FEE321 – E.C.T IIA – Oct 2020

Lecture 6: Complex frequency and the Laplace Transform (2 hrs)

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Overview

Today’s class looks at complex frequency

- Euler’s formula
- Exponential excitation - review
- Complex numbers - review
- Waveform representation
- Laplace transform – an introduction

Content

- **Euler’s formula**

Euler’s formula[1]

- Swiss mathematician Leonhard Euler
- Euler’s number, $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$
- $e \approx 2.7\ 1828\ 1828\ 45\ 90\ 45$
- e is the base of the natural logarithm, $\log_e x = \ln x$
- Exponential curve $\mathrm{y} = e^x$ has attractive properties:
  - Slope is equal to curve itself; $\frac{dy}{dx} = e^x$
  - Area under curve to any value x equals curve value at x, $\int_{-\infty}^{x} e^{x'}\, dx' = e^x$
- Properties promote its use in calculus

Euler's formula[2]

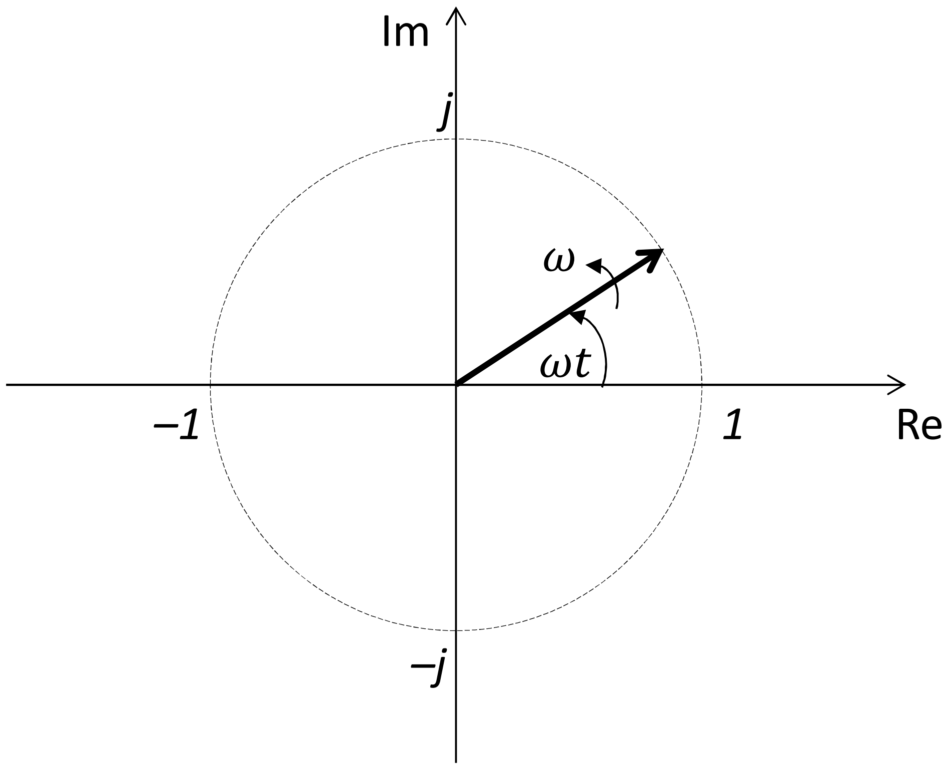
- Formula relates exponential function to trigonometric functions through a complex index
- Formula $e^{j\omega t} = \cos\omega t + j\sin\omega t$
- If index is not complex then

$$e^{\omega t} = \cosh\omega t + \sinh\omega t$$

- Magnitude: $\left|e^{j\omega t}\right| = \sqrt{\cos^2\omega t + j\sin^2\omega t} = 1$
- Angle: $\angle e^{j\omega t} = tan^{-1}\dfrac{\sin\omega t}{\cos\omega t} = \omega t$

Euler's formula[3]

- In polar form $e^{j\omega t} = 1\angle\omega t$
- On complex plane, locus of $e^{j\omega t}$ is the unit circle



