FEE321 – E.C.T IIA – Oct 2020

Lecture 11: Laplace Transform (5) (1 hr)

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Overview

Today's class continues the look at LT application

- Examples of LT application in circuit analysis

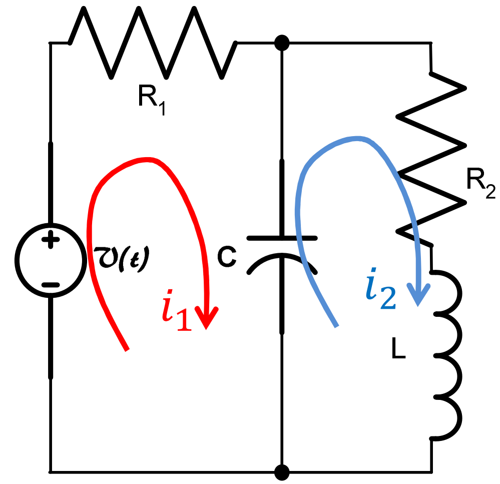
Content

- **Examples of LT application in circuit analysis (continued)**

LT application[1]

Example 9 (numerical)

A RLC circuit is supplied from an source $v(t) = 50u(t)\sin(10t)$ volts. Assume that at $t = 0$ there is zero voltage across the capacitor, and no inductor current. Let $R_1 = 5\Omega$, $R_2 = 2\Omega$, $C = 20mF$ and $L = 0.1H$



Using KVL with clockwise mesh currents, then circuit LDE are

$$\frac{1}{C}\int_{-\infty}^{t}\{i_1(t) - i_2(t)\}\,dt + R_1 i_1(t) = v(t)$$

$$\frac{1}{C}\int_{-\infty}^{t}\{i_2(t) - i_1(t)\}\,dt + R_2 i_2(t) + L\frac{di_2(t)}{dt} = 0$$

Substituting in values

$$50\int_{-\infty}^{t}\{i_1(t) - i_2(t)\}\,dt + 5i_1(t) = 50u(t)\sin 10t$$

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$$50\int_{-\infty}^{t}\{i_2(t) - i_1(t)\}\,dt + 2i_2(t) + 0.1\frac{di_2(t)}{dt} = 0$$

LT application[2]

Example 9 (numerical, cont)

Obtaining the LT

$$\mathcal{L}\left[50\int_{-\infty}^{t}\{i_1(t)-i_2(t)\}\,dt+5i_1(t)=50u(t)\sin 10t\right]$$

$$\mathcal{L}\left[50\int_{-\infty}^{t}\{i_2(t)-i_1(t)\}\,dt+2i_2(t)+0.1\frac{di_2(t)}{dt}=0\right]$$

Gives

$$\frac{v_C(0^+)}{s}+\frac{50}{s}\{I_1(s)-I_2(s)\}+5I_1(s)=\frac{500}{s^2+100}$$

$$\frac{v_C(0^+)}{s}+\frac{50}{s}\{I_2(s)-I_1(s)\}+2I_2(s)+0.1\{sI_2(s)-i_L(0^+)\}=0$$

In standard form

$$I_1(s)\left\{\frac{50}{s}+5\right\}+I_2(s)\left\{-\frac{50}{s}\right\}=\frac{500}{s^2+100}-\frac{v_C(0^+)}{s}$$

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$$I_1(s)\left\{-\frac{50}{s}\right\}+I_2(s)\left\{0.1s+2+\frac{50}{s}\right\}=0.1i_L(0^+)-\frac{v_C(0^+)}{s}$$

LT application[3]

Example 9 (numerical, cont)

Obtaining matrix equation

$$I_1(s)\left\{\frac{50}{s}+5\right\}+I_2(s)\left\{-\frac{50}{s}\right\}=\frac{500}{s^2+100}-\frac{v_C(0^+)}{s}$$

$$I_1(s)\left\{-\frac{50}{s}\right\}+I_2(s)\left\{0.1s+2+\frac{50}{s}\right\}=0.1i_L(0^+)-\frac{v_C(0^+)}{s}$$

