BO: A Unified Tool for Plasma Waves and Instabilities Analysis

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Abstract

A unified numerically solvable framework for dispersion relations with an arbitrary number of species drifting at arbitrary directions and with Krook collision is derived for linear uniform/homogenous kinetic plasma, which greatly extended the standard one [say, T. Stix, *Waves in Plasmas*, AIP Press, 1992]. The purpose of this work is to provide a kinetic plasma dispersion relation tool not only the physical model but also the numerical approach be as general/powerful as possible. As a very general application example, we give the final dispersion relations which assume further the equilibrium distribution function be bi-Maxwellian and including parallel drift, two directions of perpendicular drift (i.e., drift across magnetic field), ring beam and loss-cone. Both the electromagnetic and electrostatic versions are provided, with also the Darwin (a.k.a., magnetoinductive or magnetostatic) version. The species can be treated either magnetized or unmagnetized. Later, the equations are transformed to the matrix form be solvable by using the powerful matrix algorithm [H. S. Xie and Y. Xiao, Plasma Science and Technology, 18, 2, 97, 2016], which is the first approach can give all the important solutions of a linear kinetic plasma system without requiring initial guess for root finding and thus can be extremely useful to the community. To the best of our knowledge, the present model is the most comprehensive one in literature for the distribution function constructed bases on Maxwellian, which thus can be applied widely for study waves and instabilities in space, astrophysics, fusion and laser plasma. We limit the present work to non-relativistic case.

Keywords: Plasma physics, Kinetic dispersion relation, Waves and instabilities, Matrix eigenvalue

PROGRAM SUMMARY
Program Title: BO
Licensing provisions(please choose one): BSD 3-clause
Programming language: Matlab
Nature of problem: The linear fluid and kinetic waves and instabilities in plasma can be described by dispersion relations. The challenges are to provide a dispersion relation as general as possible and to obtain all the solutions of it, which is the goal of BO tool. The kinetic version of BO tool provides a unified numerically solvable framework for kinetic dispersion relations, which greatly extends the standard one [say, T. Stix, *Waves in Plasmas*, AIP Press, 1992], with an arbitrary number of species. It contains many new features, including anisotropic temperature/loss cone/drift in arbitrary direction/ring beam/collision, unmagnetized/magnetized, electrostatic/electromagnetic/Darwin, can support parallel wave vector $k_z \leq 0$, etc.
Solution method: Approximating the plasma dispersion function in the kinetic dispersion relation with J-pole Pade expansion. Transforming the dispersion relation to an equivalent matrix eigenvalue problem and find all the solutions using standard matrix eigenvalue library function.
Additional comments including Restrictions and Unusual features (approx. 50-250 words): Kinetic relativistic effects are not included in the present version yet.

1. Introduction

Due to the complicated evolution of charged particles and electromagnetic field, one of the most important features of plasma is the numerous waves and instabilities. The fundamental features of linear waves and instabilities

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in uniform/homogenous plasma can be described by dispersion relation and are discussed by many authors in monographes [3–5] and textbooks (cf. [6]). In standard treatment of kinetic plasma dispersion relation, the velocity space equilibrium distribution function is assumed to be $f_{s0} = f_{s0}(v_{\parallel}, v_{\perp})$, which can not treat the cases with drift across magnetic field. To include the arbitrary directions of drift and collision would significantly extend the application range of the dispersion relation, with the instabilities in the non-uniform shock [11, 14] be one of numerous of them.

In this work, we try to provide a comprehensive linear kinetic plasma dispersion relation tool to the community, which includes a number of new capabilities in both the physical model and the algorithm. The tool is largely benefited from the powerful matrix approach of PDRK solver [1], which is the first algorithm that can yield all the important kinetic solutions without initial guess for root finding. The new tool is named as BO ('wave' in Chinese), which includes the present work, the kinetic version BO-K succeeding PDRK [1] (hereafter 'BO-K' is refered equivalent to 'PDRK'), the fluid version BO-F superseding PDRF [7], and possibly more. The final goal of BO is to provide a unified, general and powerful tool for plasma waves and instabilities analysis. In this work, we describe the kinetic dispersion relation version in details, including the model derivations, the equations of the algorithm, the code and examples.

In the following sections, we first derive the most general kinetic dispersion relation equation with drift across the magnetic field and Krook collision in section 2. In section 3, we assume a very general extended Maxwellian equilibrium distribution function for all species and derive the corresponding final dispersion relation. In section 4, the corresponding equations suitable be solved by BO-K matrix algorithm are derived. In sections 5, we give some benchmark results. In section 6, we give a summary and some discussions.

2. The General Non-relativistic Dispersion Relation

Considering that the widely used magnetized kinetic dispersion relation in literature such as in Ref.[3] does not including the drift across magnetic field, we firstly derive our new models. A similar magnetized model is derived only recently in Ref.[11], but which is still not as general as the present one. We limit our study to non-relativistic model. To help the reader, we will give detailed steps of our derivations.

2.1. Starting equations

We consider only the collisionless case or with a Krook collision, the kinetic equation for each species is the Vlasov equation with Krook collision at the right hand side

$$\frac{\partial F_s}{\partial t} + \boldsymbol{v} \cdot \frac{\partial F_s}{\partial \boldsymbol{r}} + \left[\boldsymbol{a}_s + \frac{q_s}{m_s}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\right] \cdot \frac{\partial F_s}{\partial \boldsymbol{v}} = -\nu_s(F_s - F_{s0}), \tag{1}$$

where the distribution function $F_s(\boldsymbol{r}, \boldsymbol{v}, t) = n_{s0} f_s(\boldsymbol{r}, \boldsymbol{v}, t)$, and q_s, m_s and n_{s0} are the charge, mass and the number density of species s, respectively. When the collision frequency $\nu_s = 0$, the equation reduces to a collisionless case. The Maxwell equations for fields can be either

$$\partial_t \boldsymbol{E} = c^2 \nabla \times \boldsymbol{B} - \boldsymbol{J}/\epsilon_0, \tag{2}$$

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \tag{3}$$

for electromagnetic case, or

$$\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0, \tag{4}$$

$$\boldsymbol{E} = -\nabla \Phi, \tag{5}$$

for electrostatic case, where

$$\boldsymbol{J} = \sum \boldsymbol{J}_s = \sum_s q_s n_{s0} \int dv^3 \boldsymbol{v} f_s, \tag{6}$$

$$\rho = \sum \rho_s = \sum_s q_s n_{s0} \int dv^3 f_s, \tag{7}$$

$d\boldsymbol{v} = dv^3 = dv_x dv_y dv_z$ and $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light. The acceleration term $\boldsymbol{a}_s$ can be caused by other external forces such as the gravity and other (magneto-)hydro-dynamic forces due to spatial inhomogeneity, which then would cause drift motions across the magnetic field. A typical example is the low hybrid drift instability (LHDI) in space (e.g., in current sheet and shock cases) and fusion [9] plasma (e.g., in mirror and field-reversed configuration).

To extend the application range, we will also given the Darwin model [16] version. In Darwin mode, all field variables can be divided into two parts: the transverse (T, divergence free) part and the longitudinal (L, curl free) part, i.e.,

$$\boldsymbol{E} = \boldsymbol{E}_L + \boldsymbol{E}_T, \ \ \nabla\cdot\boldsymbol{E}_T = 0, \ \ \nabla\times\boldsymbol{E}_L = 0, \tag{8}$$

$$\boldsymbol{B} = \boldsymbol{B}_T, \ \ \nabla\cdot\boldsymbol{B}_T = 0, \tag{9}$$

$$\boldsymbol{J} = \boldsymbol{J}_L + \boldsymbol{J}_T, \ \ \nabla\cdot\boldsymbol{J}_T = 0, \ \ \nabla\times\boldsymbol{J}_L = 0. \tag{10}$$

The corresponding field equations are

$$\nabla\cdot\boldsymbol{E}_L = \rho/\epsilon_0, \tag{11}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{12}$$

$$\partial_t\boldsymbol{E}_L + \underbrace{\cancel{\partial_t\boldsymbol{E}_T}}_{\text{droped in Darwin model}} = c^2\nabla\times\boldsymbol{B} - \boldsymbol{J}/\epsilon_0, \tag{13}$$

$$\partial_t\boldsymbol{B} = -\nabla\times\boldsymbol{E} \ \ (\text{or}, \partial_t\boldsymbol{B} = -\nabla\times\boldsymbol{E}_L), \tag{14}$$

where for our usage we need only the last two of them. The only difference from full electromagnetic model is that the term $\partial_t\boldsymbol{E}_T$ is dropped in the Darwin model. The Darwin model which eliminates the high frequency electromagnetic wave $\omega^2 \sim k^2c^2$, is particularly interesting in theoretical study and kinetic particle-in-cell and Vlasov simulations, i.e., which can use large time steps in simulation and saves the computation resource significantly [10].

We assume the zero-order term be a homogenous system with $f_s(\boldsymbol{v}) = f_{s0}(\boldsymbol{v}) + f_{s1}(\boldsymbol{v})$, $\int dv^3 f_{s0} = 1$, $\boldsymbol{a}_s = \boldsymbol{a}_{s0}$, $\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_1$ and $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1$. The conventional derivation is further assumes $\boldsymbol{a}_{s0} = 0$ and $\boldsymbol{E}_0 = 0$, whereas we treat also $\boldsymbol{a}_{s0} \neq 0$ and $\boldsymbol{E}_0 \neq 0$ in this work.

Without the loss of generality, we assume a background magnetic field in z direction, i.e., $\boldsymbol{B}_0 = B_0\hat{z} = (0, 0, B_0)$. The zeroth-order equations

$$\Big[\boldsymbol{a}_{s0} + \frac{q_s}{m_s}(\boldsymbol{E}_0 + \boldsymbol{v}\times\boldsymbol{B}_0)\Big]\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}} = 0, \tag{15}$$

$$\boldsymbol{J}_0 = \sum_s q_s n_{s0}\boldsymbol{v}_{ds} = 0, \tag{16}$$

$$\rho_0 = \sum_s q_s n_{s0} = 0, \tag{17}$$

where $\boldsymbol{v}_{ds} = \int dv^3\boldsymbol{v} f_{s0}$. We introduce a cylindrical velocity coordinates [Note: When $\boldsymbol{v}_{ds} \neq 0$, we have assumed the system to be Galilean invariant. It is not sure how much influence to the result yet. This issue is subtle, because the Vlasov equation is Galilean invariant and the electromagnetic field equations are Lorentz invariant and the total system is neither Galilean nor Lorentz invariant.] $(v'_\perp, \phi', v'_\parallel)$ with $v'_\perp = \sqrt{(v_x - v_{dsx})^2 + (v_y - v_{dsy})^2}$ and $v'_\parallel = v_z$, where $v_{dsx} = (\frac{m_s}{q_s}a_{s0y} + E_{0y})/B_0$ and $v_{dsy} = -(\frac{m_s}{q_s}a_{s0x} + E_{0x})/B_0$. We have $v_x = v'_\perp\cos\phi' + v_{dsx}$, $v_y = v'_\perp\sin\phi' + v_{dsy}$ and $v_z = v'_\parallel$. We have also $\frac{\partial v_x}{\partial\phi'} = -v'_\perp\sin\phi'$, $\frac{\partial v_y}{\partial\phi'} = v'_\perp\cos\phi'$, $\frac{\partial v'_\perp}{\partial v_x} = \cos\phi'$ and $\frac{\partial v'_\perp}{\partial v_y} = \sin\phi'$. Eq.(15) is

$$\begin{aligned}
0 &= \Big[\boldsymbol{a}_{s0} + \frac{q_s}{m_s}(\boldsymbol{E}_0 + \boldsymbol{v}\times\boldsymbol{B}_0)\Big]\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}} \\
&= \frac{q_s}{m_s}\Big[\Big(\frac{m_s}{q_s}a_{s0x} + E_{0x} + v_yB_0\Big)\frac{\partial f_{s0}}{\partial v_x} + \Big(\frac{m_s}{q_s}a_{s0y} + E_{0y} - v_xB_0\Big)\frac{\partial f_{s0}}{\partial v_y} + \Big(\frac{m_s}{q_s}a_{s0z} + E_{0z}\Big)\frac{\partial f_{s0}}{\partial v_z}\Big] \\
&= \frac{q_sB_0}{m_s}\Big[(-v_{dsy} + v_y)\frac{\partial f_{s0}}{\partial v_x} + (v_{dsx} - v_x)\frac{\partial f_{s0}}{\partial v_y}\Big] = \frac{q_sB_0}{m_s}\Big(v'_\perp\sin\phi'\frac{\partial f_{s0}}{\partial v_x} - v'_\perp\cos\phi'\frac{\partial f_{s0}}{\partial v_y}\Big) \\
&= -\frac{q_sB_0}{m_s}\Big(\frac{\partial v_x}{\partial\phi'}\frac{\partial f_{s0}}{\partial v_x} + \frac{\partial v_y}{\partial\phi'}\frac{\partial f_{s0}}{\partial v_y}\Big) = -\frac{q_sB_0}{m_s}\frac{\partial f_{s0}}{\partial\phi'}.
\end{aligned} \tag{18}$$

Thus we have $\frac{\partial f_{s0}}{\partial \phi'} = 0$ if $B_0 \neq 0$, i.e., $f_{s0} = f_{s0}(v'_\perp, v'_\parallel)$. And, we have assume the force balance in the z-direction, i.e., $\frac{m_s}{q_s} a_{s0z} + E_{0z} = 0$.

The first order kinetic equation is (we have dropped the subscript '1')

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_s}{\partial \boldsymbol{r}} + \frac{q_s}{m_s}(\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} + \frac{q_s}{m_s}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f_{s0}}{\partial \boldsymbol{v}} = -\nu_s f_s, \tag{19}$$

and Fourier transform the equation using $\partial/\partial \boldsymbol{r} \to i\boldsymbol{k}$, $\partial/\partial t \to -i\omega$ and also $\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}$, we obtain

$$i(\omega - \boldsymbol{k} \cdot \boldsymbol{v} + i\nu_s) f_s + \omega_{cs} \frac{\partial f_s}{\partial \phi'} = \frac{q_s}{m_s}\Big[\boldsymbol{E} + \boldsymbol{v} \times \Big(\frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega}\Big)\Big] \cdot \frac{\partial f_{s0}}{\partial \boldsymbol{v}}, \tag{20}$$

where $\omega_{cs} = \frac{q_s B_0}{m_s}$ is the cyclone frequency and $\omega = \omega_r + i\omega_i$ is the complex number wave frequency. Without loss of generality, we can assume [Note: Ref.[11] assumed the across magnetic field drift $\boldsymbol{v}_{ds}$ at x direction but $\boldsymbol{k} = (k_x, k_y, k_z)$, which will limit that all species can only drift at $x - z$ direction. Later, they limit their discussion also to only $k_y = 0$. And thus their result is our result at $v_{dsy} = 0$ and $\nu_s = 0$.] the wave vector $\boldsymbol{k} = (k_x, 0, k_z) = (k \sin\theta, 0, k\cos\theta)$, which gives $k_\perp = k_x$ and $k_\parallel = k_z$. We study four cases:

1. Electromagnetic or Darwin, magnetized: $B_0 \neq 0$, $\boldsymbol{B}_1 \neq 0$.
2. Electromagnetic or Darwin, unmagnetized: $B_0 = 0$ ($\omega_{cs} = 0$), $\boldsymbol{B}_1 \neq 0$.
3. Electrostatic, magnetized: $B_0 \neq 0$, $\boldsymbol{B}_1 = 0$ ($\boldsymbol{k} \times \boldsymbol{E}_1 = 0$).
4. Electrostatic, unmagnetized: $B_0 = 0$ ($\omega_{cs} = 0$), $\boldsymbol{B}_1 = 0$ ($\boldsymbol{k} \times \boldsymbol{E}_1 = 0$).

For the convenience of theoretical study, we allow the user to choose whether a species is magnetized or unmagnetized [cf., A typical case is in shock study, say Ref.[14], where electron are magnetized and drift ions are treated unmagnetized.], i.e., say for a electromagnetic case, the different species can be either magnetized (labeled as 'm') or unmagnetized (labeled as 'u').

For unmagnetized case, Eq.(20) reduces to

$$i(\omega - \boldsymbol{k} \cdot \boldsymbol{v} + i\nu_s) f_s^u = \frac{q_s}{m_s}\Big[\boldsymbol{E} + \boldsymbol{v} \times \Big(\frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega}\Big)\Big] \cdot \frac{\partial f_{s0}}{\partial \boldsymbol{v}}, \tag{21}$$

where $\boldsymbol{k} \cdot \boldsymbol{v} = k_x v_x + k_z v_z$ or $\boldsymbol{k} \cdot \boldsymbol{v} = k_z v'_\parallel + k_x v'_\perp \cos\phi' + k_x v_{dsx}$, which gives the solution for f_s^u is

$$f_s^u = \frac{\frac{q_s}{m_s}\Big[\boldsymbol{E} + \boldsymbol{v} \times \Big(\frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega}\Big)\Big] \cdot \frac{\partial f_{s0}}{\partial \boldsymbol{v}}}{i(\omega - \boldsymbol{k} \cdot \boldsymbol{v} + i\nu_s)}. \tag{22}$$

For magnetized case, Eq.(20) reduces to

$$\frac{\partial f_s}{\partial \phi'} - i(x_s + y_s \cos\phi') f_s = \frac{q_s}{m_s \omega_{cs}}\Big[\boldsymbol{E} + \boldsymbol{v} \times \Big(\frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega}\Big)\Big] \cdot \frac{\partial f_{s0}}{\partial \boldsymbol{v}}, \tag{23}$$

where $x_s = -\frac{\omega - k_\parallel v'_\parallel - k_x v_{dsx} + i\nu_s}{\omega_{cs}}$ and $y_s = \frac{k_\perp v'_\perp}{\omega_{cs}}$. The solution for magnetized f_s^m is much complicated. Instead of the approaches using the method of characteristics in Refs.[3] and [11], we using the approach similar to Ref.[6], which solves the differential equation directly. That is to say, Eq.(23) is a first order linear differential equation of the form

$$\frac{df}{dx} + P(x) f = Q(x), \tag{24}$$

which has a general solution

$$f = e^{-\int^x P(x')dx'}\Big[\int^x Q(x') e^{\int^{x'} P(x'')dx''} dx'\Big], \tag{25}$$

where the integration factor term $\int P(x)dx$ in our case is

$$\int^{\phi'} P(\phi')d\phi'' = -i\int^{\phi'} (x_s + y_s \cos\phi'')d\phi'' = -i(x_s\phi' + y_s \sin\phi'), \tag{26}$$

and thus

$$f_s^m = e^{i(x_s\phi'+y_s\sin\phi')}\Big[\int^{\phi'} Q(\phi'')e^{-i(x_s\phi''+y_s\sin\phi'')}d\phi''\Big], \tag{27}$$

with

$$Q = \frac{q_s}{m_s\omega_{cs}}\Big[\boldsymbol{E}+\boldsymbol{v}\times\Big(\frac{\boldsymbol{k}\times\boldsymbol{E}}{\omega}\Big)\Big]\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}}. \tag{28}$$

We notice that Eq.(27) can not reduce to (22) by set $\omega_{cs}=0$, i.e., the magnetized version can not reduce to unmagnetized version by set $\omega_{cs}=0$. Thus we should treat them separately.

2.2. Electrostatic case

For the electrostatic case, $\boldsymbol{k}\times\boldsymbol{E}=0$, $\boldsymbol{E}=-\imath\boldsymbol{k}\Phi$, and only Poisson equation for field is required. We have

$$k^2\Phi = \rho/\epsilon_0, \tag{29}$$

$$\rho = \rho^u+\rho^m = \sum\rho_s^u+\sum\rho_s^m = \sum_{s=u,m} q_s n_{s0}\int dv^3 f_s. \tag{30}$$

The unmagnetized species

$$f_s^u = \frac{-\frac{q_s\Phi}{m_s}\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}}}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\nu_s)}, \tag{31}$$

where $\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}} = k_x\frac{\partial f_{s0}}{\partial v_x}+k_z\frac{\partial f_{s0}}{\partial v_z}$ or $\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}} = k_x\cos\phi'\frac{\partial f_{s0}}{\partial v'_\perp}+k_z\frac{\partial f_{s0}}{\partial v'_\parallel}$. The corresponding charge density is

$$\rho_s^u = q_s n_{s0}\int dv^3 f_s^u = -\frac{q_s^2 n_{s0}\Phi}{m_s}\int dv^3\frac{\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}}}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\nu_s)}. \tag{32}$$

The velocity space integral may not be easy for complicated f_{s0}, and it is probably better to calculate at (v_x, v_y, v_z) coordinates instead of $(v'_\perp, \phi', v'_\parallel)$.

For magnetized species

$$f_s^m = \frac{-i\Phi q_s}{m_s\omega_{cs}}e^{i(x_s\phi'+y_s\sin\phi')}\Big[\int^{\phi'}\Big(\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}}\Big)e^{-i(x_s\phi''+y_s\sin\phi'')}d\phi''\Big], \tag{33}$$

and

$$\rho_s^m = q_s n_{s0}\int dv^3 f_s^m = \frac{-iq_s^2 n_{s0}\Phi}{m_s\omega_{cs}}\int e^{i(x_s\phi'+y_s\sin\phi')}\Big[\int^{\phi'}\Big(\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}}\Big)e^{-i(x_s\phi''+y_s\sin\phi'')}d\phi''\Big]d\phi' v'_\perp dv'_\perp dv'_\parallel, \tag{34}$$

which is not yet in a useful form. Let us do some further calculations step by step. We have used the cylindrical coordinates $(v'_\perp, \phi', v'_\parallel)$ with $dv^3 = v'_\perp d\phi' dv'_\perp dv'_\parallel$, and note $\partial f_{s0}/\partial\phi' = 0$. Using $\cos\phi' = (e^{i\phi'}+e^{-i\phi'})/2$, we have

$$\int^{\phi'}\Big(\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}}\Big)e^{-i(x_s\phi''+y_s\sin\phi'')}d\phi'' = \int^{\phi'}\Big(k_x\cos\phi''\frac{\partial f_{s0}}{\partial v'_\perp}+k_z\frac{\partial f_{s0}}{\partial v'_\parallel}\Big)e^{-i(x_s\phi''+y_s\sin\phi'')}d\phi'' \tag{35}$$

$$= k_z\frac{\partial f_{s0}}{\partial v'_\parallel}\int^{\phi'} e^{-i(x_s\phi''+y_s\sin\phi'')}d\phi'' + k_x\frac{\partial f_{s0}}{\partial v'_\perp}\int^{\phi'}\frac{1}{2}(e^{i\phi''}+e^{-i\phi''})e^{-i(x_s\phi''+y_s\sin\phi'')}d\phi''.$$

Now, we use the following expansion

$$e^{-iy_s\sin\phi''} = \sum_{n=-\infty}^{\infty} J_n(y_s)e^{-in\phi''}, \tag{36}$$

where $J_n(y_s)$ it the nth order Bessel function. And we have

$$\int^{\phi'} e^{-i(x_s\phi''+y_s\sin\phi'')}d\phi'' = \sum_n J_n(y_s)\int^{\phi'} e^{-i(x_s+n)\phi''}d\phi'' = i\sum_n\frac{J_n(y_s)}{x_s+n}e^{-i(x_s+n)\phi'}, \tag{37}$$

and

$$\int^{\phi'} (e^{i\phi''} + e^{-i\phi''}) e^{-i(x_s\phi''+y_s\sin\phi'')} d\phi'' = \sum_n J_n(y_s) \int^{\phi'} [e^{-i(x_s-1+n)\phi''} + e^{-i(x_s+1+n)\phi''}] d\phi'' \tag{38}$$

$$= i\sum_n J_n(y_s)\Big[\frac{e^{-i(x_s-1+n)\phi'}}{x_s-1+n} + \frac{e^{-i(x_s+1+n)\phi'}}{x_s+1+n}\Big]. \tag{39}$$

We further use

$$e^{i(x_s\phi'+y_s\sin\phi')} = \sum_{m=-\infty}^{\infty} J_m(y_s) e^{i(x_s+m)\phi'}, \tag{40}$$

and thus in Eq.(34)

$$e^{i(x_s\phi'+y_s\sin\phi')}\Big[\int^{\phi'} \cdots d\phi''\Big] \tag{41}$$

$$= ik_z\frac{\partial f_{s0}}{\partial v'_\parallel}\sum_{m,n}\frac{J_nJ_m}{x_s+n}e^{i(m-n)\phi'} + \frac{i}{2}k_x\frac{\partial f_{s0}}{\partial v'_\perp}\sum_{m,n}J_nJ_m\Big[\frac{e^{i(m-n+1)\phi'}}{x_s+n-1} + \frac{e^{i(m-n-1)\phi'}}{x_s+n+1}\Big]$$

$$= ik_z\frac{\partial f_{s0}}{\partial v'_\parallel}\sum_{m,n}\frac{J_nJ_m}{x_s+n}e^{i(m-n)\phi'} + \frac{i}{2}k_x\frac{\partial f_{s0}}{\partial v'_\perp}\sum_{m,n}J_m\Big[J_{n+1}\frac{e^{i(m-n)\phi'}}{x_s+n} + J_{n-1}\frac{e^{i(m-n)\phi'}}{x_s+n}\Big]$$

$$= i\sum_{m,n}J_mJ_n\Big(k_z\frac{\partial f_{s0}}{\partial v'_\parallel} + \frac{n\omega_{cs}}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp}\Big)\frac{e^{i(m-n)\phi'}}{x_s+n},$$

where we have used $(J_{n+1}+J_{n-1}) = (2n/y_s)J_n$. Further integral out $\int d\phi'$ in Eq.(34) gives

$$\int e^{i(x_s\phi'+y_s\sin\phi')}\Big[\int^{\phi'} \cdots d\phi''\Big]d\phi' = i\int\sum_{m,n}J_mJ_n\Big(k_z\frac{\partial f_{s0}}{\partial v'_\parallel} + \frac{n\omega_{cs}}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp}\Big)\frac{e^{i(m-n)\phi'}}{x_s+n}d\phi' \tag{42}$$

$$= i2\pi\sum_n\frac{J_n^2}{x_s+n}\Big(k_z\frac{\partial f_{s0}}{\partial v'_\parallel} + \frac{n\omega_{cs}}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp}\Big)$$

$$= -i2\pi\sum_n\frac{\omega_{cs}J_n^2(y_s)}{\omega - k_\parallel v'_\parallel + i\nu_s - k_xv_{dsx} - n\omega_{cs}}\Big(k_z\frac{\partial f_{s0}}{\partial v'_\parallel} + \frac{n\omega_{cs}}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp}\Big),$$

where because $\int_0^{2\pi} e^{i(m-n)\phi'}d\phi' = 2\pi\delta_{m,n}$, and $\delta_{m,n}$ is the Kronecker delta. Thus we find the final form is very similar to the standard Harris dispersion relation form, except the term x_s.

Combining the magnetized and unmagnetized species ρ_s and substituting them to the Poisson equation (29), we obtain the final electrostatic dispersion relation

$$D(\omega,\boldsymbol{k}) = 1 + \sum_{s=m}\frac{\omega_{ps}^2}{k^2}\int_{-\infty}^{\infty}\int_0^{\infty}\sum_{n=-\infty}^{\infty}\frac{J_n^2(y_s)\Big(k_\parallel\frac{\partial f_{s0}}{\partial v'_\parallel} + \frac{n\omega_{cs}}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp}\Big)}{\omega - k_\parallel v'_\parallel + i\nu_s - k_xv_{dsx} - n\omega_{cs}}2\pi v'_\perp dv'_\perp dv'_\parallel$$

$$+ \sum_{s=u}\frac{\omega_{ps}^2}{k^2}\int dv^3\frac{\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial \boldsymbol{v}}}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu_s)} = 0, \tag{43}$$

where $\omega_{ps}^2 = \frac{n_{s0}q_s^2}{\epsilon_0 m_s}$.

2.3. Electromagnetic case

For electromagnetic case, $\boldsymbol{B} = (\boldsymbol{k}\times\boldsymbol{E})/\omega$, the field equations we needed are

$$-i\omega\boldsymbol{E} = ic^2\boldsymbol{k}\times\Big(\frac{\boldsymbol{k}\times\boldsymbol{E}}{\omega}\Big) - \boldsymbol{J}/\epsilon_0, \tag{44}$$

$$\boldsymbol{J} = \boldsymbol{J}^u + \boldsymbol{J}^m = \sum\boldsymbol{J}_s^u + \sum\boldsymbol{J}_s^m = \sum_{s=u,m}q_sn_{s0}\int dv^3\boldsymbol{v}f_s. \tag{45}$$

If we assume the relation between current $\boldsymbol{J}$ and electric field $\boldsymbol{E}$ be

$$\boldsymbol{J} = \boldsymbol{\sigma} \cdot \boldsymbol{E}. \tag{46}$$

which will be calculated later via the kinetic equation, we obtain from Eq.(44)

$$\boldsymbol{D}(\omega, \boldsymbol{k}) \cdot \boldsymbol{E} = 0, \tag{47}$$

where $\boldsymbol{D}$ can be expressed in terms of the dielectric tensor $\boldsymbol{K}(\omega, \boldsymbol{k})$ and gives the dispersion relation

$$|\boldsymbol{D}(\omega, \boldsymbol{k})| = |\boldsymbol{K}(\omega, \boldsymbol{k}) + (\boldsymbol{kk} - k^2\boldsymbol{I})\frac{c^2}{\omega^2}| = 0, \tag{48}$$

where $\boldsymbol{I}$ is the unit tensor, and we have used $\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) = \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{E}) - (\boldsymbol{k} \cdot \boldsymbol{k})\boldsymbol{E} = (\boldsymbol{kk} - k^2\boldsymbol{I}) \cdot \boldsymbol{E}$. And the relation to conductivity tensor $\boldsymbol{\sigma}$ is

$$\boldsymbol{K} = \boldsymbol{I} + \boldsymbol{Q} = \boldsymbol{I} - \boldsymbol{\sigma}/(i\omega\epsilon_0), \tag{49}$$

with $\boldsymbol{Q} = -\boldsymbol{\sigma}/(i\omega\epsilon_0)$.

