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HOMEWORK 1

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1 Bead on a rotating wire

A point particle of mass m is constrained to move on a parabola rotating with angular velocity ω around a z axis aligned with the opposite of a gravitationnal field of intensity g . The parabola is described in cylindrical coordinates ρ, ϕ, z by the relations $z = \alpha\rho^2$ and $\phi = \omega t$.

A Lagrangian

Following the constraints imposed on the particle, its cartesian position can be written as the following

$$x = \rho \cos(\omega t), \quad y = \rho \sin(\omega t), \quad \text{and} \quad z = \alpha\rho^2$$

leaving ρ as the only free parameter describing the motion of the particle. The cartesian velocity of the particle is then given by the time derivative of its cartesian coordinates. We have

$$\dot{x} = \dot{\rho} \cos(\omega t) - \omega\rho \sin(\omega t), \quad \dot{y} = \dot{\rho} \sin(\omega t) + \omega\rho \cos(\omega t), \quad \text{and} \quad \dot{z} = 2\alpha\rho\dot{\rho}.$$

The square magnitude of the cartesian velocity of the particle can be written in cylindrical coordinates as

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= (\dot{\rho} \cos(\omega t) - \omega\rho \sin(\omega t))^2 + (\dot{\rho} \sin(\omega t) + \omega\rho \cos(\omega t))^2 + (2\alpha\rho\dot{\rho})^2 + (2\alpha\rho\dot{\rho})^2 \\ &= \dot{\rho}^2 \cos^2(\omega t) + \omega^2 \rho^2 \sin^2(\omega t) + \dot{\rho}^2 \sin^2(\omega t) + \omega^2 \rho^2 \cos^2(\omega t) \\ &\quad + 2\dot{\rho}\rho\omega \cos(\omega t) \sin(\omega t) - 2\dot{\rho}\rho\omega \cos(\omega t) \sin(\omega t) + 4\alpha^2 \rho^2 \dot{\rho}^2 \\ &= \dot{\rho}^2 + \omega^2 \rho^2 + 4\alpha^2 \rho^2 \dot{\rho}^2. \end{aligned}$$

Following from this expression, the kinetic energy reads $T = \frac{1}{2}m(\dot{\rho}^2 + \omega^2 \rho^2 + 4\alpha^2 \rho^2 \dot{\rho}^2)$. The potential energy du to gravitationnal interaction is given by $V = mgz = mg\alpha\rho^2$. We can finally write the lagrangian

$$L = T - V = \frac{1}{2}m((1 + 4\alpha^2 \rho^2)\dot{\rho}^2 + \omega^2 \rho^2) - mg\alpha\rho^2.$$

B Time Translation Symmetry Conserved Charge

Since $\frac{\partial L}{\partial t} = 0$, Noether's theorem ensures that

$$\begin{aligned} E = \frac{\partial L}{\partial \dot{\rho}} \dot{\rho} - L &= m(1 + 4\alpha^2)\dot{\rho}^2 - \frac{1}{2}m((1 + 4\alpha^2 \rho^2)\dot{\rho}^2 + \omega^2 \rho^2) + mg\alpha\rho^2 \\ &= \frac{1}{2}m(1 + 4\alpha^2 \rho^2)\dot{\rho}^2 + \underbrace{m\rho^2 \left(g\alpha - \frac{1}{2}\omega^2 \right)}_{U_{\text{eff}}} \end{aligned}$$

is a conserved charge on physical trajectories which corresponds to an effective energy as it is associated to time translation symmetry of L . The potential-like term in E can be written as $U_{\text{eff}} = \frac{1}{2}m\rho^2(\omega_0^2 - \omega^2)$ where $\omega_0 = \sqrt{2ga}$.

C

Effective Potential

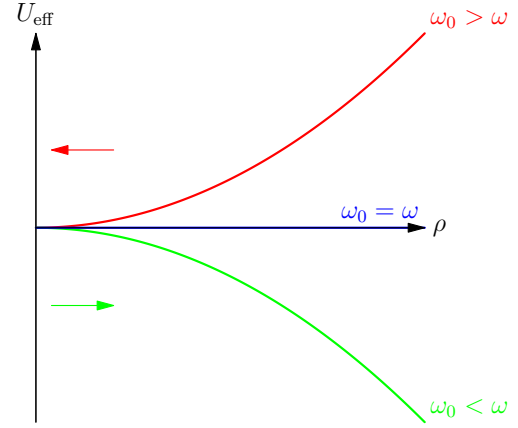
The qualitative dynamics of the system can be described with U_{eff} by considering the three following regimes:

$\omega_0 > \omega$ Gravity is stronger than rotation and the particle trapped in an harmonic potential. Note that when it oscillates through " $\rho = 0$ " it doesn't stop. Indeed, " $\rho = 0$ " is not included in the coordinate chart provided by spherical coordinates. In reality, nothing special happens at the bottom of the parabola and the particle continues to the other side;

$\omega_0 = \omega$ First consider $0 = E = \frac{1}{2}m(1 + 4\alpha^2\rho^2)\dot{\rho}^2$ which implies $\dot{\rho} = 0$. Through the constraint this means a fixed height. In this situation gravity balances the centripetal and parabola normal forces resulting in a net force accounting only for uniform rotation motion at fixed z . This is an unstable equilibrium. Indeed any perturbation will increase the energy beyond 0 and lead to the second subcase. If $E > 0$, $\frac{1}{2}m(1 + 4\alpha^2\rho^2)\dot{\rho}^2 > 0$ and $|\dot{\rho}| > 0$. This means that if $\dot{\rho} < 0$ initially, the particle will get lower on the parabola reaching $\rho = 0$ and escaping to infinity in the other branch of the parabola. Furthermore, in the effective picture (reference frame rotating with the parabola), the coefficient $(1 + 4\alpha^2\rho^2)$ acts as an effective mass. It becomes smaller near $\rho = 0$ and keeping a fixed energy means speeding up (the opposite is true if the particle moves away from $\rho = 0$);

$\omega_0 < \omega$ This situation corresponds to an unbounded growth of z . The effective potential represents the effect of the centrifugal effect pushing the particle away from the origin more than gravity is pulling it. If $E < 0$, the particle can't get arbitrarily close to $\rho = 0$ and bounces to infinity at $\rho \neq 0$ preventing the kinetic energy term from becoming negative. If $E > 0$, the particle can go through $\rho = 0$ if it was initially directed that way and fly to infinity on the other side of the parabola. Furthermore, for $\rho \rightarrow \infty$ the effective mass becomes ρ^2 and the kinetic energy behaves like $E - U_{\text{eff}} - U_{\text{eff}} \rho^2$. This means that $\rho^2 \dot{\rho}^2 \rho^2$ or, equivalently, that the velocity saturates. This happens because the kinetic energy gain becomes fully attributed to mass increase at sufficient ρ .

In each case the arrows indicate the direction of the force which is constant in each regime.



D Discussion

The total energy of the particle is given by

$$E_0 = T + V = \frac{1}{2}m((1 + 4\alpha^2\rho^2)\dot{\rho}^2 + \omega^2\rho^2) + mg\alpha\rho^2 = E + m\rho^2\omega^2.$$

The conservation of E implies $\dot{E}_0 = \dot{E} + 2m\rho\dot{\rho}\omega^2 = 2m\rho\dot{\rho}\omega^2$. From this, we see that E_0 is conserved on shell iff $\dot{\rho} = 0$ or $\rho = 0$. In the former case, ρ is a constant and this can only be realised if $\omega_0 = \omega$ or $\rho = 0$ (no effective force). In the latter case, $\rho = 0$ which is a subcase of $\dot{\rho} = 0$. If $\omega_0 < \omega$, the parabola is rotating fast enough to go beyond the gravitationnal pull of the particle towards $\rho = 0$ and the velocity of the particle grows without bounds (the motor rotating the parabola is transferring energy to the particle and E_0 is not conserved).

2 Brachistochrone

Consider a particle subject to a uniform gravitational field of strength g oriented in the negative y direction of a planar Oxy cartesian coordinate system. The particle starts at point A of coordinates x_A, y_A and ends at point B of coordinates x_B, y_B . We are interested in the curve linking A and B minimizing the travel time of the particle.

A Travel time functionnal

The first step in constructing the travel time functionnal is the parametrisation of its curve argument. Suppose the trajectory is parametrised by y so that it takes the form of a function $x(y)$. The length element at x is $dl = \sqrt{\dot{x}(y)^2 + \dot{y}^2} dy = \sqrt{1 + \dot{x}(y)^2} dy$ with $(\dot{}) = \frac{d}{dy}$. By a mass independant analogue of the conservation of energy $0 = \frac{1}{2}v^2 - gy$ (with "potential" energy zero at y_B) where v is the velocity, we have $v = \sqrt{2gy}$. For motion along an element dl of the curve, the time elapsed is $dt = dl / \sqrt{2gy}$ leading to the following travel time functionnal

$$T[x(y)] = \int_{y_A}^{y_B} dy \frac{\sqrt{1 + \dot{x}(y)^2}}{\sqrt{2gy}}.$$

B Conserved quantity

Since the integrand (effective lagrangian $L = \sqrt{1 + \dot{x}(y)^2} / \sqrt{2gy}$) of the previous functionnal is cyclic in x , the Euler-Lagrange equations imply

$$\frac{d}{dy} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0 \implies Q = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}(y)}{\sqrt{1 + \dot{x}(y)^2} \sqrt{2gy}} \quad \text{is conserved on the Brachistochrone.}$$

