FEE321 – E.C.T IIA – Oct 2020

Lecture 14: Duality and Graphs(2 hrs)

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15/01/2021

Overview

Today's class looks at

- Duality in Circuit Theory
- Topological analysis

Content

- **Duality in Circuit Theory**

Duality [1]

Kirchoff's Voltage Law

- Stated as: *The algebraic sum of voltages around each and any closed loop of a circuit must equal to zero, at all instants of time*
- Otherwise stated: *The sum of voltage rises in any closed loop in a circuit is equal to the sum of voltage drops in the loop, at all instants of time*

Kirchoff's Current Law

- Stated as: *The algebraic sum of currents at each and any node of a circuit must equal to zero, at all instants of time*
- Otherwise stated: *The sum of currents into any node in a circuit is equal to the sum of currents out of the node, at all instants of time*

- The laws are almost word for word identical, but with
  - **voltage** substituted for **current**
  - **loop** substituted for **node**
- This similarity is part of a larger pattern in the roles of voltage and current in network analysis
- This similarity with all its implications, is termed the **principal of duality**

Duality [2]

- The following is a table of some dual quantities and concepts

Quantity	Dual	Quantity	Dual
Current	Voltage	Link	Tree branch
Branch current	Branch voltage	Link current	Tree branch voltage
Mesh	Node	Tie-set	Cut-set
Mesh current	Node voltage	Short circuit	Open circuit
Loop	Node pair	Shunt paths	Series paths
Loop current	Node pair voltage	Capacitor	Inductor
Number of independent loops	Number of independent node pairs	Resistance	Conductance
		Charge	Flux linkages

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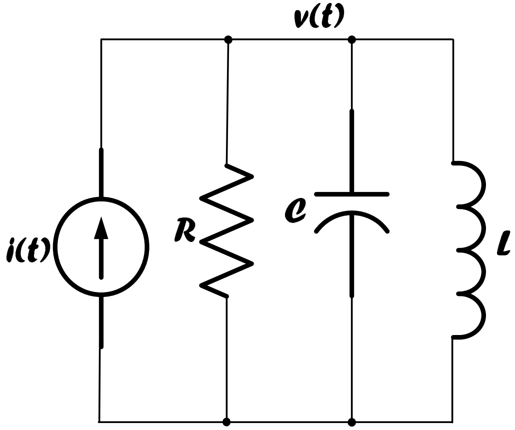
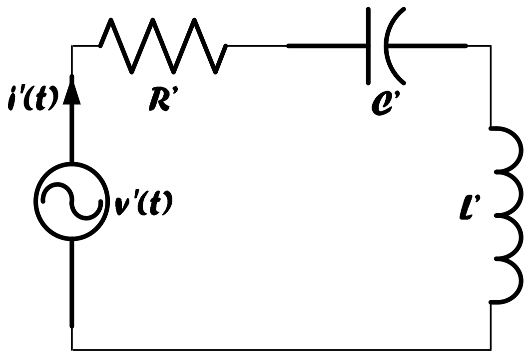
Duality [3]

- Consider the two networks below, with their equilibrium equations





$$v(t)G + C\frac{d\,v(t)}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(t)dt = i(t) \qquad i^{'}(t)R^{'} +\ L^{'}\frac{d\,i^{'}(t)}{dt} + \frac{1}{C^{'}}\int_{-\infty}^{t} i^{'}(t)dt = v^{'}(t)$$

- The two equations are mathematically equivalent, but for the letter symbols
- The solution of one equation is also the solution of the other equation
- The two networks are **duals** with the roles of the voltage and current interchanged
- Analogous terms can be identified
- Dual quantities are then easily determined from these terms
- **Dual** networks are **not identical** networks, and cannot replace one another
- Dual networks for which the coefficients in the LDE equations match, are called **inverse networks**

Duality [4]

