Impartial game: Fib Fob

Background:

In the near distant future, scientists have uncovered the secrets of time-travel. Due to the dangers of quantum fibrosity and time-space fobules, they needed to introduce a dual stabilizing axis to handle the warp fluids. This also had the added benefit of neutralizing any ψ-bananas in the nuggetspace. They called these stabilizing fluids Fib and Fob, and with this amazing new invention, modern time-space travel was born. It is best you don't question it too much, lest the ψ-bananas return. Due to the clashing nature of Fib Fob fluids, a single pilot cannot handle the travel on their own, and thus 2 co-pilots are manifested. These two co-pilots are actually individual instances of the same pilot existing within the same body, and thus only one can survive at the end.

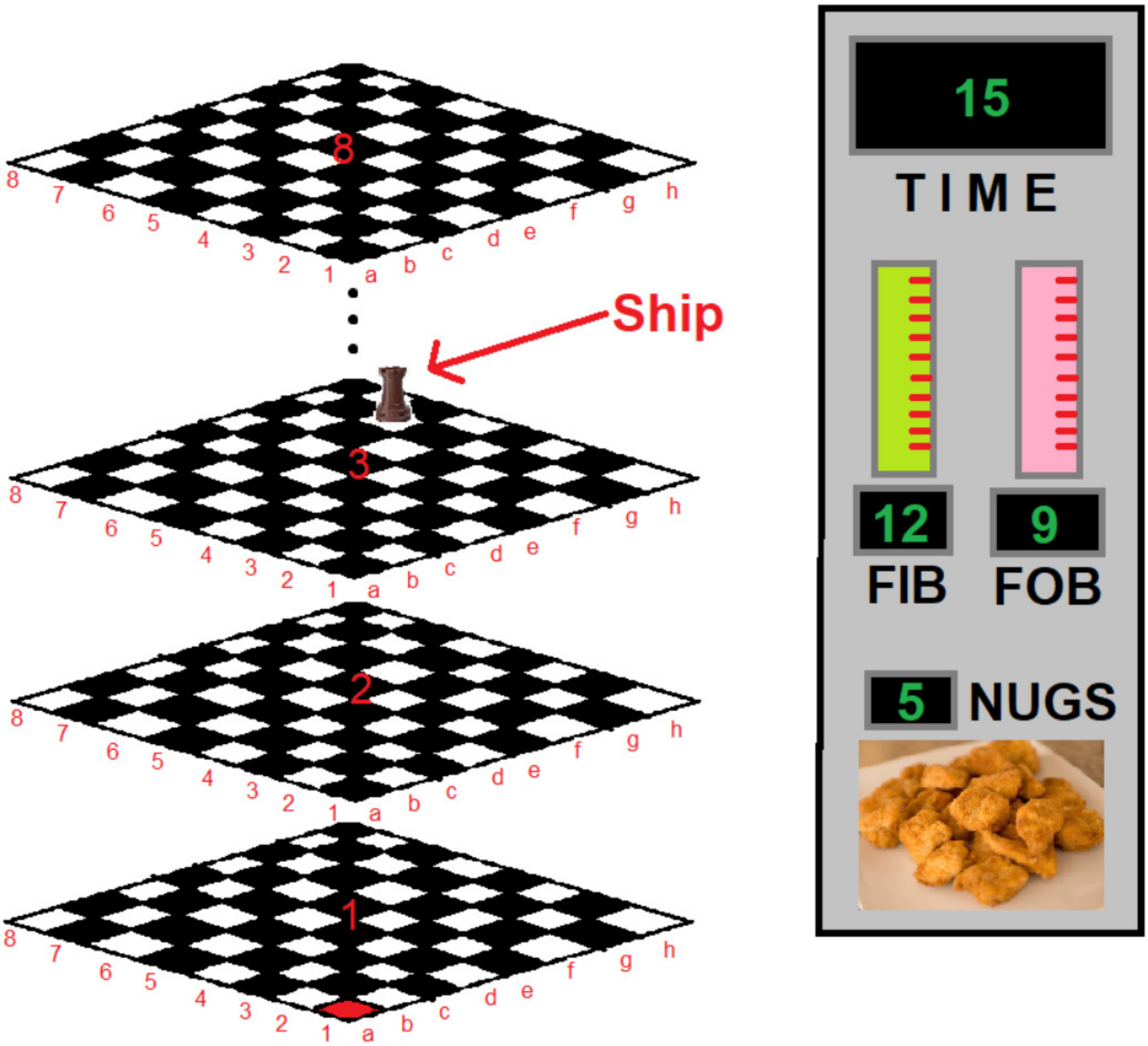
Gameplay Description:

The co-pilots navigate the ship on a 3-dimensional interface resembling 8 standard 8x8 chessboards floating atop each other in a stack. The ship is represented by a rook on the interface, with the destination being the 1A square (bottom left square) of the bottom chessboard. In addition, there is a time counter representing the number (non-negative integer) of light-years before reaching the destination, and Fib and Fob meters with a display showing how many gallons (non-negative integer) of Fib and Fob fluid remain in the tanks. The co-pilots are also equipped with some number (non-negative integer) of emergency time-nuggets, which they can use to revert the timeline in a pinch.