Now, we calculate $\boldsymbol{J}$.

For unmagnetized species

$$\begin{aligned}
\boldsymbol{J}_s^u &= q_s n_{s0} \int dv^3 \boldsymbol{v} f_s^u = \epsilon_0 \omega_{ps}^2 \int dv^3 \boldsymbol{v} \frac{\left[\boldsymbol{E} + \boldsymbol{v} \times \left(\frac{\boldsymbol{k}\times\boldsymbol{E}}{\omega}\right)\right] \cdot \frac{\partial f_{s0}}{\partial \boldsymbol{v}}}{i(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu_s)} \\
&= \frac{\epsilon_0 \omega_{ps}^2}{i\omega} \int dv^3 \boldsymbol{v} \frac{\left[(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu_s)\boldsymbol{E} - i\nu_s \boldsymbol{E} + \boldsymbol{k}(\boldsymbol{v}\cdot\boldsymbol{E})\right] \cdot \frac{\partial f_{s0}}{\partial \boldsymbol{v}}}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu_s)} \\
&= -\frac{\epsilon_0 \omega_{ps}^2}{i\omega} \boldsymbol{I} \cdot \boldsymbol{E} + \frac{\epsilon_0 \omega_{ps}^2}{i\omega} \int dv^3 \frac{\left[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0}}{\partial \boldsymbol{v}}) + (\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial \boldsymbol{v}})(\boldsymbol{vv})\right]}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu_s)} \cdot \boldsymbol{E},
\end{aligned} \tag{50}$$

where we have used $\boldsymbol{v} \times (\boldsymbol{k} \times \boldsymbol{E}) = \boldsymbol{k}(\boldsymbol{v}\cdot\boldsymbol{E}) - (\boldsymbol{v}\cdot\boldsymbol{k})\boldsymbol{E}$, and $\boldsymbol{vv}$, $\boldsymbol{v}\frac{\partial f_{s0}}{\partial \boldsymbol{v}}$ are dyadic product tensors. The result is similar to the one in Ref.[14], except our new $i\nu_s$ term. Note also that the tensor $\boldsymbol{v}\frac{\partial f_{s0}}{\partial \boldsymbol{v}} \neq \frac{\partial f_{s0}}{\partial \boldsymbol{v}}\boldsymbol{v}$.

For magnetized species

$$\begin{aligned}
\boldsymbol{J}_s^m &= q_s n_{s0} \int dv^3 \boldsymbol{v} f_s^m \\
&= \frac{q_s^2 n_{s0}}{m_s \omega_{cs}} \int e^{i(x_s\phi' + y_s \sin\phi')} \Big\{ \int^{\phi'} \left[\boldsymbol{E} + \boldsymbol{v} \times \left(\frac{\boldsymbol{k}\times\boldsymbol{E}}{\omega}\right)\right] \cdot \frac{\partial f_{s0}}{\partial \boldsymbol{v}} e^{-i(x_s\phi'' + y_s\sin\phi'')} d\phi'' \Big\} d\phi' v'_\perp dv'_\perp dv'_\parallel \\
&= \frac{q_s^2 n_{s0}}{m_s \omega_{cs} \omega} \int e^{i(x_s\phi' + y_s \sin\phi')} \Big\{ \int^{\phi'} \left[(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{E} + \boldsymbol{k}(\boldsymbol{v}\cdot\boldsymbol{E})\right] \cdot \frac{\partial f_{s0}}{\partial \boldsymbol{v}} e^{-i(x_s\phi'' + y_s\sin\phi'')} d\phi'' \Big\} d\phi' v'_\perp dv'_\perp dv'_\parallel.
\end{aligned} \tag{51}$$

We calculate firstly

$$
\begin{aligned}
&\Big[(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{E}+\boldsymbol{k}(\boldsymbol{v}\cdot\boldsymbol{E})\Big]\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}}=\Big[(\omega-k_zv'_\parallel-k_xv'_\perp\cos\phi'-k_xv_{dsx})\boldsymbol{E}+\boldsymbol{k}(\boldsymbol{v}\cdot\boldsymbol{E})\Big]\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}}\\
=\;&(\omega-k_zv'_\parallel-k_xv'_\perp\cos\phi'-k_xv_{dsx})[\frac{\partial f_{s0}}{\partial v'_\perp}(E_x\cos\phi'+E_y\sin\phi')+\frac{\partial f_{s0}}{\partial v'_\parallel}E_z]\\
&+[v'_\perp(E_x\cos\phi'+E_y\sin\phi')+v'_\parallel E_z+v_{dsx}E_x+v_{dsy}E_y](k_x\cos\phi'\frac{\partial f_{s0}}{\partial v'_\perp}+k_z\frac{\partial f_{s0}}{\partial v'_\parallel})\\
=\;&\Big\{(\omega-k_xv_{dsx})E_z+k_zv_{dsx}E_x+k_zv_{dsy}E_y+(k_zv'_\perp E_x-k_xv'_\perp E_z)\cos\phi'+k_zv'_\perp E_y\sin\phi'\Big\}\frac{\partial f_{s0}}{\partial v'_\parallel}\\
&+\Big\{(\omega-k_zv'_\parallel-k_xv_{dsx})E_y\sin\phi'+[(\omega-k_zv'_\parallel)E_x+k_xv_{dsy}E_y+k_xv'_\parallel E_z]\cos\phi'\Big\}\frac{\partial f_{s0}}{\partial v'_\perp}\\
=\;&\Big\{(k_zv'_\perp E_x-k_xv'_\perp E_z)\frac{\partial f_{s0}}{\partial v'_\parallel}+[(\omega-k_zv'_\parallel)E_x+k_xv_{dsy}E_y+k_xv'_\parallel E_z]\frac{\partial f_{s0}}{\partial v'_\perp}\Big\}\cos\phi'\\
&+\Big\{(\omega-k_zv'_\parallel-k_xv_{dsx})E_y\frac{\partial f_{s0}}{\partial v'_\perp}+k_zv'_\perp E_y\frac{\partial f_{s0}}{\partial v'_\parallel}\Big\}\sin\phi'+[(\omega-k_xv_{dsx})E_z+k_zv_{dsx}E_x+k_zv_{dsy}E_y]\frac{\partial f_{s0}}{\partial v'_\parallel}\\
=\;&(U_{s1}E_x+U_{s2}E_y+U_{s3}E_z)\cos\phi'+U_{s4}E_y\sin\phi'+(U_{s5}E_x+U_{s6}E_y+U_{s7}E_z),
\end{aligned}
\tag{52}
$$

with

$$
\begin{aligned}
&U_{s1}=[(\omega-k_zv'_\parallel)\frac{\partial f_{s0}}{\partial v'_\perp}+k_zv'_\perp\frac{\partial f_{s0}}{\partial v'_\parallel}],\quad U_{s2}=k_xv_{dsy}\frac{\partial f_{s0}}{\partial v'_\perp},\quad U_{s3}=[k_xv'_\parallel\frac{\partial f_{s0}}{\partial v'_\perp}-k_xv'_\perp\frac{\partial f_{s0}}{\partial v'_\parallel}],\\
&U_{s4}=[(\omega-k_zv'_\parallel-k_xv_{dsx})\frac{\partial f_{s0}}{\partial v'_\perp}+k_zv'_\perp\frac{\partial f_{s0}}{\partial v'_\parallel}],\quad U_{s5}=k_zv_{dsx}\frac{\partial f_{s0}}{\partial v'_\parallel},\\
&U_{s6}=k_zv_{dsy}\frac{\partial f_{s0}}{\partial v'_\parallel},\quad U_{s7}=(\omega-k_xv_{dsx})\frac{\partial f_{s0}}{\partial v'_\parallel},
\end{aligned}
\tag{53}
$$

where we have used $\boldsymbol{E}\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}}=E_x\frac{\partial f_{s0}}{\partial v_x}+E_y\frac{\partial f_{s0}}{\partial v_y}+E_z\frac{\partial f_{s0}}{\partial v_z}=\frac{\partial f_{s0}}{\partial v'_\perp}(E_x\cos\phi'+E_y\sin\phi')+\frac{\partial f_{s0}}{\partial v'_\parallel}E_z$, $\boldsymbol{v}\cdot\boldsymbol{E}=v_xE_x+v_yE_y+v_zE_z=v'_\perp(E_x\cos\phi'+E_y\sin\phi')+v'_\parallel E_z+v_{dsx}E_x+v_{dsy}E_y$, $\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial\boldsymbol{v}}=k_x\frac{\partial f_{s0}}{\partial v_x}+k_z\frac{\partial f_{s0}}{\partial v_z}=k_x\cos\phi'\frac{\partial f_{s0}}{\partial v'_\perp}+k_z\frac{\partial f_{s0}}{\partial v'_\parallel}$.

$\frac{\partial f_{s0}}{\partial v'_\perp}$ terms: $(\omega-k_zv'_\parallel-k_xv'_\perp\cos\phi'-k_xv_{dsx})(E_x\cos\phi'+E_y\sin\phi')+[v'_\perp(E_x\cos\phi'+E_y\sin\phi')+v'_\parallel E_z+v_{dsx}E_x+v_{dsy}E_y]k_x\cos\phi'=(\omega-k_zv'_\parallel-k_xv_{dsx})(E_x\cos\phi'+E_y\sin\phi')+(v'_\parallel E_z+v_{dsx}E_x+v_{dsy}E_y)k_x\cos\phi'=(\omega-k_zv'_\parallel-k_xv_{dsx})E_y\sin\phi'+[(\omega-k_zv'_\parallel)E_x+k_xv_{dsy}E_y+k_xv'_\parallel E_z]\cos\phi'$.

$\frac{\partial f_{s0}}{\partial v'_\parallel}$ terms: $(\omega-k_zv'_\parallel-k_xv'_\perp\cos\phi'-k_xv_{dsx})E_z+[v'_\perp(E_x\cos\phi'+E_y\sin\phi')+v'_\parallel E_z+v_{dsx}E_x+v_{dsy}E_y]k_z=(\omega-k_xv_{dsx})E_z+k_zv_{dsx}E_x+k_zv_{dsy}E_y+(k_zv'_\perp E_x-k_xv'_\perp E_z)\cos\phi'+k_zv'_\perp E_y\sin\phi'$.

Our Eq.(52) is the same as in Ref.[11] when set our $v_{dsy}=0$ and $v_s=0$, and their $k_y=0$.

Thus, using $\sin\phi = -i(e^{i\phi} - e^{-i\phi})/2$ and $J_{n+1} - J_{n-1} = -2J_n'$,

$$
\begin{aligned}
f_s^m &= \frac{q_s}{m_s\omega_{cs}\omega} e^{i(x_s\phi' + y_s\sin\phi')}\Big\{\int^{\phi'}\Big[(U_{s1}E_x + U_{s2}E_y + U_{s3}E_z)\frac{1}{2}(e^{i\phi''} + e^{-i\phi''}) + U_{s4}E_y\frac{-i}{2}(e^{i\phi''} - e^{-i\phi''}) \\
&\quad +(U_{s5}E_x + U_{s6}E_y + U_{s7}E_z)\Big]e^{-i(x_s\phi'' + y_s\sin\phi'')}d\phi''\Big\} \\
&= \frac{q_s}{m_s\omega_{cs}\omega}\sum_m J_m e^{i(x_s+m)\phi'}\Big\{\int^{\phi'}\Big[(U_{s1}E_x + U_{s2}E_y + U_{s3}E_z)\frac{1}{2}(e^{i\phi''} + e^{-i\phi''}) + U_{s4}E_y\frac{-i}{2}(e^{i\phi''} - e^{-i\phi''}) \\
&\quad +(U_{s5}E_x + U_{s6}E_y + U_{s7}E_z)\Big]\sum_n J_n e^{-i(x_s+n)\phi''}d\phi''\Big\} \\
&= \frac{q_s}{m_s\omega_{cs}\omega}\sum_m J_m e^{i(x_s+m)\phi'}\sum_n J_n\Big\{(U_{s1}E_x + U_{s2}E_y + U_{s3}E_z)\frac{1}{2}\Big[\frac{ie^{-i(x_s+n-1)\phi'}}{(x_s+n-1)} + \frac{ie^{-i(x_s+n+1)\phi'}}{(x_s+n+1)}\Big] \\
&\quad +U_{s4}E_y\frac{-i}{2}\Big[\frac{ie^{-i(x_s+n-1)\phi'}}{(x_s+n-1)} - \frac{ie^{-i(x_s+n+1)\phi'}}{(x_s+n+1)}\Big] + (U_{s5}E_x + U_{s6}E_y + U_{s7}E_z)\frac{ie^{-i(x_s+n)\phi'}}{(x_s+n)}\Big\} \\
&= \frac{iq_s}{m_s\omega_{cs}\omega}\sum_{m,n} J_m\frac{e^{-i(n-m)\phi'}}{(x_s+n)}\Big\{(U_{s1}E_x + U_{s2}E_y + U_{s3}E_z)\frac{n}{y_s}J_n + U_{s4}E_y iJ_n' + \\
&\quad (U_{s5}E_x + U_{s6}E_y + U_{s7}E_z)J_n\Big\},
\end{aligned}
$$

And thus we have

$$
\begin{aligned}
\boldsymbol{J}_s^m &= q_s n_{s0}\int dv^3\boldsymbol{v}f_s^m = q_s n_{s0}\int dv^3\begin{pmatrix} v_\perp'(e^{i\phi'} + e^{-i\phi'})/2 + v_{dsx} \\ -iv_\perp'(e^{i\phi'} - e^{-i\phi'})/2 + v_{dsy} \\ v_\parallel' \end{pmatrix} f_s^m \\
&= \frac{iq_s^2 n_{s0}}{m_s\omega_{cs}\omega}\sum_{m,n}\int dv^3\frac{e^{-i(n-m)\phi'}}{(x_s+n)}\Big\{(U_{s1}E_x + U_{s2}E_y + U_{s3}E_z)\frac{n}{y_s}J_n + U_{s4}E_y iJ_n' + \\
&\quad (U_{s5}E_x + U_{s6}E_y + U_{s7}E_z)J_n\Big\}\begin{pmatrix} v_\perp' mJ_m/y_s + v_{dsx}J_m \\ -iv_\perp' J_m' + v_{dsy}J_m \\ v_\parallel' J_m \end{pmatrix} \\
&= \frac{iq_s^2 n_{s0}}{m_s\omega_{cs}\omega}\sum_n\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v_\perp' dv_\perp' dv_\parallel'}{(x_s+n)}\Big\{(\frac{n}{y_s}U_{s1} + U_{s5})J_nE_x + (\frac{n}{y_s}J_nU_{s2} + iJ_n'U_{s4} + J_nU_{s6})E_y + \\
&\quad (\frac{n}{y_s}U_{s3} + U_{s7})J_nE_z\Big\}\begin{pmatrix} (v_\perp'\frac{n}{y_s} + v_{dsx})J_n \\ (-iv_\perp' J_n' + v_{dsy}J_n) \\ v_\parallel' J_n \end{pmatrix} \\
&= \frac{iq_s^2 n_{s0}}{m_s\omega_{cs}\omega}\sum_n\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v_\perp' dv_\perp' dv_\parallel'}{(x_s+n)}\boldsymbol{\Pi}_s\cdot\boldsymbol{E}, \qquad (54)
\end{aligned}
$$

with

$$
\boldsymbol{\Pi}_s = \begin{pmatrix}
(\frac{n}{y_s}U_{s1} + U_{s5})(v_\perp'\frac{n}{y_s} + v_{dsx})J_n^2 & (\frac{n}{y_s}J_nU_{s2} + iJ_n'U_{s4} + J_nU_{s6})(v_\perp'\frac{n}{y_s} + v_{dsx})J_n & (\frac{n}{y_s}U_{s3} + U_{s7})(v_\perp'\frac{n}{y_s} + v_{dsx})J_n^2 \\
(\frac{n}{y_s}U_{s1} + U_{s5})(-iv_\perp'J_n' + v_{dsy}J_n)J_n & (\frac{n}{y_s}J_nU_{s2} + iJ_n'U_{s4} + J_nU_{s6})(-iv_\perp'J_n' + v_{dsy}J_n) & (\frac{n}{y_s}U_{s3} + U_{s7})(-iv_\perp'J_n' + v_{dsy}J_n)J_n \\
(\frac{n}{y_s}U_{s1} + U_{s5})v_\parallel'J_n^2 & (\frac{n}{y_s}J_nU_{s2} + iJ_n'U_{s4} + J_nU_{s6})v_\parallel'J_n & (\frac{n}{y_s}U_{s3} + U_{s7})v_\parallel'J_n^2
\end{pmatrix}. \qquad (55)
$$

Write out each terms:

- $\Pi_{11} = (\frac{n}{y_s}U_{s1} + U_{s5})(v_\perp'\frac{n}{y_s} + v_{dsx})J_n^2 = (\frac{n}{y_s}[(\omega - k_zv_\parallel')\frac{\partial f_{s0}}{\partial v_\perp'} + k_zv_\perp'\frac{\partial f_{s0}}{\partial v_\parallel'}] + k_zv_{dsx}\frac{\partial f_{s0}}{\partial v_\parallel'})(v_\perp'\frac{n}{y_s} + v_{dsx})J_n^2$ $= J_n^2[\frac{n\omega_{cs}}{k_xv_\perp'}(\omega - k_zv_\parallel')\frac{\partial f_{s0}}{\partial v_\perp'} + k_z(\frac{n\omega_{cs}}{k_x} + v_{dsx})\frac{\partial f_{s0}}{\partial v_\parallel'}](\frac{n\omega_{cs}}{k_x} + v_{dsx})$. [agree with Umeda18, except that his four terms can cancel]
- $\Pi_{12} = (\frac{n}{y_s}J_nU_{s2} + iJ_n'U_{s4} + J_nU_{s6})(v_\perp'\frac{n}{y_s} + v_{dsx})J_n = (\frac{n}{y_s}J_nk_xv_{dsy}\frac{\partial f_{s0}}{\partial v_\perp'} + iJ_n'[(\omega - k_zv_\parallel' - k_xv_{dsx})\frac{\partial f_{s0}}{\partial v_\perp'} + k_zv_\perp'\frac{\partial f_{s0}}{\partial v_\parallel'}] +$

$J_n k_z v_{dsy} \frac{\partial f_{s0}}{\partial v'_\parallel})(v'_\perp \frac{n}{y_s} + v_{dsx})J_n$
$= J_n^2 v_{dsy}(\frac{n\omega_{cs}}{k_x} + v_{dsx})(\frac{n\omega_{cs}}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp} + k_z \frac{\partial f_{s0}}{\partial v'_\parallel}) + iJ_nJ'_n(\frac{n\omega_{cs}}{k_x} + v_{dsx})[(\omega - k_z v'_\parallel - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\perp} + k_z v'_\perp \frac{\partial f_{s0}}{\partial v'_\parallel}]$. [agree with Umeda18]

- $\Pi_{21} = (\frac{n}{y_s}U_{s1} + U_{s5})(-iv'_\perp J'_n + v_{dsy}J_n)J_n = (\frac{n}{y_s}[(\omega - k_z v'_\parallel)\frac{\partial f_{s0}}{\partial v'_\perp} + k_z v'_\perp \frac{\partial f_{s0}}{\partial v'_\parallel}] + k_z v_{dsx}\frac{\partial f_{s0}}{\partial v'_\parallel})(-iv'_\perp J'_n + v_{dsy}J_n)J_n$
$= (-iv'_\perp J_n J'_n + v_{dsy}J_n^2)[\frac{n\omega_{cs}}{k_x v'_\perp}(\omega - k_z v'_\parallel)\frac{\partial f_{s0}}{\partial v'_\perp} + \frac{n\omega_{cs}}{k_x}k_z \frac{\partial f_{s0}}{\partial v'_\parallel} + k_z v_{dsx}\frac{\partial f_{s0}}{\partial v'_\parallel}]$. [agree with Umeda18, except that his two terms can cancel]
- $\Pi_{22} = (\frac{n}{y_s}J_nU_{s2} + iJ'_nU_{s4} + J_nU_{s6})(-iv'_\perp J'_n + v_{dsy}J_n) = (\frac{n}{y_s}J_n k_x v_{dsy}\frac{\partial f_{s0}}{\partial v'_\perp} + iJ'_n[(\omega - k_z v'_\parallel - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\perp} + k_z v'_\perp \frac{\partial f_{s0}}{\partial v'_\parallel}] +$
$J_n k_z v_{dsy}\frac{\partial f_{s0}}{\partial v'_\parallel})(-iv'_\perp J'_n + v_{dsy}J_n)$
$= J_n v_{dsy}(-iv'_\perp J'_n + v_{dsy}J_n)[\frac{n\omega_{cs}}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp} + k_z\frac{\partial f_{s0}}{\partial v'_\parallel}] + iJ'_n(-iv'_\perp J'_n + v_{dsy}J_n)[(\omega - k_z v'_\parallel - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\perp} + k_z v'_\perp \frac{\partial f_{s0}}{\partial v'_\parallel}]$. [agree with Umeda18]
- $\Pi_{13} = (\frac{n}{y_s}U_{s3} + U_{s7})(v'_\perp \frac{n}{y_s} + v_{dsx})J_n^2 = (\frac{n}{y_s}[k_x v'_\parallel \frac{\partial f_{s0}}{\partial v'_\perp} - k_x v'_\perp \frac{\partial f_{s0}}{\partial v'_\parallel}] + (\omega - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\parallel})(v'_\perp \frac{n}{y_s} + v_{dsx})J_n^2$
$= J_n^2(\frac{n\omega_{cs}}{k_x} + v_{dsx})[n\omega_{cs}\frac{v'_\parallel}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp} + (\omega - n\omega_{cs} - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\parallel}]$. [agree with Umeda18]
- $\Pi_{31} = (\frac{n}{y_s}U_{s1} + U_{s5})v'_\parallel J_n^2 = (\frac{n}{y_s}[(\omega - k_z v'_\parallel)\frac{\partial f_{s0}}{\partial v'_\perp} + k_z v'_\perp \frac{\partial f_{s0}}{\partial v'_\parallel}] + k_z v_{dsx}\frac{\partial f_{s0}}{\partial v'_\parallel})v'_\parallel J_n^2$
$= v'_\parallel J_n^2[\frac{n\omega_{cs}}{k_x v'_\perp}(\omega - k_z v'_\parallel)\frac{\partial f_{s0}}{\partial v'_\perp} + \frac{n\omega_{cs}}{k_x}k_z\frac{\partial f_{s0}}{\partial v'_\parallel} + k_z v_{dsx}\frac{\partial f_{s0}}{\partial v'_\parallel}]$. [agree with Umeda18, except that his two terms can cancel]
- $\Pi_{23} = (\frac{n}{y_s}U_{s3} + U_{s7})J_n(-iv'_\perp J'_n + v_{dsy}J_n) = (\frac{n}{y_s}[k_x v'_\parallel \frac{\partial f_{s0}}{\partial v'_\perp} - k_x v'_\perp \frac{\partial f_{s0}}{\partial v'_\parallel}] + (\omega - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\parallel})J_n(-iv'_\perp J'_n + v_{dsy}J_n)$
$= (v_{dsy}J_n^2 - iv'_\perp J_n J'_n)[n\omega_{cs}\frac{v'_\parallel}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp} + (\omega - n\omega_{cs} - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\parallel}]$. [agree with Umeda18]
- $\Pi_{32} = (\frac{n}{y_s}J_nU_{s2} + iJ'_nU_{s4} + J_nU_{s6})v'_\parallel J_n = (\frac{n}{y_s}J_n k_x v_{dsy}\frac{\partial f_{s0}}{\partial v'_\perp} + iJ'_n[(\omega - k_z v'_\parallel - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\perp} + k_z v'_\perp \frac{\partial f_{s0}}{\partial v'_\parallel}] + J_n k_z v_{dsy}\frac{\partial f_{s0}}{\partial v'_\parallel})v'_\parallel J_n$
$= v'_\parallel J_n^2(\frac{n\omega_{cs}}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp} + k_z\frac{\partial f_{s0}}{\partial v'_\parallel})v_{dsy} + iv'_\parallel J_n J'_n[(\omega - k_z v'_\parallel - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\perp} + k_z v'_\perp \frac{\partial f_{s0}}{\partial v'_\parallel}]$. [agree with Umeda18]
- $\Pi_{33} = (\frac{n}{y_s}U_{s3} + U_{s7})v'_\parallel J_n^2 = (\frac{n}{y_s}[k_x v'_\parallel \frac{\partial f_{s0}}{\partial v'_\perp} - k_x v'_\perp \frac{\partial f_{s0}}{\partial v'_\parallel}] + (\omega - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\parallel})v'_\parallel J_n^2$
$= v'_\parallel J_n^2[n\omega_{cs}\frac{v'_\parallel}{v'_\perp}\frac{\partial f_{s0}}{\partial v'_\perp} + (\omega - n\omega_{cs} - k_x v_{dsx})\frac{\partial f_{s0}}{\partial v'_\parallel}]$. [agree with Umeda18]

Note: $x_s = -\frac{\omega - k_\parallel v'_\parallel - k_x v_{dsx} + i\nu_s}{\omega_{cs}}$ and $y_s = \frac{k_x v'_\perp}{\omega_{cs}}$.

In the $v_{dsy} = 0$ and $\nu_s = 0$ limit, our result reduces to exactly the same result as the Eq.(25) of Ref.[11]. By further setting $v_{dsx} = 0$, our result reduces to the standard without across magnetic field drifts one in Ref.[6] and the non-relativistic case in Ref.[3]. We should also note that when $v_{dsx,y} \neq 0$, the matrix elements Π_{ij} are not symmetric or antisymmetric any more.

The $\boldsymbol{Q}$ in electromagnetic dispersion relation is

$$\begin{aligned}
\boldsymbol{Q} &= -\frac{\boldsymbol{\sigma}}{i\omega\epsilon_0} \\
&= \sum_{s=u}\Big\{-\frac{\omega_{ps}^2}{\omega^2}\boldsymbol{I} + \frac{\omega_{ps}^2}{\omega^2}\int dv^3 \frac{\big[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0}}{\partial \boldsymbol{v}}) + (\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial \boldsymbol{v}})(\boldsymbol{v}\boldsymbol{v})\big]}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu_s)}\Big\} \\
&\quad + \sum_{s=m}\frac{\omega_{ps}^2}{\omega^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv'_\parallel}{(\omega - k_\parallel v'_\parallel - k_x v_{dsx} + i\nu_s - n\omega_{cs})}\boldsymbol{\Pi}_s.
\end{aligned} \tag{56}$$

2.4. Darwin model case

For Darwin model, we still have $\boldsymbol{B} = (\boldsymbol{k}\times\boldsymbol{E})/\omega$, and thus the calculation of current $\boldsymbol{J} = \boldsymbol{J}^u + \boldsymbol{J}^m$ is exactly the same as in Eq.(45), i.e., the kinetic solutions to the distribution function f_s and current $\boldsymbol{J}$ do not need change as in the above electromagnetic model, which simplify our derivation a lot.

We discuss the change of the field equation here. In Fourier space, we have

$$\boldsymbol{E}_T = \boldsymbol{E} - \frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E})}{k^2} = (\boldsymbol{I} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2})\cdot\boldsymbol{E}, \quad \boldsymbol{E}_L = \frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E})}{k^2} = (\frac{\boldsymbol{k}\boldsymbol{k}}{k^2})\cdot\boldsymbol{E}, \tag{57}$$

which can satisfied $\boldsymbol{k}\cdot\boldsymbol{E}_T = 0$, $\boldsymbol{k}\times\boldsymbol{E}_L = 0$ and $\boldsymbol{E} = \boldsymbol{E}_T + \boldsymbol{E}_L$. We have used the tensor relation $(\boldsymbol{U}\boldsymbol{V})\cdot\boldsymbol{W} = \boldsymbol{U}(\boldsymbol{V}\cdot\boldsymbol{W})$ and $\boldsymbol{W}\cdot(\boldsymbol{U}\boldsymbol{V}) = (\boldsymbol{W}\cdot\boldsymbol{U})\boldsymbol{V}$, where $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$ are vectors. The field equations we needed are

$$-i\omega\boldsymbol{E}_L = ic^2\boldsymbol{k}\times\Big(\frac{\boldsymbol{k}\times\boldsymbol{E}}{\omega}\Big) - \boldsymbol{J}/\epsilon_0, \tag{58}$$

i.e.,

$$\underbrace{(\frac{\boldsymbol{k}\boldsymbol{k}}{k^2})\cdot\boldsymbol{E}}_{\boldsymbol{I}\cdot\boldsymbol{E}\text{ in electromagnetic model}} + \frac{c^2}{\omega^2}(\boldsymbol{k}\boldsymbol{k} - k^2\boldsymbol{I})\cdot\boldsymbol{E} + \boldsymbol{Q}\cdot\boldsymbol{E} = 0, \tag{59}$$

or

$$\boldsymbol{I}\cdot\boldsymbol{E} + \underbrace{\Big(\frac{k^2c^2}{\omega^2} + 1\Big)}_{\frac{k^2c^2}{\omega^2}\text{ in electromagnetic model}}\Big(\frac{\boldsymbol{k}\boldsymbol{k}}{k^2} - \boldsymbol{I}\Big)\cdot\boldsymbol{E} + \boldsymbol{Q}\cdot\boldsymbol{E} = 0, \tag{60}$$

where the only change from the electromagnetic model is that the $\boldsymbol{I}\cdot\boldsymbol{E}$ term is changed to be $(\frac{\boldsymbol{k}\boldsymbol{k}}{k^2})\cdot\boldsymbol{E}$, or $\frac{k^2c^2}{\omega^2}$ term is changed to be $\frac{k^2c^2}{\omega^2} + 1$. And thus the Darwin dispersion relation is

$$|\boldsymbol{D}(\omega,\boldsymbol{k})| = |\boldsymbol{K}(\omega,\boldsymbol{k}) + \underbrace{\Big(\frac{k^2c^2}{\omega^2} + 1\Big)}_{\frac{k^2c^2}{\omega^2}\text{ in electromagnetic model}}\Big(\frac{\boldsymbol{k}\boldsymbol{k}}{k^2} - \boldsymbol{I}\Big)| = 0, \tag{61}$$

Eqs.(43), (48) and (61) with $\boldsymbol{Q}$ in (56) are our starting electrostatic, electromagnetic and Darwin dispersion relations with drift across magnetic field. The above dispersion relations are valid for arbitrary non-relativistic distribution functions. Later, we will limit our study to treat a special case of the distribution function f_{s0}.