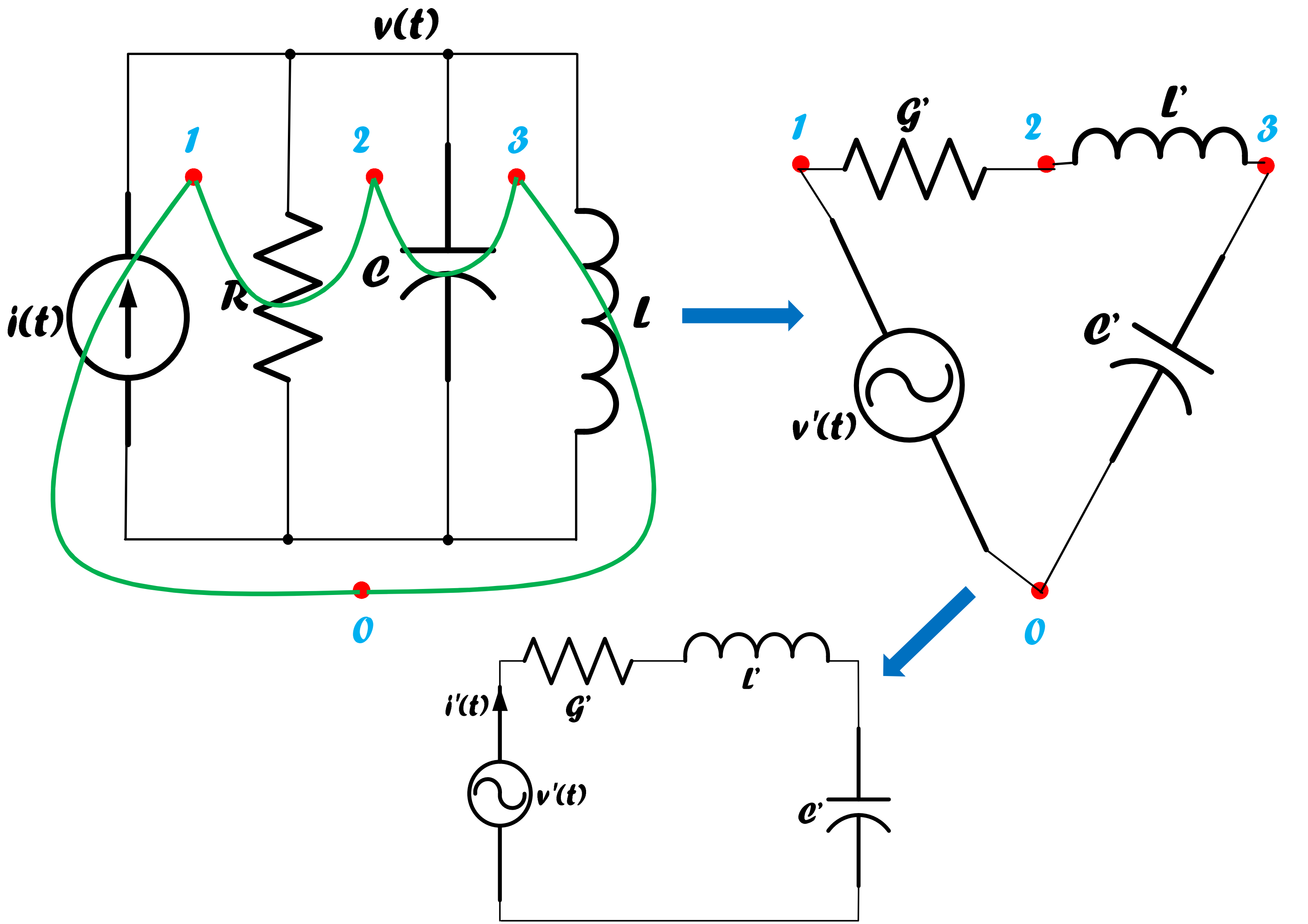
$$v(t)G + C\frac{d\,v(t)}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(t)dt = i(t) \qquad i'(t)R' + L'\frac{d\,i'(t)}{dt} + \frac{1}{C'}\int_{-\infty}^{t} i'(t)dt = v'(t)$$

- In this example, the two networks will be **inverse** networks if:

  $$\frac{i(t)}{v'(t)} = \frac{G}{R'} = \frac{C}{L'} = \frac{C'}{L} = k$$ where *k* is a constant

- Simple method exists for developing the dual of a given **planar network**
  - Inside each mesh, place a node and number it
  - Place an extra node external to the network, this is the datum node
  - Duplicate the arrangement of numbered nodes on a separate space for dual construction
  - Draw lines from node-to-node THROUGH the elements in the original network, going through only one element at a time
  - For each element traversed on the original network, connect the dual element on the dual network being constructed
  - Continue the process until ALL elements have been traversed
  - The dual of the original network will then have been obtained
- The process is illustrated using the current network examples

Duality [5]



Duality [6]

Suppose the values of the components in the original circuit were $R = 5\Omega, C = 0.1F, L = 2H$, and $i(t) = 10\sin 20t\ A$. Determine the values in the dual network that would make it an inverse network. Let $k = 2$

- Note that in the first network KCL is used, and so all terms are current terms
- In the dual network, the mathematically equivalent LDE is obtained from KVL
- Since we have that $\dfrac{i(t)}{v^{'}(t)} = \dfrac{G}{R^{'}} = \dfrac{C}{L^{'}} = \dfrac{C^{'}}{L} = k\ = 2$
- Then $\Rightarrow v^{'}(t) = \dfrac{i(t)}{2} = \dfrac{10\sin 20t}{2} = 5\sin 20t\,V$ $\qquad R^{'}\ = \dfrac{G}{2} = \dfrac{0.2}{2} = 0.1\Omega$

$$L^{'}\ = \frac{C}{2} = \frac{0.1}{2} = 0.05H \qquad C^{'}\ = 2L = 2 \times 2 = 4F$$

