Analysis 1

Paulo Witt

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1 Introduction

This is a general summary of the Analysis 1 course I attended. Prerequisites are the general knowledge of arithmetic's and proofing techniques.

2 Prerequisites

In this section I will go over proofs necessary for the following course.

2.1 Bernoulli's inequality

Theorem:

$$(1+x)^n \geq 1 + nx, n \in \mathbb{N}, x \geq -1$$

Proof:
This Proof will use induction.

Proposition:

$$n = 0$$
$$(1+x)^0 = 1 \wedge 1 + 0x = 1$$
$$\exists_{n \in \mathbb{N}} : (1+x)^n \geq 1 + x$$

Inductive step:

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx)$$
$$(1+x)(1+nx) = 1 + x + nx + nx^2 = (1 + (n+1)x) + nx^2$$
$$nx^2 \geq 0 \implies (1+x)^{n+1} \geq 1 + (n+1)x, \forall n > 0, \forall x \in \mathbb{R}$$

2.2 Binomial theorem

Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof:
Note that:

$$\binom{n}{k}\binom{n}{k-1} = \binom{n+1}{k}$$
$$\binom{n}{k} = 0, k > n, k < 0$$

Using induction.
Proposition:

$$n = 0 (a+b)^0 = 1, \sum_{k=0}^{0} \binom{n}{k} a^k b^{n-k} = \binom{0}{0} a^0 b^0 = 1$$
$$\exists_{n \in \mathbb{N}} : (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Inductive Step:

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$
$$= a \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} + b \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k}$$

Set $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} + \binom{n+1}{k} a^{n+1} b^0 = \sum_{k=0}^{n+1} \binom{n}{k} a^k b^{n+1-k}$

and $\sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} = \sum_{k=1}^{n+1} \binom{n}{k} a^k b^{n+1-k} = \sum_{k=0}^{n+1} \binom{n}{k} a^k b^{n+1-k}$

$$\sum_{k=0}^{n+1} \binom{n}{k}\binom{n}{k-1} a^k b^{n+1-k} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}$$

2.3 Inequality of the arithmetic and geometric mean

For a set of non negative numbers X:

$$\bar{x}_{\text{arithm}} = \sum_{k=1}^{n} \frac{x_k}{n} \geq \sqrt[n]{\prod_{k=1}^{n} x_k} = \bar{x}_{\text{geom}}$$

Proof:
Proposition:

$$\begin{gathered} n = 2 \\ \frac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2} \\ \left(\frac{x_1 + x_2}{2}\right)^2 \geq x_1 x_2 \\ \frac{1}{4}(x_1^2 + 2x_1x_2 + x_2^2) - x_1x_2 \geq 0 \\ \frac{1}{4}(x_1^2 - 2x_1x_2 + x_2^2) > 0, \forall_{x \neq y} \end{gathered}$$

Inductive step: $n \to 2n$

$$\begin{gathered} \sum_{k=1}^{2n} \frac{x_k}{2n} = \frac{\sum_{k=1}^{n} x_k + \sum_{k=n+1}^{2n} x_k}{2} := \frac{\bar{x}_{\text{arithm}} + \bar{y}_{\text{arithm}}}{2} \\ \frac{\bar{x}_{\text{arithm}} + \bar{y}_{\text{arithm}}}{2} \geq \sqrt{\bar{x}_{\text{arithm}} \bar{y}_{\text{arithm}}} \geq \sqrt{\bar{x}_{\text{geom}} \bar{y}_{\text{geom}}} \end{gathered}$$

Inductive step: $n \to n-1$

$$\begin{gathered} \bar{x}_{\text{arithm}} \geq \bar{x}_{\text{geom}} \\ \text{Set: } x_m = \bar{x}_{\text{arithm}}, \forall_{m>n} \\ \bar{x}_{\text{arithm}} = \sum_{k=1}^{n} \frac{x_k}{n} = \sum_{k=1}^{n} \frac{\frac{m}{n} x_k}{m} = \frac{\sum_{k=1}^{n} x_k + \frac{m-n}{n} \sum_{k=1}^{n} x_k}{m} = \\ \frac{\sum_{k=1}^{n} x_k + (m-n)\bar{x}_{\text{arithm}}}{m} = \sum_{k=1}^{m} \frac{x_k}{m} \geq \sqrt[m]{\prod_{k=1}^{n} x_k \bar{x}_{\text{arithm}}^{m-n}} \\ \bar{x}_{\text{arithm}}^{m} \geq \prod_{k=1}^{n} x_k \bar{x}_{\text{arithm}}^{m-n} \\ \bar{x}_{\text{arithm}} \geq \bar{x}_{\text{geom}} \end{gathered}$$

