FEE321 – E.C.T IIA – Oct 2020

Lecture 10: Laplace Transform (4) (2 hrs)

Lecturer: *Prof H A Ouma*

18/12/2020

Overview

Today's class continues the look at LT application

- Examples of LT application in circuit analysis
  - Characteristic equation
  - Linear Time Invariant systems (LTIS)
  - Impulse response, $h(t)$
  - Transfer function, $H(s)$

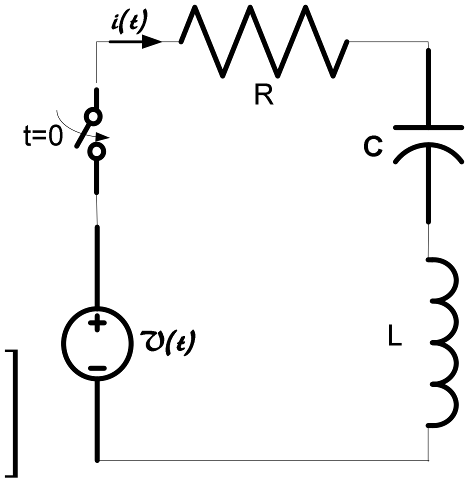
Content

- **Examples of LT application in circuit analysis (continued)**

LT application[16]

Example 3 (numerical)

A series RLC circuit is supplied from a dc source of 20 volts. The source is switched on at $t = 0$. Assume that at $t = 0$ there is zero voltage across the capacitor, and no inductor current. Let $R = 10\Omega$, $C = 0.04F$ and $L = 1.25H$



Circuit LDE is given by

$$\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) + L\frac{di(t)}{dt} = Vu(t) \quad ;t > 0$$

Substituting in values

$$\mathcal{L}\left[25\int_{-\infty}^{t} i(t)\,dt + 10i(t) + 1.25\frac{di(t)}{dt} = 20u(t)\right]$$

Expanding the integral

$$\Rightarrow \mathcal{L}\left[25\int_{-\infty}^{0} i(t)\,dt + 25\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[10i(t)] + \mathcal{L}\left[1.25\frac{di(t)}{dt}\right] = \mathcal{L}[20u(t)]$$

.

$$\Rightarrow \mathcal{L}\left[v_C(0^+) + 25\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[10i(t)] + \mathcal{L}\left[1.25\frac{di(t)}{dt}\right] = \mathcal{L}[20u(t)]$$

LT application[17]

Example 3 (numerical continued)

Starting at

$$\mathcal{L}\left[v_C(0^+) + 25\int_0^t i(t)\,dt\right] + \mathcal{L}[10i(t)] + \mathcal{L}\left[1.25\frac{di(t)}{dt}\right] = \mathcal{L}[20u(t)]$$

Applying LT

$$\Rightarrow \frac{v_C(0^+)}{s} + \frac{25I(s)}{s} + 10I(s) + 1.25[sI(s) - i(0^+)] = \frac{20}{s}$$

$$\Rightarrow I(s)\left\{\frac{25}{s} + 10 + 1.25s\right\} = \frac{20 - v_C(0^+)}{s} + 1.25i(0^+)$$

$$\Rightarrow I(s)\left\{\frac{25 + 10s + 1.25s^2}{s}\right\} = \frac{20 - v_C(0^+)}{s} + 1.25i(0^+)$$

Thus

$$\Rightarrow I(s) = \frac{20 - v_C(0^+)}{25 + 10s + 1.25s^2} + \frac{1.25s \cdot i(0^+)}{25 + 10s + 1.25s^2}$$

$$= \frac{0.8\big(20 - v_C(0^+)\big)}{s^2 + 8s + 20} + \frac{s \cdot i(0^+)}{s^2 + 8s + 20}$$

LT application[18]

Example 3 (numerical continued)

Continuing to work with **non-zero** initial conditions then

$$\Rightarrow I(s) = \frac{0.8\big(20 - v_C(0^+)\big)}{s^2 + 8s + 20} + \frac{s \cdot i(0^+)}{s^2 + 8s + 20}$$

$$= \frac{0.8\{20 - v_C(0^+)\}}{(s+4)^2 + (20-16)} + \frac{s \cdot i(0^+)}{(s+4)^2 + (20-16)}$$

$$= \frac{0.8\{20 - v_C(0^+)\} - 4i(0^+)}{(s+4)^2 + 2^2} + \frac{(s+4) \cdot i(0^+)}{(s+4)^2 + 2^2}$$

$$= 0.4\{20 - v_C(0^+) - 5i(0^+)\} \frac{2}{(s+4)^2 + 2^2} + i(0^+) \frac{s+4}{(s+4)^2 + 2^2}$$

And

$$i(t) = u(t) e^{-4t}\{[8 - 0.4v_C(0^+) - 2i(0^+)] \sin(2t) + i(0^+) \cos(2t)\} \quad A$$

.

