

# Analysis 1

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## 1 Introduction

This is a general summary of the Analysis 1 course I attended. Prerequisites are the general knowledge of arithmetic's and proofing techniques.

## 2 Prerequisites

In this section I will go over two proofs necessary for the following course.

### 2.1 Bernoulli's inequality

Theorem:

$$(1+x)^n \geq 1+nx, n \in \mathbb{N}, x \geq -1$$

Proof:

This Proof will use induction.

Proposition:

$$\begin{aligned} n &= 0 \\ (1+x)^0 &= 1 \wedge 1+0x = 1 \\ \exists_{n \in \mathbb{N}} : (1+x)^n &\geq 1+x \end{aligned}$$

Inductive step:

$$\begin{aligned} (1+x)^{n+1} &= (1+x)(1+x)^n \geq (1+x)(1+nx) \\ (1+x)(1+nx) &= 1+x+nx+nx^2 = (1+(n+1)x) + nx^2 \\ nx^2 \geq 0 &\implies (1+x)^{n+1} \geq 1+(n+1)x, \forall n > 0, \forall x \in \mathbb{R} \end{aligned}$$

## 2.2 Binomial theorem

Theorem:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Proof:

Note that:

$$\begin{aligned} \binom{n}{k} \binom{n}{k-1} &= \binom{n+1}{k} \\ \binom{n}{k} &= 0, k > n, k < 0 \end{aligned}$$

Using induction.

Proposition:

$$n = 0(a + b)^0 = 1, \sum_{k=0}^0 \binom{n}{k} a^k b^{n-k} = \binom{0}{0} a^0 b^0 = 1$$

$$\exists_{n \in \mathbb{N}} : (a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Inductive Step:

$$\begin{aligned} (a + b)^{n+1} &= (a + b)(a + b)^n = (a + b) \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \\ &= a \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} + b \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^n \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k} \end{aligned}$$

$$\text{Set } \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k} = \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k} + \binom{n+1}{k} a^{n+1} b^0 = \sum_{k=0}^{n+1} \binom{n}{k} a^k b^{n+1-k}$$

$$\text{and } \sum_{k=0}^n \binom{n}{k} a^{k+1} b^{n-k} = \sum_{k=1}^{n+1} \binom{n}{k} a^k b^{n+1-k} = \sum_{k=0}^{n+1} \binom{n}{k} a^k b^{n+1-k}$$

$$\sum_{k=0}^{n+1} \binom{n}{k} \binom{n}{k-1} a^k b^{n+1-k} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}$$

### 3 Convergence of sequences and series

#### 3.1 Definitions

Def: A sequence  $(a_n)_{n \in \mathbb{N}}$  or in short form  $(a_n)$  is a function from the natural numbers to the real or rational numbers,  $a : \mathbb{N} \rightarrow \mathbb{R}$  or  $\mathbb{Q}$ .

Def: A sequence is bounded, iff:

$$\exists c \in \mathbb{R} : \forall n \in \mathbb{N} : |a_n| < c.$$

Def: A sequence is:

strictly monotone increasing, iff:  $\forall n \in \mathbb{N} : a_n < a_{n+1}$

strictly monotone decreasing, iff:  $\forall n \in \mathbb{N} : a_n > a_{n+1}$

monotone increasing, iff:  $\forall n \in \mathbb{N} : a_n \leq a_{n+1}$

monotone decreasing, iff:  $\forall n \in \mathbb{N} : a_n \geq a_{n+1}$

Def: A sequence is convergent:

$$\begin{aligned} \forall \epsilon > 0 : \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : |a_n - a| < \epsilon &\iff \\ \lim_{n \rightarrow \infty} a_n = a &\iff \\ a_n &\xrightarrow{n \rightarrow \infty} a \end{aligned}$$

Note: The Limit is unique.

Proof:

Suppose:

$$\lim_{n \rightarrow \infty} a_n = a \wedge \lim_{n \rightarrow \infty} a_n = b \wedge |a - b| \neq 0$$

$$\text{Set: } \epsilon = \frac{|a - b|}{4}$$

$$\exists n_1 \in \mathbb{N} : \forall n > n_1 : |a_n - a| < \epsilon \wedge \exists n_2 \in \mathbb{N} : \forall n > n_2 : |a_n - b| < \epsilon$$

$$\text{Choose: } n_0 = \max\{n_1, n_2\}$$

$$|a - b| = |a + (a_n - a) + b| \leq |a_n - a| + |a_n - b|$$

$$\forall n > n_0 : |a_n - a| + |a_n - b| \leq \epsilon + \epsilon$$

$$|a - b| \leq \frac{|a - b|}{2} \perp$$

Note: Every convergent sequence is bounded.

