The Foundations: Logic and Proofs

Chapter 1, Part I: Propositional Logic

With Question/Answer Animations

Chapter Summary

- Propositional Logic
  - The Language of Propositions
  - Applications
  - Logical Equivalences
- Predicate Logic
  - The Language of Quantifiers
  - Logical Equivalences
  - Nested Quantifiers
- Proofs
  - Rules of Inference
  - Proof Methods
  - Proof Strategy

Propositional Logic

Section 1.1

Propositions

- A *proposition* is a collection of declarative sentence that is either true or false.
- Examples of propositions:
  a) Doha is the capital of Qatar.
  b) Alexandria is the capital of Egypt.
  c) $1 + 0 = 1$
  d) $0 + 0 = 2$
- Examples that are not propositions.
  a) Sit down!
  b) What time is it?
  c) $x + 1 = 2$
  d) $x + y = z$

Propositional Logic

- Constructing Propositions
  - Propositional Variables: $p, q, r, s, \ldots$
  - The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
  - Compound Propositions; constructed from logical connectives and other propositions
    - Negation $\neg$
    - Conjunction $\wedge$
    - Disjunction $\vee$ / Exclusive Or $\oplus$
    - Implication $\rightarrow$
    - Biconditional $\leftrightarrow$

Compound Propositions: Negation

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

Example: If p denotes "*I have an android smartphone.*", then $\neg p$ denotes "*it is not the case that I have an android smartphone*", which means more simply: "*I don't have an android smartphone.*"

Conjunction

- The *conjunction* of propositions p and q is denoted by $p \wedge q$.
- It's true when both p and q are true and is false otherwise and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

Disjunction

- The *disjunction* of propositions p and q is denoted by $p \vee q$.
- It's false when both p and q are false and is true otherwise and has this truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

The Connective Or in English

- In English “or” has two distinct meanings.
  - “Inclusive Or” - In the sentence “Students who have taken CS202 or Math120 may take this class,” we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \lor q$ to be true, either one or both of p and q must be true.
  - “Exclusive Or” - When reading the sentence “Soup or salad comes with this entrée,” we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, but not both.
  - “Exclusive Or” may also be written as “Either soup or salad comes with this entrée.”

The Exclusive Or

- The ***Exclusive Or*** of propositions p and q is denoted by $p \oplus q$
- It's true when ***exactly one*** of p and q is true and is false otherwise.
- "Tea or Coffee?"
- The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" or "p implies q".
- It is false when p is true and q is false, and *true* otherwise it has this truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \rightarrow q$ denotes "If I am at home then it is raining."

  That I am at home implies that it is raining.
- In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication

- In $p \rightarrow q$, p is the ***hypothesis*** (*antecedent* or *premise*) and q is the ***conclusion*** (or *consequence*)
- There does not need to be any connection between the antecedent or the consequent. The "meaning" of $p \rightarrow q$ depends only on the truth values of p and q.
- These implications are perfectly fine, but would not be used in ordinary English.
  - "If the moon is made of green cheese, then I have more money than Bill Gates."
  - "If the moon is made of green cheese then I'm on welfare."
  - "If 1 + 1 = 3, then your grandma wears combat boots." [Since 1 + 1 = 3 is F, the implication is T. We refer to the truth table.]

Understanding Implication (cont)

- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
  - "If you get 100% on the final, then you will get an A."
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true while q and hence $p \rightarrow q$ are false.

Different Ways of Expressing $p \rightarrow q$

- **if** p, **then** q
- **if** p, q
- q **when** p
- q **if** p
- q **unless** $\neg p$
- p **is sufficient for** q
- q **follows from** p
- p **implies** q
- p **only if** q
- q **is necessary for** p
- q **whenever** p
- **a sufficient condition for** q **is** p
- **a necessary condition for** p **is** q

- "If I am elected, then I will lower taxes."

Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements .
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$ (the same TF)
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$

Example: Find the converse, inverse, and contrapositive of "It raining is a sufficient condition for my not going to town."

Solution:

converse: If I do not go to town, then it is raining.

contrapositive: If I go to town, then it is not raining.

inverse: If it is not raining, then I will go to town.

Biconditional

- If p and q are propositions, then the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q."
- It's true when p and q have the *same truth values*, and is false otherwise. The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

- Some alternative ways "*p* if and only if *q*" is expressed in English:

  - ***p* is necessary and sufficient for *q***
  - **if *p* then *q* , and conversely**
  - ***p* iff *q***

Truth Tables For Compound Propositions

- Construction of a truth table:
- Rows
  - One for every possible combination of values for the atomic propositions. (in the next slide, 3 atomic propositions → 8 rows)
- Columns
  - One for each atomic propositions (on the left)
  - One for the compound propositions on (the right)
  - If needed, intermediate steps in-between.