We can now write

$$2gyQ^2(1 + \dot{x}(y)^2) = \dot{x}(y)^2 \implies \dot{x}(y) = \sqrt{\frac{2gyQ^2}{1 - 2gyQ^2}}$$

and integrate \dot{x} to get

$$\begin{aligned} x(y) &= \int dy \sqrt{\frac{2gyQ^2}{1 - 2gyQ^2}} = \int du \frac{1}{gQ^2} \frac{u^2}{\sqrt{1 - u^2}}, \quad u = \sqrt{2gyQ^2}, \quad du = \frac{2gQ^2}{2u} dy \\ &= \int du \frac{1}{gQ^2} \frac{u^2}{\sqrt{1 - u^2}} = -\frac{1}{gQ^2} \int \sin(\theta/2) d\theta \frac{\cos^2(\theta/2)}{\sqrt{1 - \cos^2(\theta/2)}} = -\frac{1}{2gQ^2} \int \text{sgn}(\sin(\theta/2)) \cos^2(\theta/2) d\theta, \quad u = \cos(\theta/2) \\ &= -\frac{1}{2gQ^2} \frac{1}{2} (\theta(y) + \sin(\theta(y))) + C, \quad \text{for } \theta \in [0, 2\pi), \sin(\theta/2) \geq 0. \end{aligned}$$

Using the new θ as an independant variable, we can write

$$\begin{aligned} x(\theta) &= -\frac{1}{2gQ^2} \frac{1}{2} (\theta + \sin(\theta)) + C \\ y(\theta) &= \frac{1}{2gQ^2} \cos^2(\theta/2) = \frac{1}{2gQ^2} \frac{1}{2} (1 + \cos(\theta)). \end{aligned}$$

To get the usual cycloid parametrisation, we set $\theta = \pi - \theta'$ and $C = \frac{\pi}{2gQ^2} \frac{1}{2}$ to obtain

$$\begin{aligned} x(\theta') &= -\frac{1}{2gQ^2} \frac{1}{2} (\pi - \theta' + \sin(\pi - \theta')) + \frac{\pi}{2gQ^2} \frac{1}{2} = \frac{1}{2gQ^2} \frac{1}{2} (\theta' - \sin(\theta')) \\ y(\theta') &= \frac{1}{2gQ^2} \frac{1}{2} (1 + \cos(\pi - \theta')) = \frac{1}{2gQ^2} \frac{1}{2} (1 - \cos(\theta')). \end{aligned}$$

where the radius of the cycloid is identified to be $a = \frac{1}{gQ^2}$. A relation between a and x_B, y_B can be obtained by considering the ratio of x and y (knowing $x_B \neq 0$) we write

$$\frac{y_B}{x_B} = \frac{1 - \cos(\theta'_B)}{\theta'_B - \sin(\theta'_B)}$$

which implicitly forms a transcendental expression for θ_B as a function of x_B, y_B . Knowing θ_B , we have

$$a = \frac{2y_B}{1 - \cos(\theta'_B(x_B, y_B))}.$$

3 A sneaky recipe for the Noether charge

A Alternate Disformation

Consider a particle described at time t by the generalized coordinate $q(t)$ evolving with respect to a lagrangian L . A disformation of the particle's history is given by $q(t) \mapsto q' = q(t) + \varepsilon \tilde{\delta}_s q(t)$ where $\delta \varepsilon$ is an infinitesimal constant. This disformation contributes to the action S with a boundary term if there exists a function R_s such that $\tilde{\delta}_s L = \frac{d}{dt} R_s$ (first order variation of L in ε).

B Variation of the Action

An alternative disformation of the history of the particle reads $q(t) \mapsto q'(t) = q(t) + \varepsilon f(t) \tilde{\delta}_s q(t)$ where f is a function vanishing at the start ($t = t_0$) and end ($t = t_1$) of the history. The action following from this disformation is