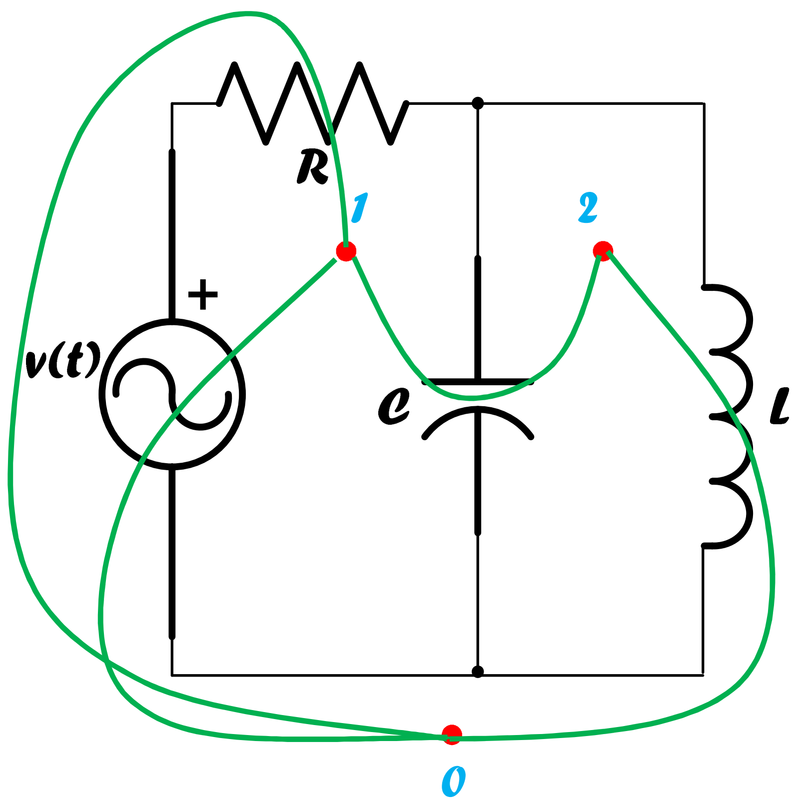
- The two equations would thus be

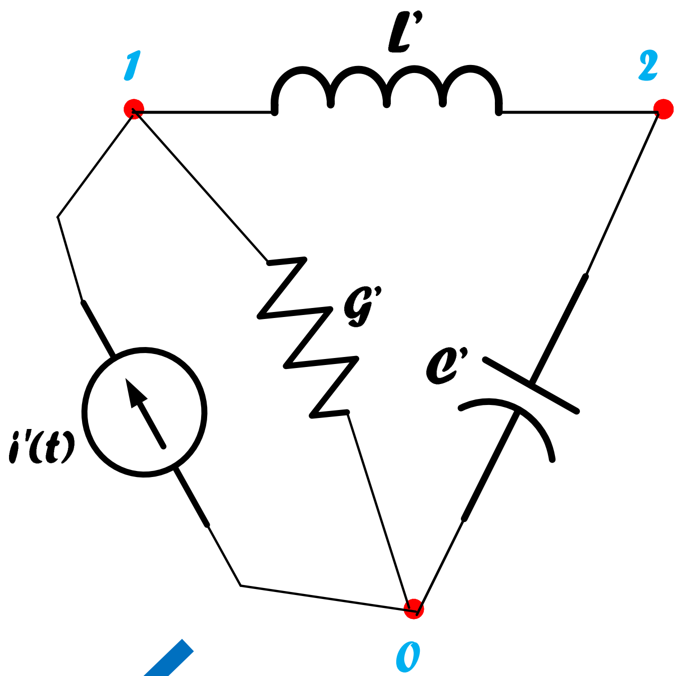
$$v(t)G + C\frac{d\,v(t)}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(t)dt = i(t) \quad \Rightarrow \quad 0.2v(t) + 0.1\frac{d\,v(t)}{dt} + 0.5\int_{-\infty}^{t} v(t)dt = 10\sin 20t$$

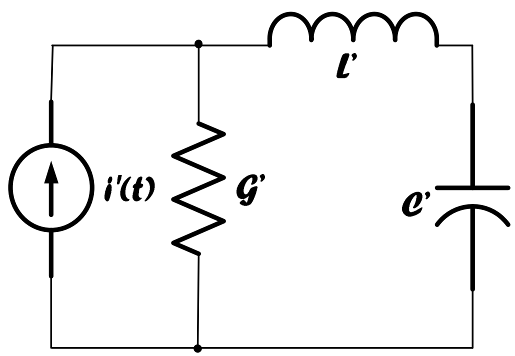
$$i^{'}(t)R^{'}\ +\ L^{'}\frac{d\,i^{'}(t)}{dt} + \frac{1}{C^{'}}\int_{-\infty}^{t} i^{'}(t)dt = v^{'}(t) \Rightarrow \quad 0.1i^{'}(t) +\ 0.05\frac{d\,i^{'}(t)}{dt} + 0.25\int_{-\infty}^{t} i^{'}(t)dt = 5\sin 20t$$

- The two equations are mathematically identical, so the solution obtained for one would be identical to the one obtained for the other

Duality [7]







$$\frac{v(t)}{i'(t)} = \frac{R}{G'} = \frac{L'}{C} = \frac{L}{C'} = k$$

Content

- Duality in Circuit Theory
- **Topological analysis**

Topological Analysis [1]

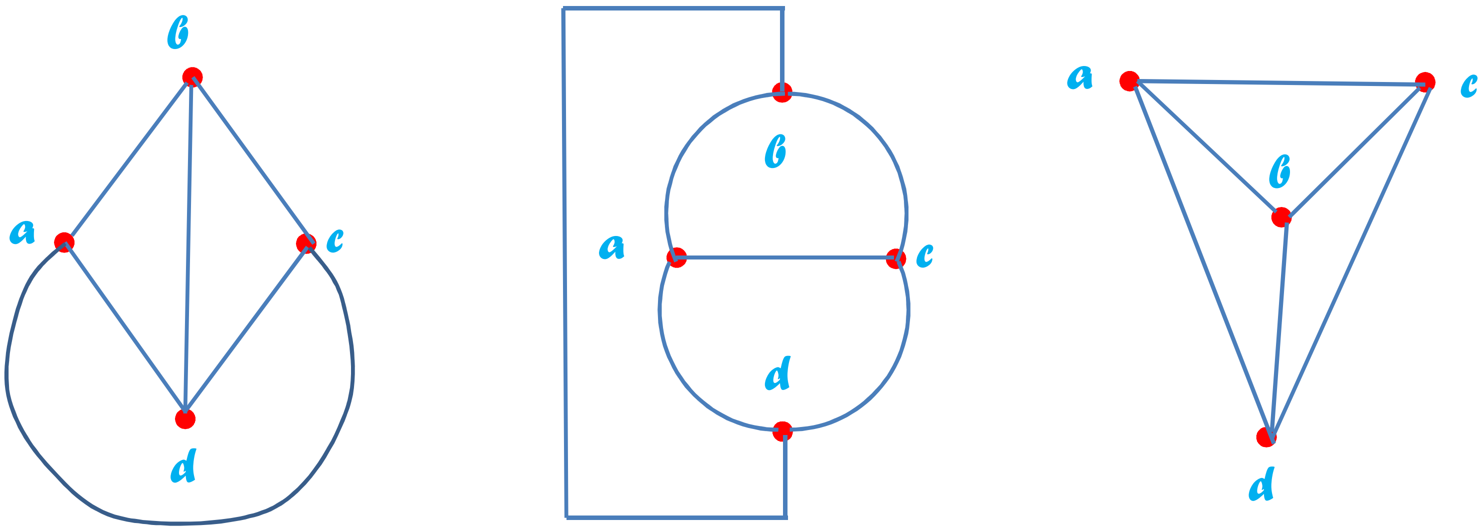
- KVL and KCL analysis work well for simple networks
- **Graph theory** however is useful for more complex networks
- Graphs may be thought of as skeletons of the networks
- Graphs are defined by their **nodes** (or vertex) and **branches**
  - Number of nodes normally denoted n
  - Number of branches normally denoted b

