3rd Assignment

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1 Group Members

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```
In [72]: %matplotlib notebook
         from __future__ import division
         import numpy as np
         import time

         nsample = 1000
         nvariables = 100
         X0=np.ones([nsample, 1])
         X1=np.random.uniform(0, 10, ([nsample,nvariables]))
         X=np.concatenate([X0, X1], axis=1)
         error=np.random.normal(0, 1, (nsample,1)) # Normal random error
         beta=np.random.uniform(-5, 5, size=([nvariables+1, 1]))
         Y=np.dot(X, beta)+error
         print(beta)
```

```
[[ 1.28069035]
 [ 2.12055007]
 [-3.35364129]
 [ 2.49180722]
 [-2.8342821 ]
 [ 2.01194948]
 [-2.85385757]
 [ 0.3348794 ]
 [ 1.54505771]
 [ 3.76039123]
 [-3.08866171]
 [-0.51838654]
 [ 4.079594  ]
 [ 3.20824974]
 [ 4.02375764]
 [ 1.82350012]
 [ 2.73740613]
```

```
[ 4.83990623]
[ 4.78336779]
[-2.28676607]
[-2.59697585]
[-4.13702962]
[-2.42724456]
[ 4.51397344]
[-4.76847074]
[ 4.55519664]
[ 4.76617981]
[-0.63032856]
[-4.11676195]
[ 1.24741552]
[-3.9683488 ]
[-1.08071633]
[-0.25369959]
[-2.47735284]
[ 2.05335703]
[ 4.89243931]
[ 2.84978551]
[ 0.31243966]
[ 4.83657486]
[ 0.43747194]
[-0.77572563]
[ 3.27644737]
[-1.55841983]
[ 0.09359911]
[ 2.1735441 ]
[ 1.83173806]
[ 4.21374777]
[ 2.44924523]
[-0.69330504]
[-1.93368367]
[ 4.97225515]
[ 2.40465098]
[-2.23907213]
[ 4.66306122]
[-2.9983419 ]
[-2.31792161]
[-4.7092036 ]
[ 2.54629324]
[ 3.39634   ]
[ 4.70833798]
[ 4.21943738]
[-3.46481252]
[-4.70657762]
[-0.37296535]
[ 0.14072207]
```

```
[ 4.26372857]
[ 4.68691278]
[ 1.20068953]
[-4.92552407]
[ 0.61007361]
[ 3.98499229]
[ 2.98152954]
[ 4.09879831]
[ 4.92827865]
[-2.10621153]
[ 3.96988303]
[ 1.05428171]
[-1.58649877]
[ 2.41262391]
[-4.68382943]
[-3.00209887]
[ 2.17824054]
[ 4.62864372]
[-3.42174215]
[ 3.24110498]
[-4.59041032]
[-0.10349687]
[ 4.74605444]
[-4.81932749]
[ 2.84527302]
[-2.46052222]
[ 0.05968899]
[ 3.07686272]
[-1.19205355]
[-1.54193042]
[-4.46625195]
[-4.83257391]
[ 0.41097987]
[-4.00972237]
[ 2.56088925]
[ 3.2564741 ]]
```

1.1 A)

Estimate the value of the regression coefficients by using the analytical solution for the least squares estimation problem.

```
In [73]: time_start = time.clock()
         beta_ls_exact=np.dot(np.dot(np.linalg.inv(np.dot(np.transpose(X),X)),np.tr
         time_elapsed = (time.clock() - time_start)
         print('time elapsed=',time_elapsed)
         print(beta_ls_exact)
```

```
('time elapsed=', 0.0023641579900868237)
[[ 1.73944484]
 [ 2.11849576]
 [-3.3627126 ]
 [ 2.49636401]
 [-2.84054722]
 [ 2.02156168]
 [-2.86232325]
 [ 0.35178602]
 [ 1.543078  ]
 [ 3.75664986]
 [-3.10144696]
 [-0.49563234]
 [ 4.07457553]
 [ 3.19734595]
 [ 4.01312126]
 [ 1.81554646]
 [ 2.73727512]
 [ 4.86453595]
 [ 4.78753664]
 [-2.28646111]
 [-2.60981044]
 [-4.13475238]
 [-2.42150444]
 [ 4.51678882]
 [-4.76576907]
 [ 4.55297872]
 [ 4.76157291]
 [-0.63205529]
 [-4.11615598]
 [ 1.24171108]
 [-3.95975354]
 [-1.08472859]
 [-0.25454118]
 [-2.45531301]
 [ 2.04235535]
 [ 4.90231666]
 [ 2.83335418]
 [ 0.28134911]
 [ 4.84884805]
 [ 0.42513108]
 [-0.79616354]
 [ 3.25057585]
 [-1.57517965]
 [ 0.0961234 ]
 [ 2.17431404]
 [ 1.82505696]
 [ 4.22074613]
```

```
[ 2.44749707]
[-0.70609706]
[-1.94542769]
[ 4.95931038]
[ 2.40542927]
[-2.24506929]
[ 4.64658585]
[-2.99305659]
[-2.30420422]
[-4.70322873]
[ 2.55568737]
[ 3.39306073]
[ 4.69860549]
[ 4.21740776]
[-3.46522597]
[-4.70731893]
[-0.38375324]
[ 0.15328952]
[ 4.26222374]
[ 4.69090512]
[ 1.21393678]
[-4.91855814]
[ 0.60281507]
[ 3.99072754]
[ 2.98359187]
[ 4.1058776 ]
[ 4.92828094]
[-2.10718704]
[ 3.9875125 ]
[ 1.06448384]
[-1.58946094]
[ 2.42858799]
[-4.70019775]
[-2.99818303]
[ 2.16842197]
[ 4.62660575]
[-3.42353357]
[ 3.24789051]
[-4.59425366]
[-0.09865654]
[ 4.73365641]
[-4.82251921]
[ 2.85609984]
[-2.43892912]
[ 0.05248573]
[ 3.066192  ]
[-1.18456813]
[-1.53919184]
```