3. The Dispersion Relation for Extend Maxwellian Distribution

The extend Maxwellian distribution here means bi-Maxwellian distribution with loss cone, parallel and perpendicular drifts and ring beam. We model it use the following equilibrium distribution function.

3.1. Equilibrium distribution function

We assume equilibrium distribution function $F_{s0}(v'_\parallel, v'_\perp) = n_{s0}f_{s0}(v'_\parallel, v'_\perp)$, with $v'_\parallel = v_z$, $v'_\perp = \sqrt{(v_x - v_{dsx})^2 + (v_y - v_{dsy})^2}$, and

$$\begin{aligned} f_{s0}(v'_\parallel, v'_\perp) &= f_{s0z}(v'_\parallel)f_{s0\perp}(v'_\perp) \\ &= \frac{1}{\pi^{3/2}v_{zts}v_{\perp ts}^2}\exp\Big[-\frac{(v'_\parallel - v_{dsz})^2}{v_{zts}^2}\Big]\Big\{\frac{r_{sa}}{A_{sa}}\exp\Big[-\frac{(v'_\perp - v_{dsr})^2}{v_{\perp ts}^2}\Big] + \frac{r_{sb}}{\alpha_sA_{sb}}\exp\Big[-\frac{(v'_\perp - v_{dsr})^2}{\alpha_sv_{\perp ts}^2}\Big]\Big\}, \end{aligned} \tag{62}$$

where $r_{sa} = \Big(\frac{1-\alpha_s\Delta_s}{1-\alpha_s}\Big)$ and $r_{sb} = \Big(\frac{-\alpha_s+\alpha_s\Delta_s}{1-\alpha_s}\Big)$, and

$$A_{s\sigma} = \exp\Big(-\frac{v_{dsr}^2}{v_{\perp ts\sigma}^2}\Big) + \frac{\sqrt{\pi}v_{dsr}}{v_{\perp ts\sigma}}\mathrm{erfc}\Big(-\frac{v_{dsr}}{v_{\perp ts\sigma}}\Big), \quad \text{for } \sigma = a, b, \tag{63}$$

with $v_{\perp tsa} = v_{\perp ts}$ and $v_{\perp tsb} = \sqrt{\alpha_s} v_{\perp ts}$.

That is, the equilibrium distribution function f_{s0} is separated to two sub-distributions: $f_{s0} = \sum_{\sigma=a,b} r_{s\sigma} f_{s0\sigma} = f_{s0z}(v'_{\parallel}) \sum_{\sigma=a,b} r_{s\sigma} f_{s0\perp\sigma}$, $f_{s0\sigma} = f_{s0z}(v'_{\parallel}) f_{s0\perp\sigma}(v'_{\perp})$, $f_{s0\perp} = \sum_{\sigma=a,b} r_{s\sigma} f_{s0\perp\sigma}$,

$$f_{s0z}(v'_{\parallel}) = \frac{1}{\pi^{1/2} v_{zts}} \exp\Big[-\frac{(v'_{\parallel} - v_{dsz})^2}{v_{zts}^2}\Big] = \frac{1}{\pi^{1/2} v_{zts}} \exp\Big[-\frac{(v_z - v_{dsz})^2}{v_{zts}^2}\Big], \tag{64}$$

and

$$\begin{aligned} f_{s0\perp\sigma}(v'_{\perp}) &= \frac{1}{\pi A_{s\sigma} v_{\perp ts\sigma}^2} \exp\Big[-\frac{(v'_{\perp} - v_{dsr})^2}{v_{\perp ts\sigma}^2}\Big] \\ &= \frac{1}{\pi A_{s\sigma} v_{\perp ts\sigma}^2} \exp\Big[-\frac{(\sqrt{(v_x - v_{dsx})^2 + (v_y - v_{dsy})^2} - v_{dsr})^2}{v_{\perp ts\sigma}^2}\Big], \end{aligned} \tag{65}$$

where v_{dsx}, v_{dsy}, v_{dsz} are the drift velocities in x (perpendicular 1), y (perpendicular 2) and z (parallel) directions, respectively. And, v_{dsr} is the perpendicular ring beam velocity, and $\mathrm{erfc}(-x) = 1 - \mathrm{erf}(-x) = 1 + \mathrm{erf}(x)$ is the complementary error function, with $\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$ and $\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. The v_{zts} and $v_{\perp ts\sigma}$ are the parallel and perpendicular thermal velocities and corresponding temperatures are $T_{zs} = \frac{1}{2} k_B m_s v_{zts}^2$ and $T_{\perp s\sigma} = \frac{1}{2} k_B m_s v_{\perp ts\sigma}^2$. We define the temperature anisotropic $\lambda_{Ts\sigma} = T_{zs}/T_{\perp s\sigma}$. The parameters Δ_s and α_s determine the depth and size of the loss-cone [Note: Actually, the user can also use two different specieses to represent the loss cone distribution, with one of them has negative density. Then, they can also have different v_{dsx}, v_{dsy} and v_{dsr}. Many complicated distribution function can be constructed based on our model, which we leave it to the user. For example, Ref.[17] constructed the shell distribution based on ring beam model.]. Here, $\Delta_s \in [0, 1]$, for max loss cone and no loss cone. If $\Delta_s = 1$ or $\alpha_s = 1$, i.e., $r_{sa} = 1$ and $r_{sb} = 0$, the above equation reduced to no loss cone case. Note $\int f_{s0} d\boldsymbol{v} = \int f_{s0z} d\boldsymbol{v} = \int f_{s0\perp} d\boldsymbol{v} = \int f_{s0\sigma} d\boldsymbol{v} = \int f_{s0\perp\sigma} d\boldsymbol{v} = 1$.

We will use the same distribution in magnetized and unmagnetized versions to simplified the notations. For unmagnetized version, to remove the troublesome integral, we study only $v_{dsr} = 0$. How to justify the assumed distribution to the realistic physics problem is left to the users. For example, the linearized equation (i.e., the dispersion relations) solver can also run when the zero order current and charge density are not zero. Thus, the user should justify whether it is reasonable.

Several $\partial f_{s0}/\partial \boldsymbol{v}$ terms are particularly useful for further discussions:

$$\begin{aligned} \frac{\partial f_{s0}}{\partial v'_{\parallel}} &= -\frac{2(v'_{\parallel} - v_{dsz})}{v_{zts}^2} f_{s0} = -2(v'_{\parallel} - v_{dsz}) \sum_{\sigma=a,b} \frac{r_{s\sigma} f_{s0\sigma}}{v_{zts}^2}, \\ \frac{\partial f_{s0}}{\partial v'_{\perp}} &= -2(v'_{\perp} - v_{dsr}) \sum_{\sigma=a,b} \frac{r_{s\sigma} f_{s0\sigma}}{v_{\perp ts\sigma}^2}, \\ \frac{\partial f_{s0}}{\partial v_x} &= -2\Big[1 - \frac{v_{dsr}}{\sqrt{(v_x - v_{dsx})^2 + (v_y - v_{dsy})^2}}\Big](v_x - v_{dsx}) \sum_{\sigma=a,b} \frac{r_{s\sigma} f_{s0\sigma}}{v_{\perp ts\sigma}^2}, \\ \frac{\partial f_{s0}}{\partial v_y} &= -2\Big[1 - \frac{v_{dsr}}{\sqrt{(v_x - v_{dsx})^2 + (v_y - v_{dsy})^2}}\Big](v_y - v_{dsx}) \sum_{\sigma=a,b} \frac{r_{s\sigma} f_{s0\sigma}}{v_{\perp ts\sigma}^2}, \\ \frac{\partial f_{s0}}{\partial v_z} &= -\frac{2(v_z - v_{dsz})}{v_{zts}^2} f_{s0} = -2(v_z - v_{dsz}) \sum_{\sigma=a,b} \frac{r_{s\sigma} f_{s0\sigma}}{v_{zts}^2}, \end{aligned}$$

where we have used that $f_{s0} = \sum_{\sigma=a,b} r_{s\sigma} f_{s0\sigma}$.

3.2. Notations

Note the definition of v_{ts}, i.e., $v_{ts} = \sqrt{\frac{2k_B T_s}{m_s}}$, not $v_{ts} = \sqrt{\frac{k_B T_s}{m_s}}$ as in Ref.[12]. Other notations: $\omega_{ps}^2 = \frac{n_{s0} q_s^2}{\epsilon_0 m_s}$, $\omega_p^2 = \sum_s \omega_{ps}^2$, $\Omega_s = \frac{q_s B_0}{m_s}$, $\lambda_{Ds}^2 = \frac{\epsilon_0 k_B T_{zs}}{n_{s0} q_s^2}$, $a_{s\sigma} = \sqrt{2} k_{\perp} \rho_{cs\sigma}$, $b_{s\sigma} = \frac{v_{dsr}}{v_{\perp ts\sigma}}$, $\rho_{cs\sigma} = \sqrt{\frac{k_B T_{s\perp\sigma}}{m_s}} \frac{1}{\Omega_s} = \frac{v_{\perp ts\sigma}}{\sqrt{2}\Omega_s}$. Note: $\Omega_s < 0$

for electron ($q_s < 0$), and thus also $\rho_{cs\sigma} < 0$ and $a_{s\sigma} < 0$. [Note the definition in previous bi-Maxwellian version: $a_s = k_\perp \rho_{cs}$, $b_s = k_\perp^2 \rho_{cs}^2$, $\rho_{cs} = \sqrt{\frac{k_B T_{s\perp}}{m_s}} \frac{1}{\Omega_s} = \frac{v_{\perp ts}}{\sqrt{2}\Omega_s}$, $\Gamma_n(b) = I_n(b)e^{-b}$, I_n is the modified Bessel function.]

We use 'ES3D' or 'ES', 'EM3D' or 'EM', 'Darwin' to represent the electrostatic, electromagnetic and Darwin version, respectively. And with suffix '-U' and '-M' to represent the corresponding unmagnetized and magnetized species.

3.3. Some integrals and functions

Here, we also clarify the correctly treat of the plasma dispersion function for $k_z \le 0$. The standard definition of plasma dispersion function $Z(\zeta)$ is

$$
\begin{aligned}
Z(\zeta) &= \frac{1}{\sqrt{\pi}} \int_C \frac{1}{x-\zeta} e^{-x^2} dx, \\
\frac{dZ}{d\zeta} &= -2(1+\zeta Z),
\end{aligned}
$$

where C is the Landau contour to analytic continuation from $Im(\zeta) > 0$ to $Im(\zeta) \le 0$. However, for our usage the above definition to plasma dispersion function with $\zeta_s = \frac{\omega - k_{cs}}{k_{ts}}$ is only correct when $k_{ts} > 0$, where for example $k_{ts} = k_\parallel v_{zts}$. To correctly capture the physics, one should be careful of the analytic continuation for both $k_{ts} > 0$ and $k_{ts} \le 0$. From standard derivation of the dispersion relations based on Laplacian transformation instead of Fourier transformation which considered the causality, say Ref.[6], we obtain the correct analytic continuation one should be

$$
Z(\zeta_s) = \begin{cases} i\sqrt{\pi} e^{-\zeta_s^2} + \frac{1}{\sqrt{\pi}} P \int_{-\infty}^{\infty} \frac{1}{x-\zeta_s} e^{-x^2} dx, & k_{ts} > 0, \\ -\frac{1}{\zeta_s}, & k_{ts} = 0, \\ -i\sqrt{\pi} e^{-\zeta^2} + \frac{1}{\sqrt{\pi}} P \int_{-\infty}^{\infty} \frac{1}{x-\zeta_s} e^{-x^2} dx, & k_{ts} < 0, \end{cases} \tag{66}
$$

where P refers to the principal value integral. The corresponding J-pole expansion should also be modified for $k_{ts} \le 0$, which will be discussed later. We can find easily that for $k_{ts} < 0$, one can use $Z(\zeta_s) = -Z(-\zeta_s)$, which will simplify our later usage.

To simplify the notation, we define also the function $Z_p(\zeta_s, k_{ts})$, and have

$$
\begin{aligned}
Z_p(\zeta_s, k_{ts}) &= \frac{1}{\sqrt{\pi}} \int_C \frac{x^p}{x-\zeta_s} e^{-x^2} dx, \\
Z_0(\zeta_s, k_{ts}) &= Z(\zeta_s) \\
Z_1(\zeta_s, k_{ts}) &= [1+\zeta_s Z_0(\zeta_s)], \\
Z_2(\zeta_s, k_{ts}) &= \zeta_s[1+\zeta_s Z_0(\zeta_s)], \\
Z_3(\zeta_s, k_{ts}) &= \frac{1}{2} + \zeta_s^2[1+\zeta_s Z_0(\zeta_s)],
\end{aligned}
$$

where we have used $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. Later, we will find by using $Z_{0,1,2,3}$ will simplify the notations a lot, and also will make the matrix linear transformation quite straightforward, say,

$$
\int_{-\infty}^{\infty} \frac{f_{sz}}{\omega_{sn} - k_z(v_\parallel - v_{dsz})} dv_\parallel = -\frac{Z(\zeta_{sn})}{k_z v_{zts}} = -\frac{Z_0(\zeta_{sn})}{k_z v_{zts}}, \tag{67}
$$

$$
\int_{-\infty}^{\infty} \frac{k_z(v_\parallel - v_{dsz}) f_{sz}}{\omega_{sn} - k_z(v_\parallel - v_{dsz})} dv_\parallel = -[1+\zeta_{sn} Z(\zeta_{sn})] = -Z_1(\zeta_{sn}), \tag{68}
$$

$$
\int_{-\infty}^{\infty} \frac{[k_z(v_\parallel - v_{dsz})]^2 f_{sz}}{\omega_{sn} - k_z(v_\parallel - v_{dsz})} dv_\parallel = -\omega_{sn}[1+\zeta_{sn} Z(\zeta_{sn})] = -k_z v_{zts} Z_2(\zeta_{sn}), \tag{69}
$$

$$
\int_{-\infty}^{\infty} \frac{[k_z(v_\parallel - v_{dsz})]^3 f_{sz}}{\omega_{sn} - k_z(v_\parallel - v_{dsz})} dv_\parallel = -\frac{k_z^2 v_{zts}^2}{2} - \omega_{sn}^2[1+\zeta_{sn} Z(\zeta_{sn})] = -k_z^2 v_{zts}^2 Z_3(\zeta_{sn}). \tag{70}
$$

For Bessel function, we have (p256 of Ref.[3]): $J_n'(x) = [J_{n-1}(x) - J_{n+1}(x)]/2$, $nJ_n(x)/x = [J_{n-1}(x) + J_{n+1}(x)]/2$, $\sum_{n=-\infty}^{\infty} J_n^2 = 1$, $\sum_{n=-\infty}^{\infty} J_n J_n' = 0$, $\sum_{n=-\infty}^{\infty} nJ_n^2 = 0$, $\sum_{n=-\infty}^{\infty}(J_n')^2 = \frac{1}{2}$, $\sum_{n=-\infty}^{\infty} \frac{n^2 J_n^2(x)}{x^2} = \frac{1}{2}$, $\sum_{n=-\infty}^{\infty} nJ_nJ_n' = 0$, $J_n(-x) = J_{-n}(x) = (-1)^n J_n(x)$, and $\sum_{m\neq 0,n=-\infty}^{\infty} J_n J_{n+m} = 0$.

We define [12]

$$
\begin{aligned}
A_n(a,b,c) &\equiv \int_0^\infty J_n^2(ay)e^{-(y-b)^2}(y-c)dy,\\
B_n(a,b,c) &\equiv \int_0^\infty J_n(ay)J_n'(ay)e^{-(y-b)^2}y(y-c)dy,\\
C_n(a,b,c) &\equiv \int_0^\infty J_n'^2(ay)e^{-(y-b)^2}y^2(y-c)dy.
\end{aligned}
$$

For our usage $a = \sqrt{2}k_\perp\rho_{cs}$, $b = \frac{v_{dsr}}{v_{\perp ts}}$ and $c = b$ or 0. Here, we calculate A_n, B_n and C_n using numerical integral. When $b = c = 0$, the above integrals reduce to the conventional Maxwellian form with modified Bessel function I_n and I_n', i.e.,

$$
\begin{aligned}
A_n(a,0,0) &= \frac{1}{2}e^{-\frac{a^2}{2}}I_n(\frac{a^2}{2}) = \frac{1}{2}\Gamma_n(\frac{a^2}{2}),\\
B_n(a,0,0) &= \frac{a}{4}e^{-\frac{a^2}{2}}[I_n'(\frac{a^2}{2}) - I_n(\frac{a^2}{2})] = \frac{a}{4}\Gamma_n'(\frac{a^2}{2}),\\
C_n(a,0,0) &= e^{-\frac{a^2}{2}}\Big[\frac{n^2}{2a^2}I_n(\frac{a^2}{2}) - \frac{a^2}{4}I_n'(\frac{a^2}{2}) + \frac{a^2}{4}I_n(\frac{a^2}{2})\Big] = \frac{n^2}{2a^2}\Gamma_n(\frac{a^2}{2}) - \frac{a^2}{4}\Gamma_n'(\frac{a^2}{2}).
\end{aligned}
$$

where $\Gamma_n(b) = e^{-b}I_n(b)$. And $\Gamma_n'(b) = (I_n' - I_n)e^{-b}$, $I_n'(b) = (I_{n+1} + I_{n-1})/2$, $I_{-n} = I_n$. Thus, for species with $v_{dsr} = 0$, we will still use the Bessel function form Γ_n and Γ_n'.

Note, we have

$$
\int_0^\infty J_n^2(y_s)f_{s0\perp\sigma}v_\perp' dv_\perp' = \frac{1}{\pi A_{s\sigma}}A_n(a_{s\sigma}, b_{s\sigma}, 0), \tag{71}
$$

$$
\int_0^\infty J_n^2(y_s)f_{s0\perp\sigma}(v_\perp' - v_{dsr})dv_\perp' = \frac{1}{\pi A_{s\sigma}}A_n(a_{s\sigma}, b_{s\sigma}, b_{s\sigma}), \tag{72}
$$

$$
\int_0^\infty J_n(y_s)J_n'(y_s)f_{s0\perp\sigma}v_\perp'^2 dv_\perp' = \frac{v_{\perp ts\sigma}}{\pi A_{s\sigma}}B_n(a_{s\sigma}, b_{s\sigma}, 0), \tag{73}
$$

$$
\int_0^\infty J_n(y_s)J_n'(y_s)f_{s0\perp\sigma}v_\perp'(v_\perp' - v_{dsr})dv_\perp' = \frac{v_{\perp ts\sigma}}{\pi A_{s\sigma}}B_n(a_{s\sigma}, b_{s\sigma}, b_{s\sigma}), \tag{74}
$$

$$
\int_0^\infty J_n'(y_s)J_n'(y_s)f_{s0\perp\sigma}v_\perp'^3 dv_\perp' = \frac{v_{\perp ts\sigma}^2}{\pi A_{s\sigma}}C_n(a_{s\sigma}, b_{s\sigma}, 0), \tag{75}
$$

$$
\int_0^\infty J_n'(y_s)J_n'(y_s)f_{s0\perp\sigma}v_\perp'^2(v_\perp' - v_{dsr})dv_\perp' = \frac{v_{\perp ts\sigma}^2}{\pi A_{s\sigma}}C_n(a_{s\sigma}, b_{s\sigma}, b_{s\sigma}). \tag{76}
$$

We have checked in Matlab, the speed of numerical integral is also fast, compared to using the Bessel function. To short the notation, we would also use such as $A_n(0) = A_n(a_{s\sigma}, b_{s\sigma}, 0)$ and $A_n(b_{s\sigma}) = A_n(a_{s\sigma}, b_{s\sigma}, b_{s\sigma})$.

For short the notations using $A_{nbs\sigma} = \frac{4}{A_{s\sigma}}A_n(a_{s\sigma}, b_{s\sigma}, b_{s\sigma})$, $B_{nbs\sigma} = \frac{4}{A_{s\sigma}}B_n(a_{s\sigma}, b_{s\sigma}, b_{s\sigma})$, $C_{nbs\sigma} = \frac{4}{A_{s\sigma}}C_n(a_{s\sigma}, b_{s\sigma}, b_{s\sigma})$, $A_{n0\sigma} = \frac{4}{A_{s\sigma}}A_n(a_{s\sigma}, b_{s\sigma}, 0)$, $B_{n0\sigma} = \frac{4}{A_{s\sigma}}B_n(a_{s\sigma}, b_{s\sigma}, 0)$ and $C_{n0\sigma} = \frac{4}{A_{s\sigma}}C_n(a_{s\sigma}, b_{s\sigma}, 0)$.

Note also: $\frac{2}{A_s}\int_0^\infty y^2 e^{-(y-b_s)^2}(y-b_s)dy = 1$ and $\frac{2}{A_s}\int_0^\infty ye^{-(y-b_s)^2}dy = 1$.

Expansion A_n and B_n at $a \to 0$ would be useful. For $|y| \ll 1$ and $n \geq 0$, $J_n(y) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+n+1)}(\frac{y}{2})^{2m+n}$, with $\Gamma(n+1) = n!$ be the Euler Γ function. Thus, we have $J_n(ay) \sim \frac{1}{n!}(\frac{ay}{2})^n - \frac{1}{(n+1)!}(\frac{ay}{2})^{n+2}$ and $J_{n\geq 1}'(ay) \sim \frac{1}{2(n-1)!}(\frac{ay}{2})^{n-1} -$

$\frac{(n+2)}{2(n+1)!}(\frac{ay}{2})^{n+1}$. To $O(a^2)$, we have

$$\begin{aligned}
A_n(a,b,b) &= \int_0^\infty J_n^2(ay)e^{-(y-b)^2}(y-b)dy \sim \begin{cases} \frac{B_s}{2}-\frac{A_s}{4}a^2, & n=0,\\ \frac{A_s}{2}a^2, & n=\pm1,\\ 0, & \text{others},\end{cases}\\
B_n(a,b,b) &\sim \int_0^\infty J_n(ay)J_n'(ay)e^{-(y-b)^2}y(y-b)dy \sim \begin{cases} -\frac{A_s}{4}a, & n=0,\\ \frac{A_s}{8}a, & n=\pm1,\\ 0, & \text{others},\end{cases}\\
A_n(a,b,0) &= \int_0^\infty J_n^2(ay)e^{-(y-b)^2}ydy \sim \begin{cases} \frac{1}{2}A_s-\frac{2b^2+3}{8}a^2A_s+\frac{1}{8}a^2B_s, & n=0,\\ \frac{2b^2+3}{16}a^2A_s-\frac{1}{16}a^2B_s, & n=\pm1,\\ 0, & \text{others},\end{cases}\\
B_n(a,b,0) &\sim \int_0^\infty J_n(ay)J_n'(ay)e^{-(y-b)^2}y^2dy \sim \begin{cases} -\frac{2b^2+3}{8}aA_s+\frac{1}{8}aB_s, & n=0,\\ \frac{2b^2+3}{16}aA_s-\frac{1}{16}aB_s, & n=\pm1,\\ 0, & \text{others},\end{cases}
\end{aligned}$$

where we have used $J_0(ay)\sim 1-(\frac{ay}{2})^2$, $J_1(ay)\sim\frac{ay}{2}$, $J_2(ay)\sim\frac{1}{2}(\frac{ay}{2})^2$, $J_{n\geq3}\sim 0$, $J_0'(ay)\sim-(\frac{ay}{2})$, $J_1'(ay)\sim\frac{1}{2}-\frac{3}{4}(\frac{ay}{2})^2$, $J_2'(ay)\sim\frac{1}{2}(\frac{ay}{2})$, $J_3'(ay)\sim\frac{1}{4}(\frac{ay}{2})^2$, $J_{n\geq4}'\sim 0$, and defined $A_s=2\int_0^\infty e^{-(y-b)^2}ydy=e^{-b^2}+\sqrt{\pi}b[\mathrm{erf}(b)+1]$, $B_s=2\int_0^\infty e^{-(y-b)^2}(y-b)dy=e^{-b^2}$.

3.4. Electrostatic dispersion relation

We derive the electrostatic dispersion relation in this subsection base on Eq.(43) and the distribution function (62). The term

$$k_\parallel\frac{\partial f_{s0}}{\partial v_\parallel'}+\frac{n\omega_{cs}}{v_\perp'}\frac{\partial f_{s0}}{\partial v_\perp'}=-2(v_\parallel'-v_{dsz})k_z\sum_{\sigma=a,b}\frac{r_{s\sigma}f_{s0\sigma}}{v_{zts}^2}-2(v_\perp'-v_{dsr})\frac{n\omega_{cs}}{v_\perp'}\sum_{\sigma=a,b}\frac{r_{s\sigma}f_{s0\sigma}}{v_{\perp tsb}^2},$$

and thus

$$\frac{v_\perp'\Big(k_\parallel\frac{\partial f_{s0}}{\partial v_\parallel'}+\frac{n\omega_{cs}}{v_\perp'}\frac{\partial f_{s0}}{\partial v_\perp'}\Big)}{\omega-k_\parallel v_\parallel'+i\nu_s-k_xv_{dsx}-n\omega_{cs}}=\sum_{\sigma=a,b}\frac{2r_{s\sigma}f_{s0\sigma}}{v_{zts}^2}\frac{-(v_\parallel'-v_{dsz})k_zv_\perp'-(v_\perp'-v_{dsr})n\omega_{cs}\lambda_{Ts\sigma}}{\omega-k_z(v_\parallel'-v_{dsz})-k_zv_{dsz}+i\nu_s-k_xv_{dsx}-n\omega_{cs}},$$

and thus the magnetized $\sum_{s=m}$ term in Eq.(43) is

$$\begin{aligned}
\sum_{s=m}\cdots &= \sum_{s=m}\frac{\omega_{ps}^2}{k^2}\int_{-\infty}^\infty\int_0^\infty\sum_{n=-\infty}^\infty J_n^2(y_s)\sum_{\sigma=a,b}\frac{4\pi r_{s\sigma}f_{s0\sigma}'}{v_{zts}^2}\frac{-v_\parallel'k_zv_\perp'-(v_\perp'-v_{dsr})n\omega_{cs}\lambda_{Ts\sigma}}{\omega-k_zv_\parallel'-k_zv_{dsz}+i\nu_s-k_xv_{dsx}-n\omega_{cs}}dv_\perp'dv_\parallel'\\
&= -\sum_{s=m}\frac{\omega_{ps}^2}{k^2}\int_{-\infty}^\infty\int_0^\infty\sum_{n=-\infty}^\infty J_n^2(y_s)\sum_{\sigma=a,b}\frac{4\pi r_{s\sigma}f_{s0\sigma}'}{v_{zts}^2}\frac{v_\parallel'k_zv_\perp'+(v_\perp'-v_{dsr})n\omega_{cs}\lambda_{Ts\sigma}}{\omega_{sn}-k_zv_\parallel'}dv_\perp'dv_\parallel'\\
&= \sum_{s=m}\frac{\omega_{ps}^2}{k^2}\int_0^\infty\sum_{n=-\infty}^\infty J_n^2(y_s)\sum_{\sigma=a,b}\frac{4\pi r_{s\sigma}f_{s0\perp\sigma}}{v_{zts}^2}\Big\{v_\perp'[1+\zeta_{sn}Z(\zeta_{sn})]+(v_\perp'-v_{dsr})n\omega_{cs}\lambda_{Ts\sigma}\frac{Z(\zeta_{sn})}{k_zv_{zts}}\Big\}dv_\perp'\\
&= \sum_{s=m}\frac{\omega_{ps}^2}{k^2v_{zts}^2}\sum_{n=-\infty}^\infty\sum_{\sigma=a,b}\frac{4r_{s\sigma}}{A_{s\sigma}}\Big\{Z_1(\zeta_{sn})A_n(a_{s\sigma},b_{s\sigma},0)+\frac{n\omega_{cs}\lambda_{Ts\sigma}}{k_zv_{zts}}Z_0(\zeta_{sn})A_n(a_{s\sigma},b_{s\sigma},b_{s\sigma})\Big\}\\
&= \sum_{s=m}\frac{\omega_{ps}^2}{k^2v_{zts}^2}\sum_{n=-\infty}^\infty\sum_{\sigma=a,b}r_{s\sigma}\Big\{Z_1(\zeta_{sn})A_{n0\sigma}+\frac{n\omega_{cs}\lambda_{Ts\sigma}}{k_zv_{zts}}Z_0(\zeta_{sn})A_{nbs\sigma}\Big\},
\end{aligned}\tag{77}$$

We find the above result is the same as in Ref.[13] ring-beam case, except that our new (1) $\omega_{sn}=\omega-k_zv_{dsz}-n\omega_{cs}-k_xv_{dsx}+i\nu_s$, and (2) the summation for loss cone.