2.4 Triangle inequality

$$| x + y | \leq | x | + | y |$$
$$| x + y |^2 = (x + y)^2 \leq (| x | + | y |)^2$$
$$x^2 + 2xy + y^2 \leq x^2 + 2 | xy | + y^2$$
$$x \leq | x |$$

3 Convergence of sequences and series

3.1 Definitions

Def: A sequence $(a_n)_{n \in \mathbb{N}}$ or in short form (a_n) is a function from the natural numbers to the real numbers, $a : \mathbb{N} \to \mathbb{R}$.
(Technically the rational numbers as the real numbers are not formally constructed, without the knwoledge of Analysis 1)

Def: A sequence is bounded, iff:

$$\exists_{c \in \mathbb{R}} : \forall_{n \in \mathbb{N}} : | a_n | < c.$$

Def: A sequence is:

$$\text{strictly monotone increasing, iff: } \forall_{n \in \mathbb{N}} : a_n < a_{n+1}$$
$$\text{strictly monotone decreasing, iff: } \forall_{n \in \mathbb{N}} : a_n > a_{n+1}$$
$$\text{monotone increasing, iff: } \forall_{n \in \mathbb{N}} : a_n \leq a_{n+1}$$
$$\text{monotone decreasing, iff: } \forall_{n \in \mathbb{N}} : a_n \leq a_{n+1}$$

Note: If a sequence is not convergent its denoted $\lim_{n \to \infty} a_n = /$.
Def: A sequence is convergent:

$$\forall_{\epsilon > 0} : \exists_{n_0 \in \mathbb{N}} : \forall_{n \geq n_0} : | a_n - a | < \epsilon \iff$$
$$\lim_{n \to \infty} a_n = a \iff$$
$$a_n \xrightarrow{n \to \infty} a$$

Note: The Limit is unique.
Proof:

Suppose:

$$\lim_{n\to\infty} a_n = a \wedge \lim_{n\to\infty} a_n = b \wedge \mid a-b \mid \neq 0$$
$$\text{Set: } \epsilon = \frac{\mid a-b \mid}{4}$$
$$\exists_{n_1\in\mathbb{N}} : \forall_{n>n_1} : \mid a_n - a \mid < \epsilon \wedge \exists_{n_2\in\mathbb{N}} : \forall_{n>n_2} : \mid a_n - b \mid < \epsilon$$
$$\text{Choose: } n_0 = \max\{n_1, n_2\}$$
$$\mid a-b \mid = \mid a + (a_n - a_n) + b \mid \leq \mid a_n - a \mid + \mid a_n - b \mid$$
$$\forall_{n>n_0} : \mid a_n - a \mid + \mid a_n - b \mid \leq \epsilon + \epsilon$$
$$\mid a-b \mid \leq \frac{\mid a-b \mid}{2} \perp$$

<u>Note:</u> Every convergent sequence is bounded.
<u>Proof:</u>

$$\text{Set: } \epsilon = 1, \exists_{n_0\in\mathbb{N}} : \forall_{n>n_0} : \mid a_n - a \mid < 1$$
$$\forall_{n\geq n_0} : \mid a_n \mid < \mid a \mid +1$$
$$c := \max\{\mid a_1 \mid, \mid a_2 \mid, ..., \mid a_{n_0-1} \mid, \mid a \mid +1\}$$

3.2 Limit theorems

For all convergent sequences $(a_n), (b_n)$ holds:

$$\forall_{(a_n),(b_n)} : \lim_{n\to\infty} a_n = a \wedge \lim_{n\to\infty} b_n = b :$$
$$\lim_{n\to\infty} a_n + b_n = a + b$$
$$\lim_{n\to\infty} a_n b_n = ab$$
$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}, a_n, a \neq 0$$