Proof:

$$\text{Set: } \epsilon = 1, \exists n_0 \in \mathbb{N} : \forall n > n_0 : |a_n - a| < 1 \quad (1)$$

$$\forall n \geq n_0 : |a_n| < |a| + 1 \quad (2)$$

$$c := \max\{|a_1|, |a_2|, \dots, |a_{n_0-1}|, |a| + 1\} \quad (3)$$

### 3.2 General arithmetic's for convergent sequences

For all convergent sequences  $(a_n), (b_n)$  holds:

$$\begin{aligned} \forall_{(a_n), (b_n)} : \lim_{n \rightarrow \infty} a_n = a \wedge \lim_{n \rightarrow \infty} b_n = b : \\ \lim_{n \rightarrow \infty} a_n + b_n = a + b \\ \lim_{n \rightarrow \infty} a_n b_n = ab \end{aligned}$$

Proof:

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n + b_n = a + b &\iff \\ \forall_{\epsilon > 0} : \exists_{n_0 \in \mathbb{N}} : \forall_{n \geq n_0} &| a_n + b_n - (a + b) | < \epsilon \\ | a_n + b_n - (a + b) | &\leq | a_n - a | + | b_n - b | \\ \exists_{n_1 \in \mathbb{N}} : \forall_{n \geq n_1} | a_n - a | < \frac{\epsilon}{2} &\wedge \exists_{n_2 \in \mathbb{N}} : \forall_{n \geq n_2} | b_n - b | < \frac{\epsilon}{2} \\ n_0 &:= \max\{n_1, n_2\} \\ \forall_{n \geq n_0} : | a_n - a | + | b_n - b | &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n b_n = ab &\iff \\ \forall_{\epsilon > 0} : \exists_{n_0 \in \mathbb{N}} : \forall_{n \geq n_0} &| a_n b_n - (ab) | < \epsilon \\ | a_n b_n - (ab) | &= | a_n b_n + (-ab_n + ab_n) - ab | = \\ | b_n(a_n - a) + a(b_n - b) | &\leq | (a_n - a)b_n | + | (b_n - b)a | = \\ | a_n - a | | b_n | + | b_n - b | &| a | \\ \text{Set: } c \text{ so } c > | b_n |, n \in \mathbb{N} & \\ \exists_{n_1 \in \mathbb{N}} : \forall_{n \geq n_1} | a_n - a | < \frac{\epsilon}{2c} &\wedge \exists_{n_2 \in \mathbb{N}} : \forall_{n \geq n_2} | b_n - b | < \frac{\epsilon}{2|a|} \\ n_0 &:= \max\{n_1, n_2\} \\ \forall_{n \geq n_0} : | a_n - a | | b_n | + | b_n - b | &| a | < | b_n | \frac{\epsilon}{2c} + | a | \frac{\epsilon}{2|a|} < \epsilon \end{aligned}$$

### 3.3 Cauchy series

Def:  $(a_n)$  is a Cauchy-series, iff

$$\forall_{\epsilon > 0} : \exists_{n_0 \in \mathbb{N}} : \forall_{n, m > n_0} : | a_n - a_m | < \epsilon$$

### 3.4 Construction of the real numbers

We are now able to define the real numbers as a Equivalence class of class of Cauchy series. We consider the set of all the Cauchy series  $X$  and define a

equivalence relation:

$$(a_n) \sim (b_n) \iff \lim_{n \rightarrow \infty} a_n - b_n = 0$$

and define the set of real numbers  $\mathbb{R}$  as all the sets of all equivalence classes  $S((a_n))$ :

$$S((a_n)) = \{(x_n) \in X, (x_n) \sim (a_n)\}$$

Showing that  $(a_n) \sim (b_n)$  is a equivalence relation:

$$\text{reflexivity: } (a_n) \sim (a_n), \forall (a_n) \in X$$

$$\text{symmetry: } (a_n) \sim (b_n) \implies (b_n) \sim (a_n), \forall (a_n), (b_n) \in X$$

$$\text{transitivity: } (a_n) \sim (b_n) \wedge (b_n) \sim (c_n) \implies (a_n) \sim (c_n), \forall (a_n), (b_n), (c_n) \in X$$

Proof:

$$\text{reflexivity: } a_n - a_n = 0 \xrightarrow{n \rightarrow \infty} 0$$

$$\text{symmetry: } a_n - b_n \xrightarrow{n \rightarrow \infty} 0 \iff -1(b_n - a_n) \xrightarrow{n \rightarrow \infty} -1(0) = 0$$

$$\text{transitivity: } a_n - b_n \xrightarrow{n \rightarrow \infty} 0 \wedge b_n - a_n \xrightarrow{n \rightarrow \infty} 0$$

$$a_n - c_n = a_n - b_n + b_n - c_n \xrightarrow{n \rightarrow \infty} 0$$

We have now shown that this is an equivalence relation and due to the arithmetic's for convergent series we show that all the axioms for the real numbers are given by this set.