The two co-pilots play alternately, and the ship starts in a random starting position with a random amount of time, time-nuggets, and Fib and Fob fluids. On a co-pilot's turn, they can do one of the following actions:
- Move the ship down one or more chessboards in the stack (It stays on the same square, just on a different chessboard. If the ship is already on the bottom chessboard, this move is not available).
- Move the ship down by one or more ranks (aka rows) on the chessboard it is on (If the ship is already on rank 1, this move is not available).
- Move the ship left by one or more files (aka columns) on the chessboard it is on. (If the ship is already on file a, this move is not available).
- Reduce the time counter by one or more (cannot be reduced past zero).
- Consume one or more time-nuggets to increase the time counter by that amount.
- Spend one gallon of either Fib or Fob fluid.
- Eject either all of the Fib fluid or all of the Fob fluid.

A co-pilot who cannot make a move loses, and is erased from reality, having never existed. The other co-pilot wins, and has safely navigated to their destination in time and space with no Fib or Fob fluid left to spare, and all their time-nuggets consumed.

The depiction below shows an example where the time is 15, Fib fluid is at 12, Fob fluid is at 9, and 5 time nuggets remaining. The ship is on square g6 of the 3rd board from the bottom. We will be finding the nimber for this specific example.



Red square is destination square

Mathematical Analysis:

We want to first show that this game is an impartial game, and we can do so by showing that it meets the 8 conditions required to be an impartial game. These conditions are:
Condition 1: There are 2 players.
Condition 2: There are several positions, with a starting position.
Condition 3: A player performs one of a set of allowable moves, which depends only on the current

position, and not on the player whose turn it is.
Condition 4: The players move alternately.
Condition 5: There is complete information.
Condition 6: There are no chance moves, is satisfied.
Condition 7: The first player with no available moves loses
Condition 8: The rules guarantee that games end.
We can easily verify that these conditions hold. Thus, this is an impartial game.

This game is a combination of other impartial games, taking heavy inspiration from Nim, Rooks, Nim Muskateers (from P-5), and Poker Nim. The stacked chessboard portion of this game is very similar to a game of Rooks. The time counter and time nuggets resembles a one-pile game of Poker Nim, with the time nuggets being the bag of B chips that can be added to the time counter. The Fib and Fob fluid meters can be seen as a two-pile game of Nim Muskateers. We can thus separate these aspects of the game and analyze them individually in order to find the nimber of the full game.

We notice that the game of Rooks within the stacked chessboard interface is essentially a 3-pile game of Nim. If we use our example above, where the ship is on square g6 of the 3rd board from the bottom, this is equivalent to $*6 + *5 + *2 \equiv *1$. The $*6$ pile is for the file g, $*5$ for the rank of 6, and $*2$ for the 3rd board from the top. We realize that if we are on the destination square (i.e. square a1 of the bottom board), then this is equivalent to $*0 + *0 + *0 \equiv *0$, a losing game.

The time counter and time nuggets are equivalent to a one pile game of Poker Nim, which is equivalent to a one pile game of Nim, as if a player adds to the time counter using some number of time nuggets, the opposing player can simply remove that number from the time counter, leaving us at the same state prior to the use of time nuggets. A one pile game of Nim is simple to analyze, and thus in the example above, the time counter and time nuggets are equivalent to $*15$.

The game of Fib and Fob fluid meters is equivalent to a two-pile game of Nim Muskateers from assignment question P-5. Thus, if there is a non-zero amount of both fluids and an even number of gallons of fluid in total, this is a losing game equivalent to $*0$. If there is a non-zero amount of both fluids and an odd number of gallons of fluid in total, this is a winning game equivalent to $*3$. If there is only one type of fluid remaining, this is a winning game equivalent to $*1$ (ejecting the rest of the fluid is the winning move). If there is no fluid remaining, this is clearly equivalent to $*0$. Thus, for our example above with 12 gallons of Fib fluid and 9 gallons of Fob fluid, we have a winning game equivalent to $*3$.

We now consider our example as a whole, i.e. the example where the time is 15, Fib fluid is at 12, Fob fluid is at 9, there are 5 time nuggets, and the ship is on square g6 of the 3rd board from the bottom. Combining our analysis from the past three paragraphs, we see that this example is equivalent to $*6 + *5 + *2 + *15 + *3 \equiv *1 + *15 + *3 \equiv *13$. Thus, the nimber for this instance of the game is 13. Thus, a winning move is to move on the $*15$ game and turn it into $*2$. We can do so by reducing the time counter by 13. Thus, the resulting game is equivalent to $*1 + *2 + *3 \equiv *0$, which is a losing game.

Thus, we see that the seemingly complicated game of Fib Fob is actually not too complicated after all, and can be reduced to smaller impartial games that are easier to analyze. Using this method of breaking the larger game down into smaller impartial games, we would be able to easily find the nimber of any instance of the Fib Fob game and determine the winning move, if one exists.