Content

- Euler's formula
- **Exponential excitation** - review

Exponential excitation

- Formula enables analysis of sinusoidal excitation through the complex exponential function
- If excitation is for example $v(t) = V\sin\omega t$,
- This is equivalent to $v(t) = \mathrm{Im}\left[Ve^{j\omega t}\right]$ since,

$$Ve^{j\omega t} = V\cos\omega t + jV\sin\omega t$$

- Analysis would then be made using complex exponential excitation, $Ve^{j\omega t}$ and for the **actual** solution the **imaginary part** of the solution is taken
- Similarly, for excitation $i(t) = I\cos\omega t = \mathrm{Re}\left[Ie^{j\omega t}\right]$, analysis would be done, and for the **actual** solution the **real part** of the solution would be taken

Content

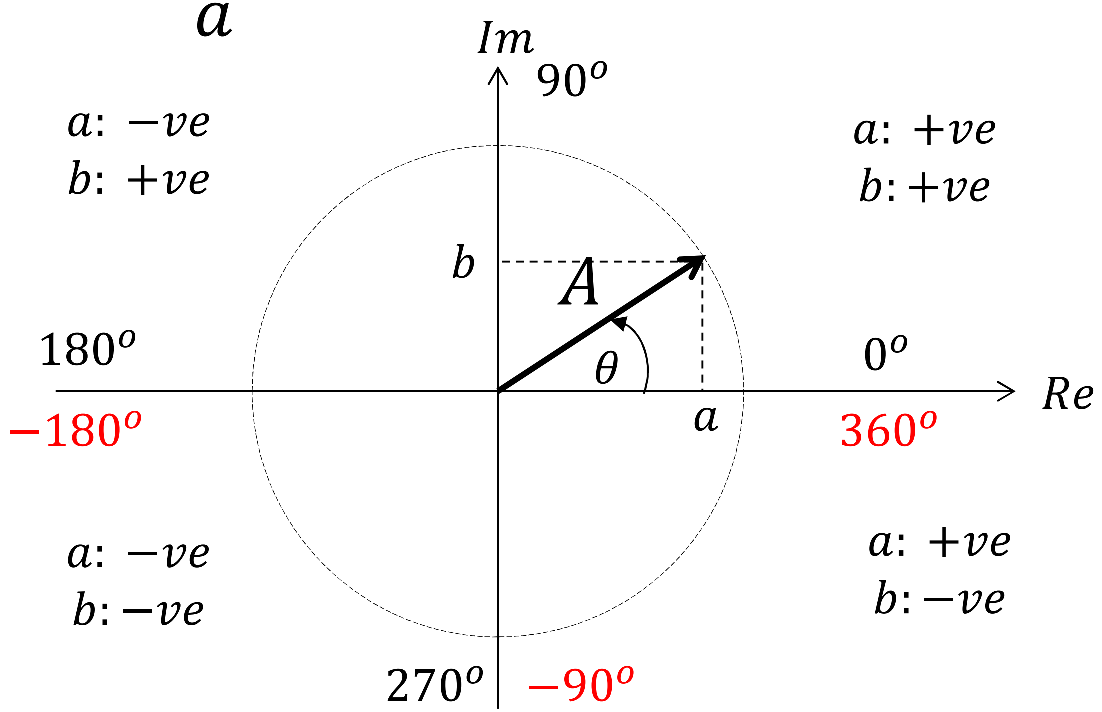
- Euler's formula
- Exponential excitation - review
- **Complex numbers** - review

Complex numbers[1]

- Complex numbers give a convenient way of analyzing AC circuits
- Sinusoidal voltages and currents are transformed into complex numbers called **phasors**
- Resistances, inductances, and capacitances are transformed into complex numbers called **impedances**
- Complex numbers may be represented in various forms
- Each form is useful under different mathematical operations
- Different forms also give different insights into the solution

Complex numbers[2]

- Rectangular form representation $x = a + jb$
- Magnitude: $|x| = \sqrt{a^2 + b^2} = A$
- Angle: $\angle x = \tan^{-1}\frac{b}{a} = \theta$
- Phasor



