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HOMEWORK 1

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Contents

1 Bead on a rotating wire

A point particle of mass m is constrained to move on a parabola rotating with angular velocity ω around a z axis aligned with the opposite of a gravitationnal field of intensity g. The parabola is described in cylindrical coordinates ρ, ϕ, z by the relations $z = \alpha\rho^2$ and $\phi = \omega t$.

A Lagrangian

Following the constraints imposed on the particle, its cartesian position can be written as the following

$$x = \rho\cos(\omega t), \quad y = \rho\sin(\omega t), \quad \text{and} \quad z = \alpha\rho^2$$

leaving ρ as the only free parameter describing the motion of the particle. The cartesian velocity of the particle is then given by the time derivative of its cartesian coordinates. We have

$$\dot{x} = \dot{\rho}\cos(\omega t) - \omega\rho\sin(\omega t), \quad \dot{y} = \dot{\rho}\sin(\omega t) + \omega\rho\cos(\omega t), \quad \text{and} \quad \dot{z} = 2\alpha\rho\dot{\rho}.$$

The square magnitude of the cartesian velocity of the particle can be written in cylindrical coordinates as

$$\begin{aligned}
\dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= (\dot{\rho}\cos(\omega t) - \omega\rho\sin(\omega t))^2 + (\dot{\rho}\sin(\omega t) + \omega\rho\cos(\omega t))^2 + (2\alpha\rho\dot{\rho})^2 + (2\alpha\rho\dot{\rho})^2 \\
&= \dot{\rho}^2\cos^2(\omega t) + \omega^2\rho^2\sin^2(\omega t) + \dot{\rho}^2\sin^2(\omega t) + \omega^2\rho^2\cos^2(\omega t) \\
&+ 2\dot{\rho}\rho\omega\cos(\omega t)\sin(\omega t) - 2\dot{\rho}\rho\omega\cos(\omega t)\sin(\omega t) + 4\alpha^2\rho^2\dot{\rho}^2 \\
&= \dot{\rho}^2 + \omega^2\rho^2 + 4\alpha^2\dot{\rho}^2\rho^2.
\end{aligned}$$

Following from this expression, the kinetic energy reads $T = \frac{1}{2}m\left(\dot{\rho}^2 + \omega^2\rho^2 + 4\alpha^2\rho^2\dot{\rho}^2\right)$. The potential energy du to gravitationnal interaction is given by $V = mgz = mg\alpha\rho^2$. We can finally write the lagrangian

$$L = T - V = \frac{1}{2}m\left(\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2 + \omega^2\rho^2\right) - mg\alpha\rho^2.$$

B Time Translation Symmetry Conserved Charge

Since $\dfrac{\partial L}{\partial t} = 0$, Noether's theorem ensures that

$$\begin{aligned}
E = \frac{\partial L}{\partial \dot{\rho}}\dot{\rho} - L &= m\left(1 + 4\alpha^2\right)\dot{\rho}\dot{\rho} - \frac{1}{2}m\left(\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2 + \omega^2\rho^2\right) + mg\alpha\rho^2 \\
&= \frac{1}{2}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2 + \underbrace{m\rho^2\left(g\alpha - \frac{1}{2}\omega^2\right)}_{U_{\text{eff}}}
\end{aligned}$$

is a conserved charge on physical trajectories which corresponds to an effective energy as it is associated to time translation symmetry of L. The potential-like term in E can be written as $U_{\text{eff}} = \frac{1}{2}m\rho^2(\omega_0^2 - \omega^2)$ where $\omega_0 = \sqrt{2ga}$.

