波粒共振二阶加热加速理论

参考 Gary1978-2001 一系列文章。以及 Fundamentals of plasma physics, by J.A. Bittencourt, The theory of plasma waves, by McGray-Hill

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1 基本的公式

考虑背景磁场和背景速度沿着 z 方向。

弗拉索夫方程：

$$\frac{\partial f_j}{\partial t}+\vec{v}\cdot\nabla f_j+\frac{e_j}{m_j}\left[\vec{E}\left(\vec{x},t\right)+\vec{v}\times\vec{B}\left(\vec{x},t\right)\right]\cdot\nabla_v f_j=0. \tag{1}$$

麦克斯韦方程组：

$$\nabla\cdot\vec{E}=\frac{\rho}{\epsilon_0}, \tag{2}$$

$$\nabla\cdot\vec{B}=0, \tag{3}$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}, \tag{4}$$

$$\nabla\times\vec{B}=\mu_0\left(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right). \tag{5}$$

其中，电荷密度 ρ 和电流密度 $\vec{J}$ 可以用分布函数表示出来：

$$\rho\left(\vec{x},t\right)=\sum_j e_j\int d^3v f_j(\vec{x},\vec{v},t), \tag{6}$$

$$\vec{J}\left(\vec{x},t\right)=\sum_j e_j\int d^3v\vec{v}f_j(\vec{x},\vec{v},t). \tag{7}$$

展开：

$$\begin{aligned}
&f_j=f_j^{(0)}\left(\vec{v}\right)+f_j^{(1)}\left(\vec{x},\vec{v},t\right)+f_j^{(2)}\left(\vec{x},\vec{v},t\right)+...,\\
&\vec{E}\left(\vec{x},t\right)=\vec{E}^{(1)}\left(\vec{x},t\right)+\vec{E}^{(2)}\left(\vec{x},t\right)+...,\\
&\vec{B}\left(\vec{x},t\right)=\vec{B}_0+\vec{B}^{(1)}\left(\vec{x},t\right)+\vec{B}^{(2)}\left(\vec{x},t\right)+...,
\end{aligned} \tag{8}$$

0 阶量是给定的，时空无关的。0 阶运动就是沿着背景磁场方向的匀速直线运动，叠加上垂直背景磁场的回旋运动。背景磁场 $\vec{B}_0=\hat{z}B_0$。高阶量应该远小于低阶量：

$$\left|g^{(i+1)}\right|\ll\left|g^{(i)}\right|. \tag{9}$$

一阶量表示振幅不变的扰动（快变），二阶量包含一阶量对应的快变扰动部分，还包含振幅的缓变部分，如图 1：

2 0th order

假设 0 阶项应该满足以上的方程，那么考虑到 0 阶分布函数时空无关，可以写出 0 阶弗拉索夫方程：

$$\left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v f_j^{(0)}=0 \tag{10}$$

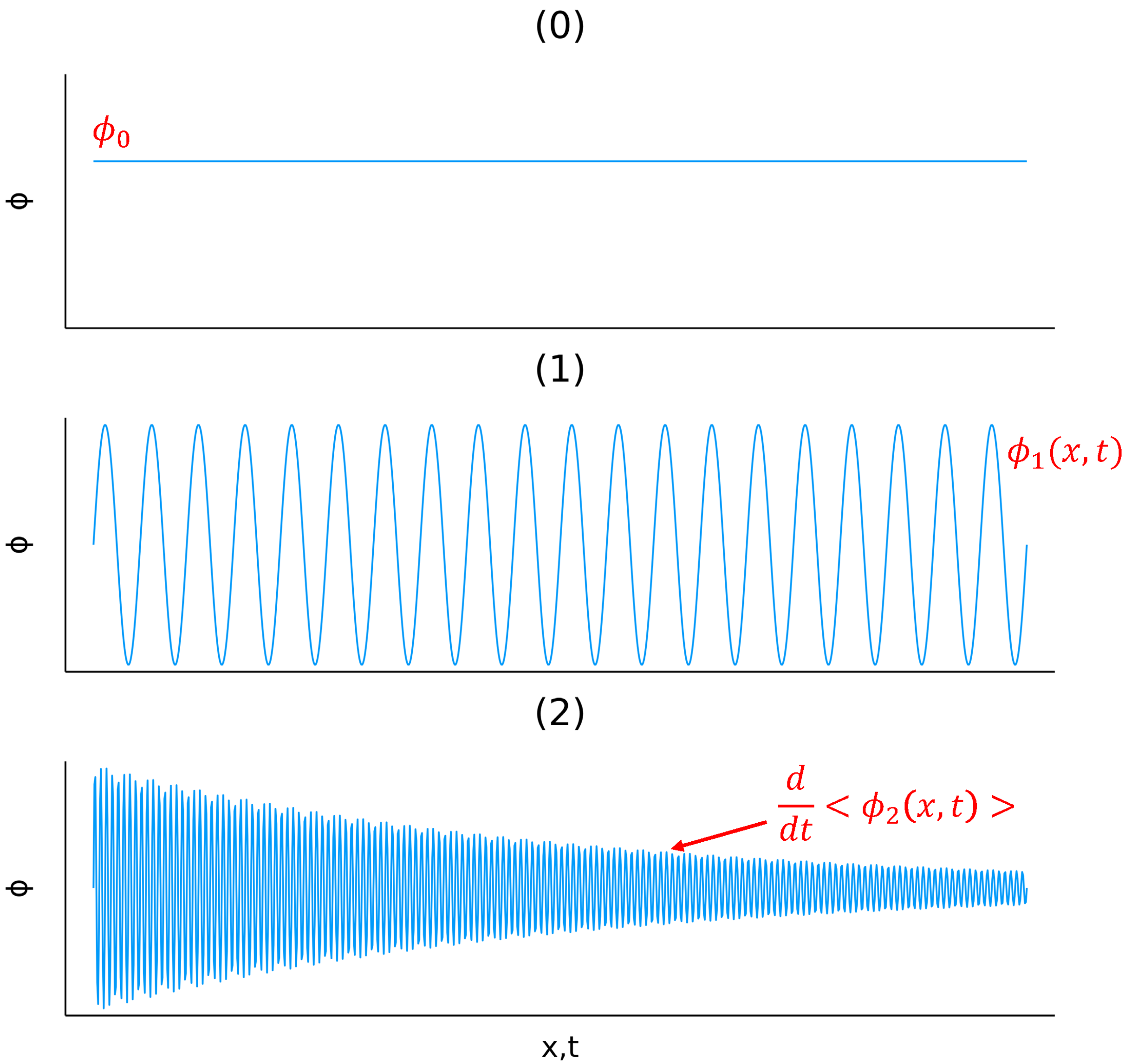

图 1: 各阶扰动的草图.

我们考虑一下 $\vec{v} \times \vec{B}_0$ 的方向：垂直于 $\vec{B}_0$ 既 z 方向，所以在 xy 平面内；垂直于 $\vec{v}$，所以应该是 xy 平面内的 $\hat{\theta}$ 方向，即与 $\vec{v}$ 垂直的角向，

$$\hat{\theta} \cdot \nabla_v f_j^{(0)} = 0$$

这代表分布函数的轴对称性，在 xy 平面内，0 阶分布函数只是垂直速度大小的函数，跟速度的方向无关，由此可以得到分布函数的形式：

$$f_j^{(0)} = f_j^{(0)}\left(v_z, v_\perp^2\right) \tag{11}$$

其中，

$$v_\perp^2 = v_x^2 + v_y^2$$

进而可以得到 0 阶分布函数的速度空间梯度满足:

$$
\begin{aligned}
\nabla_v \cdot f_j^{(0)} &= \begin{bmatrix} \frac{\partial}{\partial v_x} \\ \frac{\partial}{\partial v_y} \\ \frac{\partial}{\partial v_z} \end{bmatrix} f_j^{(0)} \\
&= \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} \frac{\partial f_j^{(0)}}{\partial (v_\perp^2/2)} + \frac{\partial f_j^{(0)}}{\partial v_z} \hat{z} \\
&= \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} f_{j,\perp}^{(0)} + f_{j,z}^{(0)} \hat{z}
\end{aligned} \tag{12}
$$

其中,

$$f_{j,z}^{(0)} = \frac{\partial f_j^{(0)}}{\partial v_z}$$

$$f_{j,\perp}^{(0)} = \frac{\partial f_j^{(0)}}{\partial (v_\perp^2/2)} = \frac{1}{v_\perp} \frac{\partial f_j^{(0)}}{\partial v_\perp}$$