<u>Proof:</u>

$$\lim_{n\to\infty} a_n + b_n = a + b \iff$$
$$\forall_{\epsilon>0} : \exists_{n_0\in\mathbb{N}} : \forall_{n\geq n_0} \mid a_n + b_n - (a+b) \mid < \epsilon$$
$$\mid a_n + b_n - (a+b) \mid \leq \mid a_n - a \mid + \mid b_n - b \mid$$
$$\exists_{n_1\in\mathbb{N}} : \forall_{n\geq n_1} \mid a_n - a \mid < \frac{\epsilon}{2} \wedge \exists_{n_2\in\mathbb{N}} : \forall_{n\geq n_2} \mid b_n - b \mid < \frac{\epsilon}{2}$$
$$n_0 := \max\{n_1, n_2\}$$
$$\forall_{n\geq n_0} : \mid a_n - a \mid + \mid b_n - b \mid < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\lim_{n\to\infty} a_n b_n = ab \iff$$
$$\forall_{\epsilon>0} : \exists_{n_0\in\mathbb{N}} : \forall_{n\geq n_0} \mid a_n b_n - (ab) \mid < \epsilon$$
$$\mid a_n b_n - (ab) \mid = \mid a_n b_n + (-ab_n + ab_n) - ab \mid =$$
$$\mid b_n(a_n - a) + a(b_n - b) \mid \leq \mid (a_n - a)b_n \mid + \mid (b_n - b)a \mid =$$
$$\mid a_n - a \mid\mid b_n \mid + \mid b_n - b \mid\mid a \mid$$
$$\text{Set: } c \text{ so } c > \mid b_n \mid, n \in \mathbb{N}$$
$$\exists_{n_1\in\mathbb{N}} : \forall_{n\geq n_1} \mid a_n - a \mid < \frac{\epsilon}{2c} \wedge \exists_{n_2\in\mathbb{N}} : \forall_{n\geq n_2} \mid b_n - b \mid < \frac{\epsilon}{\mid 2a \mid}$$
$$n_0 := \max\{n_1, n_2\}$$
$$\forall_{n\geq n_0} : \mid a_n - a \mid\mid b_n \mid + \mid b_n - b \mid\mid a \mid < \mid b_n \mid \frac{\epsilon}{2c} + \mid a \mid \frac{\epsilon}{2 \mid a \mid} < \epsilon$$

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}; a_n, a \neq 0$$
$$\mid a_n - a \mid < \epsilon \wedge \exists_{n_0\in\mathbb{N}} : \forall_{n>n_0} \mid a_n \mid > \frac{a}{2}$$
$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \left| \frac{a - a_n}{a_n a} \right| < \left| \frac{\epsilon_1}{a_n a} \right| < \frac{2\epsilon_1}{a^2} < \epsilon$$

3.3 Cauchy series

Def: (a_n) is a Cauchy-series, iff

$$\forall_{\epsilon>0} : \exists_{n_0\in\mathbb{N}} : \forall_{n,m>n_0} : \mid a_n - a_m \mid < \epsilon$$

3.4 Construction of the real numbers

We are now able to define the real numbers as a Equivalence class of class of Cauchy series. We consider the set of all the Cauchy series X and define a equivalence relation:

$$(a_n) \sim (b_n) \iff \lim_{n\to\infty} a_n - b_n = 0$$

and define the set of real numbers $\mathbb{R}$ as all the sets of all equivalence classes $S((a_n))$:

$$S((a_n)) = \{(x_n) \in X, (x_n) \sim (a_n)\}$$

Showing that $(a_n) \sim (b_n)$ is a equivalence relation:

$$\text{reflexivity: } (a_n) \sim (a_n), \forall_{(a_n)\in X}$$
$$\text{symmetry: } (a_n) \sim (b_n) \implies (b_n) \sim (a_n), \forall_{(a_n),(b_n)\in X}$$
$$\text{transitivity: } (a_n) \sim (b_n) \wedge (b_n) \sim (c_n) \implies (a_n) \sim (c_n), \forall_{(a_n),(b_n),(c_n)\in X}$$

Proof:

$$\text{reflexivity: } a_n - a_n = 0 \xrightarrow{n\to\infty} 0$$
$$\text{symmetry: } a_n - b_n \xrightarrow{n\to\infty} 0 \iff -1(b_n - a_n) \xrightarrow{n\to\infty} -1(0) = 0$$
$$\text{transitivity: } a_n - b_n \xrightarrow{n\to\infty} 0 \wedge b_n - a_n \xrightarrow{n\to\infty} 0$$
$$a_n - c_n = a_n - b_n + b_n - c_n \xrightarrow{n\to\infty} 0$$

We have now shown that this is an equivalence relation and due to the arithmetic's for convergent series it is easy to show that all the axioms for the real numbers are given by this set.

Theorem: Every convergent sequence is a Cauchy sequence.
Proof:

$$\exists n_0 \in \mathbb{N} : \forall_{n \geq n_0} :\mid a_n - a \mid < \frac{\epsilon}{2} \wedge \forall_{m \geq n_0} :\mid a_m - a \mid < \frac{\epsilon}{2}$$
$$\mid a_n - a \mid + \mid a_m - a \mid < \epsilon$$
$$\mid a_n - a \mid + \mid a_m - a \mid = \mid a_n - a \mid + \mid a - a_m \mid \geq \mid a_n - a + a - a_m \mid < \epsilon$$

Theorem: In $\mathbb{R}$ every Cauchy Sequence converges.
Proof: This follows directly from the construction of the real numbers.
Note: This holds for all complete fields.