Theorem: Every convergent sequence is a Cauchy sequence. Proof:

$$\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : |a_n - a| < \frac{\epsilon}{2} \wedge \forall m \geq n_0 : |a_m - a| < \frac{\epsilon}{2}$$

$$|a_n - a| + |a_m - a| < \epsilon$$

$$|a_n - a| + |a_m - a| = |a_n - a| + |a - a_m| \geq |a_n - a + a - a_m| < \epsilon$$

Theorem: In  $\mathbb{R}$  every Cauchy Sequence converges.

Proof: A proof will follow in the coming sections.

Note: This holds for all complete fields.

### 3.5 Subsequences and accumulation points

Def: A subsequence  $(a_{n_i})$  is a selection of elements of the sequence.

$$(a_{n_i}) = (a_{n_i})_{i \in \mathbb{N}}, \text{ with } n_{i+1} > n_i, \forall i \in \mathbb{N}$$

Accumulation Point: A number  $h$  is called an accumulation point, iff

$$\forall \epsilon > 0 : \forall n \in \mathbb{N} : \exists n_0 > n : |a_{n_0} - h| < \epsilon$$

Def: The maximum and minimum respectively of a set  $X$  is defined as:

$$\begin{aligned}\max_{x \in X} X &= a \iff \forall_{x \in X} : a \geq x \\ \min_{x \in X} X &= a \iff \forall_{x \in X} : a \leq x\end{aligned}$$

Def: The supremum and infimum of a set  $X$  is defined as:

$$\begin{aligned}\sup X &:= \max\{x \in \mathbb{R} \mid \forall_{n \in \mathbb{N}} : x \leq a_n\} \\ \inf X &:= \min\{x \in \mathbb{R} \mid \forall_{n \in \mathbb{N}} : x \geq a_n\}\end{aligned}$$

### 3.6 Infinite series

For a sequence  $(a_k)$  the sum  $\sum_{k=0}^{\infty} a_k$  is called a series.

Def: A series is called convergent, iff

$$\exists_{s \in \mathbb{R}} : \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k = s$$

For the following the sequence of partial sums  $S_n$  of a sequence  $(s_n)$  is defined as:

$$S_n := \sum_{k=0}^{\infty} s_k$$

Def: A sequence is absolutely convergent, iff  $(|S_n|)$  converges.

Theorem: If  $(S_n)$  converges  $(a_n)$  converges to zero.

Proof:

We know that  $(S_n)$  is a Cauchy sequence.

$$\begin{aligned}\forall_{\epsilon > 0} : \exists_{n_0 \in \mathbb{N}} : \forall_{n, m > n_0} : |S_n - S_m| < \epsilon \\ |S_n - S_m| &= \left| \sum_{k=m+1}^n s_k \right| < \epsilon \\ |a_n - 0| &= \left| \sum_{k=n}^n s_k \right| < \epsilon, \forall n > n_0\end{aligned}$$

Theorem: Comparison test: A series  $C_n$  is convergent, if an absolutely convergent series  $(A_n)$  exists with:

$$\forall_{n \in \mathbb{N}} : |c_n| < |a_n|$$

Proof:

$$\begin{aligned}\forall_{\epsilon > 0} : \exists_{n_0 \in \mathbb{N}} : \forall_{n, m > n_0} : |S_n - S_m| < \epsilon \\ |C_n - C_m| &< |A_n - A_m| < \epsilon\end{aligned}$$

Lemma: If  $S_n$  is absolutely convergent the two series  $S_n^+$  and  $S_n^-$  converge, with:

$$s_n^+ = \begin{cases} s_n, s_n > 0 \\ 0, s_n \leq 0 \end{cases}, s_n^- = \begin{cases} s_n, s_n < 0 \\ 0, s_n \geq 0 \end{cases}$$

$$S_n^+ = \sum_{k=0}^n s_k^+, S_n^- = \sum_{k=0}^n s_k^-$$

Proof:

$$|s_n^+| \leq |s_n|, |s_n^-| \leq |s_n|$$

Riemann rearrangement theorem: For an absolutely convergent series, if you rearrange the order of the elements ( $s_n$ ) the limit does not change.