Complex numbers[3]

- Exponential form $x = Ae^{j\theta} = A\cos\theta + jA\sin\theta$
- Equating to rectangular form:

$a = A\cos\theta$ and $b = A\sin\theta$

- Polar form (shorthand for exponential form)

$$x = A\angle\theta$$

- Conversion between the 3 forms must be mastered
- Conjugates of complex numbers differ in the sign of the angle
- If $x = Ae^{j\theta}$, then the conjugate, $x^* = Ae^{-j\theta}$

Complex numbers[4]

Algebra

- Multiplication and division
  - Use exponential form

$$\left(Ae^{j\theta_1}\right) \times \left(Be^{j\theta_2}\right) = ABe^{j\,(\theta_1+\theta_2)}$$

$$\left(Ae^{j\theta_1}\right) \div \left(Be^{j\theta_2}\right) = \frac{A}{B}e^{j\,(\theta_1-\theta_2)}$$

- Addition and subtraction
  - Use rectangular form

$$(a_1 + jb_1) \pm (a_2 + jb_2) = (a_1 \pm a_2) + j(b_1 \pm b_2)$$

Complex numbers[5]

Powers and Roots

- Use exponential form

$$\left[Ae^{j\theta}\right]^{n} = A^{n}e^{jn\theta}$$

$$\left[Ae^{j\theta}\right]^{\frac{1}{n}} = A^{\frac{1}{n}}e^{j\left(\frac{\theta+2k\pi}{n}\right)}$$

$$k = 0, 1, 2, \ldots, n-1$$

Complex numbers[6]

Phasors

- Complex number associated with a phase shifted sinusoid
- Magnitude is the effective (rms) value of the voltage or current
- Angle is the phase angle of the phase shifted sinusoid

$$v(t) = 3\sin(\omega t + 20^o) \rightarrow \quad \frac{3}{\sqrt{2}}\angle 20^o$$

- While it represents a sinusoid (real signal), a phasor is not a sinusoid as it is complex in nature
- Complex numbers ONLY phasors when they correspond to a sinusoid
- Some literature uses phasor magnitude as the sinusoid peak value
- Some also use the angle as being of a cosine rather than sine wave
- Principally we shall use rms value as magnitude, and sine wave as the base sinusoid

Complex numbers[7]

Phasors

- Sinusoids of **same frequency** can be summed using phasors
- $v(t) = 3\sin(\omega t + 20^o) + 4\sin(\omega t - 65^o)$

$$\boldsymbol{V} = \frac{3}{\sqrt{2}}\angle 20^o + \frac{4}{\sqrt{2}}\angle -65^o$$

$$= \frac{3}{\sqrt{2}}(\cos 20^o + j\sin 20^o) + \frac{4}{\sqrt{2}}(\cos(-65^o) + j\sin(-65^o))$$

$$= \frac{3}{\sqrt{2}}(0.9397 + j0.3420) + \frac{4}{\sqrt{2}}(0.4226 - j0.9063)$$

$$= 1.9934 + j0.7255 + 1.1953 - j2.5634$$

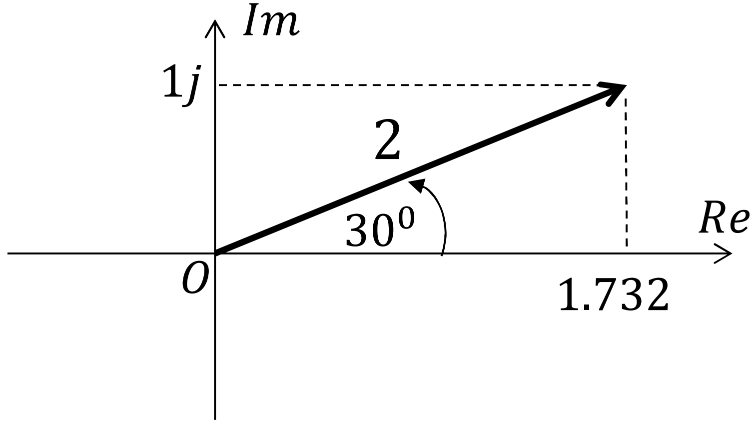
$$= 3.1887 - j1.8379 = 3.6804\angle - 29.9584^o$$

