FEE321 – E.C.T IIA – Oct 2020

Lecture 13: Laplace Transform (7) (1 hr)

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Overview

Today's class continues the look at LT application

- General Transfer Function

Content

- **General Transfer Function**

Transfer Function [1]

- **Continuous time systems** are often modelled with **LDEs with constant coefficients**
- These models are then **linear** and **time invariant**
- General equilibrium equation for the n^{th} order LTI model is $\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k}$

where

  - $x(t)$ is the **input signal** and $y(t)$ is the **output signal**
  - constants a_k, b_k, m, and n are parameters of the system

- Since

$$\mathcal{L}\left[\frac{d^k f(t)}{dt^k}\right] = s^k F(s) - s^{k-1} f(0^+) - s^{k-2} f'(0^+) - \cdots - f^{(k-1)}(0^+)$$

- And since **Initial Conditions** (IC) are set to zero when deriving the TF (else system would not be linear!)

$$\mathcal{L}\left[\frac{d^k f(t)}{dt^k}\right] = s^k F(s)$$

- Thus the general equilibrium equation above, when Laplace transformed, would give

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) Y(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0) X(s)$$

Transfer Function[2]

- The general Transfer Function (TF) is therefore

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k}$$

- $H(s)$ is a **rational function** and is obtainable for any input $x(t)$ that has a LT
- The numerator and denominator polynomials of the TF can be presented in **product-of-sums** form

$$H(s) = k\,\frac{(s+z_1)(s+z_2)(s+z_3)\ldots(s+z_m)}{(s+p_1)(s+p_2)(s+p_3)\ldots(s+p_n)} \qquad \text{where } k = \frac{b_m}{a_n}$$

- In TF of many physical systems $m < n$
- As seen before

$$H(s)_{s=-z_i}\Big| = 0$$ thus $-z_i$ are **zeros** of the TF

and

$$H(s)_{s=-p_i}\Big| = \infty$$ (i.e. undefined), thus $-p_i$ are **poles** of the TF

- When the degree of the numerator is (one) higher than that of the denominator, the network has a pole at infinity
- Zeros and/or poles occurring at the origin and infinity are referred to as **external**. The rest are **internal**

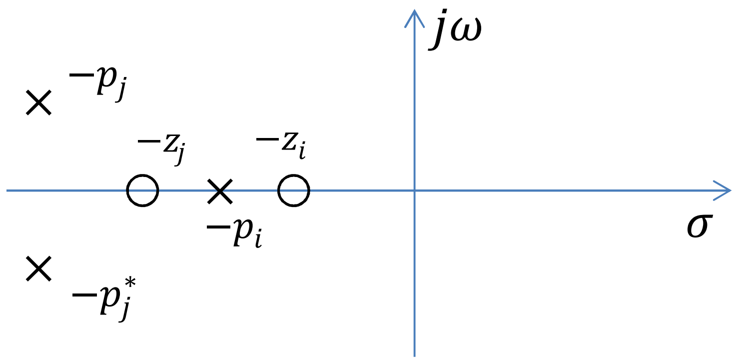
Transfer Function[3]

- To obtain the **time domain behavior** of the circuit for a given TF
  - Perform Partial Fraction Expansion of s-domain quotient of polynomials in s
  - Then obtain the Inverse Laplace Transform (ILT)

$$H(s) = k\frac{(s+z_1)(s+z_2)(s+z_3)\ldots(s+z_m)}{(s+p_1)(s+p_2)(s+p_3)\ldots(s+p_n)} \quad \Rightarrow \quad h(t) = \mathcal{L}^{-1}\left[\frac{A_1}{s+p_1}+\frac{A_2}{s+p_2}+\cdots+\frac{A_n}{s+p_n}\right]$$

$$\Rightarrow h(t) = A_1e^{-p_1t} + A_2e^{-p_2t} + \cdots + A_ne^{-p_nt}$$

- The poles and zeros may be plotted on the ***s-plane***



- For practical systems the poles must lie on the LHS half of the s-plane, as this leads to a finite response as $t \to \infty$

Transfer Function[4]

- Since $s = \sigma + j\omega$ then $e^{-p_i t} = e^{-\sigma_i t} e^{-j\omega_i t}$
- With $e^{-\sigma_i t}$representing damping (exponential decay)
- The larger the value of σ the faster the decay, i.e. the further the σ is from the y-axis