For the unmagnetized species

$$
\begin{aligned}
\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial \boldsymbol{v}} &= -2\Big[1-\frac{v_{dsr}}{\sqrt{(v_x-v_{dsx})^2+(v_y-v_{dsy})^2}}\Big]k_x(v_x-v_{dsx})\sum_{\sigma=a,b}\frac{r_{s\sigma}f_{s0\sigma}}{v_{\perp ts\sigma}^2}-2k_z(v_z-v_{dsz})\sum_{\sigma=a,b}\frac{r_{s\sigma}f_{s0\sigma}}{v_{zts}^2}\\
&= -2\sum_{\sigma=a,b}\Big[\Big(1-\frac{b_{s\sigma}}{\sqrt{x'^2+y'^2}}\Big)\frac{k_xx'}{v_{\perp ts\sigma}}+\frac{k_zz'}{v_{zts}}\Big]r_{s\sigma}f_{s0\sigma}\\
&= -2\sum_{\sigma=a,b}\Big[\Big(1-\frac{b_{s\sigma}}{\sqrt{x'^2+y'^2}}\Big)\frac{k_xx'}{v_{\perp ts\sigma}}+\frac{k_zz'}{v_{zts}}\Big]\frac{r_{s\sigma}}{A_{s\sigma}}\frac{1}{\pi^{3/2}v_{zts}v_{\perp ts\sigma}^2}e^{-z'^2-(\sqrt{x'^2+y'^2}-b_{s\sigma})^2},
\end{aligned}
$$

where we have chosen the transformation

$$
\begin{cases} x'=\frac{(v_x-v_{dsx})}{v_{\perp ts\sigma}}\\ y'=\frac{(v_y-v_{dsy})}{v_{\perp ts\sigma}}\\ z'=\frac{(v_z-v_{dsz})}{v_{zts}}\end{cases},\qquad
\begin{cases} v_x=v_{\perp ts\sigma}x'+v_{dsx}\\ v_y=v_{\perp ts\sigma}y'+v_{dsy}\\ v_z=v_{zts}z'+v_{dsz}\end{cases}. \tag{78}
$$

Further consider

$$
\begin{aligned}
\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\nu_s &= \omega-k_xv_x-k_zv_z+i\nu_s\\
&= \omega-k_xv_{\perp ts\sigma}x'-k_zv_{zts}z'-k_xv_{dsx}-k_zv_{dsz}+i\nu_s\\
&= \omega-kv_{ts\sigma}x-k_xv_{dsx}-k_zv_{dsz}+i\nu_s,
\end{aligned} \tag{79}
$$

we need a new transformation to let the two variables $k_xv_{\perp ts\sigma}x'+k_zv_{zts}z'$ change to a single variable $kv_{ts\sigma}x$, which is

$$
\begin{cases} x=\frac{k_xv_{\perp ts\sigma}}{kv_{ts\sigma}}x'+\frac{k_zv_{zts}}{kv_{ts\sigma}}z'\\ y=y'\\ z=\frac{k_zv_{zts}}{kv_{ts\sigma}}x'-\frac{k_xv_{\perp ts\sigma}}{kv_{ts\sigma}}z'\end{cases},\qquad
\begin{cases} x'=\frac{k_xv_{\perp ts\sigma}}{kv_{ts\sigma}}x+\frac{k_zv_{zts}}{kv_{ts\sigma}}z\\ y'=y\\ z'=\frac{k_zv_{zts}}{kv_{ts\sigma}}x-\frac{k_xv_{\perp ts\sigma}}{kv_{ts\sigma}}z\end{cases}, \tag{80}
$$

i.e.,

$$
\begin{cases} v_x=v_{\perp ts\sigma}\frac{k_xv_{\perp ts\sigma}}{kv_{ts\sigma}}x+v_{\perp ts\sigma}\frac{k_zv_{zts}}{kv_{ts\sigma}}z+v_{dsx}\\ v_y=v_{\perp ts\sigma}y+v_{dsy}\\ v_z=v_{zts}\frac{k_zv_{zts}}{kv_{ts\sigma}}x-v_{zts}\frac{k_xv_{\perp ts\sigma}}{kv_{ts\sigma}}z+v_{dsz}\end{cases},\qquad dv_xdv_ydv_z=v_{\perp ts\sigma}^2v_{zts}dxdydz, \tag{81}
$$

where $kv_{ts\sigma}=\sqrt{k_x^2v_{\perp ts\sigma}^2+k_z^2v_{zts}^2}=v_{zts}\sqrt{k_x^2/\lambda_{Ts\sigma}+k_z^2}$. To transform the integral to be able to use the Z function, we need e^{-x^2} term be separate. It is not easy to do so, except only when we set $v_{dsr}=0$, i.e., $b_{s\sigma}=0$ and $A_{s\sigma}=1$. At $v_{dsr}=0$ case, we can have $z'^2+(\sqrt{x'^2+y'^2}-b_{s\sigma})^2=x'^2+y'^2+z'^2=x^2+y^2+z^2$. Thus, to make life easy, we set $v_{dsr}=0$ for unmagnetized study [Note: It is also rare to meet ring beam in unmagnetized case. Thus, using $v_{dsr}=0$ for unmagnetized species will not limit too much to the application of present model. Actually, our present model is much general than Ref.[14] and the non-relativistic case in Ref.[15].].

We define $\zeta_{s\sigma}=\frac{\omega-k_xv_{dsx}-k_zv_{dsz}+i\nu_s}{kv_{ts\sigma}}$, and thus $\omega-k_xv_x-k_zv_z+i\nu_s=kv_{ts\sigma}(\zeta_{s\sigma}-x)$. The unmagnetized $\sum_{s=u}$ term in Eq.(43) is

$$
\begin{aligned}
\sum_{s=u}\cdots &= \sum_{s=u}\frac{\omega_{ps}^2}{k^2}\int dv^3\frac{-2\sum_{\sigma=a,b}\left[\frac{k_xx'}{v_{\perp ts\sigma}}+\frac{k_zz'}{v_{zts}}\right]\frac{r_{s\sigma}}{\pi^{3/2}v_{zts}v_{\perp ts\sigma}^2}e^{-(x^2+y^2+z^2)},}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\nu_s)}\\
&= \sum_{s=u}\frac{\omega_{ps}^2}{k^2}2\sum_{\sigma=a,b}\int_{-\infty}^{\infty}dxdydz\frac{\left[\frac{k^2}{kv_{ts\sigma}}x+\frac{k_xk_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)z\right]\frac{r_{s\sigma}}{\pi^{3/2}}e^{-(x^2+y^2+z^2)},}{kv_{ts\sigma}(\zeta_\sigma-x)}\\
&= \sum_{s=u}\frac{\omega_{ps}^2}{k^2}\sum_{\sigma=a,b}\frac{2r_{s\sigma}}{v_{ts\sigma}^2}\frac{1}{\pi^{3/2}}\int_{-\infty}^{\infty}dxdydz\frac{xe^{-(x^2+y^2+z^2)},}{(\zeta_\sigma-x)}\\
&= \sum_{s=u}\frac{\omega_{ps}^2}{k^2}\sum_{\sigma=a,b}\frac{2r_{s\sigma}}{v_{ts\sigma}^2}[1+\zeta_{s\sigma}Z(\zeta_{s\sigma})]=\sum_{s=u}\frac{\omega_{ps}^2}{k^2}\sum_{\sigma=a,b}\frac{2r_{s\sigma}}{v_{ts\sigma}^2}Z_1(\zeta_{s\sigma}),
\end{aligned} \tag{82}
$$

where we have used $\frac{k_x x'}{v_{\perp ts\sigma}} + \frac{k_z z'}{v_{zts}} = \frac{k_x}{v_{\perp ts\sigma}}[\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}x + \frac{k_z v_{zts}}{kv_{ts\sigma}}z] + \frac{k_z}{v_{zts}}[\frac{k_z v_{zts}}{kv_{ts\sigma}}x - \frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z] = \frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{zts}}{v_{\perp ts\sigma}}z - \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}z = \frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma} - 1)z$. We note that the $\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma} - 1)z$ term is integralled to be zero which makes the result simplified a lot.

Thus we obtain the final electrostatic dispersion relation

$$\begin{aligned} D(\omega, \boldsymbol{k}) &= 1 + \sum_{s=m} \frac{\omega_{ps}^2}{k^2 v_{zts}^2} \sum_{n=-\infty}^{\infty} \sum_{\sigma=a,b} r_{s\sigma}\Big\{Z_1(\zeta_{sn})A_{n0\sigma} + \frac{n\omega_{cs}\lambda_{Ts\sigma}}{k_z v_{zts}} Z_0(\zeta_{sn})A_{nbs\sigma}\Big\} \\ &\quad + \sum_{s=u} \frac{\omega_{ps}^2}{k^2} \sum_{\sigma=a,b} \frac{2r_{s\sigma}}{v_{ts\sigma}^2} Z_1(\zeta_{s\sigma}) = 0. \end{aligned} \tag{83}$$

The above dispersion relation Eq.(83) is very general, except requiring $v_{dsr} = 0$ for unmagnetized species.

3.5. *Electromagnetic dispersion relation*

The electromagnetic case is much complicated than the electrostatic case.

3.5.1. *Unmagnetized terms*

We start from the unmagnetized term first. We use the same transformation for $\boldsymbol{v}$ as in the unmagnetized electrostatic case. Again, we assume $v_{dsr} = 0$ for unmagnetized species, and perform calculation term by term:

$$\begin{aligned} \boldsymbol{k} \cdot \frac{\partial f_{s0}}{\partial \boldsymbol{v}} &= -2 \sum_{\sigma=a,b} \Big[\frac{k_x x'}{v_{\perp ts\sigma}} + \frac{k_z z'}{v_{zts}}\Big] \frac{r_{s\sigma}}{\pi^{3/2} v_{zts} v_{\perp ts\sigma}^2} e^{-z'^2 - (\sqrt{x'^2+y'^2} - b_{s\sigma})^2} \\ &= -2 \sum_{\sigma=a,b} \Big[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma} - 1)z\Big] \frac{r_{s\sigma}}{\pi^{3/2} v_{zts} v_{\perp ts\sigma}^2} e^{-(x^2+y^2+z^2)}, \end{aligned}$$

$$\begin{aligned} \frac{\partial f_{s0}}{\partial v_x} &= -2(v_x - v_{dsx}) \sum_{\sigma=a,b} \frac{r_{s\sigma} f_{s0\sigma}}{v_{\perp ts\sigma}^2} = -2 \sum_{\sigma=a,b} \Big[\frac{k_x}{kv_{ts\sigma}}x + \sqrt{\lambda_{Ts\sigma}}\frac{k_z}{kv_{ts\sigma}}z\Big] \frac{r_{s\sigma}}{\pi^{3/2} v_{zts} v_{\perp ts\sigma}^2} e^{-(x^2+y^2+z^2)}, \\ \frac{\partial f_{s0}}{\partial v_y} &= -2(v_y - v_{dsx}) \sum_{\sigma=a,b} \frac{r_{s\sigma} f_{s0\sigma}}{v_{\perp ts\sigma}^2} = -2 \sum_{\sigma=a,b} \frac{1}{v_{\perp ts\sigma}} y \frac{r_{s\sigma}}{\pi^{3/2} v_{zts} v_{\perp ts\sigma}^2} e^{-(x^2+y^2+z^2)}, \\ \frac{\partial f_{s0}}{\partial v_z} &= -2(v_z - v_{dsz}) \sum_{\sigma=a,b} \frac{r_{s\sigma} f_{s0\sigma}}{v_{zts}^2} = -2 \sum_{\sigma=a,b} \Big[\frac{k_z}{kv_{ts\sigma}}x - \frac{k_x}{kv_{ts\sigma}}\frac{1}{\sqrt{\lambda_{Ts\sigma}}}z\Big] \frac{r_{s\sigma}}{\pi^{3/2} v_{zts} v_{\perp ts\sigma}^2} e^{-(x^2+y^2+z^2)}. \end{aligned}$$

Define $F_{s\sigma} \equiv \frac{r_{s\sigma}}{\pi^{3/2} v_{zts} v_{\perp ts\sigma}^2} e^{-(x^2+y^2+z^2)}$ (note: $A_{s\sigma} = 1$), we have

$$\begin{aligned} (\boldsymbol{v}\frac{\partial f_{s0}}{\partial \boldsymbol{v}}) &= \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} (\frac{\partial f_{s0}}{\partial v_x}, \frac{\partial f_{s0}}{\partial v_y}, \frac{\partial f_{s0}}{\partial v_z}) \\ &= -2 \sum_{\sigma=a,b} F_{s\sigma} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} (\Big[\frac{k_x}{kv_{ts\sigma}}x + \sqrt{\lambda_{Ts\sigma}}\frac{k_z}{kv_{ts\sigma}}z\Big], \frac{1}{v_{\perp ts\sigma}}y, \Big[\frac{k_z}{kv_{ts\sigma}}x - \frac{k_x}{kv_{ts\sigma}}\frac{1}{\sqrt{\lambda_{Ts\sigma}}}z\Big]) \\ &\to -2 \sum_{\sigma=a,b} F_{s\sigma} \begin{pmatrix} \frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_x v_{dsx}}{kv_{ts\sigma}}x + \frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}z^2 & 0 & \frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_z v_{dsx}}{kv_{ts\sigma}}x - \frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}z^2 \\ \frac{k_x v_{dsy}}{kv_{ts\sigma}}x & y^2 & \frac{k_z v_{dsy}}{kv_{ts\sigma}}x \\ \frac{k_x k_z v_{zts}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_x v_{dsz}}{kv_{ts\sigma}}x - \frac{k_z}{kv_{ts\sigma}}\frac{k_x v_{zts}^2}{kv_{ts\sigma}}z^2 & 0 & \frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_z v_{dsz}}{kv_{ts\sigma}}x + \frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}z^2 \end{pmatrix} \end{aligned}$$

Note also:

$$
\begin{aligned}
(\boldsymbol{v}\frac{\partial f_{s0}}{\partial \boldsymbol{v}}) \quad &\neq \quad \begin{pmatrix} \frac{\partial f_{s0}}{\partial v_x} \\ \frac{\partial f_{s0}}{\partial v_y} \\ \frac{\partial f_{s0}}{\partial v_z} \end{pmatrix}(v_x, v_y, v_z) = -2\sum_{\sigma=a,b} F_{s\sigma}\begin{pmatrix} \Big[\frac{k_x}{kv_{ts\sigma}}x + \sqrt{\lambda_{Ts\sigma}}\frac{k_z}{kv_{ts\sigma}}z\Big] \\ \frac{1}{v_{\perp ts\sigma}}y \\ \Big[\frac{k_z}{kv_{ts\sigma}}x - \frac{k_x}{kv_{ts\sigma}}\frac{1}{\sqrt{\lambda_{Ts\sigma}}}z\Big] \end{pmatrix}(v_x, v_y, v_z) \\
&\to \quad -2\sum_{\sigma=a,b} F_{s\sigma}\begin{pmatrix} \frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_x v_{dsx}}{kv_{ts\sigma}}x + \frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}z^2 & \frac{k_x v_{dsy}}{kv_{ts\sigma}}x & \frac{k_x k_z v_{zts}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_x v_{dsz}}{kv_{ts\sigma}}x - \frac{k_z}{kv_{ts\sigma}}\frac{k_x v_{zts}^2}{kv_{ts\sigma}}z^2 \\ 0 & y^2 & 0 \\ \frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_z v_{dsx}}{kv_{ts\sigma}}x - \frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}z^2 & \frac{k_z v_{dsy}}{kv_{ts\sigma}}x & \frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_z v_{dsz}}{kv_{ts\sigma}}x + \frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}z^2 \end{pmatrix}
\end{aligned}
$$

The odd function term from y and z can be integralled to vanish, and thus we have omitted them in the above final expressions. Note

$$
\begin{cases} v_x = v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx} \\ v_y = v_{\perp ts\sigma}y + v_{dsy} \\ v_z = v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz} \end{cases}, \tag{84}
$$

and we have used:

- $[\frac{k_x}{kv_{ts\sigma}}x + \sqrt{\lambda_{Ts\sigma}}\frac{k_z}{kv_{ts\sigma}}z](v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx}) \to \frac{k_x}{kv_{ts\sigma}}x(\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}x + v_{dsx}) + \frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}z^2 = \frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_x v_{dsx}}{kv_{ts\sigma}}x + \frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}z^2$,
- $\Big[\frac{k_x}{kv_{ts\sigma}}x + \sqrt{\lambda_{Ts\sigma}}\frac{k_z}{kv_{ts\sigma}}z\Big](v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz}) \to \frac{k_x}{kv_{ts\sigma}}x(v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}x + v_{dsz}) - \frac{k_z}{kv_{ts\sigma}}\frac{k_x v_{zts}^2}{kv_{ts\sigma}}z^2 = \frac{k_x k_z v_{zts}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_x v_{dsz}}{kv_{ts\sigma}}x - \frac{k_z}{kv_{ts\sigma}}\frac{k_x v_{zts}^2}{kv_{ts\sigma}}z^2$,
- $\Big[\frac{k_z}{kv_{ts\sigma}}x - \frac{k_x}{kv_{ts\sigma}}\frac{1}{\sqrt{\lambda_{Ts\sigma}}}z\Big](v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx}) \to \frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_z v_{dsx}}{kv_{ts\sigma}}x - \frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}z^2$,
- $\Big[\frac{k_z}{kv_{ts\sigma}}x - \frac{k_x}{kv_{ts\sigma}}\frac{1}{\sqrt{\lambda_{Ts\sigma}}}z\Big](v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz}) \to \frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_z v_{dsz}}{kv_{ts\sigma}}x + \frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}z^2$.

And

$$
(\boldsymbol{k}\cdot\frac{\partial f_{s0}}{\partial \boldsymbol{v}})(\boldsymbol{vv}) \quad = \quad -2\sum_{\sigma=a,b} F_{s\sigma}\Big[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma} - 1)z\Big]\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}(v_x, v_y, v_z)
$$

with terms in $\boldsymbol{vv}$

- $v_x v_x \to [v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx}]^2$.
- $v_x v_y = v_y v_x \to [v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx}]v_{dsy}$.
- $v_x v_z = v_z v_x \to [v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx}][v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz}]$.
- $v_y v_y \to [v_{\perp ts\sigma}^2 y^2 + v_{dsy}^2]$.
- $v_y v_z = v_z v_y \to [v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz}]v_{dsy}$.
- $v_z v_z \to [v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz}]^2$.

Define $\boldsymbol{Q}_s^u = \sum_{\sigma=a,b} r_{s\sigma}\frac{\omega_{ps}^2}{\omega^2}\boldsymbol{P}_s^u$, with $\boldsymbol{P}_s^u = \Big\{-\boldsymbol{I} + \int dv^3 \frac{\Big[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}}) + (\boldsymbol{k}\cdot\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})(\boldsymbol{vv})\Big]}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu_s)}\Big\}$, and $g = \frac{1}{\pi^{3/2}}e^{-(x^2+y^2+z^2)}$. We calculate term by term, similar to

- $P^u_{s11} = \Big\{ -\boldsymbol{I} + \int dv^3 \frac{\Big[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})+(\boldsymbol{k}\cdot\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})(\boldsymbol{v}\boldsymbol{v})\Big]}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\nu_s)} \Big\}_{11} = -1 + 2\int dxdydzg \frac{-i\nu_s(\frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_x v_{dsx}}{kv_{ts\sigma}}x + \frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}z^2) + \Big[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)z\Big][v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx}]^2}{kv_{ts\sigma}(\zeta_{s\sigma}-x)} = -1 + \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s(\frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k_x v_{dsx}}{kv_{ts\sigma}}Z_1 + \frac{k_z^2 v_{zts}^2}{2k^2 v_{ts\sigma}^2}Z_0) + \frac{k^2}{kv_{ts\sigma}}(v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}})^2 Z_3 + 2\frac{k^2}{kv_{ts\sigma}}(v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}})v_{dsx}Z_2 + [\frac{1}{2}\frac{k^2}{kv_{ts\sigma}}(v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}})^2 + \frac{k^2}{kv_{ts\sigma}}v_{dsx}^2 + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)(v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}})v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}]Z_1 + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{dsx}Z_0\Big\}$
  $= -1 + \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s(\frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k_x v_{dsx}}{kv_{ts\sigma}}Z_1 + \frac{k_z^2 v_{zts}^2}{2k^2 v_{ts\sigma}^2}Z_0) + \frac{k^2 k_x^2 v_{\perp ts\sigma}^4}{k^3 v_{ts\sigma}^3}Z_3 + 2\frac{k^2 k_x v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}v_{dsx}Z_2 + [\frac{1}{2}v_{\perp ts\sigma}^2\frac{k^2 k_z^2 v_{zts}^2}{k^3 v_{ts\sigma}^3} + \frac{k^2}{kv_{ts\sigma}}v_{dsx}^2 + (\lambda_{Ts\sigma}-1)\frac{k_x^2 k_z^2 v_{\perp ts\sigma}^4}{k^3 v_{ts\sigma}^3}]Z_1 + \frac{k_x k_z^2}{k^2 v_{ts\sigma}^2}(\lambda_{Ts\sigma}-1)v_{\perp ts\sigma}^2 v_{dsx}Z_0\Big\}$,

we have others

- $P^u_{s12} = \Big\{ -\boldsymbol{I} + \int dv^3 \frac{\Big[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})+(\boldsymbol{k}\cdot\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})(\boldsymbol{v}\boldsymbol{v})\Big]}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\nu_s)} \Big\}_{12} = 0 + 2\int dxdydzg \frac{-i\nu_s 0 + \Big[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)z\Big][v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx}]v_{dsy}}{kv_{ts\sigma}(\zeta_{s\sigma}-x)} = \frac{2}{kv_{ts\sigma}}\Big\{\frac{k^2}{kv_{ts\sigma}}\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}v_{dsy}Z_2 + \frac{k^2}{kv_{ts\sigma}}v_{dsx}v_{dsy}Z_1 + \frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{dsy}Z_0\Big\}$
  $= \frac{2v_{dsy}}{kv_{ts\sigma}}\Big\{\frac{k^2 k_x v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k^2}{kv_{ts\sigma}}v_{dsx}Z_1 + \frac{1}{2}\frac{k_x k_z^2}{k^2 v_{ts\sigma}^2}(\lambda_{Ts\sigma}-1)v_{\perp ts\sigma}^2 Z_0\Big\}$,

- $P^u_{s21} = \Big\{ -\boldsymbol{I} + \int dv^3 \frac{\Big[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})+(\boldsymbol{k}\cdot\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})(\boldsymbol{v}\boldsymbol{v})\Big]}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\nu_s)} \Big\}_{21} = 0 + 2\int dxdydzg \frac{-i\nu_s\frac{k_x v_{dsy}}{kv_{ts\sigma}}x + \Big[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)z\Big][v_{\perp ts\sigma}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx}]v_{dsy}}{kv_{ts\sigma}(\zeta_{s\sigma}-x)}$
  $= \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s\frac{k_x v_{dsy}}{kv_{ts\sigma}}Z_1 + \frac{k^2}{kv_{ts\sigma}}\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}v_{dsy}Z_2 + \frac{k^2}{kv_{ts\sigma}}v_{dsx}v_{dsy}Z_1 + \frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{dsy}Z_0\Big\}$
  $= \frac{2v_{dsy}}{kv_{ts\sigma}}\Big\{-i\nu_s\frac{k_x}{kv_{ts\sigma}}Z_1 + \frac{k^2 k_x v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k^2}{kv_{ts\sigma}}v_{dsx}Z_1 + \frac{1}{2}\frac{k_x k_z^2}{k^2 v_{ts\sigma}^2}(\lambda_{Ts\sigma}-1)v_{\perp ts\sigma}^2 Z_0\Big\}$,

- $P^u_{s22} = \Big\{ -\boldsymbol{I} + \int dv^3 \frac{\Big[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})+(\boldsymbol{k}\cdot\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})(\boldsymbol{v}\boldsymbol{v})\Big]}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\nu_s)} \Big\}_{22} = -1 + 2\int dxdydzg \frac{-i\nu_s y^2 + \Big[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)z\Big][v_{\perp ts\sigma}^2 y^2 + v_{dsy}^2]}{kv_{ts\sigma}(\zeta_{s\sigma}-x)}$
  $= -1 + \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s\frac{1}{2}Z_0 + (\frac{1}{2}v_{\perp ts\sigma}^2 + v_{dsy}^2)\frac{k^2}{kv_{ts\sigma}}Z_1\Big\}$,

- $P^u_{s13} = \Big\{ -\boldsymbol{I} + \int dv^3 \frac{\Big[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})+(\boldsymbol{k}\cdot\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})(\boldsymbol{v}\boldsymbol{v})\Big]}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\nu_s)} \Big\}_{13} = 0 + 2\int dxdydzg$
  $\frac{-i\nu_s(\frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_z v_{dsx}}{kv_{ts\sigma}}x - \frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}z^2) + \Big[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)z\Big][\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx}][\frac{k_z v_{zts}^2}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz}]}{kv_{ts\sigma}(\zeta_{s\sigma}-x)}$
  $= \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s(\frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k_z v_{dsx}}{kv_{ts\sigma}}Z_1 - \frac{k_x k_z v_{\perp ts\sigma}^2}{2k^2 v_{ts\sigma}^2}Z_0) + \frac{k^2}{kv_{ts\sigma}}\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}\frac{k_z v_{zts}^2}{kv_{ts\sigma}}Z_3 + \frac{k^2}{kv_{ts\sigma}}[\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}v_{dsz} + v_{dsx}\frac{k_z v_{zts}^2}{kv_{ts\sigma}}]Z_2 + [\frac{k^2}{kv_{ts\sigma}}v_{dsx}v_{dsz} - \frac{1}{2}\frac{k^2}{kv_{ts\sigma}}v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}} - \frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}} + \frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}\frac{k_z v_{zts}^2}{kv_{ts\sigma}}]Z_1 + [\frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{dsz} - \frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{dsx}v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}]Z_0\Big\}$
  $= \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s(\frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k_z v_{dsx}}{kv_{ts\sigma}}Z_1 - \frac{k_x k_z v_{\perp ts\sigma}^2}{2k^2 v_{ts\sigma}^2}Z_0) + \frac{k^2 k_x k_z v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3}Z_3 + \frac{k^2}{kv_{ts\sigma}}[\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}v_{dsz} + v_{dsx}\frac{k_z v_{zts}^2}{kv_{ts\sigma}}]Z_2 + [\frac{k^2}{kv_{ts\sigma}}v_{dsx}v_{dsz} - \frac{1}{2}\frac{k^2 k_x k_z v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3} - \frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_x^3 k_z v_{\perp ts\sigma}^4}{k^3 v_{ts\sigma}^3} + \frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_x k_z^3 v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3}]Z_1 + [\frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_x k_z^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}v_{dsz} - \frac{1}{2}(\lambda_{Ts\sigma}-1)v_{dsx}\frac{k_x^2 k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}]Z_0\Big\}$,

- $P^u_{s31} = \Big\{ -\boldsymbol{I} + \int dv^3 \frac{\Big[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})+(\boldsymbol{k}\cdot\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})(\boldsymbol{v}\boldsymbol{v})\Big]}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\nu_s)} \Big\}_{31} = 0 + 2\int dxdydzg$
  $\frac{-i\nu_s(\frac{k_x k_z v_{zts}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_x v_{dsz}}{kv_{ts\sigma}}x - \frac{k_x k_z v_{zts}^2}{k^2 v_{ts\sigma}^2}z^2) + \Big[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)z\Big][\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}x + v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}z + v_{dsx}][\frac{k_z v_{zts}^2}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz}]}{kv_{ts\sigma}(\zeta_{s\sigma}-x)}$
  $= \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s(\frac{k_x k_z v_{zts}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k_x v_{dsz}}{kv_{ts\sigma}}Z_1 - \frac{k_x k_z v_{zts}^2}{2k^2 v_{ts\sigma}^2}Z_0) + \frac{k^2}{kv_{ts\sigma}}\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}\frac{k_z v_{zts}^2}{kv_{ts\sigma}}Z_3 + \frac{k^2}{kv_{ts\sigma}}[\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}v_{dsz} + v_{dsx}\frac{k_z v_{zts}^2}{kv_{ts\sigma}}]Z_2 + [\frac{k^2}{kv_{ts\sigma}}v_{dsx}v_{dsz} -$