Gives

$$\begin{bmatrix}\frac{50}{s}+5 & -\frac{50}{s}\\ -\frac{50}{s} & 0.1s+2+\frac{50}{s}\end{bmatrix}\begin{bmatrix}I_1(s)\\ I_2(s)\end{bmatrix}=\begin{bmatrix}\frac{500}{s^2+100}-\frac{v_C(0^+)}{s}\\ 0.1i_L(0^+)-\frac{v_C(0^+)}{s}\end{bmatrix}$$

With initial conditions set to zero

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$$\begin{bmatrix}\frac{50}{s}+5 & -\frac{50}{s}\\ -\frac{50}{s} & 0.1s+2+\frac{50}{s}\end{bmatrix}\begin{bmatrix}I_1(s)\\ I_2(s)\end{bmatrix}=\begin{bmatrix}\frac{500}{s^2+100}\\ 0\end{bmatrix}$$

LT application[4]

Example 9 (numerical, cont)

Writing terms as single fractions for easier manipulation

$$\begin{bmatrix} \frac{50+5s}{s} & -\frac{50}{s} \\ -\frac{50}{s} & \frac{0.1s^2+2s+50}{s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{500}{s^2+100} \\ 0 \end{bmatrix}$$

Using Cramer's rule

$$I_1(s) = \frac{\begin{vmatrix} \frac{500}{s^2+100} & -\frac{50}{s} \\ 0 & \frac{0.1s^2+2s+50}{s} \end{vmatrix}}{\begin{vmatrix} \frac{50+5s}{s} & -\frac{50}{s} \\ -\frac{50}{s} & \frac{0.1s^2+2s+50}{s} \end{vmatrix}} = \frac{\left(\frac{500}{s^2+100}\right)\left(\frac{0.1s^2+2s+50}{s}\right)}{\left\{\left(\frac{50+5s}{s}\right)\left(\frac{0.1s^2+2s+50}{s}\right)\right\} - \left\{\left(-\frac{50}{s}\right)\left(-\frac{50}{s}\right)\right\}}$$

$$= \frac{\left(\frac{50s^2+1000s+25000}{s(s^2+100)}\right)}{\left(\frac{5s^2+100s+2500+0.5s^3+10s^2+250s-2500}{s^2}\right)}$$

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$$= \frac{\left(\frac{50s^2+1000s+25000}{s(s^2+100)}\right)}{\left(\frac{0.5s^3+15s^2+350s}{s^2}\right)} = \frac{\left(\frac{50s^2+1000s+25000}{(s^2+100)}\right)}{0.5s^2+15s+350}$$

LT application[5]

Example 9 (numerical, cont)

$$I_1(s) = \frac{\left(\frac{50s^2 + 1000s + 25000}{(s^2 + 100)}\right)}{0.5s^2 + 15s + 350} = \frac{2(50s^2 + 1000s + 25000)}{(s^2 + 100)(s^2 + 30s + 700)} = 100\frac{(s^2 + 20s + 500)}{(s^2 + 100)(s^2 + 30s + 700)}$$

Partial fraction decomposition

$$I_1(s) = 100\frac{(s^2 + 20s + 500)}{(s^2 + 100)(s^2 + 30s + 700)} = \frac{(As + B)}{(s^2 + 100)} + \frac{(Cs + D)}{(s^2 + 30s + 700)}$$

$$As + B|_{s=j10} = (s^2 + 100)I_1(s)|_{s=j10} = 100\frac{(-100 + j200 + 500)}{(-100 + j300 + 700)} = 100\frac{(400 + j200)}{(600 + j300)}$$

$$\Rightarrow j10A + B = 100\frac{(4 + j2)}{(6 + j3)} = \frac{200}{3} \quad \Rightarrow A = 0 \quad B = \frac{200}{3}$$

$$Cs + D|_{s=-15+j\sqrt{475}} = (s^2 + 30s + 700)I_1(s)|_{s=-15+j\sqrt{475}}$$

$$C\left(-15 + j\sqrt{475}\right) + D = 100\frac{\left(-15 + j\sqrt{475}\right)^2 + 20\left(-15 + j\sqrt{475}\right) + 500}{\left(-15 + j\sqrt{475}\right)^2 + 100}$$

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$$= 100\frac{(-250 - j653.8348) + (-300 + j435.8899) + 500}{(-250 - j653.8348) + 100}$$

$$= 100\frac{(-50 - j217.9449)}{(-150 - j653.8348)} = \frac{1000}{3} \quad \Rightarrow C = 0 \quad D = \frac{1000}{3}$$