Content

- Examples of LT application in circuit analysis
  - **Characteristic equation**

LT application[19]

Characteristic equation

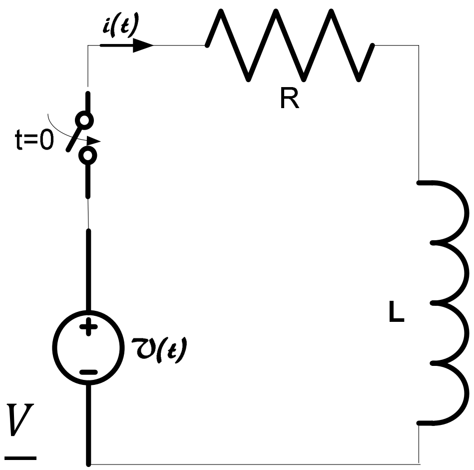
- This is an **s-domain equation** that characterizes the behavior of a circuit
- The characteristic equation depends **solely** on the circuit components
- It is affected by neither the driving **source**(s), nor the **initial conditions** of the circuit
- Obtained from the **transform equations** of the circuit
- To obtain the characteristic equation for a given circuit:
  - Determine the **s-domain equilibrium equation(s**) for the circuit, using KVL, KCL etc
  - Set the sources to zero
  - Set the initial conditions to zero
  - Obtain a single equation,
  - Write with highest power of s having a coefficient of 1
  - Write equated to zero
- The highest power of the frequency variable s, in the equation, gives the **order** of the circuit

LT application[20]

Example A

A series RL circuit is supplied from a dc source of V volts. The source is switched on at $t = 0$.

i. Dc source switched on at $t = 0 \;\Rightarrow v(t) = Vu(t)$
ii. Zero initial current before $t = 0 \;\Rightarrow i_L(0^+) = 0$



Circuit LDE is given by $L\dfrac{di(t)}{dt} + Ri(t) = Vu(t) \quad ;t > 0$

Applying LT
$$\Rightarrow \mathcal{L}\left[L\frac{di(t)}{dt}\right] + \mathcal{L}[Ri(t)] = \mathcal{L}[Vu(t)]$$

$$\Rightarrow L[sI(s) - i(0^+)] + RI(s) = \frac{V}{s} \;\Rightarrow I(s)\{sL + R\} = Li(0^+) + \frac{V}{s}$$

Thus setting initial conditions and sources to zero $\Rightarrow I(s)\{sL + R\} = 0$

Dividing through by $LI(s)$ gives the characteristic equation $\quad s + \dfrac{R}{L} = 0$

Circuit is thus a **first order** circuit

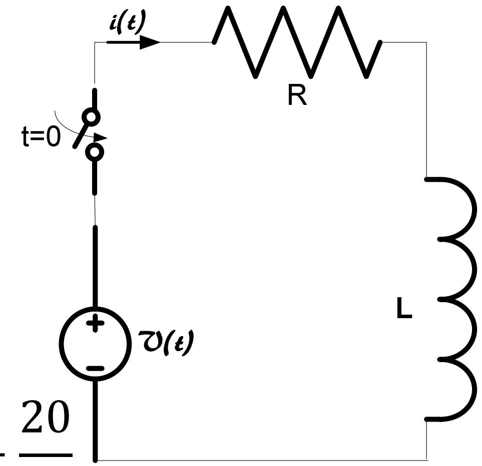
LT application[21]

Example B

A series RL circuit is supplied from a dc source of 20 volts. The source is switched on at $t = 0$. There is zero initial current in the circuit. Let $R = 5\Omega$, and $L = 0.5H$

Circuit LDE as before is given by

$$L\frac{di(t)}{dt} + Ri(t) = Vu(t) \quad ; t > 0$$



Substituting values $0.5\frac{di(t)}{dt} + 5i(t) = 20u(t) \quad ; t > 0$

Applying LT $\mathcal{L}\left[0.5\frac{di(t)}{dt} + 5i(t) = 20u(t)\right]$

$$\Rightarrow 0.5[sI(s) - i(0^+)] + 5I(s) = \frac{20}{s} \Rightarrow I(s)\{0.5s + 5\} = 0.5i(0^+) + \frac{20}{s}$$

Thus setting initial conditions and sources to zero $\Rightarrow I(s)\{0.5s + 5\} = 0$

