E.g.

  $$(T \cdot S)^{\alpha\ldots\mu\ldots}{}_{\gamma\ldots\nu\ldots} = T^{\kappa\delta\alpha\ldots}{}_{\gamma\ldots} S^{\mu\ldots}{}_{\kappa\delta\nu\ldots} \,. \tag{2.18}$$

  Here indices that are summed over, namely κ and δ are called *dummy indices* whereas the 'surviving indices' $(\alpha, \ldots)$ are called *free indices.*

  iv) Tensors are invariant objects. In particular, if a tensor is zero in one coordinate system, it is zero in all coordinate systems.

2.3 Connection

- We express laws of physics as differential equations. For this, we need to define derivatives of tensors. However, differentiation of tensors on M is problematic, c.f.,

$$\left.\frac{df}{dt}\right|_{t_0} = \lim_{s\to 0} \frac{f(t_0 + s) - f(t_0)}{s} \,.$$

On M, one might want to replace t_0 by p. However how to add s to p? And how to compare a vector at $p+\delta p$ to a vector at p when they live on a different space?

To resolve these issues one needs an additional structure. We have 3 standard possibilities:

i) *Lie derivative* (vector field U).

ii) *Exterior derivative* (forms).

iii) *Covariant derivative* (connection Γ).

Although the last possibility is by far the most complicated one, it is very important for general relativity and hence we discuss it now (please refer to Appendix A for the other two possibilities).

- We start with the following observation. For a scalar function ϕ,

$$\partial_\mu \phi(x) \tag{2.19}$$

is a $(0,1)$ tensor. However, a partial derivative of a vector transform as

$$\begin{aligned} \frac{\partial}{\partial x'^\mu} V'^\alpha(x') &= \frac{\partial}{\partial x'^\mu}\Big(\frac{\partial x'^\alpha}{\partial x^\nu} V^\nu\Big) \\ &= \underbrace{\frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x^\beta}{\partial x'^\mu} V^\nu{}_{,\beta}}_{\text{OK for (1,1) tensor}} + \underbrace{\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial^2 x'^\alpha}{\partial x^\beta \partial x^\nu} V^\nu}_{\text{problem!}} \end{aligned} \tag{2.20}$$

The second term is a problem and shows that a partial derivative of a vector does not transform as a tensor.[2]

- To deal with this problem Riemann (1830 PhD) defined the following <u>covariant derivative</u>:

$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\alpha\mu} V^\alpha\,,} \tag{2.21}$$

where Γ is a <u>connection</u> defined in such a way so that the resulting object, ∇V transforms as $(1,1)$ tensor. In its turn it implies that the connection has to transform as

$$\Gamma'^\alpha{}_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\kappa}\frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial x^\iota}{\partial x'^\gamma}\Gamma^\kappa{}_{\delta\iota} - \frac{\partial^2 x'^\alpha}{\partial x^\kappa \partial x^\delta}\frac{\partial x^\kappa}{\partial x'^\beta}\frac{\partial x^\delta}{\partial x'^\gamma}\,. \tag{2.22}$$

Any object transforming this way defines a connection.

Similarly, a covariant derivative of a 1-form is defined as

$$\boxed{\nabla_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^\gamma{}_{\beta\alpha}\omega_\gamma\,,} \tag{2.23}$$

[2]Note that the second term vanishes for the Lorentz transformation, and thence the object is a tensor in special relativity.

which by linearity extends to a covariant derivative of an arbitrary tensor as

$$\nabla_\alpha T_{\beta\ldots}{}^{\gamma\ldots} = \partial_\alpha T_{\beta\ldots}{}^{\gamma\ldots} + \Gamma^\gamma{}_{\delta\alpha} T_{\beta\ldots}{}^{\delta\ldots} + \cdots - \Gamma^\rho{}_{\beta\alpha} T_{\rho\ldots}{}^{\gamma\ldots} - \ldots . \tag{2.24}$$

Let us stress that we did not need metric to define the covariant derivative.

- Although the connection is not a tensor, its 'antisymmetrized version'

$$\boxed{T^\alpha{}_{\beta\gamma} = -2\Gamma^\alpha{}_{[\beta\gamma]} = \Gamma^\alpha{}_{\gamma\beta} - \Gamma^\alpha{}_{\beta\gamma}\,,} \tag{2.25}$$

is, and is called a torsion tensor. In GR we shall set torsion to zero (and hence have a symmetric connection.) However, in various SUGRAs and string theory, T is typically non-zero and we get an independent field equation for torsion: "spin is the source of torsion".

- Formally, the covariant derivative ∇ is a map ∇ : (k, l) tensors $\to$ $(k, l+1)$ tensors, such that:

  1. It is a derivative: linear + Leibnitz
  2. Reduces to ∂ on scalars: $\nabla_\mu f = \partial_\mu f$
  3. Commutes with contraction: $\nabla(T^{\alpha\ldots}{}_{\alpha\ldots}) = \nabla T_{\text{contr}}{}^{\ldots}{}_{\ldots}$
  4. We have

$$[\nabla_\mu, \nabla_\nu] f = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu] f = -T^\kappa{}_{\mu\nu} \nabla_\kappa f\,, \tag{2.26}$$

  where $T^\kappa{}_{\mu\nu}$ is the torsion defined above.

These properties can also be used as a definition of the covariant derivative. Let us finally show that (2.23) agrees with (2.21). Schematically for a contraction of a vector V and a 1-form ω we have

$$\nabla(V \cdot \omega) = \nabla V \cdot \omega + V \cdot \nabla \omega = (\partial V + \Gamma V) \cdot \omega + V \cdot (\partial \omega - \Gamma \omega) = \partial(V \cdot \omega)\,. \tag{2.27}$$

Here we have used the Leibnitz, and the expressions above, to obtain an expression consistent with the covariant derivative of a function :). (As always with derivatives it is enough to know how they act on a function and on a vector and everything else follows by Leibnitz and other properties.)

2.4 Metric

- Metric g is a symmetric, non-degenerate, $(0, 2)$ tensor field on M.[3]

$$g = g_{\mu\nu} dx^\mu dx^\nu = ds^2\,. \tag{2.29}$$

[3] The second equality is a really a 'misuse of language', reminding that the metric gives a scalar product of two vectors (and hence also defines distances):

$$g(V, W) = g_{\mu\nu} dx^\mu(V) dx^\nu(W) = g_{\mu\nu} V^\mu W^\nu = V \cdot W\,. \tag{2.28}$$

In particular, we consider $ds^2 = g(dx, dx)$, where dx is treated as an infinitesimal vector, and ds^2 denotes the corresponding distance.

- In GR, $g_{\mu\nu}(x)$ is a general 4×4 symmetric matrix at each x^α, with 3 positive and 1 negative eigenvalues—yielding "*pseudo-Riemannian geometry*". It has 10 components.

- Christoffel symbols. In the previous section we have discussed connection. In the presence of a metric, there exists a special connection, encoded in the so called *Christoffel symbols.* To write them down, let us introduce the following metricity tensor:

$$M_{\alpha\beta\gamma} = \nabla_\alpha g_{\beta\gamma} \,. \tag{2.30}$$

Research problem: Think about a theory in which the metricity would be dynamically determined. (Well there are actually some, see e.g. [9] :))

Theorem. Provided the torsion and metricity tensor both vanish, $M_{\alpha\beta\gamma} = 0$, $T^\alpha{}_{\beta\gamma} = 0$, there exists a unique connection given by the Christoffel symbols:

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta}) \,. \tag{2.31}$$

Proof. To prove the above theorem, we use the metricity condition and we write

$$\nabla_\alpha g_{\beta\gamma} = 0 = \partial_\alpha g_{\beta\gamma} - \Gamma^\delta{}_{\beta\alpha} g_{\delta\gamma} - \Gamma^\delta{}_{\gamma\alpha} g_{\beta\delta} = \partial_\alpha g_{\beta\gamma} - \Gamma_{\gamma\beta\alpha} - \Gamma_{\beta\gamma\alpha} \,, \tag{2.32}$$

where Γ's are a general connection here. Similarly we write $\nabla_\gamma g_{\alpha\beta} = 0$ and $\nabla_\beta g_{\gamma\alpha} = 0$. By adding the first and the last and subtracting the second, we find

$$0 = \partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} - \underbrace{(\Gamma_{\gamma\beta\alpha} + \Gamma_{\gamma\alpha\beta})}_{2\Gamma_{\gamma(\alpha\beta)}} + \underbrace{\Gamma_{\beta\alpha\gamma} - \Gamma_{\beta\gamma\alpha}}_{-T_{\beta\alpha\gamma}} + \underbrace{\Gamma_{\alpha\beta\gamma} - \Gamma_{\alpha\gamma\beta}}_{-T_{\alpha\beta\gamma}} \,. \tag{2.33}$$

By re-arranging this equation we find

$$\Gamma_{\gamma\alpha\beta} = \Gamma_{\gamma(\alpha\beta)} + \Gamma_{\gamma[\alpha\beta]} = \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}) - \frac{1}{2} K_{\alpha\beta\gamma} \,, \tag{2.34}$$

where we have defined a *cotorsion tensor*

$$K_{\alpha\beta\gamma} = T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha} \,. \tag{2.35}$$

For a vanishing torsion, $T = 0$, the cotorsion tensor also vanishes, and the connection is given by the Christoffel symbols.

- **Statement:** at a point one can always introduce coordinates so that the metric components take the form $\eta_{\mu\nu}$. The proof od this is in the following figure:

proof:
$$g_{\mu\nu}\, dx^\mu dx^\nu = dx^T G\, dx = dx^T O^T D\, O\, dx$$

$$D = \begin{pmatrix} -\lambda_0 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix} \overset{\lambda_i\text{'s}>0}{=} \begin{pmatrix} -\ell_0^2 & & & \\ & \ell_1^2 & & \\ & & \ell_2^2 & \\ & & & \ell_3^2 \end{pmatrix}$$

$$L = \begin{pmatrix} \ell^0 & & & \\ & \ell^1 & & \\ & & \ell^2 & \\ & & & \ell^3 \end{pmatrix}$$

$$= \underbrace{dx^T O^T L^T}_{dy^T}\, \eta\, \underbrace{L\, O\, dx}_{dy} = \underline{dy^T \eta\, dy} \qquad \blacksquare$$

(This is nothing else than a "diagonalization' of the symmetric matrix through the Gram–Schmidt orthonormalization process as you know it from linear algebra.)

- Principle of equivalence. At every spacetime point it is possible to chose a coordinate system (a "freely falling" or "locally inertial") such that within a sufficiently small neighbourhood the laws of physics take the same special-relativistic form as they do in Minkowski space. More specifically, one can always choose coordinates so that

$$g_{\mu\nu} = \eta_{\mu\nu}\,, \quad g_{\mu\nu,\rho} = 0 \quad \text{at a point.} \tag{2.36}$$

Proof. (not really) Instead of a proof of equation (2.36), let us provide a 'counting argument' that this may be possible. In the first step we consider the following coordinate transformation, $x \to x'$, at point p:

$$\underbrace{x'^\alpha}_{\text{new}} = A^\alpha{}_\beta \underbrace{x^\beta}_{\text{original}} + O(x^2) \quad \Leftrightarrow \quad x^\alpha = \tilde{A}^\alpha{}_\beta x'^\beta + O(x^2)\,, \tag{2.37}$$

where $\tilde{A}$ is an inverse to A. Using the transformation property for the metric, we require

$$g'_{\alpha\beta}(p) = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta} g_{\gamma\delta} = \tilde{A}^\gamma{}_\alpha \tilde{A}^\delta{}_\beta g_{\gamma\delta}(p) = \eta_{\alpha\beta}\,. \tag{2.38}$$

This requirement imposes 10 equations for 16 unknowns $\tilde{A}^\alpha{}_\beta$, or equivalently $A^\alpha{}_\beta$, and is possible to solve. (6 components of A remain undetermined, which precisely corresponds to 'Lorentz transformations' that do not change $\eta_{\mu\nu}$.)

Let us next consider the coordinate transformation to higher order:

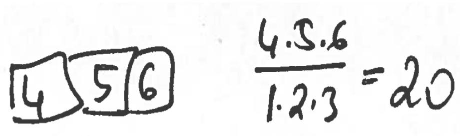
$$x'^\alpha = \underbrace{A^\alpha{}_\beta}_{\text{fixed already}}\, x^\beta + \frac{1}{2} \underbrace{B^\alpha{}_{\beta\gamma}}_{\text{new quantities}}\, x^\beta x^\gamma + O(x^3)\,. \tag{2.39}$$

Thus, we gained 40 new quantities, $B^\alpha{}_{\beta\gamma} = B^\alpha{}_{(\beta\gamma)}$ to potentially fix. We now require that

$$g'_{\mu\nu,\rho} = 0\,, \tag{2.40}$$

which represents 40 linear equations for coefficients $B^\alpha{}_{\beta\gamma}$. This has unique solution and we see that the derivatives of a metric can also be made to vanish at a point.

What would happen if we were to go to $O(x^3)$? We would then have new coefficients $C^\alpha{}_{\beta\gamma\delta} = C^\alpha{}_{(\beta\gamma\delta)}$, which represent $20 \times 4 = 80$ quantities to fix, as given by the following Young tablox:

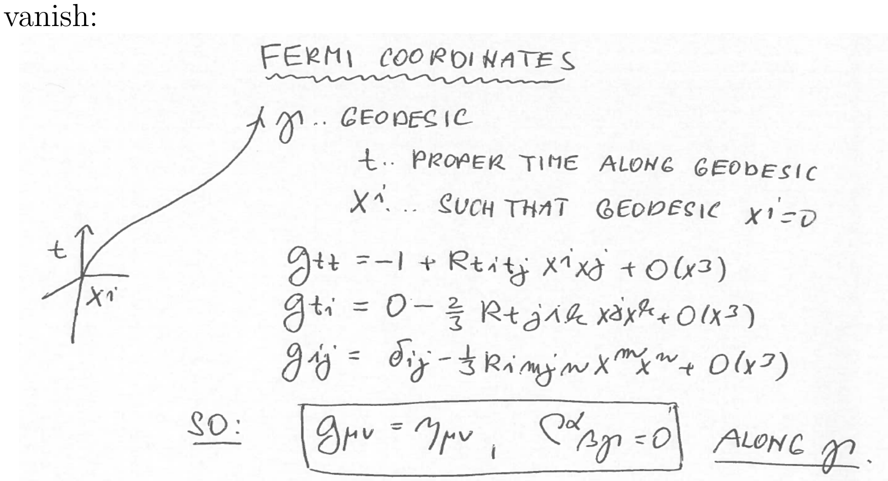


However, we now have 100 components of the second derivatives of the metric, $g_{\alpha\beta,\gamma\delta}$. This suggests that we may be able to set 80 of these to zero, leaving 20 unconstrained 2nd derivatives of the metric. These are precisely captured by a Riemann tensor

$$R^\alpha{}_{\beta\gamma\delta} = R^\alpha{}_{\beta\gamma\delta}(\partial^2 g, \partial g, g)\,, \tag{2.41}$$

discussed in the next section. [Note that in order this rank $(1,3)$ tensor has only 20 independent components, it must have lots of symmetries, you may already start wondering what they are.]

- Even more general version of the principle of equivalence is displayed in the following picture. Namely, along any geodesic one can construct a special coordinate system, called Fermi coordinates so that the metric is flat and the Christoffel's vanish:



- Invariant volume element. Let us show that the determinant of the metric, $g =$

$\det(g_{\mu\nu})$, is not a tensor. The metric transforms according to

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)\,. \tag{2.42}$$

The r.h.s. can be understood as a product of three matrices. Taking now the determinant of both sides, and applying the rule that determinant of a product of matrices is a product of determinants, we have

$$g' = \det g'_{\mu\nu} = \det \frac{\partial x^\alpha}{\partial x'^\mu} \det \frac{\partial x^\beta}{\partial x'^\nu} \det g_{\alpha\beta} = \left|\frac{\partial x'}{\partial x}\right|^{-2} g\,. \tag{2.43}$$

This is how a tensor density of weight +2 transforms. (A tensor density of <u>density weight w</u> transforms as a corresponding tensor times $|\frac{\partial x'}{\partial x}|^{-w}$.)

Another example of tensor density is the 'bare' volume element $\tilde{dV} = d^n x$. Under a coordinate change, this transforms according to

$$\tilde{dV}' = d^n x' = \left|\frac{\partial x'}{\partial x}\right| d^n x = \left|\frac{\partial x'}{\partial x}\right| \tilde{dV}\,, \tag{2.44}$$

picking up the Jacobian of the transformation. It is then easy to see that we have the following <u>invariant volume element</u>:

$$\boxed{dV = \sqrt{-g}\, d^n x\,.} \tag{2.45}$$

- <u>Levi-Civita tensor.</u> Let us introduce the permutation symbol $[\mu\nu\rho\sigma]$ by

$$[\mu\nu\rho\sigma] = \begin{cases} +1 & \text{for } [0,1,2,3] \text{ or even permutation} \\ -1 & \text{for odd permutation} \\ 0 & \text{otherwise} \end{cases}\,. \tag{2.46}$$

This is not a tensor; in any coordinate system it assumes the values ± 1 and 0. However, a Levi-Civita tensor can be introduced by

$$\boxed{\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}[\mu\nu\rho\sigma]\,, \qquad \epsilon^{\mu\nu\rho\sigma} = -\frac{1}{\sqrt{-g}}[\mu\nu\rho\sigma]\,.} \tag{2.47}$$

<u>Proof.</u> To prove this, let us consider $[\alpha\beta\gamma\delta]\frac{\partial x^\alpha}{\partial x'^{\alpha'}}\frac{\partial x^\beta}{\partial x'^{\beta'}}\frac{\partial x^\gamma}{\partial x'^{\gamma'}}\frac{\partial x^\delta}{\partial x'^{\delta'}}$. Obviously, this is completely antisymmetric in primed indices and must be therefore proportional to $[\alpha'\beta'\gamma'\delta']$,

$$[\alpha\beta\gamma\delta]\frac{\partial x^\alpha}{\partial x'^{\alpha'}}\frac{\partial x^\beta}{\partial x'^{\beta'}}\frac{\partial x^\gamma}{\partial x'^{\gamma'}}\frac{\partial x^\delta}{\partial x'^{\delta'}} = \lambda[\alpha'\beta'\gamma'\delta']\,, \tag{2.48}$$

for some λ. To find this λ, we can evaluate the previous expression for $[\alpha',\beta',\gamma',\delta'] = [0,1,2,3]$. Then, the r.h.s. equals λ and the l.h.s. becomes a determinant of the $\frac{\partial x}{\partial x'}$ matrix, so that we have

$$\lambda = \det\Bigl(\frac{\partial x}{\partial x'}\Bigr) = \frac{\sqrt{-g'}}{\sqrt{-g}}\,, \tag{2.49}$$

upon using (2.43). Plugging this back to (2.48), we see that the object $\epsilon_{\alpha\beta\gamma\delta}$ transforms as a tensor. End of proof :).

2.5 Parallel transport & geodesics

Parallel transport is a curved space generalization of "keeping a vector constant as we move it along a path".

- Consider a curve γ, parameterized by parameter λ. We define a directional covariant derivative of a vector V along γ as

$$\frac{DV^\alpha}{d\lambda} = U^\mu \nabla_\mu V^\alpha = \nabla_U V^\alpha = \dot{V}^\alpha\,, \tag{2.50}$$

  where $U^\mu = \frac{dx^\mu}{d\lambda}$ is the tangent vector to the curve γ. Generalization from a vector V to an arbitrary tensor T is straightforward.

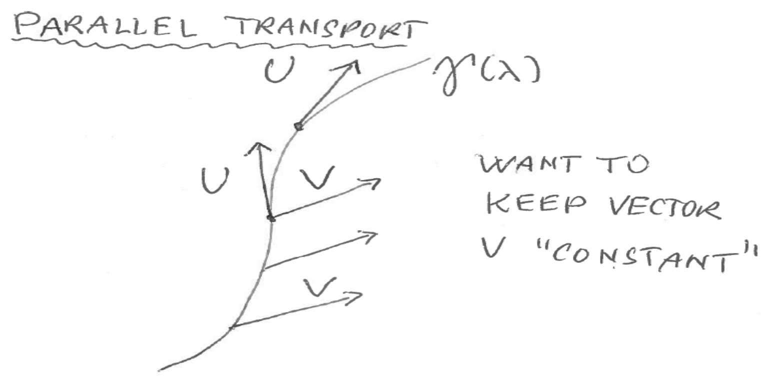
- The vector is parallel-transported along γ provided the corresponding directional derivative vanishes,

$$\boxed{\nabla_U V^\alpha = 0\,.} \tag{2.51}$$

  More explicitly, this gives

$$\frac{dV^\alpha}{d\lambda} + \Gamma^\alpha{}_{\beta\gamma} U^\beta V^\gamma = 0\,. \tag{2.52}$$

  Of course, in flat space (and Cartesian coordinates) this simply means that the components of the vector do not change and thence the vector remains "constant", see figure:



- A geodesic is a curve along which the tangent vector is parallel transported,

$$\boxed{\nabla_U U^\alpha = 0\,, \quad U^\alpha = \frac{dx^\alpha}{d\lambda}\,.} \tag{2.53}$$

  Of course, the fact whether or not a curve is geodesic should be independent of parametrization. However, a change of parameter $\lambda \to \tilde{\lambda}$ results in general in the following modification of geodesic equation:

$$\frac{DU^\alpha}{d\tilde{\lambda}} = f(\tilde{\lambda}) U^\alpha\,. \tag{2.54}$$

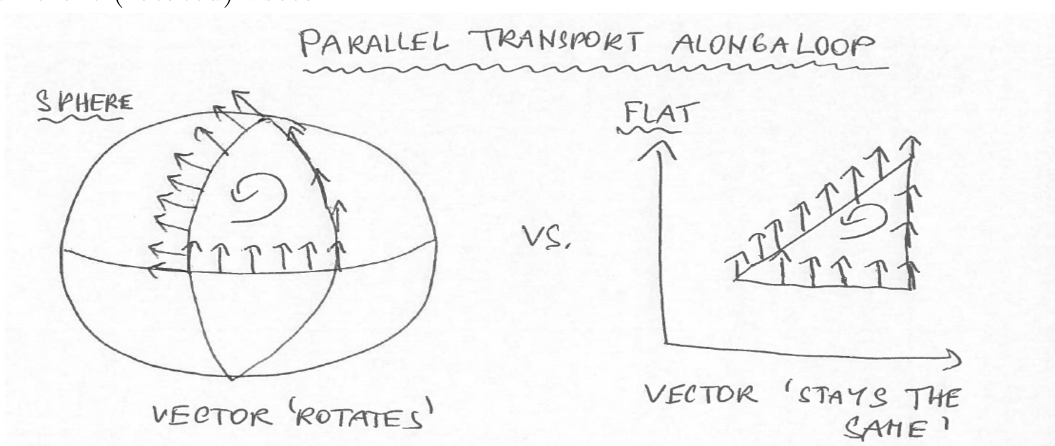
The parameter for which the r.h.s. vanishes is called an affine parameter. In what follows we shall always consider such a choice.