- $v(t) = 5.2049\sin(\omega t - 29.96^o)$

Complex numbers[8]

Phasor diagrams

- Complex plane diagrams
- Phasors shown as arrows from the origin
- Length of arrow corresponds to magnitude of phasor
- Angle of arrow with the positive real axis correspond to the phasor angle
- Rectangular coordinate plot achieves this e.g. $2e^{j30^o} = 1.7320 + j$

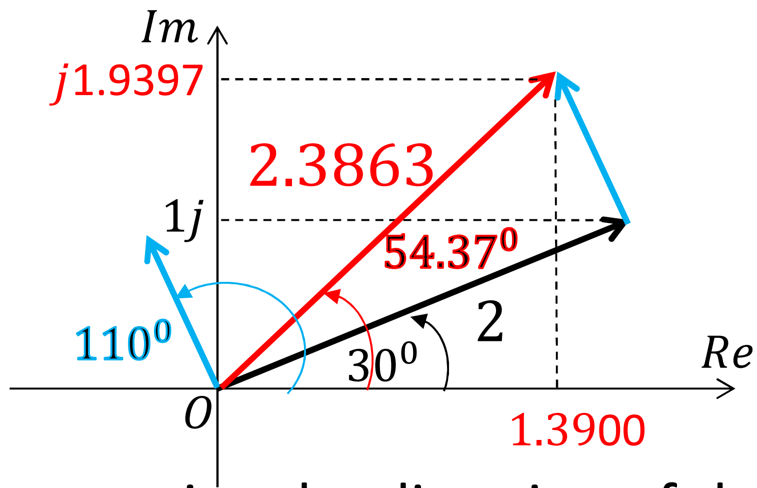


- Phasor diagrams show the relative relationship between various voltages and currents of the **same frequency**

Complex numbers[9]

Phasor diagrams

- Phasor addition achieved by placing phasors end to end and then drawing a new phasor connecting the origin to the end of the last phasor in the chain
- e.g. $1e^{j110^o} + 2e^{j30^o} = -0.3420 + j0.9397 + 1.7320 + j$
  $= 1.3900 + j1.9397$
  $= 2.3863\angle 54.37^o$



- Phasor subtraction is by reversing the direction of the arrow being subtracted and then adding as before

Complex numbers[10]

Circuit Analysis using phasors

- Time domain circuit is first transformed into a **phasor domain circuit**
- Phasor circuit has phasor currents and voltages, and component impedance values
- Circuit analysis then proceeds as in the time domain circuits, **only no calculus is involved**
- Actual values of voltages and currents are finally obtained by transforming the phasors calculated into time domain voltages and currents
- Actual components and their values are obtained by transforming the impedances calculated into time domain components

Content

- Euler's formula
- Exponential excitation - review
- Complex numbers - review
- **Waveform representation**

Waveform representation[1]

- Preceding section on phasors has hinted at waveform representation
- General voltage or current waveform may be represented in phasor form as