$\frac{1}{2}\frac{k^2}{kv_{ts\sigma}}v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}} - \frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}} + \frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}\frac{k_z v_{zts}^2}{kv_{ts\sigma}}]Z_1 + [\frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{\perp ts\sigma}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{dsz} - \frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{dsx}v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}]Z_0\Big\}$
$= \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s(\frac{k_x k_z v_{zts}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k_x v_{dsz}}{kv_{ts\sigma}}Z_1 - \frac{k_x k_z v_{zts}^2}{2k^2 v_{ts\sigma}^2}Z_0) + \frac{k^2 k_x k_z v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3}Z_3 + \frac{k^2}{kv_{ts\sigma}}[\frac{k_x v_{\perp ts\sigma}^2}{kv_{ts\sigma}}v_{dsz} + v_{dsx}\frac{k_z v_{zts}^2}{kv_{ts\sigma}}]Z_2 + [\frac{k^2}{kv_{ts\sigma}}v_{dsx}v_{dsz} - \frac{1}{2}\frac{k^2 k_x k_z v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3} - \frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_x^3 k_z v_{\perp ts\sigma}^4}{k^3 v_{ts\sigma}^3} + \frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_x k_z^3 v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3}]Z_1 + [\frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_x k_z^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}v_{dsz} - \frac{1}{2}(\lambda_{Ts\sigma}-1)v_{dsx}\frac{k_x^2 k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}]Z_0\Big\}$,

- $P^u_{s23} = \Big\{-\boldsymbol{I} + \int dv^3 \frac{\left[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}}) + (\boldsymbol{k}\cdot\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})(\boldsymbol{v}\boldsymbol{v})\right]}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu_s)}\Big\}_{23} = 0 +$
  $2\int dxdydzg\frac{-i\nu_s\frac{k_z v_{dsy}}{kv_{ts\sigma}}x + \left[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)z\right][v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz}]v_{dsy}}{kv_{ts\sigma}(\zeta_{s\sigma}-x)}$
  $= \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s\frac{k_z v_{dsy}}{kv_{ts\sigma}}Z_1 + \frac{k^2}{kv_{ts\sigma}}v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{dsy}Z_2 + \frac{k^2}{kv_{ts\sigma}}v_{dsz}v_{dsy}Z_1 - \frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}v_{dsy}Z_0\Big\}$
  $= \frac{2v_{dsy}}{kv_{ts\sigma}}\Big\{-i\nu_s\frac{k_z}{kv_{ts\sigma}}Z_1 + \frac{k^2 k_z v_{zts}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k^2}{kv_{ts\sigma}}v_{dsz}Z_1 - \frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_x^2 k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}Z_0\Big\}$,

- $P^u_{s32} = \Big\{-\boldsymbol{I} + \int dv^3 \frac{\left[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}}) + (\boldsymbol{k}\cdot\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})(\boldsymbol{v}\boldsymbol{v})\right]}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu_s)}\Big\}_{32} = 0 +$
  $2\int dxdydzg\frac{-i\nu_s 0 + \left[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)z\right][v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz}]v_{dsy}}{kv_{ts\sigma}(\zeta_{s\sigma}-x)}$
  $= \frac{2}{kv_{ts\sigma}}\Big\{\frac{k^2}{kv_{ts\sigma}}v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{dsy}Z_2 + \frac{k^2}{kv_{ts\sigma}}v_{dsz}v_{dsy}Z_1 - \frac{1}{2}\frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}v_{dsy}Z_0\Big\}$
  $= \frac{2v_{dsy}}{kv_{ts\sigma}}\Big\{\frac{k^2 k_z v_{zts}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k^2}{kv_{ts\sigma}}v_{dsz}Z_1 - \frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_x^2 k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}Z_0\Big\}$,

- $P^u_{s33} = \Big\{-\boldsymbol{I} + \int dv^3 \frac{\left[-i\nu_s(\boldsymbol{v}\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}}) + (\boldsymbol{k}\cdot\frac{\partial f_{s0\sigma}}{\partial \boldsymbol{v}})(\boldsymbol{v}\boldsymbol{v})\right]}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu_s)}\Big\}_{33} = -1 +$
  $2\int dxdydzg\frac{-i\nu_s(\frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}x^2 + \frac{k_z v_{dsz}}{kv_{ts\sigma}}x + \frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}z^2) + \left[\frac{k^2}{kv_{ts\sigma}}x + \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)z\right][v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}x - v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}z + v_{dsz}]^2}{kv_{ts\sigma}(\zeta_{s\sigma}-x)} = -1 + \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s(\frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k_z v_{dsz}}{kv_{ts\sigma}}Z_1 + \frac{k_x^2 v_{\perp ts\sigma}^2}{2k^2 v_{ts\sigma}^2}Z_0) + \frac{k^2}{kv_{ts\sigma}}(v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}})^2 Z_3 + 2\frac{k^2}{kv_{ts\sigma}}v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{dsz}Z_2 + [\frac{1}{2}\frac{k^2}{kv_{ts\sigma}}(v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}})^2 + \frac{k^2}{kv_{ts\sigma}}v_{dsz}^2 - \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{zts}\frac{k_z v_{zts}}{kv_{ts\sigma}}v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}]Z_1 - \frac{k_x k_z}{kv_{ts\sigma}}\frac{v_{\perp ts\sigma}}{v_{zts}}(\lambda_{Ts\sigma}-1)v_{zts}\frac{k_x v_{\perp ts\sigma}}{kv_{ts\sigma}}v_{dsz}Z_0\Big\}$
  $= -1 + \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s(\frac{k_z^2 v_{zts}^2}{k^2 v_{ts\sigma}^2}Z_2 + \frac{k_z v_{dsz}}{kv_{ts\sigma}}Z_1 + \frac{k_x^2 v_{\perp ts\sigma}^2}{2k^2 v_{ts\sigma}^2}Z_0) + \frac{k^2 k_z^2 v_{zts}^4}{k^3 v_{ts\sigma}^3}Z_3 + 2\frac{k^2 k_z v_{zts}^2}{k^2 v_{ts\sigma}^2}v_{dsz}Z_2 + [\frac{1}{2}\frac{k^2 k_x^2 v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3} + \frac{k^2}{kv_{ts\sigma}}v_{dsz}^2 - (\lambda_{Ts\sigma}-1)\frac{k_x^2 k_z^2 v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3}]Z_1 - (\lambda_{Ts\sigma}-1)\frac{k_x^2 k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}v_{dsz}Z_0\Big\}$,

where the argument for $Z_{0,1,2,3}$ is $\zeta_{s\sigma}$.

Note, with the '$\rightarrow$' means that to integral out the x, y and z terms, say, $y, z \rightarrow 0$, $y^3, z^3 \rightarrow 0$, $y^0, z^0 \rightarrow 1$, $y^2, z^2 \rightarrow \frac{1}{2}$, $x^p \rightarrow Z_p(\zeta_{s\sigma})$, we have used

- $(ax+bz)(cx+dz+e)^2 = ac^2x^3 + 2acdx^2z + 2acex^2 + ad^2xz^2 + 2adexz + ae^2x + bc^2x^2z + 2bcdxz^2 + 2bcexz + bd^2z^3 + 2bdez^2 + be^2z \rightarrow ac^2Z_3 + 2aceZ_2 + [\frac{1}{2}ad^2 + ae^2 + bcd]Z_1 + bdeZ_0$.
- $(ax+bz)(cx+dz+e)f = acfx^2 + adfxz + aefx + bcfxz + bdfz^2 + befz \rightarrow acfZ_2 + aefZ_1 + \frac{1}{2}bdfZ_0$.
- $(ax+bz)(cx-dz+e)f = acfx^2 - adfxz + aefx + bcfxz - bdfz^2 + befz \rightarrow acfZ_2 + aefZ_1 - \frac{1}{2}bdfZ_0$.
- $(ax+bz)(cy^2+d) = acxy^2 + adx + bcy^2z + bdz \rightarrow [\frac{1}{2}ac + ad]Z_1$.
- $(ax+bz)(cx+dz+e)(fx-gz+h) = acfx^3 - acgx^2z + achx^2 + adfx^2z - adgxz^2 + adhxz + aefx^2 - aegxz + aehx + bcfx^2z - bcgxz^2 + bchxz + bdfxz^2 - bdgz^3 + bdhz^2 + befxz - begz^2 + behz \rightarrow acfZ_3 + [ach + aef]Z_2 + [aeh - \frac{1}{2}adg - \frac{1}{2}bcg + \frac{1}{2}bdf]Z_1 + [\frac{1}{2}bdh - \frac{1}{2}beg]Z_0$.
- $(ax+bz)(cx-dz+e)^2 = ac^2x^3 - 2acdx^2z + 2acex^2 + ad^2xz^2 - 2adexz + ae^2x + bc^2x^2z - 2bcdxz^2 + 2bcexz + bd^2z^3 - 2bdez^2 + be^2z \rightarrow ac^2Z_3 + 2aceZ_2 + [\frac{1}{2}ad^2 + ae^2 - bcd]Z_1 - bdeZ_0$.

Although the above expressions are much complicated and need be carefully, they can be solved easily by BO-K matrix approach. We have also checked that, when $\nu_s = 0$, $\lambda_{Ts\sigma} = 1$ and $v_{dsz} = v_{dsy} = 0$, the result can reduce to exactly the same form as in Ref.[14] .

3.5.2. Magnetized terms

Next, we calculate the magnetized terms.

Define $\boldsymbol{Q}_s^m = \sum_{\sigma=a,b} r_{s\sigma}\frac{\omega_{ps}^2}{\omega^2}\boldsymbol{P}_s^m$, with $\boldsymbol{P}_s^m = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv'_\parallel}{(\omega-k_\parallel v'_\parallel-k_x v_{dsx}+i\nu_s-n\omega_{cs})}\boldsymbol{\Pi}_{s\sigma} = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv''_\parallel}{(\omega_{sn}-k_z v''_\parallel)}\boldsymbol{\Pi}_{s\sigma}$ and $\omega_{sn} = \omega - k_z v_{dsz} - n\omega_{cs} - k_x v_{dsx} + i\nu_s$, i.e., $\omega = \omega_{sn} + k_z v_{dsz} + n\omega_{cs} + k_x v_{dsx} - i\nu_s$. We have also

$$
\begin{aligned}
\frac{\partial f_{s0\sigma}}{\partial v'_\parallel} &= -\frac{2(v'_\parallel - v_{dsz})}{v_{zts}^2}f_{s0\sigma} = -2\frac{(v'_\parallel - v_{dsz})}{v_{zts}^2}f_{s0\sigma} = -2\frac{v''_\parallel}{v_{zts}^2}f_{s0\sigma},\\
\frac{\partial f_{s0\sigma}}{\partial v'_\perp} &= -2\frac{(v'_\perp - v_{dsr})}{v_{\perp ts\sigma}^2}f_{s0\sigma},
\end{aligned}
$$

with $v''_\parallel = v'_\parallel - v_{dsz}$, and $f_{s0\sigma} = \frac{1}{\pi^{3/2}v_{zts}v_{\perp ts\sigma}^2 A_{s\sigma}}\exp\Big[-\frac{v''^2_\parallel}{v_{zts}^2} - \frac{(v'-v_{dsr})_\perp^2}{v_{\perp ts}^2}\Big]$, i.e., $v'_\parallel = v''_\parallel + v_{dsz}$. Since we have defined A_n, B_n, C_n, we do not need transform $v'_\perp$.

Thus,

- $P_{s11}^m = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv''_\parallel}{(\omega_{sn}-k_z v''_\parallel)}\Pi_{s\sigma 11} = \sum_{n=-\infty}^{\infty}\Big\{[\frac{n\omega_{cs}}{k_x}(n\omega_{cs}+k_x v_{dsx}-i\nu_s)\frac{A_{nbs\sigma}}{v_{\perp ts\sigma}^2}\frac{Z_0}{k_z v_{zts}} + A_{n0\sigma}(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{Z_1}{v_{zts}^2}](\frac{n\omega_{cs}}{k_x} + v_{dsx})\Big\} - \sum_n A_{nbs\sigma}\frac{n\omega_{cs}}{k_x v_{\perp ts\sigma}^2}(\frac{n\omega_{cs}}{k_x} + v_{dsx})$.
- $P_{s12}^m = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv''_\parallel}{(\omega_{sn}-k_z v''_\parallel)}\Pi_{s\sigma 12} = \sum_{n=-\infty}^{\infty}\Big\{v_{dsy}(\frac{n\omega_{cs}}{k_x}+v_{dsx})(n\omega_{cs}\frac{A_{nbs\sigma}}{v_{\perp ts\sigma}^2}\frac{Z_0}{k_z v_{zts}}+A_{n0\sigma}\frac{Z_1}{v_{zts}^2})+i(\frac{n\omega_{cs}}{k_x}+v_{dsx})[(n\omega_{cs} - i\nu_s)\frac{B_{nbs\sigma}}{v_{\perp ts\sigma}}\frac{Z_0}{k_z v_{zts}} + \frac{B_{n0\sigma}v_{\perp ts\sigma}Z_1}{v_{zts}^2}]\Big\}$.
- $P_{s21}^m = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv''_\parallel}{(\omega_{sn}-k_z v''_\parallel)}\Pi_{s\sigma 21} = \sum_{n=-\infty}^{\infty}\Big\{\frac{n\omega_{cs}}{k_x}(n\omega_{cs}+k_x v_{dsx}-i\nu_s)[-i v_{\perp ts\sigma}B_{nbs\sigma}+v_{dsy}A_{nbs\sigma}]\frac{1}{v_{\perp ts\sigma}^2}\frac{Z_0}{k_z v_{zts}} + [-i v_{\perp ts\sigma}B_{n0\sigma}+v_{dsy}A_{n0\sigma}](\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{Z_1}{v_{zts}^2}\Big\}$.
- $P_{s22}^m = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv''_\parallel}{(\omega_{sn}-k_z v''_\parallel)}\Pi_{s\sigma 22} = \sum_{n=-\infty}^{\infty}\Big\{v_{dsy}[-i v_{\perp ts\sigma}B_{nbs\sigma}+v_{dsy}A_{nbs\sigma}]\frac{n\omega_{cs}}{v_{\perp ts\sigma}^2}\frac{Z_0}{k_z v_{zts}} + v_{dsy}[-i v_{\perp ts\sigma}B_{n0\sigma} + v_{dsy}A_{n0\sigma}]\frac{Z_1}{v_{zts}^2} + v_{\perp ts\sigma}[v_{\perp ts\sigma}C_{nbs\sigma}+i v_{dsy}B_{nbs\sigma}](n\omega_{cs}-i\nu_s)\frac{1}{v_{\perp ts\sigma}^2}\frac{Z_0}{k_z v_{zts}} + v_{\perp ts\sigma}[v_{\perp ts\sigma}C_{n0\sigma}+i v_{dsy}B_{n0\sigma}]\frac{Z_1}{v_{zts}^2}\Big\} - \sum_n(C_{nbs\sigma} + i\frac{v_{dsy}}{v_{\perp ts\sigma}}B_{nbs\sigma})$.
- $P_{s13}^m = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv''_\parallel}{(\omega_{sn}-k_z v''_\parallel)}\Pi_{s\sigma 13} = \sum_{n=-\infty}^{\infty}\Big\{(\frac{n\omega_{cs}}{k_x}+v_{dsx})[n\omega_{cs}(\frac{Z_1}{k_z}+v_{dsz}\frac{Z_0}{k_z v_{zts}})\frac{A_{nbs\sigma}}{v_{\perp ts\sigma}^2} + A_{n0\sigma}(v_{zts}Z_2 + (k_z v_{dsz} - i\nu_s)\frac{Z_1}{k_z})\frac{1}{v_{zts}^2}]\Big\}$.
- $P_{s31}^m = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv''_\parallel}{(\omega_{sn}-k_z v''_\parallel)}\Pi_{s\sigma 31} = \sum_{n=-\infty}^{\infty}\Big\{\frac{n\omega_{cs}}{k_x}(n\omega_{cs}+k_x v_{dsx}-i\nu_s)\frac{A_{nbs\sigma}}{v_{\perp ts\sigma}^2}(\frac{Z_1}{k_z}+v_{dsz}\frac{Z_0}{k_z v_{zts}}) + A_{n0\sigma}(\frac{n\omega_{cs}}{k_x} + v_{dsx})(\frac{Z_2}{v_{zts}}+v_{dsz}\frac{Z_1}{v_{zts}^2})\Big\}$.
- $P_{s23}^m = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv''_\parallel}{(\omega_{sn}-k_z v''_\parallel)}\Pi_{s\sigma 23} = \sum_{n=-\infty}^{\infty}\Big\{[v_{dsy}A_{nbs\sigma}-i v_{\perp ts\sigma}B_{nbs\sigma}][(\frac{Z_1}{k_z}+\frac{Z_0 v_{dsz}}{k_z v_{tsz}})\frac{n\omega_{cs}}{v_{\perp ts\sigma}^2}]+[v_{dsy}A_{n0\sigma}-i v_{\perp ts\sigma}B_{n0\sigma}][v_{zts}Z_2+ (k_z v_{dsz} - i\nu_s)\frac{Z_1}{k_z}]\frac{1}{v_{zts}^2}\Big\}$.
- $P_{s32}^m = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv''_\parallel}{(\omega_{sn}-k_z v''_\parallel)}\Pi_{s\sigma 32} = \sum_{n=-\infty}^{\infty}\Big\{[n\omega_{cs}\frac{A_{nbs\sigma}}{v_{\perp ts\sigma}^2}\frac{Z_1}{k_z}+A_{n0\sigma}\frac{k_z v_{zts}Z_2}{v_{zts}^2}]v_{dsy}+[n\omega_{cs}\frac{A_{nbs\sigma}}{v_{\perp ts\sigma}^2}\frac{Z_0}{k_z v_{zts}}+A_{n0\sigma}\frac{Z_1}{v_{zts}^2}]v_{dsy}v_{dsz}+ i[(n\omega_{cs}-i\nu_s)\frac{B_{nbs\sigma}}{v_{\perp ts\sigma}}\frac{Z_1}{k_z} + B_{n0\sigma}\frac{v_{\perp ts\sigma}Z_2}{v_{zts}}] + i v_{dsz}[(n\omega_{cs}-i\nu_s)\frac{B_{nbs\sigma}}{v_{\perp ts\sigma}}\frac{Z_0}{k_z v_{zts}} + B_{n0\sigma}\frac{v_{\perp ts\sigma}Z_1}{v_{zts}^2}]\Big\}$.
- $P_{s33}^m = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v'_\perp dv'_\perp dv''_\parallel}{(\omega_{sn}-k_z v''_\parallel)}\Pi_{s\sigma 33} = \sum_{n=-\infty}^{\infty}\Big\{[n\omega_{cs}(\frac{v_{zts}Z_2}{k_z}+v_{dsz}\frac{Z_1}{k_z})\frac{A_{nbs\sigma}}{v_{\perp ts\sigma}^2} + A_{n0\sigma}(Z_3 + (k_z v_{dsz} - i\nu_s)\frac{Z_2}{k_z v_{zts}})] + v_{dsz}[n\omega_{cs}(\frac{Z_1}{k_z}+v_{dsz}\frac{Z_0}{k_z v_{zts}})\frac{A_{nbs\sigma}}{v_{\perp ts\sigma}^2} + A_{n0\sigma}(\frac{Z_2}{v_{zts}}+(k_z v_{dsz}-i\nu_s)\frac{Z_1}{k_z v_{zts}^2})]\Big\} - \sum_n \frac{1}{2}A_{n0\sigma}$.

where the argument for $Z_{0,1,2,3}$ is ζ_{sn}, and we have used $\sum_{n=-\infty}^{\infty}J_n J'_n = 0$, $\sum_{n=-\infty}^{\infty}nJ_n^2 = 0$, $\sum_{n=-\infty}^{\infty}nJ_n J'_n = 0$. If we further use , $\sum_{n=-\infty}^{\infty}J_n^2 = 1$, $\sum_{n=-\infty}^{\infty}(J'_n)^2 = \frac{1}{2}$, $\sum_{n=-\infty}^{\infty}\frac{n^2 J_n^2(x)}{x^2} = \frac{1}{2}$, the last $\sum_n$ terms in P_{s11}^m, P_{s22}^m and P_{s33}^m yields to 1. We derive the above equation base on (define $\omega'_{sn} = \omega_{sn} - k_z v''_\parallel$, i.e., $\omega_{sn} = \omega'_{sn} + k_z v''_\parallel$)

- $v'_\perp \Pi_{s\sigma 11} = J_n^2[\frac{n\omega_{cs}}{k_x}(\omega - k_z v'_\parallel)\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + v'_\perp k_z(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{\partial f_{s0\sigma}}{\partial v'_\parallel}](\frac{n\omega_{cs}}{k_x}+v_{dsx}) \rightarrow J_n^2[\frac{n\omega_{cs}}{k_x}(\omega'_{sn}+n\omega_{cs}+k_xv_{dsx}-i\nu_s)\frac{(v'_\perp - v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp k_z(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{v''_\parallel}{v^2_{zts}}](\frac{n\omega_{cs}}{k_x}+v_{dsx}) \rightarrow J_n^2[\frac{n\omega_{cs}}{k_x}(n\omega_{cs}+k_xv_{dsx}-i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{k_zv''_\parallel}{v^2_{zts}}](\frac{n\omega_{cs}}{k_x}+v_{dsx}) + J_n^2\frac{n\omega_{cs}}{k_x}\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}}(\frac{n\omega_{cs}}{k_x}+v_{dsx})(\omega'_{sn})$.

- $v'_\perp \Pi_{s\sigma 12} = J_n^2 v_{dsy}(\frac{n\omega_{cs}}{k_x}+v_{dsx})(n\omega_{cs}\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + v'_\perp k_z\frac{\partial f_{s0\sigma}}{\partial v'_\parallel}) + iv'_\perp J_nJ'_n(\frac{n\omega_{cs}}{k_x}+v_{dsx})[(\omega - k_zv'_\parallel - k_xv_{dsx})\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + k_zv'_\perp\frac{\partial f_{s0\sigma}}{\partial v'_\parallel}] \rightarrow J_n^2v_{dsy}(\frac{n\omega_{cs}}{k_x}+v_{dsx})(n\omega_{cs}\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp k_z\frac{v''_\parallel}{v^2_{zts}}) + iv'_\perp J_nJ'_n(\frac{n\omega_{cs}}{k_x}+v_{dsx})[(\omega'_{sn}+n\omega_{cs}-i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + k_zv'_\perp\frac{v''_\parallel}{v^2_{zts}}] \rightarrow J_n^2v_{dsy}(\frac{n\omega_{cs}}{k_x}+v_{dsx})(n\omega_{cs}\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp\frac{k_zv''_\parallel}{v^2_{zts}}) + iv'_\perp J_nJ'_n(\frac{n\omega_{cs}}{k_x}+v_{dsx})[(n\omega_{cs}-i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp\frac{k_zv''_\parallel}{v^2_{zts}}] + iv'_\perp J_nJ'_n(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}}(\omega'_{sn})$.

- $v'_\perp \Pi_{s\sigma 21} = (-iv'_\perp J_nJ'_n + v_{dsy}J_n^2)[\frac{n\omega_{cs}}{k_x}(\omega - k_zv'_\parallel)\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + v'_\perp\frac{n\omega_{cs}}{k_x}k_z\frac{\partial f_{s0\sigma}}{\partial v'_\parallel} + v'_\perp k_zv_{dsx}\frac{\partial f_{s0\sigma}}{\partial v'_\parallel}] \rightarrow (-iv'_\perp J_nJ'_n + v_{dsy}J_n^2)[\frac{n\omega_{cs}}{k_x}(\omega'_{sn} + n\omega_{cs} + k_xv_{dsx} - i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp k_z(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{v''_\parallel}{v^2_{zts}}] \rightarrow (-iv'_\perp J_nJ'_n + v_{dsy}J_n^2)[\frac{n\omega_{cs}}{k_x}(n\omega_{cs}+k_xv_{dsx}-i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{k_zv''_\parallel}{v^2_{zts}}] + (-iv'_\perp J_nJ'_n + v_{dsy}J_n^2)\frac{n\omega_{cs}}{k_x}\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}}(\omega'_{sn})$.

- $v'_\perp \Pi_{s\sigma 22} = J_nv_{dsy}(-iv'_\perp J'_n + v_{dsy}J_n)(n\omega_{cs}\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + v'_\perp k_z\frac{\partial f_{s0\sigma}}{\partial v'_\parallel}) + iv'_\perp J'_n(-iv'_\perp J'_n + v_{dsy}J_n)[(\omega - k_zv'_\parallel - k_xv_{dsx})\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + k_zv'_\perp\frac{\partial f_{s0}}{\partial v'_\parallel}] \rightarrow J_nv_{dsy}(-iv'_\perp J'_n + v_{dsy}J_n)(n\omega_{cs}\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp k_z\frac{v''_\parallel}{v^2_{zts}}) + iv'_\perp J'_n(-iv'_\perp J'_n + v_{dsy}J_n)[(\omega'_{sn} + n\omega_{cs} - i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + k_zv'_\perp\frac{v''_\parallel}{v^2_{zts}}] \rightarrow J_nv_{dsy}(-iv'_\perp J'_n + v_{dsy}J_n)(n\omega_{cs}\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp\frac{k_zv''_\parallel}{v^2_{zts}}) + iv'_\perp J'_n(-iv'_\perp J'_n + v_{dsy}J_n)[(n\omega_{cs}-i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp\frac{k_zv''_\parallel}{v^2_{zts}}] + iv'_\perp J'_n(-iv'_\perp J'_n + v_{dsy}J_n)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}}(\omega'_{sn})$.

- $v'_\perp \Pi_{s\sigma 13} = J_n^2(\frac{n\omega_{cs}}{k_x}+v_{dsx})[n\omega_{cs}v'_\parallel\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + v'_\perp(\omega - n\omega_{cs} - k_xv_{dsx})\frac{\partial f_{s0\sigma}}{\partial v'_\parallel}] \rightarrow J_n^2(\frac{n\omega_{cs}}{k_x}+v_{dsx})[n\omega_{cs}(v''_\parallel + v_{dsz})\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp(\omega'_{sn} + k_zv''_\parallel + k_zv_{dsz} - i\nu_s)\frac{v''_\parallel}{v^2_{zts}}] \rightarrow J_n^2(\frac{n\omega_{cs}}{k_x}+v_{dsx})[n\omega_{cs}(v''_\parallel+v_{dsz})\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp(k_zv''_\parallel + k_zv_{dsz} - i\nu_s)\frac{v''_\parallel}{v^2_{zts}}] + J_n^2(\frac{n\omega_{cs}}{k_x}+v_{dsx})v'_\perp\frac{v''_\parallel}{v^2_{zts}}(\omega'_{sn})$.

- $v'_\perp \Pi_{s\sigma 31} = v'_\parallel J_n^2[\frac{n\omega_{cs}}{k_x}(\omega - k_zv'_\parallel)\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + v'_\perp\frac{n\omega_{cs}}{k_x}k_z\frac{\partial f_{s0\sigma}}{\partial v'_\parallel} + v'_\perp k_zv_{dsx}\frac{\partial f_{s0\sigma}}{\partial v'_\parallel}] \rightarrow (v''_\parallel + v_{dsz})J_n^2[\frac{n\omega_{cs}}{k_x}(\omega'_{sn} + n\omega_{cs} + k_xv_{dsx} - i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp k_z(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{v''_\parallel}{v^2_{zts}}] \rightarrow (v''_\parallel + v_{dsz})J_n^2[\frac{n\omega_{cs}}{k_x}(n\omega_{cs} + k_xv_{dsx} - i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{k_zv''_\parallel}{v^2_{zts}}] + (v''_\parallel + v_{dsz})J_n^2\frac{n\omega_{cs}}{k_x}\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}}(\omega'_{sn})$.

- $v'_\perp \Pi_{s\sigma 23} = (v_{dsy}J_n^2 - iv'_\perp J_nJ'_n)[n\omega_{cs}v'_\parallel\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + v'_\perp(\omega - n\omega_{cs} - k_xv_{dsx})\frac{\partial f_{s0\sigma}}{\partial v'_\parallel}] \rightarrow (v_{dsy}J_n^2 - iv'_\perp J_nJ'_n)[n\omega_{cs}(v''_\parallel + v_{dsz})\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp(\omega'_{sn} + k_zv''_\parallel + k_zv_{dsz} - i\nu_s)\frac{v''_\parallel}{v^2_{zts}}] \rightarrow (v_{dsy}J_n^2 - iv'_\perp J_nJ'_n)[n\omega_{cs}(v''_\parallel + v_{dsz})\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp(k_zv''_\parallel + k_zv_{dsz} - i\nu_s)\frac{v''_\parallel}{v^2_{zts}}] + (v_{dsy}J_n^2 - iv'_\perp J_nJ'_n)v'_\perp\frac{v''_\parallel}{v^2_{zts}}(\omega'_{sn})$.

- $v'_\perp \Pi_{s\sigma 32} = v'_\parallel J_n^2(n\omega_{cs}\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + v'_\perp k_z\frac{\partial f_{s0}}{\partial v'_\parallel})v_{dsy} + iv'_\parallel v'_\perp J_nJ'_n[(\omega - k_zv'_\parallel - k_xv_{dsx})\frac{\partial f_{s0}}{\partial v'_\perp} + k_zv'_\perp\frac{\partial f_{s0\sigma}}{\partial v'_\parallel}] \rightarrow (v''_\parallel + v_{dsz})J_n^2(n\omega_{cs}\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp k_z\frac{v''_\parallel}{v^2_{zts}})v_{dsy} + i(v''_\parallel + v_{dsz})v'_\perp J_nJ'_n[(\omega'_{sn} + n\omega_{cs} - i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + k_zv'_\perp\frac{v''_\parallel}{v^2_{zts}}] \rightarrow (v''_\parallel + v_{dsz})J_n^2(n\omega_{cs}\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp\frac{k_zv''_\parallel}{v^2_{zts}})v_{dsy} + i(v''_\parallel + v_{dsz})v'_\perp J_nJ'_n[(n\omega_{cs} - i\nu_s)\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp\frac{k_zv''_\parallel}{v^2_{zts}}] + i(v''_\parallel + v_{dsz})v'_\perp J_nJ'_n\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}}(\omega'_{sn})$.