C Effective Potential

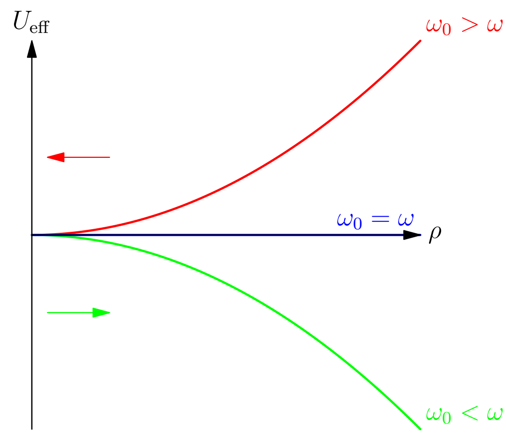
The qualitative dynamics of the system can be descriubed with U_{eff} by considering the three following regimes:

- $\omega_0 > \omega$ Gravity is stronger than rotation and the particle trapped in an harmonic potential. Note that when it oscillates through "$\rho = 0$" it doesn't stop. Indeed, "$\rho = 0$" is a not included in the coordinate chart provided by spherical coordinates. In reality, nothing spacial happens at the bottom of the parabola and the particle continues to the other side;

- $\omega_0 = \omega$ First consider $0 = E = \frac{1}{2}m\left(1+4\alpha^2\rho^2\right)\dot{\rho}^2$ which implies $\dot{\rho} = 0$. Through the constraint this means a fixed height. In this situation gravity balances the centrepetal and parabola normal forces resulting in a net force accounting only for uniform rotation motion at fixed z. This is an unstable equilibrium. Indeed any perturbation will increase the energy beyond 0 and lead to the second subcase. If $E > 0$, $\frac{1}{2}m\left(1+4\alpha^2\rho^2\right)\dot{\rho}^2 > 0$ and $|\dot{\rho}| > 0$. This means that if $\dot{\rho} < 0$ initially, the particle will get lower on the parabola reaching $\rho = 0$ and escaping to infinity in the other branch of the parabola. Furthermore, in the effective picture (reference frame rotating with the parabola), the coefficient $\left(1+4\alpha^2\rho^2\right)$ acts as an effective mass. It becomes smaller near $\rho = 0$ and keeping a fixed energy means speeding up (the opposite is true id the particle moves away from $\rho = 0$);

- $\omega_0 < \omega$ This situation corresponds an unbounded growth of z. The effective potential represents the effect of the centrifugal effect pushing the particle away from the origin more than gravity is pulling it. If $E < 0$, the particle can't get arbitrarely close to $\rho = 0$ and bounces to infinity at $\rho \neq 0$ preventing the knietic energy term from becoming negative. If $E > 0$, the particle can go trough $\rho = 0$ if it was initially directed that way and fly to infinity on the other side of the parabola. Furthermore, for $\rho \to \infty$ the effective mass becomes ρ^2 and the kinetic energy behaves like $E - U_{eff}\ - U_{eff}\ \rho^2$. This means that $\rho^2\dot{\rho}^2\ \rho^2$ or, equivalently, that the velocity saturates. This happend because the kinetic energy gain becomes fully attributed to mass increase at sufficient ρ.

In each case the arrows indicate the direction of the force which is constant in each regime.



D Discussion

The total energy of the particle is given by

$$E_0 = T + V = \frac{1}{2}m\left(\left(1+4\alpha^2\rho^2\right)\dot{\rho}^2 + \omega^2\rho^2\right) + mg\alpha\rho^2 = E + m\rho^2\omega^2.$$

The conservation of E implies $\dot{E}_0 = \dot{E} + 2m\rho\dot{\rho}\omega^2 = 2m\rho\dot{\rho}\omega^2$. From this, we see that E_0 is conserved on shell iff $\dot{\rho} = 0$ or $\rho = 0$. In the former case, ρ is a constant and this can only be realised if $\omega_0 = \omega$ or $\rho = 0$ (no effective force). In the latter case, $\rho = 0$ which is a subcase of $\dot{\rho} = 0$. If $\omega_0 < \omega$, the parabola is rotating fast enough to go beyond the gravitationnal pull of the particle towards $\rho = 0$ and the velocity of the particle grows without bounds (the motor rotating the parabola is transfering energy to the particle and E_0 is not conserved).

