Estimating the Reliability of Systems Subject to Imperfect Repair

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This study of statistical inference for repairable systems focuses on the development of estimation procedures for the life distribution F of a new system based on data on system lifetimes between consecutive repairs. The Brown–Proschan imperfect-repair model postulates that at failure the system is repaired to a condition as good as new with probability p, and is otherwise repaired to the condition just prior to failure. In treating issues of statistical inference for this model, the article first points out the lack of identifiability of the pair (p, F) as an index of the distribution of interfailure times $T_1, T_2, \ldots$. It is then shown that data pairs (T_i, Z_i) $(i = 1, 2, \ldots)$ render the parameter pair (p, F) identifiable, where Z_i is a Bernoulli variable that records the mode of repair (perfect or imperfect) following the ith failure. Under the assumption that data of the form $\{(T_i, Z_i)\}$ are drawn via inverse sampling until the occurrence of the mth perfect repair, the problem of estimating the parameter pair (p, F) of the Brown–Proschan model is studied. It is demonstrated that the nonparametric maximum likelihood estimator of F exists only in special cases, but that a neighborhood maximum likelihood estimator $\hat{F}$ (using the language of Kiefer and Wolfowitz 1956) always exists and may be derived in closed form. Under mild assumptions, the strong uniform consistency of $\hat{F}$ is demonstrated, as is the weak convergence of an appropriately scaled version of $\hat{F}$ to a Gaussian process. It is noted that these results apply to other experimental designs, such as renewal testing, and that they can be extended to the age-dependent imperfect-repair model of Block, Borges, and Savits (1985).

1. INTRODUCTION

A frequently cited example (e.g., see Ascher and Feingold 1979; Hollander and Wolfe 1973, p. 248) of data on systems under repair is the set of successive interfailure times of air conditioners on a fleet of Boeing aircraft. In the article in which he introduced the air conditioner data, Proschan (1963) tested the null hypothesis of independence against a trend alternative. Such an alternative might be motivated, for example, by the expectation that a system, degenerating as a result of successive repairs, would tend to have decreasing interfailure times. At a 5% level of significance, the Mann test for trend fails to reject the null hypothesis of independence in each of the 13 air-conditioning systems tested. The interfailure times of the air conditioner on a particular aircraft are reproduced in Table 1. For this aircraft, the Mann test for trend yields a p value of .19.

Although the Mann test for independence has good power against trend alternatives, it is well known that its power is poor against other formulations of data dependence. Among such alternatives are some that appear to be realistic models for repairable systems. A closer look at Table 1 reveals a cyclic pattern in the interfailure times. The pattern of times above (+) and below (−) the median 41.5 of these data is as follows: + + + + − − − − + + + + + − − − − + + + − − − −. The small number of runs in these data is surprising. With only 6 runs in 24 observations, the runs test has a p value of .0028, providing strong evidence against the iid assumption. We return to this data set after describing a plausible alternative to the iid hypothesis and developing estimation procedures in this alternative setting.

The iid framework assumes that a repaired system functions as well as a new system, that is, that all repairs are "perfect." Alternatively, one might reasonably hypothesize that a sequence of superficial repairs is interspersed with periodic overhauls, creating a cyclic pattern such as that apparent in Table 1. Realistic models would allow the quality of repair to vary, possibly depending on the condition of the system at failure. Using age as an indicator of the condition of the system, these features can be incorporated into an imperfect-repair model as follows. Let $T_1, T_2, \ldots$ be a sequence of interfailure times. At the ith failure the system is repaired to age A_i, with $A_i = 0$ corresponding to a perfect repair and $A_i > 0$ to a repair that is less than perfect. Further, the distribution of A_i may depend on the age $A_{i-1} + T_i$ of the system at failure. In its most general form, this imperfect-repair model does not have enough structure to shed light on the stochastic properties of systems under repair, much less allow for statistical inference from repair data.

Brown and Proschan (1983) introduced a form of this imperfect-repair model and derived many of its interesting stochastic properties. Their model also permits a reasonably comprehensive statistical treatment. The Brown–Proschan (BP) model postulates that at failure the system is repaired to a condition as good as new ($A_i = 0$) with probability p, and is otherwise repaired to its condition just prior to failure ($A_i = A_{i-1} + T_i$). The parameter of this model is the pair (p, F), where F is the distribution on $(0, \infty)$ governing the age at failure of a new system. The survival function corresponding to F is denoted by S;

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Table 1. Intervals Between Failures: Plane 7914

i	T_i	i	T_i	i	T_i	i	T_i
1	50	7	3	13	46	19	210
2	44	8	15	14	5	20	97
3	102	9	197	15	5	21	30
4	72	10	188	16	36	22	23
5	22	11	79	17	22	23	13
6	39	12	88	18	139	24	14

that is, $S(t) = \Pr(T > t) = 1 - F(t)$. Throughout this article we tacitly assume that the distribution F is continuous. Conditional distributions based on F govern the residual lifetime of an item of any given age. Implicit in the BP model is the assumption that the choice of mode of repair following any given failure depends solely on external factors, such as the availability of a replacement, and not on the condition (i.e., age) of the system at failure. The model also assumes that these external factors are stable over time; the probability of perfect repair is thus taken to be constant. An extension of the BP model, in which the probability of perfect repair depends on the age at failure, was studied by Block, Borges, and Savits (1985), and is discussed in Section 4.

A brief review of the properties of the BP model is in order. Renewals, or perfect repairs, restore a system to its condition when new. With this in mind, the repair record of systems governed by the BP model can be thought of as a sequence of independent repair epochs (i.e., periods between two successive perfect repairs), each with the same stochastic properties. To study the properties of the BP model, we restrict attention to the first repair epoch. This is reflected in the following propositions, both employed in the sequel.

Proposition 1.1 (*Brown and Proschan 1983*). Let Y_1 be the system age at the first perfect repair under the BP model with parameter (p, F). The survival function S_y of Y_1 is given by

$$S_y(t) = (S(t))^p \quad \forall\, t > 0. \tag{1.1}$$

A second result sheds light on the conditional behavior of imperfect-repair times, given the occurrence or nonoccurrence of a complete repair.

Proposition 1.2 [*Savits (1985)*]. Let $N(t)$ be the number of failures under the BP model in the time interval $[0, t]$, and let $R(t) = -\ln S(t)$ and $R_y(t) = -\ln S_y(t)$. Then,

$$E[N(t) \mid Y_1 > t] = R(t) - R_y(t)$$

$$E[N(t) \mid Y_1 = t] = R(t) - R_y(t) + 1. \tag{1.2}$$

We now turn to the treatment of statistical inference for the BP model.

2. ESTIMATING THE PARAMETER PAIR (p, F)

We first examine the identifiability of the BP model, seeking to answer the following question: Do the distributions of interfailure times have a one-to-one correspondence with the collection of possible parameter pairs (p, F)? The subject of identifiability has been studied by several authors, notably Teicher (1961) for mixtures of distributions and Sclove and Van Ryzin (1969) for convolution models. A good example of the problems caused by lack of identifiability occurs in the context of competing-risks methodology (see Tsiatis 1975). In the BP model, the immediate answer to our question is negative: (p, F) is not an identifiable parameter of the distribution of $T_1, T_2, \ldots$. To see this, denote the distribution of $T_1, T_2, \ldots$, given (p, F), by $\mathfrak{F}_{(p,F)}(\mathbf{T})$. Assume, for distinct (p_1, F_1) and (p_2, F_2), that

$$\mathfrak{F}_{(p_1, F_1)}(\mathbf{T}) = \mathfrak{F}_{(p_2, F_2)}(\mathbf{T}). \tag{2.1}$$

Since (2.1) implies that the distribution of T_1 is the same under both parameter pairs, we infer that $F_1 = F_2$ a.e.; however, (2.1) does not imply that $p_1 = p_2$. Indeed, we may only infer either that $p_1 = p_2$ or that $p_1 \neq p_2$ but $S(t) = S(s + t)/S(s)\ \forall\, s, t \geq 0$. This derivation confirms that the memoryless property of the exponential distribution is at the core of the lack of identifiability of the Brown–Proschan model. If F is exponential, each interfailure time T has the same exponential distribution, regardless of the value of p.

This development shows that the pair (p, F) is not an identifiable parameter of the distribution of $\mathbf{T}$. The main consequence of this is our inability to estimate p from a sequence of interfailure times. That F can be estimated from $\mathbf{T}$ is clear; the indicator function of the event $\{T_1 \leq t\}$ is an unbiased estimator of $F(t)$. Nevertheless, the subsequent interfailure times $T_2, T_3, \ldots$ are difficult to interpret without information on the type of repair made following each failure. Whether F can be estimated consistently from the sequence $T_1, T_2, T_3, \ldots$ is at this point an open question. Still, the ambiguity inherent in interfailure-time data when information on the mode of repair is missing suggests that estimators of F based on $T_1, T_2, T_3, \ldots$ alone will be woefully imprecise. We therefore examine what can be done when additional information is available.

If we are to augment the data $T_1, T_2, \ldots$ to ensure that the parameter (p, F) is identifiable, the most natural solution is to record the mode of repair, perfect or imperfect, in addition to the interfailure times. The augmented data are represented by the sequence of pairs (T_i, Z_i) $(i = 1, 2, \ldots)$, where Z_i is the Bernoulli variable defined as

$$\begin{aligned} Z_i &= 1 \quad \text{if the } i\text{th repair is perfect} \\ &= 0 \quad \text{if the } i\text{th repair is imperfect.} \end{aligned}$$

For each i, $P(Z_i = 1) = p$. Augmenting $\mathbf{T}$ by $\mathbf{Z}$ renders the problem of inference well defined, because the distribution of the modes of repair $Z_1, Z_2, \ldots$ is uniquely determined by p and the first failure time T_1 serves to identify F. It is important to note that in situations where the possible modes of repair are restricted to a finite set, it is realistic to expect that both $\mathbf{T}$ and $\mathbf{Z}$ would be available from standard maintenance records on a repairable system.

Throughout this section and the next, we assume that data are collected under the inverse-sampling scheme in which the system is observed until the mth perfect repair. We also assume that $p > 0$, F is continuous, and $F(0) = 0$, a restriction that precludes the possibility of instantaneous failure following repair.

Let n be the total number of repairs in the available sample, where m are perfect repairs and the remaining $n - m$ are imperfect repairs. If F is absolutely continuous with density f, the likelihood associated with the observed interfailure times $T_i = t_i$ and corresponding modes of repair $Z_i = z_i$ $(1 \le i \le n)$, is given by

$$L(p, F) = f(t_1)p^{z_1}(1-p)^{1-z_1}\frac{f(t_2 + a_1)}{S(a_1)}p^{z_2}(1-p)^{1-z_2} \times \cdots \times \frac{f(t_n + a_{n-1})}{S(a_{n-1})}p^{z_n}(1-p)^{1-z_n},$$

where a_i represents the age of the system following the ith repair, and $S = 1 - F$. We note that the sequence $\{a_i\}$ is completely determined by the (t_i, z_i) pairs. Rearranging the product and rewriting $L(p, F)$ as a function of the age x_i just prior to the ith failure, we obtain

$$L(p, F) = p^m(1-p)^{n-m}f(x_1)\frac{f(x_2)}{S(x_1)^{1-z_1}} \times \cdots \times \frac{f(x_n)}{S(x_{n-1})^{1-z_{n-1}}}. \quad (2.2)$$

It is apparent from (2.2) that the maximum likelihood estimator (MLE) $\hat{p}$ of p is m/n, the proportion of repairs that are perfect. It is convenient to rewrite the product (2.2) as a function of the ordered ages $x_{(i)}$. Here, $z_{(i)}$ denotes the mode of repair following the failure at age $x_{(i)}$. Thus $z_{(1)}, z_{(2)}, \ldots, z_{(n)}$ are not ordered, but are the so-called "induced order statistics" generated by ordering the x_i's. In the new notation,

$$L(p, F) = p^m(1-p)^{n-m}\prod_{i=1}^{n}\frac{f(x_{(i)})}{S(x_{(i-1)})^{1-z_{(i-1)}}}, \quad (2.3)$$

where $x_{(0)} = 0$ and $z_{(0)} = 1$. The variable $z_{(n)}$ does not appear in (2.3) because $z_{(n)} = 1$. This is because the largest age must occur immediately before a perfect repair, or when the experiment is terminated.

Henceforth, we restrict attention to the nonparametric framework. Finding the nonparametric MLE of F (see Kiefer and Wolfowitz 1956) is equivalent to maximizing

$$l(F) = \prod_{i=1}^{n}\frac{(F(x_{(i)}) - F(x_{(i)}^-))}{S(x_{(i-1)})^{1-z_{(i-1)}}} \quad (2.4)$$

over $F \in \mathfrak{F}_n$, the class of distribution functions with positive probability on the ages $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$. A simple example illustrates that the maximum of $l(F)$ in (2.4) is not always attainable.

Example. A system is observed until the first perfect repair, resulting in $T_1 = 1$, $Z_1 = 0$, $T_2 = 1$, and $Z_2 = 1$. The first repair is imperfect, so the two ages at failure are $X_1 = 1$ and $X_2 = 2$. Thus

$$l(F) = [F(1) - F(1^-)]\frac{[F(2) - F(2^-)]}{S(1)}. \quad (2.5)$$

To maximize (2.5), F must place its mass on the values 1 and 2; thus we write

$$l(F) = [F(1) - F(1^-)]\frac{[F(2) - F(2^-)]}{[F(2) - F(2^-)]}. \quad (2.6)$$

Letting $F(1) - F(1^-) = 1 - \varepsilon$ and $F(2) - F(2^-) = \varepsilon$, we see that the supremum of $l(F)$ in (2.6) is 1. Since $l(F)$ is undefined for $F(1) - F(1^-) = 1$, the nonparametric MLE does not exist.

In general, unless $z_{(n-1)} = 1$, the supremum of $l(F)$ in (2.4) is not attainable. Insight into this problem can be gained by considering the likelihood as a product of conditional probabilities. Let $\phi_i = S(x_{(i)})/S(x_{(i-1)})$; ϕ_i is the conditional probability of surviving beyond age $x_{(i)}$, given survival beyond age $x_{(i-1)}$. Then, among candidates for maximizing $l(F)$, that is, for distributions placing mass only at $x_{(1)}, \ldots, x_{(n)}$, we write

$$\frac{F(x_{(i)}) - F(x_{(i)}^-)}{S(x_{(i-1)})} = 1 - \phi_i$$

and

$$F(x_{(i)}) - F(x_{(i)}^-) = (1 - \phi_i)\prod_{j=1}^{i-1}\phi_j, \qquad 1 \le i \le n,$$

where the empty product is defined as 1. Thus maximizing $l(F)$ with respect to F is equivalent to maximizing $l(\boldsymbol{\phi})$ with respect to $\boldsymbol{\phi}$, where

$$l(\boldsymbol{\phi}) = \prod_{i=1}^{n}(1 - \phi_i)\,\phi_i^{k_i} \quad (2.7)$$

and $k_i = \sum_{j=i}^{n-1} z_{(j)}$ is the number of failure ages greater than $x_{(i)}$ for which the system was repaired perfectly (with the empty sum defined as 0). It is clear that $l(\boldsymbol{\phi})$ is maximized by $\hat{\boldsymbol{\phi}}$:

$$\hat{\phi}_i = \frac{k_i}{(k_i + 1)}, \qquad 1 \le i \le n - 1, \quad (2.8)$$

with $\hat{\phi}_n = 0$. If $z_{(n-1)} = 1$, the nonparametric MLE of the survival function S is thus given by

$$\begin{aligned}\hat{S}(t) &= 1, & t < x_{(1)} \\ &= \prod_{j=1}^{i}\hat{\phi}_j, & x_{(i)} \le t < x_{(i+1)}\ (i = 1, \ldots, n-1) \\ &= 0, & t \ge x_{(n)}.\end{aligned} \quad (2.9)$$

If $z_{(n-1)} = 0$, $l(\boldsymbol{\phi})$ is still maximized by $\boldsymbol{\phi}$ given in (2.8), but now $\hat{\phi}_i = 0$ for $J + 1 \le i < n$, where $J = \max\{i : z_{(i)} = 1, 1 \le i \le n - 1\}$. This implies that the corresponding estimator $\hat{S}$, given in (2.9), is equal to 0 for $x \ge x_{(J+1)}$, and that $l(\hat{F})$ in (2.4) is undefined. Furthermore, there is no member of $\mathfrak{F}_n$ maximizing $l(F)$. Thus the nonparametric MLE does not exist when $z_{(n-1)} = 0$.

Anticipating the occurrence of such anomalies, Kiefer and Wolfowitz (1956) introduced the more general concept of neighborhood MLE. In an appropriate topology, let $\Pi_\varepsilon \subset \mathfrak{F}_n$ be an ε neighborhood of $\hat{F}$. $\hat{F}$ is said to be a neighborhood MLE if for any $\varepsilon > 0$, however small, $\sup_{F \in \Pi_\varepsilon} l(F) = \sup_{F \in \mathfrak{F}_n} l(F)$. Informally, an estimator $\hat{F}$ of F is a neighborhood MLE if every neighborhood of $\hat{F}$ is a good neighborhood, in the sense that every neighborhood of F contains distributions whose likelihood is arbitrarily close to the supremum of the likelihood. It is easy to show that in the topology induced by the sup norm, the estimator $\hat{F} = 1 - \hat{S}$, where $\hat{S}$ is specified in (2.9), is a neighborhood MLE of the distribution F.

We return to the air conditioner interfailure times of Table 1. Since actual observed Z_i's are unavailable, we have assigned arbitrary (but seemingly reasonable) values to the Z_i's. Table 2 reproduces each interfailure time T_i from Table 1 along with an imputed mode of repair indicator Z_i, and each age X_i at which the system fails. For comparison, both estimated survival curves, the neighborhood MLE $\hat{S}$ and $\overline{G}_{\hat{\lambda}}(x) = e^{-x/62.14}$ [the MLE of S computed from the interfailure times alone, assuming that they are iid exponential (λ)], are graphed in Figure 1. By taking into account the imperfect repairs, the nonparametric estimator $\hat{S}$ gives a much different view of the chances of survival for a new system. In computing $\hat{S}$, small interfailure times (e.g., $T_7 = 1$, $T_{14} = 5$, $T_{15} = 5$, and $T_{23} = 13$) provide information about the tail behavior of S through the corresponding ages at failure ($X_7 = 238$, $X_{14} = 603$, $X_{15} = 608$, and $X_{23} = 512$). This results in much more optimistic estimates of the chances of survival for a new system than those obtained when the iid assumption is made.

It is easy to construct alternative nonparametric estimators for F based on the (T_i, Z_i) pairs. The most obvious estimator is the empirical distribution of the collection $\{T_i \mid Z_{i-1} = 1, 1 \le i \le n\}$, that is, the set of m interfailure times including the first failure time and the interfailure times following each perfect repair. These interfailure times are iid with distribution F, so the estimator has all of the properties of the usual empirical distribution in the iid setting. One could also restrict attention to the empirical distribution $\hat{F}_y$ of $Y_1, Y_2, \ldots, Y_m$, the times between successive perfect repairs. The corresponding estimator of the survival function, $\hat{S}_y(t)$, converges to $[S(t)]^p$ as $m \to \infty$. Thus, $\hat{S}_y^{n/m}$ is another easily computable consistent estimator of S. Neither of these estimators fully use the information contained in the (T_i, Z_i) pairs, so one might expect that they are less efficient than the neighborhood MLE, $\hat{S}$. In Section 4, the large-sample properties of these three estimators are identified and compared.

Table 2. Intervals Between Failures Augmented With Mode of Repair: Plane 7914

i	Z_i	T_i	X_i	i	Z_i	T_i	X_i
1	0	50	50	13	0	46	598
2	1	44	94	14	0	5	603
3	0	102	102	15	0	5	608
4	0	72	174	16	0	36	644
5	0	22	196	17	1	23	666
6	0	39	235	18	0	139	139
7	0	3	238	19	0	210	349
8	1	15	253	20	0	97	446
9	0	197	197	21	0	30	476
10	0	188	385	22	0	23	499
11	0	79	464	23	0	13	512
12	0	88	552	24	1	14	526

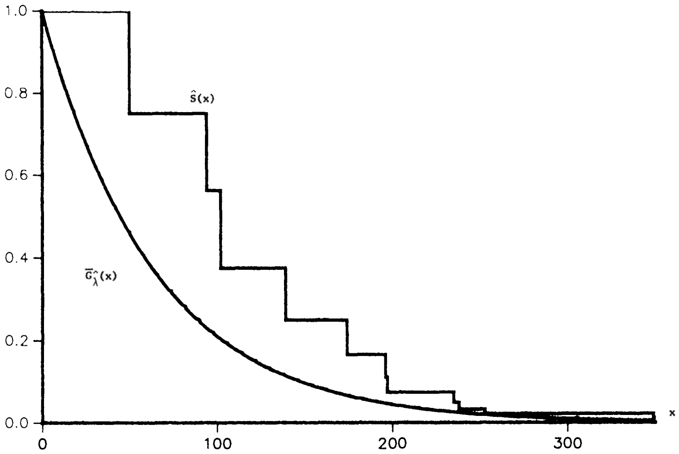

Figure 1. The Neighborhood Nonparametric MLE $\hat{S}$ and the Exponential MLE $\overline{G}_{\hat{\lambda}}$.

We examine one more intuitive estimator, $\hat{F}^*$, which has a convenient integral representation that plays a central role in studying the asymptotic properties of $\hat{F}$. To motivate the estimator $\hat{F}^*$, we establish an integral representation of the theoretical hazard function $R(t)$. This representation draws our attention to the sample analogue of R, which (as we shall see) plays an important role in further developments. The estimator $\hat{R}^*$ of R provides the desired estimator of F through the relationship $\hat{F}^* = 1 - e^{-\hat{R}^*}$. We have assumed that a system is observed until the mth perfect repair is made. Let $N_i(t)$ be the number of failures in the interval $[0, t]$ for the ith repair epoch, with $N_i(t) \equiv N_i(Y_i)$ for $t \ge Y_i$, and define $\tilde{N}(t)$ and $\hat{S}_y(t)$ by

$$\tilde{N}(t) = \frac{1}{m} \sum_{i=1}^{m} N_i(t) \tag{2.10}$$

and

$$\hat{S}_y(t) = \frac{1}{m} \sum_{i=1}^{m} I_{(t,\infty)}(Y_i), \tag{2.11}$$

where $I_A(x)$ is the indicator function of the set A. We now state and prove a useful representation of the hazard function R.

Lemma 2.1. Let F be a continuous lifetime distribution on $(0, \infty)$ governing the Brown–Proschan imperfect-repair model with parameter (p, F). Let $\tilde{N}$ and $\hat{S}_y$ be as in (2.10) and (2.11). Let $T < \infty$ satisfy $F(T) < 1$. Then, $\forall\, x \in (0, T)$,

$$R(x) = \int_0^x \{E[\hat{S}_y(u)]\}^{-1}\, dE\tilde{N}(u). \tag{2.12}$$

Proof. Fix $x \in (0, T]$. Note that $E[\tilde{N}(x)] = E[N_1(x)]$. We therefore have

$$E[\tilde{N}(x)] = E[N_1(x) \mid Y_1 > x] S_y(x) + \int_0^x E[N_1(u) \mid Y_1 = u]\, dF_y(u). \tag{2.13}$$

Using Proposition 1.2 of Section 1, we rewrite (2.13) as

$$E[\tilde{N}(x)] = [R(x) - R_y(x)]S_y(x) + \int_0^x [R(u) - R_y(u)]\, dF_y(u) + F_y(x).$$

Integrating by parts, we obtain

$$E[\tilde{N}(x)] = \int_0^x S_y(u)\, dR(u) - \int_0^x S_y(u)\, dR_y(u) + F_y(x) = \int_0^x S_y(u)\, dR(u). \tag{2.14}$$

Since $S_y = S^p$ is positive and continuous on $[0, x]$ by assumption, and $E[\tilde{N}(\cdot)]$ is increasing, the Reimann–Stieltjes integral $\int_0^x [1/S_y(u)]\, dE[\tilde{N}(u)]$ exists. Let $\varepsilon > 0$ be arbitrary, and let $0 = u_0 < u_1 < \cdots < u_n = x$ be a partition of $[0, x]$. For $\max|u_i - u_{i-1}|$ sufficiently small, we can guarantee [with the help of (2.14)] that

$$\left| \int_0^x \frac{1}{S_y(u)}\, dE[\tilde{N}(u)] - \sum_{i=0}^{n} \frac{1}{S_y(\xi_i)} \times \int_{u_{i-1}}^{u_i} S_y(u)\, dR(u) \right| < \frac{\varepsilon}{2},$$

where $\xi_i \in [u_{i-1}, u_i]\ \forall\, i$. Moreover, because of the assumed uniform continuity of S_y on $[0, x]$ we also have, for $\max_i |u_i - u_{i-1}|$ sufficiently small, that

$$\left| \sum_{i=0}^{n} \frac{1}{S_y(\xi_i)} \int_{u_{i-1}}^{u_i} S_y(u)\, dR(u) - \sum_{i=0}^{n} \frac{1}{S_y(\xi_i)} \times S_y(\xi_i)[R(u_i) - R(u_{i-1})] \right| < \frac{\varepsilon}{2}.$$

Noting that $R(0) = -\ln S(0) = 0$ and $S_y(u) = E[\hat{S}_y(u)]$, we combine the aforementioned two inequalities to obtain

$$\left| \int_0^x \{E[\hat{S}_y(u)]\}^{-1}\, dE[\tilde{N}(u)] - R(x) \right| < \varepsilon.$$

Since ε is arbitrary, the desired result follows.

The expression for $R(\cdot)$ in (2.12) suggests an ad hoc estimator of R. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(m)}$ be the order statistics corresponding to the times $Y_1, Y_2, \ldots, Y_m$ between perfect repairs. We define the estimator $\hat{R}^*$ as the sample analogue of R; that is, for $t \in (Y_{(j)}, Y_{(j+1)})$,

$$\begin{aligned} \hat{R}^*(t) &= \int_0^t [\hat{S}_y(x)]^{-1}\, d\tilde{N}(x) \\ &= \frac{1}{m} \sum_{i=1}^{m} N_i(Y_{(1)}) + \frac{1}{m-1} \sum_{i=1}^{m} [N_i(Y_{(2)}) - N_i(Y_{(1)})] \\ &\quad + \cdots + \frac{1}{m-j} \sum_{i=1}^{m} [N_i(t) - N_i(Y_{(j)})]. \end{aligned} \tag{2.15}$$

We note that, for each $k \in \{1, 2, \ldots, m - 1\}$, at the most $(m - k)$ of the differences $\{N_i(Y_{(k+1)}) - N_i(Y_{(k)})\}$ can be nonzero. Thus each term in Expression (2.15) for $\hat{R}^*$ may be viewed as an estimate of the average number of failures between successive perfect repairs. Moreover, since S is to be estimated by $\hat{S}^* = e^{-\hat{R}^*}$, we may think of each term in (2.15) as an estimate of a conditional hazard function, given the perfect repair times. That this approach to estimating R and F is efficacious is borne out in the sequel. We show that $\hat{R}^*$ is asymptotically equivalent to $\hat{R} = -\ln \hat{S}$, and that $\hat{R}^*$ (and hence $\hat{R}$) is a consistent estimator of R. Combining (2.8) and (2.9), we see that

$$\hat{R}(t) = -\ln \prod_{\mathcal{A}(t)} \frac{\sum_{k=i}^{n-1} Z_{(k)}}{\sum_{k=i}^{n-1} Z_{(k)} + 1} = \sum_{\mathcal{A}(t)} \ln \frac{\sum_{k=i}^{n-1} Z_{(k)} + 1}{\sum_{k=i}^{n-1} Z_{(k)}}, \tag{2.16}$$

where $\mathcal{A}(t) = \{i : X_{(i)} \le t\}$. To rewrite $\hat{R}$ in terms comparable to $\hat{R}^*$, note that the sum in (2.16) can be subdivided into partial sums, each with a constant summand. For $Y_{(j)} < t \le Y_{(j+1)}$,

$$\hat{R}(t) = \sum_{B_1} \ln \frac{\sum_{k=i}^{n-1} Z_{(k)} + 1}{\sum_{k=i}^{n-1} Z_{(k)}} + \sum_{B_2} \ln \frac{\sum_{k=i}^{n-1} Z_{(k)} + 1}{\sum_{k=i}^{n-1} Z_{(k)}} + \cdots + \sum_{B_t} \ln \frac{\sum_{k=i}^{n-1} Z_{(k)} + 1}{\sum_{k=i}^{n-1} Z_{(k)}},$$

where $B_k = \{i : Y_{(k-1)} < X_{(i)} \le Y_{(k)}\}$ $(k = 1, \ldots, j)$ and $B_t = \{i : Y_{(j)} < X_{(i)} \le t\}$. Thus

$$\begin{aligned} \hat{R}(t) &= m\, \tilde{N}(Y_{(1)}) \ln\left(\frac{m}{m-1}\right) + m(\tilde{N}(Y_{(2)}) - \tilde{N}(Y_{(1)})) \ln\left(\frac{m-1}{m-2}\right) \\ &\quad + \cdots + m(\tilde{N}(t) - \tilde{N}(Y_{(j)})) \ln\left(\frac{m-j}{m-j-1}\right). \end{aligned} \tag{2.17}$$

The expressions for $\hat{R}$ in (2.17) and $\hat{R}^*$ in (2.15) are very similar; it is not surprising that the two functions are asymptotically equivalent. Actually, a stronger result (needed in our proof of weak convergence) is established in the following.

Lemma 2.2. Let $T < \infty$ satisfy $F(T) < 1$. Then, as $m \to \infty$, $\sup_{0 \le t \le T} m^{1/2} |\hat{R}(t) - \hat{R}^*(t)| \to 0$ w. p. 1.

Proof. Without loss of generality, we restrict attention to the event $\{Y_{(m-1)} > T\}$, ensuring that both $\hat{R}$ and $\hat{R}^*$ are well defined in the interval $[0, T]$. Subtracting (2.15) from

(2.17) yields

$$\begin{aligned}
\hat{R}(t) - \hat{R}^*(t) &= \sum_{i=1}^{j} m(\tilde{N}(Y_{(i)}) - \tilde{N}(Y_{(i-1)})) \\
&\quad \times \left[\ln\left(\frac{m-i+1}{m-i}\right) - \frac{1}{m-i+1}\right] \\
&\quad + m(\tilde{N}(t) - \tilde{N}(Y_{(j)})) \left[\ln\left(\frac{m-j}{m-j-1}\right) - \frac{1}{m-j}\right]
\end{aligned}$$

for $Y_{(j)} < t \le Y_{(j+1)}$, where $Y_{(0)} = 0$. Applying the inequality

$$\frac{1}{m-i+1} < \ln\left(\frac{m-i+1}{m-i}\right) < \frac{1}{m-i+1} + \frac{1}{(m-i)(m-i+1)}$$

to each term in the sum, we obtain

$$\begin{aligned}
0 < \hat{R}(t) - \hat{R}^*(t) &< \sum_{i=1}^{j} m(\tilde{N}(Y_{(i)}) - \tilde{N}(Y_{(i-1)})) \cdot \frac{1}{(m-i)(m-i+1)} \\
&\quad + m(\tilde{N}(t) - \tilde{N}(Y_{(j)})) \cdot \frac{1}{(m-j-1)(m-j)}. \qquad (2.18)
\end{aligned}$$

Since $(m-j-1)^2 < (m-i)(m-i+1)$ for each $i \in \{1, \ldots, j+1\}$, we bound the sum in (2.18) as follows:

$$\begin{aligned}
|\hat{R}(t) - \hat{R}^*(t)| &< \frac{m}{(m-j-1)^2} \tilde{N}(t) \\
&= \frac{1}{m}\left(\frac{m-j}{m-j-1}\right)^2 (\hat{S}_y(t))^{-2}\, \tilde{N}(t).
\end{aligned}$$

Because $(\hat{S}_y(t))^{-2}\, \tilde{N}(t)$ is a nondecreasing function of t, it follows that

$$\sup_{0 \le t \le T} |\hat{R}(t) - \hat{R}^*(t)| < \frac{1}{m}\left(\frac{m}{m-J-1}\right)^2 (\hat{S}_y(T))^{-2}\, \tilde{N}(T), \qquad (2.19)$$

where $Y_{(J)} < T \le Y_{(J+1)}$. By the strong law of large numbers, we have (as $m \to \infty$)

$$\sup_{0 \le t \le T} m^{1/2} |\hat{R}(t) - \hat{R}^*(t)| \to 0 \quad \text{w. p. } 1,$$

since the last three terms on the right side of (2.19) converge almost surely to finite limits.

3. LARGE SAMPLE PROPERTIES OF $\hat{F}$

We first provide a straightforward but detailed argument showing strong uniform convergence of $\hat{F}$ to F.

Theorem 3.1. Let F be a continuous lifetime distribution on $(0, \infty)$ defining the Brown–Proschan imperfect-repair model, and let $\hat{F}$ be the neighborhood MLE of F given in (2.9), based on inverse sampling until the mth perfect repair. Then, $\hat{F}$ converges uniformly to F w. p. 1 as $m \to \infty$.

Proof. Let $T < \infty$ satisfy $F(T) < 1$. We first demonstrate the convergence of $\hat{R}$ to R in the interval $[0, T]$. Again, without loss of generality, we restrict attention to the event $\{Y_{(m-1)} > T\}$. Applying the triangle inequality,

$$\begin{aligned}
|\hat{R}(t) - R(t)| &\le |\hat{R}(t) - \hat{R}^*(t)| \\
&\quad + \left|\hat{R}^*(t) - \int_0^t S_y^{-1}(x)\, d\tilde{N}(x)\right| \\
&\quad + \left|\int_0^t S_y^{-1}(x)\, d\tilde{N}(x) - R(t)\right|, \qquad 0 \le t \le T. \quad (3.1)
\end{aligned}$$

Strong convergence of the first term to 0 on $[0, T]$ is a direct consequence of Lemma 2.2. Replacing $\hat{R}^*$ with its integral representation, the second term becomes

$$\begin{aligned}
\left|\hat{R}^*(t) - \int_0^t S_y^{-1}(x)\, d\tilde{N}(x)\right| &= \left|\int_0^t \hat{S}_y^{-1}(x)\, d\tilde{N}(x) - \int_0^t S_y^{-1}(x)\, d\tilde{N}(x)\right| \\
&\le \sup_{0 \le t \le T} |\hat{S}_y^{-1}(t) - S_y^{-1}(t)| \tilde{N}(T), \qquad 0 \le t \le T. \quad (3.2)
\end{aligned}$$

The Glivenko–Cantelli lemma, combined with uniform continuity of $g(x) = x^{-1}$ in the interval $[S_y(T), 1]$, implies strong uniform convergence of $\hat{S}_y^{-1}$ to S_y^{-1} on $[0, T]$. This fact, along with strong convergence of $\tilde{N}(T)$ to $E[\tilde{N}(T)]$ (by the strong law of large numbers), forces the second term in (3.1) to converge to 0 w. p. 1.

We now demonstrate that the third term on the right side of (3.1) converges strongly to 0. We note that the desired result is analogous to a well-known convergence theorem for Riemann–Stieltjes integrals, and we approach its proof in the usual way. For each integer $k \ge 1$, let $t_{0,k} \le t_{1,k} \le \cdots \le t_{k,k}$ be a partition of the interval $(0, t)$, and assume that $\max_{1 \le i \le k} |t_{i,k} - t_{i-1,k}| \to 0$ as $k \to \infty$. Using the uniform continuity of S_y^{-1} in the interval $[0, t]$ and the finiteness of $E\tilde{N}(t)$, it can be shown via the Moore–Osgood Theorem (see Olmsted 1959, p. 313) that w. p. 1,

$$\lim_{m \to \infty} \lim_{k \to \infty} g(k, m) = \lim_{k \to \infty} \lim_{m \to \infty} g(k, m), \qquad (3.3)$$

where

$$g(k, m) = \sum_{i=1}^{k} S_y^{-1}(t_{i,k})\, [\tilde{N}(t_{i,k}) - \tilde{N}(t_{i-1,k})].$$

Invoking Lemma 2.1, (3.3) may be rewritten as

$$\lim_{m \to \infty} \int_0^t S_y^{-1}(x)\, d\tilde{N}(x) = R(t) \quad \text{w. p. } 1, \qquad (3.4)$$

establishing the strong convergence of $\hat{R}$ to R on $[0, T]$. By virtue of the relationship $\hat{F}(t) = 1 - e^{-\hat{R}(t)}$, we also have that $\hat{F} \to F$ w. p. 1 on $[0, T]$. Moreover, it follows from a lemma by Chung (1974, p. 133) that this convergence is uniform on $[0, T]$. Using the continuity of F, one may choose T sufficiently large to ensure that both $\sup_{t \le T} |\hat{F}(t) - F(t)|$ and $\sup_{t \ge T} |\hat{F}(t) - F(t)|$ are bounded by an arbitrarily small $\varepsilon > 0$. Thus the uniform strong convergence of $\hat{F}$ to F obtains on the entire real line.

We now turn to the development of the large-sample distribution theory for $\hat{F}$. We first establish the weak convergence of the normalized hazard-function process $\{m^{1/2}(\hat{\hat{R}}(t) - R(t)) : 0 \le t \le T\}$, where $T < \infty$ is such that $F(T) < 1$. Let $D[0, T]$ be the space of right-continuous functions on $[0, T]$ with left-hand limits, and let $D[0, T]$ be equipped with the Skorohod topology (see Billingsley 1968, chap. 3). Using the result from Lemma 2.2 and the fact that uniform convergence in $[0, T]$ implies convergence in the Skorohod topology, we see that the processes $\{m^{1/2}(\hat{R}^*(t) - R(t)) : 0 \le t \le T\}$ and $\{m^{1/2}(\hat{\hat{R}}(t) - R(t)) : 0 \le t \le T\}$ have the same limiting distribution, which is identified in the following theorem.

Theorem 3.2. Let F be as in Theorem 3.1, and let $T < \infty$ satisfy $F(T) < 1$. Then, the process $m^{1/2}(\hat{R}^* - R)$ converges weakly to a zero-mean Gaussian process Z, with convariance structure given by

$$\operatorname{cov}(Z(s), Z(t)) = \int_0^{s \wedge t} S_y^{-1}(x)\, dR\,(x), \qquad 0 \le s, t \le T. \tag{3.5}$$

Remark. Theorem 3.2 is proven in detail in Whitaker and Samaniego (1988). An outline of the proof is as follows. Define $U_m = m^{1/2}(\hat{F}_y - F_y)$ and $V_m = m^{1/2}(\tilde{N} - E\tilde{N})$. The sequence of distributions induced by $\{(U_m, V_m)\}$ is shown to be tight, and the weak convergence of this sequence to a bivariate Gaussian process Z is established. Using the integral representations of the hazard functions R and $\hat{R}^*$ in (2.12) and (2.15), it is then shown that $m^{1/2}(\hat{R}^* - R)$ converges weakly to a zero-mean Gaussian process. We demonstrate the applicability of theorem I.2 of Anderson and Gill (1982) in our context, and derive the covariance structure of the limiting process from that result.

Using the aforementioned result, we apply the δ method to identify the limiting distribution of $\hat{F}$. Since $m^{1/2}(\hat{F} - F) = m^{1/2}(e^{-R} - e^{-\hat{R}})$, we apply the Taylor expansion $e^{-y} - e^{-x} = e^{-x}(x - y) + \frac{1}{2}e^{-z}(x - y)^2$, where z lies between x and y, to get

$$\begin{aligned} m^{1/2}(\hat{F} - F) &= m^{1/2}e^{-R}(\hat{R} - R) \\ &\quad + \tfrac{1}{2}\, m^{1/2}\, e^{-\tilde{R}}(\hat{R} - R)^2 \\ &= m^{1/2}\, S(\hat{R} - R) \\ &\quad + \tfrac{1}{2}\, m^{1/2}\, e^{-\tilde{R}}(\hat{R} - R)^2, \end{aligned} \tag{3.6}$$

where $\tilde{R}(t)$ lies between $\hat{R}(t)$ and $R(t)$ for $0 \le t \le T$. The first term on the right side of (3.6) converges weakly to $Z^*(t) = S(t)Z(t)$, a zero-mean Gaussian process. To get weak convergence of $m^{1/2}(\hat{F} - F)$ to Z^*, it remains to be shown that the second term on the right side of (3.6) converges to 0 in the supremum metric on $[0, T]$, w. p. 1. This follows since the process $m^{-1/2}e^{-\tilde{R}}$ converges to 0 in probability uniformly on $[0, T]$, and the process $[m^{1/2}(\hat{R} - R)]^2$ converges weakly to $Z^2(t)$ by Theorem 3.2. Since, w. p. 1, for each realization of the sequence $Y_1, Y_2, \ldots$ the inequality $Y_{(m-1)} > T$ is satisfied for m sufficiently large, this proves the following theorem.

Theorem 3.3. Let F be as in Theorem 3.1, and let $T < \infty$ satisfy $F(T) < 1$. Then the random process $\{m^{1/2}(\hat{F} - F) : 0 \le t \le T\}$ converges weakly to a zero-mean Gaussian process $Z^*(t)$ with covariance given by

$$S(s)S(t)\int_0^{s \wedge t} S_y^{-1}(x)\, dR\,(x), \qquad 0 \le s, t \le T. \tag{3.7}$$

4. DISCUSSION

In Section 2, we identified two simple competitors to the neighborhood nonparametric MLE as estimators of the survival function S of the Brown–Proschan model. We can now compare the performance of these three estimators in detail. Let $\hat{S}_1 = 1 - \hat{F}_m$, where $\hat{F}_m$ is the empirical cdf of "new lifetimes," that is, the m interfailure times including the first failure time and the interfailure times following each perfect repair, and let $\hat{S}_2 = \hat{S}_y^{n/m}$, where $\hat{F}_y$ is the empirical cdf of the times between successive perfect repairs.

We denote the neighborhood MLE of S by $\hat{S}_3$. Because $\hat{F}_m$ and $\hat{F}_y$ are empirical distributions, the asymptotic distributions of $\hat{S}_1$ and $\hat{S}_2$ can be obtained by standard methods. For $i = 1, 2$, $m^{1/2}(\hat{S}_i(t) - S(t))$ converges in law to a normal random variable with mean zero and asymptotic variance $\mathrm{AV}(\hat{S}_i)$, where

$$\mathrm{AV}(\hat{S}_1) = S(t)(1 - S(t)) \tag{4.1}$$

and

$$\mathrm{AV}(\hat{S}_2) = \frac{1}{p^2}\, S^{2-p}(t)(1 - S^p(t)) - (1 - p)S^2(t)R^2(t). \tag{4.2}$$

From Theorem 3.3 and Proposition 1.1, we identify the large-sample variance of $\hat{S}_3$ as

$$\begin{aligned} \frac{1}{m}\, S^2(t) \int_0^t S_y^{-1}(x)\, dR\,(x) &= \frac{1}{m}\, S^2(t) \int_{S(t)}^1 u^{-p-1}\, du \\ &= \frac{1}{mp}\, S^{2-p}(t)(1 - S^p(t)). \end{aligned} \tag{4.3}$$

From (4.3), we obtain the approximate standard error of $\hat{S}_3(t)$,

$$\hat{\sigma}_{\hat{S}_3(t)} = (mp)^{-1/2}\, \hat{S}_3^{1-p/2}(t)(1 - \hat{S}_3^p(t))^{1/2}, \tag{4.4}$$

an expression useful in forming approximate confidence intervals for $S(t)$ when m is large. Using the previous computations, we examine the asymptotic relative efficiency (ARE) of $\hat{S}_3$ with respect to the two competitors, and show that $\hat{S}_3$ is superior to both. In comparing $\hat{S}_3$ with $\hat{S}_1$, we obtain

$$\mathrm{ARE}(\hat{S}_3/\hat{S}_1) = \frac{p(1 - S(t))}{S^{1-p}(t)(1 - S^p(t))}. \tag{4.5}$$

The inequality $\mathrm{ARE}(\hat{S}_3/\hat{S}_1) \ge 1$ is a consequence of the following lemma. Plotting $\mathrm{ARE}\,(\hat{S}_3/\hat{S}_1)$ reveals that the superiority of $\hat{S}_3$ over $\hat{S}_1$ is strongest when p is small and t is in the right-hand tail of the distribution F.

Lemma 4.1. For all $(x, p) \in (0, 1)^2$,

$$p(1 - x)/x^{1-p}(1 - x^p) > 1. \tag{4.6}$$

Proof. The inequality in (4.6) is equivalent to

$$f(x) = x^{1-p} - x - p(1 - x) < 0. \tag{4.7}$$

For any fixed $p \in (0, 1)$, $f(x)$ is a strictly increasing function for $x \in [0, 1]$. Since $f(0) = -p$ and $f(1) = 0$, we have $f(x) < 0$ for $x \in (0, 1)$.

The estimators $\hat{S}_3$ and $\hat{S}_2$ admit to a similar comparison. The inequality $\mathrm{ARE}(\hat{S}_3/\hat{S}_2) \geq 1$ may be reduced to the fact that the function $g(x) = 1 - x^p - p^2x^p(\ln x)^2$ is positive for all $(p, x) \in (0, 1)^2$, a result that is easily established using arguments similar to those in the proof of Lemma 4.1.

We now turn to extensions of the results derived in Section 3. A generalization of the BBS model was introduced by Block et al. (1985). The BBS model stipulates that on failure at age t the system is repaired perfectly w. p. $p(t)$; otherwise, repair is imperfect. Thus $Z_i \mid X_i = t$ is a Bernoulli variable with parameter $p(t)$, where X_i is the system age at the ith failure. The parameter to be estimated is the pair $(p(\cdot), F)$.

To get a meaningful estimator of (p, F) in the BBS model, we need to ensure that the number of observable perfect repairs can grow to infinity. This is guaranteed by the condition

$$\int_0^\infty p(y)\, dR\,(y) = +\infty, \tag{4.8}$$

which Block et al. (1985) showed is equivalent to $\Pr(Y_1 < \infty) = 1$. Hereafter, we assume that the parameter pair $(p(\cdot), F)$ satisfies (4.8). The likelihood of observing $\mathbf{T} = \mathbf{t}$ and $\mathbf{Z} = \mathbf{z}$ [or equivalently $\mathbf{X} = \mathbf{x}$ and $\mathbf{Z} = \mathbf{z}$, given $(p(\cdot), F)$], is obtained by successive conditioning, as in Section 2.

$$L(p(\cdot), F) = \prod_{i=1}^{n} p(x_i)^{z_i}(1 - p(x_i))^{1-z_i} \times \prod_{i=1}^{n} \frac{f(x_i)}{S(x_{i-1})^{1-z_{i-1}}}, \tag{4.9}$$

where $x_0 = 0$, $z_0 = 1$, and $S = 1 - F$.

It is evident from (4.9) that the neighborhood MLE $(\hat{p}(\cdot), \hat{F})$ of the parameter pair $(p(\cdot), F)$ is given by $\hat{F} = 1 - \hat{S}$, where $\hat{S}$ is displayed in (2.9), and any function $\hat{p}:[0, \infty) \to [0, 1]$ for which $\hat{p}(x_i) = z_i$ $(i = 1, \ldots, n)$. Assuming suitable regularity conditions, the asymptotic properties of $\hat{F}$ under inverse sampling from the BBS model are identical to those of $\hat{F}$ under the BP model; that is, the strong uniform consistency of $\hat{F}$ and the weak convergence of $m^{1/2}(\hat{F} - F)$ to a zero-mean Gaussian process are also valid for the BBS model. Whitaker (1985) discusses the use of other approaches (including logistic regression) for estimating $p(\cdot)$ (see Cox 1969).

Results similar to ours may be developed for alternative sampling schemes. Consider, for example, renewal testing: Put k new systems on test, observing each for τ units of time. Because F is continuous, the last interfailure time of each system on test is, w. p. 1, a censored observation. To keep track of the censored observations, let $\delta_{i,j} = 1$ if the jth interfailure time of the ith system is uncensored, and $\delta_{i,j} = 0$ otherwise. Under the Brown–Proschan model the likelihood function, given (p, F), is

$$L(p, F) = p^m(1 - p)^{n-m} \times \prod_{i=1}^{k} \prod_{j=1}^{n_i-1} \frac{f(x_{i,j})}{S(x_{i,j-1})^{1-z_{i,j-1}}} \frac{S(x_{i,n_i})}{S(x_{i,n_i-1})^{1-z_{i,n_i-1}}}, \tag{4.10}$$

where $x_{i,j}$ is the age of the ith system just prior to the jth failure ($x_{i,0} \equiv 0$); $z_{i,j}$ is 1 or 0 if the ith system is repaired perfectly or imperfectly at the jth failure ($z_{i,0} \equiv 1$); $n_i - 1$ is the number of observed failures of the ith system; $n = \sum_{i=1}^{k} (n_i - 1)$; and m is the observed number of perfect repairs, that is, $m = \sum_{i=1}^{k} \sum_{j=1}^{n_i-1} z_{i,j}$. To keep notation consistent, x_{i,n_i} represents the age of the ith system at the renewal-testing time horizon τ rather than the age at an actual failure time. Also, define $z_{i,n_i} = 1$ for $1 \leq i \leq k$. The likelihood (4.10) can be rewritten as

$$L(p, F) = p^m(1 - p)^{n-m} \cdot \prod_{i=1}^{k} \prod_{j=1}^{n_i} \frac{f(x_{i,j})^{\delta_{i,j}} S(x_{i,j})^{1-\delta_{i,j}}}{S(x_{i,j-1})^{1-z_{i,j-1}}}.$$

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n+k)}$ be the order statistics of $x_{i,j}$ $(1 \leq i \leq k, 1 \leq j \leq n_i)$ and $z_{(1)}, \ldots, z_{(n+k)}$ and $\delta_{(1)}, \delta_{(2)}, \ldots, \delta_{(n+k)}$ be the induced order statistics of the $z_{i,j}$'s and the $\delta_{i,j}$'s. Then,

$$L(p, F) = p^m(1 - p)^{n-m} \cdot \prod_{i=1}^{n+k} \frac{f(x_{(i)})^{\delta_{(i)}} S(x_{(i)})^{1-\delta_{(i)}}}{S(x_{(i-1)})^{1-z_{(i-1)}}},$$

where $x_{(0)} \equiv 0$ and $z_{(0)} \equiv 1$. The neighborhood MLE for (p, F) is given by $\hat{p} = m/n$ and

$$\hat{S}(t) = \prod_{\{i \mid X_{(i)} \leq t\}} \left[\frac{\sum_{j=i}^{n+k-1} Z_{(j)}}{\sum_{j=i}^{n+k-1} Z_{(j)} + 1} \right]^{\delta(i)}. \tag{4.11}$$

Under this renewal-testing sampling scheme, nonparametric estimation for F, in the special case when the interfailure times of each system are iid with distribution F, was studied by Gill (1981). He showed that as $k \to \infty$, $k^{1/2}(\hat{F} - F)$ converges weakly to a zero-mean Gaussian process. The Brown–Proschan model with $p = 1$ corresponds to the iid setting. When $p = 1$, the estimator $\hat{F}$ in (4.11) reduces to the product limit estimator considered by Gill. For system lifetimes governed by the Brown–Proschan model, one may establish the weak convergence of the process $\{k^{1/2}(\hat{F} - F) \mid 0 \leq t \leq T\}$, where $F(T) < 1$, to a zero-mean Gaussian process. Details of this development are given in Whitaker (1985).

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