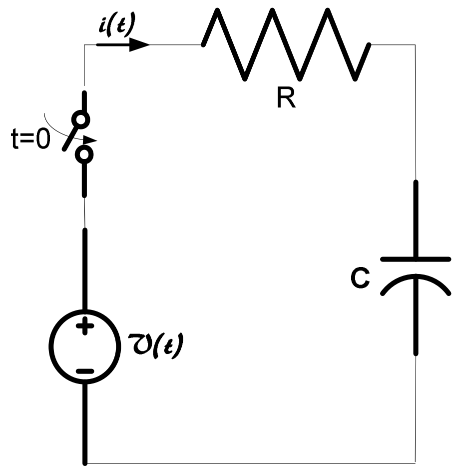
Writing in standard form gives the characteristic equation $\Rightarrow s + 10 = 0$

.

LT application[22]

Example C

A series RC circuit is supplied from a dc source of V volts. The source is switched on at $t = 0$. Assume there is zero voltage across the capacitor at $t = 0$



i. Dc source switched on at $t = 0 \;\Rightarrow v(t) = Vu(t)$
ii. Zero initial voltage $\Rightarrow v_C(0^+) = 0$

Circuit LDE is given by $\dfrac{1}{C}\displaystyle\int_{-\infty}^{t} i(t)\,dt + Ri(t) = Vu(t) \quad ; t > 0$

Applying LT

$$\Rightarrow \frac{v_C(0^+)}{s} + \frac{I(s)}{sC} + RI(s) = \frac{V}{s}$$

$$\Rightarrow I(s)\left\{\frac{1}{sC} + R\right\} = \frac{V - v_C(0^+)}{s}$$

Setting initial conditions and sources to zero, and writing in standard form

$$\Rightarrow s + \frac{1}{CR} = 0$$

.

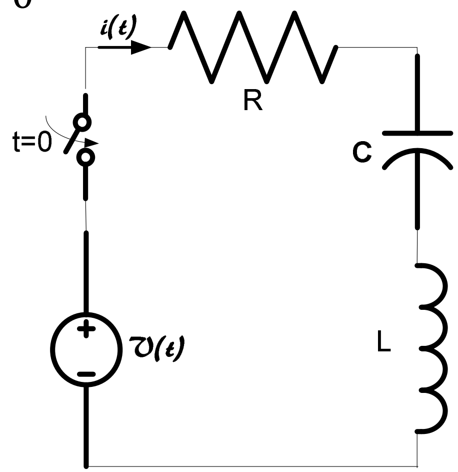
LT application[23]

Example D

A series RLC circuit is supplied from a dc source of V volts. The source is switched on at $t = 0$. At $t = 0$ there is zero voltage across the capacitor, and no inductor current

i. Dc source switched on at $t = 0 \; \Rightarrow v(t) = Vu(t), i_L(0^-) = 0$

ii. Zero initial voltage $\Rightarrow v_C(0^+) = 0$, inductor ensures $i_L(0^+) = i_L\,(0^-) = 0$



Circuit LDE is given by

$$\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) + L\frac{di(t)}{dt} = Vu(t) \quad ; t > 0$$

Applying LT

$$\Rightarrow I(s)\left\{\frac{1 + sRC + s^2LC}{sC}\right\} = \frac{V - v_C(0^+)}{s} + Li(0^+)$$

Obtaining the characteristic equation $\Rightarrow s^2 + \dfrac{R}{L}s + \dfrac{1}{LC} = 0$

This is a **second order** circuit

.

Content

- Examples of LT application in circuit analysis
  - Characteristic function
  - **Linear Time Invariant systems (LTIS)**

LT application[24]

Linear circuits

These are circuits for which:

- If
  - $y_1(t)$ is the output for an input $x_1(t)$,
  - **and** $y_2(t)$ is the output for another input $x_2(t)$,

  then
  - linear combination of the two inputs, $x(t) = \boldsymbol{a}x_1(t) + \boldsymbol{b}x_2(t)$
  - shall have an output $y(t) = \boldsymbol{a}y_1(t) + \boldsymbol{b}y_2(t)$
- The output is a **linear combination** of the individual outputs

Circuits that **do not** behave in this way are termed **non-linear circuits**

LT application[24]

Time invariant circuits

These are basically circuits whose behavior **does not change with time**,

- i.e. a response obtained for a given input at one time, will be the same response obtained for that same input when applied at a different time
- If $x(t)$ results in an output $y(t)$, then $x(t - t_o)$ results in an output $y(t - t_o)$

Circuits that do not behave in this way are termed **time varying circuits** or **non time invariant** circuits