$$\boldsymbol{X} = Ae^{\sigma t}e^{j(\omega t+\varphi)}$$

- This represents the real waveform

$$x(t) = \mathrm{Im}[\boldsymbol{X}] = Ae^{\sigma t}\sin(\omega t + \varphi)$$

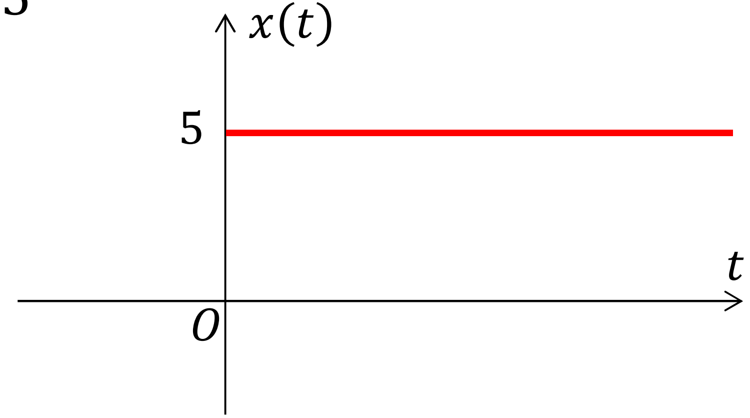
- $x(t)$takes different forms depending on the values of σ, ω, and to a lesser extent φ. For example
- When σ is negative the waveform decays with time
- When ω is zero, the waveform does not oscillate

Waveform representation[2]

- The general phasor form may be re-written as
$$\boldsymbol{X} = Ae^{\sigma t}e^{j(\omega t+\varphi)} = Ae^{j\varphi}e^{(\sigma+j\omega)t}$$
- $\sigma + j\omega = s$ is known as the complex frequency
- σ has the units of nepers/sec
- ω, as before, has the units of radians/sec
- The real waveform may thus be written as
$$x(t) = \mathrm{Im}\left[Ae^{j\varphi}e^{st}\right]$$
- The various waveform types can then be appreciated

Waveform representation[3]

- The general phasor form
$$\boldsymbol{X} = Ae^{\sigma t}e^{j(\omega t+\varphi)} = Ae^{j\varphi}e^{(\sigma+j\omega)t}$$
- Let $s = \sigma + j\omega = 0 + j0$ then
- $\boldsymbol{X} = Ae^{0t}e^{j(0t+\varphi)} = Ae^{j\varphi}$
- This represents a constant (d.c.) waveform $x(t) = \mathrm{Im}\left[Ae^{j\varphi}\right]$
- e.g. $x(t) = 5$



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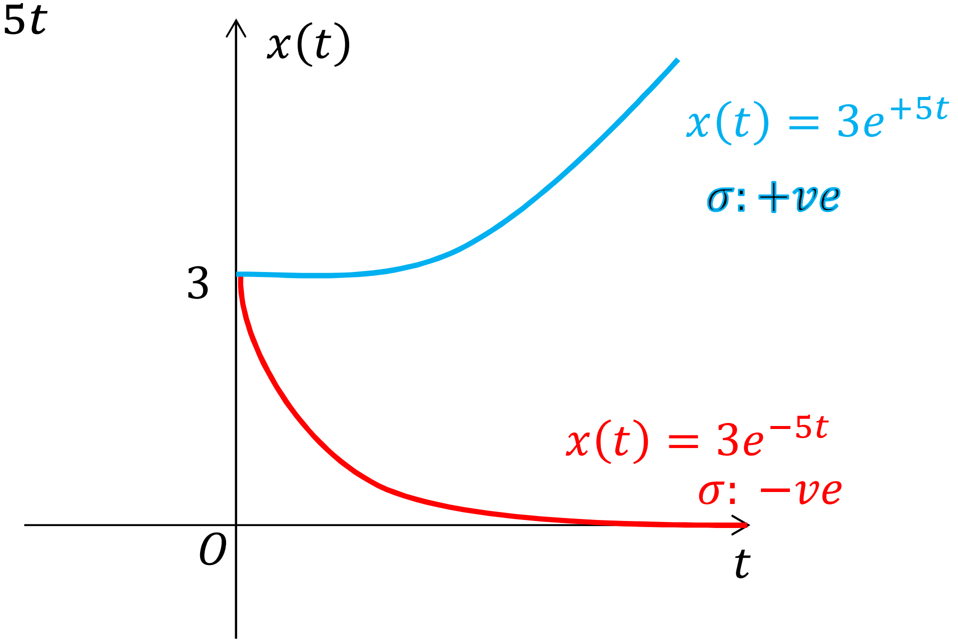
Waveform representation[4]

- The general phasor form

$$\boldsymbol{X} = Ae^{\sigma t}e^{j(\omega t+\emptyset)} = Ae^{j\emptyset}e^{(\sigma+j\omega)t}$$