双麦氏分布是一种符合这种形式的分布函数:

$$f_j^{(0)}(\vec{v}) = \frac{n_j T_{\parallel j}}{\left(2\pi v_j^2\right)^{3/2} T_{\perp j}} \cdot exp\left[-\frac{\left(v_z - v_{0j}\right)^2}{2v_j^2} - \frac{v_x^2 + v_y^2}{2v_j^2} \frac{T_{\parallel j}}{T_{\perp j}}\right] \tag{13}$$

其中 v_j 为平行方向热速度:

$$v_j^2 = \frac{k_B T_{\parallel j}}{m_j}$$

v_{0j} 表示 z 方向上 0 阶的漂移速度 $\vec{v}_{0j} = \hat{z} v_{0j}$。

3 1st order

假设所有一阶扰动量傅里叶分解后振幅不随时空变化，可以写成不同波数频率简谐波动的叠加:

$$g^{(1)}(\vec{x}, t) = g^{(1)}\left(\vec{k}, \omega\right) e^{i\left(\vec{k}\cdot\vec{x} - \omega t\right)} \tag{14}$$

弗拉索夫方程展开到一阶:

$$\frac{\partial\left(f_j^{(0)} + f_j^{(1)}\right)}{\partial t} + \vec{v} \cdot \nabla\left(f_j^{(0)} + f_j^{(1)}\right) + \frac{e_j}{m_j}\left[\vec{E}^{(1)}(\vec{x}, t) + \vec{v} \times \vec{B}_0 + \vec{v} \times \vec{B}^{(1)}(\vec{x}, t)\right] \cdot \nabla_v\left(f_j^{(0)} + f_j^{(1)}\right) = 0,$$

减去 0 阶弗拉索夫方程:

$$\frac{\partial f_j^{(1)}}{\partial t} + \vec{v} \cdot \nabla f_j^{(1)} + \frac{e_j}{m_j}\left[\vec{E}^{(1)}(\vec{x}, t) + \vec{v} \times \vec{B}^{(1)}(\vec{x}, t)\right] \cdot \nabla_v f_j^{(0)} + \frac{e_j}{m_j}\left[\vec{E}^{(1)}(\vec{x}, t) + \vec{v} \times \vec{B}_0 + \vec{v} \times \vec{B}^{(1)}(\vec{x}, t)\right] \cdot \nabla_v f_j^{(1)} = 0,$$

忽略二阶小量（这几项会放到之后二阶方程里面），一阶弗拉索夫方程（线性弗拉索夫方程）可以写为:

$$\frac{\partial f_j^{(1)}}{\partial t} + \vec{v} \cdot \nabla f_j^{(1)} + \frac{e_j}{m_j}\left(\vec{v} \times \vec{B}_0\right) \cdot \nabla_v f_j^{(1)} = -\frac{e_j}{m_j}\left[\vec{E}^{(1)}(\vec{x}, t) + \vec{v} \times \vec{B}^{(1)}(\vec{x}, t)\right] \cdot \nabla_v f_j^{(0)}. \tag{15}$$

0 阶轨道近似:

0 阶轨道的运动满足:

$$\frac{d\vec{v}}{dt} = \frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right) \tag{16}$$

$$\left(\frac{df_j}{dt}\right)_0 = \frac{\partial f_j}{\partial t} + \vec{v}\cdot\nabla f_j + \frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v f_j \tag{17}$$

这样，线性弗拉索夫方程可以写为:

$$\left(\frac{df_j^{(1)}}{dt}\right)_0 = -\frac{e_j}{m_j}\left[\vec{E}^{(1)}\left(\vec{x},t\right) + \vec{v}\times\vec{B}^{(1)}\left(\vec{x},t\right)\right]\cdot\nabla_v f_j^{(0)}, \tag{18}$$

其中 1 阶电场:

$$\vec{E}^{(1)}\left(\vec{x},t\right) = \vec{E}^{(1)}(\vec{k},\omega)e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)},$$

根据法拉第定律

$$\vec{B}^{(1)}(\vec{k},\omega) = \frac{\vec{k}}{\omega}\times\vec{E}^{(1)}(\vec{k},\omega).$$

线性弗拉索夫方程可以写为:

$$\begin{aligned}\left(\frac{df_j^{(1)}}{dt}\right)_0 &= -\frac{e_j}{m_j}e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\left[\vec{E}^{(1)} + \vec{v}\times\left(\frac{\vec{k}}{\omega}\times\vec{E}^{(1)}\right)\right]\cdot\nabla_v f_j^{(0)} \\ &= -\frac{e_j}{m_j}\vec{E}^{(1)}e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\left[1 + \frac{\cdot\vec{v}\vec{k}}{\omega} - \frac{\vec{v}\cdot\vec{k}}{\omega}\right]\cdot\nabla_v f_j^{(0)}\end{aligned} \tag{19}$$

沿着 0 阶轨道积分，可以得到 1 阶分布函数，但这里要注意积分范围的问题。如果波动是增长的，$t=-\infty$ 的时候，波动应该是 0，从而可以从 $t=-\infty$ 积分到 t；如果波动是衰减的，$t=\infty$ 的时候，波动应该是 0，应该从 t 积分到 $t=\infty$。

波动增长:

$$\begin{aligned}f_j^{(1)}\left(\vec{x},\vec{v},t\right) &= -\frac{e_j}{m_j}\int_{-\infty}^{t}\vec{E}^{(1)}e^{i\left(\vec{k}\cdot\vec{x}'-\omega t'\right)}\left[1 + \frac{\cdot\vec{v}'\vec{k}}{\omega} - \frac{\vec{v}'\cdot\vec{k}}{\omega}\right]\cdot\nabla_{v'} f_j^{(0)}\left(\vec{v}'\right)dt' \\ &= -\frac{e_j}{m_j\omega}\vec{E}^{(1)}e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\int_{-\infty}^{t}e^{i\left(\vec{k}\cdot(\vec{x}'-\vec{x})-\omega(t'-t)\right)}\left(\omega + \cdot\vec{v}'\vec{k} - \vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'} f_j^{(0)}\left(\vec{v}'\right)dt'\end{aligned} \tag{20}$$

波动衰减:

$$f_j^{(1)}\left(\vec{x},\vec{v},t\right) = \frac{e_j}{m_j\omega}\vec{E}^{(1)}e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\int_{t}^{\infty}e^{i\left(\vec{k}\cdot(\vec{x}'-\vec{x})-\omega(t'-t)\right)}\left(\omega + \cdot\vec{v}'\vec{k} - \vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'} f_j^{(0)}\left(\vec{v}'\right)dt' \tag{21}$$

对于 0 阶轨道近似，积分中的 x', v' 可以用回旋运动得到：首先，回旋运动的角速度是

$$\vec{\Omega}_j = -\frac{e_j\vec{B}_0}{m_j}$$

利用旋转矩阵，可以根据 t 时刻的垂直速度 $\vec{v}_\perp(t)$ 得到 t' 时刻的值

$$\vec{v}_\perp(t') = \begin{bmatrix} v_x(t') \\ v_y(t') \end{bmatrix} = \begin{bmatrix} cos\left[\Omega_j\left(t'-t\right)\right] & -sin\left[\Omega_j\left(t'-t\right)\right] \\ sin\left[\Omega_j\left(t'-t\right)\right] & cos\left[\Omega_j\left(t'-t\right)\right] \end{bmatrix}\vec{v}_\perp(t) \tag{22}$$

其中，

$$\Omega_j = -\frac{e_j B_0}{m_j}$$

注意，回旋角速度是带符号的，符号表示正负 z 方向。z 方向的速度：

$$v_z(t') = v_z. \tag{23}$$

回旋半径

$$\begin{aligned}\vec{r}(t') &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\vec{v}_\perp(t')}{\Omega_j} \\ &= \begin{bmatrix} sin\left[\Omega_j\left(t'-t\right)\right] & cos\left[\Omega_j\left(t'-t\right)\right] \\ -cos\left[\Omega_j\left(t'-t\right)\right] & sin\left[\Omega_j\left(t'-t\right)\right] \end{bmatrix} \frac{\vec{v}_\perp(t)}{\Omega_j}\end{aligned}$$