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are **e**_quivalent_ if they always have the same truth value.
- **Example**: Show using a truth table that the implication is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	**F**	**F**	**T**	F	T	T
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Problem

- How many rows are there in a truth table with *n* propositional variables?

Solution: 2^n We will see how to do this in Chapter 6.

- Note that this means that with n propositional variables, we can construct 2^n distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Applications of Propositional Logic

Section 1.2

Page 37

Applications of Propositional Logic: Summary

- Translating English to Propositional Logic
- System Specifications
- Boolean Searching
- Logic Circuits

Translating English Sentences

- Steps to convert an English (ambiguous) sentence to a statement in propositional logic, Translating sentences into compound statements removes the ambiguity
  - Identify atomic propositions and represent using propositional variables.
  - Determine appropriate logical connectives
- "If I go to Harry's or to the country, I will not go shopping."
  - *p*: I go to Harry's
  - q: I go to the country.
  - *r*: I will go shopping.

If *p* or *q* then not *r*.

$$(p \lor q) \rightarrow \neg r$$

Translating English Sentences *Cont.*

Example: "You can access the Internet from campus *only if* you are a computer science major or you are not a freshman."

Solution:

- *Use propositional variables to represent each sentence part and determine the appropriate logical connectives between them.*
- Let a, c, and f represent "You can access the Internet from campus," "You are a computer science major," and "You are ~~not~~ a freshman," respectively.

This sentence can be represented as: $a \rightarrow (c \lor \neg f)$.

System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"The automated reply cannot be sent when the file system is full"

Solution: One possible solution: Let p denote "The automated reply can be sent" and q denote "The file system is full."

$$q \rightarrow \neg p$$

Consistent System Specifications

Definition: A list of propositions is *consistent* if it is possible to assign truth values to the proposition variables so that each proposition is true.

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Solution: Let p denote "The diagnostic message is **not** stored in the buffer." Let q denote "The diagnostic message is retransmitted" The specification can be written as: $p \vee q$, $\neg p$, $p \rightarrow q$,. When p is false and q is true all three statements are true. So the specification is **consistent**.

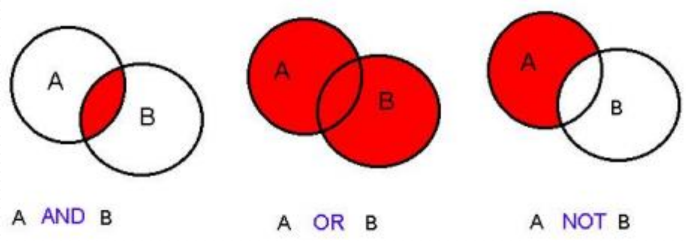
- What if "The diagnostic message is *not* retransmitted" is added.

Solution: Now we are adding $\neg q$ (false) and there is no satisfying assignment. So the specification is **not consistent**.

Boolean Search

- **Web Search**

Boolean Operators

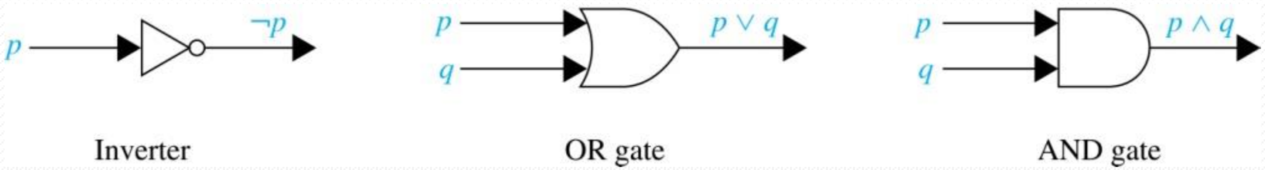


Boolean Operators can help you when you are looking for information on the web.