- Consider two parallel transported vectors V, W along γ. Then for their scalar product (given by the metric) we have:

$$\dot{V} = 0\,, \quad \dot{W} = 0 \quad \Rightarrow \quad \frac{D}{d\lambda}(V \cdot W) = \frac{D}{d\lambda}(V^\alpha W^\beta g_{\alpha\beta}) = V^\alpha W^\beta U^\gamma \nabla_\gamma \gamma_{\alpha\beta} = 0\,, \tag{2.55}$$

provided that the metricity vanishes. (This is the main reason why want the metricity to vanish.) That is, these vectors "propagate the same angle" (their scalar product remains the same).

In particular, consider a parallel transport of a vector along geodesic loop, as displayed in the following figure. Since the angle w.r.t. the tangent vector of each geodesic remains the same as we transport, at the end of loop we end up with a different (rotated) vector.



As we shall see this motivates a definition of the Riemann tensor.

- First look at Killing vectors. Let γ be a geodesic with tangent vector U, $\nabla_U U = 0$, ξ be a vector field, and C a corresponding scalar quantity linear in U:

$$C = U \cdot \xi = U^\alpha \xi_\alpha\,. \tag{2.56}$$

Let us now demand that this quantity is conserved (parallel-transported) along any geodesic, that is:

$$0 = \dot{C} = \nabla_U(U^\alpha \xi_\alpha) = (U^\beta \nabla_\beta U^\alpha)\xi_\alpha + U^\alpha U^\beta \nabla_\beta \xi_\alpha = 0 + U^\alpha U^\beta \nabla_{(\alpha}\xi_{\beta)}\,, \tag{2.57}$$

where we have used the geodesic equation and the fact that $U^\alpha U^\beta$ is symmetric in α and β. Demanding the last formula for any geodesic, the vector field ξ has to satisfy the following Killing vector equation:

$$\boxed{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0\,.} \tag{2.58}$$

As conserved quantities are associated with continuous symmetries, via Noether's theorem, such a Killing vector has to somehow encode the symmetry of the spacetime. We shall confirm this intuition in a tutorial.

2.6 Curvature

Riemann tensor describes how much the spacetime is curved, it describes the 'strength of gravity'.

- Two advanced (and vague) motivations.
  - Gravity as a gauge theory. Consider a gauge theory (electromagnetism for example) and define as usual the 'covariant derivative'

$$D_\mu = \partial_\mu + A_\mu \,, \tag{2.59}$$

where A_μ is the gauge field (understood as a connection). Consider next the commutator of the two covariant derivatives:

$$\begin{aligned} [D_\mu, D_\nu] &= (\partial_\mu + A_\mu)(\partial_\nu + A_\nu) - (\partial_\nu + A_\nu)(\partial_\mu + A_\mu) \\ &= \underbrace{\partial_\mu\partial_\nu - \partial_\nu\partial_\mu}_{0} + \underbrace{(\partial_\mu A_\nu) - (\partial_\nu A_\mu) + [A_\mu, A_\nu]}_{F_{\mu\nu}} = F_{\mu\nu} \,. \end{aligned} \tag{2.60}$$

That is, the gauge field strength emerges from a commutator of two covariant derivatives. [Note that in the case of $U(1)$ gauge theory the last term vanishes and we recover the standard definition of the Maxwell field strength tensor.]
Similarly, as you will see in Ruth's course, gravity can be described in terms of veilbeins e and the corresponding (spin) connection ω:

$$\begin{aligned} &\text{veilbeins} && e^a_\mu : \quad g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \,, \\ &\text{connection} && \omega_\mu{}^a{}_b : \quad \nabla_\mu e_b = \omega_\mu{}^a{}_b e_a \,. \end{aligned} \tag{2.61}$$

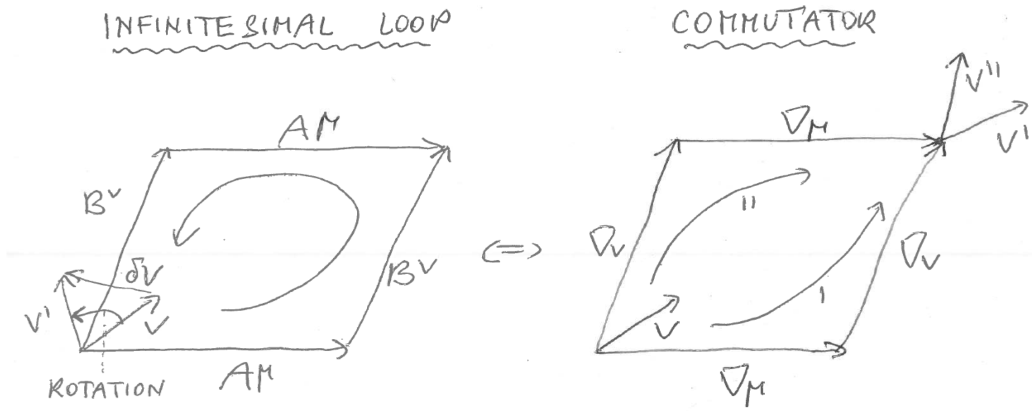
The covariant derivative is then given by

$$D_\mu = \partial_\mu + \omega_\mu \,, \tag{2.62}$$

and the corresponding field strength, by

$$[D_\mu, D_\nu]^a{}_b = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + [\omega_\mu, \omega_\nu]^a{}_b = R_{\mu\nu}{}^a{}_b \,. \tag{2.63}$$

  - Holonomy. Consider an infinitesimal loop, specified by two (infinitesimal) vectors A and B, see figure.



Then, as we parallel transport a vector V along this loop, the vector 'rotates' a bit, so that we now have $V + \delta V$. It is precisely this rotation which is measured by the curvature of the spacetime:

$$\delta V^\rho = R(A,B)^\rho{}_\sigma V^\sigma = R^\rho{}_{\sigma\beta\gamma} V^\sigma A^\beta B^\gamma \,. \tag{2.64}$$

Since everything is infinitesimal, the linearity in each argument is obvious. Moreover, the 'rotation matrix' $R(A,B)$ obeys $R(A,B) = -R(B,A)$, as switching of the two corresponds to the change of orientation of the loop.

Similarity of the previous with a commutator of two covariant derivatives is obvious: we measure the 'rotation' between transporting the vector one way with respect to transforming it other way, see above figure.

For a general manifold, the curvature is such that the rotation simply corresponds to $SO(n)$ transformation. However, there are special Riemannian manifolds for which this is a more special group – they have special holonomy – such manifolds are of great interest to mathematicians (and string theorists). Two well known examples are Kähler manifolds (with $U(n/2)$ holonomy) and their odd-dimensional cousins Sasaki manifolds. Manifolds with special holonomy admit special (Killing) spinors.

- Definition. In the absence of torsion, we define the Riemann tensor (or field strength) by[4]

$$\boxed{[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma \,.} \tag{2.65}$$

As we shall see in the next chapter, the Riemann tensor measures the tidal forces, that is the strength of gravity.

[4]Looking at the commutator as a 2nd-order differential operator one may wonder why in (2.65) it results in a multiplicative transformation. The reason is simple. The first derivative in general exists, it is given by torsion:

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\sigma{}_{\mu\nu} \nabla_\sigma V^\rho \,.$$

What about the second derivative?

By expanding the left hand side, we recover the following explicit formula in terms of the connection:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}\,. \tag{2.66}$$

It is easy to show that for a tensor with more indices, we have

$$[\nabla_\mu, \nabla_\nu] T^{\alpha\dots}{}_{\beta\dots} = R^{\alpha}{}_{\sigma\mu\nu} T^{\sigma\dots}{}_{\beta\dots} + \dots - R^{\sigma}{}_{\beta\mu\nu} T^{\alpha\dots}{}_{\sigma\dots} - \dots\,. \tag{2.67}$$

- Flatness. A spacetime is called flat if there exists a coordinate transformation that brings the metric g to the Minkowski form, $g = \eta$, everywhere.

  The teacher may tease you by giving you the following task: given a metric, find the transformation that brings it to the flat form. How do I even know such a transformation exists? The following theorem provides an answer:
  Theorem. *The spacetime is flat iff the Riemann tensor vanishes everywhere.*

- Symmetries and number of components. What are the symmetries and a number of independent components of the Riemann tensor? Let us use the Christoffel connection and write the tensor in terms of the derivatives of the metric. Namely, we introduce

$$\begin{aligned} R_{\mu\nu\lambda\rho} &\equiv g_{\mu\alpha} R^{\alpha}{}_{\nu\lambda\rho} \\ &= \frac{1}{2}(g_{\mu\rho,\nu\lambda} - g_{\mu\lambda,\nu\rho} - g_{\nu\rho,\mu\lambda} + g_{\lambda\nu,\rho,\mu}) + g_{\alpha\beta}(\Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\beta}{}_{\nu\lambda} - \Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\beta}{}_{\nu\rho})\,. \end{aligned} \tag{2.68}$$

  In the proof of the principe of equivalence in four dimensions, we have established that at a point we can always set Γ's (first derivatives of the metric) to zero, and have 20 second derivatives of the metric unspecified. The above formula then suggests that the Riemann tensor can have at most 20 independent components. Let us show how this follows from the symmetries of this tensor.

  Using the above formula, one can show the following symmetry properties: it is antisymmetric in the first two and the last two indices, and symmetric under the exchange of the two pairs:

$$R_{\mu\nu\kappa\lambda} = -R_{\mu\nu\lambda\kappa} = -R_{\nu\mu\kappa\lambda} = R_{\kappa\lambda\mu\nu}\,. \tag{2.69}$$

  We also have the following complete antisymmetry:

$$R_{[\mu\nu\kappa\lambda]} = 0\,. \tag{2.70}$$

  Together with the previous symmetries, the complete antisymmetry implies the so called cyclic property:[5]

$$R_{\mu[\nu\kappa\lambda]_c} = 0\,. \tag{2.73}$$

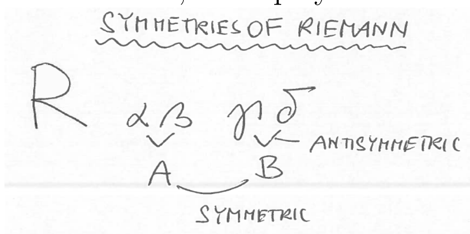
[5] It can directly be derived by using the Jacobi identity:

$$[\nabla_\mu, [\nabla_\nu, \nabla_\alpha]] + [\nabla_\alpha, [\nabla_\mu, \nabla_\nu]] + [\nabla_\nu, [\nabla_\alpha, \nabla_\mu]] = 0\,, \tag{2.71}$$

In d number of spacetime dimensions, the Riemann tensor then has at most

$$\boxed{\#_{\text{Riem}} = \frac{d^2}{12}(d^2 - 1)} \tag{2.74}$$

components, that is 20 in $d = 4$. The proof of the counting is quite entertaining. We first introduce 'capital indices A and B, each replacing the pair of antisymmetric indices, as displayed in the figure:



Each such index takes $n = \begin{pmatrix} d \\ 2 \end{pmatrix}$ values. The resultant object is symmetric in these, yilding $n(n+1)/2$ components. However, there is also the antisymmetric constraint (2.72), giving $\begin{pmatrix} d \\ 4 \end{pmatrix}$ constraints. Hence we have

$$\# = \frac{n(n+1)}{2} - \begin{pmatrix} d \\ 4 \end{pmatrix} \tag{2.75}$$

components, which should give (2.74).

- Rabbits, relations, and kins of Riemann family. Starting from a Riemann tensor, let us now define its various 'contractions' that are of great importance in general relativity. Namely, we define the (symmetric) Ricci tensor $R_{\mu\nu}$, the Ricci scalar R, and the (symmetric) Einstein tensor $G_{\mu\nu}$ by:

$$\boxed{R_{\mu\nu} = R^{\kappa}{}_{\mu\kappa\nu}\,, \quad R = R^{\mu}{}_{\mu}\,, \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\,.} \tag{2.76}$$

We can also decompose the Riemann tensor into its 'Ricci part' and 'the rest', defining the so called Weyl tensor, schematically,

$$\text{Riemann} = \text{Ricci} + \text{Weyl}\,. \tag{2.77}$$

The Weyl tensor $C^{\alpha}{}_{\beta\gamma\delta}$ has all the symmetries of the Riemann tensor, but the corresponding 'Ricci contraction' vanishes (see tutorial for more details):

$$C_{\mu\nu} = C^{\alpha}{}_{\mu\alpha\nu} = 0\,. \tag{2.78}$$

applied to a scalar:

$$\nabla_{[\mu}\nabla_{\nu}\nabla_{\alpha]}\psi = -\frac{1}{2}R^{\gamma}{}_{[\alpha\mu\nu]}\nabla_{\gamma}\psi = 0\,. \tag{2.72}$$

By applying to a vector instead, one would derive the Bianchi identity for the Riemann tensor (2.80) below.

The importance of this tensor is captured by the following:
Theorem. *The Weyl tensor vanishes iff the spacetime is conformally flat, that is there exists a coordinate system in which the metric takes the form:*

$$g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu} \,. \tag{2.79}$$

- Bianchi identities. The Riemann tensor obeys the following identity:

$$R_{\alpha\beta[\gamma\delta;\epsilon]_c} = 0 \,. \tag{2.80}$$

By contracting a few indices we arrive at a crucial Bianchi identity for the Einstein tensor (see tutorial for details):

$$\boxed{\nabla_\mu G^{\mu\nu} = 0 \,.} \tag{2.81}$$

As we shall see this has far reaching consequences for the Einstein theory.

- Uniqueness. Above we have proved that the Einstein tensor is a rank-2 tensor, that depends on the metric and its first and second derivatives, and obeys the Bianchi identity, $\nabla_\mu G^{\mu\nu} = 0$. When looking for the field equations in the previous chapter we may ask if this is the only tensor with these properties. Thanks to the metricity condition we immediately find that another such tensor is

$$\boxed{G^{\mu\nu} + \Lambda g^{\mu\nu} \,,} \tag{2.82}$$

where Λ is a constant. This term is indeed possible in the Einstein equations and we will get the right Newtonian limit provided we set Λ to be small. Of course, Λ is nothing else than the (in)famous cosmological constant, or as Einstein called it the '*biggest blunder of his life*'. To Einstein's surprise, we now believe that Λ does exist in our Universe, but is tiny tiny, to be explained by current theories (it is 10^{120} times smaller than a natural theoretical value). This is called the cosmological constant problem and you will learn about it in your Cosmology Course.

Chapter 3: General Relativity

In this chapter we want to formulate physics of gravitation as described by General Relativity. That is, we want to find relativistic geometric generalization of Newton's theory:

$$\vec{a} = -\nabla\phi\,, \quad \nabla^2\phi = 4\pi G\rho\,. \tag{3.1}$$

We have already motivated these using the physical intuition the way originally Einstein did (following Einstein's nose), they are

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,, \quad G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu}\,. \tag{3.2}$$

Let us now take the (more effective) Hilbert's point of view and approach the task via the Lagrangian formalism. We first study how the matter moves in the presence of gravity, and then how the spacetime curves in the presence of matter.

3.1 Particle in a curved spacetime

- Particle action. Similar to what happens in special relativity, the motion of particles in curved spacetime maximizes the particle's proper time, the only difference is that now this is expressed in terms of the curved metric. Hence we have

$$\boxed{S_p = -m\int d\tau = -m\int d\lambda\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,.} \tag{3.3}$$

Here, λ is a parameter parameterizing particle's trajectory, and $g_{\mu\nu} = g_{\mu\nu}(x(\lambda))$. Denoting by $\dot{} = \frac{d}{d\lambda}$, the corresponding Lagrangian reads

$$L(x^\mu, \dot{x}^\mu) = -m\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,. \tag{3.4}$$

Let us write the corresponding Euler–Lagrange equations

$$\frac{d}{d\lambda}\Big(\frac{\partial L}{\partial \dot{x}^\alpha}\Big) - \frac{\partial L}{\partial x^\alpha} = 0\,. \tag{3.5}$$

We find

$$\begin{aligned} \frac{\partial L}{\partial x^\alpha} &= \frac{m g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu}{2\sqrt{}}\,, \quad \frac{\partial L}{\partial \dot{x}^\alpha} = \frac{m g_{\mu\alpha}\dot{x}^\mu}{\sqrt{}}\,, \\ \frac{d}{d\lambda}\Big(\frac{\partial L}{\partial \dot{x}^\alpha}\Big) &= \frac{d}{d\lambda}\Big(\frac{1}{\sqrt{}}\Big) m g_{\mu\alpha}\dot{x}^\mu + \frac{m}{\sqrt{}}\big(g_{\mu\alpha,\nu}\dot{x}^\nu\dot{x}^\mu + g_{\mu\alpha}\ddot{x}^\mu\big)\,. \end{aligned} \tag{3.6}$$

Re-arranging the Euler–Lagrange equation thus gives

$$g_{\mu\alpha}\ddot{x}^{\mu}+\underbrace{\frac{1}{2}g_{\mu\alpha,\nu}\dot{x}^{\nu}\dot{x}^{\mu}+\frac{1}{2}g_{\nu\alpha,\mu}\dot{x}^{\nu}\dot{x}^{\mu}}_{g_{\mu\alpha,\nu}\dot{x}^{\nu}\dot{x}^{\mu}}-\frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^{\mu}\dot{x}^{\nu}=fg_{\mu\alpha}\dot{x}^{\mu}\,,\quad f\equiv-\sqrt{}\frac{d}{d\lambda}\left(\frac{1}{\sqrt{}}\right)\,. \tag{3.7}$$

The last 3 terms on the l.h.s. combine to the Christoffel symbol $\Gamma_{\alpha\mu\nu}$. Thus, upon multiplying both sides by $g^{\gamma\mu}$ we recover

$$\ddot{x}^{\gamma}+\Gamma^{\gamma}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=f\dot{x}^{\gamma}\,, \tag{3.8}$$

which is the geodesic equation in non-affine parametrization. Choosing further an affine parameter so that $f=0$, this is for example for λ identified with the proper time τ, $\lambda=\tau$,[1] we thus recovered the 'standard' geodesic equation

$$\boxed{\frac{D\dot{x}^{\gamma}}{d\tau}=\ddot{x}^{\gamma}+\Gamma^{\gamma}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=0\,.} \tag{3.9}$$

Of course, we can also write this in a fancy way

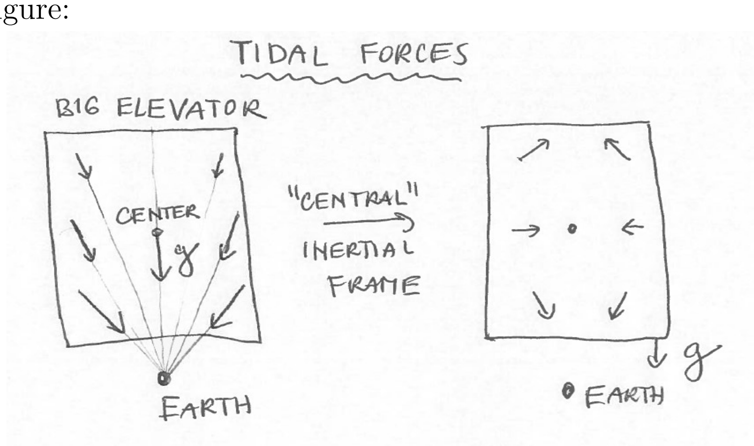
$$\boxed{u^{\alpha}\nabla_{\alpha}u^{\gamma}=0\,,\quad u^{\alpha}=\frac{dx^{\alpha}}{d\tau}\,,} \tag{3.10}$$

simply replacing in (**??**) the partial derivative ∂ with a covariant derivative ∇.

Particles in curved spacetime thus move on geodesics: curved space generalizations of 'straight lines' that maximize the proper time.

- Geodesic deviation equation.
  Q: Ho do you quantify the actual strength of gravity?
  A: By tidal forces.

  We can always remove the 'average' pull by going to the inertial frame. What remains is the 'true strength of gravity', measured by tidal forces, see the following figure:

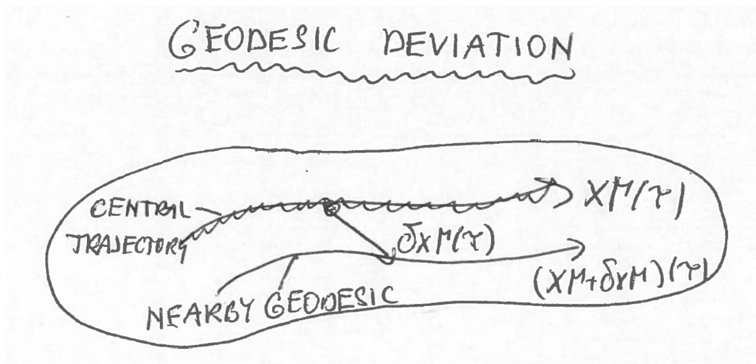


[1] In fact the affine parameter is uniquely defined up to a linear transformation $\lambda\to a\lambda+b$, with constants a and b.

In consequence the gravity makes 'muscle guys' squeezed and muscleless...

So let us calculate the tidal forces. To illustrate the idea, let us consider an extended body, a star for example in an external gravitational field, for example that of a supermassive black hole. In the first approximation, the star can be modeled by a 'swarm of particles' that move on geodesics in the black hole gravitational field. We want to find the deformation of the star due to gravitational tidal forces.

To this purpose we consider a central geodesic (for example the mass center of the star) and its nearby neighbour geodesics, as displayed in the figure.



We have the following two equations:

$$
\begin{aligned}
0 &= \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\,, \\
0 &= \frac{d^2(x^\mu+\delta x^\mu)}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}(x+\delta x)\frac{d(x^\alpha+\delta x^\alpha)}{d\tau}\frac{d(x^\beta+\delta x^\beta)}{d\tau}\,.
\end{aligned} \tag{3.11}
$$

By subtracting the two and keeping only terms linear in δx, we arrive at (non-covariant) equation

$$
\frac{d^2\delta x^\mu}{d\tau^2} + \delta x^\rho\partial_\rho\Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + 2\Gamma^\mu{}_{\alpha\beta}\dot{x}^\beta\frac{d}{d\tau}\delta x^\alpha = 0\,. \tag{3.12}
$$

To make the above equation covariant, let us study the acceleration between the geodesics:

$$
\frac{D^2\delta x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} + \Gamma^\mu u^\alpha\dot{x}^\beta\,, \quad u^\mu = \frac{D\delta x^\mu}{d\tau}\,. \tag{3.13}
$$

By expanding this, and using the geodesic equation again, we symbolically arrive at the following terms:

$$
\frac{D^2\delta x}{d\tau^2} = \frac{d^2\delta x}{d\tau^2} + 2\Gamma\dot{x}\frac{d\delta x}{d\tau} + (\partial\Gamma)\dot{x}\dot{x}\delta x + (\Gamma\Gamma-\Gamma\Gamma)\dot{x}\dot{x}\delta x\,. \tag{3.14}
$$

The first two terms can be eliminated using (3.12), to yield $-(\partial\Gamma)\dot{x}\dot{x}\delta x$. It is then not suprising that we recover the following beautiful formula:

$$
\boxed{\frac{D^2\delta x^\mu}{d\tau^2} = R^\mu{}_{\alpha\beta\nu}\delta x^\nu\dot{x}^\alpha\dot{x}^\beta\,,} \tag{3.15}
$$

called the geodesic deviation equation. Few remarks are in order:

- The above formula clearly illustrates that the Riemann tensor describes the tidal forces. Surprisingly, as we shall see later on, these are typically quite small, even on the horizon of a large black hole—we can just be passing through a horizon of a big black hole without even noticing it (behold the small one though!).
- One can understand the geodesic deviation equation as a linearization of geodesic equation around a given central geodesic. This in principle allows to find nearby geodesics around a simple central one (for example an orbit in the equatorial plane) by solving the linear geodesic equation even in the spacetime where general geodesics cannot be integrated.
- As we shall see later, the geodesic deviation equation is a key tool for describing and measuring the gravitational waves.