```
 [-4.4824774 ]
 [-4.83151957]
 [ 0.41117392]
 [-4.02762808]
 [ 2.57962647]
 [ 3.24912254]]
```

1.2 B

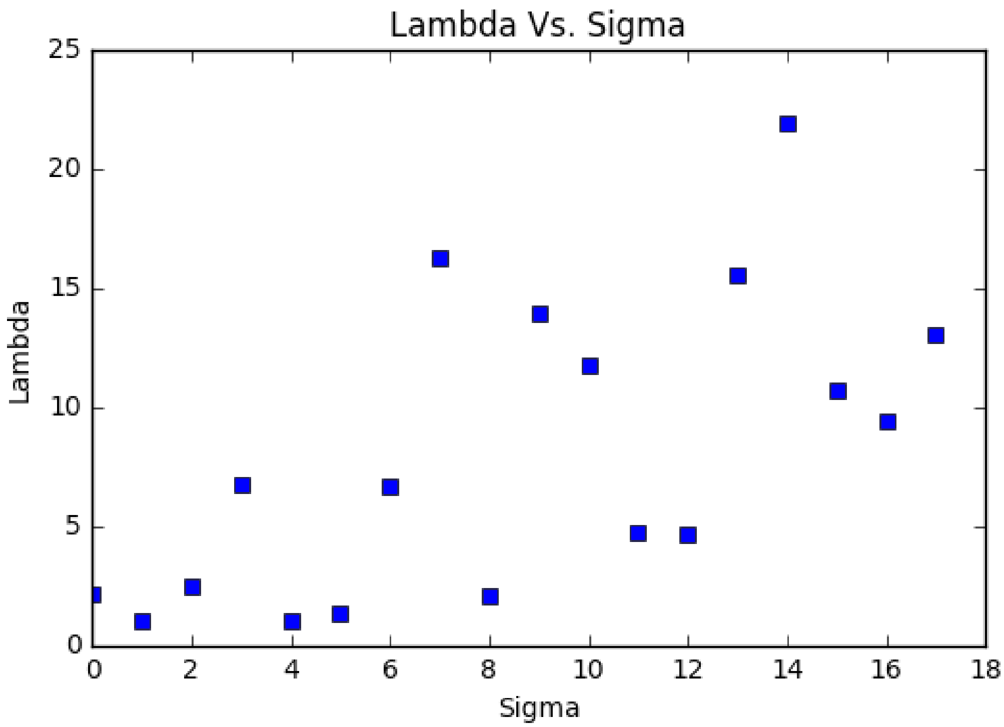
In [76]:
```
import matplotlib.pyplot as plt
%matplotlib inline
lambda_est=[]
sigma=np.arange(1,10, 0.5)

for i in sigma :
    error=np.random.normal(0, i, (nsample,1)) # Normal random error with c
    Y=np.dot(X, beta)+error
    beta_ls_exact=np.dot(np.dot(np.linalg.inv(np.dot(np.transpose(X),X)),r
    lambda_est.append((np.linalg.norm(beta-beta_ls_exact,ord=2)/np.linalg.

print lambda_est


plt.plot(lambda_est, 'bs')
plt.xlabel('Sigma')
plt.ylabel('Lambda')
plt.title('Lambda Vs. Sigma')
plt.show()
```

```
[2.0977724073092068, 1.0434809952479307, 2.4665810510725774, 6.714342056905366, 0.9
```



1.3 C)

Estimate the value of the regression coefficients (least squares) by using the tool minimize from the python package Scipy.optimize. Try at least three available solvers and compare their performance (iterations, function, gradient and hessian evaluations as well as total computational time).

```
In [77]: # c : Newton-CG
         from scipy.optimize import minimize

         def least_sq_reg_der(beta_ls,X,Y):
             beta_ls=np.matrix(beta_ls)
             pp=-2*np.dot(np.transpose(Y-np.dot(X,np.transpose(beta_ls))),X)
             return np.squeeze(np.asarray(pp))


         def least_sq_reg_hess(beta_ls,X,Y):
             return 2*np.dot(np.transpose(X),X)



         time_start = time.clock()
         res = minimize(least_sq_reg, beta_ls0, args=(X,Y), method = 'Newton-CG',ja
         time_elapsed = (time.clock() - time_start)
```

```
print('time elapsed=',time_elapsed)
print(res.x)
np.linalg.inv(least_sq_reg_hess(res.x,X,Y))
print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-res.x,ord=2)/np.
```

```
Optimization terminated successfully.
         Current function value: 84919.433779
         Iterations: 13
         Function evaluations: 15
         Gradient evaluations: 27
         Hessian evaluations: 13
('time elapsed=', 0.01405822801461909)
[-2.82008067  2.18757466 -3.33272327  2.40049448 -2.91963727  2.00327112
 -2.92131844  0.38737346  1.41670753  3.80403476 -3.04939229 -0.58693961
  4.13023388  3.2712188   4.06065034  1.86865285  2.90094534  5.10282956
  4.75282604 -2.3773995  -2.74905144 -3.94538837 -2.45767374  4.49870427
 -4.74963116  4.5581906   4.65647517 -0.59536712 -4.19275214  1.24511272
 -3.89736906 -1.04011949 -0.2648055  -2.28654196  2.16801341  4.92252474
  2.94042701  0.53332892  4.79756196  0.43628567 -0.86060568  3.35301637
 -1.41157385  0.06995832  2.14622713  1.82173066  4.29591559  2.45075603
 -0.63613901 -1.91125291  4.94260168  2.30714525 -2.39602297  4.6346139
 -3.05933576 -2.51372002 -4.71725957  2.5502426   3.42703598  4.78687509
  4.21202156 -3.54248362 -4.50429966 -0.48099936  0.13159525  4.24485933
  4.77287047  1.38593122 -4.99300658  0.74922483  3.95908503  2.79126195
  3.99975694  4.84343316 -2.07182145  3.98490325  1.19556984 -1.65011529
  2.41928262 -4.77430334 -2.80623367  2.12898233  4.6608253  -3.45711524
  3.31431846 -4.59174864 -0.0981681   4.70477634 -4.95014163  2.87792685
 -2.4599172   0.12189708  2.88425646 -1.36533708 -1.58816479 -4.39826269
 -4.80547961  0.56007547 -3.78563627  2.50914512  3.30404331]
('error=', 1.3659231966901152e-06)
```

In [78]:
```
# c: dogleg

from scipy.optimize import minimize

def least_sq_reg_der(beta_ls,X,Y):
    beta_ls=np.matrix(beta_ls)
    pp=-2*np.dot(np.transpose(Y-np.dot(X,np.transpose(beta_ls))),X)
    return np.squeeze(np.asarray(pp))


def least_sq_reg_hess(beta_ls,X,Y):
    return 2*np.dot(np.transpose(X),X)



time_start = time.clock()
```

```
        res = minimize(least_sq_reg, beta_ls0, args=(X,Y), method = 'dogleg',jac=l
        time_elapsed = (time.clock() - time_start)
        print('time elapsed=',time_elapsed)
        print(res.x)
        np.linalg.inv(least_sq_reg_hess(res.x,X,Y))
        print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-res.x,ord=2)/np.
```

```
Optimization terminated successfully.
         Current function value: 84919.433768
         Iterations: 6
         Function evaluations: 7
         Gradient evaluations: 7
         Hessian evaluations: 6
('time elapsed=', 0.008869112425600179)
[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
 -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
  4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
  4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
 -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
 -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
  2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
 -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
 -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
 -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
  4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
  4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
  3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
  2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
  3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
 -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
 -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]
('error=', 4.2990732639375794e-12)
```

```
In [79]: from scipy.optimize import minimize

         def least_sq_reg(beta_ls, X, Y):
             beta_ls=np.matrix(beta_ls)
             z=Y-X*np.transpose(beta_ls)
             return np.dot(np.transpose(z),z) #sum of the square errors, value of t
```

