2-order theory of wave-particle interacting acceleration and heating

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1 Basic formulas

Consider the background magnetic field and background velocity along the z direction.

Vlasov equation for species J:

$$\frac{\partial f_j}{\partial t} + \vec{v} \cdot \nabla f_j + \frac{e_j}{m_j} \left[\vec{E}(\vec{x}, t) + \vec{v} \times \vec{B}(\vec{x}, t) \right] \cdot \nabla_v f_j = 0. \tag{1}$$

Maxwell's equations:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \tag{2}$$

$$\nabla \cdot \vec{B} = 0, \tag{3}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{4}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right). \tag{5}$$

where the charge density ρ and the current density $\vec{J}$ can be expressed in terms of distribution functions:

$$\rho(\vec{x}, t) = \sum_j e_j \int d^3 v f_j(\vec{x}, \vec{v}, t), \tag{6}$$

$$\vec{J}(\vec{x}, t) = \sum_j e_j \int d^3 v \vec{v} f_j(\vec{x}, \vec{v}, t). \tag{7}$$

Expand:

$$\begin{aligned} & f_j = f_j^{(0)}(\vec{v}) + f_j^{(1)}(\vec{x}, \vec{v}, t) + f_j^{(2)}(\vec{x}, \vec{v}, t) + ..., \\ & \vec{E}(\vec{x}, t) = \vec{E}^{(1)}(\vec{x}, t) + \vec{E}^{(2)}(\vec{x}, t) + ..., \\ & \vec{B}(\vec{x}, t) = \vec{B}_0 + \vec{B}^{(1)}(\vec{x}, t) + \vec{B}^{(2)}(\vec{x}, t) + ..., \end{aligned} \tag{8}$$

The 0th order quantity has nothing to do with time and space. the 0th order motion is a uniform linear motion along the background magnetic field direction, plus a gyro motion perpendicular to the background magnetic field. The background magnetic field $\vec{B}_0 = \hat{z} B_0$. Higher order quantities are assumed to be much smaller than lower order quantities:

$$\left| g^{(i+1)} \right| \ll \left| g^{(i)} \right|. \tag{9}$$

The 1st order quantity represents a fluctuation with constant amplitude (fast-varying), and the 2nd order quantity contains the fast-varying part of the 1st order quantity, and also contains the slow-varying part of the amplitude, see Figure 1.

2 0th order

Assuming that the 0th order terms should satisfy the above equations, then the 0th order Vlasov equation can be written considering that the 0th order distribution function is space-time independent:

$$\left(\vec{v} \times \vec{B}_0 \right) \cdot \nabla_v f_j^{(0)} = 0. \tag{10}$$

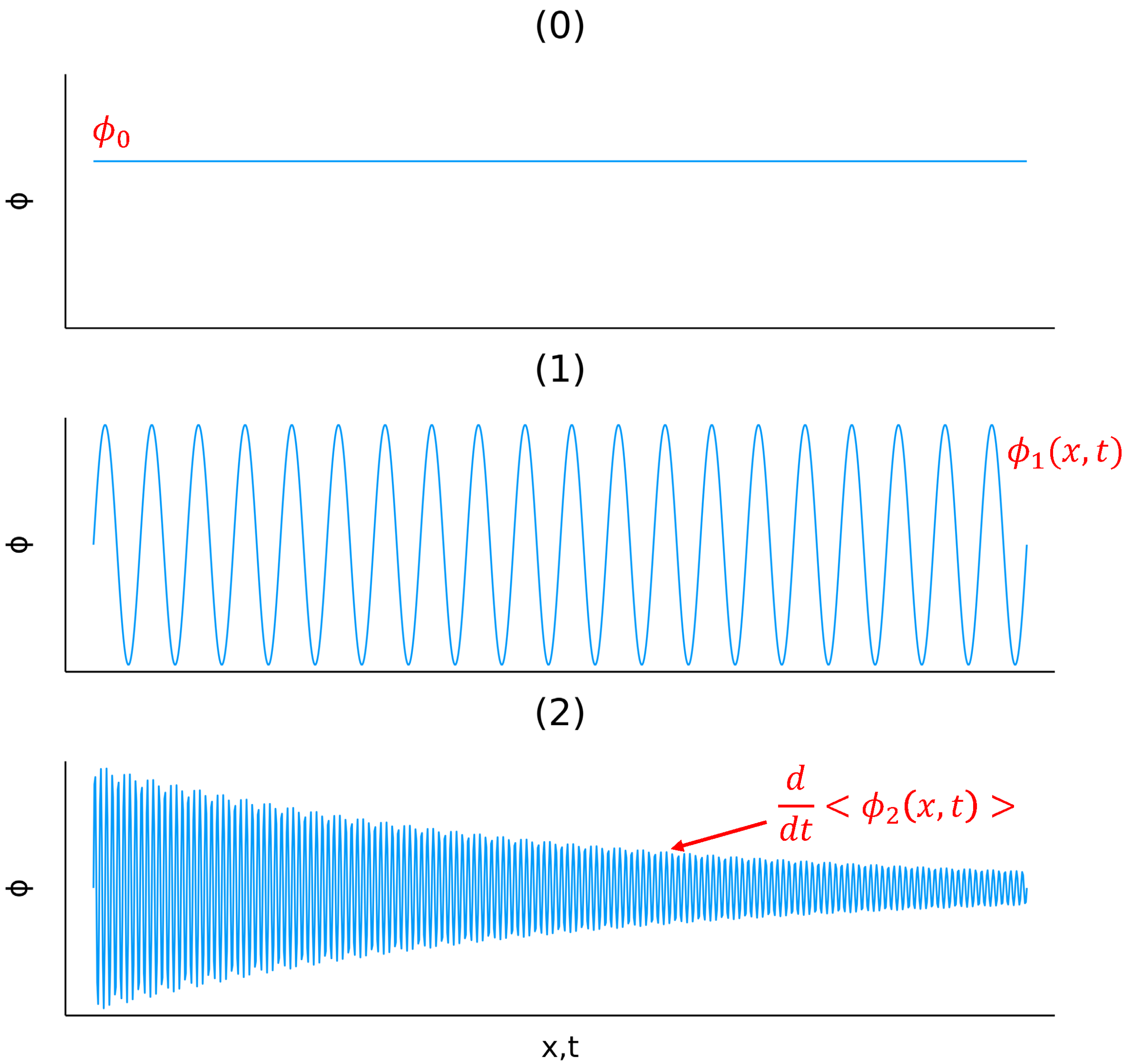

Figure 1: A sketch of each order quantity.

Let's consider the direction of $\vec{v} \times \vec{B}_0$: perpendicular to $\vec{B}_0$ (in the z direction), so in the xy plane; and perpendicular to $\vec{v}$, so that it should be in the direction of $\hat{\theta}$,

$$\hat{\theta} \cdot \nabla_v f_j^{(0)} = 0.$$

This means that the distribution function has axisymmetry, and in the xy plane, the 0th order distribution function is just a function of the magnitude of the perpendicular velocity, independent of the direction of the velocity, which gives the form of the distribution function:

$$f_j^{(0)} = f_j^{(0)} \left(v_z, v_\perp^2\right), \tag{11}$$

where,

$$v_\perp^2 = v_x^2 + v_y^2.$$

Thus, the velocity space gradient of the 0th order distribution function can be obtained to satisfy:

$$
\begin{aligned}
\nabla_v \cdot f_j^{(0)} &= \begin{bmatrix} \frac{\partial}{\partial v_x} \\ \frac{\partial}{\partial v_y} \\ \frac{\partial}{\partial v_z} \end{bmatrix} f_j^{(0)} \\
&= \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} \frac{\partial f_j^{(0)}}{\partial (v_\perp^2/2)} + \frac{\partial f_j^{(0)}}{\partial v_z} \hat{z} \\
&= \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} f_{j,\perp}^{(0)} + f_{j,z}^{(0)} \hat{z}
\end{aligned} \tag{12}
$$

where

$$
f_{j,z}^{(0)} = \frac{\partial f_j^{(0)}}{\partial v_z}
$$

$$
f_{j,\perp}^{(0)} = \frac{\partial f_j^{(0)}}{\partial (v_\perp^2/2)} = \frac{1}{v_\perp} \frac{\partial f_j^{(0)}}{\partial v_\perp}.
$$

The Bi-Maxwellian distribution is a distribution function that conforms to this form:

$$
f_j^{(0)}(\vec{v}) = \frac{n_j T_{\parallel j}}{\left(2\pi v_j^2\right)^{3/2} T_{\perp j}} \cdot exp\left[-\frac{(v_z - v_{0j})^2}{2v_j^2} - \frac{v_x^2 + v_y^2}{2v_j^2} \frac{T_{\parallel j}}{T_{\perp j}}\right] \tag{13}
$$

where v_j is the parallel thermal velocity:

$$
v_j^2 = \frac{k_B T_{\parallel j}}{m_j}
$$

v_{0j} denotes the 0th order drift velocity in the z direction $\vec{v_0}_j = \hat{z} v_{0j}$.