$$\sum_{k=0}^{\infty} s_n = \sum_{k=0}^{\infty} s_{\sigma(n)}, \sigma(n) : \mathbb{N} \rightarrow \mathbb{N}, \sigma \text{ bijective}$$

Proof:

$$\lim_{n \rightarrow \infty} S_{\sigma(n)}^+ = \lim_{n \rightarrow \infty} S_n^+$$

$$\lim_{n \rightarrow \infty} S_{\sigma(n)}^- = \lim_{n \rightarrow \infty} S_n^-$$

$$\lim_{n \rightarrow \infty} S_n^+ + S_n^- = \lim_{n \rightarrow \infty} S_n$$

$$\lim_{n \rightarrow \infty} S_{\sigma(n)}^+ + S_{\sigma(n)}^- = \lim_{n \rightarrow \infty} S_n^+ + S_n^- = \lim_{n \rightarrow \infty} S_n$$

### 3.7 Evaluation of infinite series

Comparison test: A series  $C_n$  is convergent, if an absolutely convergent series ( $A_n$ ) exists with:

$$\forall n \in \mathbb{N} : |c_n| < |a_n|$$

Generalized ratio test: A series ( $A_n$ ) absolutely converges, if

$$\exists_{q \in (0,1)} : \exists_{n_0 \in \mathbb{N}} : \forall_{n > n_0} : \left| \frac{a_{n+1}}{a_n} \right| \leq q$$

Root test:

A series ( $A_n$ ) converges absolutely, if

$$\exists_{q \in (0,1)} : \exists_{n \in \mathbb{N}} \forall_{n > n_0} : \sqrt[n]{|a_n|} \leq q$$

Leibniz test:

If ( $a_n$ ) is monoton decreasing and converges to 0, than the series  $S_n$  converges, with:

$$S_n := \sum_{k=0}^n (-1)^k a_k$$

Proof for generalized ratio test:

Lemma:  $\sum_{k=0}^n q^k, q \in (-1, 1)$  converges to  $\frac{1}{1-q}$

Proposition:

$$\sum_{k=0}^0 q^k = q^0 = 1 \wedge \frac{1 - q^{n+1}}{1 - q} = \frac{1 - 0^1}{1 - 0} = 1$$

$$\exists_{n \in \mathbb{N}} : \sum_{k=0}^n q^k = \frac{1 - q^{n+1}}{1 - q}$$

Inductive step:

$$\sum_{k=0}^{n+1} q^k = \sum_{k=0}^n q^k + q^{n+1} = \frac{1 - q^{n+1}}{1 - q} + q^{n+1} =$$

$$\frac{1 - q^{n+1} + (1 - q)q^{n+1}}{1 - q} = \frac{1 - q^{n+1} + q^{n+1} - q^{n+2}}{1 - q}$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n = \lim_{n \rightarrow \infty} \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q}$$

$$\forall_{n > n_0} \left| \frac{a_{n+1}}{a_n} \right| \leq q$$

$$\left| a_n \right| \leq q \left| a_{n-1} \right| \implies \left| a_n \right| \leq q^{n-n_0} \left| a_{n_0} \right|$$

$$\sum_{k=0}^{\infty} \left| a_k \right| \leq \left| A_{n_0} \right| + \sum_{k=0}^{\infty} \left| A_{n_0} \right| q^k = \left| A_{n_0} \right| + \left| A_{n_0} \right| \sum_{k=0}^{\infty} q^k$$

Which converges.

Proof for the root test:

$$\exists_{q \in (0,1)} : \exists n_0 \in \mathbb{N} : \forall_{n > n_0} : \sqrt[n]{\left| a_n \right|} \leq q$$

$$\sqrt[n]{\left| a_n \right|} \leq q \iff \left| a_n \right| \leq q^n$$

$$\sum_{k=0}^{\infty} \left| a_k \right| \leq \left| A_{n_0} \right| + \sum_{k=0}^{\infty} q^k$$

Proof of the Leibniz test:

$$S_{2n-1} = \sum_{k=0}^{2n-1} (-1)^k s_k = \sum_{k=0}^{n-1} (s_{2k+1} - s_{2k}) > 0$$

$$S_{2n-1} = \sum_{k=0}^{2n-1} (-1)^k s_k = s_0 + \sum_{k=1}^{n-1} (s_{2k} - s_{2k-1} - s_{n-1}) < s_0$$

We know know that  $(S_n)$  is bounded and monoton decreasing, so it must converge.



### 3.8 Construction of the exponential function