3.2 Matter in a curved spacetime

- Coupling to gravity. We have seen that the action for a particle in curved spacetime can be obtained by simply replacing the Minkowski metric by the general metric. This is an example of the minimal coupling principle. The recipe is to write the new laws in the tensorial form so that they reduce to the special relativistic laws in a local inertial frame. In practice this amounts to replacing the Minkowski metric with a general metric, and partial derivatives with covariant derivatives:
$$\eta_{\mu\nu} \to g_{\mu\nu}\,, \quad \partial_\mu \to \nabla_\mu\,. \tag{3.16}$$
Of course, this procedure is vague and not unique (as one can always add for example the curvature terms) and the final say about whether the theory is right or wrong is decided (as always in physics) by experiment.

- Action principle. To write the action for the matter fields we have to use the invariant volume element (2.45) discussed in Chapter 2, together with the principle of minimal coupling, to write
$$\boxed{S_m = \int d^dx\sqrt{-g}\mathcal{L}_m(\phi, \nabla\phi, g)\,,} \tag{3.17}$$
where $\mathcal{L}_m$ is the scalar Lagrangian density and ϕ stands for various fields. The variation gives
$$\delta S_m = \int d^dx \frac{\delta S_m}{\delta\phi}\delta\phi = \int d^dx\sqrt{-g}\Big(\frac{\partial\mathcal{L}_m}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}_m}{\partial\nabla_\mu\phi}\delta\nabla_\mu\phi\Big)\,. \tag{3.18}$$

As always, we now interchange the covariant derivative with δ, and, using the Stokes theorem, $\int_\Sigma \nabla_\mu V^\mu d^d x = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{\gamma} d^{d-1}x$, integrate by parts as follows:

$$\int A^\mu (\nabla_\mu B)\sqrt{-g}d^d x = -\int B(\nabla_\mu A^\mu)\sqrt{-g}d^d x + \text{boundary term}\,, \tag{3.19}$$

to have

$$\delta S_m = \int d^d x\sqrt{-g}\Big[\frac{\partial \mathcal{L}_m}{\partial\phi} - \nabla_\mu\Big(\frac{\partial \mathcal{L}_m}{\partial(\nabla_\mu\phi)}\Big)\Big]\delta\phi\,. \tag{3.20}$$

Thus we have derived the generalized Euler–Lagrange field equations:

$$\boxed{\frac{\delta S_m}{\delta\phi} = 0} \quad \Leftrightarrow \quad \boxed{\frac{\partial \mathcal{L}_m}{\partial\phi} - \nabla_\mu\Big(\frac{\partial \mathcal{L}_m}{\partial(\nabla_\mu\phi)}\Big) = 0\,.} \tag{3.21}$$

- Energy–momentum tensor. We can also define the following object:

$$\boxed{\delta_g S_m = -\frac{1}{2}\int d^d x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}\,,} \tag{3.22}$$

or more explicitly,

$$\boxed{T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial g^{\mu\nu}}\,,} \tag{3.23}$$

where the latter expression is true for matter Lagrangians independent of derivatives of the metric. We call the object $T_{\mu\nu}$ the (Rosenfeld's) energy momentum tensor. It is symmetric by construction.

For diffeomorphism invariant Lagrangian densities $\mathcal{L}_m$, such an energy momentum is conserved in the following sense: *If the equations of motion for matter are satisfied, the we have*

$$\boxed{\nabla_\mu T^{\mu\nu} = 0\,.} \tag{3.24}$$

(In fact, as we shall see, the conservation of energy–momentum is in many cases equivalent to the equations of motion for the matter.)

The argument for this goes as follows. Consider a diffeomorphism generated by a vector field ξ (treated as small),

$$x^\mu \to x^\mu - \xi^\mu\,. \tag{3.25}$$

This induces a variation of matter fields $\delta\phi$ as well as of the metric (see tutorial),

$$\delta g^{\mu\nu} = \mathcal{L}_\xi g^{\mu\nu} = 2\nabla^{(\mu}\xi^{\nu)}\,. \tag{3.26}$$

However, since the action is invariant by assumption, we must have

$$0 = \delta S_m = \int \frac{\delta S_m}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + \int \frac{\delta S_m}{\delta \phi} \delta \phi \,. \tag{3.27}$$

Provided the equation of motion for the matter are satisfied, $\frac{\delta S_m}{\delta \phi} = 0$, we thus have

$$\begin{aligned} 0 &= \delta S_m = \int \frac{\delta S_m}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = -\frac{1}{2} \int \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} = -\int \sqrt{-g} T_{\mu\nu} \nabla^{(\mu} \xi^{\nu)} \\ &= -\int \sqrt{-g} T_{\mu\nu} \nabla^{\mu} \xi^{\nu} = \int \sqrt{-g} \nabla^{\mu} T_{\mu\nu} \xi^{\nu} \,, \end{aligned} \tag{3.28}$$

which implies the desired result, as ξ^ν is arbitrary.

To calculate $T_{\mu\nu}$ explicitly, we can use the following two tricks (see tutorial for more details):

$$\boxed{\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \,, \quad \delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma} \,.} \tag{3.29}$$

- Example 1: Scalar field. Using the minimal coupling principle, the Lagrangian reads

$$\boxed{\mathcal{L} = -\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi) \,.} \tag{3.30}$$

The corresponding field equation is

$$\nabla^\mu \nabla_\mu \phi - \frac{dV}{d\phi} = 0 \,. \tag{3.31}$$

The energy–momentum is then

$$T_{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{2\mathcal{L}}{\sqrt{-g}} \frac{\partial(\sqrt{-g})}{\partial g^{\mu\nu}} = \nabla_\mu \phi \nabla_\nu \phi + \mathcal{L} g_{\mu\nu} \,. \tag{3.32}$$

In the flat space limit, this is the canonical energy momentum tensor for the scalar field derived from the Noether's procedure due spacetime translation invariance.

- Example 2: Electromagnetic field. The Lagrangian reads

$$\boxed{\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - e J^\nu A_\nu = -\frac{1}{16\pi} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} - e J^\nu A_\nu \,, \quad F_{\mu\nu} = 2 \nabla_{[\mu} A_{\nu]} \,.} \tag{3.33}$$

It gives rise to the Maxwell equations in curved spacetime:

$$\nabla_\mu F^{\mu\nu} = -4\pi J^\nu \,, \quad \nabla_\mu * F^{\mu\nu} = 0 \,. \tag{3.34}$$

Note that we can also write these as (see tutorial)

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) = -4\pi J^\nu\,, \quad \partial_\mu(\sqrt{-g} * F^{\mu\nu}) = 0\,. \tag{3.35}$$

In the absence of sources we get the (automatically symmetric and gauge invariant) energy–momentum tensor:

$$\boxed{T_{\mu\nu} = \frac{1}{4\pi}\Big(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2\Big)\,.} \tag{3.36}$$

It is this electromagnetic energy momentum tensor that couples to gravity (not the canonical one, which is not symmetric and not gauge invariant).

- Example 3: Perfect fluid. Perfect fluid is described by the following energy–momentum tensor:

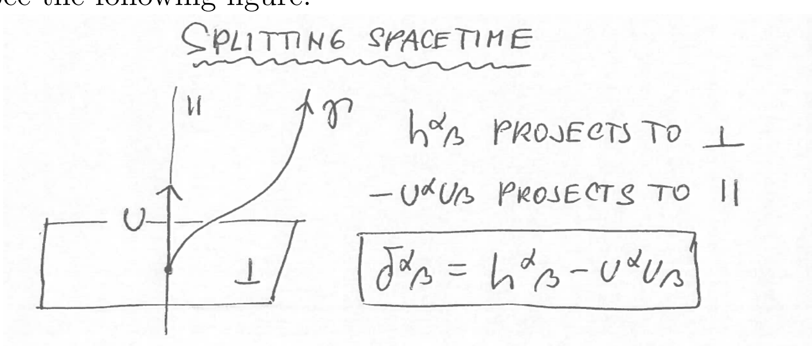
$$\boxed{T^{\mu\nu} = (\rho + P)U^\mu U^\nu + Pg^{\mu\nu}\,.} \tag{3.37}$$

Here, U^α is a unit timelike vector representing the 4-velocity of the fluid, and ρ and P are the energy density and pressure of the fluid as measured by the comoving observer (in the fluid rest frame).

Let us now look at what the conservation of this energy–momentum tensor gives us. To this purpose, we consider a projector to the perpendicular space to U:

$$\boxed{h^\alpha{}_\beta = \delta^\alpha{}_\beta + U^\alpha U_\beta\,.} \tag{3.38}$$

See the following figure:



It is easy to verify that

$$h^\alpha{}_\beta U^\beta = 0\,, \quad h^\alpha{}_\beta h^\beta{}_\gamma = h^\alpha{}_\gamma\,, \tag{3.39}$$

and thence h is indeed a projector. Similarly, one would find that $U^\alpha U_\beta$ is a projector on geodesic.

So what do we get from the conservation of $T^{\mu\nu}$? We have

$$\nabla_\mu T^{\mu\nu} = (\rho + P)_{,\mu}U^\mu U^\nu + (\rho + P)(\nabla_\mu U^\mu)U^\nu + (\rho + P)U^\mu\nabla_\mu U^\nu + P_{,\mu}g^{\mu\nu} + 0\,. \tag{3.40}$$

Projection by $h^\alpha{}_\nu$ gives

$$0 = h^\alpha{}_\nu \nabla_\mu T^{\mu\nu} = 0 + 0 + (\rho + P) \underbrace{h^\alpha{}_\nu}_{\delta^\alpha_\nu + U^\alpha U_\nu} \nabla_U U^\nu + h^{\alpha\mu} P_{,\mu} . \tag{3.41}$$

Since $U_\nu \nabla_U U^\nu = 0$, we thus found

$$\boxed{(\rho + P)\nabla_U U^\alpha = -(\nabla^\alpha P)_\perp ,} \tag{3.42}$$

which is the relativistic (ideal fluid) Navier–Stokes equation.

Similarly, projecting on geodesic we find

$$0 = U_\nu \nabla_\mu T^{\mu\nu} = -(\rho + P)_{,\mu} U^\mu - (\rho + P)\nabla_\mu U^\mu + P_{,\mu} U^\mu , \tag{3.43}$$

or

$$\boxed{\frac{d\rho}{d\tau} + (\rho + P)\nabla_\mu U^\mu = 0 ,} \tag{3.44}$$

which is the continuity equation. The standard units would be recovered upon setting $P \to P/c^2$ and thence the pressure would be suppressed.

In other words, this is a clear demonstration that the conservation of energy momentum tensor implies not only the continuity (conservation) equation (3.44) but also the equation of motion for the matter (3.42).[2]

3.3 Einstein–Hilbert action

- Action. Let us now think about how to construct the variational principle for the gravitational field itself (see Appendix **??** for how this is done for electromagnetism). To get the 2nd-order equations of motion for $g_{\mu\nu}$, we want a scalar invariant which depends on the metric and its first derivatives, $I = I(g, \partial g)$. Unfortunately there is no such thing—why? So we have to give up and take an invariant that depends also on the second derivatives of the metric. The simplest one is the Ricci scalar. This leads to the following Einstein–Hilbert action:

$$\boxed{S_{\rm EH}[g] = \frac{1}{16\pi G} \int \sqrt{-g} R(g, \partial g, \partial^2 g) .} \tag{3.45}$$

- Variation. Varying this action, and using the dirty trick that $R = R_{\mu\nu} g^{\mu\nu}$, we get

$$\delta S_{\rm EH} = \frac{1}{16\pi G} \int \delta(\sqrt{-g} R_{\alpha\beta} g^{\alpha\beta}) = \frac{1}{16\pi G} \int (R\delta\sqrt{-g} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}) . \tag{3.46}$$

[2]Similarly, one could for example show that the conservation of the electromagnetic stress tensor implies (provided sufficient algebraic generality of $F_{\mu\nu}$) the Maxwell equations.

The first two terms are easy, they give

$$R\delta\sqrt{-g}+\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}=\sqrt{-g}\Big(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\Big)\delta g^{\mu\nu}=\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}\,. \tag{3.47}$$

upon using the first identity (3.29). On the other hand, the last term, $g^{\mu\nu}\delta R_{\mu\nu}$, seems horrible. Fortunately it can be show that it only gives a boundary term [7]:

$$g^{\mu\nu}\delta R_{\mu\nu}=\nabla_\mu V^\mu\,,\quad V_\mu=\nabla^\beta(\delta g_{\mu\beta})-g^{\alpha\beta}\nabla_\mu(\delta g_{\alpha\beta})\,, \tag{3.48}$$

which does not contribute to the equations of motion. Thus we have the following Einstein equations in the absence of matter:

$$\boxed{G_{\mu\nu}=0\,.} \tag{3.49}$$

Note that, by contracting the equation with $g^{\mu\nu}$, we find that in d-dimensions we have $(1-\frac{1}{2}d)R=0$ and thence for $d\neq 2$ we must also have $R=0$ and the vacuum equations can be written as:

$$\boxed{R_{\mu\nu}=0\,.} \tag{3.50}$$

- Let us make 3 remarks.
  - Remark 1. The obtained equation is second order PDE for the metric, $G_{\mu\nu}=G_{\mu\nu}(g,\partial g,\partial^2 g)$. How is this possible? We started from the action, given by R, which already is of second order in derivatives. We should (remember the discussion in the Classical Physics course) thus have received 4th order equations of motion. However, this is not the case and the reason is simple. It can be shown that the Lagrangian density can be split to a piece that depends only on the first derivatives, plus a piece that is a total derivative (none of which is a tensor):

$$\sqrt{-g}R(g,\partial g,\partial^2 g)=\sqrt{-g}\tilde{R}(g,\partial g)+\partial_\mu\hat{R}^\mu(g,\partial g)\,, \tag{3.51}$$

    where $\delta\hat{R}^\mu=\sqrt{-g}V^\mu$. The latter term does not contribute to the equations of motion, provided we impose the corresponding boundary conditions. Such boundary conditions leave the action ill-posed (one needs to describe both, the variation of the metric as well as of its derivatives on the boundary—prescribing too many conditions for the solution of 2nd-order equations to exist.) The Einstein–Hilbert action thus gives the 'correct equations of motion' but is ill posed as a variational principle. To make the principle well-posed another boundary term, called the Gibbons–Hawking term, canceling the variations of the derivative of the metric on the boundary, has to be added (see Gravitational Physics Course).

– Remark 2. In the above variation we used the so called second-order formalism: the action was treated as a function of the metric $g_{\mu\nu}$ and contained its first and second derivatives.

Perhaps more useful is the first-order (Palatini) formalism where the action is treated as action for two fields: the metric $g_{\mu\nu}$ and the connection $\Gamma^\alpha{}_{\beta\gamma}$:

$$S_{\text{Palatini}}[g,\Gamma] = \frac{1}{16\pi G}\int \sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma)\,, \tag{3.52}$$

where we used the fact that $R_{\mu\nu}$ can be entirely written in terms of the connection and its derivatives, $R_{\mu\nu} = R_{\mu\nu}(\Gamma)$. Thus the variation w.r.t. the metric yields immediately the Einstein equations (3.49). It can then be shown that the variation w.r.t. the connection establishes that the connection is given by Christoffel symbols (this was an input in the second order formalism.)

Yet there is another 1.5 formalism (known mainly to supergravity people). One simply treats the action as action for both g and Γ but makes only variations w.r.t. g. This yields immediately Einstein equations. The variation w.r.t. the connection is not necessary as 'everybody knows' that we have Christoffel connection :).

– Remark 3: Other curvature invariants. One might think that the Ricci scalar we chose for our action is simply one and the easiest possible choice for the action, and similar to electromagnetism (see appendix **??**) we may consider other scalars, as

$$R^2\,,\quad R_{\mu\nu}R^{\mu\nu}\,,\quad R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}\,,\quad R^3\,,\quad \nabla_\mu R\nabla^\mu R\,,\dots\,, \tag{3.53}$$

and identify any function of these with the gravitational Lagrangian. However, one can show that this is not the case, as we have the following [10]:

Lovelock theorem. *In four dimensions, the Einsetin–Hilbert action is the only local action (apart from the cosmological constant and topological terms) that leads to the second order differential equations for the metric.* In other words, Einstein theory is the unique theory we can obtain from the action principle that does not have higher derivative terms.

Let us finally remark, that is is no longer true in higher dimensions, where certain combinations of the above scalars, called Euler densities, lead to the second order equations of motion, called Lovelock equations, generalizing so the form of the Einstein equations in higher dimensions.

3.4 Einstein equations

Adding matter is easy. We simply add the corresponding matter Lagrangian density. In what follows, we assume the minimal coupling to gravity, encoded in the fact that

$\mathcal{L}_m$ can depend on the metric but not on its derivatives. Then we have

$$S = S_{\rm EH}[g] + \alpha_m \int d^4x\sqrt{-g}\mathcal{L}_m\,. \tag{3.54}$$

The variation w.r.t. the metric and throwing away the boundary terms then yields

$$\delta S = \int d^4x\sqrt{-g}\Big(\frac{1}{16\pi G}G_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}\alpha_m T_{\mu\nu}\delta g^{\mu\nu}\Big) \tag{3.55}$$

Thus we recover the following Euler–Lagrange equations:

$$G^{\mu\nu} = 8\pi\alpha_m G T^{\mu\nu}\,. \tag{3.56}$$

Of course, the relative coupling between $1/(16\pi G)$ and α_m determines the 'strength of gravity'. One can fix this by demanding that the theory reduces in the weak field static limit to the Newton's theory. Namely, we have to recover the Poisson equation (3.1). As we shall see in the following section this is achieved by setting $\alpha_m = 1$. Thus we have derived the famous 1915 Einstein's field equations:[3]

$$\boxed{G^{\mu\nu} = 8\pi G T^{\mu\nu}\,.} \tag{3.57}$$

Let us make a couple of remarks.

- Consistency condition. By applying the covariant derivative ∇_μ on both sides of the Einstein equation, and using the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ valid for any metric, we arrive at the consistency condition for the Einstein equations:

  $$\boxed{\nabla_\mu T^{\mu\nu} = 0\,.} \tag{3.58}$$

  The energy momentum tensor has to be 'conserved' in order the Einstein equations make sense. However, as we have seen above, this is not really a conservation equation—one cannot in general construct any conserved charge. What this rather is is a statement about the motion of the matter. We have seen for example for the fluid that the above equation implies Navier–Stokes equations. Similarly, we would see that the conservation of a point particle energy momentum tensor would yield the geodesic equation. The above conservation is thus (more or less) equivalent to the equations of motion for the matter.

  In general relativity (contrary to the Maxwell one) the Einstein field equations thus play a double agent role: *they tell spacetime how to curve in the presence of matter* (field equations role) while at the same time *they tell matter how to move in a given curved space* (equations of motion for the matter role).

[3] Poor Einstein was scooped by Hilbert by a couple of months, after giving a seminar to him about what he is trying to do. Well, perhaps an appropriate punishment for leaving Prague before finishing his theory of gravitation :).

- How many equations? The Einstein equations are 2nd-order PDEs for the metric components $g_{\mu\nu}$ (apparently 10 components). As we are free to perform 4 coordinate transformations, only $6 = 10 - 4$ components of the metric should matter. This might suggest that only 6 out of 10 Einstein equations are 'true evolution equations' (depend on 2 time derivatives of the metric) and the remaining 4 are actually constraints. This is indeed so, as can easily be seen from the Bianchi identities.

  Let us choose coordinates where the zeroth value of the index corresponds to the time, we may write

  $$0 = \nabla_\mu G^{\mu\nu} = \partial_t G^{t\nu} + \underbrace{\Gamma^t{}_{t\mu} G^{\mu\nu} + \Gamma^\nu{}_{t\mu} G^{t\mu} + \nabla_i G^{i\nu}}_{\text{rest}} \,. \tag{3.59}$$

  Obviously, the 'rest' contains only spatial or no derivatives of $G^{\mu\nu}$ and thence contains at most 2nd-order time derivatives of the metric. However, this is an identity valid for any metric. Thus, the components $G^{t\nu}$ can at most depend on first derivatives of the metric (otherwise we cannot cancel the third time derivatives in the expression). This then means that

  $$G^{t\mu} = 8\pi G T^{t\mu} \,, \tag{3.60}$$

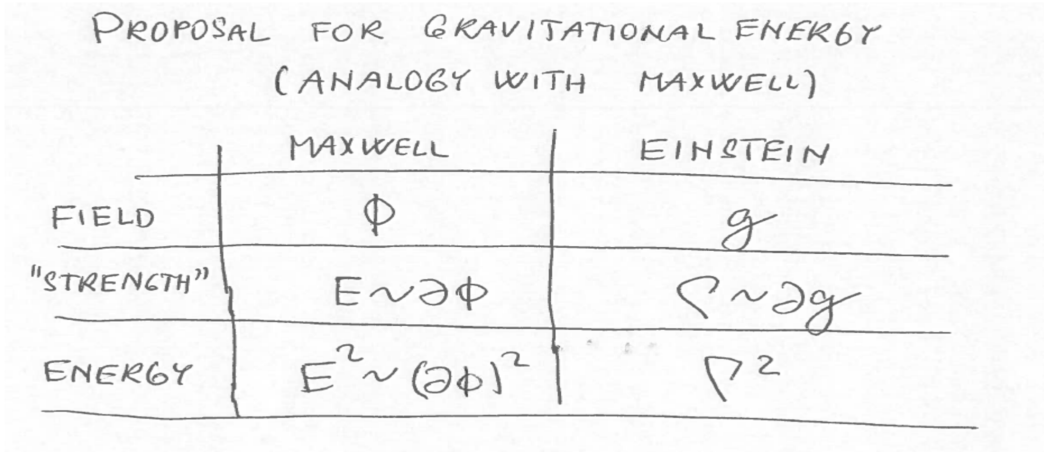
  are not true evolution equations (they do not contain second-order time derivatives of the metric) and rather are 4 constraints.