```
In [80]: # c :BFGS

         beta_ls0 = np.zeros(nvariables+1) #we need to initiate our initial point
         time_start = time.clock()
         res = minimize(least_sq_reg, beta_ls0, args=(X,Y), method = 'BFGS', optior
         time_elapsed = (time.clock() - time_start)
         print('time elapsed=',time_elapsed)
```

```
print (res.x)
print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-res.x,ord=2)/np.
```

```
Warning: Desired error not necessarily achieved due to precision loss.
         Current function value: 84919.433768
         Iterations: 131
         Function evaluations: 24515
         Gradient evaluations: 238
('time elapsed=', 1.4608064334461233)
[-2.8200985   2.18757251 -3.33272404  2.40050027 -2.9196443   2.00327308
 -2.92131936  0.3873754   1.41670726  3.80403212 -3.04939138 -0.5869397
  4.13024253  3.27121944  4.0606465   1.86865365  2.90094716  5.10282556
  4.75282723 -2.3774021  -2.74905506 -3.94539385 -2.45767775  4.49870628
 -4.74962365  4.5581867   4.65647564 -0.59536055 -4.19274501  1.2451148
 -3.89736611 -1.04011893 -0.2648137  -2.28654166  2.1680069   4.92252054
  2.94043061  0.53332426  4.79756441  0.43627895 -0.86060973  3.35301358
 -1.411572    0.06996955  2.14622612  1.82173174  4.29591722  2.45076115
 -0.63614186 -1.91125438  4.94259787  2.30714665 -2.39602388  4.63460324
 -3.05933111 -2.51371897 -4.71726272  2.55024273  3.42703609  4.78687685
  4.2120184  -3.54248668 -4.50429913 -0.48099318  0.13160623  4.24485561
  4.77287762  1.38593473 -4.99300558  0.7492287   3.95908792  2.79126178
  3.99975324  4.84343427 -2.07181555  3.98489735  1.19557716 -1.65011214
  2.41928088 -4.77430592 -2.80624141  2.12898241  4.66082837 -3.45711963
  3.31431909 -4.59174468 -0.09816652  4.70477388 -4.95014286  2.8779229
 -2.45991397  0.12189603  2.8842569  -1.36533542 -1.58816079 -4.39826433
 -4.80549213  0.56007879 -3.78564002  2.50914997  3.30404246]
('error=', 2.1148376814635847e-07)
```

```
In [81]: # c :SLSQP

         beta_ls0 = np.zeros(nvariables+1) #we need to initiate our initial point
         time_start = time.clock()
         res = minimize(least_sq_reg, beta_ls0, args=(X,Y), method = 'SLSQP', optio
         time_elapsed = (time.clock() - time_start)
         print('time elapsed=',time_elapsed)
         print (res.x)
         print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-res.x,ord=2)/np.
```

```
Optimization terminated successfully.    (Exit mode 0)
            Current function value: [[ 84919.43438612]]
            Iterations: 62
            Function evaluations: 6628
            Gradient evaluations: 62
('time elapsed=', 0.38456415182736237)
[-2.81999282  2.18757499 -3.33274337  2.40049986 -2.91964145  2.00328562
 -2.92135673  0.38740984  1.41667548  3.8040489  -3.04938633 -0.58696508
  4.13026249  3.2712273   4.06064785  1.86863405  2.90094692  5.10280418
```

```
  4.75279975 -2.37740227 -2.74908372 -3.94539617 -2.45771762  4.49873764
 -4.7495816   4.55816327  4.65648354 -0.5953383  -4.19275952  1.24510698
 -3.89736525 -1.04011663 -0.26475866 -2.28654486  2.16800036  4.92250585
  2.94045632  0.53331927  4.79756062  0.43632093 -0.86062636  3.35304649
 -1.41155463  0.07002751  2.14629284  1.82174526  4.29594552  2.45075327
 -0.63611914 -1.9112522   4.94254981  2.3071427  -2.39602753  4.63456411
 -3.05933373 -2.51372557 -4.71728558  2.55019751  3.42705259  4.78684688
  4.21203778 -3.54249077 -4.50430907 -0.48098153  0.13159908  4.24482322
  4.77287968  1.38597311 -4.99303748  0.74922122  3.95905633  2.79125096
  3.99978166  4.84342029 -2.07179286  3.98486089  1.19553417 -1.6501765
  2.41921376 -4.77430548 -2.80619574  2.12901886  4.66084734 -3.45709624
  3.31433304 -4.59175443 -0.09815162  4.70480099 -4.95015551  2.87791769
 -2.45999135  0.12189591  2.88425535 -1.36529894 -1.58814319 -4.39827985
 -4.80542618  0.56006648 -3.78563869  2.50912917  3.30408075]
('error=', 9.0791862297303001e-06)
```

In [82]:
```
# c : CG

beta_ls0 = np.zeros(nvariables+1) #we need to initiate our initial point
time_start = time.clock()
res = minimize(least_sq_reg, beta_ls0, args=(X,Y), method = 'CG', options
time_elapsed = (time.clock() - time_start)
print('time elapsed=',time_elapsed)
print (res.x)
print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-res.x,ord=2)/np.
```

```
Warning: Desired error not necessarily achieved due to precision loss.
         Current function value: 84919.438062
         Iterations: 243
         Function evaluations: 55632
         Gradient evaluations: 540
('time elapsed=', 2.9277961149491603)
[-2.85789181  2.1876595  -3.33265676  2.40058637 -2.91958424  2.0033621
 -2.92124797  0.38743794  1.41677483  3.8040492  -3.04932774 -0.58686475
  4.13034603  3.27131329  4.06073374  1.86871524  2.90101457  5.10290849
  4.75291173 -2.37733101 -2.74897404 -3.94534963 -2.45755214  4.4987615
 -4.74954624  4.55825218  4.65654622 -0.59529921 -4.19269757  1.24517746
 -3.89728324 -1.04006126 -0.26472422 -2.28641039  2.16808449  4.92261235
  2.94051853  0.5333913   4.7976289   0.43632826 -0.86052493  3.35309622
 -1.41149609  0.07004405  2.14633029  1.8218001   4.2959728   2.45083466
 -0.63607714 -1.91116057  4.94266668  2.30720999 -2.39592101  4.63465029
 -3.05928473 -2.5136702  -4.7171507   2.5503713   3.42710086  4.78695988
  4.21207411 -3.54236985 -4.50420913 -0.48090851  0.13170382  4.24491855
  4.77296519  1.3860117  -4.99294112  0.74922996  3.95915735  2.79134114
  3.99980412  4.84347789 -2.0717737   3.98497698  1.19565988 -1.65003571
  2.41931526 -4.77427489 -2.80616743  2.12906819  4.66091626 -3.45704269
  3.3143886  -4.59171885 -0.09806332  4.70488323 -4.95006486  2.87796457
```