LT application[6]

Example 9 (numerical, cont)

$$I_1(s) = \frac{(As+B)}{(s^2+100)} + \frac{(Cs+D)}{(s^2+30s+700)} \quad = \frac{200/3}{(s^2+100)} + \frac{1000/3}{(s^2+30s+700)}$$

The inverse LT can then be obtained

$$I_1(s) = \frac{20}{3} \times \frac{10}{(s^2+100)} + \frac{1000}{3\sqrt{475}} \times \frac{\sqrt{475}}{(s+15)^2 + \left(\sqrt{475}\right)^2}$$

$$\Rightarrow i_1(t) = u(t)\left\{6.6667 \sin 10t + 15.2944\, e^{-15t} \sin \sqrt{475}t\right\} \quad A$$

Note the **steady state component** and **the transient component**

The same procedure is followed to obtain the current $i_2(t)$

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LT application[7]

Example 9 (numerical, cont)

Using Cramer's rule

$$I_2(s) = \frac{\begin{vmatrix} \frac{50+5s}{s} & \frac{500}{s^2+100} \\ -\frac{50}{s} & 0 \end{vmatrix}}{\left(\frac{0.5s^3+15s^2+350s}{s^2}\right)} = \frac{\left(\frac{50}{s}\right)\left(\frac{500}{s^2+100}\right)}{\left(\frac{0.5s^3+15s^2+350s}{s^2}\right)} = \frac{\left(\frac{25000}{s(s^2+100)}\right)}{\left(\frac{0.5s^3+15s^2+350s}{s^2}\right)}$$

$$= \frac{\frac{25000}{(s^2+100)}}{0.5s^2+15s+350} = \frac{2(25000)}{(s^2+100)(s^2+30s+700)} = \frac{(As+B)}{(s^2+100)} + \frac{(Cs+D)}{(s^2+30s+700)}$$

$$As+B|_{s=j10} = (s^2+100)I_2(s)|_{s=j10} = \frac{50000}{(-100+j300+700)} = \frac{200}{3} - j\,\frac{1000}{3}$$

$$\Rightarrow A = -\frac{100}{3} \quad B = \frac{200}{3}$$

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LT application[8]

Example 9 (numerical, cont)

$$Cs + D\big|_{s=-15+j\sqrt{475}} = (s^2 + 30s + 700)I_2(s)\big|_{s=-15+j\sqrt{475}}$$

$$C\left(-15 + j\sqrt{475}\right) + D = \frac{50000}{\left(-15 + j\sqrt{475}\right)^2 + 100} = \frac{50000}{(-250 - j653.8348) + 100}$$

$$= \frac{50000}{(-150 - j653.8348)} = -\frac{50}{3} + j72.6483 \qquad \Rightarrow C = \frac{10}{3} \qquad D = \frac{100}{3}$$

$$I_2(s) = \frac{(As + B)}{(s^2 + 100)} + \frac{(Cs + D)}{(s^2 + 30s + 700)}$$

$$= \frac{(-100/3)s}{(s^2 + 100)} + \frac{200/3}{(s^2 + 100)} + \frac{(10/3)s}{(s^2 + 30s + 700)} + \frac{100/3}{(s^2 + 30s + 700)}$$

$$= \frac{(-100/3)s}{(s^2 + 100)} + \frac{20}{3} \times \frac{10}{(s^2 + 100)} + \frac{(10/3)(s + 15)}{(s + 15)^2 + \left(\sqrt{475}\right)^2} - \frac{50}{3\sqrt{475}} \frac{\sqrt{475}}{(s + 15)^2 + \left(\sqrt{475}\right)^2}$$

$$\Rightarrow i_2(t) = u(t)\left\{-33.3333 \cos 10t + 6.6667 \sin 10t + e^{-15t}\left(3.3333 \cos \sqrt{475}t - 0.7647 \sin \sqrt{475}\right)\right\} \quad A$$

LT application[9]

Example 9 (numerical, cont)