Linear Time invariant (LTI) circuits are both linear and time invariant

Most of the circuits we shall deal with in ECT shall be **LTI**

Content

- Examples of LT application in circuit analysis
  - Characteristic function
  - Linear Time Invariant systems (LTIS)
  - **Impulse response, $\boldsymbol{h(t)}$**
  - **Transfer function, $\boldsymbol{H(s)}$**

LT application[25]

Impulse response, $\boldsymbol{h(t)}$

A circuit is driven by a source, $x(t) = \delta(t)$

- The resultant circuit response is therefore a response to an impulse
- Commonly termed the **impulse response** of the circuit
- Usually denoted as $h(t)$, the impulse response is a **basic characteristic** of a given circuit
- In **LTI circuits** the impulse response enables us to work out the response of the circuit to **any other** input
- For an input $x(t) \rightleftharpoons X(s)$, resulting in an output $y(t) \rightleftharpoons Y(s)$, the two are related through the impulse response

LT application[25]

- $y(t) = x(t) * h(t) \rightleftharpoons Y(s) = X(s)H(s)$, where $h(t) \rightleftharpoons H(s)$
- Thus knowing the impulse response for a LTI circuit enables the determination of the output, or response, for any other input
- $H(s)$ is known as the **transfer function** (TF) of the LTI circuit, and is obtained with **zero initial conditions**
- **Note** that both the **input** and the **output** must be defined in order to obtain the transfer function
- The same circuit may have different TFs depending on the definition of the circuit input, and circuit output

$$H(s) = \frac{Y(s)}{X(s)} \quad \Leftrightarrow \quad Y(s) = H(s)X(s)$$

Content

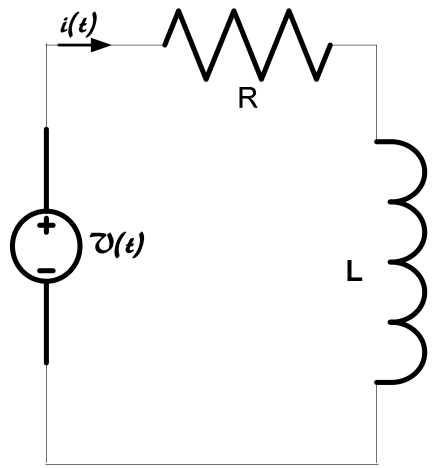
- Examples of LT application in circuit analysis
  - Characteristic function
  - Linear Time Invariant systems (LTIS)
  - Impulse response, $h(t)$
  - Transfer function, $H(s)$
- **More examples**

LT application[26]

Example 4

A series RL circuit is supplied from a source $v(t) = \delta(t)$

Circuit LDE is given by $$L\frac{di(t)}{dt} + Ri(t) = v(t) \quad ;t > 0$$



Applying LT

$$\mathcal{L}\left[L\frac{di(t)}{dt} + Ri(t) = v(t)\right]$$

$$\Rightarrow \mathcal{L}\left[L\frac{di(t)}{dt}\right] + \mathcal{L}[Ri(t)] = \mathcal{L}[v(t)]$$

$$\Rightarrow L[sI(s) - i(0^+)] + RI(s) = V(s) \quad \Rightarrow I(s)\{sL + R\} = Li(0^+) + V(s)$$

Note that the characteristic equation is given by $s + \frac{R}{L} = 0$

Thus $I(s) = \frac{Li(0^+)}{sL + R} + \frac{V(s)}{sL + R} = \frac{Li(0^+)}{sL + R} + \frac{1}{sL + R}$ since $\delta(t) \rightleftharpoons 1$

LT application[27]

Example 4 (continued)

The transform equation is then converted to time domain

$$I(s) = \frac{Li(0^+)}{sL + R} + \frac{1}{sL + R} = \frac{Li(0^+) + 1}{sL + R} = \frac{i(0^+) + \frac{1}{L}}{s + \frac{R}{L}}$$

$$\Rightarrow i(t) = u(t)\left\{i(0^+) + \frac{1}{L}\right\} e^{-\frac{R}{L}t} \quad A$$

The response is noted to be **transien**t, with **time constant** L/R

Now assume the **input** is the driving voltage, $v(t)$ and the circuit current, $i(t)$ is the **output**

The **transfer function** of the circuit, $H(s)$ would then be $\frac{I(s)}{V(s)}$, with the initial conditions set to zero

$$H(s) = \frac{1}{sL + R} = \frac{1/L}{s + R/L}$$

LT application[28]

Example 4 (continued)

- With the transfer function determined, we can then predict the circuit response to other different inputs (with zero initial conditions)
- This is because the transformed output $Y(s)$ would be given by the product of the transformed input $X(s)$ and the transfer function $H(s)$

$$H(s) = \frac{Y(s)}{X(s)} \qquad \Rightarrow Y(s) = H(s)X(s)$$

- The expression obtained for $Y(s)$ is then converted to time domain to obtain the new response