```
 -2.45985323  0.12197447  2.88435942 -1.36521473 -1.58804764 -4.39818139
 -4.8054392   0.56020827 -3.78554284  2.50927804  3.30413515]
('error=', 0.0011659627184107803)
```

In [4]:
```python
import sympy
from pandas import DataFrame
from sympy import *

# For Final report we need to put 'Current function value', 'Iteration','Fu
### Fill value instead of 1 2 3 .....
t = [[84919.433779 ,13 ,15 ,27 ,13 ,0.014 ,1.37e-6 ], #Newton-CG
     [84919.434768 , 6, 7 ,7 ,6 , 0.0089 ,4.30e-12 ], #dogleg
     [84919.433758 , 131,24515 ,238 ,"NA" ,1.46 ,2.11e-7 ], #BFGS
     [84919.434386 ,62 ,6628 ,62 ,"NA" ,0.38 ,9.08e-6 ], #SLSQP
     [84919.438062 ,243 ,55632 ,540 ,"NA" ,2.93 ,0.0012 ]] #CG


table = DataFrame(t, index=['Newton-CG', 'dogleg', 'BFGS', 'SLSQP', 'CG'],


table
```

Out[4]:
```
             Current function value  Iteration  Function evaluations  \
Newton-CG              84919.433779         13                    15
dogleg                 84919.434768          6                     7
BFGS                   84919.433758        131                 24515
SLSQP                  84919.434386         62                  6628
CG                     84919.438062        243                 55632

             Gradient evaluations Hessian evaluations  Time elapsed  \
Newton-CG                      27                  13        0.0140
dogleg                          7                   6        0.0089
BFGS                          238                  NA        1.4600
SLSQP                          62                  NA        0.3800
CG                            540                  NA        2.9300

                   Error
Newton-CG   1.370000e-06
dogleg      4.300000e-12
BFGS        2.110000e-07
SLSQP       9.080000e-06
CG          1.200000e-03
```

1.4 D)

Considering again the least squares estimation problem, estimate the value of the regression coefficients by implementing the:

i. Gradient method

ii. Newton method

iii. Quasi-Newton method

Consider a like search technique to improve the algorithm convergence, e.x., Armijo rule. Compare the performance of these algorithms (iterations, function, gradient and hessian evaluations as well as total computational time).

In [84]:
```
#definitinition of OF #objective function
def least_sq_reg(beta_ls, X, Y):
    beta_ls=np.matrix(beta_ls)
    z=Y-X*np.transpose(beta_ls)
    return np.transpose(z)*z

#definition of Gradient
def least_sq_reg_der(beta_ls,X,Y):
    beta_ls=np.matrix(beta_ls)
    pp=-2*np.transpose(Y-X*np.transpose(beta_ls))*X
    aa= np.squeeze(np.asarray(pp))
    return aa

#definition of hessian
def least_sq_reg_hess(beta_ls,X,Y):
    ss=2*np.dot(np.transpose(X),X)
    return ss
```

1.5 i) Gradient method

$\rightarrow$ From an initial iterate x_0
  $\rightarrow$ Compute search (descent) directions p_k
  $\rightarrow$ Far from the solution, compute a steplength $\alpha_k > 0$
  $\rightarrow$ Movement:

$$x_{k+1} = x_k + \alpha_k \, p_k$$

Until convergence to a local solution

In [97]:
```
### With Armijo Rule


(a,b)=X.shape
beta_lsg=np.zeros(b) #initial value for beta
#alpha= 0.0000001
n_iter=100000 #maximim nunber iteration
OF_iter=np.zeros(n_iter)
tol_iter=np.zeros(n_iter)
alpha_iter=np.zeros(n_iter)
i=0;
tol=1000; # Tolerance
epsilon=1e-3;
```

```
    time_start = time.clock()
    while (i <= n_iter-2) and (tol>epsilon):
        i=i+1
        grad=least_sq_reg_der(beta_lsg,X,Y) #this function gives us the value
        ddirect=-grad
        #######################
        #     Armijo Rule
        sigma=0.1
        beta=0.5
        alpha=1
        while (least_sq_reg(beta_lsg+alpha*ddirect,X,Y)> least_sq_reg(beta_lsg
            alpha=alpha*beta
        #######################
        beta_lsg=beta_lsg+alpha*ddirect
        OF_iter[i]=least_sq_reg(beta_lsg,X,Y) # Objective Function ---Residual
        tol=np.linalg.norm(grad,ord=2)
        tol_iter[i]=tol

    time_elapsed = (time.clock() - time_start)
    print('time elapsed=',time_elapsed)
    print('iterations',i)
    print(OF_iter[i])
    print(beta_lsg)
    print(np.transpose(beta_ls_exact))
    print('Tolerance=',tol)
    print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-beta_lsg,ord=2)/
```

```
('time elapsed=', 215.57388964362326)
('iterations', 99999)
84929.3073365
[-1.00552324  2.18336177 -3.33600171  2.39643687 -2.92246059  1.99875166
 -2.92468279  0.38429239  1.41333943  3.80317979 -3.05248791 -0.59071866
  4.12524893  3.26656723  4.05649917  1.86564815  2.8978303   5.09889964
  4.74850713 -2.38069513 -2.75260615 -3.94742467 -2.46350012  4.49601821
 -4.75297247  4.55537918  4.65282515 -0.59800657 -4.19509426  1.24210405
 -3.90135134 -1.04271539 -0.26901538 -2.29302437  2.16423839  4.91800017
  2.93592439  0.53012202  4.79433681  0.43390435 -0.86465576  3.34852329
 -1.41526794  0.06632625  2.14111084  1.81840842  4.29306879  2.44703809
 -0.63913949 -1.91576422  4.93924581  2.30399171 -2.40088197  4.63240874
 -3.06169684 -2.5159778  -4.72248466  2.54384261  3.42392076  4.7828991
  4.20934812 -3.54783455 -4.50863292 -0.48497834  0.1268127   4.24195353
  4.76853478  1.38218161 -4.99625161  0.74897886  3.95538557  2.78762389
  3.99727475  4.84105212 -2.0735365   3.98123654  1.19165168 -1.65348411
  2.41727442 -4.77572941 -2.80935509  2.12472609  4.65639364 -3.46088663
  3.31067667 -4.59263375 -0.10338909  4.6996278  -4.95383034  2.87600641
 -2.46276516  0.11816647  2.87931377 -1.37122449 -1.59362318 -4.40189991
 -4.80763108  0.55374605 -3.79003875  2.50319568  3.29974066]
```

```
[[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
  -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
   4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
   4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
  -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
  -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
   2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
  -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
  -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
  -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
   4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
   4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
   3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
   2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
   3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
  -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
  -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]]
('Tolerance=', 11.957228882608536)
('error=', 0.056169323979137355)
```

In [98]:
```
### Without Armijo Rule