- $v'_\perp \Pi_{s\sigma 33} = v'_\parallel J_n^2[n\omega_{cs}v'_\parallel\frac{\partial f_{s0\sigma}}{\partial v'_\perp} + v'_\perp(\omega - n\omega_{cs} - k_xv_{dsx})\frac{\partial f_{s0\sigma}}{\partial v'_\parallel}] \rightarrow (v''_\parallel + v_{dsz})J_n^2[n\omega_{cs}(v''_\parallel + v_{dsz})\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp(\omega'_{sn} + k_zv''_\parallel + k_zv_{dsz} - i\nu_s)\frac{v''_\parallel}{v^2_{zts}}] \rightarrow (v''_\parallel + v_{dsz})J_n^2[n\omega_{cs}(v''_\parallel + v_{dsz})\frac{(v'_\perp-v_{dsr})}{v^2_{\perp ts\sigma}} + v'_\perp(k_zv''_\parallel + k_zv_{dsz} - i\nu_s)\frac{v''_\parallel}{v^2_{zts}}] + v'_\perp J_n^2\frac{(v''_\parallel+v_{dsz})v''_\parallel}{v^2_{zts}}(\omega'_{sn})$.

where '$\rightarrow$' means we have omitted the coefficient '$-2f_{s0\sigma}$'.

We find by set $\nu_s = 0$, $v_{dsx} = v_{dsy} = 0$, these P^m_{sij} reduce exactly the same one as in Ref.[12] ring beam case. By set $\nu_s = 0$, $v_{dsy} = v_{dsr} = 0$, we have also checked that they can reduce to the one in Ref.[11] for drift across magnetic field case.

3.5.3. Final Form

The final form of the electromagnetic dispersion relation is the combine of the above unmagnetized P_s^u and magnetized P_s^m terms to $\boldsymbol{Q}$ in Eq.(56), and then to Eq.(48).

4. Transform to BO-K matrix Equation

The conventional root finding approach to solve the above dispersion relations can only give one solution at one time and heavily depends on initial guess. The KUPDAP [23] code can give multi-solutions, however which is actually using multi initial guesses and may not work well for solutions with $\omega > \omega_{ci}$ due to the singularity around cyclone frequency $\omega \sim n\omega_{cs}$. The Cauchy contour integral approach [22] can locate all the solutions in a selected complex domain, however which still can not give all the important solutions and is also difficult for complicated dispersion relation. To solve the dispersion relation using BO-K matrix approach [1], which can give all the important solution at one time, we need two further steps:

- (1) Do J-pole expansion of Z function, i.e.,

$$Z(\zeta) \simeq Z_J(\zeta) = \sum_{j=1}^{J} \frac{b_j}{\zeta - c_j}, \tag{85}$$

  where b_j and c_j are constants for given J, as given in Ref.[1] for $k_{ts} > 0$;

- (2) Do linear transformation to a equivalent matrix eigenvalue problem, e.g., $\omega \boldsymbol{M_B} \cdot \boldsymbol{X} = \boldsymbol{M_A} \cdot \boldsymbol{X}$. The standard eigenvalue library can solve all the eigenvalues of a matrix, which are the solutions of the dispersion relation.

The first step with $J = 8$ has been used well for more than thirty years in WHAMP [2] code; The second step is firstly developed in the first version of BO-K/PDRK code [1]. We derive the corresponding equations step by step.

We note: $\sum_j b_j = -1$, $\sum_j b_j c_j = 0$, $\sum_j b_j c_j^2 = -1/2$ and $\sum_{j=1}^{J} b_j c_j^3 = 0$ [8]. And also

$$\frac{1}{\omega}\frac{b}{\omega - c} = \frac{b}{c}\Big(\frac{1}{\omega - c} - \frac{1}{\omega}\Big).$$

For $\zeta_s = \frac{\omega - k_{cs}}{k_{ts}}$, we have

$$Z_0(\zeta_s, k_{ts}) = \begin{cases} Z(\zeta_s) \simeq k_{ts} \sum_{j=1}^{J} \frac{b_j}{\omega - k_{cs} - k_{ts} c_j} \\ -\frac{k_{ts}}{\omega - k_{cs}} = k_{ts} \sum_{j=1}^{J} \frac{b_j}{\omega - k_{cs}} \\ -Z(-\zeta_s) \simeq k_{ts} \sum_{j=1}^{J} \frac{b_j}{\omega - k_{cs} + k_{ts} c_j} \end{cases} = k_{ts} \sum_{j=1}^{J} \frac{b_j}{\omega - k_{cs} - |k_{ts}| c_j} = \sum_{j=1}^{J} \frac{k_{ts} b_j}{\omega - c_{sj}},$$

which is written to one compact form for both $k_{ts} > 0$ and $k_{ts} \leq 0$, with $c_{sj} = k_{cs} + |k_{ts}| c_j$. And

$$Z_1(\zeta_s) = 1 + \zeta_s Z_0(\zeta_s) \quad \simeq \quad 1 + \zeta_s \sum_{j=1}^{J} \frac{b_j}{\zeta_s - c_j} = 1 + \sum_{j=1}^{J} \Big[b_j + \frac{b_j c_j}{\zeta_s - c_j}\Big] = \sum_{j=1}^{J} \frac{b_j |k_{ts}| c_j}{\omega - c_{sj}},$$

where we have used $\sum_{j=1}^{J} b_j = -1$. And

$$\begin{aligned} Z_2(\zeta_s) &= \zeta_s(1 + \zeta_s Z_0) \simeq \frac{\omega - k_{cs}}{k_{ts}} \sum_{j=1}^{J} \frac{b_j |k_{ts}| c_j}{\omega - c_{sj}} = \frac{|k_{ts}|}{k_{ts}} \sum_{j=1}^{J} \frac{b_j c_j (\omega - k_{cs})}{\omega - c_{sj}} \\ &= \frac{|k_{ts}|}{k_{ts}} \sum_{j=1}^{J} \Big[b_j c_j + \frac{b_j c_j (c_{sj} - k_{cs})}{\omega - c_{sj}}\Big] = \frac{1}{k_{ts}} \sum_{j=1}^{J} \frac{b_j |k_{ts}|^2 c_j^2}{\omega - c_{sj}}, \end{aligned}$$

where we have used $\sum_{j=1}^{J} b_j c_j = 0$. And

$$\begin{aligned}
Z_3(\zeta_s) &= \frac{1}{2} + \zeta_s^2(1+\zeta_s Z_0) \simeq \frac{1}{2} + \frac{\omega - k_{cs}}{k_{ts}} k_{ts} \sum_{j=1}^{J} \frac{b_j c_j^2}{\omega - c_{sj}} = \frac{1}{2} + \sum_{j=1}^{J} \frac{b_j c_j^2(\omega - k_{cs})}{\omega - c_{sj}} \\
&= \frac{1}{2} + \sum_{j=1}^{J} \Big[b_j c_j^2 + \frac{b_j c_j^2 (c_{sj} - k_{cs})}{\omega - c_{sj}} \Big] = \frac{1}{k_{ts}^2} \sum_{j=1}^{J} \frac{b_j |k_{ts}|^3 c_j^3}{\omega - c_{sj}},
\end{aligned}$$

where we have used $\sum_{j=1}^{J} b_j c_j^2 = -1/2$. In the above $Z_{0,1,2,3}$ and c_{sj}, we find for $k_{ts} < 0$ it could be simply by doing the variables change with: $|k_{ts}| \to k_{ts}$ and $c_j \to -c_j$. Thus, in the later usage, to simplify the notation, all $|k_{ts}|$ is changed to be k_{ts}, and the meaning of c_j for $k_{ts} < 0$ is the $-c_j$ in the default Z_J. Thus for both $k_{ts} > 0$ and $k_{ts} \leq 0$, we have a single compact form

$$Z_p(\zeta_s) \simeq k_{ts} \sum_{j=1}^{J} \frac{b_j c_j^p}{\omega - c_{sj}},$$

with $c_{sj} = k_{cs} + k_{ts} c_j$. And the only change is that $c_j = c_{j0}$ for $k_{ts} \geq 0$ and $c_j = -c_{j0}$ for $k_{ts} < 0$, with c_{j0} be the c_j for $k_{ts} > 0$.

Typically, for magnetized species $\zeta_{sn} = \frac{\omega - k_z v_{dsz} - n\omega_{cs} - k_x v_{dsx} + i\nu_s}{k_z v_{zts}}$ and unmagnetized species $\zeta_{s\sigma} = \frac{\omega - k_x v_{dsx} - k_z v_{dsz} + i\nu_s}{k v_{ts\sigma}}$, with $k v_{ts\sigma} = \sqrt{k_x^2 v_{\perp ts\sigma}^2 + k_z^2 v_{zts}^2}$ (which always ≥ 0), we have corresponding $k_{ts} = k_z v_{zts}$ and $k v_{ts\sigma}$, respectively. For c_{sj}: $c_{snj} = k_z v_{dsz} + n\omega_{cs} + k_x v_{dsx} - i\nu_s + k_z v_{zts} c_j$, and $c_{sj\sigma} = k_z v_{dsz} + k_x v_{dsx} - i\nu_s + k v_{ts\sigma} c_j$.

4.1. The Electrostatic case

We solve Eq.(83) for electrostatic case. The corresponding linear transformation is straightforward and simple.

$$\begin{aligned}
D(\omega, \boldsymbol{k}) \simeq{}& 1 + \sum_{s=m} \frac{\omega_{ps}^2}{k^2 v_{zts}^2} \sum_{n=-\infty}^{\infty} \sum_{\sigma=a,b} r_{s\sigma} \Big\{ [1 + \zeta_{sn} \sum_{j=1}^{J} \frac{b_j}{\zeta_{sn} - c_j}] A_{n0\sigma} + \frac{n\omega_{cs}\lambda_{Ts\sigma}}{k_z v_{zts}} \sum_{j=1}^{J} \frac{b_j}{\zeta_{sn} - c_j} A_{nbs\sigma} \Big\} \\
&+ \sum_{s=u} \frac{\omega_{ps}^2}{k^2} \sum_{\sigma=a,b} \frac{2r_{s\sigma}}{v_{ts\sigma}^2} [1 + \zeta_{s\sigma} \sum_{j=1}^{J} \frac{b_j}{\zeta_{s\sigma} - c_j}] \\
={}& 1 + \sum_{s=m} \frac{\omega_{ps}^2}{k^2 v_{zts}^2} \sum_{n=-\infty}^{\infty} \sum_{\sigma=a,b} r_{s\sigma} \sum_{j=1}^{J} \frac{A_{n0\sigma} b_j c_j + \frac{A_{nbs\sigma} n\omega_{cs}\lambda_{Ts\sigma}}{k_z v_{zts}} b_j}{\zeta_{sn} - c_j} + \sum_{s=u} \frac{\omega_{ps}^2}{k^2} \sum_{\sigma=a,b} \frac{2r_{s\sigma}}{v_{ts\sigma}^2} \sum_{j=1}^{J} \frac{b_j c_j}{\zeta_{s\sigma} - c_j} \\
={}& 1 + \sum_{sn\sigma j}^{m} \frac{r_{s\sigma}\omega_{ps}^2}{k^2 v_{zts}^2} \frac{A_{n0\sigma} k_z v_{zts} b_j c_j + A_{nbs\sigma} n\omega_{cs}\lambda_{Ts\sigma} b_j}{\omega - c_{snj}} + \sum_{sj\sigma}^{u} \frac{2r_{s\sigma}\omega_{ps}^2}{k v_{ts\sigma}} \frac{b_j c_j}{\omega - c_{sj\sigma}} \\
={}& 1 + \sum_{snj}^{m} \frac{b_{snj}}{\omega - c_{snj}} + \sum_{sj\sigma}^{u} \frac{b_{sj\sigma}}{\omega - c_{sj\sigma}} = 0.
\end{aligned} \tag{86}$$

Notation: $\zeta_{sn} = \frac{\omega - k_z v_{dsz} - n\omega_{cs} - k_x v_{dsx} + i\nu_s}{k_z v_{zts}}$, $\zeta_{s\sigma} = \frac{\omega - k_x v_{dsx} - k_z v_{dsz} + i\nu_s}{k v_{ts\sigma}}$, $k v_{ts\sigma} = \sqrt{k_x^2 v_{\perp ts\sigma}^2 + k_z^2 v_{zts}^2} = v_{zts}\sqrt{k_x^2/\lambda_{Ts\sigma} + k_z^2}$.

Define: $c_{snj} = k_z v_{dsz} + n\omega_{cs} + k_x v_{dsx} - i\nu_s + k_z v_{zts} c_j$, $b_{snj} = \sum_\sigma \frac{r_{s\sigma}\omega_{ps}^2}{k^2 v_{zts}^2}(A_{n0\sigma} k_z v_{zts} b_j c_j + A_{nbs\sigma} n\omega_{cs}\lambda_{Ts\sigma} b_j)$, $c_{sj\sigma} = k_z v_{dsz} + k_x v_{dsx} - i\nu_s + k v_{ts\sigma} c_j$, and $b_{sj\sigma} = \frac{2r_{s\sigma}\omega_{ps}^2}{k v_{ts\sigma}} b_j c_j$.

The equivalent linear system can be

$$\begin{aligned}
\omega n_{sj} &= c_{sj} n_{sj} + b_{sj} E, \\
E &= -\sum_{sj} n_{sj},
\end{aligned} \tag{87}$$

or sparse matrix one

$$
\begin{aligned}
\omega n_{sj} &= c_{sj} n_{sj} + b_{sj} E, \\
\omega E &= -\sum_{sj} c_{sj} n_{sj} - \sum_{sj} b_{sj} E,
\end{aligned} \tag{88}
$$

where b_{sj} and c_{sj} is short notation for both unmagnetized and magnetized species $b_{sj\sigma,snj}$ and $c_{sj\sigma,snj}$. The symbols n_{sj} and E used here do not have direct physical meanings but are analogy to the perturbed density and electric field in the fluid derivations of plasma waves. We find the only singularity in the above final form occurs at $k = 0$, which requires $k_x = k_z = 0$. And thus the final form can be applied for arbitrary $k \neq 0$. Some other solvers in literature may meet singularity for $k_z = 0$ or $k_x = 0$, and may be incorrect for $k_z \leq 0$.

It is also obvious that the magnetized species can not reduce to unmagnetized species by set $\omega_{cs} = 0$. The major difference is $k_z v_{zts}$ in magnetized species and $k v_{ts\sigma}$ in unmagnetized species.

4.2. The Electromagnetic case

The electromagnetic case is much complicated. However, the linear transformation for $\boldsymbol{Q}^m(\omega, \boldsymbol{k})$ is still similar to the original BO-K/PDRK derivation.

To seek an equivalent linear system, the Maxwells equations

$$
\partial_t \boldsymbol{E} = c^2 \nabla \times \boldsymbol{B} - \boldsymbol{J}/\epsilon_0, \tag{89a}
$$

$$
\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \tag{89b}
$$

do not need to be changed. We only need to seek a new linear system for $\boldsymbol{J} = \boldsymbol{J}^m + \boldsymbol{J}^u = (\boldsymbol{\sigma^m} + \boldsymbol{\sigma^u}) \cdot \boldsymbol{E} = \boldsymbol{\sigma} \cdot \boldsymbol{E}$.

4.2.1. The unmagnetized terms

Considering the defination $\boldsymbol{\sigma}_s^u = -i\epsilon_0 \omega \boldsymbol{Q}_s^u = -i\epsilon_0 \sum_{\sigma=a,b} r_{s\sigma} \frac{\omega_{ps}^2}{\omega} \boldsymbol{P}_{s\sigma}^u$, after J-pole expansion, we have

- $P^u_{s\sigma 11} = -1 + \frac{2}{k v_{ts\sigma}} \Big\{ -i\nu_s \big(\frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} Z_2 + \frac{k_x v_{dsx}}{k v_{ts\sigma}} Z_1 + \frac{k_z^2 v_{zts}^2}{2k^2 v_{ts\sigma}^2} Z_0 \big) + \frac{k^2 k_x^2 v_{\perp ts\sigma}^4}{k^3 v_{ts\sigma}^3} Z_3 + 2 \frac{k^2 k_x v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} v_{dsx} Z_2 + [\frac{1}{2} v_{\perp ts\sigma}^2 \frac{k^2 k_z^2 v_{zts}^2}{k^3 v_{ts\sigma}^3} + \frac{k^2}{k v_{ts\sigma}} v_{dsx}^2 + (\lambda_{Ts\sigma} - 1) \frac{k_x^2 k_z^2 v_{\perp ts\sigma}^4}{k^3 v_{ts\sigma}^3}] Z_1 + \frac{k_x k_z^2}{k^2 v_{ts\sigma}^2} (\lambda_{Ts\sigma} - 1) v_{\perp ts\sigma}^2 v_{dsx} Z_0 \Big\} \simeq -1 + 2 \sum_{j=1}^J \frac{b_j}{\omega - c_{sj\sigma}} \Big\{ -i\nu_s \big(\frac{k_x^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} c_j^2 + \frac{k_x v_{dsx}}{k v_{ts\sigma}} c_j + \frac{k_z^2 v_{zts}^2}{2k^2 v_{ts\sigma}^2} \big) + \frac{k^2 k_x^2 v_{\perp ts\sigma}^4}{k^3 v_{ts\sigma}^3} c_j^3 + 2 \frac{k^2 k_x v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} v_{dsx} c_j^2 + [\frac{1}{2} v_{\perp ts\sigma}^2 \frac{k^2 k_z^2 v_{zts}^2}{k^3 v_{ts\sigma}^3} + \frac{k^2}{k v_{ts\sigma}} v_{dsx}^2 + (\lambda_{Ts\sigma} - 1) \frac{k_x^2 k_z^2 v_{\perp ts\sigma}^4}{k^3 v_{ts\sigma}^3}] c_j + \frac{k_x k_z^2}{k^2 v_{ts\sigma}^2} (\lambda_{Ts\sigma} - 1) v_{\perp ts\sigma}^2 v_{dsx} \Big\} = -1 + \sum_{j=1}^J \frac{p_{11sj\sigma}}{\omega - c_{sj\sigma}}$,

- $P^u_{s\sigma 12} = \frac{2 v_{dsy}}{k v_{ts\sigma}} \Big\{ \frac{k^2 k_x v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} Z_2 + \frac{k^2}{k v_{ts\sigma}} v_{dsx} Z_1 + \frac{1}{2} \frac{k_x k_z^2}{k^2 v_{ts\sigma}^2} (\lambda_{Ts\sigma} - 1) v_{\perp ts\sigma}^2 Z_0 \Big\} \simeq 2 v_{dsy} \sum_{j=1}^J \frac{b_j}{\omega - c_{sj\sigma}} \Big\{ \frac{k^2 k_x v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} c_j^2 + \frac{k^2 v_{dsx}}{k v_{ts\sigma}} c_j + \frac{1}{2} \frac{k_x k_z^2}{k^2 v_{ts\sigma}^2} (\lambda_{Ts\sigma} - 1) v_{\perp ts\sigma}^2 \Big\} = \sum_{j=1}^J \frac{p_{12sj\sigma}}{\omega - c_{sj\sigma}}$,

- $P^u_{s\sigma 21} = \frac{2 v_{dsy}}{k v_{ts\sigma}} \Big\{ -i\nu_s \frac{k_x}{k v_{ts\sigma}} Z_1 + \frac{k^2 k_x v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} Z_2 + \frac{k^2}{k v_{ts\sigma}} v_{dsx} Z_1 + \frac{1}{2} \frac{k_x k_z^2}{k^2 v_{ts\sigma}^2} (\lambda_{Ts\sigma} - 1) v_{\perp ts\sigma}^2 Z_0 \Big\} \simeq 2 v_{dsy} \sum_{j=1}^J \frac{b_j}{\omega - c_{sj\sigma}} \Big\{ -i\nu_s \frac{k_x}{k v_{ts\sigma}} c_j + \frac{k^2 k_x v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} c_j^2 + \frac{k^2}{k v_{ts\sigma}} v_{dsx} c_j + \frac{1}{2} \frac{k_x k_z^2}{k^2 v_{ts\sigma}^2} (\lambda_{Ts\sigma} - 1) v_{\perp ts\sigma}^2 \Big\} = \sum_{j=1}^J \frac{p_{21sj\sigma}}{\omega - c_{sj\sigma}}$,

- $P^u_{s\sigma 22} = -1 + \frac{2}{k v_{ts\sigma}} \Big\{ -i\nu_s \frac{1}{2} Z_0 + (\frac{1}{2} v_{\perp ts\sigma}^2 + v_{dsy}^2) \frac{k^2}{k v_{ts\sigma}} Z_1 \Big\} \simeq -1 + 2 \sum_{j=1}^J \frac{b_j}{\omega - c_{sj\sigma}} \Big\{ -i\nu_s \frac{1}{2} + (\frac{1}{2} v_{\perp ts\sigma}^2 + v_{dsy}^2) \frac{k^2}{k v_{ts\sigma}} c_j \Big\} = -1 + \sum_{j=1}^J \frac{p_{22sj\sigma}}{\omega - c_{sj\sigma}}$,

- $P^u_{s\sigma 13} = \frac{2}{k v_{ts\sigma}} \Big\{ -i\nu_s \big(\frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} Z_2 + \frac{k_z v_{dsx}}{k v_{ts\sigma}} Z_1 - \frac{k_x k_z v_{\perp ts\sigma}^2}{2k^2 v_{ts\sigma}^2} Z_0 \big) + \frac{k^2 k_x k_z v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3} Z_3 + \frac{k^2}{k v_{ts\sigma}} [\frac{k_x v_{\perp ts\sigma}^2}{k v_{ts\sigma}} v_{dsz} + v_{dsx} \frac{k_z v_{zts}^2}{k v_{ts\sigma}}] Z_2 + [\frac{k^2}{k v_{ts\sigma}} v_{dsx} v_{dsz} - \frac{1}{2} \frac{k^2 k_x k_z v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3} - \frac{1}{2} (\lambda_{Ts\sigma} - 1) \frac{k_x^3 k_z v_{\perp ts\sigma}^4}{k^3 v_{ts\sigma}^3} + \frac{1}{2} (\lambda_{Ts\sigma} - 1) \frac{k_x k_z^3 v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3}] Z_1 + [\frac{1}{2} (\lambda_{Ts\sigma} - 1) \frac{k_x k_z^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} v_{dsz} - \frac{1}{2} (\lambda_{Ts\sigma} - 1) v_{dsx} \frac{k_x^2 k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}] Z_0 \Big\} \simeq 2 \sum_{j=1}^J \frac{b_j}{\omega - c_{sj\sigma}} \Big\{ -i\nu_s \big(\frac{k_x k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} c_j^2 + \frac{k_z v_{dsx}}{k v_{ts\sigma}} c_j - \frac{k_x k_z v_{\perp ts\sigma}^2}{2k^2 v_{ts\sigma}^2} \big) + \frac{k^2 k_x k_z v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3} c_j^3 + \frac{k^2}{k v_{ts\sigma}} [\frac{k_x v_{\perp ts\sigma}^2}{k v_{ts\sigma}} v_{dsz} + v_{dsx} \frac{k_z v_{zts}^2}{k v_{ts\sigma}}] c_j^2 + [\frac{k^2}{k v_{ts\sigma}} v_{dsx} v_{dsz} - \frac{1}{2} \frac{k^2 k_x k_z v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3} - \frac{1}{2} (\lambda_{Ts\sigma} - 1) \frac{k_x^3 k_z v_{\perp ts\sigma}^4}{k^3 v_{ts\sigma}^3} + \frac{1}{2} (\lambda_{Ts\sigma} - 1) \frac{k_x k_z^3 v_{zts}^2 v_{\perp ts\sigma}^2}{k^3 v_{ts\sigma}^3}] c_j + [\frac{1}{2} (\lambda_{Ts\sigma} - 1) \frac{k_x k_z^2 v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2} v_{dsz} - \frac{1}{2} (\lambda_{Ts\sigma} - 1) v_{dsx} \frac{k_x^2 k_z v_{\perp ts\sigma}^2}{k^2 v_{ts\sigma}^2}] \Big\} = \sum_{j=1}^J \frac{p_{13sj\sigma}}{\omega - c_{sj\sigma}}$,

- $P^u_{s\sigma31} = \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s(\frac{k_xk_zv^2_{zts}}{k^2v^2_{ts\sigma}}Z_2 + \frac{k_xv_{dsz}}{kv_{ts\sigma}}Z_1 - \frac{k_xk_zv^2_{zts}}{2k^2v^2_{ts\sigma}}Z_0) + \frac{k^2k_xk_zv^2_{zts}v^2_{\perp ts\sigma}}{k^3v^3_{ts\sigma}}Z_3 + \frac{k^2}{kv_{ts\sigma}}[\frac{k_xv^2_{\perp ts\sigma}}{kv_{ts\sigma}}v_{dsz} + v_{dsx}\frac{k_zv^2_{zts}}{kv_{ts\sigma}}]Z_2 + [\frac{k^2}{kv_{ts\sigma}}v_{dsx}v_{dsz} - \frac{1}{2}\frac{k^2k_xk_zv^2_{zts}v^2_{\perp ts\sigma}}{k^3v^3_{ts\sigma}} - \frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k^3_xk_zv^4_{\perp ts\sigma}}{k^3v^3_{ts\sigma}} + \frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_xk^3_zv^2_{zts}v^2_{\perp ts\sigma}}{k^3v^3_{ts\sigma}}]Z_1 + [\frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_xk^2_zv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}v_{dsz} - \frac{1}{2}(\lambda_{Ts\sigma}-1)v_{dsx}\frac{k^2_xk_zv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}]Z_0\Big\} \simeq 2\sum_{j=1}^J \frac{b_j}{\omega-c_{sj\sigma}}\Big\{ - i\nu_s(\frac{k_xk_zv^2_{zts}}{k^2v^2_{ts\sigma}}c^2_j + \frac{k_xv_{dsz}}{kv_{ts\sigma}}c_j - \frac{k_xk_zv^2_{zts}}{2k^2v^2_{ts\sigma}}) + \frac{k^2k_xk_zv^2_{zts}v^2_{\perp ts\sigma}}{k^3v^3_{ts\sigma}}c^3_j + \frac{k^2}{kv_{ts\sigma}}[\frac{k_xv^2_{\perp ts\sigma}}{kv_{ts\sigma}}v_{dsz} + v_{dsx}\frac{k_zv^2_{zts}}{kv_{ts\sigma}}]c^2_j + [\frac{k^2}{kv_{ts\sigma}}v_{dsx}v_{dsz} - \frac{1}{2}\frac{k^2k_xk_zv^2_{zts}v^2_{\perp ts\sigma}}{k^3v^3_{ts\sigma}} - \frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k^3_xk_zv^4_{\perp ts\sigma}}{k^3v^3_{ts\sigma}} + \frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_xk^3_zv^2_{zts}v^2_{\perp ts\sigma}}{k^3v^3_{ts\sigma}}]c_j + [\frac{1}{2}(\lambda_{Ts\sigma}-1)\frac{k_xk^2_zv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}v_{dsz} - \frac{1}{2}(\lambda_{Ts\sigma}-1)v_{dsx}\frac{k^2_xk_zv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}]\Big\} = \sum_{j=1}^J \frac{p_{31sj\sigma}}{\omega-c_{sj\sigma}}$,

- $P^u_{s\sigma23} = \frac{2v_{dsy}}{kv_{ts\sigma}}\Big\{ - i\nu_s\frac{k_z}{kv_{ts\sigma}}Z_1 + \frac{k^2k_zv^2_{zts}}{k^2v^2_{ts\sigma}}Z_2 + \frac{k^2}{kv_{ts\sigma}}v_{dsz}Z_1 - \frac{1}{2}(\lambda_{Ts\sigma} - 1)\frac{k^2_xk_zv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}Z_0\Big\} \simeq 2v_{dsy}\sum_{j=1}^J \frac{b_j}{\omega-c_{sj\sigma}}\Big\{ - i\nu_s\frac{k_z}{kv_{ts\sigma}}c_j + \frac{k^2k_zv^2_{zts}}{k^2v^2_{ts\sigma}}c^2_j + \frac{k^2}{kv_{ts\sigma}}v_{dsz}c_j - \frac{1}{2}(\lambda_{Ts\sigma} - 1)\frac{k^2_xk_zv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}\Big\} = \sum_{j=1}^J \frac{p_{23sj\sigma}}{\omega-c_{sj\sigma}}$,

- $P^u_{s\sigma32} = \frac{2v_{dsy}}{kv_{ts\sigma}}\Big\{\frac{k^2k_zv^2_{zts}}{k^2v^2_{ts\sigma}}Z_2 + \frac{k^2}{kv_{ts\sigma}}v_{dsz}Z_1 - \frac{1}{2}(\lambda_{Ts\sigma} - 1)\frac{k^2_xk_zv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}Z_0\Big\} \simeq 2v_{dsy}\sum_{j=1}^J \frac{b_j}{\omega-c_{sj\sigma}}\Big\{\frac{k^2k_zv^2_{zts}}{k^2v^2_{ts\sigma}}c^2_j + \frac{k^2}{kv_{ts\sigma}}v_{dsz}c_j - \frac{1}{2}(\lambda_{Ts\sigma} - 1)\frac{k^2_xk_zv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}\Big\} = \sum_{j=1}^J \frac{p_{32sj\sigma}}{\omega-c_{sj\sigma}}$,

- $P^u_{s\sigma33} = -1 + \frac{2}{kv_{ts\sigma}}\Big\{-i\nu_s(\frac{k^2_zv^2_{zts}}{k^2v^2_{ts\sigma}}Z_2 + \frac{k_zv_{dsz}}{kv_{ts\sigma}}Z_1 + \frac{k^2_xv^2_{\perp ts\sigma}}{2k^2v^2_{ts\sigma}}Z_0) + \frac{k^2k^2_zv^4_{zts}}{k^3v^3_{ts\sigma}}Z_3 + 2\frac{k^2k_zv^2_{zts}}{k^2v^2_{ts\sigma}}v_{dsz}Z_2 + [\frac{1}{2}\frac{k^2k^2_xv^2_{zts}v^2_{\perp ts\sigma}}{k^3v^3_{ts\sigma}} + \frac{k^2}{kv_{ts\sigma}}v^2_{dsz} - (\lambda_{Ts\sigma} - 1)\frac{k^2_xk^2_zv^2_{zts}v^2_{\perp ts\sigma}}{k^3v^3_{ts\sigma}}]Z_1 - (\lambda_{Ts\sigma} - 1)\frac{k^2_xk_zv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}v_{dsz}Z_0\Big\} \simeq -1 + 2\sum_{j=1}^J \frac{b_j}{\omega-c_{sj\sigma}}\Big\{ - i\nu_s(\frac{k^2_zv^2_{zts}}{k^2v^2_{ts\sigma}}c^2_j + \frac{k_zv_{dsz}}{kv_{ts\sigma}}c_j + \frac{k^2_xv^2_{\perp ts\sigma}}{2k^2v^2_{ts\sigma}}) + \frac{k^2k^2_zv^4_{zts}}{k^3v^3_{ts\sigma}}c^3_j + 2\frac{k^2k_zv^2_{zts}}{k^2v^2_{ts\sigma}}v_{dsz}c^2_j + [\frac{1}{2}\frac{k^2k^2_xv^2_{zts}v^2_{\perp ts\sigma}}{k^3v^3_{ts\sigma}} + \frac{k^2}{kv_{ts\sigma}}v^2_{dsz} - (\lambda_{Ts\sigma} - 1)\frac{k^2_xk^2_zv^2_{zts}v^2_{\perp ts\sigma}}{k^3v^3_{ts\sigma}}]c_j - (\lambda_{Ts\sigma} - 1)\frac{k^2_xk_zv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}v_{dsz}\Big\} = -1 + \sum_{j=1}^J \frac{p_{33sj\sigma}}{\omega-c_{sj\sigma}}$.