3 1st order

Assuming that the amplitudes of all 1st order fluctuations do not vary in space and time after Fourier transform, they can be written as a superposition of simple harmonic fluctuations of different wave number and frequencies:

$$
g^{(1)}(\vec{x}, t) = g^{(1)}\left(\vec{k}, \omega\right) e^{i\left(\vec{k}\cdot\vec{x} - \omega t\right)} \tag{14}
$$

Expand the Vlasov equation to 1st order:

$$
\frac{\partial\left(f_j^{(0)} + f_j^{(1)}\right)}{\partial t} + \vec{v} \cdot \nabla\left(f_j^{(0)} + f_j^{(1)}\right) + \frac{e_j}{m_j}\left[\vec{E}^{(1)}(\vec{x}, t) + \vec{v} \times \vec{B}_0 + \vec{v} \times \vec{B}^{(1)}(\vec{x}, t)\right] \cdot \nabla_v\left(f_j^{(0)} + f_j^{(1)}\right) = 0,
$$

minus the 0th order Vlasov equation:

$$
\frac{\partial f_j^{(1)}}{\partial t} + \vec{v} \cdot \nabla f_j^{(1)} + \frac{e_j}{m_j}\left[\vec{E}^{(1)}(\vec{x}, t) + \vec{v} \times \vec{B}^{(1)}(\vec{x}, t)\right] \cdot \nabla_v f_j^{(0)} + \frac{e_j}{m_j}\left[\vec{E}^{(1)}(\vec{x}, t) + \vec{v} \times \vec{B}_0 + \vec{v} \times \vec{B}^{(1)}(\vec{x}, t)\right] \cdot \nabla_v f_j^{(1)} = 0,
$$

Ignoring the second-order minima (these terms will be put inside the second-order equations afterwards), the first-order Vlasov equation (linear Vlasov equation) can be written as:

$$
\frac{\partial f_j^{(1)}}{\partial t} + \vec{v} \cdot \nabla f_j^{(1)} + \frac{e_j}{m_j}\left(\vec{v} \times \vec{B}_0\right) \cdot \nabla_v f_j^{(1)} = -\frac{e_j}{m_j}\left[\vec{E}^{(1)}(\vec{x}, t) + \vec{v} \times \vec{B}^{(1)}(\vec{x}, t)\right] \cdot \nabla_v f_j^{(0)}. \tag{15}
$$

0th order orbital approximation[1]:

The motion of the 0th order orbit is:

$$\frac{d\vec{v}}{dt} = \frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right) \tag{16}$$

$$\left(\frac{df_j}{dt}\right)_0 = \frac{\partial f_j}{\partial t} + \vec{v}\cdot\nabla f_j + \frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v f_j \tag{17}$$

In this way, the linear Vlasov equation can be written as:

$$\left(\frac{df_j^{(1)}}{dt}\right)_0 = -\frac{e_j}{m_j}\left[\vec{E}^{(1)}\left(\vec{x},t\right) + \vec{v}\times\vec{B}^{(1)}\left(\vec{x},t\right)\right]\cdot\nabla_v f_j^{(0)}, \tag{18}$$

where

$$\vec{E}^{(1)}\left(\vec{x},t\right) = \vec{E}^{(1)}(\vec{k},\omega)e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)},$$

According to Faraday's law:

$$\vec{B}^{(1)}(\vec{k},\omega) = \frac{\vec{k}}{\omega}\times\vec{E}^{(1)}(\vec{k},\omega).$$

The linear Vlasov equation can be written as:

$$\begin{aligned}\left(\frac{df_j^{(1)}}{dt}\right)_0 &= -\frac{e_j}{m_j}e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\left[\vec{E}^{(1)} + \vec{v}\times\left(\frac{\vec{k}}{\omega}\times\vec{E}^{(1)}\right)\right]\cdot\nabla_v f_j^{(0)} \\ &= -\frac{e_j}{m_j}\vec{E}^{(1)}e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\left[1 + \frac{\cdot\vec{v}\vec{k}}{\omega} - \frac{\vec{v}\cdot\vec{k}}{\omega}\right]\cdot\nabla_v f_j^{(0)}\end{aligned} \tag{19}$$

Integrating along the 0th order orbit gives the 1st order distribution function, but here it is important to pay attention to the range of integration in order to avoid infinity. If the waves are growing, the fluctuations should be 0 at $t=-\infty$ and thus can be integrated from $t=-\infty$ to t; if the waves are decaying, the fluctuations should be zero at $t=\infty$ and should be integrated from t to $t=\infty$.

Growing waves:

$$\begin{aligned}f_j^{(1)}\left(\vec{x},\vec{v},t\right) &= -\frac{e_j}{m_j}\int_{-\infty}^{t}\vec{E}^{(1)}e^{i\left(\vec{k}\cdot\vec{x}'-\omega t'\right)}\left[1 + \frac{\cdot\vec{v}'\vec{k}}{\omega} - \frac{\vec{v}'\cdot\vec{k}}{\omega}\right]\cdot\nabla_{v'} f_j^{(0)}\left(\vec{v}'\right)dt' \\ &= -\frac{e_j}{m_j\omega}\vec{E}^{(1)}e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\int_{-\infty}^{t}e^{i\left(\vec{k}\cdot\left(\vec{x}'-\vec{x}\right)-\omega\left(t'-t\right)\right)}\left(\omega + \cdot\vec{v}'\vec{k} - \vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'} f_j^{(0)}\left(\vec{v}'\right)dt'\end{aligned} \tag{20}$$

Decaying waves:

$$f_j^{(1)}\left(\vec{x},\vec{v},t\right) = \frac{e_j}{m_j\omega}\vec{E}^{(1)}e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\int_{t}^{\infty}e^{i\left(\vec{k}\cdot\left(\vec{x}'-\vec{x}\right)-\omega\left(t'-t\right)\right)}\left(\omega + \cdot\vec{v}'\vec{k} - \vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'} f_j^{(0)}\left(\vec{v}'\right)dt' \tag{21}$$

For the 0th order orbit approximation, x', v' in the integral can be obtained in terms of gyro motion: First, the angular velocity of the gyro motion is

$$\vec{\Omega}_j = -\frac{e_j\vec{B}_0}{m_j}$$

Using the rotation matrix, one can get the value at the moment t' from the perpendicular velocity $\vec{v}_\perp(t)$ at the moment t

$$\vec{v}_\perp(t') = \begin{bmatrix} v_x(t') \\ v_y(t') \end{bmatrix} = \begin{bmatrix} cos\left[\Omega_j\left(t'-t\right)\right] & -sin\left[\Omega_j\left(t'-t\right)\right] \\ sin\left[\Omega_j\left(t'-t\right)\right] & cos\left[\Omega_j\left(t'-t\right)\right] \end{bmatrix}\vec{v}_\perp(t) \tag{22}$$

where,

$$\Omega_j = -\frac{e_j B_0}{m_j}$$

Note that the gyro angular velocity is signed, and the sign indicates the positive or negative z direction.

