Auction: Szechuan Sauce Sale

In 2017, a discontinued McDonald's sauce called "Szechuan Sauce" became a viral meme due to a popular show called Rick and Morty. As a result of the surge in popularity, McDonald's decided to revive the sauce for a one-day promotion, but due to limited supply, some jerk named Martin somehow got a hold of all of the sauce and has put 100 gallons of it up for auction.

Martin has decided that only the first 100 bidders will be able to bid, and that everyone who bids should get to try at least some of the sauce. Martin would also like his auction to be DSIC. We let $\sum b > 0$ be the summation of all the bids, and let $\sum b_{-i} > 0$ be the summation of all bids except bid b_i. We assume bids and valuations are positive. Truthful bidding in this case means bidding your valuation for 1 gallon of sauce. The allocation rule is as follows:

$$x(b) = x_i(b_i, b_{-i}) = 100\frac{b_i}{\sum b} \text{ gallons}$$

We note that this allocation rule is monotone, as we can see that the first derivative is always positive:

$$\begin{aligned} x(b) &= 100\frac{b_i}{\sum b} = 100\frac{b_i}{b_i + \sum b_{-i}} \\ \frac{dx(b)}{db_i} &= 100\frac{((1)(b_i + \sum b_{-i}) - (b_i)(1))}{(b_i + \sum b_{-i})^2} = \frac{100\sum b_{-i}}{(\sum b)^2} \end{aligned}$$

Thus, by Myerson's lemma, since we have a monotone allocation rule, there must be a unique payment rule such that (x,p) is DSIC and $p_i(b) = 0$ whenever $b_i = 0$.

Thus, suppose (x,p) is DSIC and p(0)=0. Assume $y \geq z \geq 0$ for two arbitrary bids y and z. Since (x, p) is DSIC and $y \geq z$, we can use the payment sandwich:

$$\begin{aligned} &z(x(y) - x(z)) \leq p(y) - p(z) \leq y(x(y) - x(z)) \\ &\implies \frac{z(x(y) - x(z))}{y - z} \leq \frac{p(y) - p(z)}{y - z} \leq \frac{y(x(y) - x(z))}{y - z} \\ &\lim_{y \to z} \frac{z(x(y) - x(z))}{y - z} = z * x'(z) = z\frac{100\sum b_{-i}}{(z + \sum b_{-i})^2} \\ &\lim_{y \to z} \frac{y(x(y) - x(z))}{y - z} = z * x'(z) = z\frac{100\sum b_{-i}}{(z + \sum b_{-i})^2} \end{aligned}$$

Thus, by squeeze theorem:

$$\begin{aligned} &\lim_{y \to z} \frac{p(y) - p(z)}{y - z} = p'(z) = z\frac{100\sum b_{-i}}{(z + \sum b_{-i})^2} \\ &\implies p(z) = \int z\frac{100\sum b_{-i}}{(z + \sum b_{-i})^2} dz = 100\left(\sum b_{-i}\right)\left(\ln\left(z + \sum b_{-i}\right) + \frac{\sum b_{-i}}{z + \sum b_{-i}}\right) + C \end{aligned}$$

We know anti-derivatives are unique up to a constant. We can determine this constant if we consider z=0:

$$\begin{aligned} &p(z) = p(0) = 0 \\ &p(0) = 100\left(\sum b_{-i}\right)\left(\ln\left(\sum b_{-i}\right) + 1\right) + C \\ &\implies C = -100\left(\sum b_{-i}\right)\left(\ln\left(\sum b_{-i}\right) + 1\right) \end{aligned}$$

Thus:

$$p(b_i) = 100\left(\sum b_{-i}\right)\left(\ln\left(z+\sum b_{-i}\right)+\frac{\sum b_{-i}}{z+\sum b_{-i}}\right) - 100\left(\sum b_{-i}\right)\left(\ln\left(\sum b_{-i}\right)+1\right)$$

Thus, we have derived an explicit formula for p, and we know that p must be unique since anti-derivatives are unique up to a constant, and we have determined what the constant must be.

Now, we want to prove that (x,p) is DSIC. Consider the utility function:

$$\begin{aligned} u_i(z) &= v_i x_i(z) - p_i(z) \\ &= v_i\left(100\frac{z}{z+\sum b_{-i}}\right) - 100\left(\sum b_{-i}\right)\left(\ln\left(z+\sum b_{-i}\right)+\frac{\sum b_{-i}}{z+\sum b_{-i}}\right) + 100\left(\sum b_{-i}\right)\left(\ln\left(\sum b_{-i}\right)+1\right) \end{aligned}$$

We want to maximise $u_i(z)$ to determine z, letting v_i be a constant, and show that $z = v_i$ is a global maximum. Thus:

$$u_i'(z) = \frac{100(\sum b_{-i})(v_i - z)}{(z+\sum b_{-i})^2}$$

We notice that $\frac{100(\sum b_{-i})}{(z+\sum b_{-i})^2}$ must be non-negative. Thus, when $z < v_i$, we have a non-negative slope for the utility function, and when $z > v_i$, we have a non-positive slope for the utility function. When $z = v_i$, the slope is zero. Thus, $z = v_i$ must be a global maximum. We also see that the utility is non-negative when $z = v_i$.

Thus, truthful bidding is a dominant strategy (weakly dominant with no strictly lesser utility case required), and yields a non-negative utility. Therefore, (x, p) is DSIC by definition, as Martin wanted. This auction is not ideal however. It is DSIC and can be implemented efficiently, but it is not welfare maximising, as otherwise the highest bidder should win all the sauce since they value it the most.