这样，垂直方向的粒子位置可以表示为：

$$\begin{aligned}\vec{x}_\perp(t') &= \begin{bmatrix} x(t') \\ y(t') \end{bmatrix} \\ &= \vec{x}_\perp(t) + \vec{r}(t') - \vec{r}(t) \\ &= \vec{x}_\perp(t) + \begin{bmatrix} sin\left[\Omega_j\left(t'-t\right)\right] & -\left(1-cos\left[\Omega_j\left(t'-t\right)\right]\right) \\ 1-cos\left[\Omega_j\left(t'-t\right)\right] & sin\left[\Omega_j\left(t'-t\right)\right] \end{bmatrix} \frac{\vec{v}_\perp(t)}{\Omega_j}\end{aligned} \tag{24}$$

z 方向的位置可以表示为：

$$z(t') = v_z(t'-t) + z(t) \tag{25}$$

接下来回到 1 阶分布函数

$$\begin{aligned}f_j^{(1)}\left(\vec{x},\vec{v},t\right) &= -\frac{e_j}{m_j}\int_{-\infty}^{t} \vec{E}^{(1)} e^{i\left(\vec{k}\cdot\vec{x}'-\omega t'\right)}\left[1+\frac{\cdot\vec{v}'\vec{k}}{\omega}-\frac{\vec{v}'\cdot\vec{k}}{\omega}\right]\cdot\nabla_{v'} f_j^{(0)}\left(\vec{v}'\right)dt' \\ &= -\frac{e_j}{m_j\omega} e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\int_{-\infty}^{t} e^{i\left(\vec{k}\cdot\left(\vec{x}'-\vec{x}\right)-\omega\left(t'-t\right)\right)}\vec{E}^{(1)}\left(\omega+\cdot\vec{v}'\vec{k}-\vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'} f_j^{(0)}\left(\vec{v}'\right)dt'\end{aligned}$$

其中的：

$$\begin{aligned}\vec{k}\cdot\left(\vec{x}'-\vec{x}\right)-\omega(t'-t) &= \vec{k}_\perp\cdot\left[\vec{x}_\perp(t')-\vec{x}_\perp(t)\right]+k_z\left[z(t')-z(t)\right]-\omega(t'-t) \\ &= \vec{k}_\perp\cdot\begin{bmatrix} sin\left[\Omega_j\left(t'-t\right)\right] & -\left(1-cos\left[\Omega_j\left(t'-t\right)\right]\right) \\ 1-cos\left[\Omega_j\left(t'-t\right)\right] & sin\left[\Omega_j\left(t'-t\right)\right] \end{bmatrix} \frac{\vec{v}_\perp(t)}{\Omega_j} + (k_z v_z-\omega)(t'-t) \\ &= \vec{k}_\perp^T \begin{bmatrix} sin\Omega_j\tau & -\left(1-cos\Omega_j\tau\right) \\ 1-cos\Omega_j\tau & sin\Omega_j\tau \end{bmatrix} \frac{\vec{v}_\perp}{\Omega_j} + (k_z v_z-\omega)\tau,\end{aligned} \tag{26}$$

其中，

$$\tau = t'-t. \tag{27}$$

另外，

$$
\begin{aligned}
\vec{E}^{(1)}\left(\omega+\cdot\vec{v}'\vec{k}-\vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right) &= \vec{E}^{(1)}\left(\omega-\vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)+\vec{E}^{(1)}\cdot\vec{v}'\vec{k}\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right) \\
&= \left(\omega-k_zv_z-\vec{k}_\perp\cdot\vec{v}_\perp(t')\right)\vec{E}^{(1)}\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right) \\
&+\left(E_z^{(1)}v_z+\vec{E}_\perp^{(1)}\cdot\vec{v}_\perp'\right)\left(k_z\frac{\partial f_j^{(0)}\left(\vec{v}'\right)}{\partial v_z'}+\vec{k}_\perp\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)\right) \\
&= \left(\omega-k_zv_z-\vec{k}_\perp\cdot\vec{v}_\perp(t')\right)\left(E_z^{(1)}\frac{\partial f_j^{(0)}\left(\vec{v}'\right)}{\partial v_z'}+\vec{E}_\perp^{(1)}\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)\right) \\
&+\left(E_z^{(1)}v_z+\vec{E}_\perp^{(1)}\cdot\vec{v}_\perp'\right)\left(k_z\frac{\partial f_j^{(0)}\left(\vec{v}'\right)}{\partial v_z'}+\vec{k}_\perp\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)\right) \\
&= \vec{E}_\perp^{(1)}\cdot\vec{v}_\perp'\left[\omega f_{j,\perp}^{(0)}+k_z\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)\right] \\
&- E_z^{(1)}\vec{k}_\perp\cdot\vec{v}_\perp'\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)+\omega E_z^{(1)}f_{j,z}^{(0)}
\end{aligned}
\tag{28}
$$

其中，

$$
\vec{v}_\perp'=\vec{v}_\perp(t')=\begin{bmatrix} cos\Omega_j\tau & -sin\Omega_j\tau \\ sin\Omega_j\tau & cos\Omega_j\tau \end{bmatrix}\vec{v}_\perp
$$

这样，一阶分布函数中的积分就是时空无关的，可以给出频域的 1 阶分布函数：

$$
f_j^{(1)}\left(\vec{k},\vec{v},\omega\right)=-\frac{e_j}{m_j\omega}\int_{-\infty}^{0}e^{i\left(\vec{k}\cdot(\vec{x}'-\vec{x})-\omega(t'-t)\right)}\vec{E}^{(1)}\left(\omega+\cdot\vec{v}'\vec{k}-\vec{v}'\cdot\vec{k}\right)\cdot\nabla_{v'}f_j^{(0)}\left(\vec{v}'\right)d\tau
$$

注意由于 $v_\perp$ 不随时间变化，所以实际上 $f_j^{(0)}\left(v_z,v_\perp^2\right)$ 是不随时间变化的。

我们想在垂直方向上用复数来表示，这样更方便把含有 τ 的项消掉，做一个垂直方向的坐标变换：

$$
\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ i & -i\end{bmatrix}
$$

注意这是一个非正交分解，$\hat{e}_+,\hat{e}_-$ 不是一组正交基，要求解这两个方向上的坐标，需要求逆矩阵。

$$
\begin{bmatrix}g_x \\ g_y\end{bmatrix}=\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}\begin{bmatrix}g_+ \\ g_-\end{bmatrix}\Longrightarrow\begin{bmatrix}g_+ \\ g_-\end{bmatrix}=\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}^{-1}\begin{bmatrix}g_x \\ g_y\end{bmatrix}
$$

其中，

$$
\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}^{-1}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i \\ 1 & i\end{bmatrix}
$$

另外，计算内积的时候用到的度规矩阵：

$$
\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}^{T}\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}=\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}
$$

可以得到

$$
\begin{bmatrix}v_+' \\ v_-'\end{bmatrix}=\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}^{-1}\begin{bmatrix} cos\Omega_j\tau & -sin\Omega_j\tau \\ sin\Omega_j\tau & cos\Omega_j\tau \end{bmatrix}\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}\begin{bmatrix}v_+ \\ v_-\end{bmatrix}=\begin{bmatrix}e^{-i\Omega_j\tau} & 0 \\ 0 & e^{i\Omega_j\tau}\end{bmatrix}\begin{bmatrix}v_+ \\ v_-\end{bmatrix}
$$

于是有：

$$
\begin{aligned}
\vec{E}_{\perp}^{(1)} \cdot \vec{v}_{\perp}' &= \begin{bmatrix} E_+ & E_- \end{bmatrix} \begin{bmatrix} \hat{e}_+ & \hat{e}_- \end{bmatrix}^T \begin{bmatrix} \hat{e}_+ & \hat{e}_- \end{bmatrix} \begin{bmatrix} v_+' \\ v_-' \end{bmatrix} \\
&= v_+' E_- + v_-' E_+ \\
&= e^{-i\Omega_j \tau} v_+ E_- + e^{i\Omega_j \tau} v_- E_+ \\
&= \sum_{\pm} e^{\mp i\Omega_j \tau} v_\pm E_\mp
\end{aligned} \tag{29}
$$

下面考虑 $\vec{k} \parallel \vec{B}_0$ 的情况，且考虑电磁波（电磁场扰动垂直于 $\vec{B}_0$），

3.1 波动增长

$$
\vec{k} \cdot (\vec{x}' - \vec{x}) - \omega(t' - t) = (k_z v_z - \omega)\tau, \tag{30}
$$

$$
\vec{E}^{(1)} \left(\omega + \cdot \vec{v}' \vec{k} - \vec{v}' \cdot \vec{k}\right) \cdot \nabla_{v'} f_j^{(0)}\left(\vec{v}'\right) = \vec{E}_{\perp}^{(1)} \cdot \vec{v}_{\perp}' \left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \tag{31}
$$

$$
\begin{aligned}
f_j^{(1)}\left(\vec{k}, \vec{v}, \omega\right) &= -\frac{e_j}{m_j \omega}\left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \int_{-\infty}^{0} e^{i(k_z v_z - \omega)\tau} \vec{E}_{\perp}^{(1)} \cdot \vec{v}_{\perp}' d\tau \\
&= -\frac{e_j}{m_j \omega}\left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \sum_{\pm} v_\pm E_\mp \int_{-\infty}^{0} e^{i(k_z v_z - \omega \mp \Omega_j)\tau} d\tau
\end{aligned} \tag{32}
$$