In the above, for example, $p_{11sj\sigma} = 2b_j\Big\{-i\nu_s(\frac{k^2_xv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}c^2_j + \frac{k_xv_{dsx}}{kv_{ts\sigma}}c_j + \frac{k^2_zv^2_{zts}}{2k^2v^2_{ts\sigma}}) + \frac{k^2k^2_xv^4_{\perp ts\sigma}}{k^3v^3_{ts\sigma}}c^3_j + 2\frac{k^2k_xv^2_{\perp ts\sigma}}{k^2v^2_{ts\sigma}}v_{dsx}c^2_j + [\frac{1}{2}v^2_{\perp ts\sigma}\frac{k^2k^2_zv^2_{zts}}{k^3v^3_{ts\sigma}} + \frac{k^2}{kv_{ts\sigma}}v^2_{dsx} + (\lambda_{Ts\sigma} - 1)\frac{k^2_xk^2_zv^4_{\perp ts\sigma}}{k^3v^3_{ts\sigma}}]c_j + \frac{k_xk^2_z}{k^2v^2_{ts\sigma}}(\lambda_{Ts\sigma} - 1)v^2_{\perp ts\sigma}v_{dsx}\Big\}$, and others are similar and thus we have not written them out explicitly.

We find the result is very simple by use $Z_{0,1,2,3}$, which would also make the EM3D-M case be much simpler than the previous BO-K [1] derivation. This is also why we use $Z_{0,1}$ in the ES3D case, which gives a more compact form.

Thus, we obtain the relations between $\boldsymbol{J}^u$ and $\boldsymbol{E}$, which has the following form (with $\sum_{s=u}$)

$$\begin{pmatrix} J^u_x \\ J^u_y \\ J^u_z \end{pmatrix} = -i\epsilon_0\begin{pmatrix} \frac{b^u_{11}}{\omega} + \sum_{sj\sigma}\frac{b_{sj\sigma11}}{\omega-c_{sj\sigma}} & \frac{b^u_{12}}{\omega} + \sum_{sj\sigma}\frac{b_{sj\sigma12}}{\omega-c_{sj\sigma}} & \frac{b^u_{13}}{\omega} + \sum_{sj\sigma}\frac{b_{sj\sigma13}}{\omega-c_{sj\sigma}} \\ \frac{b^u_{21}}{\omega} + \sum_{sj\sigma}\frac{b_{sj\sigma21}}{\omega-c_{sj\sigma}} & \frac{b^u_{22}}{\omega} + \sum_{sj\sigma}\frac{b_{sj\sigma22}}{\omega-c_{sj\sigma}} & \frac{b^u_{23}}{\omega} + \sum_{sj\sigma}\frac{b_{sj\sigma23}}{\omega-c_{sj\sigma}} \\ \frac{b^u_{31}}{\omega} + \sum_{sj\sigma}\frac{b_{sj\sigma31}}{\omega-c_{sj\sigma}} & \frac{b^u_{32}}{\omega} + \sum_{sj\sigma}\frac{b_{sj\sigma32}}{\omega-c_{sj\sigma}} & \frac{b^u_{33}}{\omega} + \sum_{sj\sigma}\frac{b_{sj\sigma33}}{\omega-c_{sj\sigma}} \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \tag{90}$$

with the coefficients

$$\left\{\begin{aligned} b_{sj\sigma11} &= r_{s\sigma}\omega^2_{ps}p_{11sj\sigma}/c_{sj\sigma}, & b^u_{11} &= -\sum_s^{s=u}\omega^2_{ps}[1 + \sum_\sigma r_{s\sigma}\sum_j p_{11sj\sigma}/c_{sj\sigma}], \\ b_{sj\sigma12} &= r_{s\sigma}\omega^2_{ps}p_{12sj\sigma}/c_{sj\sigma}, & b^u_{12} &= -\sum_s^{s=u}\omega^2_{ps}[\sum_\sigma r_{s\sigma}\sum_j p_{12sj\sigma}/c_{sj\sigma}], \\ b_{sj\sigma21} &= r_{s\sigma}\omega^2_{ps}p_{21sj\sigma}/c_{sj\sigma}, & b^u_{21} &= -\sum_s^{s=u}\omega^2_{ps}[\sum_\sigma r_{s\sigma}\sum_j p_{21sj\sigma}/c_{sj\sigma}], \\ b_{sj\sigma22} &= r_{s\sigma}\omega^2_{ps}p_{22sj\sigma}/c_{sj\sigma}, & b^u_{22} &= -\sum_s^{s=u}\omega^2_{ps}[1 + \sum_\sigma r_{s\sigma}\sum_j p_{22sj\sigma}/c_{sj\sigma}], \\ b_{sj\sigma13} &= r_{s\sigma}\omega^2_{ps}p_{13sj\sigma}/c_{sj\sigma}, & b^u_{13} &= -\sum_s^{s=u}\omega^2_{ps}[\sum_\sigma r_{s\sigma}\sum_j p_{13sj\sigma}/c_{sj\sigma}], \\ b_{sj\sigma31} &= r_{s\sigma}\omega^2_{ps}p_{31sj\sigma}/c_{sj\sigma}, & b^u_{31} &= -\sum_s^{s=u}\omega^2_{ps}[\sum_\sigma r_{s\sigma}\sum_j p_{31sj\sigma}/c_{sj\sigma}], \\ b_{sj\sigma23} &= r_{s\sigma}\omega^2_{ps}p_{23sj\sigma}/c_{sj\sigma}, & b^u_{23} &= -\sum_s^{s=u}\omega^2_{ps}[\sum_\sigma r_{s\sigma}\sum_j p_{23sj\sigma}/c_{sj\sigma}], \\ b_{sj\sigma32} &= r_{s\sigma}\omega^2_{ps}p_{32sj\sigma}/c_{sj\sigma}, & b^u_{32} &= -\sum_s^{s=u}\omega^2_{ps}[\sum_\sigma r_{s\sigma}\sum_j p_{32sj\sigma}/c_{sj\sigma}], \\ b_{sj\sigma33} &= r_{s\sigma}\omega^2_{ps}p_{33sj\sigma}/c_{sj\sigma}, & b^u_{33} &= -\sum_s^{s=u}\omega^2_{ps}[1 + \sum_\sigma r_{s\sigma}\sum_j p_{33sj\sigma}/c_{sj\sigma}], \\ & & c_{sj\sigma} &= k_zv_{dsz} + k_xv_{dsx} - i\nu_s + kv_{ts\sigma}c_j, \end{aligned}\right. \tag{91}$$

Considering that c_j are complex number and k_x, k_z and ν_s are real number, the singularity of the above form is also only $k = 0$.

4.2.2. The magnetized terms

Similarly to the unmagnetized case, considering the defination $\boldsymbol{\sigma}^m_s = -i\epsilon_0\omega\boldsymbol{Q}^m_s = -i\epsilon_0\sum_{\sigma=a,b} r_{s\sigma}\frac{\omega^2_{ps}}{\omega}\boldsymbol{P}^m_{s\sigma}$, after J-pole expansion, we have

- $P^m_{s\sigma 11} \simeq \sum_{n=-\infty}^{\infty}\sum_{j=1}^{J}\frac{k_z v_{zts} b_j}{\omega - c_{snj}}\Big\{[\frac{n\omega_{cs}}{k_x}(n\omega_{cs}+k_x v_{dsx}-i\nu_s)\frac{A_{nbs\sigma}}{v^2_{\perp ts\sigma}}\frac{1}{k_z v_{zts}}+A_{n0\sigma}(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{c_j}{v^2_{zts}}](\frac{n\omega_{cs}}{k_x}+v_{dsx})\Big\}-1=-1+\sum_n\sum_j\frac{p_{11snj}}{\omega-c_{snj}}$.
- $P^m_{s\sigma 12} \simeq \sum_{n=-\infty}^{\infty}\sum_{j=1}^{J}\frac{k_z v_{zts} b_j}{\omega - c_{snj}}\Big\{v_{dsy}(\frac{n\omega_{cs}}{k_x}+v_{dsx})(n\omega_{cs}\frac{A_{nbs\sigma}}{v^2_{\perp ts\sigma}}\frac{1}{k_z v_{zts}}+A_{n0\sigma}\frac{c_j}{v^2_{zts}})+i(\frac{n\omega_{cs}}{k_x}+v_{dsx})[(n\omega_{cs}-i\nu_s)\frac{B_{nbs\sigma}}{v_{\perp ts\sigma}}\frac{1}{k_z v_{zts}}+\frac{B_{n0\sigma}v_{\perp ts\sigma}c_j}{v^2_{zts}}]\Big\}=\sum_n\sum_j\frac{p_{12snj}}{\omega-c_{snj}}$.
- $P^m_{s\sigma 21} \simeq \sum_{n=-\infty}^{\infty}\sum_{j=1}^{J}\frac{k_z v_{zts} b_j}{\omega - c_{snj}}\Big\{\frac{n\omega_{cs}}{k_x}(n\omega_{cs}+k_x v_{dsx}-i\nu_s)[-iv_{\perp ts\sigma}B_{nbs\sigma}+v_{dsy}A_{nbs\sigma}]\frac{1}{v^2_{\perp ts\sigma}}\frac{1}{k_z v_{zts}}+[-iv_{\perp ts\sigma}B_{n0\sigma}+v_{dsy}A_{n0\sigma}](\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{c_j}{v^2_{zts}}\Big\}=\sum_n\sum_j\frac{p_{21snj}}{\omega-c_{snj}}$.
- $P^m_{s\sigma 22} \simeq \sum_{n=-\infty}^{\infty}\sum_{j=1}^{J}\frac{k_z v_{zts} b_j}{\omega - c_{snj}}\Big\{v_{dsy}[-iv_{\perp ts\sigma}B_{nbs\sigma}+v_{dsy}A_{nbs\sigma}]\frac{n\omega_{cs}}{v^2_{\perp ts\sigma}}\frac{1}{k_z v_{zts}}+v_{dsy}[-iv_{\perp ts\sigma}B_{n0\sigma}+v_{dsy}A_{n0\sigma}]\frac{c_j}{v^2_{zts}}+v_{\perp ts\sigma}[v_{\perp ts\sigma}C_{nbs\sigma}+iv_{dsy}B_{nbs\sigma}](n\omega_{cs}-i\nu_s)\frac{1}{v^2_{\perp ts\sigma}}\frac{1}{k_z v_{zts}}+v_{\perp ts\sigma}[v_{\perp ts\sigma}C_{n0\sigma}+iv_{dsy}B_{n0\sigma}]\frac{c_j}{v^2_{zts}}\Big\}-1=-1+\sum_n\sum_j\frac{p_{22snj}}{\omega-c_{snj}}$.
- $P^m_{s\sigma 13} \simeq \sum_{n=-\infty}^{\infty}\sum_{j=1}^{J}\frac{k_z v_{zts} b_j}{\omega - c_{snj}}\Big\{(\frac{n\omega_{cs}}{k_x}+v_{dsx})[n\omega_{cs}(\frac{c_j}{k_z}+v_{dsz}\frac{1}{k_z v_{zts}})\frac{A_{nbs\sigma}}{v^2_{\perp ts\sigma}}+A_{n0\sigma}(v_{zts}c_j^2+(k_z v_{dsz}-i\nu_s)\frac{c_j}{k_z})\frac{1}{v^2_{zts}}]\Big\}=\sum_n\sum_j\frac{p_{13snj}}{\omega-c_{snj}}$.
- $P^m_{s\sigma 31} \simeq \sum_{n=-\infty}^{\infty}\sum_{j=1}^{J}\frac{k_z v_{zts} b_j}{\omega - c_{snj}}\Big\{\frac{n\omega_{cs}}{k_x}(n\omega_{cs}+k_x v_{dsx}-i\nu_s)\frac{A_{nbs\sigma}}{v^2_{\perp ts\sigma}}(\frac{c_j}{k_z}+v_{dsz}\frac{1}{k_z v_{zts}})+A_{n0\sigma}(\frac{n\omega_{cs}}{k_x}+v_{dsx})(\frac{c_j^2}{v_{zts}}+v_{dsz}\frac{c_j}{v^2_{zts}})\Big\}=\sum_n\sum_j\frac{p_{31snj}}{\omega-c_{snj}}$.
- $P^m_{s\sigma 23} \simeq \sum_{n=-\infty}^{\infty}\sum_{j=1}^{J}\frac{k_z v_{zts} b_j}{\omega - c_{snj}}\Big\{[v_{dsy}A_{nbs\sigma}-iv_{\perp ts\sigma}B_{nbs\sigma}][(\frac{c_j}{k_z}+\frac{v_{dsz}}{k_z v_{tsz}})\frac{n\omega_{cs}}{v^2_{\perp ts\sigma}}]+[v_{dsy}A_{n0\sigma}-iv_{\perp ts\sigma}B_{n0\sigma}][v_{zts}c_j^2+(k_z v_{dsz}-i\nu_s)\frac{c_j}{k_z}]\frac{1}{v^2_{zts}}\Big\}=\sum_n\sum_j\frac{p_{23snj}}{\omega-c_{snj}}$.
- $P^m_{s\sigma 32} \simeq \sum_{n=-\infty}^{\infty}\sum_{j=1}^{J}\frac{k_z v_{zts} b_j}{\omega - c_{snj}}\Big\{[n\omega_{cs}\frac{A_{nbs\sigma}}{v^2_{\perp ts\sigma}}\frac{c_j}{k_z}+A_{n0\sigma}\frac{k_z v_{zts}c_j^2}{v^2_{zts}}]v_{dsy}+[n\omega_{cs}\frac{A_{nbs\sigma}}{v^2_{\perp ts\sigma}}\frac{1}{k_z v_{zts}}+A_{n0\sigma}\frac{c_j}{v^2_{zts}}]v_{dsy}v_{dsz}+i[(n\omega_{cs}-i\nu_s)\frac{B_{nbs\sigma}}{v_{\perp ts\sigma}}\frac{c_j}{k_z}+B_{n0\sigma}\frac{v_{\perp ts\sigma}c_j^2}{v_{zts}}]+iv_{dsz}[(n\omega_{cs}-i\nu_s)\frac{B_{nbs\sigma}}{v_{\perp ts\sigma}}\frac{1}{k_z v_{zts}}+B_{n0\sigma}\frac{v_{\perp ts\sigma}c_j}{v^2_{zts}}]\Big\}=\sum_n\sum_j\frac{p_{32snj}}{\omega-c_{snj}}$.
- $P^m_{s\sigma 33} \simeq \sum_{n=-\infty}^{\infty}\sum_{j=1}^{J}\frac{k_z v_{zts} b_j}{\omega - c_{snj}}\Big\{[n\omega_{cs}(\frac{v_{zts}c_j^2}{k_z}+v_{dsz}\frac{c_j}{k_z})\frac{A_{nbs\sigma}}{v^2_{\perp ts\sigma}}+A_{n0\sigma}(c_j^3+(k_z v_{dsz}-i\nu_s)\frac{c_j^2}{k_z v_{zts}})]+v_{dsz}[n\omega_{cs}(\frac{c_j}{k_z}+v_{dsz}\frac{1}{k_z v_{zts}})\frac{A_{nbs\sigma}}{v^2_{\perp ts\sigma}}+A_{n0\sigma}(\frac{c_j^2}{v_{zts}}+(k_z v_{dsz}-i\nu_s)\frac{c_j}{k_z v^2_{zts}})]\Big\}-1=-1+\sum_n\sum_j\frac{p_{33snj}}{\omega-c_{snj}}$.

In the above, say, $p_{11snj}=k_z v_{zts}b_j\Big\{[\frac{n\omega_{cs}}{k_x}(n\omega_{cs}+k_x v_{dsx}-i\nu_s)\frac{A_{nbs\sigma}}{v^2_{\perp ts\sigma}}\frac{1}{k_z v_{zts}}+A_{n0\sigma}(\frac{n\omega_{cs}}{k_x}+v_{dsx})\frac{c_j}{v^2_{zts}}](\frac{n\omega_{cs}}{k_x}+v_{dsx})\Big\}$, and other terms are similar.

It is thus easy to find that after J-pole expansion, the relations between $\boldsymbol{J}^m$ and $\boldsymbol{E}$ has the following form (with $\sum_{s=m}$)

$$\begin{pmatrix} J^m_x \\ J^m_y \\ J^m_z \end{pmatrix} = -i\epsilon_0 \begin{pmatrix} \frac{b^m_{11}}{\omega}+\sum_{snj}\frac{b_{snj11}}{\omega-c_{snj}} & \frac{b^m_{12}}{\omega}+\sum_{snj}\frac{b_{snj12}}{\omega-c_{snj}} & \frac{b^m_{13}}{\omega}+\sum_{snj}\frac{b_{snj13}}{\omega-c_{snj}} \\ \frac{b^m_{21}}{\omega}+\sum_{snj}\frac{b_{snj21}}{\omega-c_{snj}} & \frac{b^m_{22}}{\omega}+\sum_{snj}\frac{b_{snj22}}{\omega-c_{snj}} & \frac{b^m_{23}}{\omega}+\sum_{snj}\frac{b_{snj23}}{\omega-c_{snj}} \\ \frac{b^m_{31}}{\omega}+\sum_{snj}\frac{b_{snj31}}{\omega-c_{snj}} & \frac{b^m_{32}}{\omega}+\sum_{snj}\frac{b_{snj32}}{\omega-c_{snj}} & \frac{b^m_{33}}{\omega}+\sum_{snj}\frac{b_{snj33}}{\omega-c_{snj}} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \tag{92}$$

with the coefficients

$$\left\{\begin{array}{c} b_{snj11}=\sum_\sigma r_{s\sigma}\omega_{ps}^2 p_{11snj}/c_{snj},\ b^m_{11}=-\sum_s^{s=m}\omega_{ps}^2\sum_\sigma r_{s\sigma}[\sum_n A_{nbs\sigma}\frac{n\omega_{cs}}{k_x v^2_{\perp ts\sigma}}(\frac{n\omega_{cs}}{k_x}+v_{dsx})+\sum_{nj}p_{11snj}/c_{snj}], \\ b_{snj12}=\sum_\sigma r_{s\sigma}\omega_{ps}^2 p_{12snj}/c_{snj}, \quad b^m_{12}=-\sum_s^{s=m}\omega_{ps}^2[\sum_\sigma r_{s\sigma}\sum_{nj}p_{12snj}/c_{snj}], \\ b_{snj21}=\sum_\sigma r_{s\sigma}\omega_{ps}^2 p_{21snj}/c_{snj}, \quad b^m_{21}=-\sum_s^{s=m}\omega_{ps}^2[\sum_\sigma r_{s\sigma}\sum_{nj}p_{21snj}/c_{snj}], \\ b_{snj22}=\sum_\sigma r_{s\sigma}\omega_{ps}^2 p_{22snj}/c_{snj},\ b^m_{22}=-\sum_s^{s=m}\omega_{ps}^2\sum_\sigma r_{s\sigma}[\sum_n(C_{nbs\sigma}+i\frac{v_{dsy}}{v_{\perp ts\sigma}}B_{nbs\sigma})+\sum_{nj}p_{22snj}/c_{snj}], \\ b_{snj13}=\sum_\sigma r_{s\sigma}\omega_{ps}^2 p_{13snj}/c_{snj}, \quad b^m_{13}=-\sum_s^{s=m}\omega_{ps}^2[\sum_\sigma r_{s\sigma}\sum_{nj}p_{13snj}/c_{snj}], \\ b_{snj31}=\sum_\sigma r_{s\sigma}\omega_{ps}^2 p_{31snj}/c_{snj}, \quad b^m_{31}=-\sum_s^{s=m}\omega_{ps}^2[\sum_\sigma r_{s\sigma}\sum_{nj}p_{31snj}/c_{snj}], \\ b_{snj23}=\sum_\sigma r_{s\sigma}\omega_{ps}^2 p_{23snj}/c_{snj}, \quad b^m_{23}=-\sum_s^{s=m}\omega_{ps}^2[\sum_\sigma r_{s\sigma}\sum_{nj}p_{23snj}/c_{snj}], \\ b_{snj32}=\sum_\sigma r_{s\sigma}\omega_{ps}^2 p_{32snj}/c_{snj}, \quad b^m_{32}=-\sum_s^{s=m}\omega_{ps}^2[\sum_\sigma r_{s\sigma}\sum_{nj}p_{32snj}/c_{snj}], \\ b_{snj33}=\sum_\sigma r_{s\sigma}\omega_{ps}^2 p_{33snj}/c_{snj}, \quad b^m_{33}=-\sum_s^{s=m}\omega_{ps}^2\sum_\sigma r_{s\sigma}[\sum_n\frac{1}{2}A_{n0\sigma}+\sum_{nj}p_{33snj}/c_{snj}], \\ c_{snj}=k_z v_{dsz}+n\omega_{cs}+k_x v_{dsx}-i\nu_s+k_z v_{zts}c_j. \end{array}\right. \tag{93}$$

In numerical test, considering the cut of summation n, i.e., $\sum_n = \sum_{n=-N}^{N}$ with $N \neq \infty$, we find the above original form of b^m_{11}, b^m_{22} and b^m_{33}, are better than the below form [to undetstand]

$$\begin{cases} b^m_{11} = -\sum_s^{s=m} \omega_{ps}^2[1 + \sum_\sigma r_{s\sigma} \sum_{nj} p_{11snj}/c_{snj}], \\ b^m_{22} = -\sum_s^{s=m} \omega_{ps}^2[1 + \sum_\sigma r_{s\sigma} \sum_{nj} p_{22snj}/c_{snj}], \\ b^m_{33} = -\sum_s^{s=m} \omega_{ps}^2[1 + \sum_\sigma r_{s\sigma} \sum_{nj} p_{33snj}/c_{snj}]. \end{cases} \tag{94}$$

It is readily to see that all the singularities from $\frac{1}{k_z}$ in $\boldsymbol{P}^m_{s\sigma}$ are removable. The $\frac{n\omega_{cs}}{k_x}$ singularities at $k_x = 0$ in $P^m_{s\sigma 11}$, $P^m_{s\sigma 12}$, $P^m_{s\sigma 21}$, $P^m_{s\sigma 13}$, $P^m_{s\sigma 31}$ are also removable. Thus, the overall equations have no singularity and will not meet numerical difficulty. In the solver, to short the code, if $k_x\rho_s < k_\delta$ we set $k_x\rho_s = k_\delta$ for magnetized species in EM version. For example, we can set $k_\delta = 10^{-30}$.

Combining Eqs. (89), (90) and (92), the equivalent linear system for electromagnetic dispersion relation can be obtained as

$$\begin{cases} \omega v^{s=m}_{snjx} &= c_{snj}v_{snjx} + b_{snj11}E_x + b_{snj12}E_y + b_{snj13}E_z, \\ \omega v^{s=u}_{sj\sigma x} &= c_{sj\sigma}v_{sj\sigma x} + b_{sj\sigma 11}E_x + b_{sj\sigma 12}E_y + b_{sj\sigma 13}E_z, \\ \omega j_x &= b_{11}E_x + b_{12}E_y + b_{13}E_z, \\ iJ_x\epsilon_0 &= j_x + \sum_{snj}^{s=m} v_{snjx} + \sum_{sj\sigma}^{s=u} v_{sj\sigma x}, \\ \omega v^{s=m}_{snjy} &= c_{snj}v_{snjy} + b_{snj21}E_x + b_{snj22}E_y + b_{snj23}E_z, \\ \omega v^{s=u}_{sj\sigma y} &= c_{sj\sigma}v_{sj\sigma y} + b_{sj\sigma 21}E_x + b_{sj\sigma 22}E_y + b_{sj\sigma 23}E_z, \\ \omega j_y &= b_{21}E_x + b_{22}E_y + b_{23}E_z, \\ iJ_y/\epsilon_0 &= j_y + \sum_{snj}^{s=m} v_{snjy} + \sum_{sj\sigma}^{s=u} v_{sj\sigma y}, \\ \omega v^{s=m}_{snjz} &= c_{snj}v_{snjz} + b_{snj31}E_x + b_{snj32}E_y + b_{snj33}E_z, \\ \omega v^{s=u}_{sj\sigma z} &= c_{sj\sigma}v_{sj\sigma z} + b_{sj\sigma 31}E_x + b_{sj\sigma 32}E_y + b_{sj\sigma 33}E_z, \\ \omega j_z &= b_{31}E_x + b_{32}E_y + b_{33}E_z, \\ iJ_z/\epsilon_0 &= j_z + \sum_{snj}^{s=m} v_{snjz} + \sum_{sj\sigma}^{s=u} v_{sj\sigma z}, \\ \omega E_x &= c^2k_zB_y - iJ_x/\epsilon_0, \\ \omega E_y &= -c^2k_zB_x + c^2k_xB_z - iJ_y/\epsilon_0, \\ \omega E_z &= -c^2k_xB_y - iJ_z/\epsilon_0, \\ \omega B_x &= -k_zE_y, \\ \omega B_y &= k_zE_x - k_xE_z, \\ \omega B_z &= k_xE_y, \end{cases} \tag{95}$$

which yields a sparse matrix eigenvalue problem, where $b_{11} = b^m_{11} + b^u_{11}$ and so on. The symbols such as v_{snjx}, $j_{x,y,z}$ and $J_{x,y,z}$ used here do not have direct physical meanings but are analogy to the perturbed velocity and current density in the fluid derivations of plasma waves. The elements of the eigenvector $(E_x, E_y, E_z, B_x, B_y, B_z)$ still represent the original perturbed electric and magnetic fields. Thus, the polarization of the solutions can also be obtained in a straightforward manner. The dimension of the matrix is $N_N = 3\times(N_{SmNJ} + N_{SuJ} + 1) + 6 = 3\times\{[S_m\times(2\times N+1) + S_u\times 2]\times J + 1\} + 6$. And another good aspect of the final BO-K matrix equation is that it is valid for arbitrary real number of k_x and k_z, i.e., $\theta \in [0, 2\pi]$ and the only requirement is $k \neq 0$.