3.5 Subsequences and accumulation points

Def: A subsequence (a_{n_i}) is a selection of elements of the sequnce.

$$(a_{n_i}) = (a_{n_i})_{i\in\mathbb{N}}, \text{with } n_{i+1} > n_i, \forall i \in \mathbb{N}$$

Accumulation Point: A number h is called an accumulation point, iff

$$\forall \epsilon > 0 : \forall n \in \mathbb{N} : \exists n_0 > n :\mid a_{n_0} - h \mid < \epsilon$$

Def: The maximum and minimum respectively of a set X is defined as:

$$\max_{x\in X} X = a \iff \forall_{x\in X} : a \geq x$$
$$\min_{x\in X} X = a \iff \forall_{x\in X} : a \leq x$$

Def: The supremum and infimum of a set X is defined as:

$$\sup X := max\{x \in \mathbb{R} \mid \forall_{n\in\mathbb{N}} : x \leq a_n\}$$
$$\inf X := min\{x \in \mathbb{R} \mid \forall_{n\in\mathbb{N}} : x \geq a_n\}$$

Bolzano-Weierstrass theorem: Every bounded sequence has a convergent subsequence.

Proof:

$$s := \inf(x_n), S := \sup(x_n)$$
$$I_1 = \left[s, \frac{s+S}{2}\right], I_2 = \left[\frac{s+S}{2}, S\right],$$
choose I so that it contains infinitely many elements and repeat.

3.6 Infinite series

For a sequence (a_k) the sum $\sum_{k=0}^{\infty} a_k$ is called a series.
Def: A series is called convergent, iff

$$\exists_{s\in\mathbb{R}} : \lim_{n\to\infty} \sum_{k=0}^{n} a_k = s$$

For the following the sequence of partial sums S_n of a sequence (s_n) is defined as:

$$S_n := \sum_{k=0}^{\infty} s_k$$

Def: A sequence is absolutely convergent, iff $(\mid S_n \mid)$ converges.
Theorem: If (S_n) converges (a_n) converges to zero.
Proof:
We know that (S_n) is a Cauchy sequence.

$$\forall_{\epsilon>0} : \exists_{n_0\in\mathbb{N}} : \forall_{n,m>n_0} : \mid S_n - S_m \mid < \epsilon$$
$$\mid S_n - S_m \mid = \mid \sum_{k=m+1}^{n} s_n \mid < \epsilon$$
$$\mid a_n - 0 \mid = \mid \sum_{k=n}^{n} s_n \mid < \epsilon, \forall n > n_0$$

Theorem: Comparison test: A series C_n is convergent, if an absolutely convergent series (A_n) exists with:

$$\forall_{n\in\mathbb{N}} : \mid c_n \mid < \mid a_n \mid$$

Proof:

$$\forall_{\epsilon>0} : \exists_{n_0\in\mathbb{N}} : \forall_{n,n>n_0} : \mid S_n - S_m \mid < \epsilon$$
$$\mid C_n - C_m \mid < \mid A_n - A_m \mid < \epsilon$$

Lemma: If S_n is absolutely convergent the two series S_n^+ and S_n^- converge, with:

$$s_n^+ = \begin{cases} s_n, s_n > 0 \\ 0, s_n \leq 0 \end{cases}, s_n^- = \begin{cases} s_n, s_n < 0 \\ 0, s_n \geq 0 \end{cases}$$

$$S_n^+ = \sum_{k=0}^{n} s_n^+, S_n^- = \sum_{k=0}^{n} s_n^-$$

Proof:

$$\mid s_n^+ \mid \leq \mid s_n \mid, \mid s_n^- \mid \leq \mid s_n \mid$$

Riemann rearrangement theorem: For an absolutely convergent series, if you rearrange the order of the elements (s_n) the limit does not change.

$$\sum_{k=0}^{\infty} s_n = \sum_{k=0}^{\infty} s_{\sigma(n)}, \sigma(n) : \mathbb{N} \to \mathbb{N}, \sigma \text{ bijective}$$

Proof:

$$\lim_{n\to\infty} S_{\sigma(n)}^+ = \lim_{n\to\infty} S_n^+$$
$$\lim_{n\to\infty} S_{\sigma(n)}^- = \lim_{n\to\infty} S_n^-$$
$$\lim_{n\to\infty} S_n^+ + S_n^- = \lim_{n\to\infty} S_n$$
$$\lim_{n\to\infty} S_{\sigma(n)}^+ + S_{\sigma(n)}^- = \lim_{n\to\infty} S_n^+ + S_n^- = \lim_{n\to\infty} S_n$$