$$\begin{aligned}
 & S(q(t) + \varepsilon f(t) \tilde{\delta}_s q(t)) \\
 &= \int_{t_0}^{t_1} dt \, L(q(t) + \varepsilon f(t) \tilde{\delta}_s q(t), \dot{q}(t) + \varepsilon \frac{d}{dt}(f(t) \tilde{\delta}_s q(t)), t) \\
 &= S(q(t)) + \varepsilon \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} f(t) \tilde{\delta}_s q(t) + \frac{\partial L}{\partial \dot{q}} (\dot{f}(t) \tilde{\delta}_s q(t) + f(t) \tilde{\delta}_s \dot{q}(t)) \right] + O(\varepsilon^2) \\
 &= S(q(t)) + \varepsilon \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} f(t) \tilde{\delta}_s q(t) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \tilde{\delta}_s q(t) \right) f(t) + \frac{\partial L}{\partial \dot{q}} f(t) \tilde{\delta}_s \dot{q}(t) \right] + \left[\frac{\partial L}{\partial \dot{q}} f(t) \tilde{\delta}_s q(t) \right]_{t_0}^{t_1} + O(\varepsilon^2) \\
 &= S(q(t)) + \varepsilon \int_{t_0}^{t_1} dt \left[\left(\frac{\partial L}{\partial q} \tilde{\delta}_s q(t) + \frac{\partial L}{\partial \dot{q}} \tilde{\delta}_s \dot{q}(t) \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \tilde{\delta}_s q(t) \right) \right] f(t) + O(\varepsilon^2) \\
 &= S(q(t)) + \varepsilon \int_{t_0}^{t_1} dt \left[\frac{d}{dt} R_s - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \tilde{\delta}_s q(t) \right) \right] f(t) + O(\varepsilon^2)
 \end{aligned}$$

and its first order variation reads $\delta_s^f S = - \int_{t_0}^{t_1} dt \dot{Q}_s f$ with $Q_s = \frac{\partial L}{\partial \dot{q}} \tilde{\delta}_s q(t) - R_s$. Using integration by parts we get

$$\delta_s^f S = \int_{t_0}^{t_1} dt \, Q_s \dot{f} + [\dot{Q}_s f]_{t_0}^{t_1} = \int_{t_0}^{t_1} dt \, Q_s \dot{f}.$$

C Conserved quantity

Since $f(t) \tilde{\delta}_s q(t)$ vanishes on the end points of the history, it can be interpreted the usual $\tilde{\delta} q(t)$ used in the action principle leading to on-shell histories. This means that the first order variation of S vanishes independently of f on-shell. We have $0 = \delta_s^f S = - \int_{t_0}^{t_1} dt \dot{Q}_s f \implies \dot{Q}_s = 0$. Note that since \dot{f} excludes constant functions except $f = 0$, $\int_{t_0}^{t_1} dt \, Q_s \dot{f}$ can't be used to conclude $Q_s = 0$ (\dot{f} is not arbitrary).

D

Application to time translations

Consider the Lagrangian $L = m\dot{q}^2/2 - V(q)$ describing a particle of mass m in a potential V . The infinitesimal disformation of an history due to time translation is given by $q'(t) = q(t + \varepsilon) = q(t) + \varepsilon\dot{q}$ is associated to $\tilde{\delta}_s q = \dot{q}$. Here, we consider an alternate disformation given by $q'(t) = q(t + \varepsilon) = q(t) + f(t)\varepsilon\dot{q}$ leading to the following variation of the action:

$$\begin{aligned}
& S(q(t) + \varepsilon f(t)\tilde{\delta}_s q(t)) \\
&= \int_{t_0}^{t_1} dt L(q(t) + \varepsilon f(t)\tilde{\delta}_s q(t), \dot{q}(t) + \varepsilon \frac{d}{dt}(f(t)\tilde{\delta}_s q(t)), t) + O(\varepsilon^2) \\
&= S + \varepsilon \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial q} f(t)\tilde{\delta}_s q(t) + \frac{\partial L}{\partial \dot{q}} (\dot{f}(t)\tilde{\delta}_s q(t) + f(t)\tilde{\delta}_s \dot{q}(t)) \right) + O(\varepsilon^2) \\
&= S + \varepsilon \int_{t_0}^{t_1} dt \left(-V'(q)\tilde{\delta}_s q(t) + m\dot{q}\tilde{\delta}_s \dot{q}(t) - \frac{d}{dt}(m\dot{q}\tilde{\delta}_s q(t)) \right) f(t) + O(\varepsilon^2) \\
&= S + \varepsilon \int_{t_0}^{t_1} dt \left(-V'(q)\dot{q}(t) + m\dot{q}(t)\ddot{q}(t) - \frac{d}{dt}(m\dot{q}^2) \right) f(t) + O(\varepsilon^2) \\
&= S + \varepsilon \int_{t_0}^{t_1} dt \frac{d}{dt} \left(-V(q) + \frac{1}{2}m\dot{q}^2 - m\dot{q}^2 \right) f(t) + O(\varepsilon^2).
\end{aligned}$$

Since $f(t)$ is arbitrary, the first order vanishing of the variation of the action on-shell implies

$$0 = \frac{d}{dt} \left(V(q) + \frac{1}{2}m\dot{q}^2 \right).$$

References

- [1] Aldo Riello. *Fourteen Lectures in CLASSICAL PHYSICS*. 2023.