设 $\omega = \omega_r + i\gamma$, $\gamma > 0$，有

$$
\begin{aligned}
\int_{-\infty}^{0} e^{i(k_z v_z - \omega \mp \Omega_j)\tau} d\tau &= \int_{-\infty}^{0} e^{i(k_z v_z - \omega_r \mp \Omega_j)\tau} e^{\gamma\tau} d\tau \\
&= \frac{1}{i(k_z v_z - \omega \mp \Omega_j)} \left[e^{i(k_z v_z - \omega_r \mp \Omega_j)\tau} e^{\gamma\tau}\right]_{-\infty}^{0} \\
&= \frac{-i}{k_z v_z - \omega \mp \Omega_j}
\end{aligned}
$$

最后，

$$
f_j^{(1)}\left(\vec{k}, \vec{v}, \omega\right) = \frac{i e_j}{m_j \omega}\left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \sum_{\pm} v_\pm E_\mp \frac{1}{k_z v_z - \omega \mp \Omega_j} \tag{33}
$$

3.2 波动衰减

$$
\begin{aligned}
f_j^{(1)}\left(\vec{k}, \vec{v}, \omega\right) &= \frac{e_j}{m_j \omega}\left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \int_{0}^{\infty} e^{i(k_z v_z - \omega)\tau} \vec{E}_{\perp}^{(1)} \cdot \vec{v}_{\perp}' d\tau \\
&= \frac{e_j}{m_j \omega}\left[\omega f_{j,\perp}^{(0)} + k_z \left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] \sum_{\pm} v_\pm E_\mp \int_{0}^{\infty} e^{i(k_z v_z - \omega \mp \Omega_j)\tau} d\tau
\end{aligned} \tag{34}
$$

设 $\omega = \omega_r + i\gamma$, $\gamma < 0$，有

$$
\begin{aligned}
\int_{0}^{\infty} e^{i(k_z v_z - \omega \mp \Omega_j)\tau} d\tau &= \int_{0}^{\infty} e^{i(k_z v_z - \omega_r \mp \Omega_j)\tau} e^{\gamma\tau} d\tau \\
&= \frac{1}{i(k_z v_z - \omega \mp \Omega_j)} \left[e^{i(k_z v_z - \omega_r \mp \Omega_j)\tau} e^{\gamma\tau}\right]_{0}^{\infty} \\
&= \frac{i}{k_z v_z - \omega \mp \Omega_j}
\end{aligned}
$$

最后，

$$f_j^{(1)}\left(\vec{k},\vec{v},\omega\right)=\frac{ie_j}{m_j\omega}\left[\omega f_{j,\perp}^{(0)}+k_z\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)\right]\sum_{\pm}v_\pm E_\mp\frac{1}{k_zv_z-\omega\mp\Omega_j} \tag{35}$$

也就是说不论波动增长还是衰减，一阶分布函数的结果是一样的。

3.3 $\gamma=0$

这里不详细讨论波动不增长也不衰减的情况，因为实际的热等离子体中的波动不会一直 $\gamma=0$。这种情况为了使一阶分布函数不是时间的函数，只有

$$\omega=k_zv_z\mp\Omega_j$$

对应离子回旋波/线性阿尔芬波。

注意，以上的增长衰减都是线性简化之下的，增长率都是线性理论对应的。

3.4 波动增长和衰减时的色散关系

根据法拉第定律

$$\vec{B}^{(1)}(\vec{k},\omega)=\frac{\vec{k}}{\omega}\times\vec{E}^{(1)}(\vec{k},\omega).$$

代入安培定律：

$$i\vec{k}\times\vec{B}^{(1)}=\mu_0\vec{J}^{(1)}-\frac{i\omega}{c^2}\vec{E}^{(1)},$$

得到色散关系

$$\vec{E}^{(1)}\left(1-\frac{k^2c^2}{\omega^2}\right)+\frac{i\vec{J}^{(1)}}{\omega\epsilon_0}=0. \tag{36}$$

电流是各成分贡献的：

$$\vec{J}=\sum_j e_j\vec{\Gamma}_j,$$

其中，$\vec{\Gamma}_j$ 是流密度：

$$\vec{\Gamma}_j=\int d^3v\vec{v}f_j$$

欧姆定律：

$$e_j\vec{\Gamma}_j=-i\epsilon_0\frac{k^2c^2}{\omega}\boldsymbol{S_j}\vec{E}$$

代入一阶分布函数后得到：

$$\begin{aligned}
e_j\vec{\Gamma}_j^{(1)}&=e_j\int d^3v\vec{v}f_j^{(1)}\\
&=\frac{ie_j^2}{m_j\omega}\int d^3v\vec{v}\left[\omega f_{j,\perp}^{(0)}+k_z\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)\right]\sum_{\pm}v_\pm E_\mp\frac{1}{k_zv_z-\omega\mp\Omega_j}
\end{aligned}$$

z 方向的积分结果是 0（因为是 v_x,v_y 的奇函数），考虑垂直方向

$$\begin{aligned}
e_j\vec{\Gamma}_{j,\perp}^{(1)}&=e_j\int d^3v\vec{v}_\perp f_j^{(1)}\\
&=\sum_{\pm}E_\mp\frac{ie_j^2}{m_j\omega}\int d^3v\vec{v}_\perp\left[\omega f_{j,\perp}^{(0)}+k_z\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)\right]\frac{v_\pm}{k_zv_z-\omega\mp\Omega_j}\\
&=\sum_{\pm}E_\mp\frac{ie_j^2}{m_j\omega}\int d^3v v_\pm\left(v_+\hat{e}_++v_-\hat{e}_-\right)\left[\omega f_{j,\perp}^{(0)}+k_z\left(f_{j,z}^{(0)}-v_zf_{j,\perp}^{(0)}\right)\right]\frac{1}{k_zv_z-\omega\mp\Omega_j}
\end{aligned}$$

考虑到

$$
\begin{aligned}
v_+v_+ &= (v_x - iv_y)^2/2 = (v_x^2 - v_y^2 - 2iv_xv_y)/2 \\
v_-v_- &= (v_x + iv_y)^2/2 = (v_x^2 - v_y^2 + 2iv_xv_y)/2 \\
v_+v_- &= (v_x + iv_y)(v_x - iv_y)/2 = (v_x^2 + v_y^2)/2 = \frac{v_\perp^2}{2}
\end{aligned}
$$

由于 0 阶分布函数轴对称，有

$$
\int d^3v(v_x^2 - v_y^2)\left[\omega f_{j,\perp}^{(0)} + k_z\left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right]\frac{1}{k_zv_z - \omega \mp \Omega_j} = 0
$$

下面积分中积分项对于 v_x 或 v_y 是奇函数，所以积分为 0：

$$
\int d^3v v_x v_y\left[\omega f_{j,\perp}^{(0)} + k_z\left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right]\frac{1}{k_zv_z - \omega \mp \Omega_j} = 0
$$

所以

$$
\begin{aligned}
e_j\vec{\Gamma}_{j,\perp}^{(1)} &= \sum_{\pm} E_\mp \frac{ie_j^2}{m_j\omega}\int d^3v v_\pm v_\mp \hat{e}_\mp\left[\omega f_{j,\perp}^{(0)} + k_z\left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right]\frac{1}{k_zv_z - \omega \mp \Omega_j} \\
&= \sum_{\pm} E_\mp \frac{ie_j^2}{m_j\omega}\int d^3v \frac{v_\perp^2}{2}\hat{e}_\mp\left[\omega f_{j,\perp}^{(0)} + k_z\left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right]\frac{1}{k_zv_z - \omega \mp \Omega_j} \\
&= -i\epsilon_0\frac{k^2c^2}{\omega}\sum_{\pm} E_\mp \frac{-e_j^2}{m_j\epsilon_0k^2c^2}\int d^3v \frac{v_\perp^2}{2}\hat{e}_\mp\left[\omega f_{j,\perp}^{(0)} + k_z\left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right]\frac{1}{k_zv_z - \omega \mp \Omega_j} \\
&= -i\epsilon_0\frac{k^2c^2}{\omega}\sum_{\pm} E_\mp \frac{-\omega_j^2}{2k^2c^2n_j}\int d^3v v_\perp^2\hat{e}_\mp\left[\omega f_{j,\perp}^{(0)} + k_z\left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right]\frac{1}{k_zv_z - \omega \mp \Omega_j} \\
&= \sum_{\pm} e_j\vec{\Gamma}_{j,\pm}^{(1)}\hat{e}_\pm,
\end{aligned}
$$

其中等离子体频率 $\omega_j = \sqrt{\frac{n_je_j^2}{\epsilon_0m_j}}$.