3.7 Evaluation of infinite series

Comparison test: A series C_n is convergent, if an absolutely convergent series (A_n) exists with:

$$\forall_{n\in\mathbb{N}} : \mid c_n \mid < \mid a_n \mid$$

Generalized ratio test: A series (A_n) absolutely converges, if

$$\exists_{q\in(0,1)} : \exists_{n_0\in\mathbb{N}} : \forall_{n>n_0} : \mid \frac{a_{n+1}}{a_n} \mid \leq q$$

Root test:
A series (A_n) converges absolutely, if

$$\exists_{q\in(0,1)} : \exists_{n\in\mathbb{N}} \forall_{n>n_0} : \sqrt[n]{\mid a_n \mid} \leq q$$

Leibniz test:
If (a_n) is monoton decreasing and converges to 0, than the series S_n converges, with:

$$S_n := \sum_{k=0}^{n} (-1)^k a_k$$

Proof for generalized ratio test:
Lemma: $\sum_{k=0}^{n} q^k, q \in (-1,1)$ converges to $\frac{1}{1-q}$
Proposition:

$$\sum_{k=0}^{0} q^k = q^0 = 1 \land \frac{1-q^{n+1}}{1-q} = \frac{1-0^1}{1-0} = 1$$

$$\exists_{n\in\mathbb{N}} : \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

Inductive step:

$$\sum_{k=0}^{n+1} q^k = \sum_{k=0}^{n} q^k + q^{n+1} = \frac{1-q^{n+1}}{1-q} + q^{n+1} =$$
$$\frac{1-q^{n+1}+(1-q)q^{n+1}}{1-q} = \frac{1-q^{n+1}+q^{n+1}-q^{n+2}}{1-q}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} = \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q}$$

Generalized ratio test:

$$\forall_{n>n_0} \mid \frac{a_{n+1}}{a_n} \mid \leq q$$
$$\mid a_n \mid \leq q \mid a_{n-1} \mid \Longrightarrow \mid a_n \mid \leq q^{n-n_0} \mid a_{n_0} \mid$$
$$\sum_{k=0}^{\infty} \mid a_k \mid \leq \mid A_{n_0} \mid + \sum_{k=0}^{\infty} \mid A_{n_0} \mid q^k = \mid A_{n_0} \mid + \mid A_{n_0} \mid \sum_{k=0}^{\infty} q^k$$

Which converges.
Proof for the root test:

$$\exists_{q\in(0,1)} : \exists n_0 \in \mathbb{N} : \forall_{n>n_0} : \sqrt[n]{\mid a_n \mid} \leq q$$
$$\sqrt[n]{a_n} \leq q \iff \mid a_n \mid \leq q^n$$
$$\sum_{k=0}^{\infty} \mid a_k \mid \leq A_{n_0} + \sum_{k=0}^{\infty} q^k$$

Proof of the Leibniz test:

$$S_{2n-1} = \sum_{k=0}^{2n-1} (-1)^k s_k = \sum_{k=0}^{n-1} (s_{2k+1} - s_{2k}) > 0$$
$$S_{2n-1} = \sum_{k=0}^{2n-1} (-1)^k s_k = s_0 + \sum_{k=1}^{n-1} (s_{2k} - s_{2k-1}) - s_{n-1} < s_0$$

We know know that (S_n) is bounded and monoton decreasing, so it must converge.

3.8 Construction of the exponential function

The exponential function is defined as:

$$E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We will now show that this function converges for all $x \in \mathbb{R}$. Using the generalized ratio test:

$$e_n(x) = \frac{x^k}{k!}$$

$$\frac{\mid e_{k+1} \mid}{\mid e_k \mid} = \frac{\mid x^{k+1} \mid k!}{\mid x^k \mid (k+1)!} = \frac{\mid x \mid}{k+1}$$

$$\text{Set: } k_0 > x-1 : \forall_{k>k_0} : \frac{x}{k+1} \leq \frac{x}{k_0+1} < 1$$

A different expression of the exponential function:

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

<u>Proof:</u>
First step:

$$\left(1+\frac{x}{n}\right)^n \leq \sum_{k=0}^{n} \frac{x^k}{k!} \leq \left(1+\frac{x}{n}\right)^{n+1}$$

