FEE321 – E.C.T IIA – Oct 2020

Lecture 12: Laplace Transform (6) (2 hrs)

Lecturer: *Prof H A Ouma*

08/01/2021

Overview

Today's class continues the look at LT application

- Transform circuit elements
- Transform circuits

Content

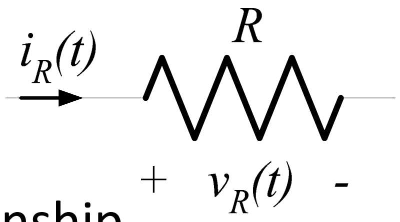
- **Transform circuit elements**

Transform elements[1]

- Time domain circuit elements may be transformed to **s-domain elements**
- These are then used in **s-domain models** and **analysis**
- Transform circuit elements include the possible initial conditions of the elements
- Initial voltage for the capacitor
- Initial current for the inductor
- Element transformation is through their time domain VI relationships
- Solution obtained by use of these complete models therefore give circuit responses that take care of the complete circuit state; both steady state and transient state
- s-domain solutions are then normally transformed back to time domain solutions as seen earlier

Transform elements[2]

Resistor



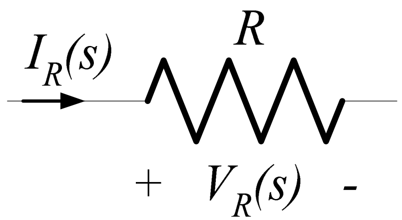
- VI relationship

$$v_R(t) = i_R(t)R$$

- Applying Laplace transform to the equation gives

$$V_R(s) = I_R(s)R$$

- Thus in s-domain the resistor model is



- The time domain resistor would therefore be replaced with the s-domain equivalent in the s-domain model

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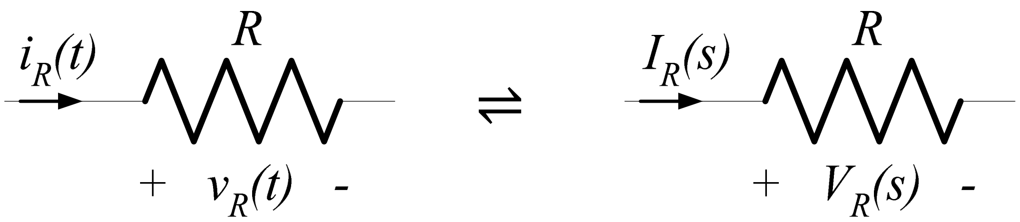
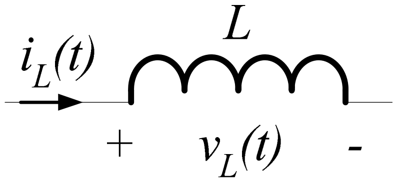
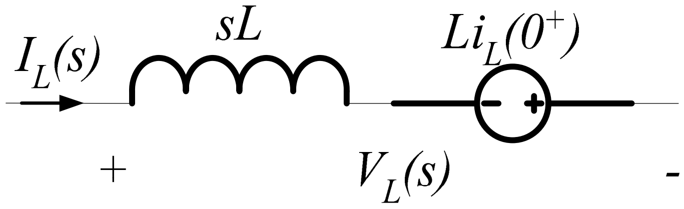


Transform elements[3]

Inductor 

- VI relationship $v_L(t) = L\dfrac{di_L(t)}{dt}$
- Applying Laplace transform to the equation gives $V_L(s) = L\{sI_L(s) - i_L(0^+)\}$
  $= sLI_L(s) - Li_L(0^+)$
- Looking at the expression as a KVL expression, it indicates a voltage drop due to the current through the element *plus* a **constant voltage rise** in series
- Thus possible s-domain inductor model is 

- The time domain inductor would therefore be replaced with the s-domain equivalent in the s-domain model

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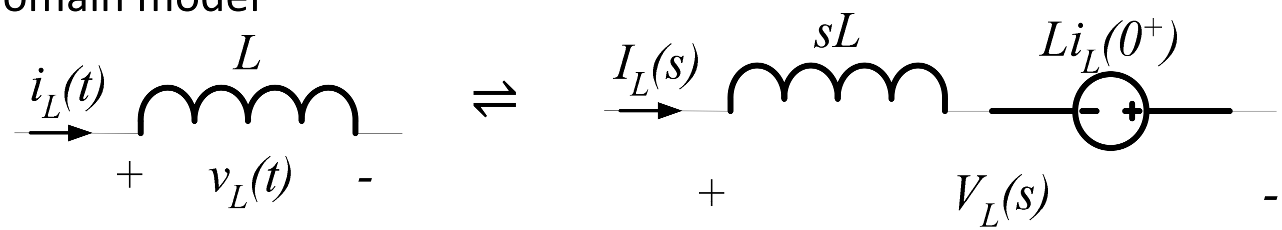
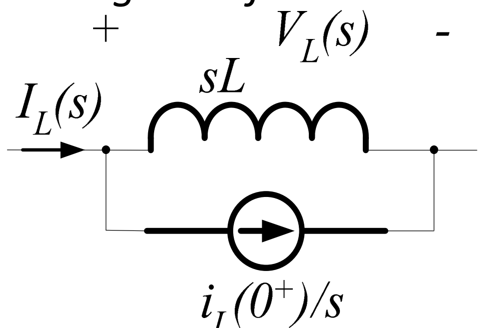


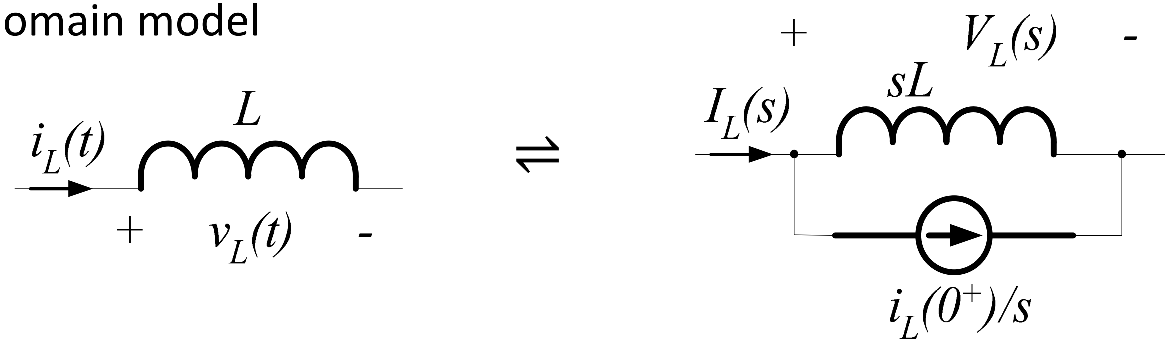
Transform elements[4]

Inductor

- The transform VI relationship may be rearranged as a current expression
- Thus $V_L(s) = L\{sI_L(s) - i_L(0^+)\} \quad \Rightarrow I_L(s) = \dfrac{V_L(s)}{sL} + \dfrac{i_L(0^+)}{s}$
- Looking at the expression as a KCL expression, it indicates node having a current through the element *plus* a **constant current source** *feeding out of* the node
- Thus another possible s-domain inductor model is