- Suppose in the Example 4 circuit our **input $\boldsymbol{x(t)}$** was $v(t) = Vu(t)$ and the **output $\boldsymbol{y(t)}$** still defined as the current $i(t)$

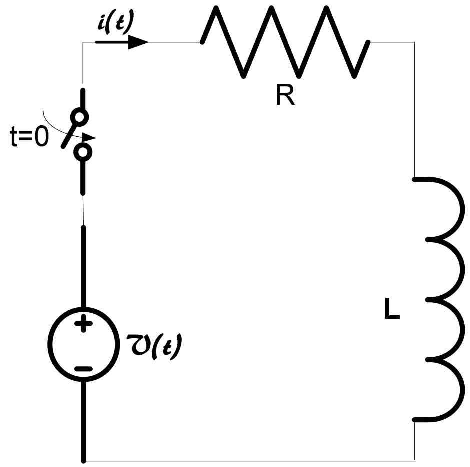
- Then $V(s) = V/s$ and $\Rightarrow I(s) = H(s)V(s) = \dfrac{V/L}{s(s+R/L)}$

- Using PFD we find that $\Rightarrow I(s) = \dfrac{V}{R}\left(\dfrac{1}{s} - \dfrac{1}{s+\frac{R}{L}}\right) \quad \Rightarrow i(t) = u(t)\dfrac{V}{R}\left\{1 - e^{-\frac{R}{L}t}\right\}$

LT application[29]

Remember **Example 1** **result – the solution corresponds**

A series RL circuit is supplied from a dc source of V volts. The source is switched on at $t = 0$.



i. Dc source switched on at $t = 0 \;\Rightarrow v(t) = Vu(t)$
ii. Zero initial current before $t = 0 \;\Rightarrow i_L(0^+) = 0$

.
.
.

Solution was

$$i(t) = u(t)\left\{\frac{V}{R}u(t) + \left[i(0^+) - \frac{V}{R}\right]e^{-\frac{R}{L}t}\right\} \quad A$$

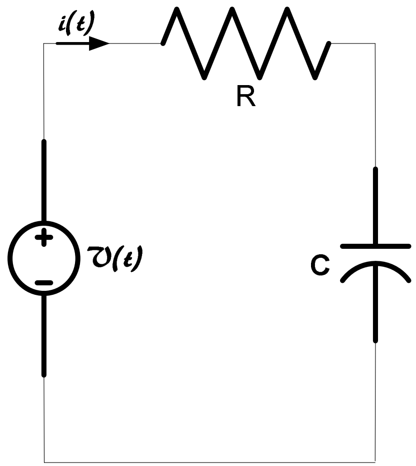
.
.
.

If the zero initial conditions are applied $i(t) = u(t)\left\{\frac{V}{R}\left(u(t) - e^{-\frac{R}{L}t}\right)\right\} \quad A$

LT application[30]

Example 5

A series RC circuit is supplied from a source $v(t) = \delta(t)$



Circuit LDE is given by

$$\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) = v(t) \quad ;t > 0$$

Applying LT $\quad \mathcal{L}\left[\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) = v(t)\right]$

$$\Rightarrow \mathcal{L}\left[\frac{1}{C}\int_{-\infty}^{0} i(t)\,dt + \frac{1}{C}\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[Ri(t)] = \mathcal{L}[v(t)]$$

$$\Rightarrow \mathcal{L}\left[v_C(0^+) + \frac{1}{C}\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[Ri(t)] = \mathcal{L}[v(t)]$$

LT application[31]

Example 5 (continued)

$$\Rightarrow \mathcal{L}\left[v_C(0^+) + \frac{1}{C}\int_0^t i(t)\,dt\right] + \mathcal{L}[Ri(t)] = \mathcal{L}[v(t)]$$

$$\Rightarrow \frac{v_C(0^+)}{s} + \frac{I(s)}{sC} + RI(s) = V(s) \quad \Rightarrow I(s)\left\{\frac{1}{sC} + R\right\} = 1 - \frac{v_C(0^+)}{s}$$

$$\Rightarrow I(s) = \frac{sC\left(1 - \frac{v_C(0^+)}{s}\right)}{sCR + 1} \quad = \frac{\frac{s}{R}\left(1 - \frac{v_C(0^+)}{s}\right)}{s + \frac{1}{CR}}$$