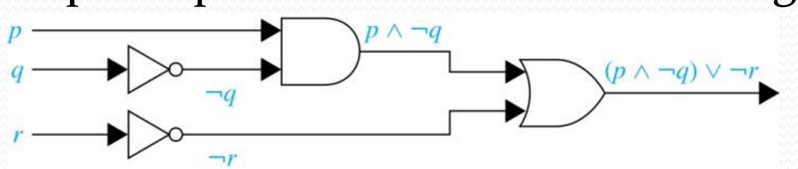
- "NEW *AND* MEXICO *AND* UNIVERSITIES"
- "(NEW *AND* MEXICO *OR* ARIZONA) *AND* UNIVERSITIES"
- "(MEXICO *AND* UNIVERSITIES) *NOT* NEW"

Logic Circuits (Studied in depth in Chapter 12)

- Electronic circuits; each input/output signal can be viewed as a 0 or 1.
  - 0 represents **False** & 1 represents **True**
- Complicated circuits are constructed from three basic circuits called gates.

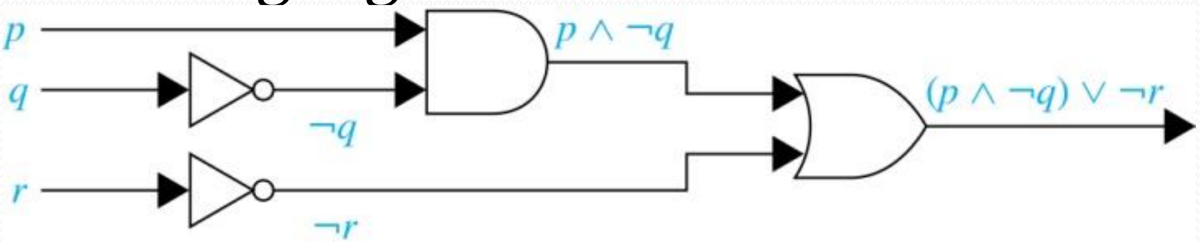


  - The inverter (**NOT gate**)takes an input bit and produces the negation of that bit.
  - The **OR gate** takes two input bits and produces the value equivalent to the disjunction of the two bits.
  - The **AND gate** takes two input bits and produces the value equivalent to the conjunction of the two bits.
- More complicated digital circuits can be constructed by combining these basic circuits to produce the desired output given the input signals by building a circuit for each piece of the output expression and then combining them. For example:



Logic Circuits (Studied in depth in Chapter 12)

Example: Determine the output for the combinatorial circuit in the following Figure.



- **Solution**:
- We see that the AND gate takes input of p and ¬q, the output of the inverter with input q, and produces $p \wedge \neg q$.
- Next, we note that the OR gate takes input $p \wedge \neg q$ and $\neg r$, the output of the inverter with input r, and produces the final output $(p \wedge \neg q) \vee \neg r$

Propositional Equivalences

Section 1.3

Properties of Propositions

Tautologies, Contradictions, and Contingencies

- A *tautology* is a proposition which is always true.
  - Example: $p \vee \neg p$
- A *contradiction* is a proposition which is always false.
  - Example: $p \wedge \neg p$
- A *contingency* is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logically Equivalent

- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ denote p and q are equivalent, where p and q are compound propositions.
- Two compound propositions p and q are equivalent *if and only if* the columns in a truth table giving their truth values agree.
- This truth table show $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

Example: Show that ¬(p ∨ q) and ¬p ∧¬q are logically equivalent

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan
1806-1871

Solution: This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

De Morgan's Laws

Example: Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent

Solution: This truth table shows that the compound propositions are logically equivalent. Because the truth values of $\neg p \vee q$ and $p \rightarrow q$ agree.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Key Logical Equivalences

- Identity Laws: $p \wedge T \equiv p$, $p \vee F \equiv p$
- Domination Laws: $p \vee T \equiv T$, $p \wedge F \equiv F$
- Idempotent laws: $p \vee p \equiv p$, $p \wedge p \equiv p$
- Double Negation Law: $\neg(\neg p) \equiv p$
- Negation Laws: $p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$
- Implication equivalence: $p \rightarrow q \equiv q \vee \neg p$

Key Logical Equivalences (*cont*)

- Commutative Laws: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$
- Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
  $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- Distributive Laws: $(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$
  $(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$
- Absorption Laws: $p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \lor q$

$p \to q \equiv \neg q \to \neg p$

$p \lor q \equiv \neg p \to q$

$p \land q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \land \neg q$

$(p \to q) \land (p \to r) \equiv p \to (q \land r)$

$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$

$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$

$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative law} \\
& && \text{for disjunction} \\
&\equiv (\neg p \wedge \neg q) && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by associative and} \\
& && \text{commutative laws} \\
& && \text{laws for disjunction} \\
&\equiv T \vee T && \text{by truth tables} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$