(a,b)=X.shape
beta_lsg=np.zeros(b) #initial value for beta
alpha= 0.0000001
n_iter=100000 #maximim nunber iteration
OF_iter=np.zeros(n_iter)
tol_iter=np.zeros(n_iter)
alpha_iter=np.zeros(n_iter)
i=0;
tol=1000; # Tolerance
epsilon=1e-3;

time_start = time.clock()
while (i <= n_iter-2) and (tol>epsilon):
    i=i+1
    grad=least_sq_reg_der(beta_lsg,X,Y) #this function gives us the value
    ddirect=-grad
    beta_lsg=beta_lsg+alpha*ddirect
    OF_iter[i]=least_sq_reg(beta_lsg,X,Y) # Objective Function ---Residual
    tol=np.linalg.norm(grad,ord=2)
    tol_iter[i]=tol

time_elapsed = (time.clock() - time_start)
print('time elapsed=',time_elapsed)
print('iterations',i)
print(OF_iter[i])
```

```python
print(beta_lsg)
print(np.transpose(beta_ls_exact))
print('Tolerance=',tol)
print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-beta_lsg,ord=2)/
```

```
('time elapsed=', 14.0880259515543)
('iterations', 99999)
84942.527301
[-0.0449488   2.18113141 -3.33773806  2.39428447 -2.92395271  1.99635687
 -2.92646459  0.38265903  1.41155527  3.80272724 -3.05412837 -0.59272041
  4.12260421  3.26410322  4.05430245  1.86405585  2.89617906  5.09682015
  4.74621892 -2.38243967 -2.75448725 -3.94850098 -2.46658356  4.49459399
 -4.75474652  4.55389168  4.65089142 -0.59940854 -4.19633916  1.24050891
 -3.90346229 -1.04409118 -0.27124091 -2.29645735  2.16224217  4.91560592
  2.93353771  0.52842552  4.79262692  0.43264598 -0.86679887  3.34614498
 -1.41722575  0.06439638  2.13840169  1.8166479   4.29155963  2.44506596
 -0.64072767 -1.91815287  4.93747005  2.30232025 -2.40345492  4.63124573
 -3.06295048 -2.51717478 -4.72525025  2.54045333  3.4222703   4.78079204
  4.20793326 -3.55066681 -4.51092836 -0.48708925  0.12427388  4.240416
  4.76623454  1.38019354 -4.99797124  0.74884532  3.95342441  2.78569681
  3.99596141  4.83978978 -2.07444881  3.9792973   1.18957239 -1.65527045
  2.41621101 -4.77648427 -2.81100466  2.1224716   4.65404478 -3.46288209
  3.30874722 -4.59310568 -0.106155    4.69690238 -4.95578365  2.87499064
 -2.46427576  0.11619089  2.87669574 -1.37434324 -1.59651609 -4.40382578
 -4.80876468  0.55039243 -3.7923686   2.50004233  3.29746211]
[[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
  -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
   4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
   4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
  -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
  -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
   2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
  -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
  -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
  -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
   4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
   4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
   3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
   2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
   3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
  -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
  -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]]
('Tolerance=', 16.639447266111439)
('error=', 0.085955155074071907)
```

In [ ]:

1.6 ii) Newtons method

→ From an initial iterate x_0

→ Compute search (descent) directions $p_k = -(\nabla^2 f(x_k))^{-1}\nabla f(x_k)$, whenever $\nabla^2 f(x_k)$ is non-singular.

→ Far from the solution, compute a steplength $\alpha_k > 0$

→ Movement:

$$x_{k+1} = x_k + \alpha_k \, p_k$$

Until convergence to a local solution

In [99]:
```
### With Armijo Rule

(a,b)=X.shape
beta_lsg=np.zeros(b) #initial value for beta
#alpha= 0.0000001
n_iter=100000 #maximim nunber iteration
OF_iter=np.zeros(n_iter)
tol_iter=np.zeros(n_iter)
alpha_iter=np.zeros(n_iter)
i=0;
tol=1000; # Tolerance
epsilon=1e-3;

time_start = time.clock()
while (i <= n_iter-2) and (tol>epsilon):
    i=i+1
    grad=least_sq_reg_der(beta_lsg,X,Y)
    hess=least_sq_reg_hess(beta_lsg,X,Y) #this function gives us the value
    ddirect=-(np.dot(np.linalg.inv(hess), grad))
    ########################
    #     Armijo Rule
    sigma=0.1
    beta=0.5
    alpha=1
    while (least_sq_reg(beta_lsg+alpha*ddirect,X,Y)> least_sq_reg(beta_lsg
        alpha=alpha*beta
    ########################
    beta_lsg=beta_lsg+alpha*ddirect
    OF_iter[i]=least_sq_reg(beta_lsg,X,Y)
    tol=np.linalg.norm(hess,ord=2)
    tol_iter[i]=tol

time_elapsed = (time.clock() - time_start)
print('time elapsed=',time_elapsed)
print('iterations',i)
print(OF_iter[i])
print(beta_lsg)
print(np.transpose(beta_ls_exact))
```

```
    print('Tolerance=',tol)
    print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-beta_lsg,ord=2)/
```

```
('time elapsed=', 293.5455931344186)
('iterations', 99999)
84919.4337684
[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
 -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
  4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
  4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
 -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
 -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
  2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
 -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
 -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
 -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
  4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
  4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
  3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
  2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
  3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
 -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
 -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]
[[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
  -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
   4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
   4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
  -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
  -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
   2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
  -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
  -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
  -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
   4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
   4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
   3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
   2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
   3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
  -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
  -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]]
('Tolerance=', 4972171.1853834139)
('error=', 4.2435594497630421e-12)
```

In [100]:
```
### Without Armijo Rule

(a,b)=X.shape
beta_lsg=np.zeros(b) #initial value for beta
```

```python
alpha= 0.0000001
n_iter=100000 #maximim nunber iteration
OF_iter=np.zeros(n_iter)
tol_iter=np.zeros(n_iter)
alpha_iter=np.zeros(n_iter)
i=0;
tol=1000; # Tolerance
epsilon=1e-3;

time_start = time.clock()
while (i <= n_iter-2) and (tol>epsilon):
    i=i+1
    grad=least_sq_reg_der(beta_lsg,X,Y)
    hess=least_sq_reg_hess(beta_lsg,X,Y) #this function gives us the valu
    ddirect=-(np.dot(np.linalg.inv(hess), grad))
    beta_lsg=beta_lsg+alpha*ddirect
    OF_iter[i]=least_sq_reg(beta_lsg,X,Y)
    tol=np.linalg.norm(hess,ord=2)
    tol_iter[i]=tol

time_elapsed = (time.clock() - time_start)
print('time elapsed=',time_elapsed)
print('iterations',i)
print(OF_iter[i])
print(beta_lsg)
print(np.transpose(beta_ls_exact))
print('Tolerance=',tol)
print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-beta_lsg,ord=2)
```