2 Brachistochrone

Consider a particle subject to a uniform gravitationnal field of strength g oriented in the negative y direction of a planar Oxy cartesian coordinate system. The particle starts at point A of coordinates x_A, y_A and ends at point B of coordinates x_B, y_B. We are interested in the curve linking A and B minimizing the travel time of the particle.

A Travel time functionnal

The first step in constructing the travel time functionnal is the parametrisation of its curve argument. Suppose the trajectory is parametrised by y so that it takes the form of a function $x(y)$. The length element at x is $\mathrm{d}l = \sqrt{\dot{x}(y)^2 + \dot{y}^2}\mathrm{d}y = \sqrt{1+\dot{x}(y)^2}\mathrm{d}y$ with $\dot{()} = \frac{d}{dy}$. By a mass independant analogue of the conservation of energy $0 = \frac{1}{2}v^2 - gy$ (with "potential" energy zero at y_B) where v is the velocity, we have $v = \sqrt{2gy}$. For motion along an element $\mathrm{d}l$ of the curve, the time elapsed is $\mathrm{d}t = \mathrm{d}l/\sqrt{2gy}$ leading to te following travel time functionnal

$$T[x(y)] = \int_{y_A}^{y_B} \mathrm{d}y \frac{\sqrt{1+\dot{x}(y)^2}}{\sqrt{2gy}}.$$

B Conserved quantity

Since the integrand (effective lagrangian $L = \sqrt{1+\dot{x}(y)^2}/\sqrt{2gy}$) of the previous functionnal is cyclic in x, the Euler-Lagrange equations imply

$$\frac{d}{dy}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \implies Q = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}(y)}{\sqrt{1+\dot{x}(y)^2}\sqrt{2gy}} \quad \text{is conserved on the Brachistochrone.}$$

We can now write

$$2gyQ^2(1+\dot{x}(y)^2) = \dot{x}(y)^2 \implies \dot{x}(y) = \sqrt{\frac{2gyQ^2}{1-2gyQ^2}}$$

and integrate $\dot{x}$ to get

$$\begin{aligned}
x(y) &= \int \mathrm{d}y \sqrt{\frac{2gyQ^2}{1-2gyQ^2}} = \int \mathrm{d}u \frac{1}{gQ^2}\frac{u^2}{\sqrt{1-u^2}}, \quad u = \sqrt{2gyQ^2},\ \mathrm{d}u = \frac{2gQ^2}{2u}\mathrm{d}y \\
&= \int \mathrm{d}u \frac{1}{gQ^2}\frac{u^2}{\sqrt{1-u^2}} = -\frac{1}{gQ^2}\int \sin(\theta/2)\mathrm{d}\theta \frac{\cos^2(\theta/2)}{\sqrt{1-\cos^2(\theta/2)}} = -\frac{1}{2gQ^2}\int \mathrm{sgn}(\sin(\theta/2))\cos^2(\theta/2)\mathrm{d}\theta, \quad u = \cos(\theta/2) \\
&= -\frac{1}{2gQ^2}\frac{1}{2}(\theta(y) + \sin(\theta(y))) + C, \quad \text{for } \theta \in [0, 2\pi),\ \sin(\theta/2) \geq 0.
\end{aligned}$$

Using the new θ as an independant variable, we can write

$$\begin{aligned}
x(\theta) &= -\frac{1}{2gQ^2}\frac{1}{2}(\theta + \sin(\theta)) + C \\
y(\theta) &= \frac{1}{2gQ^2}\cos^2(\theta/2) = \frac{1}{2gQ^2}\frac{1}{2}(1+\cos(\theta)).
\end{aligned}$$