- Let $s = \sigma + j\omega = \sigma + j0$ then
- $\boldsymbol{X} = Ae^{\sigma t}e^{j(0t+\varphi)} = Ae^{\sigma t}e^{j\varphi}$
- This represents an exponential waveform $x(t) = \mathrm{Im}\left[Ae^{\sigma t}e^{j\varphi}\right]$
- e.g. $x(t) = 3e^{\pm 5t}$



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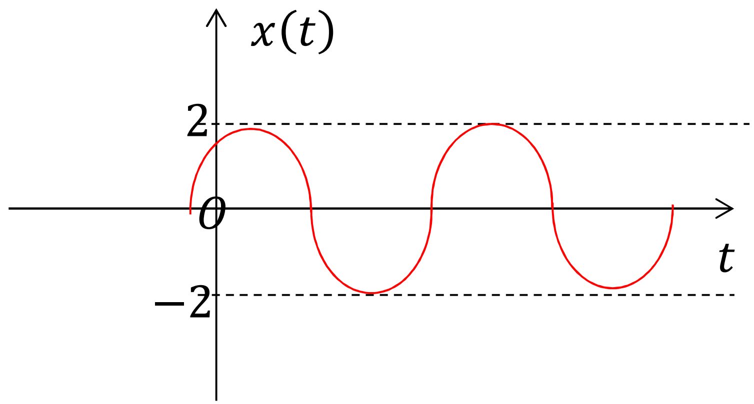
Waveform representation[5]

- The general phasor form

$$\boldsymbol{X} = Ae^{\sigma t}e^{j(\omega t+\varphi)} = Ae^{j\varphi}e^{(\sigma+j\omega)t}$$

- Let $s = \sigma + j\omega = 0 + j\omega$ then
- $\boldsymbol{X} = Ae^{0t}e^{j(\omega t+\varphi)} = Ae^{j\omega t}e^{j\varphi}$
- This represents a sinusoidal waveform $x(t) = \mathrm{Im}\left[Ae^{j\omega t}e^{j\varphi}\right]$
- e.g. $x(t) = 2\sin(10t + 20^o)$



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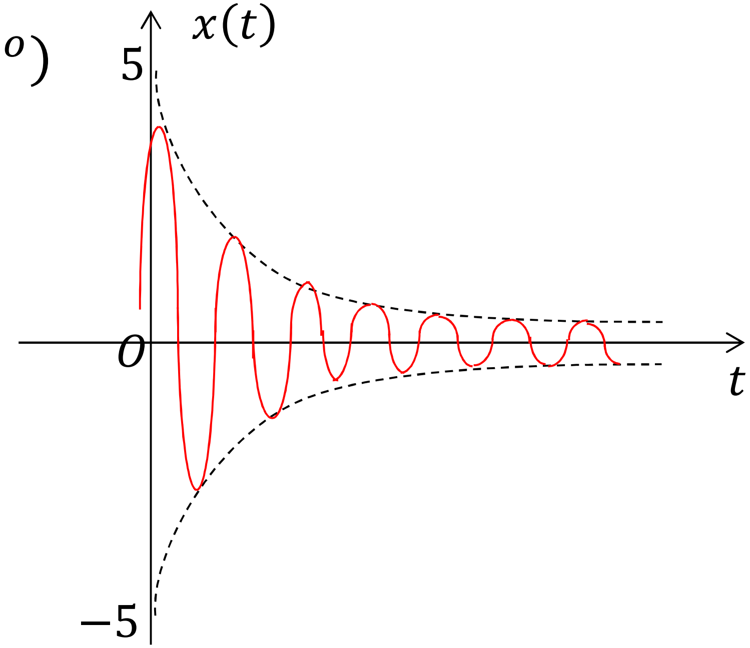
Waveform representation[6]