- Conservation laws. As suggested above, the conservation of the energy momentum,

  $$\nabla_\mu T^{\mu\nu} = 0 \,, \tag{3.61}$$

  does not really present any conservation law. Mathematically, this is because there is no Stokes theorem for symmetric fields (and $T^{\mu\nu}$ is symmetric) that would turn the above equation to an integral over the boundary as we have done with the electric current. Physically this makes sense, the spacetime contains energy which is composed of the energy of matter plus the energy of gravitational field. However, the above law only concerns the matter and the corresponding energy cannot be conserved.

  Could we add the energy of the gravitational field? From the following picture it is obvious that we immediately run into a problem:



PROPOSAL FOR GRAVITATIONAL ENERGY
(ANALOGY WITH MAXWELL)

	MAXWELL	EINSTEIN
FIELD	ϕ	g
"STRENGTH"	$E \sim \partial\phi$	$\Gamma \sim \partial g$
ENERGY	$E^2 \sim (\partial\phi)^2$	Γ^2

Γ^2 is not a tensor and thence the gravitational energy 'depends' on the frame of reference—it is not localized. (This is a first hint that gravity is holographic.)

Let us not probe this problem any deeper at the moment and rather ask a question: Do we really expect something to be conserved in general spacetime? The answer according to the Noether's theorem is NO! (We do not in general have any continuous symmetries.) Only in the presence of continuous symmetries (described by Killing vectors) we should expect to have some conserved quantities. And this is indeed so. Let ξ^μ be a Killing vector,

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\,. \tag{3.62}$$

Then we can construct, starting from the matter field energy momentum tensor, the following current:

$$\boxed{j^\mu = T^{\mu\nu}\xi_\nu\,.} \tag{3.63}$$

We have

$$\nabla_\mu j^\mu = \nabla_\mu (T^{\mu\nu}\xi_\nu) = (\nabla_\mu T^{\mu\nu})\xi_\nu + T^{\mu\nu}\nabla_{(\mu}\xi_{\nu)} = 0\,. \tag{3.64}$$

The above current is thus conserved,

$$\boxed{\nabla_\mu j^\mu = 0\,,} \tag{3.65}$$

and being a vector (we have the corresponding Stokes theorem) defines a conserved quantity.[4]

- A remark on energy conditions. How to find solutions of Einstein equations? The Pooh Bear is a bear of very little brains. So he might think to do it Pooh's way: Take any metric and calculate the corresponding $G_{\mu\nu}$, then impose the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{3.66}$$

[4]Of course in flat space we have all Killing vectors of Minkowski space. In particular, the spacetime translations guarantee the existence of 4 currents $j^\mu{}_{(\nu)} = T^\mu{}_\nu\,,\ \nu = 0,1,2,3,$ which give the conservation of energy and momentum.

to find the corresponding source $T_{\mu\nu}$ for this metric. In this way Pooh can obtain solutions such as wormholes, warp drives, ...

The problem with this is that so obtained $T_{\mu\nu}$ is typically "crazy" and the solutions are then considered not physical. To limit this, we impose "reasonable assumptions" on $T_{\mu\nu}$, called the energy conditions. For example, one such condition, called the weak energy condition requires that $T_{\mu\nu}t^\mu t^\nu \geq 0$ for any timelike t^μ. Diagonalizing the energy momentum tensor in the orthonormal basis, this translates to $\rho \geq 0$ and $\rho + P_i > 0$, where ρ is the energy density in this frame and P_i are the principal pressures. Other conditions one may impose are: null, strong, dominant, and so on. The aim is to guarantee that the matter has some reasonable properties, such as the energy density is positive, and so are the pressures, the energy flux is timelike, and so on.

Where do they come from? These conditions cannot be a property of matter itself. If they were, they would be described by QFT and thence one could find them in the Weinberg's books.[5] They are not there, which proves the point. So are they a property of how matter can interact with gravity? Can they be derived from some fundamental principles? These are all unknown at the moment.

Are they always satisfied? At least we know the answer to this one: NO! For example, the inflation necessarily violates the strong energy condition, the Hawking radiation violates the area increase theorem and thence the null energy condition, and so on (typically quantum matter violates these, at least to some level—we thence have the quantum version of energy conditions which could be satisfied, perhaps, maybe...)

One thing is sure, (whether or not they are derivable from some fundamental principle) if we abandon the energy conditions completely, anything is possible and Pooh's way can be used to find whatever the crazy solutions we may imagine.

3.5 Linearized gravity

Let us now make a contact with the Newtonian theory in the non-relativistic limit, as well as study the first-order post-Newtonian expansion. We will look at the limits of both, the field equations and the geodesic equation:

$$G^{\mu\nu} = 8\pi T^{\mu\nu}\,, \quad \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,. \tag{3.67}$$

Post-Newtonian Expansions play nowadays an important role for understanding the gravitational waves produced in various violent events in our Universe.

[5]And if anyone knows anything about anything, it's Weinberg who knows something about something, said Bear to himself—or my name is not Winnie-the-Pooh—he said—which it is, he added—so there you are.

Linearized Gravity

- Let us first look at the linearized Einstein equations. That is, we linearize the gravitational field around the Minkowski metric:

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)\,,} \tag{3.68}$$

where everywhere we shall neglect terms of the order $O(h^2)$ and higher. Then we have

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu}\,, \\ \Gamma^{\mu}{}_{\alpha\beta} &= \frac{1}{2}\eta^{\mu\sigma}(h_{\sigma\beta,\alpha} + h_{\alpha\sigma,\beta} - h_{\alpha\beta,\sigma})\,. \end{aligned} \tag{3.69}$$

For example, the first one just follows from $(\eta - h)(\eta + h) = \delta + h - h + O(h^2)$. In this way we find (only the $\partial\Gamma$ terms contribute to Riemann)

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma})\,, \tag{3.70}$$

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu})\,, \tag{3.71}$$

$$G_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h)\,. \tag{3.72}$$

Here $\Box$ is the flat space wave operator, $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, and $h = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation. Introducing further

$$\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\,, \quad \Leftrightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}\,,} \tag{3.73}$$

with $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu}$, and exploiting the gauge freedom in a diffeomorphism,

$$x^\mu \to x^\mu - \xi^\mu\,, \tag{3.74}$$

where ξ^μ is an infinitesimal vector field (of the order of the metric perturbation), yields the following change of components of the linearized metric (see tutorial):

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,, \tag{3.75}$$

we can impose the following <u>Lorenz (De Donder, or harmonic) gauge condition</u>:

$$\boxed{\partial_\mu\bar{h}^{\mu\nu} = 0\,.} \tag{3.76}$$

Then we find that $G^{\mu\nu} = -\frac{1}{2}\Box\bar{h}^{\mu\nu}$ and hence the *linearized Einstein equations* read

$$\boxed{\Box\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}\,.} \tag{3.77}$$

- Note that, in the absence of sources, Eqs. (3.76) and (3.77) are Pauli–Fierz (1939) equations for a massless spin-2 particle in flat spacetime. Thus, the general relativity can be viewed as a theory of massless spin-2 particle which undergoes a nonlinear self-interaction. Note, however, that this only makes sense in the perturbative sense around the flat metric and cannot be given a precise meaning in a general curved space. We shall return back to this observation if time permits.

- Let us finally remind that the general structure of $T^{\mu\nu}$ is as follows:

$$T_{\mu\nu} = \left(\begin{array}{c|c} \text{energy density} & \text{momentum density} \\ \hline & \\ & \text{Pressure} \\ & \end{array} \right) . \tag{3.78}$$

Newtonian limit

$$\boxed{\textit{Newtonian limit} = \text{weak \& static gravitational field, slow motion, small pressures.}}$$

- In this limit, the energy density dominates all the components of $T_{\mu\nu}$ and hence we have

$$T_{\mu\nu} = \begin{pmatrix} \rho & \cdot \\ \cdot & \cdot \end{pmatrix} , \tag{3.79}$$

- Writing the linearized equations (3.77), and since $h_{\alpha\beta,0} = 0$, we recover the *Poisson equation*

$$\boxed{\nabla^2\phi = 4\pi\rho\,, \quad \phi = -\frac{1}{4}\bar{h}_{00}\,,} \tag{3.80}$$

from the $(0,0)$ component, while the other equations give

$$\nabla^2\bar{h}^{\mu\nu} = 0 \quad (\mu,\nu) \neq (0,0)\,, \tag{3.81}$$

and yield a trivial solution (provided we require nice behavior at infinity). Using (3.68) and (3.73) we thus find

$$\boxed{g_{00} = -1 - 2\phi\,, \quad g_{0i} = 0\,, \quad g_{ij} = (1-2\phi)\delta_{ij}\,.} \tag{3.82}$$

In particular, for a spherically symmetric spacetime (for example a star) we have

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dr^2 + r^2 d\Omega^2)\,, \quad \phi = -\frac{M}{r}\,, \tag{3.83}$$

c.f. the Schwarzschild solution discussed in the next chapter. Our approximation is valid as long as

$$\frac{\phi}{c^2} \approx \frac{GM}{c^2 r} \ll 1\,. \tag{3.84}$$

For a surface of the following objects we find the following values of this ratio: proton: 10^{-39}, Earth: 10^{-9}, Sun: 10^{-6}, neutron star: $10^{-2} - 10^{-1}$, black hole: $10^{-1} - 1$ ($r = r_g = \frac{2GM}{c^2}$).

- Geodesic equation. The Newtonian limit of the geodesic equation has already been discuss in Chapter 1, where we have shown that we recover the *Newton's law*:

$$\boxed{\frac{d^2x^i}{dt^2} = \frac{1}{2}h_{00,}{}^{i} = -\nabla^i\phi\,.} \tag{3.85}$$

Beyond Newton: gravitomagnetism

Let us now study the lowest order effects due to the motion of particles and sources, still neglecting the pressures, approximating to linear order in velocities.

- Our energy momentum is now

$$T_{\mu\nu} = \begin{pmatrix} \rho & \vec{j} \\ \vec{j} & 0 \end{pmatrix}\,, \quad j_\mu = T_{0\mu} = (\rho, \vec{j})\,. \tag{3.86}$$

  Denoting further

$$A^\mu = -\frac{1}{4}\bar{h}^{0\mu} = (\phi, \vec{A})\,, \tag{3.87}$$

  the $(\mu, 0)$ component of the linearized equations (3.77), together with the gauge condition (3.76), yield

$$\boxed{\Box A^\mu = -4\pi j^\mu\,, \quad \partial_\mu A^\mu = 0\,.} \tag{3.88}$$

  These have the form of wave equations, equivalent to Maxwell's equations, and are in this context called the *gravitomagnetic equations*.

- If we further assume that $h_{\alpha\beta,0} = 0$, the spatial components $\bar{h}^{ij}$ obey the sourceless Laplace equation which, provided the conditions at infinity, yields $\bar{h}^{ij} = 0$. The solution is then

$$\boxed{g_{00} = -1 - 2\phi\,, \quad g_{0i} = -4A_i\,, \quad g_{ij} = (1 - 2\phi)\delta_{ij}\,.} \tag{3.89}$$

- Geodesic equation now reads (keeping the linear but not quadratic in velocity terms)

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{00}\left(\frac{dt}{d\tau}\right)^2 + 2\Gamma^\mu{}_{0i}\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0\,. \tag{3.90}$$

  Since $A_{\mu,0} = 0$, we can define

$$E_i = -A_{0,i}\,, \quad F_{ij} = \partial_i A_j - \partial_j A_i\,, \tag{3.91}$$

  in analogy with electromagnetism. Then we find

$$\Gamma^i{}_{00} = -E^i\,, \quad \Gamma^i{}_{0j} = 2F^i{}_j\,, \tag{3.92}$$

and the geodesic equation gives

$$\boxed{\frac{d^2x^i}{dt^2} = E^i - 4F^i{}_j v^j\,,} \tag{3.93}$$

which is almost the *Lorenz force law* for $q = m$.

- So we recovered the 'magnetic properties' of relativistic gravity. Of course, in an analogy with Maxwell's theory, we expect that relativistic theory of gravity also propagates its own dof in terms of the gravitational waves. These will be studied in the next chapter.

Chapter 4: Applications

4.1 Gravitational waves

In this section we shall seek the gravitational wave solutions. We first show that they propagate 2 degrees of freedom. We then concentrate on the power of these waves produced in an isolated system, a binary system for example, deriving so the famous quadrupole formula.

Harmonic gauge and 2 degrees of freedom

- Linearized gravity. In the previous section we have shown that the linearized gravity equations can be written as

$$\boxed{\Box\bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu}\,, \quad \partial_\mu \bar{h}^{\mu\nu} = 0\,,} \tag{4.1}$$

where the latter is called the harmonic, or de Donder, or transverse trace reverse (since $h = -\bar{h}$) gauge, and

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\,, \quad \Leftrightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}\,. \tag{4.2}$$

It is easy to show that the harmonic gauge can be written in the following 3 equivalent ways:

$$\partial_\mu \bar{h}^{\mu\nu} = 0 \quad \Leftrightarrow \quad \Gamma^\mu{}_{\alpha\beta}g^{\alpha\beta} = 0 \quad \Leftrightarrow \quad \nabla_\alpha \nabla^\alpha x^\mu = 0\,, \tag{4.3}$$

where the last is understood as a (covariant) wave operator acting on 4 scalars x^μ, thence the name harmonic gauge.

- Vacuum. To seek gravitational wave solutions, we switch off the matter sources, $T^{\mu\nu} = 0$. Thus we have

$$\Box\bar{h}^{\mu\nu} = 0 \quad \Rightarrow \quad \Box\bar{h} = 0 \quad \Rightarrow \quad \boxed{\Box h^{\mu\nu} = 0\,.} \tag{4.4}$$

The second equation is a contraction of the first and the last follows from (4.2). The plane wave ansatz (as always we have to in the end take the real part)

$$\boxed{h_{\mu\nu} = \epsilon_{\mu\nu}\exp(ix^\alpha k_\alpha)\,,} \tag{4.5}$$

where $\epsilon_{\mu\nu}$ is the so called polarization tensor, is a solution of the above Einstein equation provided

$$\boxed{k^2 = 0\,,} \tag{4.6}$$

that is the wave propagates at the speed of light.

- Fixing polarization tensor. In the ansatz (4.5), the polarization tensor has 10 components. Our aim now is to show that we can fix a gauge so that only 2 independent components survive, giving 2 true degrees of freedom of gravitational waves.

  The harmonic gauge implies that

  $$k^\mu \epsilon_{\mu\nu} - \frac{1}{2}\epsilon k_\nu = 0\,, \quad \epsilon = \epsilon^\mu{}_\mu\,. \tag{4.7}$$

  This yields 4 constraints on the polarization tensor $\epsilon_{\mu\nu}$. However, we still have not fixed the gauge completely. Namely, we can still perform additional coordinate transformations,

  $$x^\mu \to x^\mu - \xi^\mu\,, \quad h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\,, \tag{4.8}$$

  such that they preserve the harmonic gauge, namely $\nabla^2 x^\mu = 0$, yielding

  $$\nabla^2 \xi^\mu = \Box \xi^\mu \equiv \eta^{\alpha\beta}\partial_\alpha \partial_\beta \xi^\mu = 0\,, \tag{4.9}$$

  where the latter formula is valid since ξ^μ is already small. Expanding ξ in the 'same fourier mode' as the gravity wave we have

  $$\xi_\mu = \tilde{\xi}_\mu e^{ix^\alpha k_\alpha}\,, \tag{4.10}$$

  and the above residual gauge condition (4.9) is automatically satisfied for any $\tilde{\xi}^\mu$ as $k^2 = 0$ already. We can thus use the residual freedom

  $$\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + ik_\mu \tilde{\xi}_\nu + ik_\nu \tilde{\xi}_\mu\,, \tag{4.11}$$

  to set additional 4 components of the polarization tensor equal to zero.

  In particular, for a wave propagating in the x-direction, that is $k^\mu = (k, k, 0, 0)$, we can achieve (try if you do not believe) that the polarization tensor takes the following form:

  $$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_{22} & \epsilon_{23} \\ 0 & 0 & \epsilon_{23} & -\epsilon_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_+ & \epsilon_\times \\ 0 & 0 & \epsilon_\times & -\epsilon_+ \end{pmatrix}. \tag{4.12}$$

  Because of this form of the polarization tensor, the resultant gauge is called the transverse–traceless (TT) gauge. The 2 degrees of freedom of a gravitational wave are now manifest, encoded in ϵ_+ and $\epsilon_\times$ polarizations. The wave propagates at the speed of light in the x-direction, affecting only the y and z components of the metric, thence the wave is transverse.

  The above 'extraction' of the true gravitational degrees of freedom is more elegant using the scalar–vector–tensor (SVT) decomposition. You shall discuss this in the Cosmology Exploration Course, see also [1]. Another, very explicit approach is to use the lightcone gauge, see tutorial.

Gravitational wave in action

The effect of a gravitational wave passing by in the x-direction is best illustrated on how it affects a 'ring' of stationary particles in the $y-z$ plane. Of course, the motion of the nearby particles is described by the geodesic deviation equation,

$$\frac{D^2 \delta x^\mu}{d\tau^2} = R^\mu{}_{\alpha\beta\nu} \dot{x}^\alpha \dot{x}^\beta \delta x^\nu \,. \tag{4.13}$$

For the center of the ring we have

$$\dot{x}^\alpha = (1, 0, 0, 0) \,. \tag{4.14}$$

Moreover, using the linearized formula for the Riemann tensor, (3.70), we find

$$R_{\mu tt\nu} = \frac{1}{2}(\partial_t^2 h_{\mu\nu} + \partial_\nu \partial_\mu h_{tt} - \partial_\nu \partial_t h_{\mu t} - \partial_\mu \partial_t h_{\nu t}) = \frac{1}{2}\partial_t^2 h_{\mu\nu} \,, \tag{4.15}$$

upon using the above form of $h_{\mu\nu}$, (4.12). Thus, the geodesic deviation equation gives

$$\boxed{\frac{\partial^2 \delta x^\mu}{\partial t^2} = \frac{1}{2}\delta x^\nu \frac{\partial^2}{\partial t^2} h^\mu{}_\nu \,.} \tag{4.16}$$

Obviously, as the gravitational wave passes by in the x-direction, only δy and δz will be non-trivial, the wave is <u>perpendicular</u>. Let us consider the two polarizations separately. Considering first the <u>+ polarization</u>, that is $\epsilon_\times = 0$ and $\epsilon_+ \neq 0$, the above equation yields

$$\partial_t^2 \delta y = \frac{1}{2}\delta y \partial_t^2 (\epsilon_+ e^{ik\cdot x}) \,, \quad \partial_t^2 \delta z = -\frac{1}{2}\delta z \partial_t^2 (\epsilon_+ e^{ik\cdot x}) \,, \tag{4.17}$$

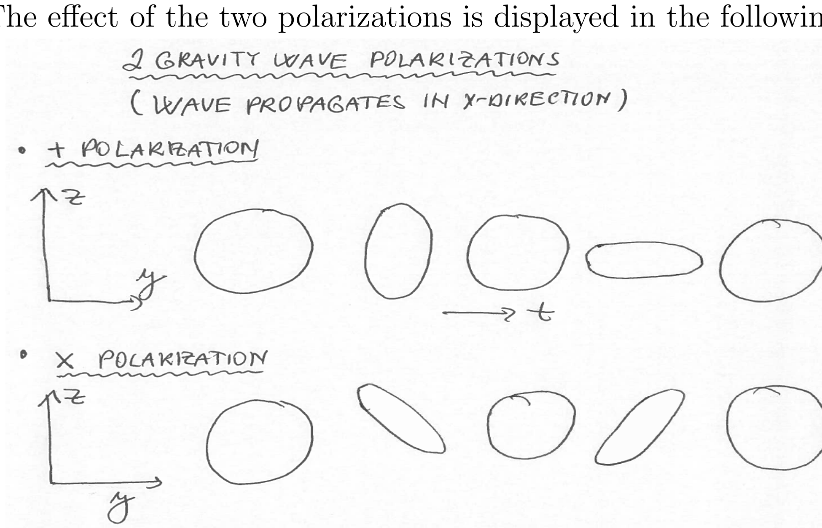
which has the following perturbative solution:

$$\delta y = (1 + \frac{1}{2}\epsilon_+ e^{ik\cdot x})\delta y(0) \,, \quad \delta z = (1 - \frac{1}{2}\epsilon_+ e^{ik\cdot x})\delta z(0) \,. \tag{4.18}$$

Similarly, for the <u>$\times$ polarization</u>, we would recover

$$\delta y = \delta y(0) + \frac{1}{2}\epsilon_\times e^{ik\cdot x}\delta z(0) \,, \quad \delta z = \delta z(0) + \frac{1}{2}\epsilon_\times e^{ik\cdot x}\delta y(0) \,. \tag{4.19}$$

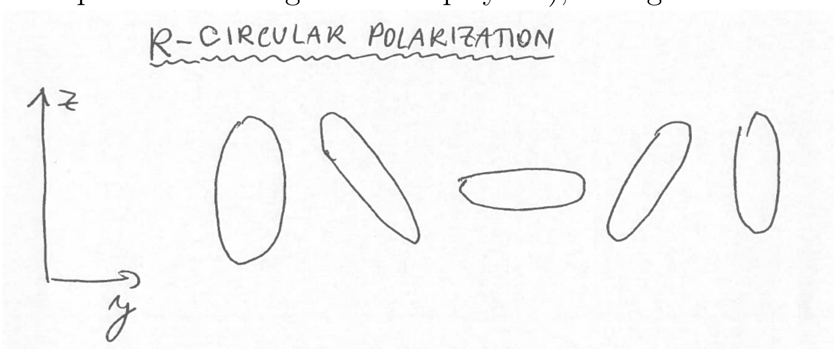
The effect of the two polarizations is displayed in the following figure:



Alternatively, one might consider circularly polarized modes,

$$\epsilon_R = \frac{1}{\sqrt{2}}(\epsilon_+ + i\epsilon_\times)\,, \quad \epsilon_L = \frac{1}{\sqrt{2}}(\epsilon_+ - i\epsilon_\times)\,, \tag{4.20}$$

which rotate the particles shape in a right-handed (left-handed) sense (with the individual particles moving on little epicycles), see figure

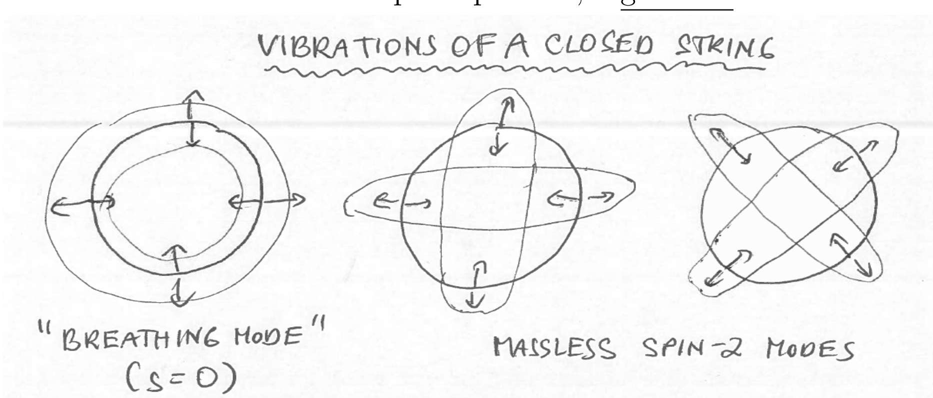


Why spin 2?