Obtaining **characteristic equation** from equilibrium equations in standard form

$$I_1(s)\left\{\frac{50}{s}+5\right\}+I_2(s)\left\{-\frac{50}{s}\right\}=\frac{500}{s^2+100}-\frac{v_C(0^+)}{s}$$

$$I_1(s)\left\{-\frac{50}{s}\right\}+I_2(s)\left\{0.1s+2+\frac{50}{s}\right\}=0.1i_L(0^+)-\frac{v_C(0^+)}{s}$$

Set initial conditions and sources to zero

$$I_1(s)\left\{\frac{50}{s}+5\right\}+I_2(s)\left\{-\frac{50}{s}\right\}=0 \quad \Rightarrow I_1(s)=I_2(s)\left\{\frac{10}{10+s}\right\}$$

$$I_1(s)\left\{-\frac{50}{s}\right\}+I_2(s)\left\{0.1s+2+\frac{50}{s}\right\}=0 \quad \Rightarrow I_2(s)\left\{\frac{10}{10+s}\right\}\left\{-\frac{50}{s}\right\}+I_2(s)\left\{0.1s+2+\frac{50}{s}\right\}=0$$

$$\Rightarrow \left\{\frac{10}{10+s}\right\}\left\{-\frac{50}{s}\right\}+\left\{0.1s+2+\frac{50}{s}\right\}=0 \quad \Rightarrow \{-500\}+s\{10+s\}\left\{0.1s+2+\frac{50}{s}\right\}=0$$

$$\Rightarrow \{-500\}+\{10+s\}\{0.1s^2+2s+50\}=0 \Rightarrow \{-500\}+\{s^2+20s+500\}+\{0.1s^3+2s^2+50s\}=0$$

$$\Rightarrow 0.1s^3+3s^2+70s=0 \qquad \Rightarrow s^3+30s^2+700s=0 \quad \Rightarrow s^2+30s+70=0$$

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LT application[10]

Example 9 (numerical, cont)

Suppose the input of the circuit were defined as the voltage source and the output were defined as the voltage across R_2. The **transfer function** of the circuit could then be determined.

Since the current was determined with zero initial conditions, it is ready for direct use

$$H(s) = \frac{Y(s)}{X(s)} \quad = \frac{I_2(s) \times R_2}{V(s)} \quad = \frac{50000 \times 2}{(s^2+100)(s^2+30s+700)} \times \frac{s^2+100}{500}$$

$$= \frac{200}{(s^2+30s+700)}$$

Impulse response of the circuit would then be determined by inverse LT of $H(s)$

$$H(s) = \frac{200}{(s^2+30s+700)} \quad = \frac{200}{\sqrt{475}} \times \frac{\sqrt{475}}{(s+15)^2 + \left(\sqrt{475}\right)^2}$$

$$\Rightarrow h(t) = u(t)\left\{9.1766 e^{-15t} \sin \sqrt{475}t\right\}$$

Using the transfer function the response to other types of input can be determined as seen before

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LT application[11]

Critical Frequencies

- Values of $\boldsymbol{s}$ that take $H(s)$ to zero are known as **zeros** of $H(s)$
- These would be **roots of the numerator polynomial** of $H(s)$

- Values of $\boldsymbol{s}$ that take $H(s)$ to infinity are known as **poles** of $H(s)$.
- These would be **the roots of the denominator polynomial** of $H(s)$
- Poles and zeros are known as **critical frequencies**
- **Internal** poles and zeros have values, $0 < s < \infty$
- **External** poles and zeros occur at $s = 0$ or $s = \infty$

- Poles are such that the output of the circuit will be unbounded, whatever the input $Y(s) = H(s)X(s) \quad \Rightarrow Y(s) = \infty \cdot X(s) \ = \infty$
- Zeros are such that the output of the circuit would be zero, irrespective of the input value $Y(s) = H(s)X(s) \quad \Rightarrow Y(s) = 0 \cdot X(s) \ = 0$

- Poles and zeros of a circuit give insight as to the behavior of the circuit
- In the immediate previous example $H(s) = \dfrac{200}{(s^2 + 30s + 700)}$
- External zero at $s = \infty$
- Poles at $s = -15 + j\sqrt{475}$ and $s = -15 - j\sqrt{475}$

Summary

Today's class looked at more Laplace transform application

- Examples of LT application in circuit analysis

QUESTIONS?