- The general phasor form
$$\boldsymbol{X} = Ae^{\sigma t}e^{j(\omega t+\varphi)} = Ae^{j\varphi}e^{(\sigma+j\omega)t}$$
- Let $s = \sigma + j\omega$ then
- $\boldsymbol{X} = Ae^{\sigma t}e^{j(\omega t+\varphi)}$
- This represents an exponentially varying sinusoidal waveform $x(t) = \mathrm{Im}\left[Ae^{\sigma t}e^{j(\omega t+\varphi)}\right]$
- e.g. $x(t) = 5e^{-3t}\sin(10t + 40^o)$



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Waveform representation[6]

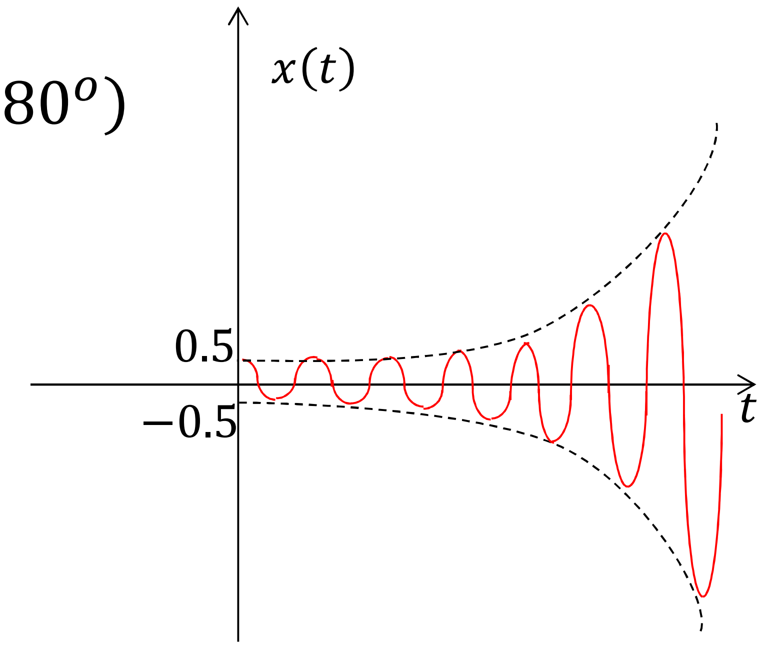
- The general phasor form

$$\boldsymbol{X} = Ae^{\sigma t}e^{j(\omega t+\varphi)} = Ae^{j\varphi}e^{(\sigma+j\omega)t}$$

- Let $s = \sigma + j\omega$ then

$$\boldsymbol{X} = Ae^{\sigma t}e^{j(\omega t+\varphi)}$$

- This represents an exponentially varying sinusoidal waveform $x(t) = \mathrm{Im}\left[Ae^{\sigma t}e^{j(\omega t+\varphi)}\right]$
- e.g. $x(t) = \frac{1}{2}e^{3t}\sin(10t + 80^o)$



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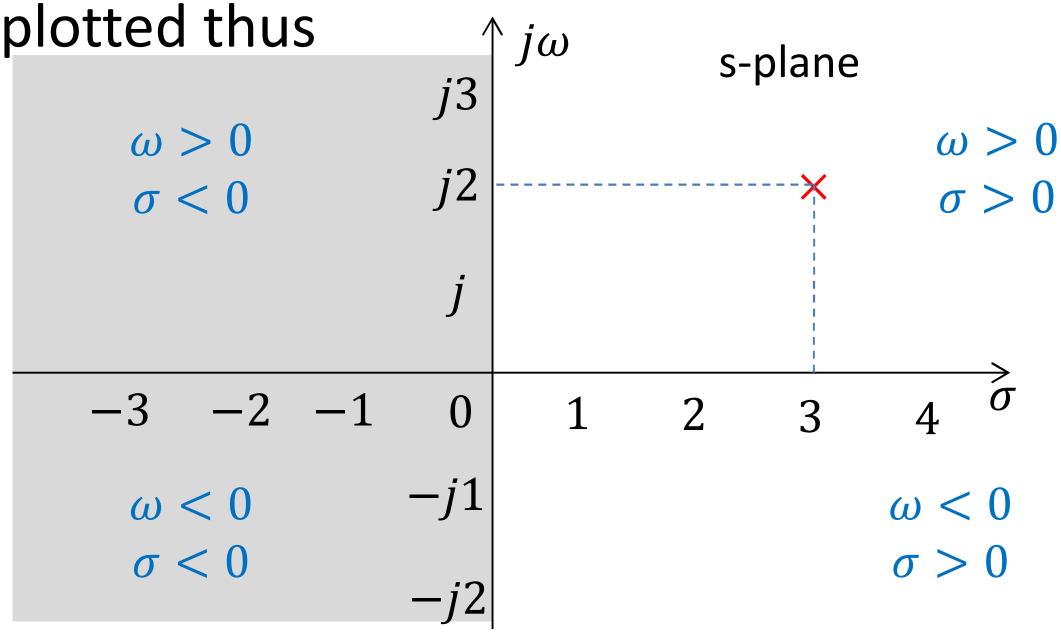
Waveform representation[7]