- Procedure for obtaining a network graph is to
  - Identify and label the nodes on the original network; with only one passive element per branch
  - Duplicate the arrangement of numbered nodes on a separate space for graph construction
  - On the graph, replace the passive elements with lines
  - Replace the independent sources with their **internal resistance**

- **Oriented or directed graphs** are obtained by indicating reference directions for current flow on each branch on the graph
- **Rank** of a graph is $n - 1$

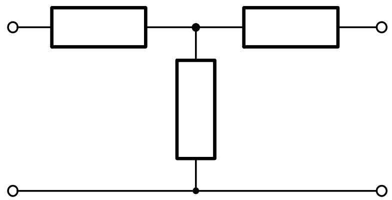
Topological Analysis [2]

- Graphs are used to describe the topological properties of networks
- Topology deals with propertied of networks unaffected by
  - Stretching
  - Twisting
  - Otherwise distorting the size and shape of the network
- Schematic of a circuit is often nothing like the wiring of the circuit and graphs enable the equivalence to be more obvious
- The following topologies appear different but are actually the same

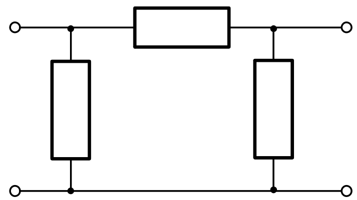


Topological Analysis [3]

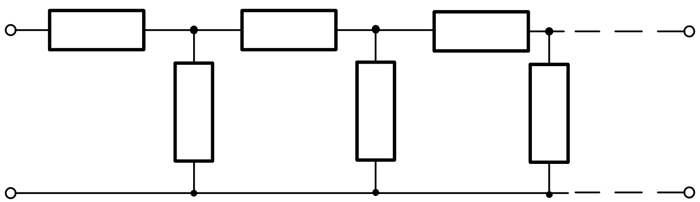
- Some often encountered topologies are:
  - T-network (also Y- or star-)
  - Π-network (also ∇-)
  - Ladder network
  - Bridged-T-network
  - Lattice network
  - Bridge network