```
('time elapsed=', 192.15463624136464)
('iterations', 99999)
84611989.3714
[-0.0280601   0.0217665  -0.03316083  0.02388514 -0.02905066  0.0199327
 -0.02906733  0.00385441  0.01409633  0.03785038 -0.03034165 -0.00584009
  0.04109619  0.03254885  0.04040371  0.01859323  0.02886462  0.05077346
  0.04729095 -0.02365531 -0.02735328 -0.03925694 -0.02445406  0.04476243
 -0.04725908  0.04535427  0.04633225 -0.00592388 -0.0417181   0.01238898
 -0.03877906 -0.01034925 -0.00263491 -0.02275125  0.02157182  0.04897941
  0.02925748  0.00530661  0.04773609  0.00434101 -0.00856313  0.03336271
 -0.01404524  0.0006962   0.0213551   0.01812635  0.04274467  0.02438524
 -0.00632965 -0.01901711  0.04917918  0.02295627 -0.0238406   0.04611462
 -0.03044055 -0.02501167 -0.04693708  0.02537509  0.03409924  0.04762975
  0.04190987 -0.03524798 -0.04481808 -0.00478591  0.00130949  0.0422366
  0.04749046  0.01379014 -0.04968074  0.00745488  0.03939319  0.02777324
  0.03979782  0.0481925  -0.02061471  0.03965     0.01189607 -0.01641873
  0.02407201 -0.04750467 -0.02792229  0.02118352  0.04637556 -0.03439857
  0.0329777  -0.04568817 -0.00097676  0.04681282 -0.04925426  0.02863553
 -0.02447631  0.00121287  0.02869855 -0.01358518 -0.01580231 -0.04376303
```

```
 -0.04781497  0.00557282 -0.03766738  0.02496621  0.03287545]
[[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
  -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
   4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
   4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
  -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
  -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
   2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
  -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
  -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
  -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
   4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
   4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
   3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
   2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
   3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
  -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
  -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]]
('Tolerance=', 4972171.1853834139)
('error=', 99.501828333603413)
```

1.7 iii) Quasi-Newton method

In [101]:
```
### With Armijo Rule


        (a,b)=X.shape
        beta_quasi=np.zeros(b) #initial value for beta
        #alpha= 0.0000001
        n_iter=100000 #maximim nunber iteration
        OF_iter=np.zeros(n_iter)
        tol_iter=np.zeros(n_iter)
        alpha_iter=np.zeros(n_iter)
        i=0;
        tol=1000; # Tolerance
        epsilon=1e-5;



        time_start = time.clock()
        while (i <= n_iter-2) and (tol>epsilon):
            i=i+1
            grad=least_sq_reg_der(beta_quasi,X,Y)
            if (i==1):
                grad=least_sq_reg_der(beta_quasi,X,Y)
                B=least_sq_reg_hess(beta_quasi,X,Y)
            else:
```

```
                grad_previous=grad
                grad=least_sq_reg_der(beta_quasi,X,Y)
                y= grad - grad_previous
                s= beta_quasi - beta_quasi_previous

                a=np.dot(B, s)
                b=np.transpose(np.dot(B, s))
                c=np.dot(np.dot(np.transpose(s), B), s)
                d=np.dot(y, np.transpose(y))
                e=np.dot(np.transpose(y), s)

                ### Page 44 Formulas
                B= B + np.dot(y-a, np.transpose(y-a))/np.dot(np.transpose(y-a),s)
                # B= B - (np.dot(a,b)/c)+(d/e)    (BFGS Formula)
            ddirect=-(np.dot(np.linalg.inv(B), grad))

            ########################
            #     Armijo Rule
            sigma=0.1
            beta=0.5
            alpha=1
            while (least_sq_reg(beta_quasi+alpha*ddirect,X,Y)> least_sq_reg(beta_
                alpha=alpha*beta
            ########################
            beta_quasi_previous= beta_quasi
            beta_quasi=beta_quasi_previous + alpha*ddirect
            OF_iter[i]=least_sq_reg(beta_quasi,X,Y)
            tol=np.linalg.norm(grad,ord=2)
            tol_iter[i]=tol

        time_elapsed = (time.clock() - time_start)
        print('time elapsed=',time_elapsed)
        print('iterations',i)
        print(OF_iter[i])
        print(beta_quasi)
        print(np.transpose(beta_ls_exact))
        print('Tolerance=',tol)
        print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-beta_quasi,ord=
```

```
('time elapsed=', 0.0035460236395010725)
('iterations', 2)
84919.4337684
[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
 -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
  4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
  4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
 -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
 -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
```

```
  2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
 -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
 -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
 -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
  4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
  4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
  3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
  2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
  3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
 -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
 -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]
[[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
  -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
   4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
   4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
  -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
  -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
   2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
  -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
  -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
  -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
   4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
   4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
   3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
   2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
   3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
  -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
  -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]]
('Tolerance=', 2.987721920840585e-06)
('error=', 6.429578978306502e-13)
```

```
In [102]: ### Without Armijo Rule


          (a,b)=X.shape
          beta_quasi=np.zeros(b) #initial value for beta
          alpha= 0.0000001
          n_iter=100000 #maximim nunber iteration
          OF_iter=np.zeros(n_iter)
          tol_iter=np.zeros(n_iter)
          alpha_iter=np.zeros(n_iter)
          i=0;
          tol=1000; # Tolerance
          epsilon=1e-5;
```

```python
time_start = time.clock()
while (i <= n_iter-2) and (tol>epsilon):
    i=i+1
    grad=least_sq_reg_der(beta_quasi,X,Y)
    if (i==1):
        grad=least_sq_reg_der(beta_quasi,X,Y)
        B=least_sq_reg_hess(beta_quasi,X,Y)
    else:
        grad_previous=grad
        grad=least_sq_reg_der(beta_quasi,X,Y)
        y= grad - grad_previous
        s= beta_quasi - beta_quasi_previous

        a=np.dot(B, s)
        b=np.transpose(np.dot(B, s))
        c=np.dot(np.dot(np.transpose(s), B), s)
        d=np.dot(y, np.transpose(y))
        e=np.dot(np.transpose(y), s)

        ### Page 44 Formulas

        B= B + np.dot(y-a, np.transpose(y-a))/np.dot(np.transpose(y-a),s)

        # B= B - (np.dot(a,b)/c)+(d/e)    (BFGS Formula)
    ddirect=-(np.dot(np.linalg.inv(B), grad))

    beta_quasi_previous= beta_quasi
    beta_quasi=beta_quasi_previous + alpha*ddirect
    OF_iter[i]=least_sq_reg(beta_quasi,X,Y)
    tol=np.linalg.norm(grad,ord=2)
    tol_iter[i]=tol

time_elapsed = (time.clock() - time_start)
print('time elapsed=',time_elapsed)
print('iterations',i)
print(OF_iter[i])
print(beta_quasi)
print(np.transpose(beta_ls_exact))
print('Tolerance=',tol)
print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-beta_quasi,ord=
```