The expression may be re-arranged as follows

$$\Rightarrow I(s) = \frac{\frac{1}{R}s}{s + \frac{1}{CR}} - \frac{\frac{1}{R}v_C(0^+)}{s + \frac{1}{CR}} \quad = \frac{\frac{1}{R}\left(s + \frac{1}{CR}\right)}{s + \frac{1}{CR}} - \frac{\frac{1}{CR^2} + \frac{1}{R}v_C(0^+)}{s + \frac{1}{CR}}$$

$$= \frac{1}{R} - \frac{\frac{1}{CR^2} + \frac{1}{R}v_C(0^+)}{s + \frac{1}{CR}}$$

LT application[32]

Example 5 (continued)

And the time domain current can therefore be obtained,

$$I(s) = \frac{1}{R} - \frac{\frac{1}{CR^2} + \frac{1}{R}v_C(0^+)}{s + \frac{1}{CR}} \quad = \frac{1}{R}\left\{1 - \frac{\frac{1}{CR} + v_C(0^+)}{s + \frac{1}{CR}}\right\}$$

$$\rightleftharpoons \quad i(t) = \frac{u(t)}{R}\left\{\delta(t) - \left[\frac{1}{CR} + v_c(0^+)\right]e^{-\frac{t}{CR}}\right\}$$

Note that this is a transient with a high value at the start due to the impulse function

If the output is again taken as the circuit current and the input as the driving voltage, the transfer function of the circuit would be obtained as

$$\Rightarrow I(s)\left\{\frac{1}{sC} + R\right\} = V(s) - \frac{v_C(0^+)}{s} \quad \Rightarrow H(s) = \frac{I(s)}{V(s)} = \frac{sC}{1 + sRC} \quad = \frac{1}{R} \cdot \frac{s}{s + \frac{1}{RC}}$$

Characteristic equation $\quad s + \frac{1}{RC} = 0$

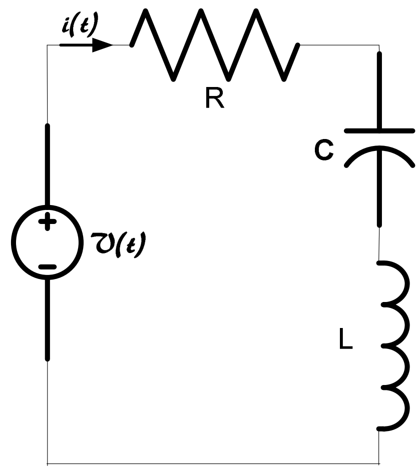
LT application[33]

Example 6

A series RLC circuit is supplied from a source $v(t) = \delta(t)$

Circuit LDE is given by

$$\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) + L\frac{di(t)}{dt} = v(t) \quad ; t > 0$$



Applying LT

$$\mathcal{L}\left[\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) + L\frac{di(t)}{dt} = v(t)\right]$$

$$\Rightarrow \mathcal{L}\left[v_C(0^+) + \frac{1}{C}\int_{0}^{t} i(t)\,dt\right] + \mathcal{L}[Ri(t)] + \mathcal{L}\left[L\frac{di(t)}{dt}\right] = \mathcal{L}[v(t)]$$

. $$\Rightarrow \frac{v_C(0^+)}{s} + \frac{I(s)}{sC} + RI(s) + L[sI(s) - i(0^+)] = V(s)$$

LT application[32]

Example 6 (continued)

Making $I(s)$ the subject

$$\frac{v_C(0^+)}{s}+\frac{I(s)}{sC}+RI(s)+L[sI(s)-i(0^+)]=V(s)\qquad \Rightarrow I(s)\left\{\frac{1}{sC}+R+sL\right\}=V(s)-\frac{v_C(0^+)}{s}+Li(0^+)$$

$$\Rightarrow I(s)\left\{\frac{1+sRC+s^2LC}{sC}\right\}=V(s)-\frac{v_C(0^+)}{s}+Li(0^+)$$

$$\Rightarrow I(s)=\frac{sC\big(1+Li(0^+)\big)}{1+sRC+s^2LC}-\frac{Cv_C(0^+)}{1+sRC+s^2LC}=\frac{\frac{1}{L}s\{1+Li(0^+)\}}{s^2+s\frac{R}{L}+\frac{1}{LC}}+\frac{\frac{1}{L}v_c(0^+)}{s^2+s\frac{R}{L}+\frac{1}{LC}}$$

$$=\frac{\frac{1}{L}s\{1+Li(0^+)\}}{\left(s+\frac{R}{2L}\right)^2+\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}+\frac{\frac{1}{L}v_C(0^+)}{\left(s+\frac{R}{2L}\right)^2+\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}$$

Thus

$$=\frac{\frac{1}{L}s\{1+Li(0^+)\}}{\left(s+\frac{R}{2L}\right)^2+\left\{\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}\right\}^2}+\frac{\frac{1}{L}v_C(0^+)}{\left(s+\frac{R}{2L}\right)^2+\left\{\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}\right\}^2}$$

.