The spin of a quantized field is directly related to the transformation properties of the field under spatial rotations. Namely, spin s is related to the angle θ under which the polarization modes are invariant by

$$\boxed{s = 360°/\theta\,.} \tag{4.21}$$

Since the above described polarizations are invariant under rotations of 180°, the associated field is that of a massless spin-2 particle, a graviton.



Following [1] let us also note that the fundamental vibrations of a closed string, see above figure, contain the spin-0 (dilaton) and spin-2 (graviton) modes. The string theory predicts the scalar–tensor theory of gravity. If it should be applicable to our Universe, there must be a mechanism by which the scalar becomes massive and thence has not been observed.

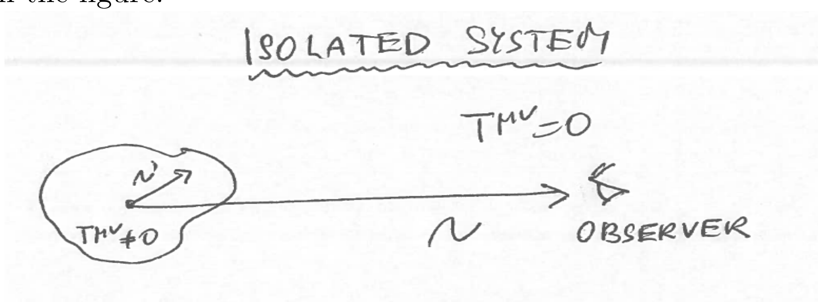
Gravitational radiation from an isolated system

- To describe how the gravitational waves are produced by sources, let us return back to the full linearized equations (4.1). As discussed in the Classical Physics Course, the particular solution of the inhomogeneous wave equation is given by the following retarded potential:

$$\boxed{\bar{h}^{\mu\nu}(t,\vec{r}) = 4G\int d^3r'\frac{T^{\mu\nu}(t_R,\vec{r}')}{|\vec{r}-\vec{r}'|}\,, \quad t_R = t-|\vec{r}-\vec{r}'|\,,} \tag{4.22}$$

c.f. (**??**). (Only sources on the past light cone can influence the gravitational field.)

- Let us now consider and isolated, far away, and slowly moving source, as displayed in the figure:



For such system we have

$$|\vec{r}'| \ll |\vec{r}|\,, \quad \omega \ll \frac{1}{|\vec{r}'|}\,, \tag{4.23}$$

where ω is a characteristic frequency of the source.

We then get the following for the Fourier transform of the above solution in time coordinate:

$$\begin{aligned}
\tilde{h}_{\mu\nu}(\omega,\vec{r}) &= \frac{1}{\sqrt{2\pi}}\int dte^{i\omega t}\bar{h}_{\mu\nu} = \frac{4G}{\sqrt{2\pi}}\int dtd^3r'e^{i\omega t}\frac{T_{\mu\nu}(t_R,\vec{r}')}{|\vec{r}-\vec{r}'|} \\
&\approx \frac{4G}{\sqrt{2\pi}}\int dt_R d^3r'e^{i\omega t_R+i\omega|\vec{r}-\vec{r}'|}\frac{T_{\mu\nu}(t_R,\vec{r}')}{|\vec{r}-\vec{r}'|} = 4G\int d^3r'e^{i\omega|\vec{r}-\vec{r}'|}\frac{\tilde{T}_{\mu\nu}(\omega,\vec{r}')}{|\vec{r}-\vec{r}'|} \\
&\approx 4G\frac{e^{i\omega r}}{r}\int d^3r'\tilde{T}_{\mu\nu}(\omega,\vec{r}')\,,
\end{aligned} \tag{4.24}$$

where the third and the last formulae result from the leading order asymptotic expansion.

- Due to the Lorenz condition, we can concentrate on the spatial components, as we have

$$\partial_\mu\bar{h}^{\mu\nu}=0 \quad \Rightarrow \quad \tilde{h}^{t\mu} = \frac{i}{\omega}\partial_i\tilde{h}^{i\mu} \quad \Rightarrow \quad \tilde{h}^{tj} = \frac{i}{\omega}\partial_i\tilde{h}^{ij}\,, \quad \tilde{h}^{tt} = -\frac{1}{\omega^2}\partial_i\partial_j\tilde{h}^{ij}\,, \tag{4.25}$$

that is, $\tilde{h}^{t\mu}$ are entirely determined in terms of $\tilde{h}^{ij}$.

- Let us now use the conservation of the energy momentum tensor,

$$\partial'_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad -\partial'_k \tilde{T}^{k\mu} = i\omega \tilde{T}^{t\mu} \,. \tag{4.26}$$

Thus we have

$$\int d^3r' \tilde{T}^{ij}(\omega, \vec{r}') = \int \partial'_k (x'^i \tilde{T}^{kj}) d^3r' - \int x'^i (\partial'_k \tilde{T}^{kj}) d^3r' = 0 + i\omega \int x'^i \tilde{T}^{tj} d^3r' \,, \tag{4.27}$$

upon throwing away the boundary term (which certainly vanished for an isolated system) and by using (4.26). Let us now define the quadrupole moment tensor as

$$\boxed{I_{ij} = \int x'^i x'^j T^{tt}(t, \vec{r}') d^3r' \,.} \tag{4.28}$$

Repeating the above trick, we get

$$\begin{aligned} \int d^3r' \tilde{T}^{ij}(\omega, \vec{r}') &= \frac{i\omega}{2} \int (x'^i \tilde{T}^{tj} + x'^j \tilde{T}^{ti}) d^3r' \\ &= \frac{i\omega}{2} \int (\partial'_l (x'^i x'^j \tilde{T}^{tl}) - x'^i x'^j (\partial'_l \tilde{T}^{tl})) d^3r' = -\frac{\omega^2}{2} \tilde{I}^{ij}(\omega) \end{aligned} \tag{4.29}$$

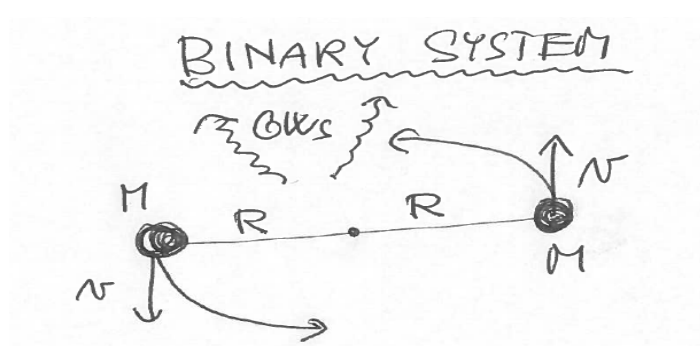
Thus we found

$$\tilde{h}_{ij}(\omega, \vec{r}) = -2G\omega^2 \frac{e^{i\omega r}}{r} \tilde{I}_{ij}(\omega) \quad \Leftrightarrow \quad \boxed{\bar{h}_{ij}(t, \vec{r}) = \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}\Big|_{t=t_R} \,,} \tag{4.30}$$

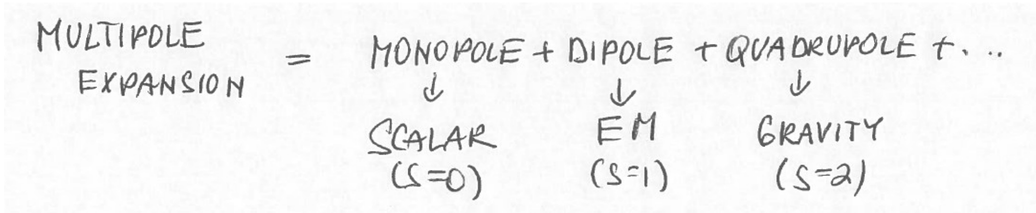
which is the famous (metric perturbation) quadrupole formula.

- In particular, you will derive in your tutorial that for a binary star system displayed below, one finds

$$\bar{h}_{ij}(t, \vec{r}) = \frac{8GM}{r} \Omega^2 R^2 \begin{pmatrix} -\cos 2\Omega t_R & -\sin 2\Omega t_R & 0 \\ -\sin 2\Omega t_R & \cos 2\Omega t_R & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \qquad \Omega = \sqrt{\frac{GM}{4R^3}} \,. \tag{4.31}$$



- Spin & radiation. Let us finally remark one important feature. We have found that the lowest multipole that yields the radiation mode in gravity is the quadrupole. This is directly related to the spin-2 of the graviton. On the other hand for the spin-1 (electromagnetism) there is already radiation produced by the dipole. One would also found that the scalar field already has a monopole radiation. *Which multipole radiates is directly related to the spin of the corresponding field.*



Typically, higher the multipole, weaker the signal, which is the reason as to why the gravitational radiation is much weaker than the electromagnetic one and why it is hard to detect gravitational waves and why it took 100 years to observe them after they were predicted. (Compared to this, the discovery of the Higgs was obviously just a baby problem :).)

Energy loss due to gravitational radiation

- Gravitational energy momentum pseudotensor. Our aim is to estimate how much energy is irradiated in the gravitational waves, for example by a binary star system. To this purpose we have to first construct the energy momentum tensor for the gravitational field. However, as discussed in the previous chapter and following the analogy with electromagnetism, there cannot be any such tensor, as we expect this to be a quadratic in derivatives of metric quantity,

$$T \propto (\partial g)^2 \propto \Gamma^2 \,, \tag{4.32}$$

which is not a tensor. Note, however, that this is a Lorentz tensor and so around 'flat space' it should give meaningful results—it should allow us to study how much energy is carried by weak gravitational waves.

There are a number of ways for how to construct such an energy momentum pseudotensor. Since the result is not unique, there are many such tensors (related by the divergence of some superpotential). Conceptually perhaps most clear is the Landau–Lifshitz one [3], see below. Let us, however, first construct a 'simpler' pseudotensor, using the perturbation theory [1].

- Perturbations. To find the energy momentum pseudotensor we shall pertubatively solve the vacuum Einstein equations,

$$G^{\mu\nu} = 0 \,, \tag{4.33}$$

order by order, in small $h_{\mu\nu}$ expansion. That is, we expand

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} + O(h^3)\,, \tag{4.34}$$

where $h^{(2)}_{\mu\nu}$ is of the order $O(h^2)$.

– *Zeroth order:* This order is trivial, we have

$$g_{\mu\nu} = \eta_{\mu\nu} \quad \Rightarrow \quad G^{\mu\nu}(\eta) = 0 \tag{4.35}$$

is automatically satisfied.

– *Linear order:* This is what we have done previously, at the linear order we have

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \Rightarrow \quad G^{\mu\nu}(\eta + h) = G^{\mu\nu}_{\text{lin}}(h) + O(h^2)\,. \tag{4.36}$$

The vacuum Einstein equations are now

$$G^{\mu\nu}_{\text{lin}}(h) = 0\,, \tag{4.37}$$

and determine $h_{\mu\nu}$. At the same time we have

$$\partial_\mu G^{\mu\nu}_{\text{lin}} = 0\,, \tag{4.38}$$

as a consequence of the full Bianchi identity, specified to a linear order. This is true for any metric!

– *Quadratic order:* We now expand the metric as in (4.34), and find

$$G^{\mu\nu}(\eta + h + h^{(2)}) = G^{\mu\nu}_{\text{lin}}(h) + G^{\mu\nu}_{\text{lin}}(h^{(2)}) + G^{\mu\nu}_{\text{quadr}}(h) + O(h^3)\,. \tag{4.39}$$

The first term vanishes from previous and the Einstein equations at the quadratic order yield

$$G^{\mu\nu}_{\text{lin}}(h^{(2)}) = 8\pi G \tau^{\mu\nu}\,, \qquad \boxed{\tau^{\mu\nu} = -\frac{1}{8\pi G} G^{\mu\nu}_{\text{quadr}}(h)\,.} \tag{4.40}$$

$\tau^{\mu\nu}$ is the energy momentum tensor for the first order perturbations (which we already solved for in the previous order, and so the r.h.s. is known). Such $\tau^{\mu\nu}$ is symmetric, and conserved,

$$\boxed{\partial_\mu \tau^{\mu\nu} = 0\,,} \tag{4.41}$$

on behalf of (4.38). This is our energy momentum pseudotensor.

- So what is it? It turns out (as is almost obvious) that $G^{\mu\nu}_{\mathrm{quadr}}(h)$ is just the Einstein tensor expanded on a linear perturbation to the quadratic order:

$$G^{\mu\nu}(\eta + h) = G^{\mu\nu}_{\mathrm{lin}}(h) + G^{\mu\nu}_{\mathrm{quadr}}(h) + O(h^3)\,. \tag{4.42}$$

The expression for it is pretty nasty. To simplify this, one 'coarse grains' the spacetime, by averaging over the spacetime oscillations. This avoids the problem of localization of gravitational energy at a point. From a practical point of view we can then integrate by parts, throwing away the boundary terms

$$\langle A \partial_\mu B \rangle = -\langle (\partial_\mu A) B \rangle\,. \tag{4.43}$$

One then recovers the following expression

$$\boxed{\tau_{\mu\nu} = \frac{1}{32\pi G} \langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} - \frac{1}{2} \partial_\mu h \partial_\nu h - \partial_\rho h^{\rho\sigma} \partial_\mu h_{\nu\sigma} - \partial_\rho h^{\rho\sigma} \partial_\nu h_{\mu\sigma} \rangle\,.} \tag{4.44}$$

Of course, this formula significantly simplifies for the TT gauge, in which case only the first term survives, yielding

$$\tau_{\mu\nu} = \frac{1}{32\pi G} \langle \partial_\mu h^{TT}_{\rho\sigma} \partial_\nu h^{TT\,\rho\sigma} \rangle\,. \tag{4.45}$$

- Landau–Lifshitz pseudotensor. The idea of construction is very simple:
  - In any reference frame we have the Bianchi identity

$$\nabla_\mu G^{\mu\nu} = 0\,. \tag{4.46}$$

    This is not a conservation law for anything.
  - Consider now a freely falling frame (at a point) so that all Γ's vanish. Then the above reduces to

$$\partial_\mu G^{\mu\nu} = 0\,. \tag{4.47}$$

    To satisfy this identically, we introduce a superpotential $S^{\kappa\mu\nu}$, which is symmetric in the last two indices and antisymmetric in the first two, by

$$G^{\mu\nu} \equiv \partial_\kappa S^{\kappa\mu\nu}\,. \tag{4.48}$$

    Of course, from (4.48) one can find $S^{\kappa\mu\nu}$ explicitly, see Landau's book [3].
  - Let's now go back to any frame (putting back the Γ's), and define $\tau^{\mu\nu}$ by

$$\boxed{\partial_\kappa S^{\kappa\mu\nu} - G^{\mu\nu} \equiv 8\pi G \tau^{\mu\nu}\,,} \tag{4.49}$$

    which is the Landau–Lifshitz pseudotensor. From construction this is symmetric in μ and ν, and is schematically given by Γ^2. Moreover, we have

$$\partial_\kappa S^{\kappa\mu\nu} = 8\pi G (T^{\mu\nu} + \tau^{\mu\nu})\,, \tag{4.50}$$

upon using the Einstein equations. Applying ∂_μ on this we recover

$$\boxed{\partial_\mu(T^{\mu\nu} + \tau^{\mu\nu}) = 0\,.} \tag{4.51}$$

The meaning is simple: total energy of matter plus gravitational field is conserved. Note that we did not use any linearization in this derivation.

- Power formula. Let us finally calculate the rate of energy loss from a system emitting gravitational radiation according to the quadrupole formula (4.30). The system looses energy with the following power:

$$P = \frac{\Delta E}{dt} = \int_{S^2_\infty} \tau_{0\mu} r^2 n^\mu d\Omega\,, \tag{4.52}$$

where n^μ is the normalized normal to S^2_∞. The corresponding calculation is a bit lengthy, see [1] for details, however, the result is extremely simple:

$$\boxed{P = \frac{G}{5c^5}\Big\langle \frac{d^3Q_{ij}}{dt^3}\frac{d^3Q^{ij}}{dt^3}\Big\rangle\,,} \tag{4.53}$$

which is known as (power) quadrupole formula.[1] Here Q_{ij} is a traceless part of the quadrupole,

$$\boxed{Q_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}I_{kl}\,.} \tag{4.55}$$

In particular, you will show in your tutorial that for our binary system this yields

$$P = \frac{2}{5}\frac{G^4M^5}{R^5c^5}\,. \tag{4.56}$$

Of course, we can get a crude estimate for an object of mass M and length L rotating at frequency ω as

$$Q \sim ML^2 \quad \Rightarrow \quad P = \frac{\omega^6 G M^2 L^4}{c^5}\,. \tag{4.57}$$

In particular, this yields

  - 10^{-52} Watts for a one-meter rod rotating at frequency of one Hertz.
  - 5 kilowatts for Jupiter on its orbit around the Sun.

[1] This is to be compared to the electromagnetic dipole formula,

$$P = \frac{2}{3c^3}\ddot{d}_i\ddot{d}_i\,, \tag{4.54}$$

where $d^i = ex^i$ stands for the electric dipole moment.

– 10^{48} Watts for a spinning pulsar binary, PSR1913+16, observed by Hulse and Taylor in 1974. They measured the decreasing orbital period, which precisely matched the energy loss due to the gravitational radiation—one Nobel Prize for General Relativity, and the most precise measurement of those days. Of course, we have now another Nobel Prize, awarded for the direct observation of gravitational waves produced in the binary black hole collision in 2015.

4.2 Schwarzschild solution

So far we only discussed the perturbative solutions to the Einstein equations. Let us now study some exact ones :)

- Schwarzschild metric. A month after Einstein published his theory, Karl Schwarzschild (1916) found a first (apart from Minkowski) exact solution of the full non-linear vacuum Einstein equations,
$$R_{\mu\nu} = 0\,. \tag{4.58}$$
The solution reads
$$\boxed{ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2M}{r}\,, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2\,.} \tag{4.59}$$
Here M is a free parameter (integration constant).

- Bikhoff's theorem: *The above solution is the most general spherically symmetric solution to vacuum Einstein field equations.*

  Observing carefully the solution, we see that it is static, that is, it does not depend on time t and we have a timelike Killing vector $\xi = \partial_t$. As we shall see this is a consequence of spherical symmetry (spherically symmetric plus vacuum implies static). In particular, this clearly shows that we cannot have spherically symmetric gravitational waves—why?

  Proof. To prove the above theorem, let us write the most general spherically symmetric metric element. In the area gauge where the sphere of radius r has surface $4\pi r^2$, this reads (see [1] for all bloody details):
$$ds^2 = -e^{2\psi} f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2m(r,t)}{r}\,, \quad \psi = \psi(r,t)\,. \tag{4.60}$$
The Einstein field equations then give [5]
$$\frac{\partial m}{\partial r} = 4\pi r^2(-T^t_t)\,, \quad \frac{\partial m}{\partial t} = -4\pi r^2(-T^t_r)\,, \quad \frac{\partial \psi}{\partial r} = \frac{4\pi r}{f}(-T^t_t + T^r_r)\,. \tag{4.61}$$
In vacuum we therefore find $m = M = \text{const.}$, and $\psi = \psi(t)$ which can be set to one by redefining time.

- Into the abyss. At far distances, $r \to \infty$, the metric approaches that of a Minkowski metric in spherical coordinates, the spacetime is asymptotically flat. On the other hand, as we decrease the radius, strange things start happening. In particular, the metric is no longer well defined at

$$\boxed{r = r_+ = 2M\,.} \tag{4.62}$$

  As we shall see, this simply corresponds to a coordinate singularity and gives rise to, provided the spacetime can be trusted all the way to these small radia, a black hole horizon; the spacetime then describes a black hole.

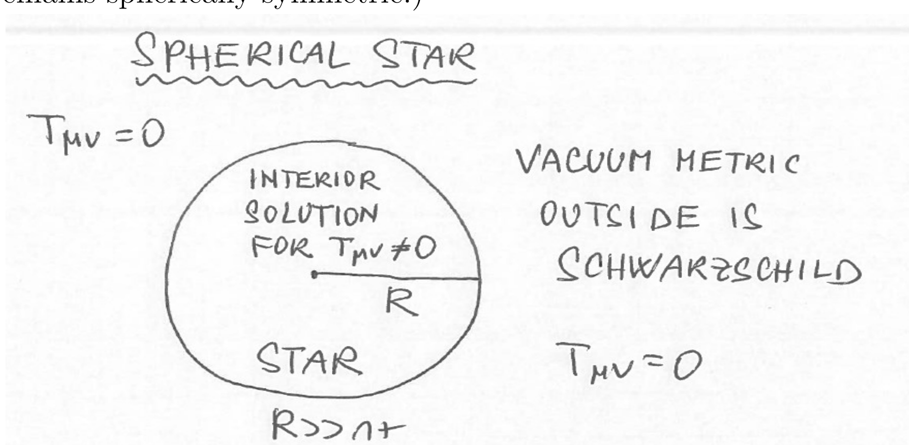
  Finally, we reach another problem at $r = 0$. One can show that this is a true singularity where the 'smooth structure' of the manifold breaks down. A simple check of this statement is to write down the Kretschmann scalar,

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6}\,. \tag{4.63}$$

  Since this is a scalar (and thence observer independent quantity—can you write another one?) and it diverges as $r \to 0$, we see there is a real problem at $r = 0$. In other words, Einstein's gravity predicts its own doom.

- What does it describe?