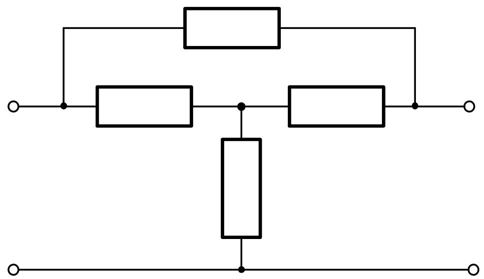
T-network



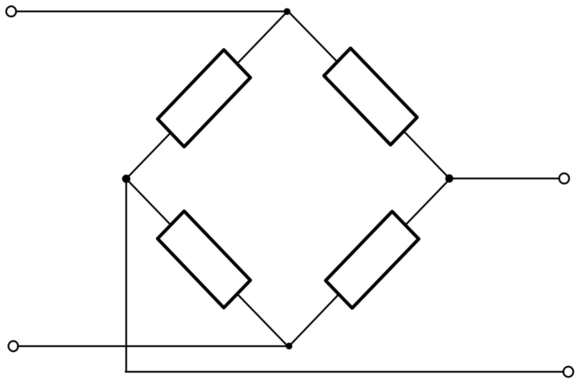
Π-network



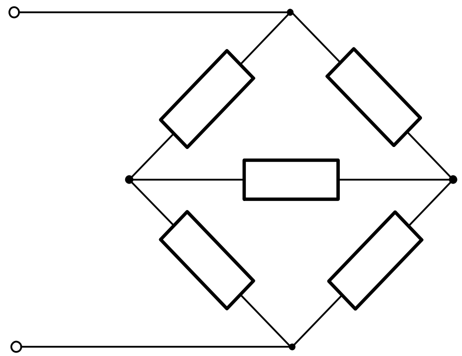
Ladder-network



Bridged-T-network



Lattice network



Bridge-network

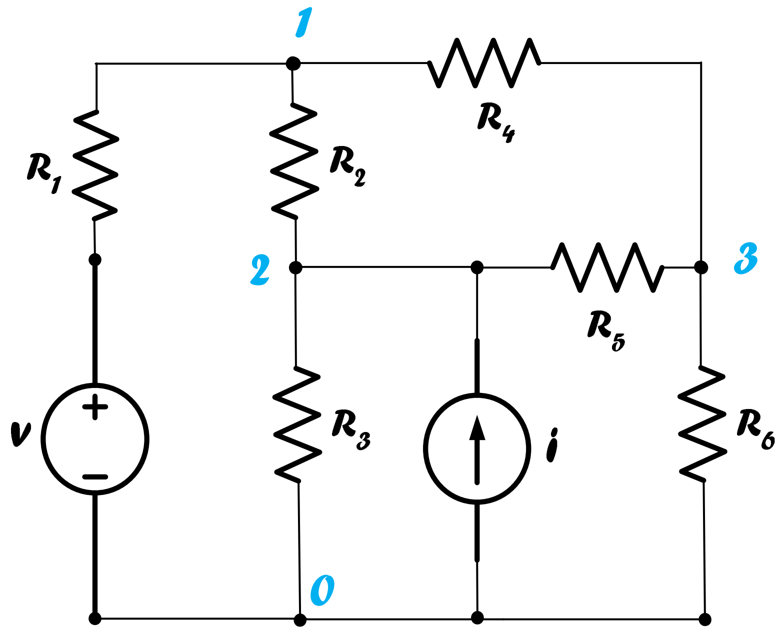
Topological Analysis [4]

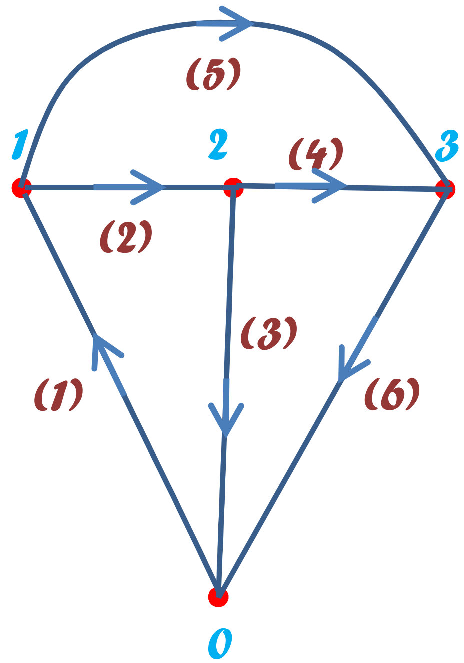
- **Balanced** or balanceable vs **unbalanced** networks
- Network graphs may have more than one separate part e.g. in magnetically coupled circuits
- Graphs that can be drawn on paper without any crossing lines is said to be **planar**; Otherwise **non-planar**
- Two nodes used in defining a particular voltage variable is called a **node pair**
- A closed path in a graph formed by a number of connected branches is called a **loop** or **mesh**
- A **subgraph** of a given graph formed by removing branches from the original graph
- Important subgraph is the **tree**
- A tree, of a **connected graph** of n nodes and b branches, has the properties
  - Contains all the nodes of the graph
  - No node is isolated
  - Contains no closed paths
  - Contains $n-1$ branches
- Branches left out of the graph to form a tree, are called **links** or **chords**; # $b-(n-1)$
- Tree branches are called **twigs**; # $n-1$

Topological Analysis [5]

- Determine the graph of the given network. Solve the network using graphical methods
- Assume values for the resistors and sources as follows:

$R_1 = R_4 = 3\Omega\,;\, R_2 = R_5 = 4\Omega\,;\, R_3 = 2\Omega\,;\, v = 10\ V;\, i = 4\ A$





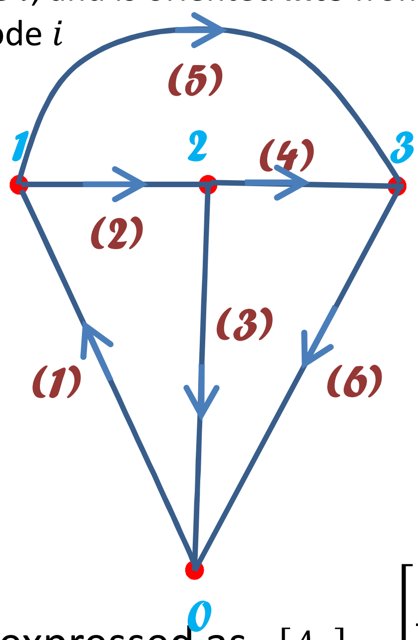
Final diagram is of an oriented or directed graph

Topological Analysis [6]

- Branches which meet at a node are said to be **incident** at the node
- Branch incidence is represented in an **Incidence Matrix**
- The Incidence Matrix completely defines the oriented graph
- **Complete** or **Augmented** Incidence Matrix, **[A_a]** for a graph of *n* nodes and *b* branches is a rectangular matrix of order *n x b*
- Elements of the matrix, a_{ij} have the following values
  - $a_{ij} = 1$, if branch (j) is incident at node i, and is oriented **away** from node i
  - $a_{ij} = -1$, if branch (j) is incident at node i, and is oriented **into** from node i
  - $a_{ij} = 0$, if branch (j) is **not** incident at node i
- For the graph in the example

branches

nodes		(1)	(2)	(3)	(4)	(5)	(6)
	0	1	0	-1	0	0	-1
	1	-1	1	0	0	1	0
	2	0	-1	1	1	0	0
	3	0	0	0	-1	-1	1



- Complete Incident Matrix is therefore expressed as $[A_a] = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$

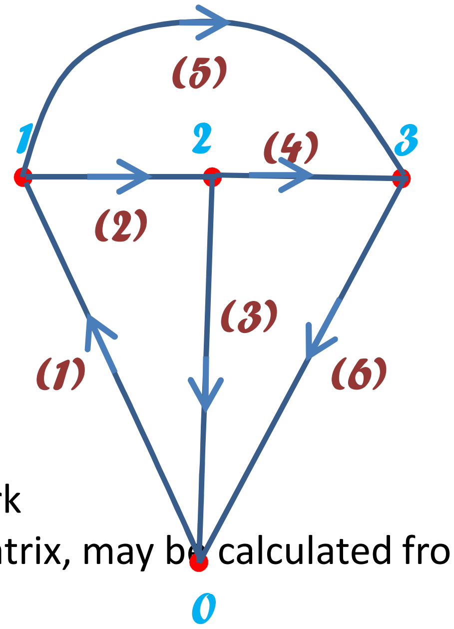
Topological Analysis [7]

- Note that since each branch is associated with only **two** nodes, each column has **only two non-zero** values
- The non-zero values in every column are 1 and -1, indicating the branch leaves one node and enters another node
- **Sum of values** in every column is therefore **zero**
- Any row of the matrix can be written as a **linear combination** of the remaining rows
- Rows are therefore **not independent** and matrix is not very useful mathematically as its determinant is zero
- Matrix is adapted to the **(Reduced) Incidence Matrix, [A]** by dropping the datum node row

$$[A_a] = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$[A] = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$



- Reduced Incidence Matrix is the one used in solving the network
- **Number of possible trees**, N_t for the graph of the incidence matrix, may be calculated from $N_t = |[A][A]^T|$ where $[A]^T$ is the transpose of $[A]$

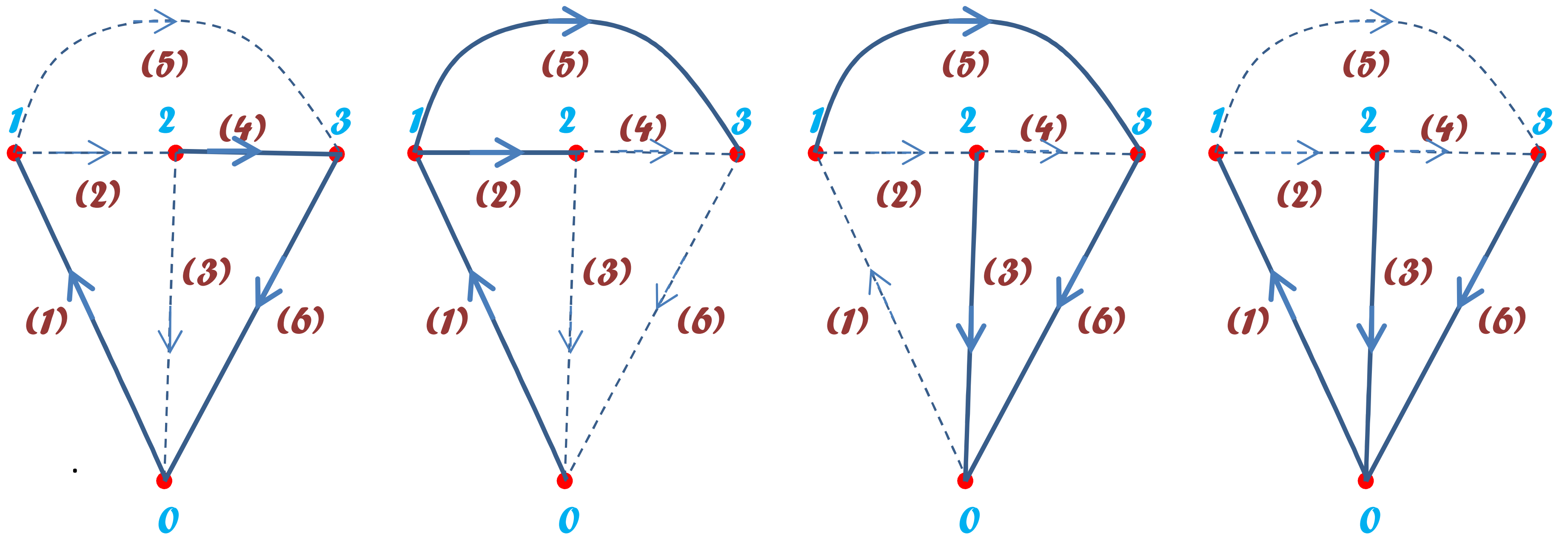
Topological Analysis [8]

- In this example

$$N_t = |[A][A]^T| = \left| \begin{bmatrix} -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \right|$$

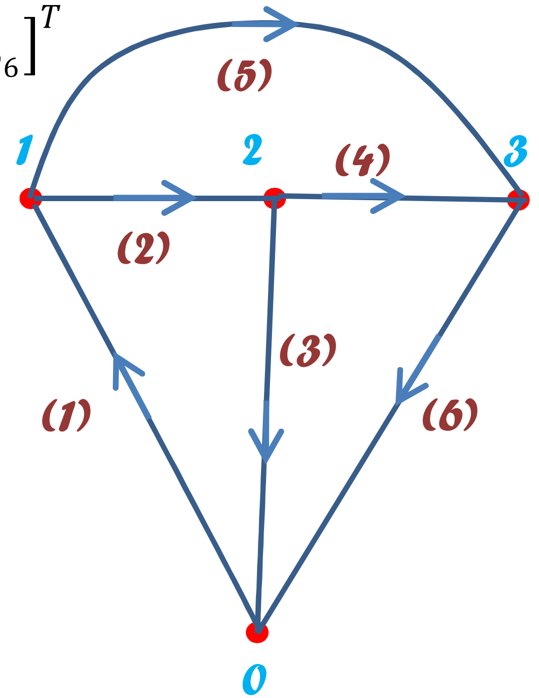
$$= 3(9-1) + (-3-1) - (1+3) \;\; = 16$$

- There are therefore **16 possible trees** for this network graph
- A few of them are shown here (links are shown dotted)



Topological Analysis [9]

- Incidence Matrix facilitates the determination of **branch voltages** from the **node voltages**
- If $[v_b]$ is the branch voltage matrix, and $[v_n]$ is the node voltage matrix, then $$[v_b] = [A]^T [v_n]$$
- In the current example $$[v_b] = \begin{bmatrix} v_{b_1} & v_{b_2} & v_{b_3} & v_{b_4} & v_{b_5} & v_{b_6} \end{bmatrix}^T$$ $$[v_n] = \begin{bmatrix} v_{n_1} & v_{n_2} & v_{n_3} \end{bmatrix}^T$$



- Therefore

$$\begin{bmatrix} v_{b_1} \\ v_{b_2} \\ v_{b_3} \\ v_{b_4} \\ v_{b_5} \\ v_{b_6} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{n_1} \\ v_{n_2} \\ v_{n_3} \end{bmatrix} = \begin{bmatrix} -v_{n_1} \\ v_{n_1} - v_{n_2} \\ v_{n_2} \\ v_{n_2} - v_{n_3} \\ v_{n_1} - v_{n_3} \\ v_{n_6} \end{bmatrix}$$

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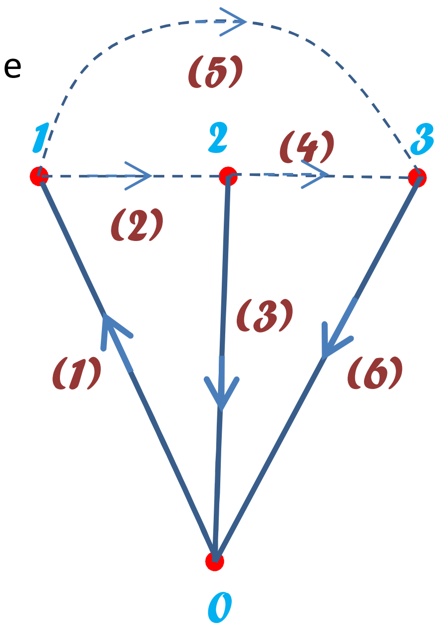
Topological Analysis [10]

- When links are replaced **one-at-a-time** on a tree of the graph, they form loops called **fundamental circuits** or **f-circuits**
- The **set of branches** forming an f-circuit is called a **tie-set**
- The f-circuit **loop current direction** is chosen to **coincide** with the **link direction**
- Number of f-circuits is equal to the number of links i.e. *b-(n-1)*
- The **Tie-set Matrix**, **[B]** for a graph of *n* nodes and *b* branches is a rectangular matrix of order *b-(n-1) x b*
- Elements of the matrix, b_{ij} have the following values
  - $b_{ij} = 1$, if branch (j) is in the f-circuit loop i, and their orientation is the same
  - $b_{ij} = -1$, if branch (j) is in the f-circuit loop i, and their orientation differ
  - $b_{ij} = 0$, if branch (j) is **not** in the f-circuit loop i



For the tree of the graph alongside

- f-ccts are numbered by link
- Ordered numerically
- Set starts with link

f-cct (2): { (2), (1), (3)}
f-cct (4): { (4), (3), (6)}
f-cct (5): { (5), (1), (6)}

branches

f-cct		**(1)**	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
	(2)	**1**	**1**	**1**	**0**	**0**	**0**
	(4)	**0**	**0**	**-1**	**1**	**0**	**1**
	(5)	**1**	**0**	**0**	**0**	**1**	**1**

- Tie-Set Matrix is therefore expressed as $[B] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

Topological Analysis [11]

- Tie-set Matrix facilitates the determination of **branch currents** from the **loop currents**
- If $[i_b]$ is the branch current matrix, and $[i_l]$ is the loop current matrix, then

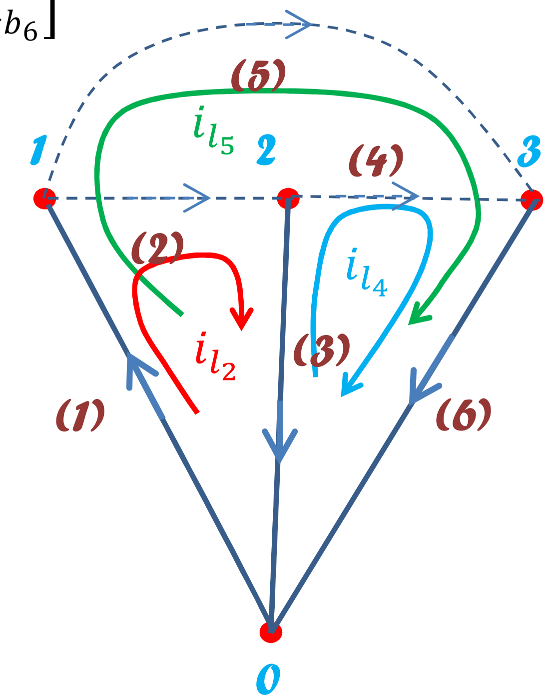
$$[i_b] = [B]^T [i_l]$$

- In the current example

$$[i_b] = \begin{bmatrix} i_{b_1} & i_{b_2} & i_{b_3} & i_{b_4} & i_{b_5} & i_{b_6} \end{bmatrix}^T$$

