# FEE321 – E.C.T IIA – Oct 2020

# Lecture 8: Laplace Transform (2) (2 hrs)

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27/11/2020

# Overview

Today's class continues the look at LT and its application

- Sample LT calculation (continued)
- Sample LT pairs
- Examples of LT application in circuit analysis

# Content

- **Sample LT calculation (continued)**

# Laplace transform[9]

**Examples**

2. Exponential function, $f(t) = e^{-at}$

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} e^{-at}e^{-st}dt = \int_0^{\infty} e^{-(s+a)t}dt$$

$$= -\frac{1}{s+a}\left\{e^{-(s+a)t}\right\}_0^{\infty} = -\frac{1}{s+a}\left\{e^{-(\sigma+a)t}e^{-j\omega t}\right\}_0^{\infty}$$

$$= \frac{1}{s+a}\left\{1 - e^{-(\sigma+a)\infty}e^{-j\omega\infty}\right\}$$

Exponential terms only vanish under the condition, $(\sigma + a) > 0$

$$= \frac{1}{s+a} \qquad \text{ROC: } (Re[s+a] = \sigma + a) > 0$$

Thus

$$e^{-at} \rightleftharpoons \frac{1}{s+a}$$

# Laplace transform[10]

**Examples**

3. Linear combination, $f(t) = a_1 f_1(t) + a_2 f_2(t)$

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} \{a_1 f_1(t) + a_2 f_2(t)\}e^{-st}dt$$

$$= \int_0^{\infty} \{a_1 f_1(t)\}e^{-st}dt + \int_0^{\infty} \{a_2 f_2(t)\}e^{-st}dt$$

$$= a_1 \int_0^{\infty} f_1(t)e^{-st}dt + a_2 \int_0^{\infty} f_2(t)e^{-st}dt$$

$$= a_1 F_1(s) + a_2 F_2(s)$$

Thus $\quad a_1 f_1(t) + a_2 f_2(t) \rightleftharpoons a_1 F_1(s) + a_2 F_2(s)$

# Laplace transform[11]

**Examples**

4. Complex shift, $f_1(t) = e^{-at}f(t)$

$$\mathcal{L}[f_1(t)] = \int_0^{\infty} f_1(t)e^{-st}dt = \int_0^{\infty} e^{-at}f(t)e^{-st}dt$$

$$= \int_0^{\infty} f(t)\, e^{-(s+a)t}dt$$

$$= F(s+a)$$

Thus $e^{-at}f(t) \rightleftharpoons F(s+a)$

# Laplace transform[12]

**Examples**

5. Real shift, $f_1(t) = f(t - t_o)u(t - t_o)$

$$\mathcal{L}[f_1(t)] = \int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty f(t - t_o)u(t - t_o)e^{-st}dt$$

$$= \int_{t_o}^\infty f(t - t_o)e^{-st}dt \quad = \int_0^\infty f(\alpha)e^{-s(\alpha + t_o)}d\alpha$$

; with $\alpha = t - t_o$

$$= \int_0^\infty f(\alpha)e^{-\alpha s}e^{-st_o}d\alpha \quad = e^{-st_o}\int_0^\infty f(\alpha)e^{-\alpha s}d\alpha$$

$$= e^{-st_o}F(s)$$

Thus $\quad f(t - t_o)u(t - t_o) \rightleftharpoons e^{-st_o}F(s)$

# Laplace transform[13]

**Examples**

6. Dirac delta, $f(t) = \delta(t)$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^\infty \delta(t)e^{-st}dt$$

$$= \int_0^\infty \delta(t)e^{-s\times 0}dt$$

$$= \int_0^\infty \delta(t)dt$$

$$= 1$$

Thus $\delta(t) \rightleftharpoons 1$

# Content

- Sample LT calculation (continued)
- **Sample LT pairs**

# Laplace transform[14]

**Some LT pairs**

| $f(t),\ t \geq 0$ | $F(s)$ | $ROC$ |
|---|---|---|
| $\delta(t)$ | $1$ | $\forall s$ |
| $u(t)$ | $\frac{1}{s}$ | $Re(s) > 0$ |
| $t$ | $\frac{1}{s^2}$ | $Re(s) > 0$ |
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ | $Re(s) > 0$ |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ | $Re(s) > 0$ |
| $e^{-at} \cos \omega t$ | $\frac{(s+a)}{(s+a)^2+\omega^2}$ | $Re(s) > -a$ |
| $\sinh \omega t$ | $\frac{\omega}{s^2-\omega^2}$ | $Re(s) > 0$ |
| $\cosh \omega t$ | $\frac{s}{s^2-\omega^2}$ | $Re(s) > 0$ |