- And $e^{\pm j\omega_i t}$representing sinusoidal oscillation
- The larger the value of ω the greater the oscillation frequency, i.e. the further from the x-axis

- Whenever the poles are complex, they appear in conjugate pairs
- When n is even then the poles may be real or complex conjugate pairs
- When n is odd, at least one pole is real, the others may be real or complex conjugate pairs

- Given all the poles and zeros of a transfer function, $H(s)$ and one value of the $H(s)$ at a value of s that is not a critical frequency, it is possible to determine the expression for $H(s)$

Transfer Function[5]

A certain TF has zeros at $s = 0, -4, -\infty$, and poles at $s = -2, \pm j5$. At $s = 1$, the value of the TF is $\frac{145}{78}$. Determine the TF and obtain its time domain equivalent

- Generally $H(s) = k\dfrac{(s + z_1)(s + z_2)(s + z_3)\dots(s + z_m)}{(s + p_1)(s + p_2)(s + p_3)\dots(s + p_n)}$

- From the given critical frequencies

$$H(s) = k\frac{(s + 0)(s + 4)}{(s + 2)(s + j5)(s - j5)} = k\frac{s(s + 4)}{(s + 2)(s^2 + 25)}$$

- It is evident that $H(\infty) = 0$, and thus a zero exists at infinity

- Since $H(1) = k\dfrac{1(1 + 4)}{(1 + 2)(1 + 25)} = \dfrac{145}{78}$ $\Rightarrow k = \dfrac{145}{78} \times \dfrac{26 \times 3}{5} = 29$

- Thus $H(s) = 29\dfrac{s(s + 4)}{(s + 2)(s^2 + 25)}$

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Transfer Function[6]

- Partial fraction expansion would give

$$H(s) = 29\frac{s(s+4)}{(s+2)(s^2+25)} = \frac{A}{(s+2)} + \frac{Bs+C}{(s^2+25)} \quad \Rightarrow A = -4 \quad B = 33 \quad C = 50$$

- Time domain response would then be $h(t) = u(t)\{-4e^{-2t} + 33\cos 5t + 10\sin 5t\}$
- **Steady state response** and **transient response** components are noted in the $h(t)$

- Suppose the response for an input $x(t) = 17e^{-3t}$was required
- Transform pair $x(t) = 17e^{-3t} \quad \rightleftharpoons \quad X(s) = \dfrac{17}{s+3}$

- Thus since $Y(s) = H(s)X(s)$ then the transformed response would be

$$Y(s) = H(s) \times X(s) \quad = 29\frac{s(s+4)}{(s+2)(s^2+25)} \times \frac{17}{s+3} \quad = 493\frac{s(s+4)}{(s+2)(s+3)(s^2+25)}$$

- Using partial fraction expansion

$$Y(s) = 493\frac{s(s+4)}{(s+2)(s+3)(s^2+25)} = \frac{A}{(s+2)} + \frac{B}{(s+3)} + \frac{Cs+D}{(s^2+25)}$$

- There are 4 constants to be determined

- Expansion $\Rightarrow A = -68 \quad B = 43.5 \quad C = 24.5 \quad D = 487.5$

- Time response to $x(t)$ is then $y(t) = u(t)\{-68e^{-2t} + 43.5e^{-3t} + 24.5\cos 5t + 97.5\sin 5t\}$

Transfer Function[7]

- Transfer functions are often plotted against frequency
- Two plots are often useful for analysis
  - Magnitude plot , i.e. $|H(j\omega)| \quad vs \;\; \omega$
  - Phase plot, i.e. $\angle H(j\omega) \quad vs \;\; \omega$
- On the magnitude plot, the zero values would take the TF to zero, i.e. giving **zero points** on the frequency axis
- On the magnitude plot, poles would take the TF to infinity i.e. poles lead to vertical **asymptotes**
- At times it is possible to plot $H(j\omega) \quad vs \;\; \omega$

Transfer Function[8]

The circuit transfer function below has a pole at $s = -3$. Determine

- (i) **all** its critical frequencies **and** plot them
- (ii) the impulse response of the circuit
- (iii) circuit's response to the input $x(t) = 5\exp(-8t)$

$$H(s) = 8\frac{s^2 + 6s + 8}{s^3 + 9s^2 + 23s + 15}$$

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Summary

Today's class looked at Laplace transform application

- General Transfer Function

QUESTIONS?