- The complex frequency may be plotted on the s-plane
- The s-plane is very similar to the complex number plane
- Values of the complex frequency are plotted the same way complex numbers are plotted on the complex plane
- e.g. $s = 3 + j2$ is plotted thus



- Note $\sigma \leq 0$ region for practical systems

Content

- Euler’s formula
- Exponential excitation - review
- Complex numbers - review
- Waveform representation
- **Laplace transform - an introduction**

Laplace transform[1]

- Solutions using the complex exponential method provide **steady state** solutions for circuits
- **Transient solutions** may be obtained by using the **Laplace transform**
- Laplace transform (LT) moves analysis from the **time domain** to the **s-domain**
- In so doing, calculus operations are traded for arithmetic operations
- Analysis in the s-domain also offers insight into the circuit characteristics
- Continuous time physical systems are usually modelled with linear differential equations (LDEs), with constant coefficients
- Laplace transform of the LDEs gives a good description of the characteristics of the LDEs, and thus the physical system

Laplace transform[2]

- Transformed LDEs are algebraic and thus simpler to manipulate and solve
- Solution is in terms of the transform variable s, and thus the inverse LT is needed to convert the solution to a function of time
- Laplace transform, $F(s)$ of a function of time, $f(t)$ is given by

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

- This is the **bilateral** or **two sided Laplace transform**, taking analysis into the s-domain
- The inverse LT is given by the complex inversion integral

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$$

- Minimum value of c for a given transform is called termed the **abscissa of absolute convergence**
- The two equations form the **bilateral LT pair**
- Shorthand designation of the bilateral LT pair, $f(t) \rightleftharpoons F_b(s)$
- Sometimes indicated as $f(t) \overset{\mathcal{L}_b}{\rightleftharpoons} F_b(s)$

Laplace transform[3]

- In many applications $f(t) = 0$ for $t < 0$, and a more useful **unilateral LT** is defined

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

- The inverse LT integral remains the same

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$$

- The two equations form the **(unilateral) LT pair**
- The unilateral LT is the form that will be used in our ECT studies
- Shorthand designation of the LT pair, $f(t) \rightleftharpoons F(s)$
- Sometimes $f(t) \overset{\mathcal{L}}{\rightleftharpoons} F(s)$

- The complex inversion integral is hardly ever used due to the difficulty in evaluating it
- Use of tables of transform pairs is the norm

Laplace transform[4]

- Conditions for the existence of the **unilateral LT**
- A given function $f(t)$, must be an **exponential order function**, for it to have an unilateral LT

i.e. real constants M and α exist such that $|f(t)| < Me^{\alpha t}$ for all $t > t_o$,

- Some common properties of the LT are considered next
- It is important to be able to prove these properties
- With repeated use some of these properties may become obvious
- List of properties given is not necessarily exhaustive and student should read further and obtain their library of properties to use as required for s-domain analysis

Summary

Today's class looked at complex frequency

- Euler's formula
- Exponential excitation - review
- Complex numbers - review
- Waveform representation
- Laplace transform -introduction

QUESTIONS?