To get the usual cycloid paramatrisation, we set $\theta = \pi - \theta'$ and $C = \frac{\pi}{2gQ^2}\frac{1}{2}$ to obtain

$$
\begin{aligned}
x(\theta') &= -\frac{1}{2gQ^2}\frac{1}{2}(\pi - \theta' + \sin(\pi - \theta')) + \frac{\pi}{2gQ^2}\frac{1}{2} = \frac{1}{2gQ^2}\frac{1}{2}(\theta' - \sin(\theta')) \\
y(\theta') &= \frac{1}{2gQ^2}\frac{1}{2}(1 + \cos(\pi - \theta')) = \frac{1}{2gQ^2}\frac{1}{2}(1 - \cos(\theta')).
\end{aligned}
$$

where the radius of the cycloid is identified to be $a = \dfrac{1}{gQ^2}$. A relation between a and x_B, y_B can be obtained by considering the ratio of x and y (knowing $x_B \neq 0$) we write

$$
\frac{y_B}{x_B} = \frac{1 - \cos(\theta'_B)}{\theta'_B - \sin(\theta'_B)}
$$

which implicitly forms a transcendental expression for θ'_B as a function of x_B, y_B. Knowing θ'_B, we have

$$
a = \frac{2y_B}{1 - \cos(\theta'_B(x_B, y_B))}.
$$

3 A sneaky recipe for the Noether charge

A Alternate Disformation

Consider a particle described at time t by the generalized coordinate $q(t)$ evolving with respect to a lagrangian L. A disformation of the particle's history is given by $q(t) \mapsto q' = q(t) + \varepsilon\tilde{\delta}_s q(t)$ where $\delta\varepsilon$ is an infinitesimal constant. This disformation contributes to the action S with a boundary term if there exists a function R_s such that $\tilde{\delta}_s L = \frac{d}{dt}R_s$ (first order variation of L in ε).

B Variation of the Action

An alternative disformation of the history of the particle reads $q(t) \mapsto q'(t) = q(t) + \varepsilon f(t)\tilde{\delta}_s q(t)$ where f is a function vanishing at the start ($t = t_0$) and end ($t = t_1$) of the history. The action following from this disformation is

$$\begin{aligned}
&S(q(t) + \varepsilon f(t)\tilde{\delta}_s q(t)) \\
&= \int_{t_0}^{t_1} \mathrm{d}t\; L(q(t) + \varepsilon f(t)\tilde{\delta}_s q(t), \dot{q}(t) + \varepsilon\frac{d}{dt}(f(t)\tilde{\delta}_s q(t)), t) \\
&= S(q(t)) + \varepsilon\int_{t_0}^{t_1} \mathrm{d}t\left[\frac{\partial L}{\partial q} f(t)\tilde{\delta}_s q(t) + \frac{\partial L}{\partial \dot{q}}(\dot{f}(t)\tilde{\delta}_s q(t) + f(t)\tilde{\delta}_s \dot{q}(t))\right] + O(\varepsilon^2) \\
&= S(q(t)) + \varepsilon\int_{t_0}^{t_1} \mathrm{d}t\left[\frac{\partial L}{\partial q} f(t)\tilde{\delta}_s q(t) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\tilde{\delta}_s q(t)\right) f(t) + \frac{\partial L}{\partial \dot{q}} f(t)\tilde{\delta}_s \dot{q}(t)\right] + \left[\frac{\partial L}{\partial \dot{q}} f(t)\tilde{\delta}_s q(t)\right]_{t_0}^{t_1} + O(\varepsilon^2) \\
&= S(q(t)) + \varepsilon\int_{t_0}^{t_1} \mathrm{d}t\left[\left(\frac{\partial L}{\partial q}\tilde{\delta}_s q(t) + \frac{\partial L}{\partial \dot{q}}\tilde{\delta}_s \dot{q}(t)\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\tilde{\delta}_s q(t)\right)\right] f(t) + O(\varepsilon^2) \\
&= S(q(t)) + \varepsilon\int_{t_0}^{t_1} \mathrm{d}t\left[\frac{d}{dt}R_s - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\tilde{\delta}_s q(t)\right)\right] f(t) + O(\varepsilon^2)
\end{aligned}$$

and its first order variation reads $\delta_s^f S = -\int_{t_0}^{t_1} \mathrm{d}t \dot{Q}_s f$ with $Q_s = \frac{\partial L}{\partial \dot{q}}\tilde{\delta}_s q(t) - R_s$. Using integration by parts we get

$$\delta_s^f S = \int_{t_0}^{t_1} \mathrm{d}t\; Q_s \dot{f} + \left[\dot{Q}_s f\right]_{t_0}^{t_1} = \int_{t_0}^{t_1} \mathrm{d}t\; Q_s \dot{f}.$$