$$[i_l] = \begin{bmatrix} i_{l_2} & i_{l_4} & i_{l_5} \end{bmatrix}^T$$

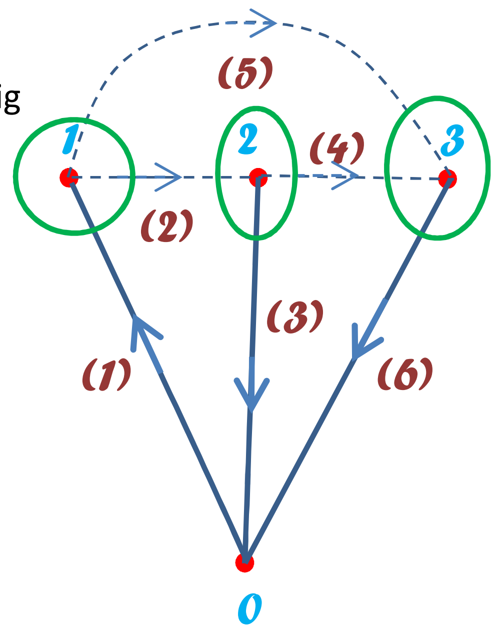
- Therefore



$$\begin{bmatrix} i_{b_1} \\ i_{b_2} \\ i_{b_3} \\ i_{b_4} \\ i_{b_5} \\ i_{b_6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} i_{l_2} \\ i_{l_4} \\ i_{l_5} \end{bmatrix} = \begin{bmatrix} i_{l_2} + i_{l_5} \\ i_{l_2} \\ i_{l_2} - i_{l_4} \\ i_{l_4} \\ i_{l_2} \\ i_{l_4} + i_{l_5} \end{bmatrix}$$

Topological Analysis [12]

- On the tree of a graph, if we draw a set of **closed surfaces** (**cuts**), each of which **cuts only ONE twig** and a **MINIMUM number of links**, each closed surface will describe a **cut-set**
- A **fundamental cut-set** (f-cutset) is the set of **twigs and links** cut by such closed surface
- There shall be as many f-cutsets as there are twigs i.e. *n-1*
- The f-cutset orientation is chosen to **coincide** with the **twig direction** with respect to the closed surface
- The **Cut-set Matrix**, **[Q]** for a graph of *n* nodes and *b* branches is a rectangular matrix of order *(n-1) x b*
- Elements of the matrix, q_{ij} have the following values
  - $q_{ij} = 1$, if branch (j) is in the f-cutset i, and their orientation is same as twig
  - $q_{ij} = -1$, if branch (j) is in the f-cutset i, and their orientation differs from twig
  - $q_{ij} = 0$, if branch (j) is **not** in the f-cutset i



For the tree of the graph alongside

- f-cutsets are numbered by twig
- Ordered numerically
- Set starts with cut twig

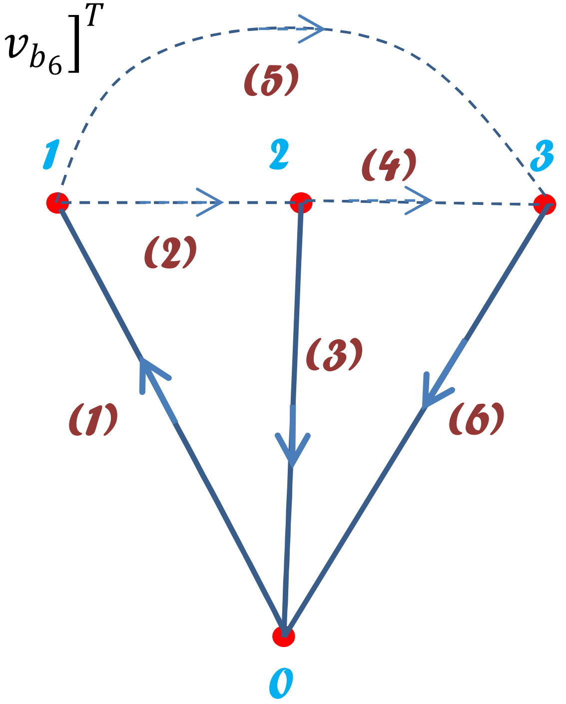
f-cutset (1): { (1), (2), (5)}
f-cutset (3): { (3), (2), (4)}
f-cutset (6): { (6), (4), (5)}

		branches					
		(1)	(2)	(3)	(4)	(5)	(6)
f-cutset	(1)	**1**	**-1**	**0**	**0**	**-1**	**0**
	(3)	**0**	**-1**	**1**	**1**	**0**	**0**
	(6)	**0**	**0**	**0**	**-1**	**-1**	**1**

- Cut-Set Matrix is therefore expressed as $[Q] = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$

Topological Analysis [13]

- Cut-set Matrix facilitates the determination of **branch voltages** from the **twig voltages**
- If $[v_b]$ is the branch voltage matrix, and $[v_t]$ is the twig voltage matrix, then
$$[v_b] = [Q]^T[v_t]$$
- In the current example $[v_b] = \begin{bmatrix} v_{b_1} & v_{b_2} & v_{b_3} & v_{b_4} & v_{b_4} & v_{b_6} \end{bmatrix}^T$
$$[v_t] = \begin{bmatrix} v_{t_1} & v_{t_3} & v_{t_6} \end{bmatrix}^T$$



- Therefore

$$\begin{bmatrix} v_{b_1} \\ v_{b_2} \\ v_{b_3} \\ v_{b_4} \\ v_{b_5} \\ v_{b_6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{t_1} \\ v_{t_3} \\ v_{t_6} \end{bmatrix} = \begin{bmatrix} v_{t_1} \\ -v_{t_1} - v_{t_3} \\ v_{t_3} \\ v_{t_3} - v_{t_6} \\ -v_{t_1} - v_{t_6} \\ v_{t_6} \end{bmatrix}$$

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Summary

Today's class looked at Laplace transform application

- Duality in Circuit Theory
- Topological analysis

QUESTIONS?