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{n^k} = \sum_{k=0}^{n} \frac{n!x^k}{(n-k)!k!n^k} \leq \sum_{k=0}^{n} \frac{x^k}{k!}$$

$$\left(1+\frac{x}{n}\right)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{x^k}{n^k} = 1 + \sum_{k=1}^{n+1} \binom{n+1}{k} \frac{x^k}{n^k} = 1 + \sum_{k=0}^{n} \binom{n+1}{k+1} \frac{x^{k+1}}{n^{k+1}} =$$

$$1 + \sum_{k=0}^{n} \frac{(n+1)!x^{k+1}}{(k+1)!(n-k)!n^{k+1}} \geq 1 + \sum_{k=0}^{n} \frac{(n-(k-1))^{k+1}}{n^{k+1}} \frac{x^{k+1}}{(k+1)!} =$$

$$1 + \sum_{k=0}^{n} \left(1 - \frac{(k+1)(k-1)}{n}\right) \frac{x^{k+1}}{(k+1)!} =$$

$$1 + \sum_{k=0}^{n} \frac{x^{k+1}}{(k+1)!} + \sum_{k=0}^{n} - \frac{(k+1)(k-1)}{n} \frac{x^{k+1}}{(k+1)!} = \sum_{k=0}^{n+1} \frac{x^k}{k!} - \frac{1}{n}\left(-x^n + 0 + x^n - \frac{x^k}{n!}\right) \geq$$

$$\sum_{k=0}^{n} \frac{x^k}{k!}$$

Showing that: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n+1}$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = a$$
$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n+1} = b$$
$$\lim_{n\to\infty}\frac{\left(1+\frac{x}{n}\right)^{n+1}}{\left(1+\frac{x}{n}\right)^{n+1}} = \lim_{n\to\infty} 1+\frac{x}{n} = 1$$
$$\lim_{n\to\infty}\frac{b}{a} = 1 \iff a = b$$

<u>Functional equation:</u>

$$\exp(x+y) = \exp(x)\exp(y)$$
$$\left(1+\frac{x}{n}\right)\left(1+\frac{y}{n}\right)^n = \left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n =$$
$$\left(1+\frac{x+y}{n}\right)^n\left(1+\frac{xy}{n(n+x+y)}\right)^n \xrightarrow{n\to\infty}$$
$$\text{Note that:}\frac{xy}{n(n+x+y)} \xrightarrow{n\to\infty} = 0 \wedge \exp(0) = \sum_{k=0}^{\infty}\frac{0^k}{k!} = 1$$
$$\exp(x+y)\exp(0) = \exp(x+y)$$

<u>Lemma:</u>

$$\exp(-x) = \frac{1}{\exp(x)}$$
$$\exp(-x)\exp(x) = exp(x-x) = 1 \iff \exp(-x) = \frac{1}{\exp(x)}$$

<u>Lemma:</u>

$$\exp(x) > 0$$
$$\forall_{x\geq 0} : \exp(x) \geq 1+x$$
$$\forall_{x<0} : \exp(x) = \frac{1}{\exp(-x)}$$

Lemma:

$$\lim_{x\to\infty} \exp(x) = \infty, \exp(x) \geq x, \forall_{x>0}$$

$$\lim_{x\to-\infty} \exp(x) = 0, \exp(x) \leq \frac{1}{x}, \forall_{x<0}$$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

$$\exp(x) > \frac{x^{+1}}{(n+1)!}$$

$$\frac{x^n}{\exp(x)} \leq \frac{(n+1)!}{x} \xrightarrow{x\to\infty} 0$$

3.9 Definition of the logarithm:

The logarithmic function is defined as:

$$\log(x) = \exp^{-1}(x)$$

3.10 Limits of functions and continuity

Def: An interval I is called open in $\mathbb{R}$, iff

$$\forall_{x\in I} \exists_{\delta>0} : [x-\delta, x+\delta] \subset I$$

Def: An interval I is called closed, iff it's not open.
Note: This will become severely important in Analysis 2 .

Def: Let $f : I \to \mathbb{R}$ and I be open. g is called limit in a or $\lim_{x\to a} f(x)$, iff

$$(\mathrm{I})\forall_{(x_n)\subset I\setminus\{a\}} : \lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = g \iff$$
$$(\mathrm{II})\forall_{\epsilon>0} : \exists_{\delta>0} : \forall_{x\in I\setminus\{a\}} :\mid x-a \mid< \delta \implies \mid f(x) - g \mid< \epsilon$$

Proof:

(I) $\implies$ (II) For an arbitrary sequence (x_n) choose: $\forall_{\epsilon>0} :\mid f(x_{n_0}) - g \mid< \epsilon : \delta =\mid x_{n_0} - a \mid$

(II) $\implies$ (I) Let (x_n) be an arbitrary sequence with: $\lim\limits_{n\to\infty} x_n = a$

$\forall_{\epsilon>0} : \forall_{\delta>0} : \exists_{n_0\in\mathbb{N}} : \forall_{n>n_0} :\mid x_n - a \mid< \delta \implies \mid f(x_n) - g \mid< \epsilon \implies \lim\limits_{n\to\infty} f(x_n) = g$

Theorem: Let I be open and $f, g : I \to \mathbb{R}$, then:

$$\lim_{x\to a} f(x) = f, \lim_{x\to a} g(x) = g$$

$$\lim_{x\to a} f(x) + g(x) = f + g$$

$$\lim_{x\to a} f(x)g(x) = fg$$

$$\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{f}, f(x) \neq 0$$

This follows trivially from the limit theorems.