LT application[33]

Example 6 (continued)

Writing in recognizable forms $I(s) = \dfrac{\frac{1}{L}s\{1 + Li(0^+)\}}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2} + \dfrac{\frac{1}{L}v_C(0^+)}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2}$

$$I(s) = \frac{\frac{1}{L}\{1 + Li(0^+)\}\left\{s + \frac{R}{2L}\right\}}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2} + \frac{\frac{1}{L}v_C(0^+) - \frac{R}{2L^2}\{1 + Li(0^+)\}}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2}$$

$$= \frac{1}{L}\{1 + Li(0^+)\} \times \frac{\left\{s + \frac{R}{2L}\right\}}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2} +$$

.

$$\frac{\frac{1}{L}\left(v_C(0^+) - \frac{R}{2L}\{1 + Li(0^+)\}\right)}{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}} \times \frac{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}}{\left(s + \frac{R}{2L}\right)^2 + \left\{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right\}^2}$$

.

LT application[34]

Example 6 (continued)

Thus inverting

$$i(t)=\frac{u(t)}{L}\{1+Li(0^{+})\}e^{-\frac{R}{2L}t}\cos\left(\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}\,t\right)\;+$$

$$\left\{\frac{\frac{u(t)}{L}\left(v_C(0^{+})-\frac{R}{2L}\{1+Li(0^{+})\}\right)}{\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}}\right\}e^{-\frac{R}{2L}t}\sin\left(\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}\,t\right)$$

Or

$$i(t)=u(t)e^{-\frac{R}{2L}t}\left\{\frac{1}{L}+i(0^{+})\right\}\cos\left(\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}\,t\right)\;+$$

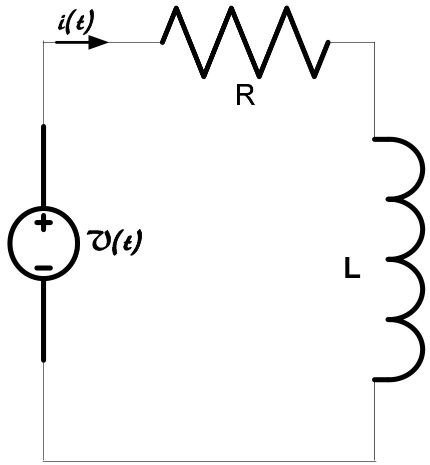
$$\left\{\frac{u(t)e^{-\frac{R}{2L}t}\left(v_C(0^{+})-\frac{R}{2}\{1+Li(0^{+})\}\right)}{.\;\sqrt{\left(\frac{L}{C}-\frac{R^2}{4}\right)}}\right\}\sin\left(\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}\,t\right)$$

LT application[35]

Example 7

A series RL circuit is supplied from a source $v(t) = 2\sin 5t$, Assume $R = 10\Omega$ and $L = 2H$

Circuit LDE is given by $L\dfrac{di(t)}{dt} + Ri(t) = v(t) \quad ; t > 0$



$$\Rightarrow L[sI(s) - i(0^+)] + RI(s) = V(s)$$

$$\Rightarrow I(s)\{sL + R\} = Li(0^+) + V(s)$$

$$I(s) = \frac{Li(0^+)}{sL + R} + \frac{V(s)}{sL + R} = \frac{2i(0^+)}{2s + 10} + \frac{2 \times 5}{(s^2 + 25)(2s + 10)}$$

$$= \frac{i(0^+)}{s + 5} + \frac{5}{(s^2 + 25)(s + 5)} \quad = \frac{i(0^+)}{s + 5} + \frac{As + B}{s^2 + 25} + \frac{C}{s + 5}$$

Thus

- $$As + B|_{s=j5} = \left.\frac{5}{s + 5}\right|_{s=j5} = \frac{5}{5 + j5} = 0.5 - j0.5 \quad \Rightarrow A = -0.1, B = 0.5$$
- $$C = \left.\frac{5}{s^2 + 25}\right|_{s=-5} = \frac{5}{25 + 25} = 0.1$$

LT application[36]

Example 7 (continued)

The transform current is therefore $I(s) = \dfrac{i(0^+)}{s+5} + \dfrac{-0.1s + 0.5}{s^2+25} + \dfrac{0.1}{s+5}$

$$I(s) = \frac{\{i(0^+) + 0.1\}}{s+5} - 0.1\frac{s}{s^2+25} + 0.1\frac{5}{s^2+25}$$

These are easily recognized for inversion

$$i(t) = u(t)(\{i(0^+) + 0.1\}e^{-5t} - 0.1\{\cos 5t - \sin 5t\}) \quad A$$

First term is the **transient**, while the second is the **steady state** current

We may also express the steady state in purely cosine or purely sine form

$$i(t) = u(t)(\{i(0^+) + 0.1\}e^{-5t} - 0.1421\cos(5t + 45^o)) \quad A$$

$$i(t) = u(t)(\{i(0^+) + 0.1\}e^{-5t} - 0.1421\sin(5t - 45^o)) \quad A$$

.