```
('time elapsed=', 77.44199172950175)
('iterations', 99999)
84612255.8376
[-0.69455439  0.02330877 -0.03196133  0.02538503 -0.02802235  0.02159238
 -0.02782555  0.00498215  0.01533958  0.03817249 -0.0292014  -0.00445133
  0.04292589  0.03425616  0.04192465  0.01969262  0.03000818  0.05220902
  0.04888016 -0.02244421 -0.02604743 -0.0385179  -0.02231664  0.04574423
```

```
 -0.04602987  0.04638269  0.04767248 -0.00495406 -0.04085386  0.01351047
 -0.03731758 -0.00940024 -0.00109407 -0.02037651  0.02295996  0.05064143
  0.03090196  0.00648642  0.04892168  0.0052158  -0.00708273  0.03500935
 -0.01268554  0.00202509  0.02323332  0.01934208  0.04378862  0.02574653
 -0.00521638 -0.01736145  0.05041185  0.02412116 -0.02206341  0.04692197
 -0.02956954 -0.0241941  -0.0450268   0.02771753  0.03523912  0.04909869
  0.04288969 -0.03328658 -0.04322499 -0.00332134  0.00306589  0.04330142
  0.04908182  0.01516352 -0.04848864  0.00754634  0.04075071  0.02911294
  0.04070567  0.04907051 -0.01998648  0.04099637  0.01332996 -0.01517363
  0.0248067  -0.04697969 -0.02677797  0.02275512  0.04800963 -0.03300447
  0.0343062  -0.04535301  0.00093947  0.04869848 -0.04790353  0.02933167
 -0.02343911  0.00257443  0.03050908 -0.0114265  -0.0137986  -0.04242246
 -0.04702582  0.00790206 -0.03604466  0.02715162  0.03446269]
[[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
  -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
   4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
   4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
  -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
  -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
   2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
  -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
  -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
  -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
   4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
   4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
   3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
   2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
   3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
  -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
  -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]]
('Tolerance=', 27512897.560719263)
('error=', 41.810947541672022)
```

In [5]:
```python
import sympy
from pandas import DataFrame
from sympy import *

# For Final report we need to put 'Time elapsed', 'Iteration','Tolerance',
### Fill value instead of 1 2 3 .....
t = [[215.57 ,99999 ,11.96 ,0.05617 ], #Gradient method (Armijo Rule)
     [14.09 ,99999 ,16.64 ,0.08595 ], #Gradient method
     [293.55 ,99999 ,4972171.18 ,4.24e-12 ], #Newton method (Armijo Rule)
     [192.15 ,99999 ,4972171.18 ,99.15 ], #Newton method
     [0.0035 ,2 ,2.99e-6 ,6.43e-13 ], #Quasi Newton method (Armijo Rule)
     [77.44 ,99999 ,27512897.56 ,41.81 ]] #Quasi Newton method
```

```
table = DataFrame(t, index=['Gradient method (Armijo Rule)', 'Gradient meth
                  columns=['Time elapsed', 'Iteration', 'Tolerance', 'Error
table
```

```
Out[5]:                                  Time elapsed  Iteration     Tolerance  \
        Gradient method (Armijo Rule)        215.5700      99999  1.196000e+01
        Gradient method                       14.0900      99999  1.664000e+01
        Newton method (Armijo Rule)          293.5500      99999  4.972171e+06
        Newton method                        192.1500      99999  4.972171e+06
        Quasi Newton method (Armijo Rule)      0.0035          2  2.990000e-06
        Quasi Newton method                   77.4400      99999  2.751290e+07

                                                Error
        Gradient method (Armijo Rule)    5.617000e-02
        Gradient method                  8.595000e-02
        Newton method (Armijo Rule)      4.240000e-12
        Newton method                    9.915000e+01
        Quasi Newton method (Armijo Rule) 6.430000e-13
        Quasi Newton method              4.181000e+01
```

1.8 E)

Estimate the value of the regression coefficients y implementing the coordinate gradient method and the stochastic gradient method. Compare their performance with the algorithms in c).

```
In [88]: #definition of partial gradient
         def least_sq_reg_der_par(beta_ls,X,Y,i):
             beta_ls=np.matrix(beta_ls)
             pp=-2*(Y[i]-X[i,]*np.transpose(beta_ls))*X[i,]
             aa= np.squeeze(np.asarray(pp))
             return aa
```

1.8.1 Coordinate gradient

```
In [89]: import time
         import random

         (a,b)=X.shape
         #alpha= 0.0000001
         beta_coor=np.zeros(b) #initial value for beta
         n_iter=100000 #maximim nunber iteration
         OF_iter=np.zeros(n_iter)
         tol_iter=np.zeros(n_iter)
         alpha_iter=np.zeros(n_iter)
         i=0;
         tol=1000;
         epsilon=1e-3;
```

```
time_start = time.clock()
while (i <= n_iter-2) and (tol>epsilon):
    i=i+1
    grad = least_sq_reg_der(beta_coor,X,Y)
    ddirect =np.zeros(b)
    j = random.randint(0, b-1)
    ddirect[j] = - grad[j]
    ###################################
    #       Armijo Rule-----Choose an appropiate alpha
    sigma = 0.1
    beta = 0.5
    alpha = 1
    while (least_sq_reg(beta_coor + alpha*ddirect , X, Y)> least_sq_reg(be
          alpha = alpha*beta
    ###################################
    beta_coor = beta_coor + alpha*ddirect
    OF_iter[i] = least_sq_reg(beta_coor, X, Y)
    tol = np.linalg.norm(grad,ord=2)
    tol_iter[i] = tol
    alpha_iter[i] = alpha

time_elapsed = (time.clock() - time_start)
print('time elapsed=',time_elapsed)
print('iterations',i)
print(OF_iter[i])
print(beta_coor)
print(np.transpose(beta_ls_exact))
print('Tolerance=',tol)
print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-beta_coor,ord=2)
```

```
('time elapsed=', 163.64487752970308)
('iterations', 99999)
84919.7410451
[-2.50021396  2.18682373 -3.33330932  2.39975782 -2.92014081  2.00246299
 -2.92190745  0.38682466  1.41610014  3.80388697 -3.04994025 -0.58761597
  4.12933115  3.27038748  4.05991182  1.86812078  2.90038461  5.10213145
  4.7520726  -2.37795869 -2.74969627 -3.94575757 -2.45870403  4.49820917
 -4.75009063  4.55768429  4.65587883 -0.59582511 -4.19314339  1.24460461
 -3.89808893 -1.04057926 -0.26554226 -2.28768565  2.16739383  4.92170907
  2.93963913  0.53277756  4.79699291  0.43585696 -0.86131784  3.3522031
 -1.41222325  0.06931246  2.14531845  1.82115116  4.29541405  2.45009805
 -0.63666568 -1.91204923  4.94201387  2.30657644 -2.39688656  4.63435576
 -3.05973144 -2.51412631 -4.71816209  2.54908238  3.42649047  4.78616836
  4.2115354  -3.54344122 -4.50507367 -0.48169823  0.13078017  4.24435255
  4.7720993   1.38526205 -4.99358747  0.74918188  3.95849775  2.79061004
  3.99931219  4.84301543 -2.07213891  3.98423849  1.19489186 -1.65072046
  2.41891899 -4.77456556 -2.80681004  2.1282199   4.66004788 -3.45777118
  3.31367115 -4.59191388 -0.09910009  4.70386052 -4.95080481  2.87758379
```