Def: A function f is continuous in a, iff $\lim_{x \to a} = f(a)$.
Def: A function f is continuous on an interval I, iff $\forall_{a \in I} \lim_{x \to a} = f(a)$
Theorem: For two continuous functions on I f, g: $f + g, fg, f \circ g$ are also continuous.
Proof:

$$f + g, \text{follows from the limit theorems}$$
$$fg, \text{follows from the limit theorems}$$
$$f \circ g, \forall_{(x_n)} : \lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} g(x_n) = g(a) \implies \lim_{n\to\infty} f(g(x_n)) = f(g(a))$$

Note: For a continuous and bijectiv function f and f^{-1} is also bijective, then f^{-1} is also continuous.
Proof:

$$\forall_{\epsilon>0} : \exists_{\delta>0} :\mid x - a \mid< \delta \implies \mid f(x) - f(a) \mid< \epsilon$$
$$\text{Set:}\forall_{\epsilon>0} : \exists_{n_0 \in \mathbb{N}} : \frac{1}{n_0} < \epsilon : \epsilon_1 = \min \left\{ \frac{1}{n}, \delta \right\}$$
$$\forall_{\epsilon_1>0} : \exists_{\epsilon>0} :\mid x - a \mid< \epsilon \implies \mid f^{-1}(x) - f^{-1}(a) \mid< \epsilon_1$$

Intermediate value theorem: Let $f : [a, b] \to \mathbb{R}$ and f be continuous on $[a, b]$:

$$f(a) < c < f(b) \implies \exists_{\phi \in (a,b)} : f(\phi) = c$$

Proof:

$$M := \{x \in [a, b] \mid f(x) \le c\}, a \in M \neq \emptyset$$
$$\phi := \sup M, \text{this exists because } M \text{ is bounded by } b$$
$$(x_n) \subset M : \lim_{n\to\infty} x_n = \phi \implies \lim_{n\to\infty} f(x_n) = f(\phi) \le c$$
$$N := \{x \in [a, b] \mid f(x) \ge c\}, b \in N \neq \emptyset$$
$$(y_n) \subset N : \lim_{n\to\infty} y_n = \phi \implies \lim_{n\to\infty} f(y_n) = f(\phi) \ge c$$

Max-Min theorem: Let f be continuous on $[a, b]$, then $\exists_{x_0, x_1 \in [a,b]} f(x_0) = \max_{x\in[a,b]} f(x) \wedge f(x_1) = \min_{x\in[a,b]} f(x)$
Proof:
Lemma: f is bounded on $[a, b]$

$$\text{Proposition: f is not bounded} \lim_{x\to x_0} f(x) = /, x_0 \in \mathbb{R}$$
$$\lim_{x\to x_0} f(x) = f(x) \perp$$

$$m := \sup f(x), m - \frac{1}{n} \le f(x_n) \le m$$
$$x_n \xrightarrow{n\to\infty} x_0 \implies f(x_n) \xrightarrow{n\to\infty} m$$
The minimum is constructed in the same way.

Def: A function is uniform continuous on $[a.b]$, iff

$$\forall_{\epsilon>0} : \forall_{x_0\in[a,b]} : \exists_{\delta>0} :\mid x - x_0 \mid < \delta \implies \mid f(x) - f(x_0) \mid < \epsilon$$

3.11 Derivatives

Def: A function $f : I \to \mathbb{R}$ is called differentiable in x, iff $f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ converges.
Note: A more generalized definition would be:

$$\lim_{n\to\infty} \frac{f(x_n) - f(x) - L(x_n - x)}{\mid x - x_n \mid} = 0,$$
$$L(a+b) = L(a) + L(b) \wedge L(\lambda a) = \lambda L(a)$$

Theorem: Let f be differentiable in x_0,l then f is continuous in x_0.
Proof:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \implies \lim_{h\to 0} f(x+h) - f(x) = 0 \implies \lim_{h\to 0} f(x+h) = f(x)$$

Lemma: The derivative is linear:

$$(f(x) + g(x))' = f'(x) + g'(x)$$
$$(\lambda f(x))' = \lambda f'(x)$$

Proof:

$$(f(x) + g(x))' = f'(x) + g'(x)$$
$$\lim_{h\to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} =$$
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)$$
$$(\lambda f(x))' = \lambda f'(x)$$
$$\lim_{h\to 0} \frac{\lambda f(x+h) - f(x)}{h} = f'(x)$$

