Challenging Novel Many and Multi-Objective Bound Constrained Benchmark Problems

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1 Challenging Test Problems for Many and Multi Objective Optimization

In this work, we construct a set of ten test problems based on the following general formulation:

$$
\begin{array}{lll}
\text{minimize} \;\; f_1 & = & r_1 \times \alpha_1(x) \times (1 + g_1(x)) \\
\text{minimize} \;\; f_2 & = & r_2 \times \alpha_2(x) \times (1 + g_2(x)) \\
\cdots\cdots & & \\
\text{minimize} \;\; f_{M-1} & = & r_{M-1} \times \alpha_{M-1}(x) \times (1 + g_{M-1}(x)) \\
\text{minimize} \;\; f_M & = & r_M \times \alpha_M(x) \times (1 + g_M(x))
\end{array}
\tag{1}
$$

where

- $x = (x_1, \ldots, x_N)$ is the decision vector in $[0,1]^N$, and $(f_1, \ldots, f_M)$ is the objective vector in R^M;
- $r_i, i = 1, \ldots, M$, is the scaling factor for the i-th objective function;
- $\alpha_i, i = 1, \ldots, M$, is the shape function for the i-th objective function;
- $g_i \geq 0, i = 1, \ldots, M$, is the distance function for the i-th objective function.

Within the above toolkit, the challenging problem difficulties in multiobjective optimization, such as objective scalability, multi-modality, complicated Pareto sets [1], disconnectedness, degeneracy [2, 3], and bias [4], are taken into account. Compared with DTLZ and WFG test problems [5, 6], our test problems are more challenging and easy to provide the analytical form of their PS sets.

- MaOP1
  - shape function:

$$
\begin{array}{lll}
\alpha_1 & = & 1 - x_1 x_2 \cdots x_{M-1} \\
\alpha_2 & = & 1 - x_1 x_2 \cdots (1 - x_{M-1}) \\
\cdots\cdots & & \\
\alpha_{M-1} & = & 1 - x_1(1 - x_2) \\
\alpha_M & = & 1 - (1 - x_1)
\end{array}
\tag{2}
$$

    difficulty: inverse of a simplex

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– distance function:

$$g_i(x) = \frac{1}{N}\sum_{j=M}^{N}\left((x_j - 0.5)^2 + 1 - \cos(20\pi(x_j - 0.5))\right)$$

difficulty: many local Pareto fronts

– scaling factor:

$$r_i = 0.1 + 10 \times (i-1), i = 1, \dots, M$$

difficulty: objective scalability

– the PF and the PS:

$$PF = \{F(x) \in R^M | f_1/r_1 + f_2/r_2 + \cdots + f_M/r_M = M - 1\}$$

and

$$PS = \{x \in R^N | x_i \in [0,1], i = 1, \dots, M-1, x_i = 0.5, i = M, \dots, N\}$$

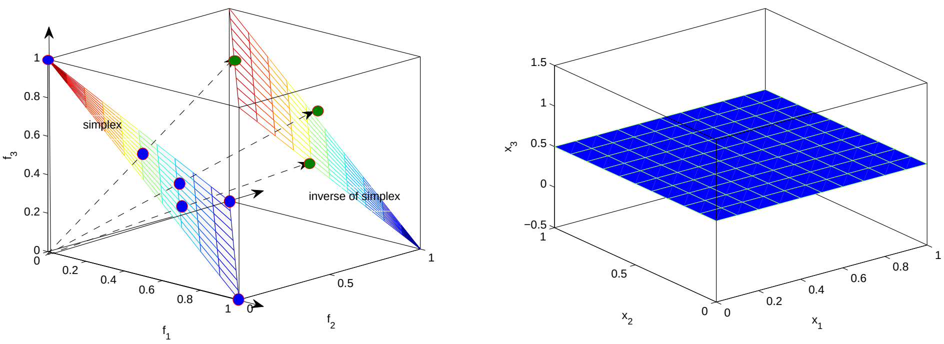

Figure 1: The PF of MaOP1: inverse of a simplex (left) and the PS of MaOP1: a linear plane (right)

- MaOP2

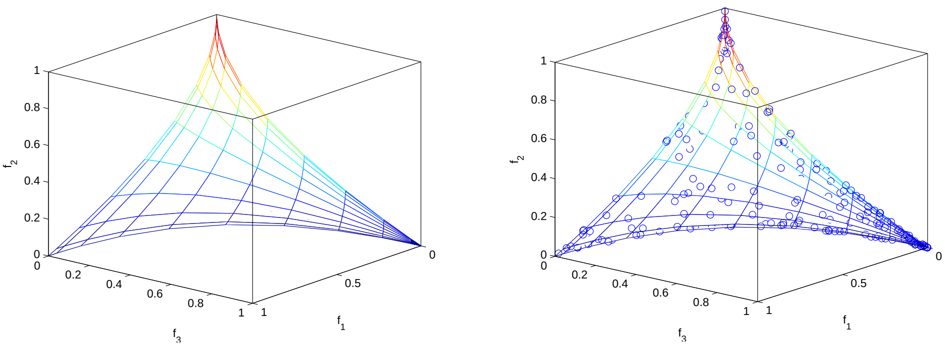

Figure 2: The PF of MaOP2: the distribution of 200 random solutions shows the bias towards the boundary of PF.

– shape function:

$$\begin{aligned}
\alpha_1 &= [\cos(0.5x_1\pi)\cos(0.5x_2\pi)\cdots\cos(0.5x_{M-2}\pi)\cos(0.5x_{M-1}\pi))]^{p_1} \\
\alpha_2 &= [\cos(0.5x_1\pi)\cos(0.5x_2\pi)\cdots\cos(0.5x_{M-2}\pi)\sin(0.5x_{M-1}\pi))]^{p_2} \\
&\cdots\cdots \\
\alpha_{M-1} &= [\cos(0.5x_1\pi)\sin(0.5x_2\pi)]^{p_{M-1}} \\
\alpha_M &= [\sin(0.5x_1\pi)]^{p_M}
\end{aligned} \tag{3}$$

where

$$p_i = \begin{cases} 4 & \text{if } \mathrm{mod}(i,2) = 1 \\ 2 & \text{otherwise.} \end{cases} \quad i = 1, \ldots, M$$

difficulty: bias near the PF boundary

– distance function

$$g_i(x) = 200 \times \sum_{j=M}^{N} (x_j - y_j)^2, i = 1, \ldots, M$$

with

$$y_j = \begin{cases} 0.5 & \text{if } \mathrm{mod}(j,5) \neq 0 \\ \prod_{k=1}^{M-1} \sin(0.5x_k\pi) & \text{otherwise} \end{cases}$$

difficulty: nonlinear PS shape (interacting variables along the PS)

– scaling factor: $r_i = 1, i = 1, \ldots, M$

– the PF and the PS:

$$PF = \{F(x) \in R^M | f_1^{q_1} + f_2^{q_2} + \cdots + f_M^{q_M} = 1\}$$

with

$$q_j = \begin{cases} 1 & \text{if } j \text{ is even} \\ 1/2 & \text{otherwise} \end{cases} \quad j = 1, \ldots, M$$

and

$$\begin{aligned} PS = \{x_j \in [0,1] | j \in \{1, \ldots, M-1\}\} \cup \{x_j = 0.5 | j \in \{M, \ldots, N\} \text{ and } \mathrm{mod}(j,5) \neq 0\} \\ \cup \{x_j = \prod_{k=1}^{M-1} \sin(0.5x_k\pi) | j = \{M, \ldots, N\} \text{ and } \mathrm{mod}(j,5) = 0\} \end{aligned}$$

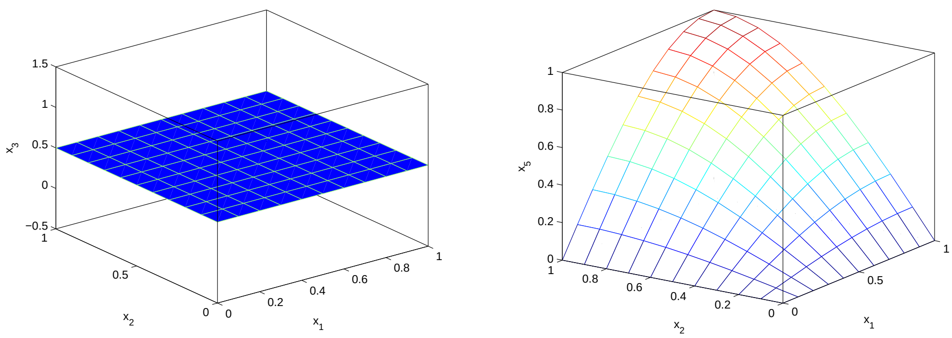

Figure 3: The PSs of MaOP2: linear PS shape in $x_1 - x_2 - x_3$ (left), nonlinear PS shape in $x_1 - x_2 - x_5$(right)

- MaOP3

  – shape function:

$$\begin{array}{lcl} \alpha_1 & = & \cos(0.5x_1\pi)\cos(0.5x_2\pi)\cdots\cos(0.5x_{M-2}\pi)\cos(0.5x_{M-1}\pi)) \\ \alpha_2 & = & \cos(0.5x_1\pi)\cos(0.5x_2\pi)\cdots\cos(0.5x_{M-2}\pi)\sin(0.5x_{M-1}\pi)) \\ \cdots\cdots & & \\ \alpha_{M-1} & = & \cos(0.5x_1\pi)\sin(0.5x_2\pi) \\ \alpha_M & = & \sin(0.5x_1\pi) \end{array} \quad (4)$$

  – distance function:

$$g_i(x) = i \times \sum_{j=M}^{N} |x_j - y_j|^{0.1}$$

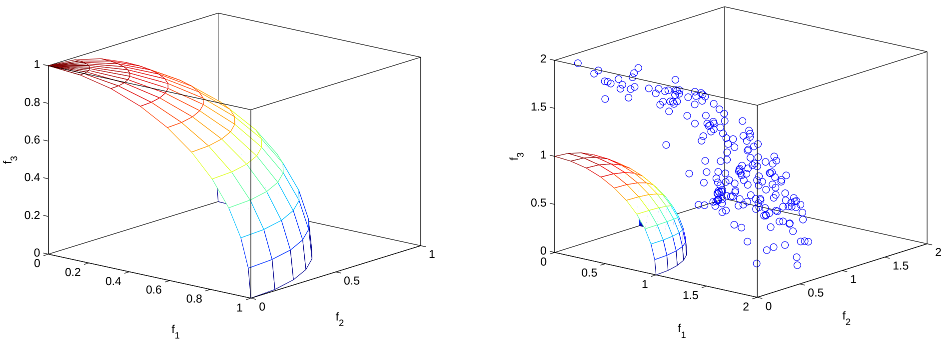

Figure 4: The PF of MaOP3: the distribution of 200 random solutions.

with

$$y_j = \begin{cases} 0.5 & \text{if } \mathrm{mod}(j,5) \neq 0 \\ \prod_{k=1}^{M-1} \sin(0.5x_k\pi) & \text{otherwise} \end{cases}$$

difficulty: bias in convergence towards the PF

- scaling factor: $r_i = 1, i = 1, \dots, M$
- the PF and the PS:

$$PF = \{F(x) \in [0,1]^M | f_1^2 + f_2^2 + \cdots + f_M^2 = 1\}$$

and the PS is the same as that of MaOP2.

- MaOP4

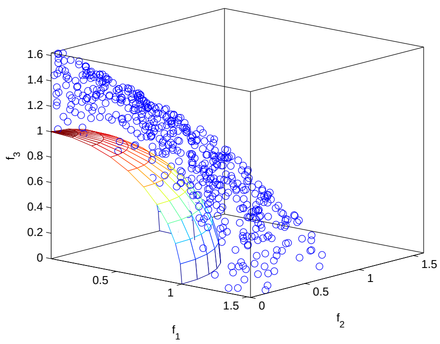

Figure 5: The PF of MaOP4: the distribution of 200 random solutions shows the bias on the convergence towards the PF with $x_1 = 0$ or $x_1 = 1$.

- shape function:

$$\begin{array}{lcl} \alpha_1 & = & \cos(0.5x_1\pi)\cos(0.5x_2\pi)\cdots\cos(0.5x_{M-2}\pi)\cos(0.5x_{M-1}\pi)) \\ \alpha_2 & = & \cos(0.5x_1\pi)\cos(0.5x_2\pi)\cdots\cos(0.5x_{M-2}\pi)\sin(0.5x_{M-1}\pi)) \\ \cdots\cdots & & \\ \alpha_{M-1} & = & \cos(0.5x_1\pi)\sin(0.5x_2\pi) \\ \alpha_M & = & \sin(0.5x_1\pi) \end{array} \tag{5}$$

- distance function:

$$g_i(x) = 20 \times \sin(\pi x_1) \times \sum_{j=M}^{N} [-0.9(x_j - y_j)^2 + |x_j - y_j|^{0.6}]$$

with

$$y_j = \begin{cases} 0.5 & \text{if } \mathrm{mod}(j,5) \neq 0 \\ \prod_{k=1}^{M-1} \sin(0.5 x_k \pi) & \text{otherwise} \end{cases}$$

difficulty: bias at $x_1 = 0$ and $x_1 = 1$, and nonlinear PS shape

- scaling factor: $r_i = 1, i = 1, \ldots, M$
- the PF and the PS:
  the PF and the PS is the same as that of MaOP3.

- MaOP5

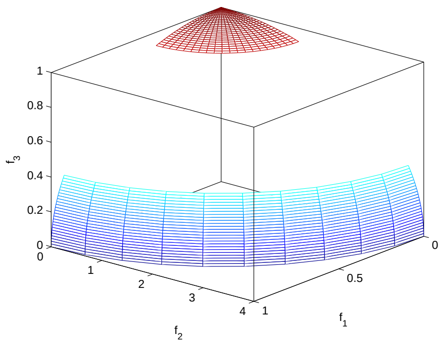

Figure 6: The PF of MaOP5.

- shape function:

$$\begin{array}{lcl} \alpha_1 & = & \cos(0.5x_1\pi)\cos(0.5x_2\pi) \\ \alpha_2 & = & \cos(0.5x_1\pi)\sin(0.5x_2\pi) \\ \alpha_3 & = & \sin(0.5x_1\pi) \\ \cdots\cdots & & \\ \alpha_i & = & \frac{i}{M} \times \alpha_1(x_1,x_2) + (1-\frac{i}{M}) \times \alpha_2(x_1,x_2) + \sin(\frac{0.5i\pi}{M})\alpha_3(x_1,x_2), \quad i = 4,\ldots,M \end{array} \tag{6}$$

  difficulty: highly correlated objectives

- distance function:

$$g_i(x) = 10 \times \begin{cases} \max(0, -1.4\cos(2x_1\pi)) + \sum_{j=3}^{N}(x_j - x_1x_2)^2 & i = 1,2,3 \\ \exp((x_i - x_1x_2)^2) - 1 & i = 4,\ldots,M \end{cases}$$

  difficulty: disconnected PF and nonlinear PS shapes

- scaling factor: $r_i = 1, i = 1, 3 \ldots, M$ and $r_2 = 4$.
- the PF and the PS:

$$PF = \{(f_1,\ldots,f_M) | f_1^2 + f_2^2/16 + f_3^2 = 1, f_2 \in [0, 4\sin(\pi/8)] \cup [4\sin(3\pi/8), 4]\}$$

- MaOP6

- shape function:

$$\begin{array}{lcl} \alpha_1 & = & x_1x_2 \\ \alpha_2 & = & x_1(1-x_2) \\ \alpha_3 & = & (1-x_1) \\ \cdots\cdots & & \\ \alpha_i & = & \frac{i}{M} \times \alpha_1(x_1,x_2) + (1-\frac{i}{M}) \times \alpha_2(x_1,x_2) + \sin(\frac{0.5i\pi}{M})\alpha_3(x_1,x_2), \quad i = 4,\ldots,M \end{array} \tag{7}$$

  difficulty: highly correlated objectives

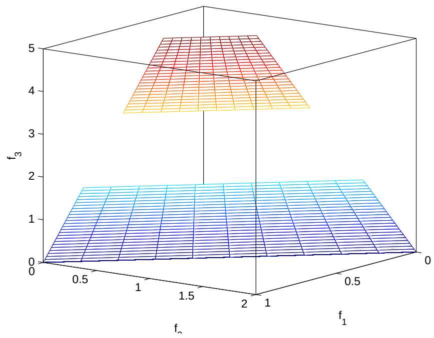

Figure 7: The PF of MaOP6

– distance function:

$$g_i(x) = \begin{cases} 10 \times (\max(0, 1.4\sin(4x_1\pi)) + \sum_{j=1}^{3} |x_j - x_1 x_2|^2) & i = 1, 2, 3 \\ 10 \times (\max(0, 1.4\sin(4x_1\pi)) + \exp(|x_i - x_1 x_2|^2) - 1) & i = 4, \ldots, N \end{cases}$$

difficulty: disconnected PF and nonlinear PS shapes

– scaling factor: $r_1 = 1, r_2 = 2, r_3 = 6,\ r_i = 1, i = 4 \ldots, M$.

- MaOP7

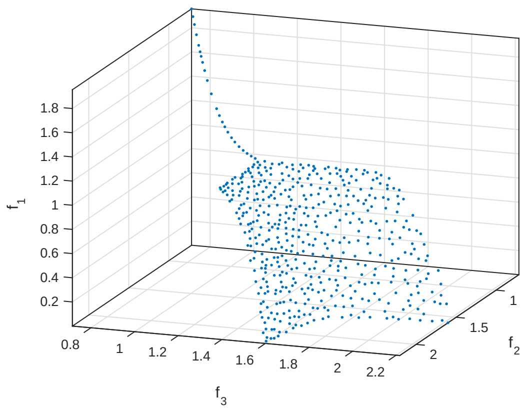

Figure 8: The PF of MaOP7 in $f_1 - f_2 - f_3$.

– shape function:

$$\begin{aligned} \alpha_1 &= (-1) \times (2x_1 - 1)^3 + 1 \\ \alpha_{2k}(x_1, x_k) &= x_1 + \tau 2x_k + \tau|2x_k - 1|^{0.5+x_1} \\ \alpha_{2k+1}(x_1, x_k) &= x_1 - \tau(2x_k - 2) + \tau|2x_k - 1|^{0.5+x_1} \\ & \qquad\qquad k = 2, \ldots, T+1 \\ \alpha_M &= \begin{cases} 1 - \alpha_1 & \text{if } M \text{ is even} \\ \alpha_{2T+1} & \text{otherwise} \end{cases} \end{aligned}$$

with the constant $\tau = \sqrt{2}/2$.
difficulty: local PF degeneracy

– distance function:

$$g_i(x) = 100 \times \sum_{j=T+2}^{N} (x_j - y_j)^2, i = 1, \ldots, M$$

with

$$y_j = \begin{cases} 0.5 & \text{if } \mathrm{mod}(j,5) \neq 0 \\ \prod_{k=1}^{T+1} \sin(0.5 x_k \pi) & \text{otherwise} \end{cases}$$

difficulty: nonlinear PS shape

- scaling factor: $r_i = 1, i = 1, \ldots, M$.
- the PS and the PF:

$$PS = \Omega_1 \cup \Omega_2$$

with

$$\Omega_1 = \left\{ x \in [0,1]^n \middle| x_1 \in [0, 0.5], x_k = 0.5, k = 2, \ldots, T+1 \right\}$$

$$\Omega_2 = \left\{ x \in [0,1]^n \middle| x_1 \in (0.5, 1], x_k \in \left[\frac{1 - e(x_1)}{2}, \frac{1 + e(x_1)}{2} \right], k = 2, \ldots, T+1 \right\}$$

with

$$e(x_1) = (0.5 + x_1)^{\frac{1}{0.5 - x_1}}$$

The distribution of $PF = F(PS)$ in $f_1 - f_2 - f_3$ is plotted in Fig.8.

- MaOP8
  - shape function:

$$\begin{array}{lcl} \alpha_1 & = & (-1) \times (2x_1 - 1)^3 + 1 \\ \alpha_{2k} & = & x_1 + \tau 2x_k + \tau|2x_k - 1|^{1-\sin(4x_1\pi)} \\ \alpha_{2k+1} & = & x_1 - \tau(2x_k - 2) + \tau|2x_k - 1|^{1-\sin(4x_1\pi)} \\ & & \hfill k = 2, \ldots, T \\ \alpha_M & = & \begin{cases} 1 - \alpha_1 & \text{if } M \text{ is even} \\ \alpha_{2T+1} & \text{otherwise} \end{cases} \end{array}$$

with the constant $\tau = \sqrt{2}/2$.
difficulty: local PF degeneracy

  - distance function: $g_i, i = 1, \ldots, M$, is the same as that in MaOP7.
  - scaling factor: $r_i = 1, i = 1, \ldots, M$.
  - the PS and the PF:

$$PS = \Omega_1 \cup \Omega_2$$

with

$$\Omega_1 = \left\{ x \in [0,1]^n \middle| x_1 \in [0, 0.25] \cup [0.5, 0.75], x_k = 0.5, k = 2, \ldots, T+1 \right\}$$

$$\Omega_2 = \left\{ x \in [0,1]^n \middle| x_1 \in (0.25, 0.5) \cup (0.75, 1], x_k \in \left[\frac{1 - e(x_1)}{2}, \frac{1 + e(x_1)}{2} \right], k = 2, \ldots, T+1 \right\}$$

with

$$e(x_1) = (1 - \sin(4\pi x_1))^{\frac{1}{\sin(4\pi x_1)}}$$

The distribution of $PF = F(PS)$ in $f_1 - f_2 - f_3$ is plotted in Fig.9.

- MaOP9
  - shape function:

$$\begin{array}{lcl} \alpha_1 & = & (-1) \times (2x_1 - 1)^3 + 1 \\ \alpha_{2k} & = & x_1 + \tau 2x_k + \tau|2(2x_k - \lfloor 2x_k \rfloor) - 1|^{0.5+x_1} \\ \alpha_{2k+1} & = & x_1 - \tau(2x_k - 2) + \tau|2(2x_k - \lfloor 2x_k \rfloor) - 1|^{0.5+x_1} \\ & & \hfill k = 2, \ldots, T \\ \alpha_M & = & \begin{cases} 1 - \alpha_1 & \text{if } M \text{ is even} \\ \alpha_{2T+1} & \text{otherwise} \end{cases} \end{array}$$

with the constant $\tau = \sqrt{2}/2$.
difficulty: local PF degeneracy and disconnected PF

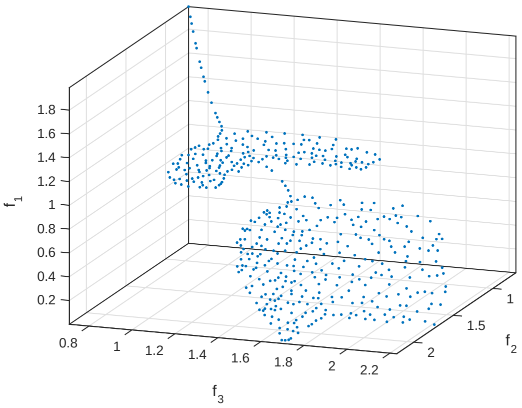

Figure 9: The PF of MaOP8 in $f_1 - f_2 - f_3$.

– distance function: $g_i, i = 1, \ldots, M$, is the same as that in MaOP7.

– scaling factor: $r_i = 1, i = 1, \ldots, M$.

– the PS and the PF:

$$PS = \Omega_1 \cup \Omega_2 \cup \Omega_3$$

where

$$\Omega_1 = \left\{ x \in [0,1]^n \middle| x_1 \in [0, 0.5], x_k \in \{0.25, 0.75\}, k = 2, \ldots, T+1 \right\}$$

$$\Omega_2 = \left\{ x \in [0,1]^n \middle| x_1 \in (0.5, 1], x_k \in \left[\frac{1 - e(x_1)}{4}, \frac{1 + e(x_1)}{4}\right], k = 2, \ldots, T+1 \right\}$$

$$\Omega_3 = \left\{ x \in [0,1]^n \middle| x_1 \in (0.5, 1], x_k \in \left[\frac{3 - e(x_1)}{4}, \frac{3 + e(x_1)}{4}\right], k = 2, \ldots, T+1 \right\}$$

with

$$e(x_1) = (0.5 + x_1)^{\frac{1}{0.5 - x_1}}$$

The distribution of $PF = F(PS)$ in $f_1 - f_2 - f_3$ is plotted in Fig.10.

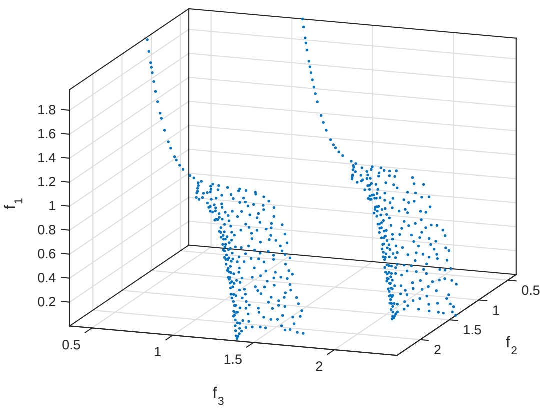

Figure 10: The PF of MaOP9 in $f_1 - f_2 - f_3$.

- MaOP10

– shape function:

$$\begin{array}{lcl} \alpha_1 & = & (-1) \times (2x_1 - 1)^3 + 1 \\ \alpha_{2k} & = & x_1 + \tau 2x_k + \tau |2(2x_k - \lfloor 2x_k \rfloor) - 1|^{p(x_1, x_k)} \\ \alpha_{2k+1} & = & x_1 - \tau(2x_k - 2) + \tau |2(2x_k - \lfloor 2x_k \rfloor) - 1|^{p(x_1, x_k)} \\ & & \qquad\qquad\qquad\qquad\qquad k = 2, \dots, T \\ \alpha_M & = & \left\{ \begin{array}{ll} 1 - \alpha_1 & \text{if } M \text{ is even} \\ \alpha_{2T+1} & \text{otherwise} \end{array} \right. \end{array}$$

with

$$p(x_1, x_k) = \left\{ \begin{array}{ll} 0.5 + x_1 & \text{if } \mathrm{mod}(\lfloor 2x_k \rfloor, 2) = 0 \\ 1.5 - x_1 & \text{otherwise} \end{array} \right.$$

with the constant $\tau = \sqrt{2}/2$.
difficulty: local PF degeneracy and disconnected PF

– distance function: $g_i, i = 1, \dots, M$, is the same as that in MaOP7.
– scaling factor: $r_i = 1, i = 1, \dots, M$.
– the PS and the PF:

$$PS = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$$

where

$$\Omega_1 = \left\{ x \in [0,1]^n \middle| x_1 \in [0, 0.5], x_k = 0.25, k = 2, \dots, T+1 \right\}$$

$$\Omega_2 = \left\{ x \in [0,1]^n \middle| x_1 \in [0.5, 1], x_k = 0.75, k = 2, \dots, T+1 \right\}$$

$$\Omega_3 = \left\{ x \in [0,1]^n \middle| x_1 \in [0, 0.5), x_k \in \left[\frac{1 - e_1(x_1)}{4}, \frac{1 + e_1(x_1)}{4} \right], k = 2, \dots, T+1 \right\}$$

$$\Omega_4 = \left\{ x \in [0,1]^n \middle| x_1 \in (0.5, 1], x_k \in \left[\frac{3 - e_2(x_1)}{4}, \frac{3 + e_2(x_1)}{4} \right], k = 2, \dots, T+1 \right\}$$

with

$$e_1(x_1) = (0.5 + x_1)^{\frac{1}{0.5 - x_1}} e_2(x_1) = (1.5 - x_1)^{\frac{1}{x_1 - 0.5}}$$

The distribution of $PF = F(PS)$ in $f_1 - f_2 - f_3$ is plotted in Fig.11.

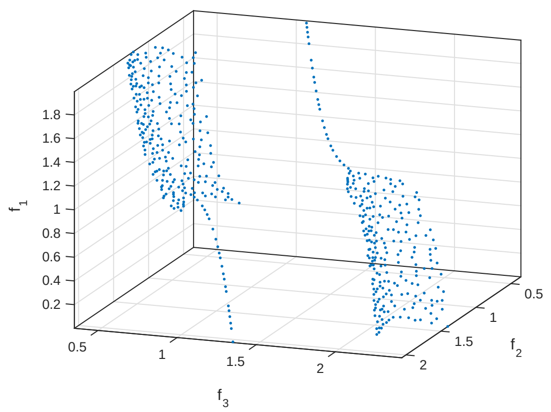

Figure 11: The PF of MaOP10 in $f_1 - f_2 - f_3$.

2 Recommended Experimental Settings

2.1 The settings of test instances and major parameters in algorithms

The number of objectives M in MaOP1-MaOP10 is set as $M = 3, 5, 8, 10$. The number of variables in all ten test problems can be N =20 (small scale) or N =50 (large scale). The population size in each algorithm is set to $pop = 100 \times M$. The total number of function evaluations is set to $pop \times 500$ for $N = 20$ and $pop \times 1000$ for $N = 50$.

2.2 Performance indicator

$$IGD(S_{pf}, S) = \frac{1}{|S_{pf}|} \sum_{y \in S_{pf}} dist(y, S)$$

where $dist(y, S)$ measures the distance between the objective vector y and the set S of objective vectors obtained by some algorithm with following formulation:

$$dist(y, S) = \min_{z \in S} ||y - z||_2$$

References

[1] H. Li and Q. Zhang. Multiobjective optimization problems with complicated Pareto sets, MOEA/D and NSGA-II. *IEEE Trans. on Evolutionary Computation*, 12(2):284–302, 2009.

[2] H. Li, Q. Zhang, and J.D. Deng. Multiobjective Test Problems with Complicated Pareto Fronts: Difficulties in Degeneracy. *Congress on Evolutionary Computation*, pages 2156–2163, 2014.

[3] Hai-Lin Liu, Lei Chen, Qingfu Zhang, and Kalyanmoy Deb. Adaptively allocating search effort in challenging many-objective optimization problems. *IEEE Trans. on Evolutionary Computation*, in press, 2017.

[4] H. Li, Q. Zhang, and J.D. Deng. Biased multiobjective optimization and decomposition algorithm. *IEEE Trans. Cybernetics*, 47(1):52–66, 2017.

[5] K. Deb, L. Thiele, M. Laumanns, and E. Zitzler. Scalable Test Problems for Evolutionary Multi-Objective Optimization. Technical Report Technical report 112, Computer Engineering and Networks Laboratory, ETH Zuric, 2001.

[6] S. Huband, P. Hingston, L. Barone, and R. L. While. A review of multiobjective test problems and a scalable test problem toolkit. *IEEE Trans. on Evolutionary Computation*, 10(5):477–506, 2006.