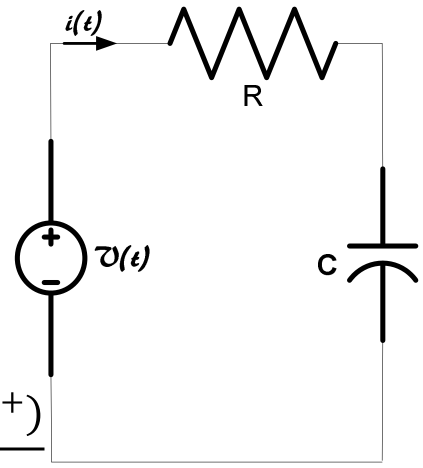
LT application[37]

Example 8

A series RC circuit is supplied from a source $v(t) = \cos 10t$

$R = 2\Omega$ and $C = 0.5F$

Circuit LDE is given by $\dfrac{1}{C}\displaystyle\int_{-\infty}^{t} i(t)\,dt + Ri(t) = v(t) \quad ; t > 0$



$$\Rightarrow \mathcal{L}\left[v_C(0^+) + \frac{1}{C}\int_0^t i(t)\,dt\right] + \mathcal{L}[Ri(t)] = \mathcal{L}[v(t)]$$

$$\Rightarrow \frac{v_C(0^+)}{s} + \frac{I(s)}{sC} + RI(s) = V(s) \quad \Rightarrow I(s)\left\{\frac{1}{sC} + R\right\} = V(s) - \frac{v_C(0^+)}{s}$$

$$\Rightarrow I(s) = \frac{sC\left(V(s) - \frac{v_C(0^+)}{s}\right)}{sCR + 1} \quad = \frac{\frac{s}{R}V(s)}{s + \frac{1}{CR}} - \frac{\left(\frac{v_C(0^+)}{R}\right)}{s + \frac{1}{CR}} \quad \text{with} \quad V(s) = \frac{s}{s^2 + 100}$$

$$= \frac{\frac{s}{2}\left(\frac{s}{s^2 + 100}\right)}{s + \frac{1}{0.5 \times 2}} - \frac{\left(\frac{v_C(0^+)}{2}\right)}{s + \frac{1}{0.5 \times 2}} \quad = \frac{\frac{s^2}{2}}{(s^2 + 100)(s + 1)} - \frac{\left(\frac{v_C(0^+)}{2}\right)}{s + 1}$$

LT application[38]

Example 8 (continued)

Partial fraction decomposition is required for the first term

$$I(s) = \frac{0.5s^2}{(s^2+100)(s+1)} - \frac{0.5v_C(0^+)}{s+1} = \frac{As+B}{s^2+100} + \frac{C}{s+1} - \frac{0.5v_C(0^+)}{s+1}$$

$$As+B|_{s=j5} = \left.\frac{0.5s^2}{s+1}\right|_{s=j10} = \frac{-50}{1+j10} = -0.4950 + j4.950 \quad \Rightarrow A = 0.99, B = -0.4950$$

$$C = \left.\frac{0.5s^2}{s^2+100}\right|_{s=-1} = \frac{0.5}{1+100} = 0.00495$$

Transform equation is therefore

$$I(s) = \left(\frac{0.99s}{s^2+100} - \frac{0.4950}{s^2+100}\right) + 0.00495\frac{1}{s+1} - \frac{0.5v_C(0^+)}{s+1}$$

The time domain current

$$i(t) = u(t)\{0.99\cos 10t - 0.0495\sin 10t + [0.00495 - 0.5v_C(0^+)]e^{-t}\}$$

.

$$= 0.9912u(t)\cos(10t + 2.86^o) + \{0.00495 - 0.5v_C(0^+)\}u(t)e^{-t}$$

Summary

Today's class looked at Laplace transform application

- Examples of LT application in circuit analysis
  - Characteristic function
  - Linear Time Invariant systems (LTIS)
  - Impulse response, $h(t)$
  - Transfer function, $H(s)$

QUESTIONS?