可以得到

$$
e_j\Gamma_{j,\pm}^{(1)} = -i\epsilon_0\frac{k^2c^2}{\omega}E_\pm\frac{-\omega_j^2}{2k^2c^2n_j}\int d^3v v_\perp^2\left[\omega f_{j,\perp}^{(0)} + k_z\left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right]\frac{1}{k_zv_z - \omega \pm \Omega_j}
$$

所以，

$$
e_j\Gamma_{j,\pm}^{(1)} = -i\epsilon_0\frac{k^2c^2}{\omega}S_{j,\pm}E_\pm,
$$

其中，电导率

$$
S_{j,\pm}(\vec{k},\omega) = \frac{-\omega_j^2}{2k^2c^2n_j}\int d^3v v_\perp^2\left[\omega f_{j,\perp}^{(0)} + k_z\left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right]\frac{1}{k_zv_z - \omega \pm \Omega_j} \tag{37}
$$

色散关系可以用电导率表示：

$$
\omega^2 - k^2c^2 + \sum_j k^2c^2S_{j,\pm}(\vec{k},\omega) = 0. \tag{38}
$$

对于双麦氏分布：

$$
f_j^{(0)}(v_z, v_\perp^2) = \frac{n_jT_{\parallel j}}{\left(2\pi v_j^2\right)^{3/2}T_{\perp j}}\cdot exp\left[-\frac{(v_z - v_{0j})^2}{2v_j^2} - \frac{v_\perp^2}{2v_j^2}\frac{T_{\parallel j}}{T_{\perp j}}\right]
$$

$$dv^3 = 2\pi v_\perp dv_z dv_\perp$$

$$f_{j,\perp}^{(0)} = -\frac{1}{v_j^2}\frac{T_{\parallel j}}{T_{\perp j}} f_j^{(0)}$$

$$f_{j,z}^{(0)} = -\frac{v_z - v_{0j}}{v_j^2} f_j^{(0)}$$

那么

$$\left[\omega f_{j,\perp}^{(0)} + k_z\left(f_{j,z}^{(0)} - v_z f_{j,\perp}^{(0)}\right)\right] = \left[(k_z v_z - \omega)\frac{1}{v_j^2}\frac{T_{\parallel j}}{T_{\perp j}} - k_z\frac{v_z - v_{0j}}{v_j^2}\right] f_j^{(0)}$$

电导率可以写为:

$$S_{j,\pm}(\vec{k},\omega) = \frac{-\pi\omega_j^2}{k^2c^2 n_j}\int_{-\infty}^{\infty}\frac{dv_z}{k_z v_z - \omega \pm \Omega_j}\left[(k_z v_z - \omega)\frac{1}{v_j^2}\frac{T_{\parallel j}}{T_{\perp j}} - k_z\frac{v_z - v_{0j}}{v_j^2}\right]\int_0^\infty dv_\perp v_\perp^3 f_j^{(0)} \tag{39}$$

先对垂直速度进行积分:

$$\int dv_\perp v_\perp^3 f_j^{(0)} = \frac{n_j T_{\parallel j}}{\left(2\pi v_j^2\right)^{3/2} T_{\perp j}} exp\left(-\frac{(v_z - v_{0j})^2}{2v_j^2}\right)\int_0^\infty dv_\perp v_\perp^3 exp\left[-\frac{v_\perp^2}{2v_j^2}\frac{T_{\parallel j}}{T_{\perp j}}\right]$$

其中,

$$\int_0^\infty dv_\perp v_\perp^3 exp\left[-\frac{v_\perp^2}{2v_j^2}\frac{T_{\parallel j}}{T_{\perp j}}\right] = \frac{1}{2\left(\frac{1}{2v_j^2}\frac{T_{\parallel j}}{T_{\perp j}}\right)^2} = \frac{2v_j^4 T_\perp^2}{T_\parallel^2}$$

从而

$$S_{j,\pm}(\vec{k},\omega) = \frac{-\omega_j^2}{k^2c^2}\frac{v_j T_{\perp j}}{(2\pi)^{1/2} T_{\parallel j}}\int_{-\infty}^{\infty}\frac{dv_z}{k_z v_z - \omega \pm \Omega_j}\left[(k_z v_z - \omega)\frac{1}{v_j^2}\frac{T_{\parallel j}}{T_{\perp j}} - k_z\frac{v_z - v_{0j}}{v_j^2}\right] exp\left(-\frac{(v_z - v_{0j})^2}{2v_j^2}\right)$$

令 $\xi = \frac{v_z - v_{0j}}{\sqrt{2}v_j}$, 则

$$S_{j,\pm}(\vec{k},\omega) = \frac{\omega_j^2}{k^2c^2}\pi^{-1/2}\int_{-\infty}^{\infty}\frac{d\xi}{\xi - \frac{\omega - k_z v_{0j} \mp \Omega_j}{\sqrt{2}k_z v_j}}\left[-\xi\left(1 - \frac{T_{\perp j}}{T_{\parallel j}}\right) + \frac{\omega - k_z v_{0j}}{\sqrt{2}k_z v_j}\right] exp\left(-\xi^2\right)$$

对于 $Im\zeta > 0$ 的情况，plasma dispersion function:

$$Z(\zeta) = \pi^{-1/2}\int_{-\infty}^{\infty} dt\frac{e^{-t^2}}{t - \zeta} \tag{40}$$

其导数为 (分部积分后可以得到):

$$Z'(\zeta) = -\pi^{-1/2}\int_{-\infty}^{\infty} dt\frac{2t}{t - \zeta}e^{-t^2} \tag{41}$$

为了凑成这种形式，我们设

$$\zeta_j = sign(k_z\gamma)\frac{\omega - k_z v_{0j}}{\sqrt{2}k_z v_j}$$

$$\zeta_{j,\pm} = sign(k_z\gamma)\frac{\omega - k_z v_{0j} \mp \Omega_j}{\sqrt{2}k_z v_j}$$

$$x = sign(k_z\gamma)\xi$$

则

$$
\begin{aligned}
S_{j,\pm}(\vec{k},\omega) &= \frac{\omega_j^2}{k^2c^2}\pi^{-1/2} sign(k_z\gamma)\int_{\xi=-\infty}^{\xi=\infty}\frac{dx}{x-\zeta_{j,\pm}}\left[-x\left(1-\frac{T_{\perp j}}{T_{\parallel j}}\right)+\zeta_j\right] exp\left(-x^2\right)\\
&= \frac{\omega_j^2}{k^2c^2}\pi^{-1/2}\int_{-\infty}^{\infty}\frac{dx}{x-\zeta_{j,\pm}}\left[-x\left(1-\frac{T_{\perp j}}{T_{\parallel j}}\right)+\zeta_j\right] exp\left(-x^2\right)\\
&= \frac{\omega_j^2}{k^2c^2}\left[\zeta_j Z(\zeta_{j,\pm})+\left(1-\frac{T_{\perp j}}{T_{\parallel j}}\right)\frac{Z'(\zeta_{j,\pm})}{2}\right]
\end{aligned}
$$

4 2nd order

弗拉索夫方程展开到二阶：

$$
\begin{aligned}
&\frac{\partial\left(f_j^{(0)}+f_j^{(1)}+f_j^{(2)}\right)}{\partial t}+\vec{v}\cdot\nabla\left(f_j^{(0)}+f_j^{(1)}+f_j^{(2)}\right)+\\
&\frac{e_j}{m_j}\left[\vec{E}^{(1)}+\vec{E}^{(2)}+\vec{v}\times\vec{B}_0+\vec{v}\times\vec{B}^{(1)}+\vec{v}\times\vec{B}^{(2)}\right]\cdot\nabla_v\left(f_j^{(0)}+f_j^{(1)}+f_j^{(2)}\right)=0,
\end{aligned}
$$