Velocity in the z direction:

$$v_z(t') = v_z. \tag{23}$$

Gyro radio:

$$\begin{aligned}
\vec{r}(t') &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\vec{v}_\perp(t')}{\Omega_j} \\
&= \begin{bmatrix} sin\left[\Omega_j\left(t'-t\right)\right] & cos\left[\Omega_j\left(t'-t\right)\right] \\ -cos\left[\Omega_j\left(t'-t\right)\right] & sin\left[\Omega_j\left(t'-t\right)\right] \end{bmatrix} \frac{\vec{v}_\perp(t)}{\Omega_j}
\end{aligned}$$

Thus, the particle position in the xy plane can be expressed as:

$$\begin{aligned}
\vec{x}_\perp(t') &= \begin{bmatrix} x(t') \\ y(t') \end{bmatrix} \\
&= \vec{x}_\perp(t) + \vec{r}(t') - \vec{r}(t) \\
&= \vec{x}_\perp(t) + \begin{bmatrix} sin\left[\Omega_j\left(t'-t\right)\right] & -\left(1-cos\left[\Omega_j\left(t'-t\right)\right]\right) \\ 1-cos\left[\Omega_j\left(t'-t\right)\right] & sin\left[\Omega_j\left(t'-t\right)\right] \end{bmatrix} \frac{\vec{v}_\perp(t)}{\Omega_j}
\end{aligned} \tag{24}$$

The position in the z direction can be expressed as:

$$z(t') = v_z(t'-t) + z(t) \tag{25}$$

Go back to the 1st order distribution function,

$$\begin{aligned}
f_j^{(1)}\left(\vec{x},\vec{v},t\right) &= -\frac{e_j}{m_j}\int_{-\infty}^{t} \vec{E}^{(1)} e^{i\left(\vec{k}\cdot\vec{x}'-\omega t'\right)} \left[1 + \frac{\cdot\vec{v}'\vec{k}}{\omega} - \frac{\vec{v}'\cdot\vec{k}}{\omega}\right] \cdot \nabla_{v'} f_j^{(0)}\left(\vec{v}'\right) dt' \\
&= -\frac{e_j}{m_j\omega} e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)} \int_{-\infty}^{t} e^{i\left(\vec{k}\cdot\left(\vec{x}'-\vec{x}\right)-\omega\left(t'-t\right)\right)} \vec{E}^{(1)} \left(\omega + \cdot\vec{v}'\vec{k} - \vec{v}'\cdot\vec{k}\right) \cdot \nabla_{v'} f_j^{(0)}\left(\vec{v}'\right) dt'
\end{aligned}$$

where

$$\begin{aligned}
\vec{k}\cdot\left(\vec{x}'-\vec{x}\right) - \omega(t'-t) &= \vec{k}_\perp \cdot \left[\vec{x}_\perp(t') - \vec{x}_\perp(t)\right] + k_z\left[z(t') - z(t)\right] - \omega(t'-t) \\
&= \vec{k}_\perp \cdot \begin{bmatrix} sin\left[\Omega_j\left(t'-t\right)\right] & -\left(1-cos\left[\Omega_j\left(t'-t\right)\right]\right) \\ 1-cos\left[\Omega_j\left(t'-t\right)\right] & sin\left[\Omega_j\left(t'-t\right)\right] \end{bmatrix} \frac{\vec{v}_\perp(t)}{\Omega_j} + (k_z v_z - \omega)(t'-t) \\
&= \vec{k}_\perp^T \begin{bmatrix} sin\Omega_j\tau & -\left(1-cos\Omega_j\tau\right) \\ 1-cos\Omega_j\tau & sin\Omega_j\tau \end{bmatrix} \frac{\vec{v}_\perp}{\Omega_j} + (k_z v_z - \omega)\tau,
\end{aligned} \tag{26}$$

$$\tau = t' - t. \tag{27}$$

and

$$\begin{aligned}
\vec{E}^{(1)}\left(\omega+\cdot\vec{v}'\vec{k}-\vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right) &= \vec{E}^{(1)}\left(\omega-\vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)+\vec{E}^{(1)}\cdot\vec{v}'\vec{k}\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)\\
&= \left(\omega-k_zv_z-\vec{k}_\perp\cdot\vec{v}_\perp(t')\right)\vec{E}^{(1)}\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)\\
&+\left(E_z^{(1)}v_z+\vec{E}_\perp^{(1)}\cdot\vec{v}'_\perp\right)\left(k_z\frac{\partial f_j^{(0)}\left(\vec{v}'\right)}{\partial v'_z}+\vec{k}_\perp\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)\right)\\
&= \left(\omega-k_zv_z-\vec{k}_\perp\cdot\vec{v}_\perp(t')\right)\left(E_z^{(1)}\frac{\partial f_j^{(0)}\left(\vec{v}'\right)}{\partial v'_z}+\vec{E}_\perp^{(1)}\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)\right)\\
&+\left(E_z^{(1)}v_z+\vec{E}_\perp^{(1)}\cdot\vec{v}'_\perp\right)\left(k_z\frac{\partial f_j^{(0)}\left(\vec{v}'\right)}{\partial v'_z}+\vec{k}_\perp\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)\right)\\
&= \vec{E}_\perp^{(1)}\cdot\vec{v}'_\perp\left[\omega f_{j,\perp}^{(0)}+k_z\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)\right]\\
&- E_z^{(1)}\vec{k}_\perp\cdot\vec{v}'_\perp\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)+\omega E_z^{(1)}f_{j,z}^{(0)} \qquad (28)
\end{aligned}$$

where,

$$\vec{v}'_\perp=\vec{v}_\perp(t')=\begin{bmatrix}cos\Omega_j\tau & -sin\Omega_j\tau\\ sin\Omega_j\tau & cos\Omega_j\tau\end{bmatrix}\vec{v}_\perp$$

In this way, the integral in the 1st order distribution function is space-time independent and gives the 1st order distribution function in the frequency domain:

$$f_j^{(1)}\left(\vec{k},\vec{v},\omega\right)=-\frac{e_j}{m_j\omega}\int_{-\infty}^{0}e^{i\left(\vec{k}\cdot(\vec{x}'-\vec{x})-\omega(t'-t)\right)}\vec{E}^{(1)}\left(\omega+\cdot\vec{v}'\vec{k}-\vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)d\tau$$

Note that since $v_\perp$ does not vary over time, in fact $f_j^{(0)}\left(v_z,v_\perp^2\right)$ does not vary over time.

We want to use complex numbers in the perpendicular direction so that it's easier to eliminate the term containing τ. Do a coordinate transformation in the perpendicular direction:

$$\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$$

Note that this is a non-orthogonal decomposition, and that $\hat{e}_+$, $\hat{e}_-$ is not a set of orthogonal basis, and that the coordinate transformation requires solving for the inverse matrix.

$$\begin{bmatrix}g_x\\ g_y\end{bmatrix}=\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}\begin{bmatrix}g_+\\ g_-\end{bmatrix}\Longrightarrow\begin{bmatrix}g_+\\ g_-\end{bmatrix}=\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}^{-1}\begin{bmatrix}g_x\\ g_y\end{bmatrix}$$

where,

$$\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}^{-1}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}$$

Also, the Metric matrix is used to compute the inner product:

$$\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}^T\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

Then,

$$\begin{bmatrix}v'_+\\ v'_-\end{bmatrix}=\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}^{-1}\begin{bmatrix}cos\Omega_j\tau & -sin\Omega_j\tau\\ sin\Omega_j\tau & cos\Omega_j\tau\end{bmatrix}\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}\begin{bmatrix}v_+\\ v_-\end{bmatrix}=\begin{bmatrix}e^{-i\Omega_j\tau} & 0\\ 0 & e^{i\Omega_j\tau}\end{bmatrix}\begin{bmatrix}v_+\\ v_-\end{bmatrix}$$

So,

$$\begin{aligned}\vec{E}_\perp^{(1)} \cdot \vec{v}'_\perp &= \begin{bmatrix} E_+ & E_- \end{bmatrix} \begin{bmatrix} \hat{e}_+ & \hat{e}_- \end{bmatrix}^T \begin{bmatrix} \hat{e}_+ & \hat{e}_- \end{bmatrix} \begin{bmatrix} v'_+ \\ v'_- \end{bmatrix} \\ &= v'_+ E_- + v'_- E_+ \\ &= e^{-i\Omega_j \tau} v_+ E_- + e^{i\Omega_j \tau} v_- E_+ \\ &= \sum_{\pm} e^{\mp i\Omega_j \tau} v_\pm E_\mp \end{aligned} \tag{29}$$

Consider the case that $\vec{k} \parallel \vec{B}_0$, and consider electromagnetic waves (EMF fluctuations perpendicular to $\vec{B}_0$).

3.1 Growing waves

$$\vec{k} \cdot (\vec{x}' - \vec{x}) - \omega(t' - t) = (k_z v_z - \omega)\tau, \tag{30}$$

$$\vec{E}^{(1)} \left(\omega + \cdot \vec{v}' \vec{k} - \vec{v}' \cdot \vec{k} \right) \cdot \nabla_{v'} f_j^{(0)} \left(\vec{v}' \right) = \vec{E}_\perp^{(1)} \cdot \vec{v}'_\perp \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)} \right) \right] \tag{31}$$

$$\begin{aligned} f_j^{(1)} \left(\vec{k}, \vec{v}, \omega \right) &= -\frac{e_j}{m_j \omega} \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)} \right) \right] \int_{-\infty}^0 e^{i(k_z v_z - \omega)\tau} \vec{E}_\perp^{(1)} \cdot \vec{v}'_\perp d\tau \\ &= -\frac{e_j}{m_j \omega} \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)} \right) \right] \sum_{\pm} v_\pm E_\mp \int_{-\infty}^0 e^{i(k_z v_z - \omega \mp \Omega_j)\tau} d\tau \end{aligned} \tag{32}$$

Assuming $\omega = \omega_r + i\gamma$, $\gamma > 0$,

$$\begin{aligned} \int_{-\infty}^0 e^{i(k_z v_z - \omega \mp \Omega_j)\tau} d\tau &= \int_{-\infty}^0 e^{i(k_z v_z - \omega_r \mp \Omega_j)\tau} e^{\gamma \tau} d\tau \\ &= \frac{1}{i(k_z v_z - \omega \mp \Omega_j)} \left[e^{i(k_z v_z - \omega_r \mp \Omega_j)\tau} e^{\gamma \tau} \right]_{-\infty}^0 \\ &= \frac{-i}{k_z v_z - \omega \mp \Omega_j} \end{aligned}$$

Finally,

$$f_j^{(1)} \left(\vec{k}, \vec{v}, \omega \right) = \frac{i e_j}{m_j \omega} \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)} \right) \right] \sum_{\pm} v_\pm E_\mp \frac{1}{k_z v_z - \omega \mp \Omega_j} \tag{33}$$

3.2 Decaying waves

$$\begin{aligned} f_j^{(1)} \left(\vec{k}, \vec{v}, \omega \right) &= \frac{e_j}{m_j \omega} \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)} \right) \right] \int_0^\infty e^{i(k_z v_z - \omega)\tau} \vec{E}_\perp^{(1)} \cdot \vec{v}'_\perp d\tau \\ &= \frac{e_j}{m_j \omega} \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)} \right) \right] \sum_{\pm} v_\pm E_\mp \int_0^\infty e^{i(k_z v_z - \omega \mp \Omega_j)\tau} d\tau \end{aligned} \tag{34}$$

Assuming $\omega = \omega_r + i\gamma$, $\gamma < 0$

$$\begin{aligned} \int_0^\infty e^{i(k_z v_z - \omega \mp \Omega_j)\tau} d\tau &= \int_0^\infty e^{i(k_z v_z - \omega_r \mp \Omega_j)\tau} e^{\gamma \tau} d\tau \\ &= \frac{1}{i(k_z v_z - \omega \mp \Omega_j)} \left[e^{i(k_z v_z - \omega_r \mp \Omega_j)\tau} e^{\gamma \tau} \right]_0^\infty \\ &= \frac{i}{k_z v_z - \omega \mp \Omega_j} \end{aligned}$$

Finally,

$$f_j^{(1)}\left(\vec{k},\vec{v},\omega\right)=\frac{ie_j}{m_j\omega}\left[\omega f_{j,\perp}^{(0)}+k_z\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)\right]\sum_{\pm}v_\pm E_\mp\frac{1}{k_zv_z-\omega\mp\Omega_j}\tag{35}$$

This means that the 1st order distribution function are the same regardless of whether the fluctuation grows or decays.

3.3 Dispersion relation

Still, we consider $\vec{k}\parallel\vec{B}_0$. Faraday's law

$$\vec{B}^{(1)}(\vec{k},\omega)=\frac{\vec{k}}{\omega}\times\vec{E}^{(1)}(\vec{k},\omega).$$

Substitute Ampere's law:

$$i\vec{k}\times\vec{B}^{(1)}=\mu_0\vec{J}^{(1)}-\frac{i\omega}{c^2}\vec{E}^{(1)},$$

Dispersion relation:

$$\vec{E}^{(1)}\left(1-\frac{k^2c^2}{\omega^2}\right)+\frac{i\vec{J}^{(1)}}{\omega\epsilon_0}=0.\tag{36}$$

The current is contributed by each component:

$$\vec{J}=\sum_j e_j\vec{\Gamma}_j,$$

where $\vec{\Gamma}_j$ is the flow density:

$$\vec{\Gamma}_j=\int d^3v\vec{v}f_j.$$

Ohm's Law:

$$e_j\vec{\Gamma}_j=-i\epsilon_0\frac{k^2c^2}{\omega}\boldsymbol{S_j}\vec{E}$$

Substituting in the 1st order distribution function yields:

$$\begin{aligned}e_j\vec{\Gamma}_j^{(1)}&=e_j\int d^3v\vec{v}f_j^{(1)}\\&=\frac{ie_j^2}{m_j\omega}\int d^3v\vec{v}\left[\omega f_{j,\perp}^{(0)}+k_z\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)\right]\sum_{\pm}v_\pm E_\mp\frac{1}{k_zv_z-\omega\mp\Omega_j}\end{aligned}$$

The integration in the z direction results in 0 (since it is an odd function of v_x,v_y).

Consider the perpendicular direction,

$$\begin{aligned}e_j\vec{\Gamma}_{j,\perp}^{(1)}&=e_j\int d^3v\vec{v}_\perp f_j^{(1)}\\&=\sum_{\pm}E_\mp\frac{ie_j^2}{m_j\omega}\int d^3v\vec{v}_\perp\left[\omega f_{j,\perp}^{(0)}+k_z\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)\right]\frac{v_\pm}{k_zv_z-\omega\mp\Omega_j}\\&=\sum_{\pm}E_\mp\frac{ie_j^2}{m_j\omega}\int d^3v v_\pm\left(v_+\hat{e}_++v_-\hat{e}_-\right)\left[\omega f_{j,\perp}^{(0)}+k_z\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)\right]\frac{1}{k_zv_z-\omega\mp\Omega_j}.\end{aligned}$$

note that

$$
\begin{aligned}
v_+ v_+ &= (v_x - iv_y)^2/2 = (v_x^2 - v_y^2 - 2iv_x v_y)/2 \\
v_- v_- &= (v_x + iv_y)^2/2 = (v_x^2 - v_y^2 + 2iv_x v_y)/2 \\
v_+ v_- &= (v_x + iv_y)(v_x - iv_y)/2 = (v_x^2 + v_y^2)/2 = \frac{v_\perp^2}{2}.
\end{aligned}
$$

Since the 0th order distribution function is axisymmetric, we have

$$
\int d^3v (v_x^2 - v_y^2) \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \frac{1}{k_z v_z - \omega \mp \Omega_j} = 0
$$

The integral function following is an odd function of v_x and v_y, so the integral is 0:

$$
\int d^3v v_x v_y \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \frac{1}{k_z v_z - \omega \mp \Omega_j} = 0
$$

thus,

$$
\begin{aligned}
e_j \vec{\Gamma}_{j,\perp}^{(1)} &= \sum_{\pm} E_{\mp} \frac{ie_j^2}{m_j \omega} \int d^3v v_\pm v_\mp \hat{e}_\mp \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \frac{1}{k_z v_z - \omega \mp \Omega_j} \\
&= \sum_{\pm} E_{\mp} \frac{ie_j^2}{m_j \omega} \int d^3v \frac{v_\perp^2}{2} \hat{e}_\mp \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \frac{1}{k_z v_z - \omega \mp \Omega_j} \\
&= -i\epsilon_0 \frac{k^2 c^2}{\omega} \sum_{\pm} E_{\mp} \frac{-e_j^2}{m_j \epsilon_0 k^2 c^2} \int d^3v \frac{v_\perp^2}{2} \hat{e}_\mp \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \frac{1}{k_z v_z - \omega \mp \Omega_j} \\
&= -i\epsilon_0 \frac{k^2 c^2}{\omega} \sum_{\pm} E_{\mp} \frac{-\omega_j^2}{2k^2 c^2 n_j} \int d^3v v_\perp^2 \hat{e}_\mp \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \frac{1}{k_z v_z - \omega \mp \Omega_j} \\
&= \sum_{\pm} e_j \vec{\Gamma}_{j,\pm}^{(1)} \hat{e}_\pm,
\end{aligned}
$$

the plasma frequency $\omega_j = \sqrt{\frac{n_j e_j^2}{\epsilon_0 m_j}}$.

Then,

$$
e_j \Gamma_{j,\pm}^{(1)} = -i\epsilon_0 \frac{k^2 c^2}{\omega} E_\pm \frac{-\omega_j^2}{2k^2 c^2 n_j} \int d^3v v_\perp^2 \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \frac{1}{k_z v_z - \omega \pm \Omega_j}
$$

It can be written as

$$
e_j \Gamma_{j,\pm}^{(1)} = -i\epsilon_0 \frac{k^2 c^2}{\omega} S_{j,\pm} E_\pm,
$$

Where conductivity

$$
S_{j,\pm}(\vec{k}, \omega) = \frac{-\omega_j^2}{2k^2 c^2 n_j} \int d^3v v_\perp^2 \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \frac{1}{k_z v_z - \omega \pm \Omega_j}. \tag{37}
$$

The dispersion relation can be expressed in terms of conductivity:

$$
\omega^2 - k^2 c^2 + \sum_j k^2 c^2 S_{j,\pm}(\vec{k}, \omega) = 0. \tag{38}
$$

For Bi-Maxwellian distribution,

$$f_j^{(0)}(v_z, v_\perp^2) = \frac{n_j T_{\parallel j}}{\left(2\pi v_j^2\right)^{3/2} T_{\perp j}} \cdot exp\left[-\frac{(v_z - v_{0j})^2}{2v_j^2} - \frac{v_\perp^2}{2v_j^2}\frac{T_{\parallel j}}{T_{\perp j}}\right]$$

$$dv^3 = 2\pi v_\perp dv_z dv_\perp$$
$$f_{j,\perp}^{(0)} = -\frac{1}{v_j^2}\frac{T_{\parallel j}}{T_{\perp j}} f_j^{(0)}$$
$$f_{j,z}^{(0)} = -\frac{v_z - v_{0j}}{v_j^2} f_j^{(0)}$$

then

$$\left[\omega f_{j,\perp}^{(0)} + k_z\left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] = \left[(k_z v_z - \omega)\frac{1}{v_j^2}\frac{T_{\parallel j}}{T_{\perp j}} - k_z \frac{v_z - v_{0j}}{v_j^2}\right] f_j^{(0)}$$

Conductivity can be written as:

$$S_{j,\pm}(\vec{k},\omega) = \frac{-\pi\omega_j^2}{k^2c^2 n_j}\int_{-\infty}^{\infty}\frac{dv_z}{k_z v_z - \omega \pm \Omega_j}\left[(k_z v_z - \omega)\frac{1}{v_j^2}\frac{T_{\parallel j}}{T_{\perp j}} - k_z\frac{v_z - v_{0j}}{v_j^2}\right]\int_0^\infty dv_\perp v_\perp^3 f_j^{(0)} \tag{39}$$

Integrate the perpendicular velocity first:

$$\int dv_\perp v_\perp^3 f_j^{(0)} = \frac{n_j T_{\parallel j}}{\left(2\pi v_j^2\right)^{3/2} T_{\perp j}} exp\left(-\frac{(v_z - v_{0j})^2}{2v_j^2}\right)\int_0^\infty dv_\perp v_\perp^3 exp\left[-\frac{v_\perp^2}{2v_j^2}\frac{T_{\parallel j}}{T_{\perp j}}\right]$$

where

$$\int_0^\infty dv_\perp v_\perp^3 exp\left[-\frac{v_\perp^2}{2v_j^2}\frac{T_{\parallel j}}{T_{\perp j}}\right] = \frac{1}{2\left(\frac{1}{2v_j^2}\frac{T_{\parallel j}}{T_{\perp j}}\right)^2} = \frac{2v_j^4 T_\perp^2}{T_\parallel^2}$$

Thus

$$S_{j,\pm}(\vec{k},\omega) = \frac{-\omega_j^2}{k^2c^2}\frac{v_j T_{\perp j}}{(2\pi)^{1/2} T_{\parallel j}}\int_{-\infty}^{\infty}\frac{dv_z}{k_z v_z - \omega \pm \Omega_j}\left[(k_z v_z - \omega)\frac{1}{v_j^2}\frac{T_{\parallel j}}{T_{\perp j}} - k_z\frac{v_z - v_{0j}}{v_j^2}\right] exp\left(-\frac{(v_z - v_{0j})^2}{2v_j^2}\right)$$

We define $\xi = \frac{v_z - v_{0j}}{\sqrt{2}v_j}$ then

$$S_{j,\pm}(\vec{k},\omega) = \frac{\omega_j^2}{k^2c^2}\pi^{-1/2}\int_{-\infty}^{\infty}\frac{d\xi}{\xi - \frac{\omega - k_z v_{0j} \mp \Omega_j}{\sqrt{2}k_z v_j}}\left[-\xi\left(1 - \frac{T_{\perp j}}{T_{\parallel j}}\right) + \frac{\omega - k_z v_{0j}}{\sqrt{2}k_z v_j}\right] exp\left(-\xi^2\right)$$

For $Im\zeta > 0$ plasma dispersion function

$$Z(\zeta) = \pi^{-1/2}\int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t - \zeta} \tag{40}$$

Its derivative is (which can be obtained after integration by parts)

$$Z'(\zeta) = -\pi^{-1/2}\int_{-\infty}^{\infty} dt \frac{2t}{t-\zeta} e^{-t^2} \tag{41}$$

To write into this form, we set

$$\zeta_j = sign(k_z\gamma)\frac{\omega - k_z v_{0j}}{\sqrt{2}k_z v_j}$$
$$\zeta_{j,\pm} = sign(k_z\gamma)\frac{\omega - k_z v_{0j} \mp \Omega_j}{\sqrt{2}k_z v_j}$$
$$x = sign(k_z\gamma)\xi$$

thus, conductivity can be written as

$$\begin{aligned}
S_{j,\pm}(\vec{k},\omega) &= \frac{\omega_j^2}{k^2c^2}\pi^{-1/2} sign(k_z\gamma)\int_{\xi=-\infty}^{\xi=\infty} \frac{dx}{x-\zeta_{j,\pm}}\left[-x\left(1-\frac{T_{\perp j}}{T_{\parallel j}}\right)+\zeta_j\right] exp\left(-x^2\right) \\
&= \frac{\omega_j^2}{k^2c^2}\pi^{-1/2}\int_{-\infty}^{\infty} \frac{dx}{x-\zeta_{j,\pm}}\left[-x\left(1-\frac{T_{\perp j}}{T_{\parallel j}}\right)+\zeta_j\right] exp\left(-x^2\right) \\
&= \frac{\omega_j^2}{k^2c^2}\left[\zeta_j Z(\zeta_{j,\pm}) + \left(1-\frac{T_{\perp j}}{T_{\parallel j}}\right)\frac{Z'(\zeta_{j,\pm})}{2}\right]
\end{aligned}$$

4 2nd order

Expand the Vlasov equation to second order:

$$\begin{aligned}
&\frac{\partial\left(f_j^{(0)}+f_j^{(1)}+f_j^{(2)}\right)}{\partial t} + \vec{v}\cdot\nabla\left(f_j^{(0)}+f_j^{(1)}+f_j^{(2)}\right) + \\
&\frac{e_j}{m_j}\left[\vec{E}^{(1)}+\vec{E}^{(2)}+\vec{v}\times\vec{B}_0+\vec{v}\times\vec{B}^{(1)}+\vec{v}\times\vec{B}^{(2)}\right]\cdot\nabla_v\left(f_j^{(0)}+f_j^{(1)}+f_j^{(2)}\right) = 0,
\end{aligned}$$

Minus the 0th order Vlasov equation Eq.10 and the 1st order Vlasov equation Eq.15 to get:

$$\begin{aligned}
&\frac{\partial f_j^{(2)}}{\partial t} + \vec{v}\cdot\nabla f_j^{(2)} + \frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v f_j^{(2)} = \\
&-\frac{e_j}{m_j}\left[\vec{E}^{(1)}+\vec{E}^{(2)}+\vec{v}\times\vec{B}^{(1)}+\vec{v}\times\vec{B}^{(2)}\right]\cdot\nabla_v\left(f_j^{(1)}+f_j^{(2)}\right) - \frac{e_j}{m_j}\left[\vec{E}^{(2)}+\vec{v}\times\vec{B}^{(2)}\right]\cdot\nabla_v f_j^{(0)},
\end{aligned}$$

Preserve second-order minima:

$$\frac{\partial f_j^{(2)}}{\partial t} + \vec{v}\cdot\nabla f_j^{(2)} + \frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v f_j^{(2)} = -\frac{e_j}{m_j}\left[\vec{E}^{(1)}+\vec{v}\times\vec{B}^{(1)}\right]\cdot\nabla_v f_j^{(1)} - \frac{e_j}{m_j}\left[\vec{E}^{(2)}+\vec{v}\times\vec{B}^{(2)}\right]\cdot\nabla_v f_j^{(0)}, \tag{42}$$

We are concerned with the evolution of the mean distribution as a whole, not local perturbations, so take spatial averaging:

$$\frac{\partial\left\langle f_j^{(2)}\right\rangle}{\partial t} + \frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v\left\langle f_j^{(2)}\right\rangle = -\frac{e_j}{m_j}\left\langle\left[\vec{E}^{(1)}+\vec{v}\times\vec{B}^{(1)}\right]\cdot\nabla_v f_j^{(1)}\right\rangle - \frac{e_j}{m_j}\left[\left\langle\vec{E}^{(2)}\right\rangle+\vec{v}\times\left\langle\vec{B}^{(2)}\right\rangle\right]\cdot\nabla_v f_j^{(0)},$$

Faraday's law.

$$\left\langle\nabla\times\vec{E}^{(2)}\right\rangle = -\frac{\partial\left\langle\vec{B}^{(2)}\right\rangle}{\partial t} \Longrightarrow \frac{\partial\left\langle\vec{B}^{(2)}\right\rangle}{\partial t} = 0$$

This means that $\left\langle \vec{B}^{(2)} \right\rangle$ does not vary with time and can be set to 0 (by changing $\vec{B}_0$). Thus, the 2nd order Vlasov equation after spatial averaging is:

$$\frac{\partial \left\langle f_j^{(2)} \right\rangle}{\partial t} + \frac{e_j}{m_j} \left(\vec{v} \times \vec{B}_0 \right) \cdot \nabla_v \left\langle f_j^{(2)} \right\rangle = -\frac{e_j}{m_j} \left\langle \left[\vec{E}^{(1)} + \vec{v} \times \vec{B}^{(1)} \right] \cdot \nabla_v f_j^{(1)} \right\rangle - \frac{e_j}{m_j} \left\langle \vec{E}^{(2)} \right\rangle \cdot \nabla_v f_j^{(0)}, \quad (43)$$

We then integrate to obtain the governing equations for the velocity moments.

4.1 First-order moment

$$\begin{aligned} m_j \int d^3 v \vec{v} \frac{\partial \left\langle f_j^{(2)} \right\rangle}{\partial t} &= m_j \frac{\partial}{\partial t} \left\langle \int d^3 v \vec{v} f_j^{(2)} \right\rangle \\ &= \frac{\partial \left\langle \vec{P}_j^{(2)} \right\rangle}{\partial t} \end{aligned}$$

$$\begin{aligned} e_j \int dv^3 \vec{v} \left(\vec{v} \times \vec{B}_0 \right) \cdot \nabla_v \left\langle f_j^{(2)} \right\rangle &= e_j \frac{\left\langle \vec{P}_j^{(2)} \right\rangle}{m_j} \times \vec{B}_0 \\ &= \vec{\Omega}_j \times \left\langle \vec{P}_j^{(2)} \right\rangle \end{aligned}$$

$$-e_j \int dv^3 \vec{v} \left\langle \left[\vec{v} \times \vec{B}^{(1)} \right] \cdot \nabla_v f_j^{(1)} \right\rangle = e_j \left\langle \vec{\Gamma}_j \times \vec{B}^{(1)} \right\rangle$$

$$\begin{aligned} -e_j \int dv^3 \vec{v} \vec{E}^{(1)} \cdot \nabla_v f_j^{(1)} &= -e_j \int dv^3 v_k E_i^{(1)} \frac{\partial f_j^{(1)}}{\partial v_i} \\ &= -e_j \int dv^3 v_k \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} \\ &= e_j \int dv^3 f_j^{(1)} E_k^{(1)} \\ &= e_j \vec{E}^{(1)} \int dv^3 f_j^{(1)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} -e_j \int dv^3 \vec{v} \vec{E}^{(2)} \cdot \nabla_v f_j^{(0)} &= -e_j \int dv^3 v_k E_i^{(2)} \frac{\partial f_j^{(0)}}{\partial v_i} \\ &= -e_j \int dv^3 v_k \frac{\partial f_j^{(0)} E_i^{(2)}}{\partial v_i} \\ &= e_j \int dv^3 f_j^{(0)} E_k^{(2)} \\ &= e_j \vec{E}^{(2)} \int dv^3 f_j^{(0)} \\ &= e_j n_j \vec{E}^{(2)} \end{aligned}$$

Thus,

$$\frac{\partial \left\langle \vec{P}_j^{(2)} \right\rangle}{\partial t} + \vec{\Omega}_j \times \left\langle \vec{P}_j^{(2)} \right\rangle - e_j n_j \left\langle \vec{E}^{(2)} \right\rangle = e_j \left\langle \vec{\Gamma}_j \times \vec{B}^{(1)} \right\rangle \quad (44)$$

4.2 Sencond-order moment

$$m_j \int d^3v \left(\vec{v}-\vec{v}_{0j}\right)^2 \frac{\partial \left\langle f_j^{(2)} \right\rangle}{\partial t} = m_j \frac{\partial}{\partial t} \left\langle \int d^3v \left(\vec{v}-\vec{v}_{0j}\right)^2 f_j^{(2)} \right\rangle$$
$$= 3n_j \frac{\partial \left\langle T_j^{(2)} \right\rangle}{\partial t}$$

$$\begin{aligned}
e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v f_j^{(2)} &= e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \epsilon_{lmn} v_l B_m^{(0)} \frac{\partial f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 \left(v^2 - 2\vec{v}\cdot\vec{v}_{0j}\right) \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 v^2 \frac{\partial v_l f_j^{(2)}}{\partial v_n} - 2e_j \epsilon_{lmn} B_m^{(0)} v_{0j,k} \int dv^3 v_k \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= -2e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 v_l v_n f_j^{(2)} + 2e_j \epsilon_{lmn} B_m^{(0)} v_{0j,n} \int dv^3 v_l f_j^{(2)} \\
&= 0 + 2\frac{e_j}{m_j} \vec{v}_{0j} \cdot \left(\vec{P}^{(2)} \times \vec{B}^{(0)}\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \left(\vec{v}\times\vec{B}^{(1)}\right)\cdot\nabla_v f_j^{(1)} &= e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \epsilon_{lmn} v_l B_m^{(1)} \frac{\partial f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 \left(v^2 - 2\vec{v}\cdot\vec{v}_{0j}\right) \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 v^2 \frac{\partial v_l f_j^{(1)}}{\partial v_n} - 2e_j \epsilon_{lmn} B_m^{(1)} v_{0j,k} \int dv^3 v_k \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= -2e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 v_l v_n f_j^{(1)} + 2e_j \epsilon_{lmn} B_m^{(1)} v_{0j,n} \int dv^3 v_l f_j^{(1)} \\
&= 2\vec{v}_{0j} \cdot \left(e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)}\right)
\end{aligned}$$

$$
\begin{aligned}
e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \vec{E}^{(1)} \cdot \nabla_v f_j^{(1)} &= e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 E_i^{(1)} \frac{\partial f_j^{(1)}}{\partial v_i} \\
&= e_j \int dv^3 \left(v^2 - 2\vec{v}\cdot\vec{v}_{0j}\right) \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} \\
&= e_j \int dv^3 v^2 \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} - 2e_j \int dv^3 \vec{v}\cdot\vec{v}_{0j} \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} \\
&= -2e_j \int dv^3 f_j^{(1)} E_i^{(1)} v_i + 2e_j v_{0j,i} \int dv^3 f_j^{(1)} E_i^{(1)} \\
&= -2e_j \vec{E}^{(1)} \cdot \vec{\Gamma}_j^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \vec{E}^{(2)} \cdot \nabla_v f_j^{(0)} &= e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 E_i^{(2)} \frac{\partial f_j^{(0)}}{\partial v_i} \\
&= e_j \int dv^3 \left(v^2 - 2\vec{v}\cdot\vec{v}_{0j}\right) \frac{\partial f_j^{(0)} E_i^{(2)}}{\partial v_i} \\
&= e_j \int dv^3 v^2 \frac{\partial f_j^{(0)} E_i^{(2)}}{\partial v_i} - 2e_j \int dv^3 \vec{v}\cdot\vec{v}_{0j} \frac{\partial f_j^{(0)} E_i^{(2)}}{\partial v_i} \\
&= -2e_j \int dv^3 f_j^{(0)} E_i^{(2)} v_i + 2e_j v_{0j,i} \int dv^3 f_j^{(0)} E_i^{(2)} \\
&= -2e_j n_j \vec{E}^{(2)} \cdot \vec{v}_{0j} + 2e_j n_j \vec{v}_{0j} \cdot \vec{E}^{(2)} \\
&= 0
\end{aligned}
$$

The governing equation for the second order temperature is obtained:

$$
n_j \frac{\partial \left\langle T_j^{(2)} \right\rangle}{\partial t} = \frac{2}{3} e_j \left\langle \vec{\Gamma}_j^{(1)} \cdot \vec{E}^{(1)} \right\rangle - \frac{2}{3} \vec{v}_{0j} \cdot \left\langle e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)} \right\rangle \tag{45}
$$

$$
\begin{aligned}
m_j \int d^3v \left(v_\parallel - v_{0\parallel j}\right)^2 \frac{\partial \left\langle f_j^{(2)} \right\rangle}{\partial t} &= m_j \frac{\partial}{\partial t} \left\langle \int d^3v \left(v_\parallel - v_{0\parallel j}\right)^2 f_j^{(2)} \right\rangle \\
&= n_j \frac{\partial \left\langle T_{\parallel j}^{(2)} \right\rangle}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \left(\vec{v} \times \vec{B}_0\right) \cdot \nabla_v f_j^{(2)} &= e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \epsilon_{lmn} v_l B_m^{(0)} \frac{\partial f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 \left(v_\parallel^2 - 2 v_\parallel v_{0\parallel j}\right) \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 v_\parallel^2 \frac{\partial v_l f_j^{(2)}}{\partial v_n} - 2 e_j \epsilon_{lmn} B_m^{(0)} v_{0j,\parallel} \int dv^3 v_\parallel \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= -2 e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 v_l v_\parallel f_j^{(2)} \delta_{n,\parallel} + 2 e_j \epsilon_{lmn} B_m^{(0)} v_{0\parallel j} \int dv^3 v_l f_j^{(2)} \delta_{n,\parallel} \\
&= -2 e_j \int dv^3 \left(\vec{v} \times \vec{B}_0\right) \cdot \vec{v}_\parallel f_j^{(2)} + 2 \frac{e_j}{m_j} v_{0\parallel j} \cdot \left(\vec{P}^{(2)} \times \vec{B}^{(0)}\right)_\parallel \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \left(\vec{v} \times \vec{B}^{(1)}\right) \cdot \nabla_v f_j^{(1)} &= e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \epsilon_{lmn} v_l B_m^{(1)} \frac{\partial f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 \left(v_\parallel^2 - 2 v_\parallel v_{0\parallel j}\right) \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 v_\parallel^2 \frac{\partial v_l f_j^{(1)}}{\partial v_n} - 2 e_j \epsilon_{lmn} B_m^{(1)} v_{0\parallel j} \int dv^3 v_\parallel \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= -2 e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 v_l v_\parallel f_j^{(1)} \delta_{n,\parallel} + 2 e_j \epsilon_{lmn} B_m^{(1)} v_{0j,\parallel} \int dv^3 v_l f_j^{(1)} \delta_{n,\parallel} \\
&= -2 e_j \int dv^3 \left(\vec{v} \times \vec{B}^{(1)}\right)_\parallel v_\parallel f_j^{(1)} + 2 v_{0\parallel j} \cdot \left(e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)}\right)_\parallel \\
&= -2 e_j \int dv^3 \left(\vec{v} \times \vec{B}^{(1)}\right) \cdot \vec{v}_\parallel f_j^{(1)} + 2 \vec{v}_{0j} \cdot \left(e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)}\right) \\
&= -2 e_j \vec{B}^{(1)} \cdot \int dv^3 \left(\vec{v}_\parallel \times \vec{v}\right) f_j^{(1)} + 2 \vec{v}_{0j} \cdot \left(e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)}\right) \\
&= 2 e_j \vec{B}^{(1)} \cdot \int dv^3 \left(\vec{v}_\perp \times \vec{v}\right) f_j^{(1)} + 2 \vec{v}_{0j} \cdot \left(e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \vec{E}^{(1)} \cdot \nabla_v f_j^{(1)} &= e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 E_i^{(1)} \frac{\partial f_j^{(1)}}{\partial v_i} \\
&= e_j \int dv^3 \left(v_\parallel^2 - 2 v_\parallel v_{0\parallel j}\right) \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} \\
&= e_j \int dv^3 v_\parallel^2 \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} - 2 e_j \int dv^3 v_\parallel v_{0\parallel j} \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} \\
&= -2 e_j \int dv^3 f_j^{(1)} E_\parallel^{(1)} v_\parallel + 2 e_j v_{0\parallel j} \int dv^3 f_j^{(1)} E_\parallel^{(1)} \\
&= -2 \left\langle e_j \Gamma_\parallel^{(1)} E_\parallel^{(1)} \right\rangle \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \vec{E}^{(2)} \cdot \nabla_v f_j^{(0)} &= e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 E_i^{(2)} \frac{\partial f_j^{(0)}}{\partial v_i} \\
&= e_j \int dv^3 v_\parallel^2 \frac{\partial f_j^{(0)} E_i^{(2)}}{\partial v_i} - 2 e_j \int dv^3 v_\parallel v_{0\parallel j} \frac{\partial f_j^{(0)} E_i^{(2)}}{\partial v_i} \\
&= -2 e_j \int dv^3 f_j^{(0)} E_\parallel^{(2)} v_\parallel + 2 e_j v_{0\parallel j} \int dv^3 f_j^{(0)} E_\parallel^{(2)} \\
&= -2 e_j n_j \vec{E}^{(2)} \cdot \vec{v}_{0j} + 2 e_j n_j \vec{v}_{0j} \cdot \vec{E}^{(2)} \\
&= 0
\end{aligned}
$$

Thus, the governing equation for the second-order parallel temperature is:

$$
n_j \frac{\partial \left\langle T_{\parallel j}^{(2)} \right\rangle}{\partial t} = -2 \vec{v}_{0j} \cdot \left\langle e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)} \right\rangle - 2 e_j \left\langle \vec{B}^{(1)} \cdot \int dv^3 \left(\vec{v}_\perp \times \vec{v}\right) f_j^{(1)} \right\rangle \tag{46}
$$

Governing equation for the second-order perpendicular temperature is:

$$
\begin{aligned}
n_j \frac{\partial \left\langle T_{\perp j}^{(2)} \right\rangle}{\partial t} &= \frac{3}{2} n_j \frac{\partial \left\langle T_j^{(2)} \right\rangle}{\partial t} - \frac{1}{2} n_j \frac{\partial \left\langle T_{\parallel j}^{(2)} \right\rangle}{\partial t} \\
&= e_j \left\langle \vec{\Gamma}_j^{(1)} \cdot \vec{E}^{(1)} \right\rangle + e_j \left\langle \vec{B}^{(1)} \cdot \int dv^3 \left(\vec{v}_\perp \times \vec{v}\right) f_j^{(1)} \right\rangle
\end{aligned} \tag{47}
$$

4.3 Summarize the governing equations for second-order speed and temperature

The variables in the equations are in x, t space and not in the frequency domain.

$$
\frac{\partial \left\langle \vec{P}_j^{(2)} \right\rangle}{\partial t} + \vec{\Omega}_j \times \left\langle \vec{P}_j^{(2)} \right\rangle - e_j n_j \left\langle \vec{E}^{(2)} \right\rangle = e_j \left\langle \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)} \right\rangle
$$

$$
n_j \frac{\partial \left\langle T_j^{(2)} \right\rangle}{\partial t} = \frac{2}{3} e_j \left\langle \vec{\Gamma}_j^{(1)} \cdot \vec{E}^{(1)} \right\rangle - \frac{2}{3} \vec{v}_{0j} \cdot \left\langle e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)} \right\rangle
$$

$$
n_j \frac{\partial \left\langle T_{\parallel j}^{(2)} \right\rangle}{\partial t} = -2 \vec{v}_{0j} \cdot \left\langle e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)} \right\rangle - 2 e_j \left\langle \vec{B}^{(1)} \cdot \int dv^3 \left(\vec{v}_\perp \times \vec{v}\right) f_j^{(1)} \right\rangle
$$

$$n_j \frac{\partial \left\langle T_{\perp j}^{(2)} \right\rangle}{\partial t} = e_j \left\langle \vec{\Gamma}_j^{(1)} \cdot \vec{E}^{(1)} \right\rangle + e_j \left\langle \vec{B}^{(1)} \cdot \int dv^3 \left(\vec{v}_\perp \times \vec{v}\right) f_j^{(1)} \right\rangle$$

Now let's consider 1st order fluctuating variables are all in the frequency domain (after wavelet transform or Fourier transform).