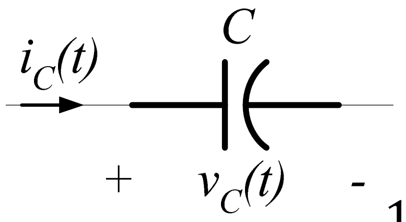


- The time domain inductor could therefore be replaced with the s-domain equivalent in the s-domain model

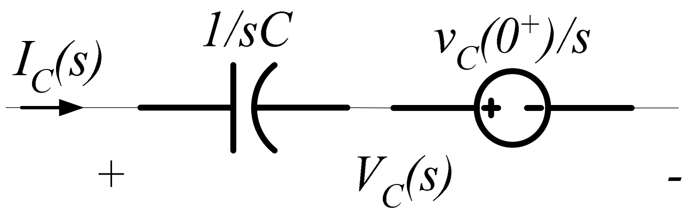


Transform elements[5]

Capacitor



- VI relationship $v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t)\,dt$
- Applying Laplace transform to the equation gives $V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0^+)}{s}$
- Looking at the expression as a KVL expression, it indicates a voltage drop due to the current through the element *plus* a **constant voltage drop** in series
- Thus possible s-domain capacitor model is



- The time domain inductor would therefore be replaced with the s-domain equivalent in the s-domain model

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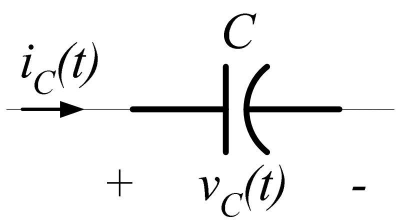
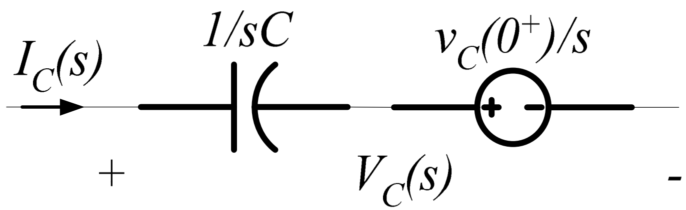


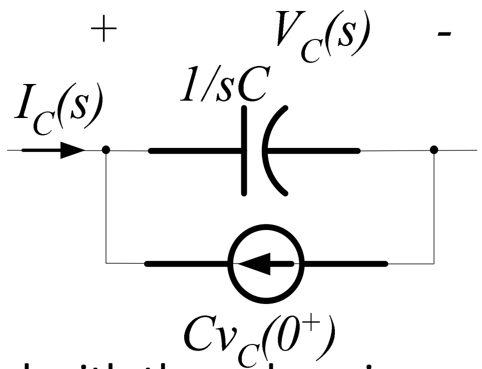
$\rightleftharpoons$



Transform elements[6]

Capacitor

- The transform VI relationship may be rearranged as a current expression
- Thus $V_C(s) = \dfrac{I_C(s)}{sC} + \dfrac{v_C(0^+)}{s} \quad \Rightarrow\ I_C(s) = sCV_C(s) - Cv_C(0^+)$
- Looking at the expression as a KCL expression, it indicates node having a current through the element *plus* a **constant current source** *feeding into* the node
- Thus another possible s-domain capacitor model is



- The time domain inductor could therefore be replaced with the s-domain equivalent in the s-domain model

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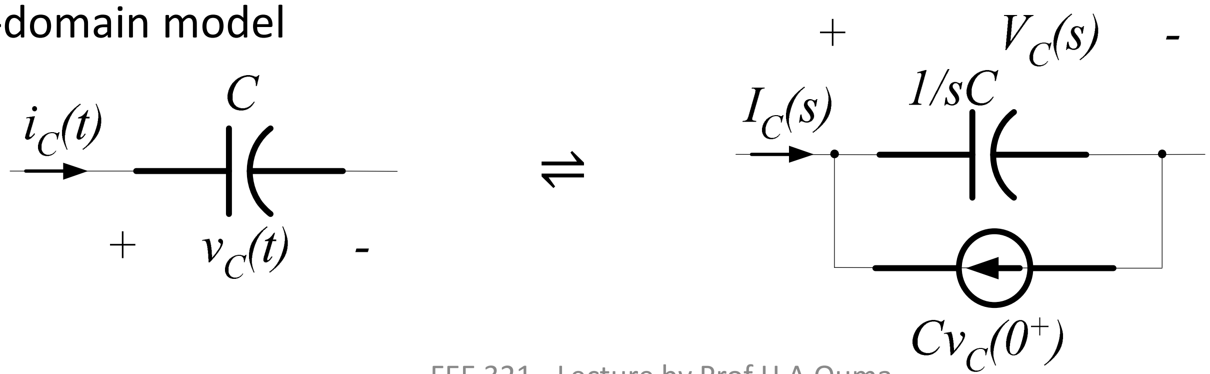


Transform elements[7]

Other notes on the transform circuit

- Sources transform into the s-domain with only the value now being the LT of the time domain value
  - For example a voltage source of value $v(t) = V_o \sin \omega t$ would transform into a voltage source with value $V(s) = \frac{\omega V_o}{s^2+\omega^2}$
- Type of source remains the same after transformation
  - For example, a VCCS remains a VCCS even after transformation, retaining its symbol but now having an s-domain value
- An s-domain circuit model is obtained by transforming all the time-domain elements
- The element interconnections remain the same,
- As do the branches and nodes of the circuit
- There can be **no mixing** of time domain elements with s-domain elements in **either** model
- Note that s-domain units for the quantities are generally different from the time domain units
- Current has the unit **ampere-seconds**
- Voltage has the unit **volt-second**
- Resistors R , capacitors $\left(\frac{1}{sC}\right)$ and inductors sL, **all** have the unit of **ohms** in the s-domain model
- For the inductor and capacitor models, choice of model depends on whether KCL or KVL is being used

Content

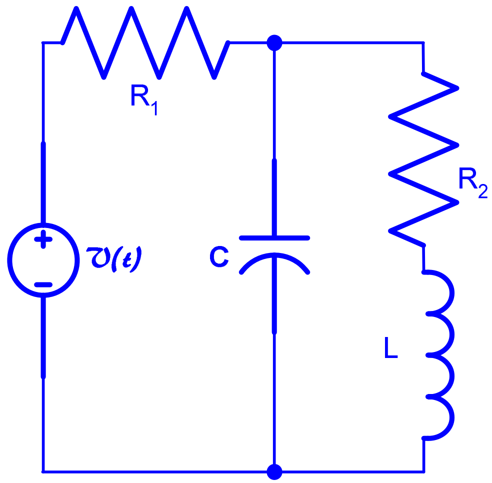
- Transform circuit elements
- **Transform circuits**

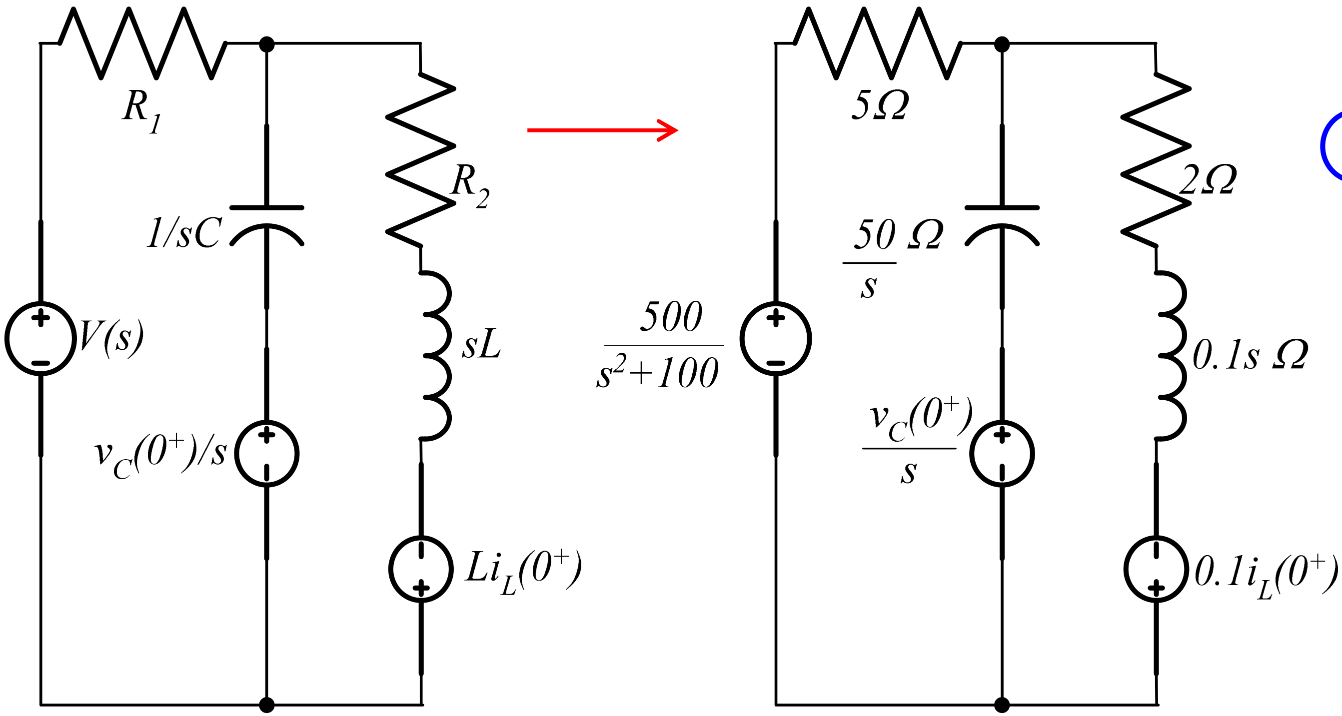
Transform circuit [1]

Example 1 (numerical)

A RLC circuit is supplied from an source $v(t) = 50u(t)\sin(10t)$ volts. Determine the transform circuit. Let $R_1 = 5\Omega$, $R_2 = 2\Omega$, $C = 20mF$ and $L = 0.1H$.

- Assuming we want to use **KVL** in solution, the **voltage source models** for the inductor and capacitor should be chosen
- Transform circuit is thus





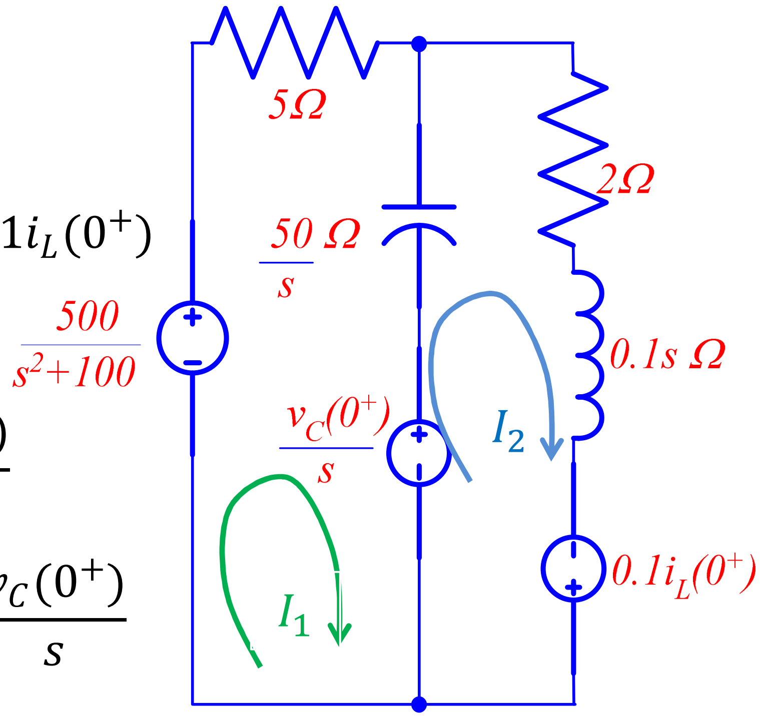
Transform circuit[2]

Example 1 (numerical, cont)

Using KVL with clockwise mesh currents, then circuit LDE are

$$\frac{50}{s}\{I_1(s) - I_2(s)\} + 5I_1(s) = \frac{500}{s^2+100} - \frac{v_C(0^+)}{s}$$

$$\frac{50}{s}\{I_2(s) - I_1(s)\} + 2I_2(s) + 0.1sI_2(s) = \frac{v_C(0^+)}{s} + 0.1i_L(0^+)$$



In standard form

$$I_1(s)\left\{\frac{50}{s} + 5\right\} + I_2(s)\left\{-\frac{50}{s}\right\} = \frac{500}{s^2+100} - \frac{v_C(0^+)}{s}$$

$$I_1(s)\left\{-\frac{50}{s}\right\} + I_2(s)\left\{0.1s + 2 + \frac{50}{s}\right\} = 0.1i_L(0^+) - \frac{v_C(0^+)}{s}$$

This is the same result obtained in the last lesson

Transform circuit[3]

Example 1 (numerical, cont)

Obtaining matrix equation

$$I_1(s)\left\{\frac{50}{s}+5\right\}+I_2(s)\left\{-\frac{50}{s}\right\}=\frac{500}{s^2+100}-\frac{v_C(0^+)}{s}$$

$$I_1(s)\left\{-\frac{50}{s}\right\}+I_2(s)\left\{0.1s+2+\frac{50}{s}\right\}=0.1i_L(0^+)-\frac{v_C(0^+)}{s}$$

Gives

$$\begin{bmatrix}\frac{50}{s}+5 & -\frac{50}{s}\\ -\frac{50}{s} & 0.1s+2+\frac{50}{s}\end{bmatrix}\begin{bmatrix}I_1(s)\\ I_2(s)\end{bmatrix}=\begin{bmatrix}\frac{500}{s^2+100}-\frac{v_C(0^+)}{s}\\ 0.1i_L(0^+)-\frac{v_C(0^+)}{s}\end{bmatrix}$$

With initial conditions set to zero

.

$$\begin{bmatrix}\frac{50}{s}+5 & -\frac{50}{s}\\ -\frac{50}{s} & 0.1s+2+\frac{50}{s}\end{bmatrix}\begin{bmatrix}I_1(s)\\ I_2(s)\end{bmatrix}=\begin{bmatrix}\frac{500}{s^2+100}\\ 0\end{bmatrix}$$

Transform circuit[4]

Example 1 (numerical, cont)

Writing terms as single fractions for easier manipulation

$$\begin{bmatrix} \frac{50+5s}{s} & -\frac{50}{s} \\ -\frac{50}{s} & \frac{0.1s^2+2s+50}{s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{500}{s^2+100} \\ 0 \end{bmatrix}$$

Using Cramer's rule

$$I_1(s) = \frac{\begin{vmatrix} \frac{500}{s^2+100} & -\frac{50}{s} \\ 0 & \frac{0.1s^2+2s+50}{s} \end{vmatrix}}{\begin{vmatrix} \frac{50+5s}{s} & -\frac{50}{s} \\ -\frac{50}{s} & \frac{0.1s^2+2s+50}{s} \end{vmatrix}} = \frac{\left(\frac{500}{s^2+100}\right)\left(\frac{0.1s^2+2s+50}{s}\right)}{\left\{\left(\frac{50+5s}{s}\right)\left(\frac{0.1s^2+2s+50}{s}\right)\right\} - \left\{\left(-\frac{50}{s}\right)\left(-\frac{50}{s}\right)\right\}}$$

$$= \frac{\left(\frac{50s^2+1000s+25000}{s(s^2+100)}\right)}{\left(\frac{5s^2+100s+2500+0.5s^3+10s^2+250s-2500}{s^2}\right)}$$

.

$$= \frac{\left(\frac{50s^2+1000s+25000}{s(s^2+100)}\right)}{\left(\frac{0.5s^3+15s^2+350s}{s^2}\right)} = \frac{\left(\frac{50s^2+1000s+25000}{(s^2+100)}\right)}{0.5s^2+15s+350}$$

Transform circuit[5]

Example 1 (numerical, cont)

$$I_1(s) = \frac{\left(\frac{50s^2 + 1000s + 25000}{(s^2+100)}\right)}{0.5s^2 + 15s + 350} = \frac{2(50s^2 + 1000s + 25000)}{(s^2+100)(s^2+30s+700)} = 100\frac{(s^2+20s+500)}{(s^2+100)(s^2+30s+700)}$$

Partial fraction decomposition

$$I_1(s) = 100\frac{(s^2+20s+500)}{(s^2+100)(s^2+30s+700)} = \frac{(As+B)}{(s^2+100)} + \frac{(Cs+D)}{(s^2+30s+700)}$$

$$As + B|_{s=j10} = (s^2+100)I_1(s)|_{s=j10} = 100\frac{(-100+j200+500)}{(-100+j300+700)} = 100\frac{(400+j200)}{(600+j300)}$$

$$\Rightarrow j10A + B = 100\frac{(4+j2)}{(6+j3)} = \frac{200}{3} \quad \Rightarrow A = 0 \quad B = \frac{200}{3}$$

$$Cs + D|_{s=-15+j\sqrt{475}} = (s^2+30s+700)I_1(s)|_{s=-15+j\sqrt{475}}$$

$$C\left(-15+j\sqrt{475}\right) + D = 100\frac{\left(-15+j\sqrt{475}\right)^2 + 20\left(-15+j\sqrt{475}\right) + 500}{\left(-15+j\sqrt{475}\right)^2 + 100}$$

.

$$= 100\frac{(-250 - j653.8348) + (-300 + j435.8899) + 500}{(-250 - j653.8348) + 100}$$

$$= 100\frac{(-50 - j217.9449)}{(-150 - j653.8348)} = \frac{100}{3} \quad \Rightarrow C = 0 \quad D = \frac{100}{3}$$

Transform circuit[6]

Example 1 (numerical, cont)

$$I_1(s) = \frac{(As+B)}{(s^2+100)} + \frac{(Cs+D)}{(s^2+30s+700)} \quad = \frac{200/3}{(s^2+100)} + \frac{100/3}{(s^2+30s+700)}$$

The inverse LT can then be obtained

$$I_1(s) = \frac{20}{3} \times \frac{10}{(s^2+100)} + \frac{100}{3\sqrt{475}} \times \frac{\sqrt{475}}{(s+15)^2 + \left(\sqrt{475}\right)^2}$$

$$\Rightarrow i_1(t) = u(t)\left\{6.6667 \sin 10t + 1.5294\, e^{-15t} \sin \sqrt{475}t\right\} \quad A$$

Note the **steady state component** and **the transient component**

The same procedure is followed to obtain the current $i_2(t)$

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Transform circuit[7]

Example 1 (numerical, cont)

Using Cramer's rule

$$I_2(s) = \frac{\begin{vmatrix} \frac{50+5s}{s} & \frac{500}{s^2+100} \\ -\frac{50}{s} & 0 \end{vmatrix}}{\left(\frac{0.5s^3+15s^2+350s}{s^2}\right)} = \frac{\left(\frac{50}{s}\right)\left(\frac{500}{s^2+100}\right)}{\left(\frac{0.5s^3+15s^2+350s}{s^2}\right)} = \frac{\left(\frac{25000}{s(s^2+100)}\right)}{\left(\frac{0.5s^3+15s^2+350s}{s^2}\right)}$$

$$= \frac{\frac{25000}{(s^2+100)}}{0.5s^2+15s+350} = \frac{2(25000)}{(s^2+100)(s^2+30s+700)} = \frac{(As+B)}{(s^2+100)} + \frac{(Cs+D)}{(s^2+30s+700)}$$

$$As+B|_{s=j10} = (s^2+100)I_2(s)|_{s=j10} = \frac{50000}{(-100+j300+700)} = \frac{200}{3} - j\,\frac{100}{3}$$

$$\Rightarrow A = -\frac{10}{3} \qquad B = \frac{200}{3}$$

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Transform circuit[8]

Example 1 (numerical, cont)

$$Cs + D|_{s=-15+j\sqrt{475}} = (s^2 + 30s + 700)I_2(s)|_{s=-15+j\sqrt{475}}$$

$$C\left(-15 + j\sqrt{475}\right) + D = \frac{50000}{\left(-15 + j\sqrt{475}\right)^2 + 100} = \frac{50000}{(-250 - j653.8348) + 100}$$

$$= \frac{50000}{(-150 - j653.8348)} = -\frac{50}{3} + j72.6483 \qquad \Rightarrow C = \frac{10}{3} \qquad D = \frac{100}{3}$$

$$I_2(s) = \frac{(As + B)}{(s^2 + 100)} + \frac{(Cs + D)}{(s^2 + 30s + 700)}$$

$$= \frac{(-10/3)s}{(s^2 + 100)} + \frac{200/3}{(s^2 + 100)} + \frac{(10/3)s}{(s^2 + 30s + 700)} + \frac{100/3}{(s^2 + 30s + 700)}$$

$$= \frac{(-10/3)s}{(s^2 + 100)} + \frac{20}{3} \times \frac{10}{(s^2 + 100)} + \frac{(10/3)(s + 15)}{(s + 15)^2 + \left(\sqrt{475}\right)^2} - \frac{50}{3\sqrt{475}} \frac{\sqrt{475}}{(s + 15)^2 + \left(\sqrt{475}\right)^2}$$

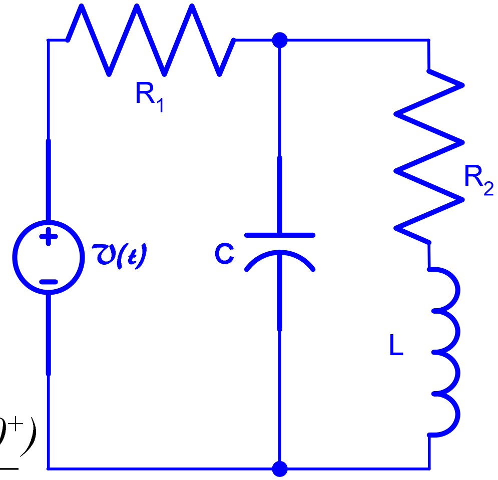
- $\Rightarrow i_2(t) = u(t)\left\{-3.3333 \cos 10t + 6.6667 \sin 10t + e^{-15t}\left(3.3333 \cos \sqrt{475}t - 0.7647 \sin \sqrt{475}\, t\right)\right\} \; A$

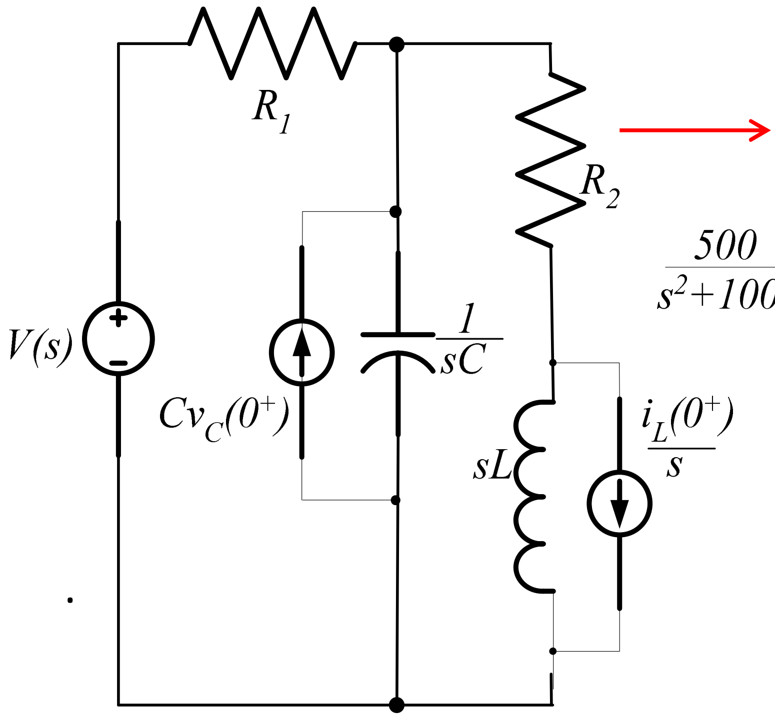
Transform circuit[9]

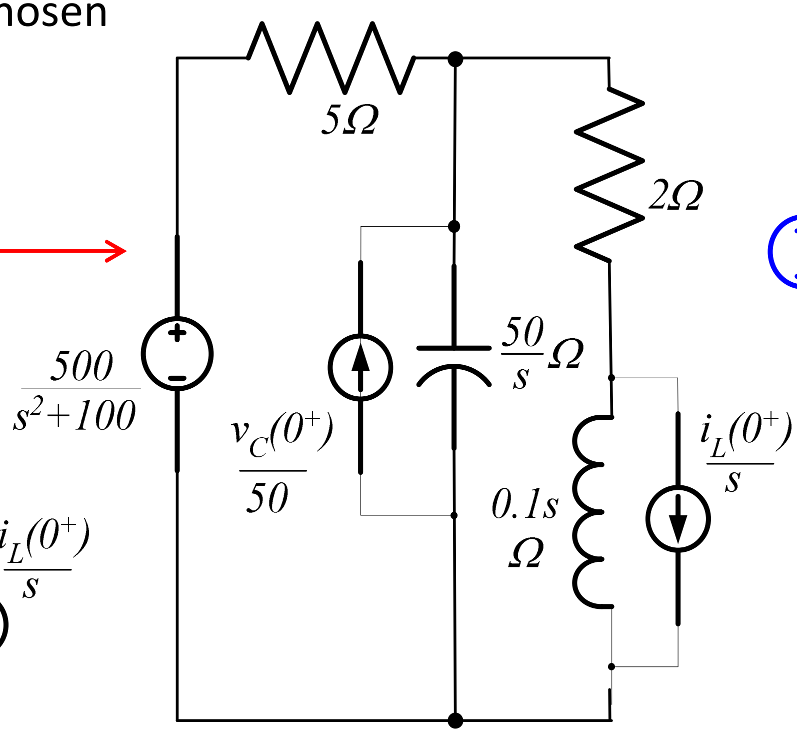
Example 2 (numerical)

A RLC circuit is supplied from an source $v(t) = 50u(t)\sin(10t)$ volts. Determine the transform circuit. Let $R_1 = 5\Omega$, $R_2 = 2\Omega$, $C = 20mF$ and $L = 0.1H$.

- Assuming we want to use **KCL** in solution, the **current source models** for the inductor and capacitor should be chosen
- Transform circuit is thus

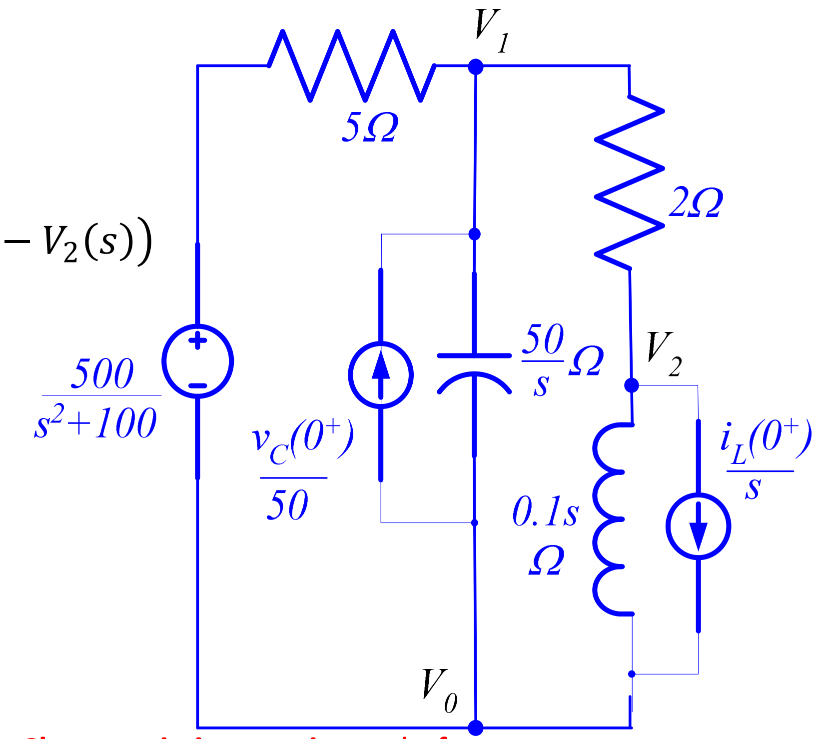






Transform circuit[10]

Example 2 (numerical, cont)



Using KCL, then circuit LDE are

$$0 = \frac{1}{5}\left(V_1(s) - \frac{500}{s^2+100}\right) - \frac{v_C(0^+)}{50} + V_1(s)\frac{s}{50} + \frac{1}{2}\left(V_1(s) - V_2(s)\right)$$

$$0 = \frac{1}{2}\left(V_2(s) - V_1(s)\right) + \frac{V_2(s)}{0.1s} + \frac{i_L(0^+)}{s})$$

In standard form

$$\frac{100}{s^2+100} + \frac{v_C(0^+)}{50} = V_1(s)\left(\frac{1}{5} + \frac{1}{2} + \frac{s}{50}\right) + V_2(s)\left(-\frac{1}{2}\right)$$

$$-\frac{i_L(0^+)}{s} = V_1(s)\left(-\frac{1}{2}\right) + V_2(s)\left(\frac{10}{s} + \frac{1}{2}\right))$$

Check that the same **Characteristic equation** as before is obtained from this system of KCL equations

Matrix equation

$$\begin{bmatrix} \frac{100}{s^2+100} + \frac{v_C(0^+)}{50} \\ -\frac{i_L(0^+)}{s} \end{bmatrix} = \begin{bmatrix} \frac{s+35}{50} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{s+20}{2s} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix}$$

Cramer's rule may then be used to solve the circuit

Transform circuit[11]

Example 2 (numerical, cont)

Assuming zero initial conditions

$$\begin{bmatrix} \dfrac{100}{s^2+100} \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{s+35}{50} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{s+20}{2s} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix}$$

Using Cramer's rule

$$V_1(s) = \frac{\begin{vmatrix} \dfrac{100}{s^2+100} & -\dfrac{1}{2} \\ 0 & \dfrac{s+20}{2s} \end{vmatrix}}{\begin{vmatrix} \dfrac{s+35}{50} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{s+20}{2s} \end{vmatrix}} = \frac{\left(\dfrac{100}{s^2+100}\right)\left(\dfrac{s+20}{2s}\right)}{\left(\dfrac{s+35}{50}\right)\left(\dfrac{s+20}{2s}\right) - \dfrac{1}{4}} = \frac{\left(\dfrac{100s+2000}{2s(s^2+100)}\right)}{\left(\dfrac{s^2+20s+35s+700-25s}{100s}\right)}$$

$$= \frac{\left(\dfrac{100s+2000}{2s(s^2+100)}\right)}{\left(\dfrac{s^2+30s+700}{100s}\right)} = \frac{5000(s+20)}{(s^2+100)(s^2+30s+700)}$$

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$$= \frac{(As+B)}{(s^2+100)} + \frac{(Cs+D)}{(s^2+30s+700)}$$

Transform circuit[12]

Example 2 (numerical, cont)

$$V_1(s) = \frac{5000(s+20)}{(s^2+100)(s^2+30s+700)} \quad = \frac{(As+B)}{(s^2+100)} + \frac{(Cs+D)}{(s^2+30s+700)}$$

$$As+B|_{s=j10} = (s^2+100)V_1(s)|_{s=j10} \;= \frac{5000(j10+20)}{(-100+j300+700)} = \frac{500(2+j)}{(6+j3)} = \frac{500}{3}$$

$$\Rightarrow A = 0 \qquad B = \frac{500}{3}$$

$$Cs+D|_{s=-15+j\sqrt{475}} = (s^2+30s+700)V_1(s)|_{s=-15+j\sqrt{475}}$$

$$C\left(-15+j\sqrt{475}\right)+D = 5000\frac{\left(-15+j\sqrt{475}\right)+20}{\left(-15+j\sqrt{475}\right)^2+100} \;= 5000\frac{\left(5+j\sqrt{475}\right)}{(-250-j653.8348)+100}$$

$$= 5000\frac{(5+j21.7945)}{(-150-j653.8348)} \;= \frac{500}{3} \qquad \Rightarrow C = 0 \qquad D = \frac{500}{3}$$

$$V_1(s) = \frac{5000(s+20)}{(s^2+100)(s^2+30s+700)} \quad = \frac{500/3}{(s^2+100)} + \frac{500/3}{(s^2+30s+700)}$$

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Transform circuit[13]

Example 2 (numerical, cont)

$$V_1(s) = \frac{5000(s+20)}{(s^2+100)(s^2+30s+700)} \quad = \frac{500/3}{(s^2+100)} + \frac{500/3}{(s^2+30s+700)}$$

The inverse LT can then be obtained

$$V_1(s) = \frac{50}{3} \times \frac{10}{(s^2+100)} + \frac{500}{3\sqrt{475}} \times \frac{\sqrt{475}}{(s+15)^2 + \left(\sqrt{475}\right)^2}$$

$$\Rightarrow v_1(t) = u(t)\left\{16.6667 \sin 10t + 7.6472\, e^{-15t} \sin \sqrt{475}t\right\} \quad V$$

Note the **steady state component** and **the transient component**

The same procedure is followed to obtain $v_2(t)$

.

Transform circuit[14]

Example 2 (numerical, cont)

Using Cramer's rule

$$V_2(s) = \frac{\begin{vmatrix} \frac{s+35}{50} & \frac{100}{s^2+100} \\ -\frac{1}{2} & 0 \end{vmatrix}}{\begin{vmatrix} \frac{s+35}{50} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{s+20}{2s} \end{vmatrix}} = \frac{\frac{1}{2}\left(\frac{100}{s^2+100}\right)}{\left(\frac{s^2+30s+700}{100s}\right)} = \frac{\frac{50}{(s^2+100)}}{\left(\frac{s^2+30s+700}{100s}\right)}$$

$$= \frac{5000s}{(s^2+100)(s^2+30s+700)}$$

$$= \frac{(As+B)}{(s^2+100)} + \frac{(Cs+D)}{(s^2+30s+700)}$$

$$As+B|_{s=j10} = (s^2+100)V_2(s)|_{s=j10} = \frac{j50000}{(-100+j300+700)} = \frac{j500}{(6+j3)} = \frac{100}{3} + j\frac{200}{3}$$

- $\Rightarrow A = \frac{20}{3} \quad B = \frac{100}{3}$

Transform circuit[15]

Example 2 (numerical, cont)

$$Cs + D|_{s=-15+j\sqrt{475}} = (s^2 + 30s + 700)I_2(s)|_{s=-15+j\sqrt{475}}$$

$$C\left(-15 + j\sqrt{475}\right) + D = \frac{j50000}{\left(-15 + j\sqrt{475}\right)^2 + 100} = \frac{j50000}{(-250 - j653.8348) + 100}$$

$$= \frac{j50000}{(-150 - j653.8348)} = -72.6483 - j\frac{50}{3} \quad \Rightarrow C = -0.7647$$
$$D = -84.1191$$

$$V_2(s) = \frac{(As + B)}{(s^2 + 100)} + \frac{(Cs + D)}{(s^2 + 30s + 700)}$$

$$= \frac{(20/3)s}{(s^2 + 100)} + \frac{100/3}{(s^2 + 100)} + \frac{-0.7647s}{(s^2 + 30s + 700)} + \frac{-84.1191}{(s^2 + 30s + 700)}$$

$$= \frac{(20/3)s}{(s^2 + 100)} + \frac{10}{3} \times \frac{10}{(s^2 + 100)} - \frac{0.7647(s + 15)}{(s + 15)^2 + \left(\sqrt{475}\right)^2} - \frac{72.6843}{\sqrt{475}} \frac{\sqrt{475}}{(s + 15)^2 + \left(\sqrt{475}\right)^2}$$

$$\Rightarrow v_2(t) = u(t)\left\{6.6667\cos 10t + 3.3333\sin 10t - e^{-15t}\left(0.7647\cos\sqrt{475}t + 3.3350\sin\sqrt{475}t\right)\right\} \quad V$$

Transform circuit[16]

Example 2 (numerical, cont)

- To **include** initial conditions (IC) carry out the solution again **but only for initial conditions**; then add the solution to the ones already obtained

$$\begin{bmatrix} \frac{v_C(0^+)}{50} \\ -\frac{i_L(0^+)}{s} \end{bmatrix} = \begin{bmatrix} \frac{s+35}{50} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{s+20}{2s} \end{bmatrix} \begin{bmatrix} V_1'(s) \\ V_2'(s) \end{bmatrix}$$

- Using Cramer's rule

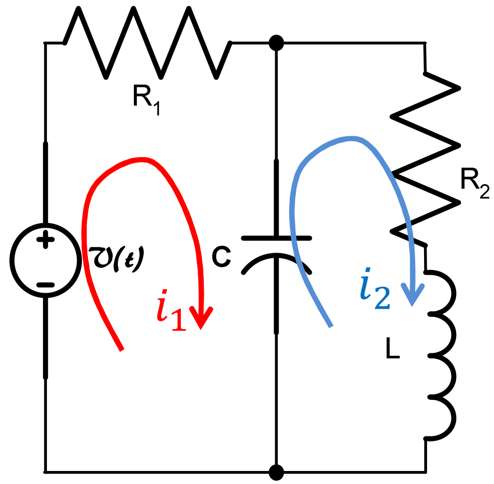
$$V_1'(s) = \frac{\begin{vmatrix} \frac{v_C(0^+)}{50} & -\frac{1}{2} \\ -\frac{i_L(0^+)}{s} & \frac{s+20}{2s} \end{vmatrix}}{\begin{vmatrix} \frac{s+35}{50} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{s+20}{2s} \end{vmatrix}} \qquad V_2'(s) = \frac{\begin{vmatrix} \frac{s+35}{50} & \frac{v_C(0^+)}{50} \\ -\frac{1}{2} & -\frac{i_L(0^+)}{s} \end{vmatrix}}{\begin{vmatrix} \frac{s+35}{50} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{s+20}{2s} \end{vmatrix}}$$

- The obtained time domain solutions are then added to the time domain solutions earlier obtained
- The result is the same as solving with the LHS column matrix with IC in place

Complex Exponential Solution[1]

Example 1 (numerical)

A RLC circuit is supplied from an source $v(t) = 50u(t)\sin(10t)$ volts. Assume that at $t = 0$ there is zero voltage across the capacitor, and no inductor current. Let $R_1 = 5\Omega$, $R_2 = 2\Omega$, $C = 20mF$ and $L = 0.1H$



Using KVL with clockwise mesh currents, then circuit LDE are

$$\frac{1}{C}\int_{-\infty}^{t}\{i_1(t) - i_2(t)\}\,dt + R_1 i_1(t) = v(t)$$

$$\frac{1}{C}\int_{-\infty}^{t}\{i_2(t) - i_1(t)\}\,dt + R_2 i_2(t) + L\frac{di_2(t)}{dt} = 0$$

Substituting in values

$$50\int_{-\infty}^{t}\{i_1(t) - i_2(t)\}\,dt + 5i_1(t) = 50u(t)\sin 10t$$

$$50\int_{-\infty}^{t}\{i_2(t) - i_1(t)\}\,dt + 2i_2(t) + 0.1\frac{di_2(t)}{dt} = 0$$

Complex Exponential Solution[2]

Example 1 (numerical, cont)

At steady state quantities have same form as the source

$v(t) = 50u(t)\sin 10t = Im\left[50u(t)e^{j10t}\right] \quad \Rightarrow i_1(t) = Im\left[I_1e^{j10t}\right] \quad i_2(t) = Im\left[I_2e^{j10t}\right]$

Substituting into equations

$$Im\left[50\int_{-\infty}^{t}\{I_1 - I_2\}e^{j10t}\,dt + 5I_1e^{j10t} = 50u(t)e^{j10t}\right]$$

$$Im\left[50\int_{-\infty}^{t}\{I_2 - I_1\}e^{j10t}\,dt + 2I_2e^{j10t} + 0.1\,\frac{dI_2e^{j10t}}{dt} = 0\right]$$

Which gives

$$Im\left[\frac{50\{I_1 - I_2\}}{j10}e^{j10t} + 5I_1e^{j10t} = 50u(t)e^{j10t}\right] \qquad \Rightarrow \frac{50\{I_1 - I_2\}}{j10} + 5I_1 = 50u(t)$$

$$Im\left[\frac{50\{I_2 - I_1\}}{j10}e^{j10t} + 2I_2e^{j10t} + jI_2e^{j10t} = 0\right] \qquad \Rightarrow \frac{50\{I_2 - I_1\}}{j10} + 2I_2 + jI_2 = 0$$

In standard form

$$\{5 - j5\}I_1 + \{j5\}I_2 = 50u(t)$$

$$\{j5\}I_1 + \{2 - j4\}I_2 = 0$$

$$\Rightarrow \begin{bmatrix} 5 - j5 & j5 \\ j5 & 2 - j4 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 50u(t) \\ 0 \end{bmatrix}$$

.

Complex Exponential Solution[3]

Example 1 (numerical, cont)

$$\begin{bmatrix} 5-j5 & j5 \\ j5 & 2-j4 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 50u(t) \\ 0 \end{bmatrix}$$

Using Cramer's rule

$$I_1 = \frac{\begin{vmatrix} 50u(t) & j5 \\ 0 & 2-j4 \end{vmatrix}}{\begin{vmatrix} 5-j5 & j5 \\ j5 & 2-j4 \end{vmatrix}} = \frac{50u(t)(2-j4)}{(5-j5)(2-j4)-(j5)(j5)} = \frac{100u(t)(1-j2)}{(-10-j30)+25} = \frac{100u(t)(1-j2)}{(15-j30)}$$

$$= \frac{20u(t)}{3} \qquad \Rightarrow i_1(t) = Im\left[\frac{20u(t)}{3}e^{j10t}\right] = 6.6667u(t)\sin 10t \;\; A$$

Similarly

$$I_2 = \frac{\begin{vmatrix} 5-j5 & 50u(t) \\ j5 & 0 \end{vmatrix}}{\begin{vmatrix} 5-j5 & j5 \\ j5 & 2-j4 \end{vmatrix}} = \frac{(-j5)50u(t)}{(15-j30)} = \frac{-j250u(t)}{(15-j30)} = \frac{u(t)(20-j10)}{3} = u(t)7.4536\angle-26.57^o$$

$$\Rightarrow i_2(t) = Im\left[u(t)7.4536\angle-26.57^o e^{j10t}\right] = 7.4536u(t)\sin(10t-26.57^o)$$

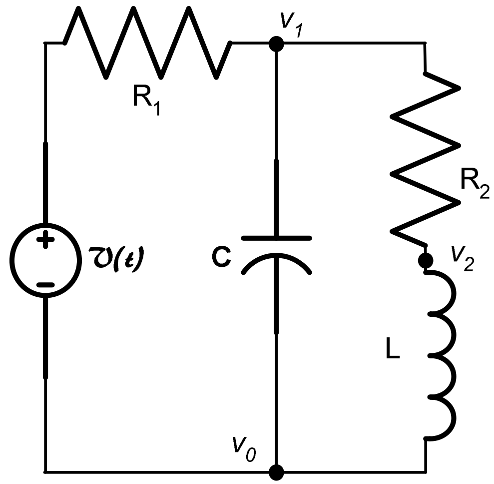
$$= u(t)\{6.6664\sin 10t - 3.3339\cos 10t\} \;\; A$$

These solutions may be confirmed to be the same as the **steady state components** of the currents earlier calculated using LT

Complex Exponential Solution[4]

Example 2 (numerical)

A RLC circuit is supplied from an source $v(t) = 50u(t)\sin(10t)$ volts. Assume that at $t = 0$ there is zero voltage across the capacitor, and no inductor current. Let $R_1 = 5\Omega$, $R_2 = 2\Omega$, $C = 20mF$ and $L = 0.1H$



Using KCL with nodes as labelled, then circuit LDE are

$$\frac{\{v_1(t) - v(t)\}}{R_1} + C\frac{dv_1(t)}{dt} + \frac{\{v_1(t) - v_2(t)\}}{R_2} = 0$$

$$\frac{\{v_2(t) - v_1(t)\}}{R_2} + \frac{1}{L}\int_{-\infty}^{t} v_2(t)\,dt = 0$$

Substituting in values

$$\frac{1}{5}\{v_1(t) - v(t)\} + 0.02\frac{dv_1(t)}{dt} + \frac{1}{2}\{v_1(t) - v_2(t)\} = 0$$

.

$$\frac{1}{2}\{v_2(t) - v_1(t)\} + 10\int_{-\infty}^{t} v_2(t)\,dt = 0$$

Complex Exponential Solution[5]

Example 1 (numerical, cont)

At steady state quantities have same form as the source

$v(t) = 50u(t)\sin 10t \;= Im\left[50u(t)e^{j\,10t}\right] \quad \Rightarrow v_1(t) = Im\left[V_1 e^{j\,10t}\right] \;\; v_2(t) = Im\left[V_2 e^{j\,10t}\right]$

Substituting into equations

$$Im\left[\frac{1}{5}\{V_1 - 50u(t)\}e^{j\,10t} + 0.02\frac{dV_1 e^{j\,10t}}{dt} + \frac{1}{2}\{V_1 - V_2\}e^{j\,10t} = 0\right]$$

$$Im\left[\frac{1}{2}\{V_2 - V_1\}e^{j\,10t} + 10\int_{-\infty}^{t} V_2 e^{j\,10t}\,dt = 0\right]$$

Which gives

$$Im\left[\frac{1}{5}\{V_1 - 50u(t)\}e^{j\,10t} + j0.2V_1 e^{j\,10t} + \frac{1}{2}\{V_1 - V_2\}e^{j\,10t} = 0\right]$$

$$\Rightarrow \frac{1}{5}\{V_1 - 50u(t)\} + j0.2V_1 + \frac{1}{2}\{V_1 - V_2\} = 0$$

$$Im\left[\frac{1}{2}\{V_2 - V_1\}e^{j\,10t} + \frac{10}{j10}V_2 e^{j\,10t} = 0\right] \;\Rightarrow \frac{1}{2}\{V_2 - V_1\} - jV_2 = 0$$

In standard form

. $\{0.7 + j0.2\}V_1 + \{-0.5\}V_2 = 10u(t)$

$\{-0.5\}V_1 + \{0.5 - j\}V_2 = 0$

$$\Rightarrow \begin{bmatrix} 0.7 + j0.2 & -0.5 \\ -0.5 & 0.5 - j \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 10u(t) \\ 0 \end{bmatrix}$$

Complex Exponential Solution[3]

Example 1 (numerical, cont)

Using Cramer's rule

$$\begin{bmatrix} 0.7 + j0.2 & -0.5 \\ -0.5 & 0.5 - j \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 10u(t) \\ 0 \end{bmatrix}$$

$$V_1 = \frac{\begin{vmatrix} 10u(t) & -0.5 \\ 0 & 0.5 - j \end{vmatrix}}{\begin{vmatrix} 0.7 + j0.2 & -0.5 \\ -0.5 & 0.5 - j \end{vmatrix}} = \frac{u(t)(5 - j10)}{(0.7 + j0.2)(0.5 - j) - 0.25} = \frac{u(t)(5 - j10)}{(0.55 - j0.6) - 0.25} = \frac{u(t)(5 - j10)}{(0.3 - j0.6)}$$

$$= \frac{50u(t)}{3} \quad \Rightarrow v_1(t) = Im\left[\frac{50u(t)}{3} e^{j10t}\right] = 16.6667u(t)\sin 10t \quad V$$

Similarly

$$V_2 = \frac{\begin{vmatrix} 0.7 + j0.2 & 10u(t) \\ -0.5 & 0 \end{vmatrix}}{\begin{vmatrix} 0.7 + j0.2 & -0.5 \\ -0.5 & 0.5 - j \end{vmatrix}} = \frac{5u(t)}{(0.3 - j0.6)} = \frac{(10 + j20)u(t)}{3} = u(t)7.4536\angle 63.43^o$$

$$\Rightarrow v_2(t) = Im\left[u(t)7.4536\angle 63.43^o e^{j10t}\right] = 7.4536u(t)\sin(10t + 63.43^o)$$

$$= u(t)\{3.3339\sin 10t + 6.6664\cos 10t\} \quad V$$

These solutions may be confirmed to be the same as the **steady state components** of the voltages earlier calculated using LT

Summary

Today's class looked at Laplace transform application

- Transform circuit elements
- Transform circuits

QUESTIONS?