```
 -2.46041758  0.12123455  2.88339249 -1.36639218 -1.58913823 -4.39890042
 -4.80587062  0.55896863 -3.78642918  2.50807425  3.30328032]
[[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
  -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
   4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
   4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
  -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
  -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
   2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
  -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
  -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
  -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
   4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
   4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
   3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
   2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
   3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
  -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
  -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]]
('Tolerance=', 4.4303189821277034)
('error=', 0.0098751677873833519)
```

1.9 Stochastic gradient

```
In [90]: (a,b)=X.shape
         #alpha= 0.0000001
         beta_sto=np.zeros(b) #initial value for beta
         n_iter=100000 #maximum nunber iteration
         OF_iter=np.zeros(n_iter)
         tol_iter=np.zeros(n_iter)
         alpha_iter=np.zeros(n_iter)
         i=0;
         tol=1000;
         epsilon=1e-3;

         time_start = time.clock()
         while (i <= n_iter-2) and (tol>epsilon):
             i=i+1
             j = random.randint(0, b-1)
             grad_par = least_sq_reg_der_par(beta_sto,X,Y,j)
             grad = least_sq_reg_der(beta_sto,X,Y)
             ddirect = - grad_par
             ###################################
             #      Armijo Rule-----Choose an appropiate alpha
             sigma = 0.1
             beta = 0.5
             alpha = 1
```

```
            while (least_sq_reg(beta_sto + alpha*ddirect , X, Y)> least_sq_reg(bet
                    alpha = alpha*beta
            ###################################
            beta_sto = beta_sto + ddirect*alpha
            OF_iter[i] = least_sq_reg(beta_sto, X, Y)
            tol = np.linalg.norm(grad,ord=2)
            tol_iter[i] = tol
            alpha_iter[i] = alpha

        time_elapsed = (time.clock() - time_start)
        print('time elapsed=',time_elapsed)
        print('iterations',i)
        print(OF_iter[i])
        print(beta_sto)
        print(np.transpose(beta_ls_exact))
        print('Tolerance=',tol)
        print('error=',np.linalg.norm(np.transpose(beta_ls_exact)-beta_sto,ord=2)/
```

```
('time elapsed=', 322.0782561465603)
('iterations', 99999)
2113446.74323
[ 0.10969587  2.15459106 -2.73794796  2.43062366 -1.81939819  1.55858974
 -2.06132185  1.17113482  0.49903371  1.57226603 -0.64545193 -0.10494674
  5.42318687  1.38397385  2.23523869  2.42252443  2.59244987  3.03017531
  4.35250852 -0.45845149 -2.30684351 -3.81494188 -1.9625614   2.64530825
 -2.50652389  4.22243716  3.39618464  1.15481196 -3.14793245  2.51914073
 -1.94143724 -0.90108007 -0.03820853 -1.42390156  2.09482173  1.95039804
 -1.77012151  1.55584506  2.69934517  1.4177997  -1.7876631   1.38743046
 -0.57572693  2.07897314  0.04608848  0.83539717  4.01770176  1.29246787
 -0.06485886 -1.26403007  3.41106878  1.65507843 -1.80252834  3.02580618
 -1.95996077 -0.67287535 -3.8303346   1.53249861  2.95217127  3.50176353
  3.30527248 -1.89845257 -1.97434118  0.15292702 -1.0444904   2.73142989
  2.0422938   1.27693678 -3.35879968  1.12025885  2.81241255  0.76902383
  2.01769281  0.77575851  1.53522836  2.43482166 -0.96953795 -1.15269728
  2.71898709 -2.98207682 -1.19521613  0.38061522  2.15831357 -2.96984636
  2.2403703  -0.82964194  1.08526317  4.11447167 -1.7634749   0.71495591
 -1.09210182  0.74986064  2.98311974  0.95349042  1.22215274 -1.3047014
 -4.55340843 -0.87160734 -2.48210947  3.10061311  2.04836621]
[[-2.82009166  2.18757262 -3.33272408  2.40050035 -2.91964431  2.00327309
  -2.92131935  0.38737545  1.41670728  3.80403217 -3.04939149 -0.58693972
   4.13024247  3.27121943  4.06064646  1.86865365  2.9009471   5.1028255
   4.75282718 -2.37740208 -2.74905513 -3.94539386 -2.45767779  4.49870622
  -4.74962366  4.55818668  4.65647563 -0.59536061 -4.19274504  1.24511478
  -3.8973661  -1.04011885 -0.26481368 -2.28654173  2.16800687  4.92252054
   2.94043056  0.5333243   4.79756441  0.43627901 -0.86060974  3.35301358
  -1.41157203  0.06996948  2.14622611  1.8217317   4.29591723  2.45076112
  -0.63614187 -1.91125437  4.94259785  2.30714669 -2.39602395  4.63460325
  -3.05933115 -2.51371899 -4.71726272  2.55024272  3.42703614  4.7868769
```

```
   4.21201842 -3.5424867  -4.50429917 -0.48099319  0.1316062   4.24485554
   4.7728776   1.38593473 -4.99300561  0.74922867  3.95908786  2.79126176
   3.99975326  4.84343426 -2.07181554  3.98489739  1.19557716 -1.65011212
   2.41928082 -4.7743059  -2.80624142  2.12898243  4.66082827 -3.45711962
   3.31431908 -4.59174474 -0.0981666   4.70477382 -4.95014286  2.87792284
  -2.45991397  0.12189604  2.88425693 -1.36533547 -1.58816079 -4.39826434
  -4.8054921   0.56007872 -3.78564008  2.50915003  3.30404243]]
('Tolerance=', 259606.19719997552)
('error=', 0.74201546127562212)
```

In [6]:
```
import sympy
from pandas import DataFrame
from sympy import *

# For Final report we need to put 'Time elapsed', 'Iteration','Tolerance',
### Fill value instead of 1 2 3 .....
t = [[163.64 ,99999 ,4.43 ,0.0099 ], #Coordinate gradient
     [322.08 ,99999 ,259606.20 ,0.7420 ]] #Stochastic gradient


table = DataFrame(t, index=['Coordinate gradient', 'Stochastic gradient'],

table
```

Out[6]:

	Time elapsed	Iteration	Tolerance	Error
Coordinate gradient	163.64	99999	4.43	0.0099
Stochastic gradient	322.08	99999	259606.20	0.7420

In [ ]: