# FEE321 – E.C.T IIA – Oct 2020

# Lecture 9: Laplace Transform (3) (1 hr)

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# Overview

Today's class continues the look at LT application

- Examples of LT application in circuit analysis (continued)

# Content

- **Examples of LT application in circuit analysis (continued)**

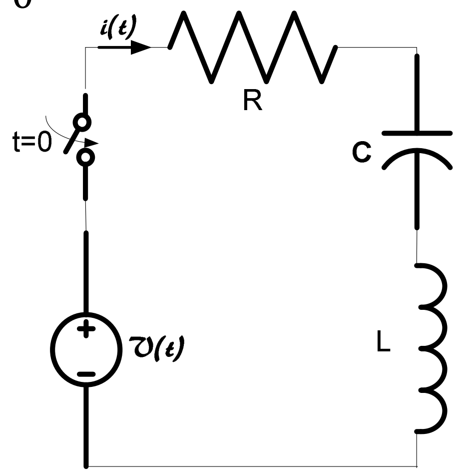
# LT application[10]

**Example 3**

A series $RLC$ circuit is supplied from a dc source of $V$ volts. The source is switched on at $t = 0$. At $t = 0$ there is zero voltage across the capacitor, and no inductor current

i. Dc source switched on at $t = 0 \Rightarrow v(t) = Vu(t), i_L(0^-) = 0$

ii. Zero initial voltage $\Rightarrow v_C(0^+) = 0$, inductor ensures $i_L(0^+) = i_L\ (0^-) = 0$




Circuit LDE is given by

$$\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) + L\frac{di(t)}{dt} = Vu(t) \quad ; t > 0$$

Applying LT $\mathcal{L}\left[\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) + L\frac{di(t)}{dt} = Vu(t)\right]$

$$\Rightarrow \mathcal{L}\left[\frac{1}{C}\int_{-\infty}^{0} i(t)\,dt + \frac{1}{C}\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[Ri(t)] + \mathcal{L}\left[L\frac{di(t)}{dt}\right] = \mathcal{L}[Vu(t)]$$

$$. \Rightarrow \mathcal{L}\left[v_C(0^+) + \frac{1}{C}\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[Ri(t)] + \mathcal{L}\left[L\frac{di(t)}{dt}\right] = \mathcal{L}[Vu(t)]$$

# LT application[11]

**Example 3 (continued)**

Expanding the integral term

$$\Rightarrow \mathcal{L}\left[v_C(0^+) + \frac{1}{C}\int_0^t i(t)\,dt\right] + \mathcal{L}[Ri(t)] + \mathcal{L}\left[L\frac{di(t)}{dt}\right] = \mathcal{L}[Vu(t)]$$

Applying LT

$$\Rightarrow \frac{v_C(0^+)}{s} + \frac{I(s)}{sC} + RI(s) + L[sI(s) - i(0^+)] = \frac{V}{s}$$

$$\Rightarrow I(s)\left\{\frac{1}{sC} + R + sL\right\} = \frac{V - v_C(0^+)}{s} + Li(0^+)$$

$$\Rightarrow I(s)\left\{\frac{1 + sRC + s^2LC}{sC}\right\} = \frac{V - v_C(0^+)}{s} + Li(0^+)$$

Thus

$$\Rightarrow I(s) = \frac{C\big(V - v_C(0^+)\big)}{1 + sRC + s^2LC} + \frac{sCLi(0^+)}{1 + sRC + s^2LC}$$

$$= \frac{CV}{1 + sRC + s^2LC}$$ when $i(0^+), v_C(0^+) = 0$

# LT application[12]

**Example 3 (continued)**

i. The transform equation then needs to be converted back to time domain
ii. Terms need to be organized in the forms for which LT pairs exist or LT properties may be applied, otherwise the ILT integral has to be used

$$I(s) = \frac{CV}{1 + sRC + s^2LC} \quad = \frac{V/L}{\frac{1}{LC} + s\frac{R}{L} + s^2} \quad = \frac{V/L}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

A completing the square in the denominator term, so it looks more like the transform pairs that we know, gives us

$$I(s) = \frac{V/L}{\left(s + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

This is now clearly relates to $e^{-at}\sin(\omega t) \rightleftharpoons \frac{\omega}{(s+a)^2 + \omega^2}$

The expression may therefore be rewritten as

$$I(s) = \frac{V/L}{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}} \times \frac{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2}$$

.

# LT application[13]

**Example 3 (continued)**

We can then do the time domain expression

$$i(t) = u(t)\left\{\frac{\frac{V}{L}}{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}} \times e^{-\frac{R}{2L}t}\sin\left(\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\,t\right)\right\}$$

$$= u(t)\left\{\frac{V}{\sqrt{\left(\frac{L}{C} - \frac{R^2}{4}\right)}} \times e^{-\frac{R}{2L}t}\sin\left(\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\,t\right)\right\} = \frac{Vu(t)e^{-\frac{R}{2L}t}}{\sqrt{\left(\frac{L}{C} - \frac{R^2}{4}\right)}}\sin\left(\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\,t\right) \quad A$$

- We notice that in this case we shall have a **decaying sinusoid** current response
- And this despite there being no sinusoidal input
- The response is due to the exchange of stored energy between the capacitor and the inductor; the magnetic field and the electric field
- The rate of decay is determined by $\frac{R}{2L}$
- For the same value of $L$, the larger the value of $R$, the faster the decay
- Since the resistor is the one that dissipates the energy, this makes sense
- The **circuit time constant**, the time it takes the amplitude to reduce to 1/e, is $\frac{2L}{R}$ seconds
- At steady state, inductor is a short circuit, capacitor is open circuit and no current flows

# LT application[14]

**Example 3 (continued)**

- Consider

$$i(t) = u(t)\left\{\frac{V/L}{\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}} \times e^{-\frac{R}{2L}t}\sin\left(\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}\,t\right)\right\}$$

- If the term under the square root is zero, what happens?
- We have to go back to the s-domain expression to answer this question

$$I(s) = \frac{V/L}{\left(s+\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)} \quad \dashrightarrow \quad I(s) = \frac{V/L}{\left(s+\frac{R}{2L}\right)^2}$$

- This would now correspond to $u(t)te^{-at} \rightleftharpoons \dfrac{1}{(s+a)^2}$

- Note that this is obtained by either applying **complex shift** property to the LT of $t \cdot u(t)$ **or** applying the **multiplication by $t$** property to the LT of $u(t)e^{-at}$

- Under this special condition therefore, the circuit current response shall be $i(t) = \dfrac{V}{L}t \cdot u(t)e^{-\frac{R}{2L}t}$

- As $t \to 0$, the **initial value** property of LT gives us $i(t) = 0$, since $sI(s) \to 0$ as $s \to \infty$

- As $t \to \infty$, the **final value** property of LT gives us $i(t) = 0$, since $sI(s) \to 0$ as $s \to 0$

# LT application[15]

**Example 3 (continued)**

If the initial conditions were not zero then

$$\Rightarrow I(s) = \frac{C\left(V - v_C(0^+)\right)}{1 + sRC + s^2LC} + \frac{sCLi(0^+)}{1 + sRC + s^2LC} = \frac{\frac{1}{L}\{V - v_C(0^+)\}}{s^2 + s\frac{R}{L} + \frac{1}{LC}} + \frac{s \cdot i(0^+)}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

$$= \frac{\frac{1}{L}\{V - v_C(0^+)\}}{\left(s + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} + \frac{s \cdot i(0^+)}{\left(s + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

$$= \frac{\frac{1}{L}\{V - v_C(0^+)\} - \frac{R}{2L}i(0^+)}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2} + \frac{\left(s + \frac{R}{2L}\right) \cdot i(0^+)}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2}$$

$$= \frac{\frac{1}{L}\left\{V - v_C(0^+) - \frac{R}{2}i(0^+)\right\}}{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}} \times \frac{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2} + i(0^+)\frac{\left(s + \frac{R}{2L}\right)}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2}$$

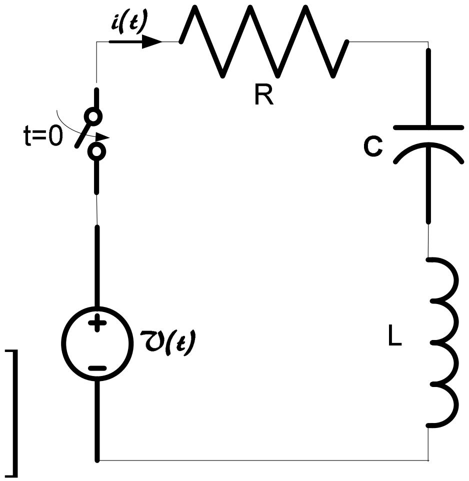
And

$$i(t) = u(t)\left\{\frac{\left\{V - v_C(0^+) - \frac{R}{2}i(0^+)\right\}}{\sqrt{\left(\frac{L}{C} - \frac{R^2}{4}\right)}} \times e^{-\frac{R}{2L}t}\sin\left(\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}t\right) + i(0^+) \times e^{-\frac{R}{2L}t}\cos\left(\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}t\right)\right\} \quad A$$

# LT application[16]

**Example 3 (numerical)**

A series $RLC$ circuit is supplied from a dc source of $20$ volts. The source is switched on at $t = 0$. Assume that at $t = 0$ there is zero voltage across the capacitor, and no inductor current. Let $R = 10\Omega$, $C = 0.04F$ and $L = 1.25H$




Circuit LDE is given by

$$\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) + L\frac{di(t)}{dt} = Vu(t) \quad ; t > 0$$

Substituting in values

$$\mathcal{L}\left[25\int_{-\infty}^{t} i(t)\,dt + 10i(t) + 1.25\frac{di(t)}{dt} = 20u(t)\right]$$

Expanding the integral

$$\Rightarrow \mathcal{L}\left[25\int_{-\infty}^{0} i(t)\,dt + 25\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[10i(t)] + \mathcal{L}\left[1.25\frac{di(t)}{dt}\right] = \mathcal{L}[20u(t)]$$

.

$$\Rightarrow \mathcal{L}\left[v_C(0^+) + 25\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[10i(t)] + \mathcal{L}\left[1.25\frac{di(t)}{dt}\right] = \mathcal{L}[20u(t)]$$

# LT application[17]

**Example 3 (numerical continued)**

Starting at

$$\mathcal{L}\left[v_C(0^+) + 25\int_0^t i(t)\,dt\right] + \mathcal{L}[10i(t)] + \mathcal{L}\left[1.25\frac{di(t)}{dt}\right] = \mathcal{L}[20u(t)]$$

Applying LT

$$\Rightarrow \frac{v_C(0^+)}{s} + \frac{25I(s)}{s} + 10I(s) + 1.25[sI(s) - i(0^+)] = \frac{20}{s}$$

$$\Rightarrow I(s)\left\{\frac{25}{s} + 10 + 1.25s\right\} = \frac{20 - v_C(0^+)}{s} + 1.25i(0^+)$$

$$\Rightarrow I(s)\left\{\frac{25 + 10s + 1.25s^2}{s}\right\} = \frac{20 - v_C(0^+)}{s} + 1.25i(0^+)$$

Thus

$$\Rightarrow I(s) = \frac{20 - v_C(0^+)}{25 + 10s + 1.25s^2} + \frac{1.25s \cdot i(0^+)}{25 + 10s + 1.25s^2}$$

$$= \frac{0.8\big(20 - v_C(0^+)\big)}{s^2 + 8s + 20} + \frac{s \cdot i(0^+)}{s^2 + 8s + 20}$$

# LT application[18]

**Example 3 (numerical continued)**

Continuing to work with **non-zero** initial conditions then

$$\Rightarrow I(s)=\frac{0.8\big(20-v_C(0^+)\big)}{s^2+8s+20}+\frac{s\cdot i(0^+)}{s^2+8s+20}$$

$$=\frac{0.8\{20-v_C(0^+)\}}{(s+4)^2+(20-16)}+\frac{s\cdot i(0^+)}{(s+4)^2+(20-16)}$$

$$=\frac{0.8\{20-v_C(0^+)\}-4i(0^+)}{(s+4)^2+2^2}+\frac{(s+4)\cdot i(0^+)}{(s+4)^2+2^2}$$

$$=0.4\{20-v_C(0^+)-5i(0^+)\}\frac{2}{(s+4)^2+2^2}+i(0^+)\frac{s+4}{(s+4)^2+2^2}$$

And

$$i(t)=u(t)e^{-4t}\{[8-0.4v_C(0^+)-2i(0^+)]\sin(2t)+\ i(0^+)\cos(2t)\}\quad A$$

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# Summary

Today's class looked at Laplace transform application

- Examples in circuit analysis

# QUESTIONS?