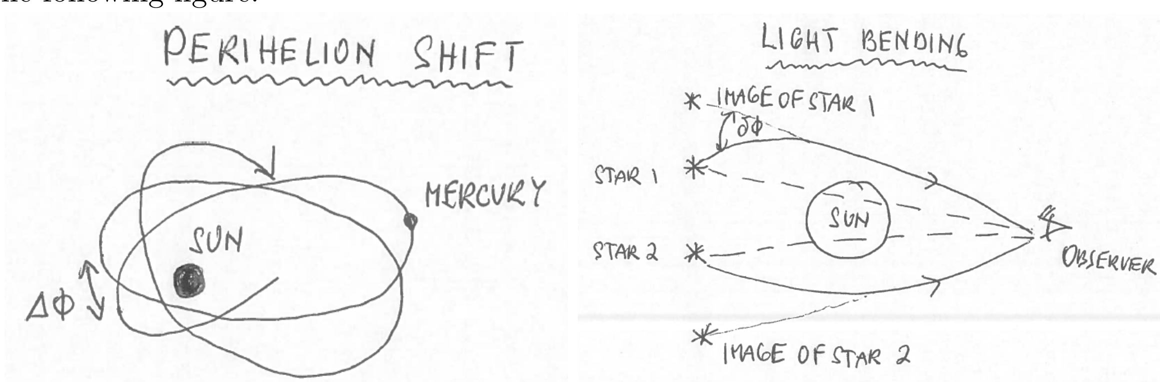
  - Gravitational field outside of stars, planets, and so on... Due to Bikhoff's theorem, it is obvious that a metric outside of a spherical body (a star for example) has to be described by Schwarzschild metric. Since typically the star radius $R \gg r_+$ we do not have to worry about the coordinate singularity at $r = r_+$ or about the true singularity at $r = 0$ as the corresponding solution has to simply be replaced by an 'interior solution' describing a star. Moreover, by employing the weak field approximation far away from the object, we can identify M with the 'Newtonian' mass of the body. (Interestingly the body maybe time dependent, for example a collapsing object, but the metric outside is still described by the Schwarzschild geometry, as long as it remains spherically symmetric.)



– Black hole. Another possibility is to understand the solution as is all the way till small radia. In this case, we have described a new 'object', a black hole. It took a lot of effort to convince astronomers that such objects can exist in our Universe (they are a tough crowd). At present, there is a little doubt that black holes indeed exist in our Universe. Based on their size, we can distinguish 3 cases: i) primordial black holes that are (if they exist) tiny tiny ii) solar mass black hole that formed by a gravitational collapse of massive stars and iii) supermassive black holes at the centers of galaxies. Only the latter two have been observed by now.

Perihelion shift & light bending

- Einstein's triumph. An immediate triumph of Einstein's theory (Einstein waited to show this till the next day after finding his equations) was to explain the perihelion shift of Mercury. This was well known to astronomers and amounts to (after taking into account various effects) $\Delta\phi$=43 arsec/century, as displayed in the following figure:



  Surprisingly, however impressive this result was, it did not really convince the community that Einstein's theory is The One (after all the effect was known for a long time and people suspected that Einstein fiddled with the theory to get the right result).

  However, Einstein received his fame a few years later due to his prediction of the light bending by the Sun, which was then subsequently measured by Eddington. Einstein became a new prototype of a genius and a celebrity overnight.

- Geodesics. To account for both these effects, it is enough to study the test particle and light motion in the gravitational field of the Sun, described by the Schwarzschild metric. Since such a metric possess spherical symmetry, the motion is planar and one can w.l.o.g. choose

$$\theta = \pi/2 \tag{4.64}$$

  plane. Morever, since the metric is static ($k = \partial_t$) and axisymmetric ($m = \partial_\phi$), we have 2 Killing integrals of motion, which together with the normalization of the 4-velocity render the motion completely integrable. Namely, we have, using $u^\mu = \frac{dx^\mu}{d\lambda} = (\dot t, \dot r, 0, \dot\phi)$,

$$\begin{aligned} E &= -k_\mu u^\mu = -g_{tt}\dot t = f\dot t\,, \\ L &= m_\mu u^\mu = g_{\phi\phi}\dot\phi = r^2\dot\phi\,, \\ -\kappa &= u^2 = -f\dot t^2 + \dot r^2/f + r^2\dot\phi^2\,, \end{aligned} \tag{4.65}$$

where $\kappa = 1$ for timelike and $\kappa = 0$ for null geodesics. Plugging the first two equations into the last one, we have

$$-\kappa = -f\frac{E^2}{f^2} + \dot r^2/f + r^2\frac{L^2}{r^4}\,, \tag{4.66}$$

which, after re-arranging gives

$$\boxed{\frac{1}{2}\dot r^2 + V(r) = \frac{1}{2}E^2\,, \quad V(r) = \frac{1}{2}f\Big(\kappa + \frac{L^2}{r^2}\Big) = \frac{\kappa}{2} - \frac{\kappa M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}\,.} \tag{4.67}$$

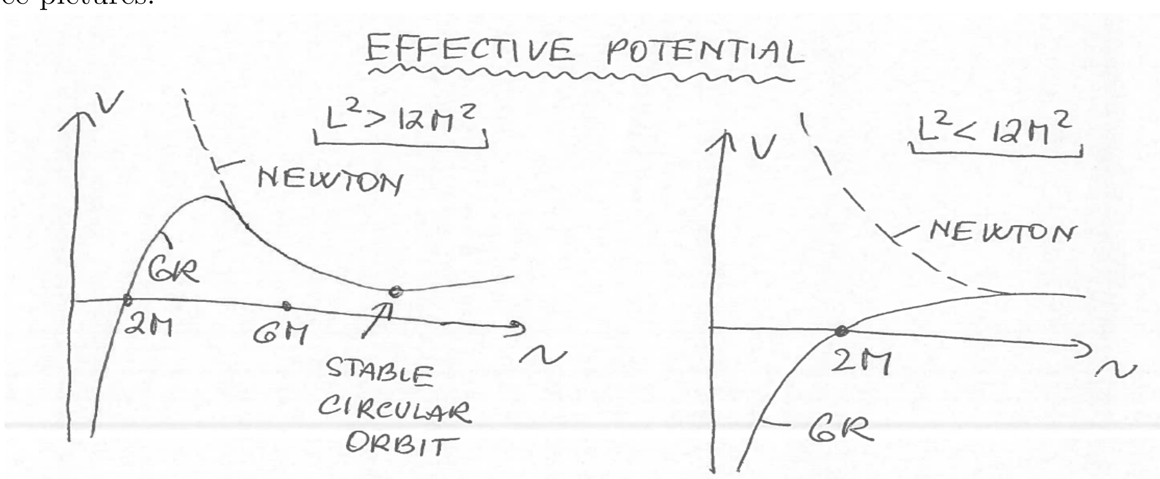
In what follows we discuss the massive particles, while in your homework you shall study the bending of light. So we set $\kappa = 1$ and sketch the form of the effective potential V. We observe that it has an inflection point where

$$V' = 0 = V'' \quad \Leftrightarrow \quad L^2 = 12M^2\,,\ r = 6M\,. \tag{4.68}$$

For larger L, there are two extrema, a minimum at $r > 6M$ and a maximum at $3M < r < 6M$; they correspond to stable and unstable circular orbits. That is the Inner Most Circular Orbit (ISCO) is located at

$$\boxed{R_{\text{ISCO}} = 6M\,.} \tag{4.69}$$

For smaller L no extrema exist and the particle necessarily falls into a singularity, see pictures.



For comparison, we have also plotted the Newtonian potential

$$V = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2}\,. \tag{4.70}$$

It misses the last term in (4.67), which is the GR correction and allows the approach to $r = 0$ only for zero angular momentum $L = 0$.

Let us now concentrate on bound orbits. It can be shown that they are no longer ellipses, and apart from a circular trajectory they are not closed, they undergo a perihelion shift. The corresponding equation can be solved perturbatively and I refer you for this fun to Carroll's book [1]. However, to derive the Mercury perihelion shift we can use the following shortcut [7].

The trajectory is nearly circular and far from the Sun, $r \approx r_c \gg M$. By solving $V' = 0$ and taking the larger root, we find

$$L^2 = \frac{M r_c^2}{r_c - 3M} \,. \tag{4.71}$$

The radial oscillations around the circular trajectory have a frequency, given by

$$\omega_r^2 = V''(r_c) = \frac{M(r_c - 6M)}{r_c^3(r_c - 3M)} \,, \tag{4.72}$$

upon using (4.71). On the other hand, the angular frequency is

$$\omega_\phi^2 = \frac{L^2}{r_c^4} = \frac{M}{r_c^2(r_c - 3M)} \,. \tag{4.73}$$

The failure of ω_r to equal ω_ϕ means that in GR the orbits do not close. The corresponding precession frequency is

$$\omega_p = \omega_\phi - \omega_r \approx \frac{3M^{3/2}}{r_c^{5/2}} \,, \tag{4.74}$$

which yields the corresponding angle per orbit, using the Keplerian $T^2 = 4\pi^2 r_c^3/M$,

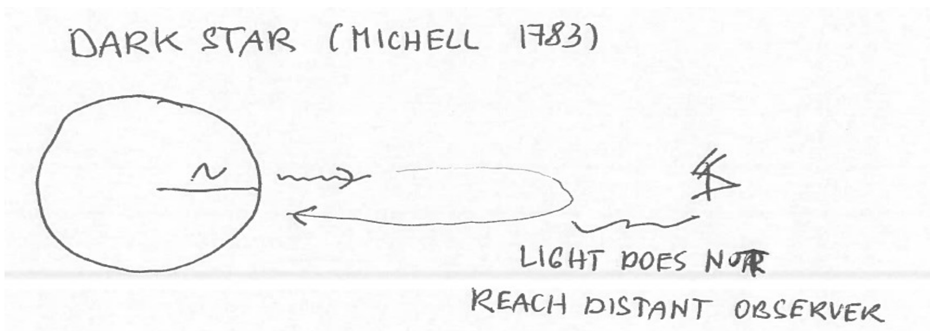
$$\boxed{\Delta\phi = T\omega_p = \frac{6\pi M^2}{L^2} \,,} \tag{4.75}$$

or 43 arsec/century.

4.3 Black holes

Dark stars

Even in the pre-relativity physics there is a notion of a 'dark star', studied first by John Michell in 1783. A dark star is a star of large mass whose surface escape velocity exceeds the speed of light, rendering it dark, as displayed in the figure.



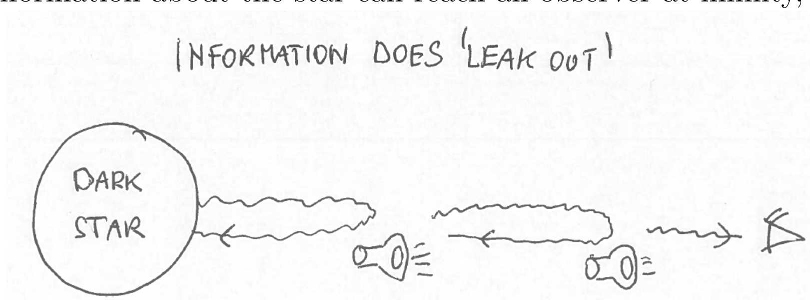
Using the naive Newtonian physics, treating the light as a massive particle propagating at the speed of light yields

$$\frac{1}{2}mc^2 - \frac{GmM}{r} = 0\,, \tag{4.76}$$

if light just reaches an observer at infinity with zero velocity. Thus we find for the radius of the dark star

$$r = r_+ = \frac{2GM}{c^2}\,. \tag{4.77}$$

As we shall see this is precisely the gravitational radius of a black hole. Note however that the interpretation is completely different, and so is the fact whether or not the information about the star can reach an observer at infinity, see the following picture:



What can we learn from Rindler?

Before we study the real black holes let us look at several properties of the Rindler space associated with an accelerated observer—let's demystify black holes :).

- Rindler horizon. Tthe Rindler space is equipped with the metric

$$\boxed{ds^2 = -(1+aX)^2 dT^2 + dX^2\,.} \tag{4.78}$$

Obviously, the coordinate T is free to run from minus to plus infinity, $T \in (-\infty, \infty)$. However, the metric becomes 'ill defined', we have a 'singularity', at $X = X_+$, where

$$\boxed{X_+ = -\frac{1}{a}\,,} \tag{4.79}$$

which is a 'boundary of the Rindler space' known as the Rindler (accelerated) horizon; our coordinates only cover $X > X_+$ patch of the spacetime.

- Surface of infinite redshift. Consider two observers in the Rindler space, Alice and Bob, at constant X. Then their proper times are gravitationally redshifted as

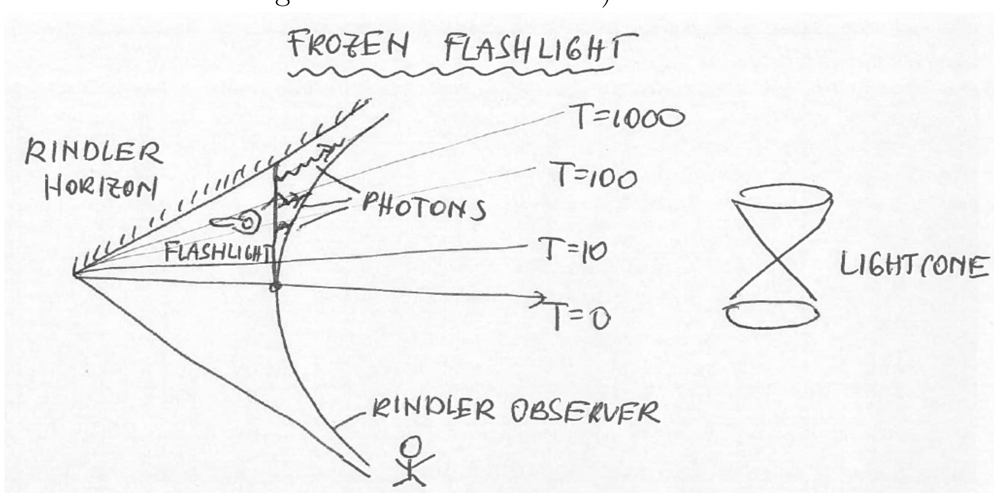
$$\frac{\nu_A}{\nu_B} = \frac{d\tau_B}{d\tau_A} = \frac{\sqrt{-g_{TT}}|_B}{\sqrt{-g_{TT}}|_A} = \frac{1+aX_B}{1+aX_A}\,. \tag{4.80}$$

If Bob 'is located on horizon', $X_B \to X_+$, we find

$$\frac{\nu_A}{\nu_B} \to 0\,, \tag{4.81}$$

that is Alice would see Bob as infinitely red-shifted, that is completely dark. (Bob's watch goes infinitely slower than that of Alice.) Rindler horizon is thus a surface of infinite redshift.

- Frozen flashlight. Let us now consider the following experiment. The Rindler ($X = 0$) observer throws a flashlight towards the Rindler horizon. As we are really in flat space (the light cones are those of Minkowski) the light travels 'under 45' degrees. As is obvious from the following picture, the observer will only see the flashlight to reach the horizon at $T = \infty$. (Rindler's Universe has to end before the flashlight reaches the horizon.)



In other words, the observer sees more and more redshifted flashlight (which is darker and darker) and sees its fall slowing down as it approaches the horizon—the flashlight gets 'frozen' on the Rindler horizon.

- Maximal extension. So what is happening on the horizon? Is there a singularity? Is it a place where all frozen objects accumulate? Whenever you encounter such a surface, try to direct some geodesics at it and see if they can pass through, the best is to use the null geodesics. These are given by

$$ds^2 = 0 = -(1+aX)^2\dot{T}^2 + \dot{X}^2 \quad \Rightarrow \quad T = \pm\frac{1}{a}\lg(1+aX) + \text{const.}, \tag{4.82}$$

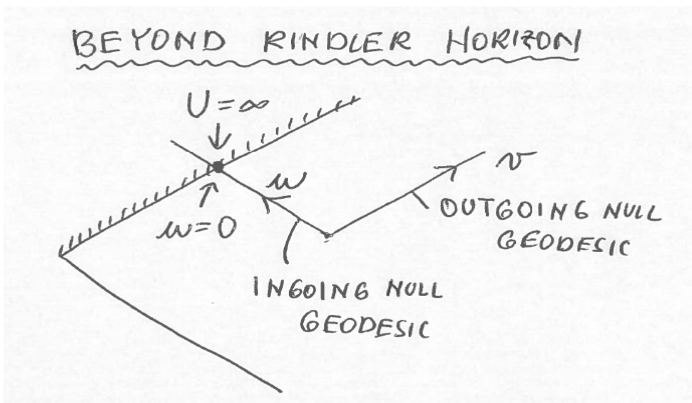
corresponding to outgoing and ingoing geodesics. Thus we may define null coordinates

$$U = T - \frac{1}{a}\lg(1 + aX)\,, \quad V = T + \frac{1}{a}\lg(1 + aX)\,, \tag{4.83}$$

in which the metric takes the form

$$ds^2 = -e^{V-U}dUdV\,. \tag{4.84}$$

This still only covers the Rindler patch as $U, V \in (-\infty, \infty)$. (The future horizon is located at $U = \infty$.) Can we extend below X_+? To this purpose, let us find the affine parameters parameterizing these geodesics. It can be show that they are $\lambda_{\rm out} = e^V$ for the outgoing geodesic and $\lambda_{\rm in} = -e^{-U}$ for the ingoing one, see figure.



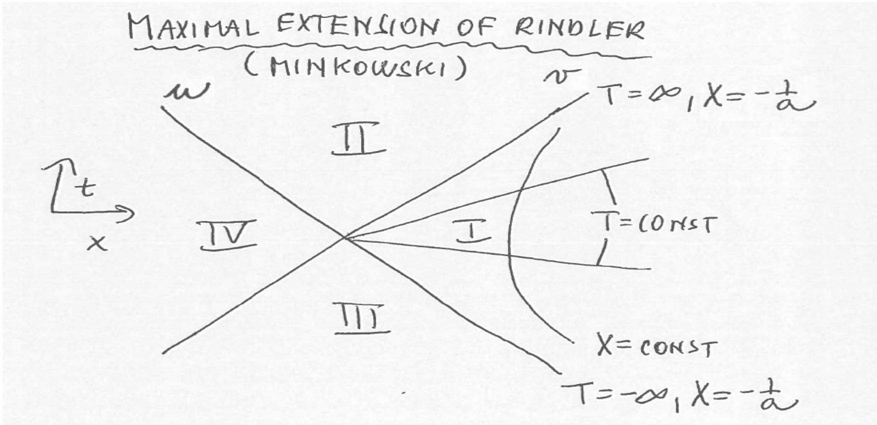
So we introduce new coordinates by

$$u = \lambda_{\rm in} = -e^{-U}\,, \quad v = \lambda_{\rm out} = e^V \quad \Rightarrow \quad ds^2 = -dudv\,. \tag{4.85}$$

The horizon is now located at a finite affine parameter $u = 0$ and the geodesic can be extended through the horizon (the geodesics were incomplete in the original Rindler frame). Introducing finally $t = (u + v)/2$ and $x = (v - u)/2$ brings the form of the metric into the Minkowski form,

$$ds^2 = -dt^2 + dx^2\,, \tag{4.86}$$

and we obtained the maximal extension of the Rindler space, displayed in the figure.



It is now obvious that the Rindler horizon simply corresponds to $t = \pm x$ and there is no singularity whatsoever, only the Rindler coordinates were singular there, that is $X = X_+$ is simply a coordinate singularity. Rindler coordinates simply only covered region I, the whole spacetime has 4 regions: I, II, III, and IV.

- Fate of flashlight. So let us finally ask: is something special happening with the flashlight, as it freely falls towards the Rindler horizon? The answer is NO, the flashlight moves on a geodesic in Minkowski space and as such it does not see the Rindler horizon at all. It simply passes through it and continues its journey behind the horizon. From a perspective of the Rindler observer, this however happens after infinite elapse of his time and he never sees the flashlight to go through the horizon.
- What about BHs? As we shall see, all these conclusions remain the same in the black hole spacetime, with a tiny difference, there is a true singularity hiding underneath the black hole horizon.

Schwarzschild black hole

- Black hole horizon. The Schwarzschild black hole spacetime is described by the Schwarzschild metric,

$$\boxed{ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2M}{r}\,, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2\,,} \tag{4.87}$$

which we now take seriously also for small r (there is no surface of a star shielding small r regions.) We know already that there is a true curvature singularity at $r = 0$. We also have an apparent singularity at $r = r_+$, where

$$\boxed{r_+ = 2M\,.} \tag{4.88}$$

It is straightforward to show (following Rindler analogy) that this is a surface of infinite redshift where objects get 'frozen'. For this reason, the objects like this were first called frozen stars, until 1968 when Wheeler coined the term black hole. The surface $r = r_+$ was then consequently called the black hole horizon. It denotes the 'boundary of black hole'.

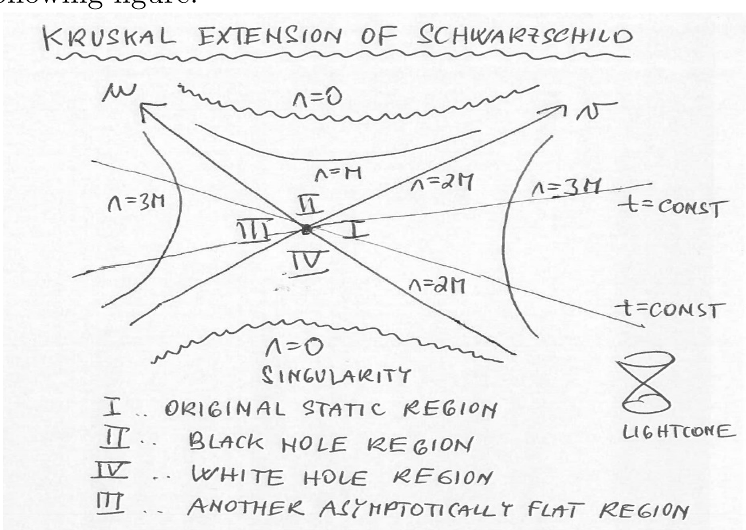
- Kruskal extension. As shown by Kruskal in 1960, this is again a coordinate singularity similar to what happens in the case of Rindler horizon. By following null geodesics that penetrate the horizon and switching to coordinates associate with the affine parameter along these geodesics, u and v,

$$u = -\exp\Big(\frac{r^*-t}{4M}\Big)\,, \quad v = \exp\Big(\frac{r^*+t}{4M}\Big)\,, \quad r^* = \int \frac{dr}{f} = r + 2M\lg\Big|\frac{r}{2M}-1\Big|\,, \tag{4.89}$$

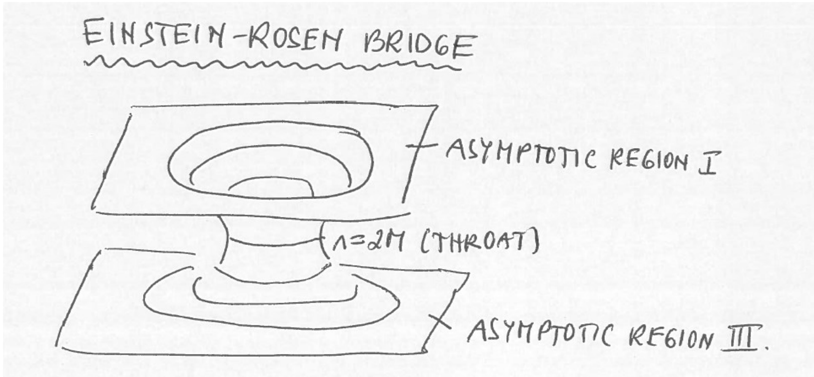
we arrive at the following metric

$$ds^2 = -\frac{32M^3}{r}\exp\Big(\frac{-r}{2M}\Big)dudv + r^2 d\Omega^2\,, \tag{4.90}$$

which is manifestly regular at $r = r_+$. Using these Kruskal coordinates, one discovers the maximal extension of the Schwarzschild spacetime, displayed in the following figure:



- Einstein–Rosen bridge. The geometry of the surface t =const. and $\theta = \pi/2$, in the Kuskal extension can be embedded in the 3-dimensional Euclidean space E^3, to get the following picture.