# Content

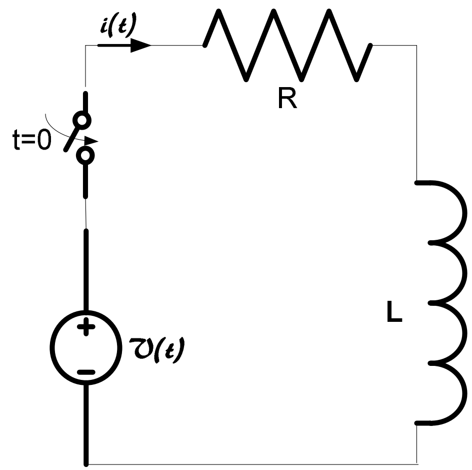
- Sample LT calculation (continued)
- Sample LT pairs
- **Examples of LT application in circuit analysis**

# LT application[1]

**Example 1**

A series $RL$ circuit is supplied from a dc source of $V$ volts. The source is switched on at $t = 0$.

i. Dc source switched on at $t = 0 \;\Rightarrow v(t) = Vu(t)$
ii. Zero initial current before $t = 0 \;\Rightarrow i_L(0^+) = 0$




Circuit LDE is given by $L\dfrac{di(t)}{dt} + Ri(t) = Vu(t) \quad ; t > 0$

Applying LT $\mathcal{L}\left[L\dfrac{di(t)}{dt} + Ri(t) = Vu(t)\right]$

$$\Rightarrow \mathcal{L}\left[L\frac{di(t)}{dt}\right] + \mathcal{L}[Ri(t)] = \mathcal{L}[Vu(t)]$$

$$\Rightarrow L[sI(s) - i(0^+)] + RI(s) = \frac{V}{s} \;\Rightarrow I(s)\{sL + R\} = Li(0^+) + \frac{V}{s}$$

Thus $I(s) = \dfrac{Li(0^+)}{sL + R} + \dfrac{V}{s(sL + R)}$

# LT application[2]

**Example 1 (continued)**

i. The transform equation then needs to be converted back to time domain
ii. Partial fraction decomposition is used
iii. Terms are organized in the forms for which LT pairs exist or LT properties may be applied

$$\Rightarrow I(s) = \frac{Li(0^+)}{sL+R} + \frac{V}{s(sL+R)} = \frac{i(0^+)}{s+\frac{R}{L}} + \frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)} = I_1(s) + I_2(s)$$

Since $e^{-at} \rightleftharpoons \dfrac{1}{s+a} \quad \Rightarrow \quad I_1(s) = \dfrac{i(0^+)}{s+\frac{R}{L}} \rightleftharpoons i_1(t) = i(0^+)e^{-\frac{R}{L}t}$

Using partial fraction decomposition for the second term

$$I_2(s) = \frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)} = \frac{A}{s} + \frac{B}{s+\frac{R}{L}} \qquad \Rightarrow A = s\left(\frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)}\right)\Bigg|_{s=0} = \frac{V}{R}$$

.

$$\Rightarrow B = \left(s+\frac{R}{L}\right)\left(\frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)}\right)\Bigg|_{s=-\frac{R}{L}} = -\frac{V}{R}$$

# LT application[3]

**Example 1 (continued)**

$$\therefore \quad I_2(s) = \frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)} = \frac{\frac{V}{R}}{s} - \frac{\frac{V}{R}}{s+\frac{R}{L}} \quad \rightleftharpoons \; i_2(t) = \frac{V}{R}\left(u(t) - e^{-\frac{R}{L}t}\right)$$

$$\Rightarrow I(s) = \frac{i(0^+)}{s+\frac{R}{L}} + \frac{\frac{V}{R}}{s} - \frac{\frac{V}{R}}{s+\frac{R}{L}} \quad \rightleftharpoons \; i(t) = u(t)\left\{i(0^+)e^{-\frac{R}{L}t} + \frac{V}{R}\left(u(t) - e^{-\frac{R}{L}t}\right)\right\}$$

Expression may be re-arranged thus

$$i(t) = u(t)\left\{\frac{V}{R}u(t) + \left[i(0^+) - \frac{V}{R}\right]e^{-\frac{R}{L}t}\right\} \quad A$$

The first term in the curly brackets is the **steady state** circuit current

The second term in the curly brackets is the **transient** circuit current

NB: At steady state, for dc excitation the inductor appears as a **short circuit**

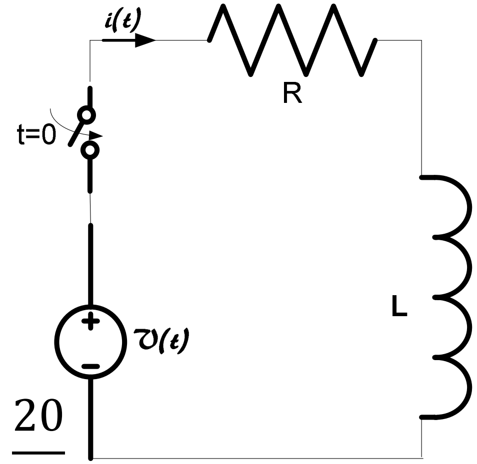
The unit step MUST always be included in these LT solutions as $t > 0$ is a condition for LT

If the zero initial conditions are applied $i(t) = u(t)\left\{\frac{V}{R}\left(u(t) - e^{-\frac{R}{L}t}\right)\right\} \quad A$

# LT application[4]

**Example 1b (Numerical)**

A series $RL$ circuit is supplied from a dc source of $20$ volts. The source is switched on at $t = 0$. There is zero initial current in the circuit. Let $R = 5\Omega$, and $L = 0.5H$




Circuit LDE as before is given by

$$L\frac{di(t)}{dt} + Ri(t) = Vu(t) \quad ; t > 0$$

Substituting values $0.5\dfrac{di(t)}{dt} + 5i(t) = 20u(t) \quad ; t > 0$

Applying LT $\mathcal{L}\left[0.5\dfrac{di(t)}{dt} + 5i(t) = 20u(t)\right]$

$$\Rightarrow 0.5[sI(s) - i(0^+)] + 5I(s) = \frac{20}{s} \Rightarrow I(s)\{0.5s + 5\} = 0.5i(0^+) + \frac{20}{s}$$

Thus $I(s) = \dfrac{0.5i(0^+)}{0.5s + 5} + \dfrac{20}{s(0.5s + 5)} = \dfrac{i(0^+)}{s + 10} + \dfrac{40}{s(s + 10)} = I_1(s) + I_2(s)$

The first term $I_1(s)$ is of a form we recognize and can immediately be inverted. The second term $I_2(s)$ needs to be decomposed into two fractions

$$I_2(s) = \frac{40}{s(s + 10)} = \frac{A}{s} + \frac{B.}{s + 10}$$

# LT application[5]

**Example 1b (continued)** $I_2(s) = \dfrac{40}{s(s+10)} = \dfrac{A}{s} + \dfrac{B}{s+10}$

$$\Rightarrow A = s\left(\frac{40}{s(s+10)}\right)\Bigg|_{s=0} = 4 \quad \text{and} \quad B = (s+10)\left(\frac{40}{s(s+10)}\right)\Bigg|_{s=-10} = -4$$

$$I_2(s) = \frac{40}{s(s+10)} = \frac{4}{s} - \frac{4}{s+10} \qquad \text{and} \qquad I(s) = \frac{4}{s} - \frac{\{4 - i(0^+)\}}{s+10}$$

$$e^{-at} \rightleftharpoons \frac{1}{s+a} \qquad \text{and} \qquad u(t) \rightleftharpoons \frac{1}{s}$$

$$i(t) = u(t)\{4u(t) - [4 - i(0^+)]e^{-10t}\} \;\; A$$

.

Note the **steady state** component and the **transient** component

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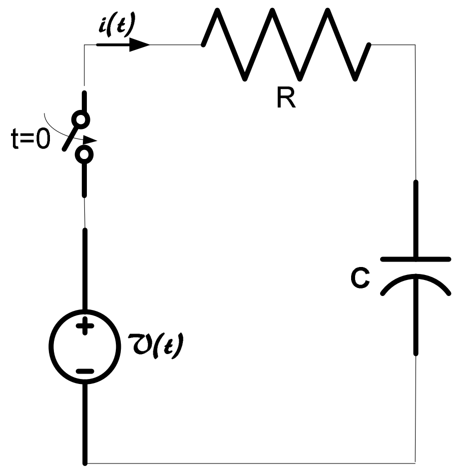
# LT application[6]

**Example 2**

A series $RC$ circuit is supplied from a dc source of $V$ volts. The source is switched on at $t = 0$. Assume there is zero voltage across the capacitor at $t = 0$

i. Dc source switched on at $t = 0 \Rightarrow v(t) = Vu(t)$

ii. Zero initial voltage $\Rightarrow v_C(0^+) = 0$




Circuit LDE is given by $\dfrac{1}{C}\displaystyle\int_{-\infty}^{t} i(t)\,dt + Ri(t) = Vu(t) \quad ; t > 0$

Applying LT $\mathcal{L}\left[\dfrac{1}{C}\displaystyle\int_{-\infty}^{t} i(t)\,dt + Ri(t) = Vu(t)\right]$

$$\Rightarrow \mathcal{L}\left[\frac{1}{C}\int_{-\infty}^{0} i(t)\,dt + \frac{1}{C}\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[Ri(t)] = \mathcal{L}[Vu(t)]$$

$$\Rightarrow \mathcal{L}\left[v_C(0^+) + \frac{1}{C}\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[Ri(t)] = \mathcal{L}[Vu(t)]$$

Thus $\Rightarrow \dfrac{v_C(0^+)}{s} + \dfrac{I(s)}{sC} + RI(s) = \dfrac{V}{s}$ $\qquad \Rightarrow I(s)\left\{\dfrac{1}{sC} + R\right\} = \dfrac{V - v_C(0^+)}{s}$

$$\Rightarrow I(s) = \frac{sC\big(V - v_C(0^+)\big)}{s(sCR + 1)}$$

# LT application[7]

**Example 2 (continued)**

i. The transform equation then needs to be converted back to time domain
ii. Partial fraction decomposition not needed this time
iii. Let us first assume the initial conditions were not zero, to see complete solution
iv. Terms are organized in the forms for which LT pairs exist or LT properties may be applied

$$I(s) = \frac{C\big(V - v_C(0^+)\big)}{sCR + 1} = \frac{\frac{1}{R}\big(V - v_C(0^+)\big)}{s + \frac{1}{CR}}$$

Since $e^{-at} \rightleftharpoons \dfrac{1}{s+a} \quad \Rightarrow \quad \dfrac{\frac{1}{R}\big(V - v_C(0^+)\big)}{s + \frac{1}{CR}} \rightleftharpoons \dfrac{1}{R}\big(V - v_C(0^+)\big)e^{-\frac{1}{CR}t}$

The current is therefore $i(t) = u(t)\left\{\dfrac{1}{R}\big(V - v_C(0^+)\big)e^{-\frac{1}{CR}t}\right\}\ A$

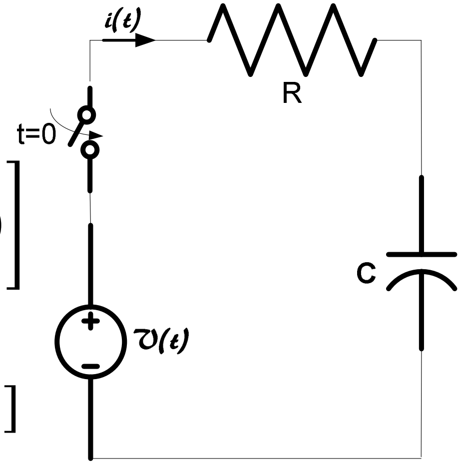
- Only **transient** current exists
- Makes sense because in **steady state** the capacitor appears as an **open circuit** under dc excitation, and no current flows
- Again the multiplication by $u(t)$ is mandatory due to definition of LT

For zero initial conditions $i(t) = \dfrac{Vu(t)}{R}e^{-\frac{1}{CR}t}\quad A$

# LT application[8]

**Example 2b (numerical)**

A series $RC$ circuit is supplied from a dc source of $20$ volts. The source is switched on at $t = 0$. There is zero voltage across the capacitor at $t = 0$. Let $R = 5\Omega$, and $\mathrm{C} = 0.1F$




Circuit LDE is given by $\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) = Vu(t) \quad ; t > 0$

Substituting and applying LT $\mathcal{L}\left[10\int_{-\infty}^{t} i(t)\,dt + 5i(t) = 20u(t)\right]$

$$\Rightarrow \mathcal{L}\left[10\int_{-\infty}^{0} i(t)\,dt + 10\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[5i(t)] = \mathcal{L}[20u(t)]$$

$$\Rightarrow \mathcal{L}\left[v_C(0^+) + 10\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[5i(t)] = \mathcal{L}[20u(t)]$$

.

# LT application[9]

**Example 2b (numerical)**

Thus

$$\Rightarrow \frac{v_C(0^+)}{s} + \frac{10I(s)}{s} + 5I(s) = \frac{20}{s} \qquad \Rightarrow I(s)\left\{\frac{1}{0.1s} + 5\right\} = \frac{20 - v_C(0^+)}{s}$$

$$\Rightarrow I(s) = \frac{0.1s\big(20 - v_C(0^+)\big)}{s(0.5s+1)} = \frac{0.2\big(20 - v_C(0^+)\big)}{s+2} = \frac{\big(4 - 0.2v_C(0^+)\big)}{s+2}$$

This form is easily recognized and is easily inverted to give

$$i(t) = u(t)\{4 - 0.2v_C(0^+)\}e^{-2t} \quad A$$

As before it is noted that only the **transient component** exists in this response

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# Summary

Today's class looked at Laplace transform and application

- Sample LT calculation (continued)
- Sample LT pairs
- Examples of LT application in circuit analysis

# QUESTIONS?