Rules for differentiation:

$$(c)' = 0$$
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = 0$$

$$(f(x)g(x))' = f(x)g'(x) + f'(x)g(x)$$
$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} =$$
$$\lim_{h\to 0} \frac{f(x+h)g(x+h) + f(x)g(x+h) - f(x)g(x+h) - f(x)g(x)}{h} =$$
$$\lim_{h\to 0} \frac{g(x+h)(f(x+h) - f(x)) + f(x)(g(x+h) - g(x))}{h}$$

$$(f \circ g(x))' = f'(g(x))g'(x)$$
$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$
$$\lim_{x \to x_0} g(x) = z_0$$
$$D(z, z_0) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0}, z \neq z_0 \\ u'(z), z = z_0 \end{cases}$$

$$(c)' = 0$$
$$(x^n)' = nx^{n-1}$$
$$(exp(x))' = exp(x)$$
$$(f(x)g(x))' = f(x)g'(x) + f'(x)g(x)$$
$$(f \circ g(x))' = f'(g(x))g'(x)$$

<u>Def:</u> Let f be differentiable. f has a local maximum [minimum] in a, iff:

$$\exists_{\delta>0} : \forall_{x \in [a,b]} \mid x - a \mid < \delta \implies f(x) \leq f(a)$$
$$[\exists_{\delta>0} : \forall_{x \in [a,b]} \mid x - a \mid < \delta \implies f(x) \geq f(a)]$$

<u>Lemma:</u> Let f be differentiable and have a local maximum or minimum in a, then $f'(a) = 0$.
<u>Proof:</u>

Without loss of generality: let a be a local maximum

$$\forall_{x \in [a-\delta, a]} : \frac{f(x) - f(a)}{x - a} \leq 0$$
$$\forall_{x \in [a, a+\delta]} : \frac{f(x) - f(a)}{x - a} \geq 0$$

Let $(x_n) \subset [a - \delta, a + \delta]$ converge to a

$$\lim_{n \to \infty} \frac{f(x_n) - f(x)}{x_n - x} = 0$$

<u>Rolle's theorem:</u> Let f be differentiable on (a, b) and continuous on $[a, b]$ and $f(a) = f(b)$:

$$\exists_{\phi \in [a,b]} : f'(\phi) = 0$$

<u>Proof:</u>

Using the Max-Min-theorem:

$$\exists_{x_0, x_1 \in [a,b]} f(x_0) = \max_{x \in [a,b]} f(x) \wedge f(x_1) = \min_{x \in [a,b]} f(x)$$

If a is the local minimum then it cant be the local maximum except:

$$f(x) = c \implies f'(x) = 0$$

Without loss of generality assume that $x_0 \neq a, b$

$$\implies x_0 \text{ is local maximum} \implies f'(x_0) = 0$$

1. mean value theorem for differential calculus: Let f be differentiable on (a, b) and continuous on $[a, b]$, then:

$$\exists_{\phi\in(a,b)} : f'(\phi) = \frac{f(b) - f(a)}{b - a}$$

Proof:

$$h(x) := f(x) - \frac{f(b) - f(a)}{b - a}(b - a)$$
$$h(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = f(a)$$
$$h(a) = f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = f(a) = h(b)$$
$$\text{Rolle's theorem:}_{\phi\in(a,b)} : h'(\phi) = 0$$
$$h'(\phi) = f'(\phi) - \frac{f(b) - f(a)}{b - a} = 0$$

Theorem: Let a differentiable function be monotone increasing on (a, b), $\iff$ $f'(x) \geq 0$

Proof:

$$\frac{f(y) - f(x)}{y - x} \geq 0 \implies \lim_{y\to x} \frac{f(y) - f(x)}{y - x} \geq 0$$
$$\exists_{\phi\in(a,b)} : f'(\phi) = \frac{f(b) - f(a)}{b - a} \implies$$
$$f(b) - f(a) = f'(\phi)(b - a) \implies f(b) \geq f(a), \text{ because} f'(\phi) \geq 0$$

2. mean value theorem for differential calculus: Let f, g be differentiable on (a, b) and continuous on $[a, b]$, then

$$\exists_{\in(a,b)} : g'(\phi)(f(b) - f(a)) = f'(\phi)(g(b) - g(a))$$

Proof:

$$h(x) := (f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - f(b))$$
$$h(a) = h(b) = 0$$
$$\text{Using Rolle's theorem: } \exists_{\phi\in(a,b)} : h'(\phi) = 0$$
$$h'(x) = g'(\phi)(f(b) - f(a)) - f'(\phi)(g(b) - g(a))$$