减去 0 阶弗拉索夫方程 10和 1 阶弗拉索夫方程 15，得到：

$$
\begin{aligned}
&\frac{\partial f_j^{(2)}}{\partial t}+\vec{v}\cdot\nabla f_j^{(2)}+\frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v f_j^{(2)}=\\
&-\frac{e_j}{m_j}\left[\vec{E}^{(1)}+\vec{E}^{(2)}+\vec{v}\times\vec{B}^{(1)}+\vec{v}\times\vec{B}^{(2)}\right]\cdot\nabla_v\left(f_j^{(1)}+f_j^{(2)}\right)-\frac{e_j}{m_j}\left[\vec{E}^{(2)}+\vec{v}\times\vec{B}^{(2)}\right]\cdot\nabla_v f_j^{(0)},
\end{aligned}
$$

保留到二阶小量：

$$
\frac{\partial f_j^{(2)}}{\partial t}+\vec{v}\cdot\nabla f_j^{(2)}+\frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v f_j^{(2)}=-\frac{e_j}{m_j}\left[\vec{E}^{(1)}+\vec{v}\times\vec{B}^{(1)}\right]\cdot\nabla_v f_j^{(1)}-\frac{e_j}{m_j}\left[\vec{E}^{(2)}+\vec{v}\times\vec{B}^{(2)}\right]\cdot\nabla_v f_j^{(0)}, \tag{42}
$$

我们关心的是分布整体的演化，而不是局部的扰动，求空间平均：

$$
\frac{\partial\left\langle f_j^{(2)}\right\rangle}{\partial t}+\frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v\left\langle f_j^{(2)}\right\rangle=-\frac{e_j}{m_j}\left\langle\left[\vec{E}^{(1)}+\vec{v}\times\vec{B}^{(1)}\right]\cdot\nabla_v f_j^{(1)}\right\rangle-\frac{e_j}{m_j}\left[\left\langle\vec{E}^{(2)}\right\rangle+\vec{v}\times\left\langle\vec{B}^{(2)}\right\rangle\right]\cdot\nabla_v f_j^{(0)},
$$

考虑法拉第定律，

$$
\left\langle\nabla\times\vec{E}^{(2)}\right\rangle=-\frac{\partial\left\langle\vec{B}^{(2)}\right\rangle}{\partial t}\Longrightarrow\frac{\partial\left\langle\vec{B}^{(2)}\right\rangle}{\partial t}=0
$$

也就是说 $\left\langle\vec{B}^{(2)}\right\rangle$ 不随时间变化，可以设为 0（通过改变 $\vec{B}_0$）。这样，空间平均后的二阶弗拉索夫方程为：

$$
\frac{\partial\left\langle f_j^{(2)}\right\rangle}{\partial t}+\frac{e_j}{m_j}\left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v\left\langle f_j^{(2)}\right\rangle=-\frac{e_j}{m_j}\left\langle\left[\vec{E}^{(1)}+\vec{v}\times\vec{B}^{(1)}\right]\cdot\nabla_v f_j^{(1)}\right\rangle-\frac{e_j}{m_j}\left\langle\vec{E}^{(2)}\right\rangle\cdot\nabla_v f_j^{(0)}, \tag{43}
$$

积分后可以求速度的各阶矩。

4.1 一阶矩

$$
\begin{aligned}
m_j \int d^3 v \vec{v} \frac{\partial \left\langle f_j^{(2)} \right\rangle}{\partial t} &= m_j \frac{\partial}{\partial t} \left\langle \int d^3 v \vec{v} f_j^{(2)} \right\rangle \\
&= \frac{\partial \left\langle \vec{P}_j^{(2)} \right\rangle}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \vec{v} \left(\vec{v} \times \vec{B}_0 \right) \cdot \nabla_v \left\langle f_j^{(2)} \right\rangle &= e_j \frac{\left\langle \vec{P}_j^{(2)} \right\rangle}{m_j} \times \vec{B}_0 \\
&= \vec{\Omega}_j \times \left\langle \vec{P}_j^{(2)} \right\rangle
\end{aligned}
$$

$$
-e_j \int dv^3 \vec{v} \left\langle \left[\vec{v} \times \vec{B}^{(1)} \right] \cdot \nabla_v f_j^{(1)} \right\rangle = e_j \left\langle \vec{\Gamma}_j \times \vec{B}^{(1)} \right\rangle
$$

$$
\begin{aligned}
-e_j \int dv^3 \vec{v} \vec{E}^{(1)} \cdot \nabla_v f_j^{(1)} &= -e_j \int dv^3 v_k E_i^{(1)} \frac{\partial f_j^{(1)}}{\partial v_i} \\
&= -e_j \int dv^3 v_k \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} \\
&= e_j \int dv^3 f_j^{(1)} E_k^{(1)} \\
&= e_j \vec{E}^{(1)} \int dv^3 f_j^{(1)} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
-e_j \int dv^3 \vec{v} \vec{E}^{(2)} \cdot \nabla_v f_j^{(0)} &= -e_j \int dv^3 v_k E_i^{(2)} \frac{\partial f_j^{(0)}}{\partial v_i} \\
&= -e_j \int dv^3 v_k \frac{\partial f_j^{(0)} E_i^{(2)}}{\partial v_i} \\
&= e_j \int dv^3 f_j^{(0)} E_k^{(2)} \\
&= e_j \vec{E}^{(2)} \int dv^3 f_j^{(0)} \\
&= e_j n_j \vec{E}^{(2)}
\end{aligned}
$$

从而得到一阶矩的方程：

$$
\frac{\partial \left\langle \vec{P}_j^{(2)} \right\rangle}{\partial t} + \vec{\Omega}_j \times \left\langle \vec{P}_j^{(2)} \right\rangle - e_j n_j \left\langle \vec{E}^{(2)} \right\rangle = e_j \left\langle \vec{\Gamma}_j \times \vec{B}^{(1)} \right\rangle \tag{44}
$$

4.2 二阶矩

$$
\begin{aligned}
m_j \int d^3 v \left(\vec{v}-\vec{v}_{0j}\right)^2 \frac{\partial \left\langle f_j^{(2)} \right\rangle}{\partial t} &= m_j \frac{\partial}{\partial t} \left\langle \int d^3 v \left(\vec{v}-\vec{v}_{0j}\right)^2 f_j^{(2)} \right\rangle \\
&= 3 n_j \frac{\partial \left\langle T_j^{(2)} \right\rangle}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \left(\vec{v}\times\vec{B}_0\right)\cdot\nabla_v f_j^{(2)} &= e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \epsilon_{lmn} v_l B_m^{(0)} \frac{\partial f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 \left(v^2 - 2\vec{v}\cdot\vec{v}_{0j}\right) \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 v^2 \frac{\partial v_l f_j^{(2)}}{\partial v_n} - 2 e_j \epsilon_{lmn} B_m^{(0)} v_{0j,k} \int dv^3 v_k \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= -2 e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 v_l v_n f_j^{(2)} + 2 e_j \epsilon_{lmn} B_m^{(0)} v_{0j,n} \int dv^3 v_l f_j^{(2)} \\
&= 0 + 2 \frac{e_j}{m_j} \vec{v}_{0j} \cdot \left(\vec{P}^{(2)} \times \vec{B}^{(0)}\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \left(\vec{v}\times\vec{B}^{(1)}\right)\cdot\nabla_v f_j^{(1)} &= e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \epsilon_{lmn} v_l B_m^{(1)} \frac{\partial f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 \left(v^2 - 2\vec{v}\cdot\vec{v}_{0j}\right) \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 v^2 \frac{\partial v_l f_j^{(1)}}{\partial v_n} - 2 e_j \epsilon_{lmn} B_m^{(1)} v_{0j,k} \int dv^3 v_k \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= -2 e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 v_l v_n f_j^{(1)} + 2 e_j \epsilon_{lmn} B_m^{(1)} v_{0j,n} \int dv^3 v_l f_j^{(1)} \\
&= 2 \vec{v}_{0j} \cdot \left(e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \vec{E}^{(1)} \cdot \nabla_v f_j^{(1)} &= e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 E_i^{(1)} \frac{\partial f_j^{(1)}}{\partial v_i} \\
&= e_j \int dv^3 \left(v^2 - 2\vec{v}\cdot\vec{v}_{0j}\right) \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} \\
&= e_j \int dv^3 v^2 \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} - 2e_j \int dv^3 \vec{v}\cdot\vec{v}_{0j} \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} \\
&= -2e_j \int dv^3 f_j^{(1)} E_i^{(1)} v_i + 2e_j v_{0j,i} \int dv^3 f_j^{(1)} E_i^{(1)} \\
&= -2e_j \vec{E}^{(1)} \cdot \vec{\Gamma}_j^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 \vec{E}^{(2)} \cdot \nabla_v f_j^{(0)} &= e_j \int dv^3 \left(\vec{v}-\vec{v}_{0j}\right)^2 E_i^{(2)} \frac{\partial f_j^{(0)}}{\partial v_i} \\
&= e_j \int dv^3 \left(v^2 - 2\vec{v}\cdot\vec{v}_{0j}\right) \frac{\partial f_j^{(0)} E_i^{(2)}}{\partial v_i} \\
&= e_j \int dv^3 v^2 \frac{\partial f_j^{(0)} E_i^{(2)}}{\partial v_i} - 2e_j \int dv^3 \vec{v}\cdot\vec{v}_{0j} \frac{\partial f_j^{(0)} E_i^{(2)}}{\partial v_i} \\
&= -2e_j \int dv^3 f_j^{(0)} E_i^{(2)} v_i + 2e_j v_{0j,i} \int dv^3 f_j^{(0)} E_i^{(2)} \\
&= -2e_j n_j \vec{E}^{(2)} \cdot \vec{v}_{0j} + 2e_j n_j \vec{v}_{0j} \cdot \vec{E}^{(2)} \\
&= 0
\end{aligned}
$$