Obviously, we have two connected asymptotically flat regions, connected by the Einstein–Rosen bridge. The situation reminds that of a wormhole. However, this is only a spatial geometry and no observer can travel through due to the causal structure of the spacetime. Interestingly, the length of the throat (neck) of a wormhole like this in AdS space was recently conjectured to describe the holographic complexity [11].

- Remark on astrophysical black holes.
  - Origin. Astrophysical black holes were formed by gravitational collapse. Consequently the regions III and IV disappear (they are replaced by the interior solution of the collapsing object).
  - Kerr. Astrophysical black holes are often fast rotating. In the isolated (vacuum) case the corresponding geometry is known as the Kerr geometry and was found by Kerr in 1963. This geometry contains only 2 parameters: rotation parameter a and mass M:

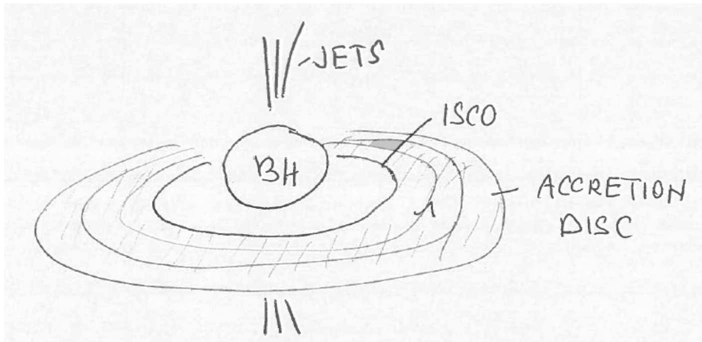
$$M\,, \quad J = Ma\,. \tag{4.91}$$

    All multipole moments are already determined by these (black hole has no hair). In principle, one could also include electric charge Q. However, this is believed not astrophysically important as the black hole would get discharged by the surrounding charged plasma.
  - Accretion disc. No astrophysical black hole is completely isolated, typically it sucks the nearby plasma or the atmosphere of a companion binary star. The material is heated up through friction and slowly inspirals towards the black hole in the form of an accretion disc. However, as we derived in the previous section, no circular orbits exist below the ISCO, which for Schwarzschild is (see previous section)[2]

$$r_{\mathrm{ISCO}} = 6M\,. \tag{4.92}$$

[2] For Kerr the formula is much more complicated and depends on whether the orbit is prograde (smaller ISCO) or retrograde (bigger ISCO up to $9M$ for an extremal black hole).

Once the matter goes below this it simply 'falls' into the black hole. On the way a significant amount of the rest energy is irradiated away due friction (6 percent for Schwarzshild and up to 42 percent for Kerr). The situation is displayed in the following picture:



It is the aim of the Event Horizon Telescope to observe the inner edge of the accretion disc of Sagittarius supermassive black hole and to determine if the ISCO is as predicted by General Relativity.

– Jets. For rotating black holes, jets may form as well, perhaps through a process known as the Blandford–Znajek effect. For these reasons the supermassive black holes are the brightest objects in the sky, called quasars, blazars, and so on :).

– Gravitational waves. Perhaps the most exciting is the recent observation of black hole collisions and the corresponding gravitational wave production discussed in the previous section.

Chapter 5: Advanced Topics

5.1 Black Hole Thermodynamics

Motivation

Let us study some properties of the Schwarzschild black hole:

$$\boxed{ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,,} \qquad \boxed{f = 1 - \frac{2m}{r}\,,} \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2\,. \tag{5.1}$$

- Asymptotic mass (conserved energy)

Noether's theorem: symmetry $\Leftrightarrow$ conserved quantity

described by Killing fields

$$\boxed{\nabla_{(\mu} k_{\nu)} = \nabla_\mu k_\nu + \nabla_\nu k_\mu = 0}$$

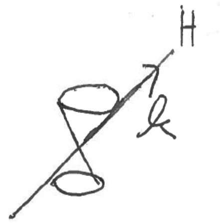
k^μ geometry "does not change"

spec: $k = \partial_t$ $\uparrow\uparrow^{t}$ spacetime is static $\Rightarrow$ energy is conserved

This leads to the following prescription for the *Komar mass*:

$$\boxed{M = -\frac{1}{8\pi}\int_{S^2_\infty} *dk = m\,.} \tag{5.2}$$

- Surface gravity. The black hole horizon($r = r_+ = 2M$) is a *Killing horizon*: a null surface generated by Killing field $k = \partial_t$.



$$k^\mu \nabla_\mu k^\nu |_H = \kappa k^\nu |_H$$

surface gravity

It can be shown that κ reads

$$\kappa = \frac{f'(r_+)}{2} = \frac{1}{2}\frac{2M}{r_+^2} = \frac{1}{4M} = \frac{1}{2r_+} \,. \tag{5.3}$$

Note that this 'coincides' with Newtonian acceleration evaluated on the black hole horizon:

$$\kappa = \frac{M}{r_+^2} = \frac{M}{(2M)^2} = \frac{1}{4M} \,. \tag{5.4}$$

- Horizon area. Taking the $dt = 0 = dr$, the induced spatial metric 'on the horizon' is $d\gamma^2 = r_+^2 d\Omega^2$. The area then reads

$$A = \int \sqrt{\det \gamma} d\theta d\varphi = \int r_+^2 \sin\theta d\theta d\varphi = 4\pi r_+^2 \,. \tag{5.5}$$

- Observation: Calculating the following differentials:

$$dM = \frac{dr_+}{2} \,, \quad dA = 8\pi r_+ dr_+ \,, \tag{5.6}$$

we find that

$$\boxed{dM = \frac{\kappa}{2\pi}\frac{dA}{4} \,.} \tag{5.7}$$

Laws of Black Hole Mechanics

Bardeen, Carter and Hawking (1973) proved the following 4 laws of black hole mechanics. For a stationary, charged, and rotating black hole with mass M, angular momentum J, and charge Q, we have:

- **Zeroth law:** The surface gravity κ is constant on the black hole horizon.
- **First law:**

$$\boxed{dM = \frac{\kappa}{2\pi}\frac{dA}{4} + \underbrace{\Omega dJ + \Phi dQ}_{\text{work terms}} \,.} \tag{5.8}$$

Here, Ω is the angular velocity of the black hole horizon, and Φ is its 'electrostatic potential'.

- **Second law:** Classically, the area of the horizon never decreases (provided the null energy condition holds).
$$\boxed{dA \geq 0\,.} \tag{5.9}$$

- **Third law:** It is impossible to reduce κ to zero in a finite number of steps.

We would like to compare these to the laws of thermodynamics. In particular, the first law to
$$dE = TdS + \text{work terms}\,. \tag{5.10}$$
However, there is a problem: Classical black holes act as ultimate sponges: no heat can flow out, they are at absolute zero temperature. So we cannot have $\kappa \propto T$.

Black Hole Thermodynamics

- Wheeler's cup of tea: "If you throw a cup of tea to the black hole, where did its entropy go?" Based on analyzing this question Bekenstein proposed
$$\boxed{S \propto A\,.} \tag{5.11}$$

- Hawking 1974. When quantum effects are taken into account, black holes radiate away as black body with
$$\boxed{T = \frac{\hbar\kappa}{2\pi k_B}\,, \quad S = \frac{A}{4\hbar G_N}\,.} \tag{5.12}$$

  Derivation used QFT in curved space. Hawking basically showed "stimulated emission". The problem with his derivation is that due to the bluehift near the horizon, the test field approximation breaks down and we cannot really trust the result. However, since then the same result has been reproduced by many other approaches, e.g: Euclidean path integral, tunneling, string theory, LQG.

Euclidean Trick

We shall use the following two facts:

- Thermal Green functions have periodicity in imaginary Euclidean time $\boxed{\tau = it}$:
$$\boxed{G(\tau) = G(\tau + \beta)\,, \quad \beta = 1/T\,.} \tag{5.13}$$

  Conversely, periodicity of G defines a thermal state. Green functions of quantum fields in the vicinity of black holes have this property (as seen by a static observer). What about gravitational field itself?

- Partition function. One can calculate the gravitational partition function in the WKB (semiclassical) approximation as

$$\boxed{Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}\,,} \tag{5.14}$$

where g_c stands for the metric(s) describing the classical solution(s). Note that the Euclidean action S_E consists of two terms: the Einstein–Hilbert action and the Gibbons–Hawking term:

$$S_E = \int_\Omega \frac{d^4x\sqrt{g}R}{16\pi G} + \int_{\partial\Omega} \frac{d^3x\epsilon\sqrt{h}K}{8\pi G}\,, \tag{5.15}$$

where $\epsilon = -1$ for spacelike and $\epsilon = 1$ for timelike boundary. The second term is needed to ensure well-posed variational principle (it kills the unwanted boundary terms in the case of a compact manifold). Here K stands for the extrinsic curvature and the second integral is over the boundary. Once the partition function is determined, we can calculate the free energy

$$F = -\frac{1}{\beta}\log Z \approx \frac{S_E}{\beta}\,, \tag{5.16}$$

which knows everything about thermodynamics. In particular, the entropy is given by

$$S = -\frac{\partial F}{\partial T}\,. \tag{5.17}$$

Let us explicitly demonstrate these on two examples, that of the Rindler space and the Schwarzschild black hole.

Rindler space

- The Rindler space reads

$$ds^2 = -(1+aX)^2 dT^2 + dX^2 + dy^2 + dz^2\,, \tag{5.18}$$

where we restored the additional two flat directions. Let us now Wick rotate, introducing the Euclidean time, $\tau = iT$. Then we have the following Euclidean metric:

$$ds_E^2 = (1+aX)^2 d\tau^2 + dX^2 + dy^2 + dz^2\,. \tag{5.19}$$

To simplify this, let us introduce a new coordinate ρ, by

$$\rho = \frac{1+aX}{a} \quad \Rightarrow \quad d\rho = dX\,, \tag{5.20}$$

so that

$$ds_E^2 = a^2\rho^2 d\psi^2 + d\rho^2 + \cdots = \rho^2 d\varphi^2 + d\rho^2\,, \tag{5.21}$$

upon introducing a new angle coordinate, $\varphi = a\tau$. This looks like a flat space written in polar coordinates, provided the angle φ has a period 2π, otherwise there is a conical singularity at $\rho = 0$, which corresponds to the original Rindler horizon. The reasoning now goes as follows: since the Rindler horizon was originally non-singular, we expect it to be non-singular again. This is achieved by setting (we want to avoid conical singularity)

$$\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \tau \sim \tau + \underbrace{2\pi/a}_{\beta} \quad \Leftrightarrow \quad \boxed{T = \frac{a}{2\pi}\,,} \tag{5.22}$$

which is the famous Unruh temperature. That is, an accelerated observer sees a thermal bath at a temperature proportional to his acceleration. If you accelerate really fast, you can cook a chicken.

- Let us next calculate the partition function and derive the entropy of the Rindler horizon. To this purpose we have to calculate the classical action, (5.15), evaluated for the Rindler Euclidean metric (5.19). Of course, on shell $R = 0$ and the first term vanishes. It is the second term that determines the value of the action. This is done as follows. We introduce a boundary at $X = X_0$ =const., calculate the contribution of the second term and then let $X_0 \to \infty$. The boundary has a normal $n^\mu = (0, 1, 0, 0)$ and the corresponding extrinsic curvature is

$$K = \nabla_\mu n^\mu = \frac{1}{\sqrt{g}}(\sqrt{g}n^\mu)_{,\mu} = \frac{a}{1 + aX_0}\,. \tag{5.23}$$

At the same time we have that the boundary metric has the following determinant $\sqrt{h} = 1 + aX_0$. We thus have

$$S_E = -\frac{8\pi}{a}\beta \underbrace{\int dydz}_{A} = -\frac{a\beta A}{8\pi} \quad \Rightarrow \quad F = \frac{S_E}{\beta} = -\frac{a}{2\pi}\frac{A}{4} = -T\frac{A}{4}\,. \tag{5.24}$$

Thus we have

$$\boxed{S = -\frac{\partial F}{\partial T} = \frac{A}{4}\,,} \tag{5.25}$$

which is the Bekenstein result. Note also thatt $F = M - TS = -TS$ as the energy of the spacetime is zero. Let us now do similar calculation for Schwarzschild.

Schwarzschild black hole

- The Euclideanized Schwarzschild solution ($\tau = it$) is

$$ds^2 = fd\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,. \tag{5.26}$$

Near the horizon we may expand

$$f = \underbrace{f(r_+)}_{0} + \underbrace{(r - r_+)}_{\Delta r}\underbrace{f'(r_+)}_{2\kappa} + \cdots = 2\kappa\Delta r\,. \tag{5.27}$$

Therefore, the near horizon limit of the 'Euclidean Schwarzschild solution' takes the following form:

$$ds^2 = 2\kappa\Delta r d\tau^2 + \frac{dr^2}{2\kappa\Delta r} + r_+^2 d\Omega^2\,. \tag{5.28}$$

We can now introduce a new coordinate ρ by

$$d\rho^2 = \frac{dr^2}{2\kappa\Delta r} \quad\Leftrightarrow\quad d\rho = \frac{dr}{\sqrt{2\kappa\Delta r}} \quad\Leftrightarrow\quad \Delta r = \frac{\kappa}{2}\rho^2\,, \tag{5.29}$$

getting

$$ds^2 = \kappa^2\rho^2 d\tau^2 + d\rho^2 + r_+^2 d\Omega^2\,. \tag{5.30}$$

The first two terms just look like what we had for the Rindler space, (5.21). By repeating the same steps we just conclude that

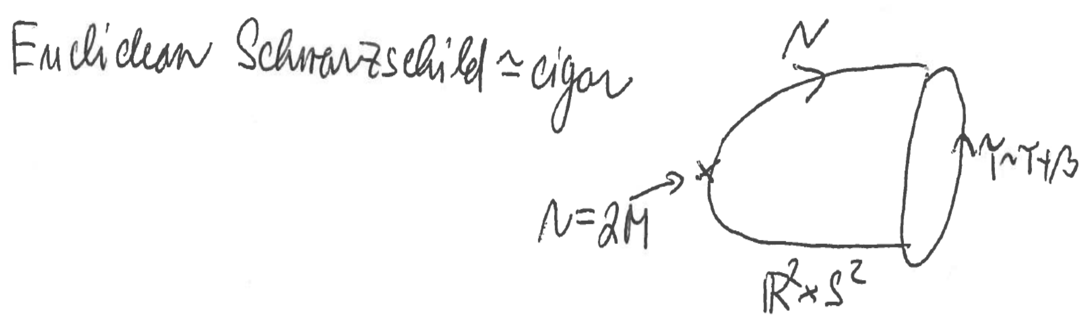
$$\boxed{T = \frac{\kappa}{2\pi}\,,} \tag{5.31}$$

and in particular,

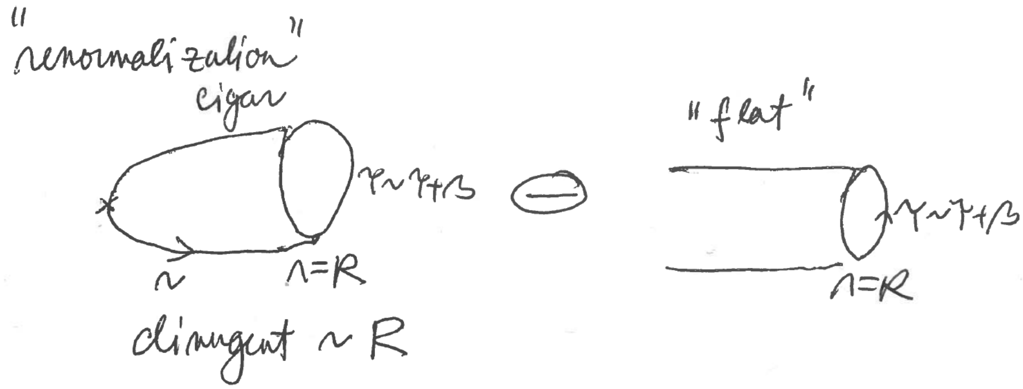
$$\boxed{T = \frac{1}{8\pi M}} \tag{5.32}$$

for the Schwarzschild solution.

- Let us now calculate the action. To this purpose we notice that the Euclidean Schwarzschild solution (5.26) looks like a surface of a *cigar*, where each point corresponds to a suppressed sphere:



Again, we have $R = 0$ and the first term vanishes. It is the second term that determines the value of the action. We introduce a boundary at $r = R_0$, calculate the contribution of the second term and then let $R_0 \to \infty$. Unfortunately, the second term then diverges and we have to renormalize as displayed in the figure (see Gravitational Physics course for more details):



So we get

$$S_E = S_E(\text{cigar}) - S_E(\text{flat}) = \frac{\beta M}{2} \,. \tag{5.33}$$

Free energy then becomes

$$F = -\frac{1}{\beta} \log Z = M/2 \,, \tag{5.34}$$

giving the following entropy:

$$S = -\frac{\partial F}{\partial T} = \left| T = \frac{1}{8\pi M} \right| = \frac{1}{16\pi T^2} = \pi r_+^2 = \frac{A}{4} \,, \tag{5.35}$$

confirming Bekenstein's result.

5.2 Hawking evaporation and information paradox

- Hawking radiation is a kinematic effect. (One needs equivalence principle, vacuum fluctuations, but the Einstein equations are not required.) This opens a possibility for observing this effect in '*analogue systems*', e.g. surface water waves.

- Black holes do not radiate as true black body as some waves 'scatter back' to the horizon. For this reason the corresponding distribution reads

$$\langle n_\omega \rangle = \frac{\Gamma_\omega}{e^{\omega/T} - 1} \,, \tag{5.36}$$

where Γ_ω is the so called *greybody factor*.

Correspondingly, the black hole looses mass according to the 'effective' Stefan–Boltzmann law

$$\frac{dM}{dt} \propto -\sigma T^4 A \propto -\frac{1}{M^2} \,, \tag{5.37}$$

so that it would completely evaporate in (M_S denoting the mass of the Sun)

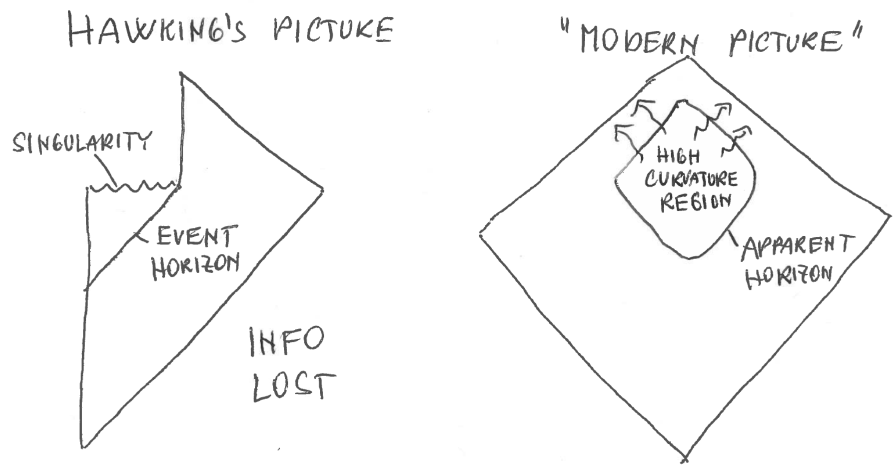
$$t_{\text{evap}} \approx \left(\frac{M}{M_S}\right)^3 \times 10^{71}\, s\,, \tag{5.38}$$

see tutorial for more details.

- Note also that since
$$T = \frac{1}{8\pi M}\,, \tag{5.39}$$
smaller the black hole is the hotter it is. This implies that the evaporation accelerates and towards the end we can observe 'black hole explosions' (CERN?)

  This also means that Schwarzschild black hole has a *negative specific heat*:
$$C = T\frac{\partial S}{\partial T} = -\frac{1}{8\pi T^2}\,. \tag{5.40}$$
(This is quite typical for self-gravitating systems. For example, a sattelite as it falls it increases its kinetic energy; a gravothermal catastrophe described by Lynden Bell.)

- Black hole information paradox: to be written!



Appendix A: More on Differential Geometry

A.1 Lie derivative

- Differentiation of tensors on M is problematic, c.f.,

$$\frac{df}{dt}\Big|_{t_0} = \lim_{s\to 0} \frac{f(t_0+s)-f(t_0)}{s}\,.$$

  On M, one might want to replace t_0 by p. However how to add s to p? And how to compare a vector at $p+\delta p$ to a vector at p when they live on a different space?

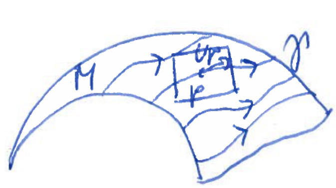
- To resolve these issues one needs an additional structure. We have 3 standard possibilities:

  i) *Lie derivative* (vector field U).

  ii) *Exterior derivative* (forms).

  iii) *Covariant derivative* (connection Γ).

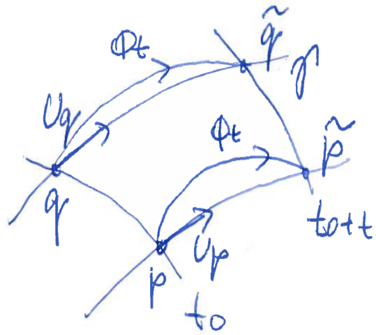
- A vector field U defines its <u>*integral curves*</u> on M (their tangent vector coincides with U_p for all $p \in M$):



proof: in coordinates x^μ

$$\frac{dx^\mu}{dt} = U^\mu(x) \quad \text{has unique sol.}$$

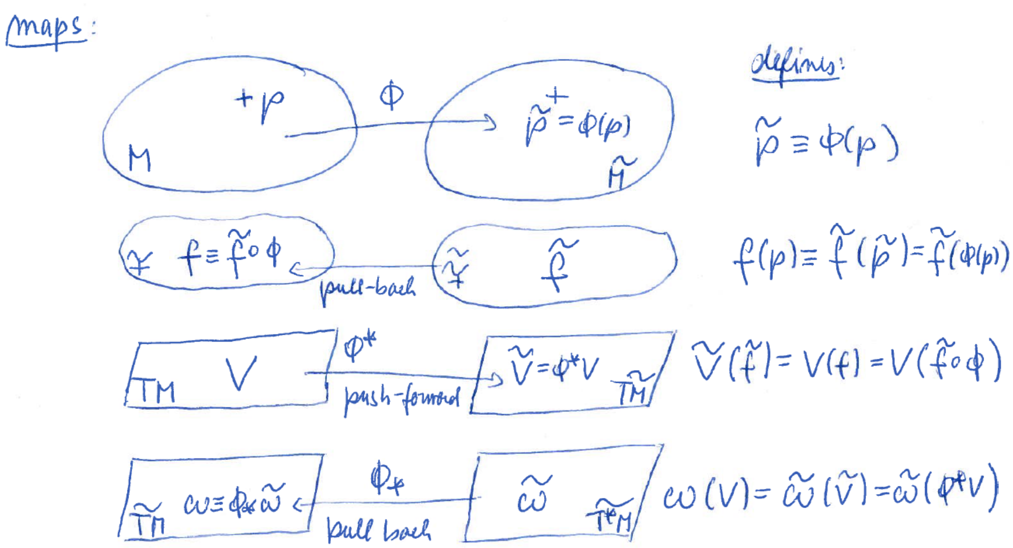
  This defines a map $\phi_t : M \to M$ by $\phi_t : \gamma(t_0) \to \gamma(t_0+t)$.



Φ_t: • t continuous
• $\Phi_0 = \mathrm{Id}$, $\Phi_{t+s} = \Phi_t \circ \Phi_s$, $\Phi_{-t} = \Phi_t^{-1}$
defines 1-parametric group (Lie) of diffeomorphisms

- Maps between manifolds. Let M and $\tilde{M}$ be two manifolds and $\phi : M \to \tilde{M}$ a smooth map. The we find the following induced maps:

maps:



Note that for a given type of the object, the map always works only one way (one cannot define it other way). However, a very special case happens when ϕ is a diffeomorphism.

- *A diffeomorphism* is a map $\phi : M \to \tilde{M}$: 1-1, onto, ϕ^{-1} is smooth. In this case, one can use ϕ^{-1} to define above maps *both ways*. In particular one can define an "overloaded operation":

$$\phi^* : \text{tensors on } M \to \text{tensors on } \tilde{M}\,. \tag{A.1}$$

- **Definition.** Let ϕ_t be a 1-parametric group of diffeomorphisms generated by a vector field U. Then the *Lie derivative* $\mathcal{L}_U$ w.r.t. U is defined as

$$\boxed{\mathcal{L}_U T\Big|_p = \lim_{t\to 0} \frac{T|_p - \phi_t^* T|_p}{t}\,.} \tag{A.2}$$

"WHAT WAS THERE MINUS WHAT I TRANSPORTED THERE".

Example:

$$\mathcal{L}_U f = \lim_{t\to 0}\frac{f(t_0)-\tilde{f}(t_0)}{t} = \Big|\tilde{f}(t_0)=f(t_0-t)\Big| = \frac{df}{dt} = \frac{dx^\mu}{dt}\frac{\partial f}{\partial x^\mu} = U^\mu\frac{\partial f}{\partial x^\mu} = U(f)\,, \tag{A.3}$$

where we have used the definition of the integral curve.

- Properties:

  i) $\mathcal{L}_U$ maps (r,s) tensors to (r,s) tensors.

  ii) $\mathcal{L}_U$ is linear and preserves contraction.

  iii) Leibnitz: $\mathcal{L}_U(T\otimes S) = (\mathcal{L}_U T)\otimes S + T\otimes(\mathcal{L}_U S)$.

  iv) We have the following expressions for the Lie derivative of a function f and a vector V:

  $$\mathcal{L}_U f = Uf = U^\mu\frac{\partial f}{\partial x^\mu}\,,\quad \mathcal{L}_U V = [U,V] = UV - VU\,,$$

  where the latter is called a *Lie bracket*.

  v) For components of a general tensor we then find

  $$\mathcal{L}_U T^{\alpha\dots}{}_{\beta\dots} = U^\gamma\frac{\partial}{\partial x^\gamma}T^{\alpha\dots}{}_{\beta\dots} - T^{\gamma\dots}{}_{\beta\dots}\frac{\partial}{\partial x^\gamma}U^\alpha + \dots + T^{\alpha\dots}{}_{\gamma\dots}\frac{\partial}{\partial x^\beta}U^\gamma\,. \tag{A.4}$$

The explicit expressions for the Lie derivative are easily obtained by using the *passive* rather than the *active* approach to diffeomorphisms. In the passive approach we simply interpret the associated 'map' as a coordinate transformation.

- Symmetries. The Lie derivative plays a key role for defining symmetries. Namely, it may happen that for a given tensor field T one can find such a vector field U so that

$$\mathcal{L}_U T^{\alpha\dots}{}_{\beta\dots} = 0\,. \tag{A.5}$$

This means that vector U describes a special direction in the manifold along which the tensor T 'stays the same'—it describes a *symmetry* of a given tensor field T. You may be familiar with a particular case of symmetries of the metric, $\mathcal{L}_U g_{\alpha\beta} = 0$, called *isometries*. We shall see some other examples soon in this course.

A.2 Differential forms

- **Definition.** A *differential p-form* ω is a totally antisymmetric tensor of type $(0,p)$, that is,

$$\omega_{\alpha_1\dots\alpha_p} = \omega_{[\alpha_1\dots\alpha_p]} = \frac{1}{p!}\sum_{\mathrm{perm}\,\pi}\mathrm{sign}(\pi)\omega_{\alpha_{\pi(1)}\dots\alpha_{\pi(p)}}\,. \tag{A.6}$$

Hence, a differential form is antisymmetric under exchange of any 2 indices. We shall denote Λ_x^p a vector space of p-forms at x. One can show that it has a dimensionality $dim\Lambda_x^p = \binom{n}{p}$.

- **Definition.** A <u>*wedge product*</u> $\wedge : \Lambda_x^p \times \Lambda_x^q \to \Lambda_x^{p+q}$:

$$(\omega \wedge \nu)_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q} = \frac{(p+q)!}{p!q!} \omega_{[\alpha_1 \dots \alpha_p} \nu_{\beta_1 \dots \beta_q]} \,. \tag{A.7}$$

That is, $\omega \wedge \nu$ is a $(p+q)$-form. It obeys

$$\omega \wedge \nu = (-1)^{pq} \nu \wedge \omega \,. \tag{A.8}$$

Since dx^α is a coordinate basis of 1-forms, general p-form can be written as

$$\boxed{\omega = \frac{1}{p!} \omega_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p} \,.} \tag{A.9}$$

- For any vector V, we define an <u>*inner derivative*</u> $i_V : \Lambda^p \to \Lambda^{p-1}$:

$$i_V \omega = V \rfloor \omega = V \cdot \omega = \omega(V, \dots) : \qquad (V \rfloor \omega)_{\alpha_1 \dot{\alpha}_{p-1}} = V^\beta \omega_{b \alpha_1 \dots \alpha_{p-1}} \,. \tag{A.10}$$

<u>Properties of inner derivative:</u>

i) i_V is linear
ii) i_V is linear in V : $i_{fV+gW} = f i_V + g i_W$.
iii) graded Leibnitz rule: For $\omega \in \Lambda^p$ we have

$$i_V(\omega \wedge \nu) = (i_V \omega) \wedge \nu + (-1)^p \omega \wedge i_V \nu \,. \tag{A.11}$$

iv)

$$i_v i_W + i_W i_V = 0 \quad \text{spec.} \quad i_V^2 = 0 \,. \tag{A.12}$$

- **Definition.** <u>*Exterior derivative*</u> $d : \Lambda^p \to \Lambda^{p+1}$ is defined as follows:

i) On a function f we have $d : f \to df = \frac{\partial f}{\partial x^\alpha} dx^\alpha$.
ii) On a p-form ω we then have

$$d : \omega \to d\omega = \frac{1}{p!} d\omega_{\alpha_1 \dots \alpha_p} \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p} \,. \tag{A.13}$$

That is $(d\omega)_{\alpha_1 \dots \alpha_{p+1}} = (p+1) \partial_{[\alpha_1} \omega_{\alpha_2 \dots \alpha_{p+1}]}$.
Note that we have $d^2 = 0$. Conversely, a p-form α is called <u>*closed*</u> when $d\alpha = 0$. It is called <u>*exact*</u> when $\alpha = d\beta$. Any closed form α can be *locally* written as $\alpha = d\beta$ but not *globally*.[1]

[1]Dimension of a vector space of closed p-forms modulo the exact p-forms equals p-th <u>*Betti number*</u> of the manifold and is a topological quantity.

- *Cartan's lemma.* For a vector field V and a p-form ω, we have the following identity:

$$\boxed{\mathcal{L}_V \omega = V \lrcorner\, d\omega + d(V \lrcorner\, \omega)\,.} \tag{A.14}$$

In particular, this implies that

$$\mathcal{L}_V df = d\mathcal{L}_V f\,. \tag{A.15}$$

- Integration. A p-form ω can be integrated over a p-dimensional (sub)manifold. Writing $\omega = f dx^1 \wedge \ldots dx^p$ we then define

$$\int_{O_p} \omega = \int_{\psi(O_p)} f dx^1 \ldots dx^p \quad \text{where r.h.s. is defined as Lebesgue integral.} \tag{A.16}$$

Note that this definition is independent of coordinates, as we have

$$\omega = f' dx'^1 \wedge \cdots \wedge dx'^p\,, \quad f' = f \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right). \tag{A.17}$$

Stokes theorem. The following identity is valid

$$\boxed{\int_\Omega d\omega = \int_{\partial\Omega} \omega\,.} \tag{A.18}$$

Many of the differential vector identities you know are a special case of this beautiful theorem.

If you want to know more about differential geometry, I refer you to an excellent (but concise) book [12].

Appendix B: Perturbative Approach to GR

So far we have discussed the standard geometric approach to GR. Let us now introduce this theory as a self-consistent special relativistic field theory of massless spin-2 particle.

Basic Idea

- The field describing a spin-2 particle in a Minkowski spacetime $\eta_{\mu\nu}$ is a symmetric rank-2 Lorentz tensor $h_{\mu\nu}$.
- Let's propose the following EOM for the free field:

$$\partial^2 h^{\mu\nu} = 0\,. \tag{B.1}$$

  Problem: as shown by Weyl, we further need to impose

$$\partial^\mu (h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h) = 0\,, \quad h = h^\mu{}_\mu = 0\,, \tag{B.2}$$

  to remove spin-1 and spin-0 helicity states and to ensure the positive definite energy.

- Matter couples to gravity through its energy momentum tensor

$$\partial^2 h^{\mu\nu} = \chi T^{\mu\nu}_m\,, \tag{B.3}$$

  where χ is a coupling constant. On account of the constraints such $T^{\mu\nu}_m$ has to be symmetric and divergence free.

- We would like to find a gauge theory, which allows one to impose the constraints (B.2). Gauge-invariant equations of motion are

$$\boxed{D^{\mu\nu}(h) = \chi T^{\mu\nu}_m\,,} \tag{B.4}$$

  and have to reduce to (B.3) upon gauge fixing. We also have to have

$$\boxed{\partial_\mu D^{\mu\nu}(h) = 0\,,} \tag{B.5}$$

  as the off-shell Bianchi identity.

Fierz–Pauli Theory

To construct the above gauge theory let us i) write the most general Lorentz invariant action quadratic in $\partial_\rho h_{\mu\nu}$ and constraint it by imposing the gauge identity (B.5), $\partial_\mu D^{\mu\nu}(h) = 0$ and ii) search for its gauge invariance.

i) Writing

$$\mathcal{L} = \frac{1}{4}\partial^\rho h^{\mu\nu}\partial_\rho h_{\mu\nu} + a\partial^\mu h^{\nu\rho}\partial_\nu h_{\mu\rho} + b\partial^\mu h\partial^\nu h_{\nu\mu} + c\partial^\mu h\partial_\mu h\,, \tag{B.6}$$

where we normalized the "kinetic term", by imposing (B.5), we arrive at the *Fierz–Pauli action*

$$\boxed{S_{FP} = \int d^dx\Big(\frac{1}{4}\partial^\rho h^{\mu\nu}\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial^\mu h^{\nu\rho}\partial_\nu h_{\mu\rho} + \frac{1}{2}\partial^\mu h\partial^\nu h_{\nu\mu} - \frac{1}{4}\partial^\mu h\partial_\mu h\Big)\,.} \tag{B.7}$$

With this action we have

$$\delta S_{FP} = -\frac{1}{2}\int d^dx D^{\mu\nu}\delta h_{\mu\nu}\,, \tag{B.8}$$

where

$$D^{\mu\nu} = \partial^2 h^{\mu\nu} + \partial^\mu\partial^\nu h - 2\partial_\lambda\partial^{(\mu}h^{\nu)\lambda} - \eta^{\mu\nu}(\partial^2 h - \partial_\lambda\partial_\sigma h^{\lambda\sigma})\,. \tag{B.9}$$

ii) If $\delta h_{\mu\nu}$ is a gauge transformation we must have

$$\delta S_{FP} \propto \int d^dx D^{\mu\nu}\delta h_{\mu\nu} \propto \int d^dx \partial_\mu D^{\mu\nu}\epsilon_\nu\,. \tag{B.10}$$

That is, the gauge parameter $\epsilon_\mu(x)$ is a local Lorentz vector and we have a gauge symmetry:

$$\boxed{\delta_\epsilon h_{\mu\nu} = -2\partial_{(\mu}\epsilon_{\nu)}\,.} \tag{B.11}$$

This gauge freedom allows one to remove $2d$ dof, leaving

$$\# = \frac{1}{2}d(d+1) - 2d = \frac{1}{2}d(d-3)\ \text{dof}\,, \tag{B.12}$$

of a massless spin-2 particle in d dimensions.

There are 2 popular gauge choices: i) *transverse, traceless gauge:*

$$\partial_\mu h^{\mu\nu} = 0\,, \quad h = 0\,, \tag{B.13}$$

yielding $D_{\mu\nu}(h) = \partial^2 h_{\mu\nu} = 0$ and *De Donder* or *harmonic gauge*

$$\partial_\mu \bar{h}^{\mu\nu} = 0\,, \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\,, \tag{B.14}$$

yielding $D_{\mu\nu}(h) = \partial^2\bar{h}_{\mu\nu} = 0$.

Coupling to Matter

- Writing the metric as

$$\gamma_{\mu\nu} = \eta_{\mu\nu} + \chi h_{\mu\nu} \,, \tag{B.15}$$

we can expand

$$\begin{aligned} S &= S_{FP} + S_m[\phi,\eta] + \frac{\chi}{2}\int d^dx h_{\mu\nu} \underbrace{2\frac{\delta S_m[\phi,\gamma]}{\delta\gamma_{\mu\nu}}\Big|_{\gamma=\eta}}_{T_m^{\mu\nu}} \\ & \quad +\chi^2 \int d^dx d^dx' h_{\mu\nu}(x) h_{\rho\sigma}(x') \frac{\delta^2 S_m}{\delta\gamma_{\mu\nu}\delta\gamma_{\rho,\sigma}}\Big|_{\gamma=\eta} + \dots \,. \end{aligned} \tag{B.16}$$

Truncating at first order we obtain "minimal coupling to gravity"

$$\boxed{\mathcal{L} = \mathcal{L}_{FP} + \mathcal{L}_m(\phi) + \frac{1}{2}\chi h_{\mu\nu} T_m^{\mu\nu}(\phi) \,,} \tag{B.17}$$

which upon variation w.r.t. $\delta h_{\mu\nu}$ yields (B.4).

- *Problems.* This theory has several problems:

  1. Predicts wrong perihelion shift for Mercury.
  2. Is no longer *self-consistent.* Due to the presence of interaction term, the new energy momentum tensor of matter, $T_m^{\mu\nu}(\phi,h)$, is no longer conserved.
  3. Moreover, gravity should also couple to itself through its own energy momentum tensor $t^{\mu\nu}$ (Lorentz tensor), see below. So the consistency would require

  $$\boxed{D^{\mu\nu}(h) = \chi(T_m^{\mu\nu}(\phi,h) + t^{\mu\nu}) \,,} \tag{B.18}$$

  while on-shell we have

  $$\boxed{\partial_\mu(T_m^{\mu\nu}(\phi,h) + t^{\mu\nu}) = 0 \,.} \tag{B.19}$$

  That is, only the total energy-momentum tensor of matter plus gravity plus interaction should be conserved. Note also that, following the principle of equivalence, we use the same universal coupling constant χ, both for coupling to matter and self-coupling to gravitational field.

  Theorem (Weinberg, Boulware, Deser). *Quantum massless spin-2 theory can have a Lorentz invariant S-matrix only if it couples to the total energy momentum, including the gravitational energy momentum whose form in the infrared limit is the one predicted by general relativity.*

  In other words, any interacting quantum theory of a spin-2 particles coincides with general relativity in the infrared limit.

- One can show that a first correction to (B.17) of the type

$$\mathcal{L}^{(1)} = \frac{1}{2}\chi h^{\mu\nu}\mathcal{L}^{(1)}_{\mu\nu} \tag{B.20}$$

can be found so that the perihelion calculation works. However, the theory is still not self-consistent: $t^{\mu\nu}$ obtained is not invariant under the same gauge transformations as modified Lagrangian; an infinite series of corrections is required.

Deser's argument

It can be shown that general relativity can be obtained by resumming the infinite perturbative series, using the following trick.

- Let us start with the 1st-order actin of two (off-shell) independent fields: the auxiliary field $\Gamma_{\mu\nu}{}^{\rho}$ and the 'gravitational field' $\varphi^{\mu\nu}$:

$$\boxed{S^{(1)}_{FP}[\varphi,\Gamma] = \frac{1}{\chi^2}\int d^dx\Big(-\chi\varphi^{\mu\nu}2\partial_{[\mu}\Gamma_{\rho]\nu}{}^{\rho} + \eta^{\mu\nu}2\Gamma_{\lambda[\mu}{}^{\rho}\Gamma_{\rho]\nu}{}^{\lambda}\Big)\,.} \tag{B.21}$$

The variation $\delta\Gamma_{\mu\nu}{}^{\rho}$ gives an algebraic equation for Γ's, which in terms of the new field

$$h_{\mu\nu} = \varphi_{\mu\nu} - \frac{1}{d-2}\eta_{\mu\nu}\varphi\,, \tag{B.22}$$

yields the standard definition of Christoffel's: $\Gamma_{\rho\mu\nu} = \frac{1}{2}\chi(\partial_\rho h_{\mu\nu} + \partial_\mu h_{\nu\rho} - \partial_\nu h_{\rho\mu})$. Substituting to EOM for $\varphi^{\mu\nu}$ then gives

$$\frac{\delta S^{(1)}}{\delta\varphi^{\mu\nu}} = -\frac{1}{2}\Big(D_{\mu\nu}(h) - \frac{1}{d-2}\eta_{\mu\nu}D(h)\Big) = 0\,. \tag{B.23}$$

The action is thence equivalent to the FP action.

- We want to find a correction $S^{(2)}$ so that

$$\boxed{D_{\mu\nu}(h) = \chi t_{\mu\nu}\,,} \tag{B.24}$$

where $t_{\mu\nu}$ is the energy momentum tensor of $\varphi^{\mu\nu}$ in $S^{(1)}$:

$$t_{\alpha\beta} = -\frac{2}{\sqrt{|\gamma|}}\frac{\delta S^{(1)}}{\delta\gamma^{\alpha\beta}}\Big|_{\gamma=\eta} = -\frac{2}{\chi}\Gamma_{\lambda[\alpha}{}^{\rho}\Gamma_{\rho]\beta}{}^{\lambda} + \text{total derivative}\,, \tag{B.25}$$

where we treated $\varphi^{\alpha\beta}$ as energy density already containing $\sqrt{\gamma}$. (Note that such gravitational energy-momentum can be 'expected' based on an analogy with EM field.) This gives

$$\boxed{S^{(2)} = \frac{1}{\chi^2}\int d^dx\big(-2\chi\varphi^{\alpha\beta}\Gamma_{\lambda[\alpha}{}^{\rho}\Gamma_{\rho]\beta}{}^{\lambda}\big)\,.} \tag{B.26}$$

This is a cubic in fields term that does not contribute to the energy momentum, as there is no η to be replaced by γ.

- In terms of a new field $g^{\mu\nu}$

$$\sqrt{|g|}g^{\mu\nu} = \eta^{\mu\nu} - \chi\varphi^{\mu\nu}\,, \tag{B.27}$$

  the EOM for Γ's yield the Christoffel expression. Note that this is an infinite series in terms of the original variable $\varphi^{\mu\nu}$ (we use inverse metric). This means that the correction Lagrangian (B.26) gives an infinite series of corrections to the original theory in terms of the original field $\varphi^{\mu\nu}$. At the same time the EOM for $\varphi^{\mu\nu}$ yields the Einstein equation

$$\boxed{R_{\mu\nu}(g) = 0\,,} \tag{B.28}$$

  which is equivalent to (B.24).

- *Final remarks*:

  1. In geometric approach there is no general covariant energy momentum tensor of gravitational field. There is only Lorentz covariant energy momentum tensor $t \propto \Gamma^2$ (embedded in the Ricci tensor). This obscures the physical interpretation of vacuum solutions—these are filled with gravitational field that acts as a source for itself.
  2. Theory now has a gauge symmetry

$$\delta_\epsilon g_{\mu\nu} = -2\nabla_{(\mu}\epsilon_{\nu)}\,. \tag{B.29}$$

  3. To obtain the standard Einstein–Hilbert action, one has to add a total derivative term

$$S^{(0)} = \frac{1}{\chi^2}\int d^dx\big(2\eta^{\mu\nu}\partial_{[\mu}\Gamma_{\rho]\nu}{}^{\rho}\big)\,. \tag{B.30}$$

  4. The FP action can be extended to any background which is an Einstein space. (We want to characterize massless spin-2 by the presence of a gauge symmetry that can remove the extra dof. For the covariant FP action need Einstein equations for the background to show such a gauge symmetry.)

Bibliography

[1] S. M. Carroll, *Spacetime and geometry. An introduction to general relativity.* 2004.

[2] J. B. Hartle, *Gravity: An introduction to einstein's general relativity*, 2003.

[3] L. D. Landau and E. Lifshitz, *Statistical physics, part i*, 1980.

[4] A. Zee, *Einstein gravity in a nutshell.* Princeton University Press, 2013.

[5] E. Poisson, *A relativist's toolkit: the mathematics of black-hole mechanics.* Cambridge university press, 2004.

[6] T. Ortín, *Gravity and strings.* Cambridge University Press, 2004.

[7] R. M. Wald, *General relativity.* University of Chicago press, 2010.

[8] C. Rovelli, *Quantum gravity.* Cambridge university press, 2007.

[9] M. de Cesare, M. Sakellariadou and P. Vitale, *Noncommutative gravity with self-dual variables*, 1806.04666.

[10] D. Lovelock, *The einstein tensor and its generalizations*, *Journal of Mathematical Physics* **12** (1971) 498–501.

[11] L. Susskind, *Computational Complexity and Black Hole Horizons*, *Fortsch. Phys.* **64** (2016) 44–48, [1403.5695].

[12] M. Göckeler and T. Schücker, *Differential geometry, gauge theories, and gravity.* Cambridge University Press, 1989.