C Conserved quantity

Since $f(t)\tilde{\delta}_s q(t)$ vanishes on the end points of the history, it can be interpreted the usual $\tilde{\delta} q(t)$ used in the action principle leading to on-shell histories. This means that teh first order variation of S vanishes independently of f on-shell. We have $0 = \delta_s^f S = -\int_{t_0}^{t_1} \mathrm{d}t \dot{Q}_s f \implies \dot{Q}_s = 0$. Note that since $\dot{f}$ excludes constant functions except $f = 0$, $\int_{t_0}^{t_1} \mathrm{d}t\; Q_s \dot{f}$ can't be used to conclude $Q_s = 0$ ($\dot{f}$ is not arbitrary).

D Application to time translations

Consider the Lagrangian $L = m\dot{q}^2/2 - V(q)$ describing a particle of mass m in a potential V. The infinitesimal disformation of an history due to time translation is given by $q'(t) = q(t+\varepsilon) = q(t) + \varepsilon\dot{q}$ is associated to $\tilde{\delta}_s q = \dot{q}$. Here, we consider an altenrate disformation given by $q'(t) = q(t+\varepsilon) = q(t) + f(t)\varepsilon\dot{q}$ leading to the following variation of the action:

$$
\begin{aligned}
&S(q(t) + \varepsilon f(t)\tilde{\delta}_s q(t)) \\
&= \int_{t_0}^{t_1} \mathrm{d}t\ L(q(t) + \varepsilon f(t)\tilde{\delta}_s q(t), \dot{q}(t) + \varepsilon \frac{d}{dt}(f(t)\tilde{\delta}_s q(t)), t) + O(\varepsilon^2) \\
&= S + \varepsilon \int_{t_0}^{t_1} \mathrm{d}t\ \left(\frac{\partial L}{\partial q} f(t)\tilde{\delta}_s q(t) + \frac{\partial L}{\partial \dot{q}}(\dot{f}(t)\tilde{\delta}_s q(t) + f(t)\tilde{\delta}_s \dot{q}(t))\right) + O(\varepsilon^2) \\
&= S + \varepsilon \int_{t_0}^{t_1} \mathrm{d}t\ \left(-V'(q)\tilde{\delta}_s q(t) + m\dot{q}\tilde{\delta}_s \dot{q}(t) - \frac{d}{dt}\left(m\dot{q}\tilde{\delta}_s q(t)\right)\right) f(t) + O(\varepsilon^2) \\
&= S + \varepsilon \int_{t_0}^{t_1} \mathrm{d}t\ \left(-V'(q)\dot{q}(t) + m\dot{q}(t)\ddot{q}(t) - \frac{d}{dt}\left(m\dot{q}^2\right)\right) f(t) + O(\varepsilon^2) \\
&= S + \varepsilon \int_{t_0}^{t_1} \mathrm{d}t\ \frac{d}{dt}\left(-V(q) + \frac{1}{2}m\dot{q}^2 - m\dot{q}^2\right) f(t) + O(\varepsilon^2).
\end{aligned}
$$

Since $f(t)$ is arbitrary, the first order vanishing of the variation of the action on-shell implies

$$
0 = \frac{d}{dt}\left(V(q) + \frac{1}{2}m\dot{q}^2\right).
$$

References

[1] Aldo Riello. *Fourteen Lectures in CLASSICAL PHYSICS*. 2023.