得到二阶温度的控制方程:

$$
n_j \frac{\partial \left\langle T_j^{(2)} \right\rangle}{\partial t} = \frac{2}{3} e_j \left\langle \vec{\Gamma}_j^{(1)} \cdot \vec{E}^{(1)} \right\rangle - \frac{2}{3} \vec{v}_{0j} \cdot \left\langle e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)} \right\rangle \tag{45}
$$

$$
\begin{aligned}
m_j \int d^3v \left(v_\parallel - v_{0\parallel j}\right)^2 \frac{\partial \left\langle f_j^{(2)} \right\rangle}{\partial t} &= m_j \frac{\partial}{\partial t} \left\langle \int d^3v \left(v_\parallel - v_{0\parallel j}\right)^2 f_j^{(2)} \right\rangle \\
&= n_j \frac{\partial \left\langle T_{\parallel j}^{(2)} \right\rangle}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \left(\vec{v} \times \vec{B}_0\right) \cdot \nabla_v f_j^{(2)} &= e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \epsilon_{lmn} v_l B_m^{(0)} \frac{\partial f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 \left(v_\parallel^2 - 2 v_\parallel v_{0\parallel j}\right) \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 v_\parallel^2 \frac{\partial v_l f_j^{(2)}}{\partial v_n} - 2 e_j \epsilon_{lmn} B_m^{(0)} v_{0j,\parallel} \int dv^3 v_\parallel \frac{\partial v_l f_j^{(2)}}{\partial v_n} \\
&= -2 e_j \epsilon_{lmn} B_m^{(0)} \int dv^3 v_l v_\parallel f_j^{(2)} \delta_{n,\parallel} + 2 e_j \epsilon_{lmn} B_m^{(0)} v_{0\parallel j} \int dv^3 v_l f_j^{(2)} \delta_{n,\parallel} \\
&= -2 e_j \int dv^3 \left(\vec{v} \times \vec{B}_0\right) \cdot \vec{v}_\parallel f_j^{(2)} + 2 \frac{e_j}{m_j} v_{0\parallel j} \cdot \left(\vec{P}^{(2)} \times \vec{B}^{(0)}\right)_\parallel \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \left(\vec{v} \times \vec{B}^{(1)}\right) \cdot \nabla_v f_j^{(1)} &= e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \epsilon_{lmn} v_l B_m^{(1)} \frac{\partial f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 \left(v_\parallel^2 - 2 v_\parallel v_{0\parallel j}\right) \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 v_\parallel^2 \frac{\partial v_l f_j^{(1)}}{\partial v_n} - 2 e_j \epsilon_{lmn} B_m^{(1)} v_{0\parallel j} \int dv^3 v_\parallel \frac{\partial v_l f_j^{(1)}}{\partial v_n} \\
&= -2 e_j \epsilon_{lmn} B_m^{(1)} \int dv^3 v_l v_\parallel f_j^{(1)} \delta_{n,\parallel} + 2 e_j \epsilon_{lmn} B_m^{(1)} v_{0j,\parallel} \int dv^3 v_l f_j^{(1)} \delta_{n,\parallel} \\
&= -2 e_j \int dv^3 \left(\vec{v} \times \vec{B}^{(1)}\right)_\parallel v_\parallel f_j^{(1)} + 2 v_{0\parallel j} \cdot \left(e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)}\right)_\parallel \\
&= -2 e_j \int dv^3 \left(\vec{v} \times \vec{B}^{(1)}\right) \cdot \vec{v}_\parallel f_j^{(1)} + 2 \vec{v}_{0j} \cdot \left(e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)}\right) \\
&= -2 e_j \vec{B}^{(1)} \cdot \int dv^3 \left(\vec{v}_\parallel \times \vec{v}\right) f_j^{(1)} + 2 \vec{v}_{0j} \cdot \left(e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)}\right) \\
&= 2 e_j \vec{B}^{(1)} \cdot \int dv^3 \left(\vec{v}_\perp \times \vec{v}\right) f_j^{(1)} + 2 \vec{v}_{0j} \cdot \left(e_j \vec{\Gamma}_j^{(1)} \times \vec{B}^{(1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 \vec{E}^{(1)} \cdot \nabla_v f_j^{(1)} &= e_j \int dv^3 \left(v_\parallel - v_{0\parallel j}\right)^2 E_i^{(1)} \frac{\partial f_j^{(1)}}{\partial v_i} \\
&= e_j \int dv^3 \left(v_\parallel^2 - 2 v_\parallel v_{0\parallel j}\right) \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} \\
&= e_j \int dv^3 v_\parallel^2 \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} - 2 e_j \int dv^3 v_\parallel v_{0\parallel j} \frac{\partial f_j^{(1)} E_i^{(1)}}{\partial v_i} \\
&= -2 e_j \int dv^3 f_j^{(1)} E_\parallel^{(1)} v_\parallel + 2 e_j v_{0\parallel j} \int dv^3 f_j^{(1)} E_\parallel^{(1)} \\
&= -2 \left\langle e_j \Gamma_\parallel^{(1)} E_\parallel^{(1)} \right\rangle \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
e_j\int dv^3\left(v_\parallel - v_{0\parallel j}\right)^2\vec{E}^{(2)}\cdot\nabla_v f_j^{(0)} &= e_j\int dv^3\left(v_\parallel - v_{0\parallel j}\right)^2 E_i^{(2)}\frac{\partial f_j^{(0)}}{\partial v_i}\\
&= e_j\int dv^3 v_\parallel^2\frac{\partial f_j^{(0)}E_i^{(2)}}{\partial v_i} - 2e_j\int dv^3 v_\parallel v_{0\parallel j}\frac{\partial f_j^{(0)}E_i^{(2)}}{\partial v_i}\\
&= -2e_j\int dv^3 f_j^{(0)}E_\parallel^{(2)}v_\parallel + 2e_j v_{0\parallel j}\int dv^3 f_j^{(0)}E_\parallel^{(2)}\\
&= -2e_j n_j\vec{E}^{(2)}\cdot\vec{v}_{0j} + 2e_j n_j\vec{v}_{0j}\cdot\vec{E}^{(2)}\\
&= 0
\end{aligned}
$$

这样，二阶平行温度的控制方程为：

$$
n_j\frac{\partial\left\langle T_{\parallel j}^{(2)}\right\rangle}{\partial t} = -2\vec{v}_{0j}\cdot\left\langle e_j\vec{\Gamma}_j^{(1)}\times\vec{B}^{(1)}\right\rangle - 2e_j\left\langle\vec{B}^{(1)}\cdot\int dv^3\left(\vec{v}_\perp\times\vec{v}\right)f_j^{(1)}\right\rangle \tag{46}
$$

二阶垂直温度的控制方程为：

$$
\begin{aligned}
n_j\frac{\partial\left\langle T_{\perp j}^{(2)}\right\rangle}{\partial t} &= \frac{3}{2}n_j\frac{\partial\left\langle T_j^{(2)}\right\rangle}{\partial t} - \frac{1}{2}n_j\frac{\partial\left\langle T_{\parallel j}^{(2)}\right\rangle}{\partial t}\\
&= e_j\left\langle\vec{\Gamma}_j^{(1)}\cdot\vec{E}^{(1)}\right\rangle + e_j\left\langle\vec{B}^{(1)}\cdot\int dv^3\left(\vec{v}_\perp\times\vec{v}\right)f_j^{(1)}\right\rangle
\end{aligned} \tag{47}
$$