One can use linear velocity fluctuations $\delta\tilde{\vec{v}}_j^{(1)}(k)$, linear magnetic field fluctuations $\delta\tilde{\vec{b}}^{(1)}(k)$, linear electric field fluctuations $\delta\tilde{\vec{e}}^{(1)}(k)$ to calculate the change rate of 2nd order temperature and velocity. Neglecting 2nd order electric field and the perpendicular 2nd order velocity,

$$m_j \frac{\partial \vec{v}_j^{(2)}}{\partial t} = e_j \int dk \frac{1}{2} Re\{\delta\tilde{\vec{v}}_j^{(1)} \times \delta\tilde{\vec{b}}^{(1)*}\} \tag{48}$$

$$\frac{\partial T_j^{(2)}}{\partial t} = \frac{2}{3} e_j \int dk \frac{1}{2} Re\{\delta\tilde{\vec{v}}_j^{(1)} \cdot \delta\tilde{\vec{e}}^{(1)}\} - \frac{2}{3} e_j \vec{v}_{0j} \cdot \int dk \frac{1}{2} Re\{\delta\tilde{\vec{v}}_j^{(1)} \times \delta\tilde{\vec{b}}^{(1)}\} \tag{49}$$

$$\begin{aligned}
\delta\vec{v}_j^{(1)}(k) &: linear\, fluctuating\, velocity\, of\, species\, j \\
\delta\vec{b}^{(1)}(k) &: linear\, fluctuating\, magnetic\, field \\
\delta\vec{e}^{(1)}(k) &: linear\, fluctuating\, electric\, field \\
\vec{v}_{0j} &: background\, velocity\, of\, specieds\, j \\
\frac{\partial \vec{v}_j^{(2)}}{\partial t} &: 2nd\, order\, acceleration\, of\, specieds\, j \\
\frac{\partial T_j^{(2)}}{\partial t} &: 2nd\, order\, temperature\, change\, rate\, of\, specieds\, j
\end{aligned}$$

For $\vec{k} \parallel \vec{B}_0$. We can express the terms in the control equation in terms of conductivity and power spectral density[2, 3]. Define,

$$\epsilon_\pm = \frac{1}{2} \epsilon_0 E_\pm^* E_\pm \tag{50}$$

$$\begin{aligned}
e_j \vec{\Gamma}_j^{(1)} \cdot \vec{E}^{(1)*} &= -i\epsilon_0 \frac{k^2 c^2}{|\omega|^2} \omega^* \sum_\pm S_{j,\pm} E_\pm \hat{e}_\pm \cdot \sum_\pm E_\pm^* \hat{e}_\pm^* \\
&= -i\epsilon_0 \frac{k^2 c^2}{|\omega|^2} \omega^* \begin{bmatrix} S_{j,+} E_+ & S_{j,-} E_- \end{bmatrix} \begin{bmatrix} \hat{e}_+ & \hat{e}_- \end{bmatrix}^T \begin{bmatrix} \hat{e}_- & \hat{e}_+ \end{bmatrix} \begin{bmatrix} E_+^* \\ E_-^* \end{bmatrix} \\
&= -i\epsilon_0 \frac{k^2 c^2}{|\omega|^2} \omega^* \begin{bmatrix} S_{j,+} E_+ & S_{j,-} E_- \end{bmatrix} \begin{bmatrix} E_+^* \\ E_-^* \end{bmatrix} \\
&= -i\epsilon_0 \frac{k^2 c^2}{|\omega|^2} \omega^* \left(S_{j,+} E_+ E_+^* + S_{j,-} E_- E_-^* \right) \\
&= -2i \frac{k^2 c^2}{|\omega|^2} \omega^* \left(S_{j,+} \epsilon_+ + S_{j,-} \epsilon_- \right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{2} Re\left(e_j \vec{\Gamma}_j^{(1)} \cdot \vec{E}^{(1)*} \right) &= Re\left(-i \frac{k^2 c^2}{|\omega|^2} \omega^* \left(S_{j,+} \epsilon_+ + S_{j,-} \epsilon_- \right) \right) \\
&= \frac{k^2 c^2}{|\omega|^2} Im\left[\omega^* \left(S_{j,+} \epsilon_+ + S_{j,-} \epsilon_- \right) \right]
\end{aligned}$$

$$
\begin{aligned}
e_j\vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)*} &= e_j\vec{\Gamma}_j^{(1)} \times \left(\frac{\vec{k}}{\omega^*} \times \vec{E}^{(1)*}\right) \\
&= \frac{\vec{k}}{\omega^*}\left(e_j\vec{\Gamma}_j^{(1)} \cdot \vec{E}^{(1)*}\right) - \vec{E}^{(1)*}\left(e_j\vec{\Gamma}_j^{(1)} \cdot \frac{\vec{k}}{\omega^*}\right) \\
&= -2i\frac{k^2c^2}{|\omega|^2}\vec{k}\left(S_{j,+}\epsilon_+ + S_{j,-}\epsilon_-\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{1}{2}Re\left(e_j\vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)*}\right) &= Re\left(-i\frac{k^2c^2}{|\omega|^2}\vec{k}\left(S_{j,+}\epsilon_+ + S_{j,-}\epsilon_-\right)\right) \\
&= \vec{k}\frac{k^2c^2}{|\omega|^2}Im\left(S_{j,+}\epsilon_+ + S_{j,-}\epsilon_-\right)
\end{aligned}
$$

$$
\begin{aligned}
e_j\vec{B}^{(1)*} \cdot \int dv^3\left(\vec{v}_\perp \times \vec{v}\right)f_j^{(1)} &= e_j\left(\frac{\vec{k}}{\omega^*} \times \vec{E}^{(1)*}\right) \cdot \int dv^3\left(\vec{v}_\perp \times \vec{v}\right)f_j^{(1)} \\
&= e_j\int dv^3\left(\frac{\vec{k}}{\omega^*} \times \vec{E}^{(1)*}\right) \cdot \left(\vec{v}_\perp \times \vec{v}\right)f_j^{(1)} \\
&= \frac{e_j}{\omega^*}\int dv^3\vec{k} \cdot \left(\vec{E}^{(1)*} \times \left(\vec{v}_\perp \times \vec{v}\right)\right)f_j^{(1)} \\
&= -\frac{e_j}{\omega^*}\vec{E}^{(1)*} \cdot \int dv^3\vec{v}_\perp k_z v_z f_j^{(1)} \\
&= -\frac{e_j}{\omega^*}\vec{E}^{(1)*} \cdot \int_{-\infty}^{\infty} dv_z k_z v_z \int_0^{\infty} dv_\perp 2\pi v_\perp \vec{v}_\perp f_j^{(1)} \\
&= -\frac{\vec{E}^{(1)*}}{\omega^*} \cdot \sum_\pm\left(\omega \mp \Omega_j\right)e_j\Gamma_{j,\pm}\hat{e}_\pm - \frac{\vec{E}^{(1)*}}{\omega^*} \cdot \left[-i\epsilon_0\frac{\omega_j^2}{|\omega|^2}\omega^*\left(\omega - k_z v_{0j}\right)\sum_\pm E_\pm\hat{e}_\pm\right] \\
&= 2i\frac{k^2c^2}{|\omega|^2}\left[\left(\omega - \Omega_j\right)S_{j,+}\epsilon_+ + \left(\omega + \Omega_j\right)S_{j,-}\epsilon_-\right] + 2i\frac{\omega_j^2}{|\omega|^2}\left(\omega - k_z v_{0j}\right)\left(\epsilon_+ + \epsilon_-\right)
\end{aligned}
$$

We have,

$$
\frac{1}{2}Re\left[e_j\vec{B}^{(1)*} \cdot \int dv^3\left(\vec{v}_\perp \times \vec{v}\right)f_j^{(1)}\right] = -\frac{k^2c^2}{|\omega|^2}Im\left[\left(\omega - \Omega_j\right)S_{j,+}\epsilon_+ + \left(\omega + \Omega_j\right)S_{j,-}\epsilon_-\right] - \frac{\gamma\omega_j^2}{|\omega|^2}\left(\epsilon_+ + \epsilon_-\right)
$$

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