4.3. The Darwin case

Based on the electromagnetic result, the Darwin model case is straightforward, where the linear system for $\boldsymbol{J} = \boldsymbol{J}^m + \boldsymbol{J}^u = (\boldsymbol{\sigma^m} + \boldsymbol{\sigma^u}) \cdot \boldsymbol{E} = \boldsymbol{\sigma} \cdot \boldsymbol{E}$ is the same to electromagnetic case. We only need to modify the linear system of the Maxwells equations, which are also straightforward

$$\omega(\frac{\boldsymbol{kk}}{k^2}) \cdot \boldsymbol{E} = -c^2\boldsymbol{k} \times \boldsymbol{B} - i\boldsymbol{J}/\epsilon_0, \tag{96a}$$

$$\omega\boldsymbol{I} \cdot \boldsymbol{B} = \boldsymbol{k} \times \boldsymbol{E}, \tag{96b}$$

and the matrix eigenvalue problem becomes $\omega\boldsymbol{M_B} \cdot \boldsymbol{X} = \boldsymbol{M}_A \cdot \boldsymbol{X}$, where $\boldsymbol{M}_A$ is still the same as the electromagnetic one from Eq.(95) and $\boldsymbol{M}_B$ changes from unit matrix $\boldsymbol{I}_{N_N\times N_N}$ to

$$\boldsymbol{M}_B = \begin{pmatrix} \boldsymbol{I}_{(N_N-6)\times(N_N-6)} & & \\ & (\frac{\boldsymbol{kk}}{k^2})_{3\times 3} & \\ & & \boldsymbol{I}_{3\times 3} \end{pmatrix}. \tag{97}$$

Though $(\frac{\boldsymbol{kk}}{k^2})_{3\times3}$ may not be full rank matrix, the standard eigenvalue library, such as 'eig()' in Matlab, can solve the eigenvalue problem well.

4.4. The polarizations

The matrix solver can obtain $(E_x, E_y, E_z, B_x, B_y, B_z)$ directly [Note: In principle, BO-K matrix can also obtain (E_x, E_y, E_z) as in standard 3×3 matrix $\boldsymbol{D} \cdot \boldsymbol{E} = 0$. To obtain group velocity $\boldsymbol{v}_g = d\omega/d\boldsymbol{k}$ or do ray tracing, we may also need $\partial D/\partial\omega$ and $\partial D/\partial\boldsymbol{k}$.] from the matrix eigenvalue problem. Considered that the magnitude of the wave has no meaning for a linear system, we should do normalizations. We set $|E| = 1mV/m$ and $E_x = Re(E_x)$.

Some other useful: electric field energy $U_E = \frac{\epsilon_0}{2}\boldsymbol{E}\cdot\boldsymbol{E}^*$, magnetic field energy $U_B = \frac{1}{2\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}^*$, energy flux Poynting vector $\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}^*$. $P = E_y/iE_x$.

5. Benchmark

There exist numerous applications of this newly developed updated version BO-K tool, we only show some typical benchmarks to the reader to get a flavor of it.

The first benchmark is the ring beam case in Ref.[12], which is to make sure the function A_n, B_n and C_n are treated correctly in our model. The results are shown in Fig.1, with very good agreement with Min's [17] code. The second benchmark shown in Fig.2 is the mixed of magnetized and unmagnetized species in Ref.[14] for the instabilities driven by perpendicular beam in shock, which also show good agreement. And, the treatment of ion to be magnetized species also shows close result to the unmagnetized ion model, which implies that the unmagnetized ion assumption is valid in that case. This also gives us confidence of the validity of our magnetized model, since that the equations are totally different but yield similar solution as should be. The third benchmark shown in Fig.3 is for the Darwin model in Ref.[16], which is the same as the one solved using accurate Z function with conventional iterative root finding in Fig.5 of Ref.[16]. Here, we have also shown other branches and $k_z < 0$ branches. The symmetry between $k_z < 0$ and $k_z > 0$ solutions implies that the Z function is treated correctly for $k_z < 0$ in this new solver.

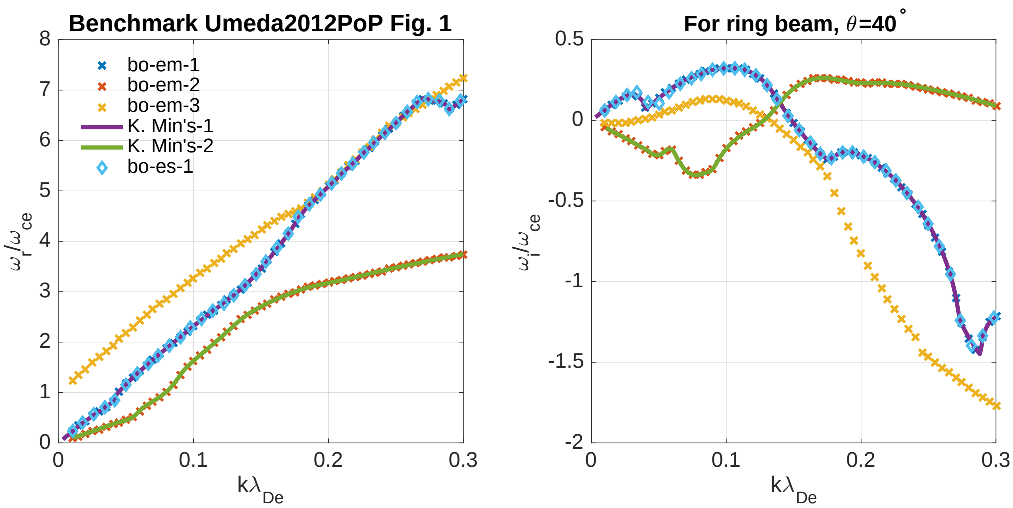

Figure 1: Benchmark the electromagnetic ring beam dispersion relation between BO-K and Min's code [17] with the parameters in Ref.[12] Fig.1 for $\theta = 40°$, which shows very good agreement. BO-K gives all the three unstable branches, whereas the third branch 'bo-em-3' could easily be missed by the iterative root finding approaches used in Min's [17] or Umeda's [12] codes. One branch of the electrostatic BO-K solution is also shown for reference, which is close to the electromagnetic one implies that this branch is essentially an electrostatic mode. The results also agree well with the PIC simulation result in Ref.[12].

The benchmark parameters for above cases are listed below.

- Defaultly we use SI unit: $c = 2.99792458e8$, $\epsilon_0 = 8.854187817e-12$, $k_B = 1.38064852e-23$, $q_e = 1.60217662e-19$, $m_p = 1.6726219e-27$.

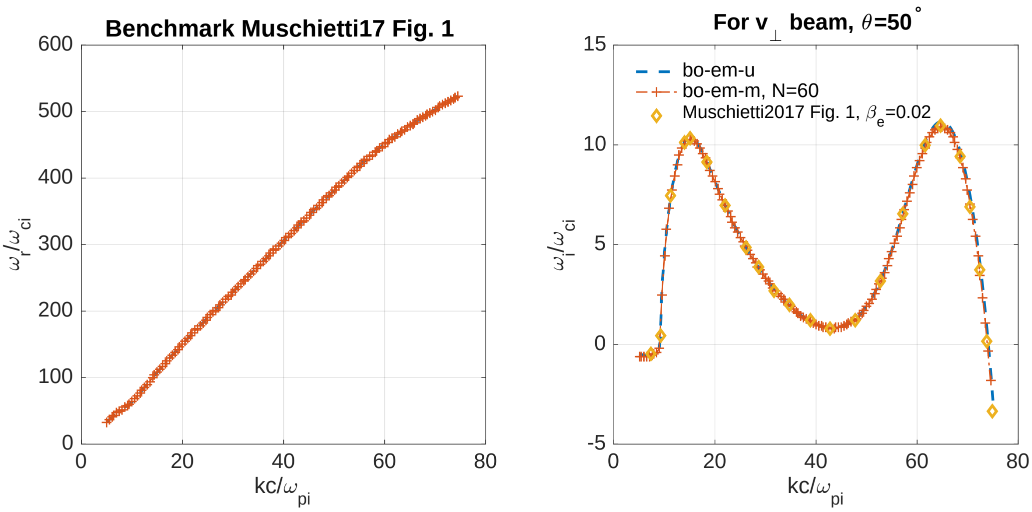

Figure 2: Benchmark the electromagnetic penpendicular beam dispersion relation between BO-K and Ref.[14] Fig. 1 with the parameters $\theta = 50°$ and $\beta_e = 0.02$, which shows very good agreement. Ref.[14] assumes unmagnetized ions and magnetized electron. 'bo-em-u' uses the same assumption; whereas 'bo-em-m' uses also magnetized ions with $N = 60$ where the $\sum_n$ has convergent. The agreement between '-U' and '-M' versions also implies that the unmagnetized ions is a valid assumption for this case.

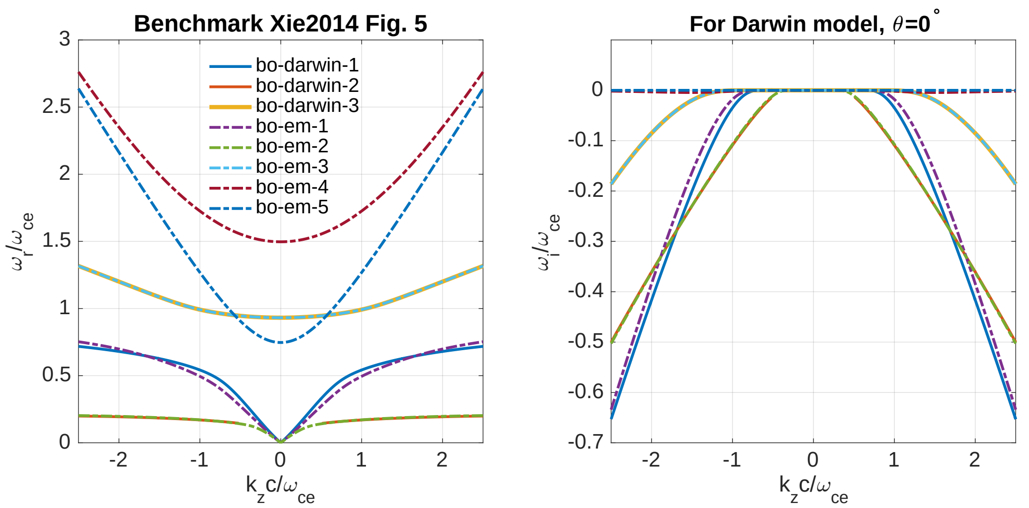

Figure 3: Compare the electromagnetic model and Darwin model in BO-K with the parameters in Ref.[16] Fig. 5, $\theta = 0°$ and $\omega_{ce}/\omega_{pe} = 1.2$, $c/v_{zte} = 5/\sqrt{2}$, $m_i/m_e = 4$, $T_i = T_e$. The $k_z < 0$ solutions are also shown, which shows good symmetry to the $k_z > 0$ solutions. The results show that Darwin model is a good approximate model in this case, and the slight deviations agree with the theoretical expectation in Ref.[16].

- Parameters for Ref.[12] benchmark: $B_0 = 96.24E-9$. $q_s = [-1,-1]$, $m_s = [5.447e-4, 5.447e-4]$, $n_s = [1e5, 9e5]$, $T_{zs} = T_{\perp s} = [5.1, 5.1]$, $v_{dsz}/c = [0.1, 0]$, $v_{dsr}/c = [0.05, 0]$.
- Parameters for Ref.[14] benchmark: $q_s = [1, 1, -1]$, $m_s = [1, 1, 1.1111e-3]$, $n_s = [0.8, 0.2, 1.0]$, $T_{zs} = T_{\perp s} = [0.111, 0.111, 0.01]$, $v_{dsx}/c = [-8.333e-3, 3.333e-2, 0]$. $B_0 = 1$, $c = 300$, $\mu_0 = 1.0$, $k_B = 1.0$, $q_e = 1.0$, $m_p = 1.0$.

The purpose of the present work is to provide the foundation of this new tool. And thus, the applications to new examples would be discussed in other works.

6. Summary and Discussion

In summary, a powerful new kinetic dispersion relation tool is developed, which greatly extends both the physical models and numerical capacities of other works in literature. The advantages of this new version of BO-K tool is that it contains many new features (anisotropic temperature/loss cone/drift in arbitrary direction/ring beam/collision, unmagnetized/magnetized, electrostatic/electromagnetic/Darwin, etc) and can be widely applied. And compared to some other solvers, the $k_z = 0$ ($\theta = \pi/2$) and $k_x = 0$ ($\theta = 0$) cases are not singular in BO-K. Furthermore, the $k_z < 0$ modes are also correctly treated. The most attractive feature is that it does not require initial guess for root finding and thus will not miss solutions. Thus, we think that this is a unified tool what exactly dreamed of by the plasma community.

The limitation of the present BO-K approach is that it can only be used for cases when the 2D velocity integral are decoupled, i.e., usually required $f_{s0}(\boldsymbol{v}) = f_z(v'_\parallel) f_\perp(v'_\perp)$. Typically, the present BO-K approach can not be used to oblique propagation κ-distribution function [19, 20] and relativistic [3] or other arbitrary distribution[18] cases. However, extensions are possible. For example, for κ-distribution, we can use the similar J-pole for corresponding Z_κ function [Noet: The κ-distribution $Z_\kappa(\zeta) = \sum_{j=1}^{J} \frac{b_j}{(\zeta-c_j)^j}$, which is slightly different from the standard one $Z(\zeta) = \sum_{j=1}^{J} \frac{b_j}{\zeta-c_j}$ and thus the corresponding transformation is also slightly different.] of parallel velocity integral and use Gaussian quadrature for perpendicular velocity integral, which will yield a 2D J-pole expansion and then can be transformed to a equivalent linear matrix eigenvalue problem, which is solvable though the corresponding matrix dimension could be much larger than the Maxwellian one, depends on how many nodes are used for perpendicular velocity integral. And thus, with 2D J-pole expansion, the BO-K approach can also be used for other distributions and non-uniform magnetic field drift wave case such as in gyrokinetic model [21]. The discussion of these extensions are outside the scope of the present work. Besides the conventional iterative root finding approaches, it is still required to develop a new powerful algorithm, say, similar to BO-K approach, to solve the more challenged relativistic kinetic plasma dispersion relation generally.

7. Acknowledgments

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Appendix A. The BO-K Solver

The update version of BO-K/PDRK code includes more features, but still consists of two parts: the main program and the input data file. The input file “bo.in” has the follow structure (the case in Fig.2)

```
qs(e)   ms(mp)      ns(m^-3)    Tzs(eV)   Tps(eV)   alphas   Deltas   vdsz/c    vdsx/c     vdsy/c    vdsr/c    nu_s    m_or_u(1/0)
1       1.0         0.8e0       0.111     0.111     1.0      1.0      0.0       -8.333e-3  0.0       0.0       0.0     0
1       1.0         0.2e0       0.111     0.111     1.0      1.0      0.0       3.333e-2   0.0       0.0       0.0     0
-1      1.1111e-3   1.0e0       0.010     0.010     1.0      1.0      0.0       0.0        0.0       0.0       0.0     1
```

One can add the corresponding parameters for additional species. Here, 'qs(e)', ' ms(mp) ', 'ns(m^-3)', 'Tzs(eV) ', 'Tps(eV) ', 'alphas', 'Deltas', 'vdsz/c', 'vdsx/c ', ' vdsy/c', ' vdsr/c ', 'nu_s ', 'm_or_u(1/0) ', are the charge in electron charge unit q_s/e, mass in proton mass unit m_s/m_p, density n_{s0}, parallel temperature T_{zs}, perpendicular temperature $T_{\perp s}$, loss cone parameters α_s and Δ_s, drift velocities v_{dsz}/c, v_{dsx}/c, v_{dsy}/c, ring beam velocity v_{dsr}/c, collision frequency ν_s, and whether the species is magnetized, respectively. Normalizations and other parameters can be set in BO-K main program. The code can be found in CPC Program Library, and more information can also be found at: http://hsxie.me/codes/pdrk/ or https://github.com/hsxie/pdrk/.

Due to many solutions exist in the results, we have also provided module to search solutions in same branch automatically which could be convenient to the user. One can refer the user manual along with the code. For example, the results shown in the Figs.1-3 are only selected several branches from all the solutions.

Appendix B. Relation to the dispersion relation of drift instabilities in inhomogeneous plasma

In conventional derivation [3–5] of drift instabilities in inhomogeneous magnetized plasma, the distribution function is assumed to depend on space inhomogeneous. We take the electrostatic density inhomogeneous case as in Chap. 4 of Ref.[5] for example to compare with our model. The conventional derivation, i.e., Eqs.(4.2.1) and (4.2.2) in Ref.[5], which ignores all terms of order ϵ_n^2 and valids only for $\epsilon_n \ll k$ and $\epsilon_n\rho_{cs} \ll 1$, gives

$$f_{s0} = \frac{1}{\sqrt{\pi}v_{ts}^3}\Big[1+\epsilon_n\Big(x+\frac{v_y-v_{gs}}{\omega_{cs}}\Big)\Big]e^{-\frac{v_z^2}{v_{ts}{}^2}}e^{-\frac{(v_y-v_{gs})^2+v_x^2}{v_{ts}{}^2}}, \quad \boldsymbol{k}=(0,k_y,k_z), \tag{B.1}$$

$$\begin{aligned} D(\omega,\boldsymbol{k}) &= 1+\sum_s\frac{\omega_{ps}^2}{2k^2v_{zts}^2}\Big\{1+\sum_{n=-\infty}^{\infty}\Big[\frac{\omega-k_y(v_{ns}+v_{gs})}{|k_z|v_{ts}}+\frac{n\omega}{|k_z|v_{zts}}\frac{\epsilon_n}{k_y}\Big]Z(\zeta_{sn})\Gamma_n(\frac{a_s^2}{2})\Big\} \\ &\simeq 1+\sum_s\frac{\omega_{ps}^2}{2k^2v_{zts}^2}\Big\{1+\sum_{n=-\infty}^{\infty}\frac{\omega-k_\perp(v_{ns}+v_{gs})}{|k_z|v_{ts}}Z(\zeta_{sn})\Gamma_n(\frac{a_s^2}{2})\Big\}=0, \end{aligned} \tag{B.2}$$

with $v_{ns}=\frac{\epsilon_n v_{ts}^2}{\omega_{cs}}$, $v_{gs}=\frac{g}{\omega_{cs}}$ and $\zeta_{sn}=\frac{\omega-k_\perp v_{gs}-n\omega_{cs}}{|k_z|v_{ts}}$, and where we have ignored the $\frac{\epsilon_n}{k_y}\ll 1$ term. Here, ϵ_n is parameter for space inhomogeneous and $g=|\boldsymbol{g}|$ is for force such as gravity. The corresponding dispersion relation in our model is

$$f_{s0} = \frac{1}{\sqrt{\pi}v_{ts}^3}e^{-\frac{v_z^2}{v_{ts}{}^2}}e^{-\frac{(v_x-v_{dsx})^2+v_y^2}{v_{ts}{}^2}}, \quad \boldsymbol{k}=(k_x,0,k_z), \tag{B.3}$$

$$\begin{aligned} D(\omega,\boldsymbol{k}) &= 1+\sum_s\frac{\omega_{ps}^2}{2k^2v_{zts}^2}\sum_{n=-\infty}^{\infty}\Big[Z_1(\zeta_{sn})+\frac{n\omega_{cs}}{k_zv_{zts}}Z_0(\zeta_{sn})\Big]\Gamma_n(\frac{a_s^2}{2}) \\ &= 1+\sum_s\frac{\omega_{ps}^2}{2k^2v_{zts}^2}\Big\{1+\sum_{n=-\infty}^{\infty}\frac{\omega-k_\perp v_{dsx}}{|k_z|v_{ts}}Z(\zeta_{sn})\Gamma_n(\frac{a_s^2}{2})\Big\}=0, \end{aligned} \tag{B.4}$$

with $\zeta_{sn}=\frac{\omega-k_\perp v_{dsx}-n\omega_{cs}}{|k_z|v_{ts}}$, and we have used $\sum_{n=-\infty}^{\infty}\Gamma_n=1$. If we consider the perpendicular drift in our model is due to space inhomogeneous $-\frac{\nabla p}{n_{s0}}$ and gravity $m\boldsymbol{g}$, with $p=n_{s0}(y)k_BT_{s0}$, $n_{s0}(y)=n_{s0}(1-\epsilon_ny)$ and $\boldsymbol{g}=g\hat{y}$, by using force balance and $\boldsymbol{F}_s=m_s\boldsymbol{g}-\frac{\nabla(n_{s0}k_BT_{s0})}{n_{s0}}=(m_sg+\epsilon_nm_sv_{ts}^2)\hat{y}$, we have $v_{dsx}=\frac{F_s}{q_sB_0}=v_{gs}+\frac{\epsilon_nv_{ts}^2}{\omega_{cs}}=v_{gs}+v_{ns}$. We noticed that Eqs.(B.2) and (B.4) are exactly the same, except that in the conventional derivation (B.2) the density inhomogeneous drift velocity v_{ns} in not included in the argument of Z function ζ_{sn}.

That is to say, our model can be used to study the drift modes in inhomogeneous plasma although we assume the derivation is under homogeneous plasma assumption. It is also not easy to say which of the the two derivations, i.e., the conventional one and the present one, is more accurate to describe the drift modes in inhomogeneous plasma, because that in our derivation the inhomogeneity contributes to the drift velocity and all other steps are rigorous, whereas the conventional derivation ignores high order terms in more than one steps to obtain the final dispersion relation.

Table C.1: The more accurate coefficients c_j and b_j for $J = 4$, $J = 8$, $J = 12$, $J = 16$ and $J = 24$ under J-pole Pade approximations of $Z(\zeta)$, where the asterisk denotes complex conjugation, and $\delta_Z = |Z_J(\zeta) - Z(\zeta)|$ is the maximum error between $Z_J(\zeta)$ and $Z(\zeta)$ for $\zeta = x + iy$ with $x \in [0, 50]$ and $y = -0.1$.

$J = 4$ $(I = 5, K = 3)$	b_1=- 1.0467968598346571444 - 2.1018525680357924343i b_2=0.54679685983465714437 + 0.037196505239893094435i $b(3:4)=b^*(1:2)$	c_1=0.37861161238699661757 - 1.3509435854325440801i c_2=1.2358876534356917085 - 1.2149821325576149719i $c(3:4)=-c^*(1:2)$	$\delta_Z = 3 \times 10^{-3}$
$J = 8$ $(I = 10, K = 6)$	b_1=- 5.5833741816150427087 - 11.208550459628098648i b_2=- 0.7399178112200519477 - 0.83951828462027428396i b_3=- 0.017340112270400811857 - 0.04630643962629377424i b_4=5.8406321051054954683 - 0.95360275132203964347i $b(5:8)=b^*(1:4)$	c_1=0.27393621805538084727 - 1.9417870375760945628i c_2=- 1.4652340919391423883 - 1.7896202996033145873i c_3=2.2376877251342932158 - 1.6259410241203623422i c_4=- 0.83925396636792203153 - 1.8919952115314256943i $c(5:8)=-c^*(1:4)$	$\delta_Z = 6 \times 10^{-6}$
$J = 12$ $(I = 16, K = 8)$	b_1=- 47.913598578418315281 - 106.98699311451399461i b_2=- 20.148858425809293248 + 12.874749056250453631i b_3=- 4.5311004339957471789E-3 + 6.3311756354943215316E-4i b_4=0.2150040123642351701 + 0.20042340981056393122i b_5=0.43131038679231352184 - 4.1505366661190555077i b_6=66.920673705505055584 + 20.747375125403268524i $b(7:12)=b^*(1:6)$	c_1=0.22536708628380726987 - 2.4862558428460328565i c_2=1.1590491549279069691 - 2.4061921257040740764i c_3=2.9785703941315209704 - 2.0490809954949754985i c_4=2.2568587892309227294 - 2.2080229126485700572i c_5=1.6738373878120108271 - 2.3235155478934783777i c_6=0.68229440981712468 - 2.4598334422617114946i $c(7:12)=-c^*(1:6)$	$\delta_Z = 8 \times 10^{-9}$
$J = 12$ $(I = 12, K = 12)$	b_1=- 10.020983259474214017 - 14.728932929429874883i b_2=- 0.58878169153449514493 + 0.19067303610080007359i b_3=- 0.27475707659732384029 + 3.617920717493884482i b_4=4.5713742777499515344E-4 + 2.7155393843737098852E-4i b_5=0.017940627032508378515 - 0.036436053276701248142i b_6=10.366124263145749629 - 2.5069048649816145967i $b(7:12)=b^*(1:6)$	c_1=0.22660012611958088508 - 2.0716877594897791206i c_2=- 1.7002921516300350075 - 1.882247422161272446i c_3=1.1713932508560117853 - 1.9772503319208541098i c_4=3.0666201126826972102 - 1.5900208259325997176i c_5=2.3073274904105782764 - 1.7546732543728200654i c_6=0.68720052490601906567 - 2.0402885259758440187i $c(7:12)=-c^*(1:6)$	$\delta_Z = 5 \times 10^{-8}$
$J = 16$ $(I = 18, K = 14)$	b_1=- 86.416592794839804566 - 147.57960545984972964i b_2=- 22.962540986214500398 + 46.211318219085729914i b_3=- 8.8757833558787660662 - 11.561957978688249474i b_4=- 0.025134802434111256483 + 0.19730442150379382482i b_5=- 5.6462830661756538039E-3 - 2.7884991898011769583E-3i b_6=2.8262945845046458372E-5 + 2.6335348714810255537E-5i b_7=2.3290098166119338312 - 0.57238325918028725167i b_8=115.45666014287557906 - 2.8617578808752183449i $b(9:16)=b^*(1:8)$	c_1=0.19664397441136646085 - 2.5854046363167904821i c_2=1.0004276870893045112 - 2.5277610669350594581i c_3=1.4263380087098663429 - 2.4694803409658086505i c_4=2.382753075769737514 - 2.2903917960623787648i c_5=2.9566517643704010427 - 2.1658992556376956217i c_6=- 3.6699741330155866185 - 2.0087276133120462601i c_7=1.8818356204685089975 - 2.3907395820644127768i c_8=0.59330036294742852232 - 2.5662607006180515205i $c(9:16)=-c^*(1:8)$	$\delta_Z = 3 \times 10^{-11}$
$J = 24$ $(I = 24, K = 24)$	b_1=- 579.77656932346560644 - 844.01436313629880827i b_2=- 179.52530851977905732 - 86.660002027244731382i b_3=- 52.107235029274485215 + 453.3246806707749413i b_4=- 2.1607927691932962178 + 0.63681255371973499384i b_5=- 0.018283386874895507814 - 0.21941582055233427677i b_6=- 6.819511737162705016E-5 + 3.2026091897256872621E-4i b_7=- 2.8986123310445793648E-6 - 9.9510625011385493369E-7i b_8=2.3382228949223867744E-9 - 4.0404517369565098657E-9i b_9=0.01221466589423530596 + 0.00097890737323377354166i b_{10}=7.3718296773233126912 - 12.575687057120635407i b_{11}=44.078424019374375065 - 46.322124026599601416i b_{12}=761.62579175738689742 + 185.11797721443392707i $b(13:24)=b^*(1:12)$	c_1=0.16167711630587375808 - 2.9424665391729649011i c_2=1.1509135876493567245 - 2.874554296549015316i c_3=0.81513635269214329287 - 2.9085569383176322447i c_4=2.2362950589041724111 - 2.7033607074680388479i c_5=2.6403561313404041541 - 2.6228400297078984517i c_6=3.5620497451197056658 - 2.4245607245823420556i c_7=4.1169251257106753931 - 2.3036541720854573609i c_8=4.8034117493360317933 - 2.1592490859689535413i c_9=3.0778922349246567316 - 2.5301774598854448463i c_{10}=- 1.8572088635240765004 - 2.7720571884094886584i c_{11}=1.496988132246689338 - 2.8290855580900544693i c_{12}=- 0.48636891219330428093 - 2.9311741817223824196i $c(13:24)=-c^*(1:12)$	$\delta_Z < 2 \times 10^{-13}$

Appendix C. More J-pole coefficients of $Z(\zeta)$ function

Ref.[1] has provided the typical J-pole coefficients of $Z(\zeta)$ function with $J = 4, 8, 12$. Here, following Ref.[1], we update them to more accurate data and even beyond the double precision (16 digit number) number with the help of symbolic computation. In the Table C.1, I means we keep I equations for $\zeta \to 0$, and K means we keep K equations for $\zeta \to \infty$, as described in Ref.[1].

Appendix D. The Possible Numerical Inaccuracy

Since BO-K solves the dispersion relation use matrix eigenvalue approach, the computational time to obtain all solutions for a $N_N \times N_N$ dimensions matrix is $O(N_N^\alpha)$, with usually $\alpha \sim 2.7$. Thus, the numerical $\sum_n$ with cut off to $n \in [-N, N]$ is crucial. For larger N, the matrix dimension could be large and the computation will slower. Usually $N \leq 10$ is sufficient for many cases and the computation is very fast, which can yield all solutions for a typical single k run in less than 1 second. For larger N, say $N > 50$, the sparse matrix approach to search solutions around some initial guesses are still fast and thus the BO-K tool has not limitation on this.

However, there exists several possible numerical inaccuracies, which should be careful of:

- (1) The J-pole expansion of $Z(\zeta)$ may give some artificial growing mode with growth rate $\gamma \sim 10^{-6}$ for $J = 8$, which however can be distinguished and reduced by using $J = 12, 16, 24$, etc.
- (2) The cut off in $\sum_{n=-\infty}^{\infty} \to \sum_{n=-N}^{N}$, which should be checked by using larger N to make sure the results are convergent.
- (3) Rand off error of eigen solver library, due to that the default numerical data is double precision (16 digit number) in the solver. This in principle can be solved by using high precision digit data.
- (4) Others, such as singularity in Darwin matrix, degree of accuracy of functions I_n, A_n, B_n and C_n, etc.

The above inaccuracies mainly occur at extremely small k and large k. The third inaccuracy is not easy to distinguish at this moment, and however, we have not met this problem for most benchmarks.

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