整理二阶速度、温度的控制方程（方程中变量是 x,t 空间的而非频域的）：

$$
\frac{\partial\left\langle\vec{P}_j^{(2)}\right\rangle}{\partial t} + \vec{\Omega}_j\times\left\langle\vec{P}_j^{(2)}\right\rangle - e_j n_j\left\langle\vec{E}^{(2)}\right\rangle = e_j\left\langle\vec{\Gamma}_j^{(1)}\times\vec{B}^{(1)}\right\rangle
$$

$$
n_j\frac{\partial\left\langle T_j^{(2)}\right\rangle}{\partial t} = \frac{2}{3}e_j\left\langle\vec{\Gamma}_j^{(1)}\cdot\vec{E}^{(1)}\right\rangle - \frac{2}{3}\vec{v}_{0j}\cdot\left\langle e_j\vec{\Gamma}_j^{(1)}\times\vec{B}^{(1)}\right\rangle
$$

$$
n_j\frac{\partial\left\langle T_{\parallel j}^{(2)}\right\rangle}{\partial t} = -2\vec{v}_{0j}\cdot\left\langle e_j\vec{\Gamma}_j^{(1)}\times\vec{B}^{(1)}\right\rangle - 2e_j\left\langle\vec{B}^{(1)}\cdot\int dv^3\left(\vec{v}_\perp\times\vec{v}\right)f_j^{(1)}\right\rangle
$$

$$
n_j\frac{\partial\left\langle T_{\perp j}^{(2)}\right\rangle}{\partial t} = e_j\left\langle\vec{\Gamma}_j^{(1)}\cdot\vec{E}^{(1)}\right\rangle + e_j\left\langle\vec{B}^{(1)}\cdot\int dv^3\left(\vec{v}_\perp\times\vec{v}\right)f_j^{(1)}\right\rangle
$$

下面考虑平行传播的情况，把各项用电导率和功率谱密度表示出来（下面出现的一阶量是频域的）：
设

$$
\epsilon_\pm = \frac{1}{2}\epsilon_0 E_\pm^* E_\pm \tag{48}
$$

$$
\begin{aligned}
e_j\vec{\Gamma}_j^{(1)}\cdot\vec{E}^{(1)*} &= -i\epsilon_0\frac{k^2c^2}{|\omega|^2}\omega^*\sum_{\pm}S_{j,\pm}E_{\pm}\hat{e}_{\pm}\cdot\sum_{\pm}E_{\pm}^*\hat{e}_{\pm}^* \\
&= -i\epsilon_0\frac{k^2c^2}{|\omega|^2}\omega^*\begin{bmatrix}S_{j,+}E_+ & S_{j,-}E_-\end{bmatrix}\begin{bmatrix}\hat{e}_+ & \hat{e}_-\end{bmatrix}^T\begin{bmatrix}\hat{e}_- & \hat{e}_+\end{bmatrix}\begin{bmatrix}E_+^*\\ E_-^*\end{bmatrix} \\
&= -i\epsilon_0\frac{k^2c^2}{|\omega|^2}\omega^*\begin{bmatrix}S_{j,+}E_+ & S_{j,-}E_-\end{bmatrix}\begin{bmatrix}E_+^*\\ E_-^*\end{bmatrix} \\
&= -i\epsilon_0\frac{k^2c^2}{|\omega|^2}\omega^*\left(S_{j,+}E_+E_+^* + S_{j,-}E_-E_-^*\right) \\
&= -2i\frac{k^2c^2}{|\omega|^2}\omega^*\left(S_{j,+}\epsilon_+ + S_{j,-}\epsilon_-\right)
\end{aligned}
$$

从而

$$
\begin{aligned}
\frac{1}{2}Re\left(e_j\vec{\Gamma}_j^{(1)}\cdot\vec{E}^{(1)*}\right) &= Re\left(-i\frac{k^2c^2}{|\omega|^2}\omega^*\left(S_{j,+}\epsilon_+ + S_{j,-}\epsilon_-\right)\right) \\
&= \frac{k^2c^2}{|\omega|^2}Im\left[\omega^*\left(S_{j,+}\epsilon_+ + S_{j,-}\epsilon_-\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
e_j\vec{\Gamma}_j^{(1)}\times\vec{B}^{(1)*} &= e_j\vec{\Gamma}_j^{(1)}\times\left(\frac{\vec{k}}{\omega^*}\times\vec{E}^{(1)*}\right) \\
&= \frac{\vec{k}}{\omega^*}\left(e_j\vec{\Gamma}_j^{(1)}\cdot\vec{E}^{(1)*}\right) - \vec{E}^{(1)*}\left(e_j\vec{\Gamma}_j^{(1)}\cdot\frac{\vec{k}}{\omega^*}\right) \\
&= -2i\frac{k^2c^2}{|\omega|^2}\vec{k}\left(S_{j,+}\epsilon_+ + S_{j,-}\epsilon_-\right)
\end{aligned}
$$

从而

$$
\begin{aligned}
\frac{1}{2}Re\left(e_j\vec{\Gamma}_j^{(1)}\times\vec{B}^{(1)*}\right) &= Re\left(-i\frac{k^2c^2}{|\omega|^2}\vec{k}\left(S_{j,+}\epsilon_+ + S_{j,-}\epsilon_-\right)\right) \\
&= \vec{k}\frac{k^2c^2}{|\omega|^2}Im\left(S_{j,+}\epsilon_+ + S_{j,-}\epsilon_-\right)
\end{aligned}
$$

$$
\begin{aligned}
e_j\vec{B}^{(1)*}\cdot\int dv^3\left(\vec{v}_\perp\times\vec{v}\right)f_j^{(1)} &= e_j\left(\frac{\vec{k}}{\omega^*}\times\vec{E}^{(1)*}\right)\cdot\int dv^3\left(\vec{v}_\perp\times\vec{v}\right)f_j^{(1)}\\
&= e_j\int dv^3\left(\frac{\vec{k}}{\omega^*}\times\vec{E}^{(1)*}\right)\cdot\left(\vec{v}_\perp\times\vec{v}\right)f_j^{(1)}\\
&= \frac{e_j}{\omega^*}\int dv^3\vec{k}\cdot\left(\vec{E}^{(1)*}\times\left(\vec{v}_\perp\times\vec{v}\right)\right)f_j^{(1)}\\
&= -\frac{e_j}{\omega^*}\vec{E}^{(1)*}\cdot\int dv^3\vec{v}_\perp k_z v_z f_j^{(1)}\\
&= -\frac{e_j}{\omega^*}\vec{E}^{(1)*}\cdot\int_{-\infty}^{\infty}dv_z k_z v_z\int_0^\infty dv_\perp 2\pi v_\perp\vec{v}_\perp f_j^{(1)}\\
&= -\frac{\vec{E}^{(1)*}}{\omega^*}\cdot\sum_\pm\left(\omega\mp\Omega_j\right)e_j\Gamma_{j,\pm}\hat{e}_\pm-\frac{\vec{E}^{(1)*}}{\omega^*}\cdot\left[-i\epsilon_0\frac{\omega_j^2}{|\omega|^2}\omega^*\left(\omega-k_z v_{0j}\right)\sum_\pm E_\pm\hat{e}_\pm\right]\\
&= 2i\frac{k^2c^2}{|\omega|^2}\left[\left(\omega-\Omega_j\right)S_{j,+}\epsilon_++\left(\omega+\Omega_j\right)S_{j,-}\epsilon_-\right]+2i\frac{\omega_j^2}{|\omega|^2}\left(\omega-k_z v_{0j}\right)\left(\epsilon_++\epsilon_-\right)
\end{aligned}
$$

有

$$
\frac{1}{2}Re\left[e_j\vec{B}^{(1)*}\cdot\int dv^3\left(\vec{v}_\perp\times\vec{v}\right)f_j^{(1)}\right]=-\frac{k^2c^2}{|\omega|^2}Im\left[\left(\omega-\Omega_j\right)S_{j,+}\epsilon_++\left(\omega+\Omega_j\right)S_{j,-}\epsilon_-\right]-\frac{\gamma\omega_j^2}{|\omega|^2}\left(\epsilon_++\epsilon_-\right)
$$