Computational Fluid Dynamics
(AE 706)

Computer Assignment 4:

Numerical Solution of Shock Tube Problem using van Leer Flux Vector Splitting Method

By

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Abstract:

This report presents a numerical study of the shock tube problem using the van Leer Flux Vector Splitting (FVS) method to solve the 1D Euler equations. The shock tube problem is a classic test case in computational fluid dynamics (CFD) where the goal is to simulate the evolution of waves and discontinuities resulting from a sudden release of high-pressure gas into a low-pressure region separated by a diaphragm. The study compares the numerical results obtained using the van Leer method with exact solutions to validate the accuracy of the numerical simulation. The initial conditions for the shock tube problem consist of two uniform states separated by the diaphragm. One side of the tube has higher pressure compared to the other, leading to the formation of shock waves, contact surfaces, and rarefaction waves. The exact solution is known for this problem, allowing for a rigorous comparison with the numerical results.

The numerical simulation utilizes a grid-based approach with N = 201 equidistant grid points spanning the domain [0, 1]. The Euler equations in conservative form are discretized using the van Leer FVS method, which involves computing split fluxes based on the local Mach number at each grid point. The simulation is advanced in time using a stable time step calculated from the CFL condition to ensure numerical stability. The main objective of this study is to plot and analyze the variations of pressure, temperature, velocity, and Mach number inside the shock tube at a specified time of 0.75 × 10^-3 seconds after the diaphragm rupture. The results obtained from the numerical simulation are compared with exact solution data provided in the `exact-solution-shock-tube.xlsx` file to evaluate the accuracy and performance of the implemented numerical method. The report concludes with discussions on the observed results, including insights into the behavior of shock waves and wave interactions within the shock tube. The comparison between numerical and exact solutions provides valuable validation and verification of the numerical method's effectiveness in capturing the complex flow physics of the shock tube problem.

Problem Statement:

Given the initial conditions as shown in Figure 1, find out the variations of normalized numerical values of pressure, temperature, velocity and Mach number inside the shock tube at 0.75×10^{-3} seconds after the diaphragm is ruptured. Apply van-Leer (VL) Flux Vector Splitting method [1] to obtain the numerical solution. Compare the numerical results with the exact solution data uploaded (as exact-solution-shock-tube.xlsx) in moodle

Domain Specification and Discretization:

Consider N=201 equally spaced grid points in the domain x = [0, 1]. You may consider, one additional grid (ghost) points on either sides for applying boundary conditions. Thus, i=0 and i=N+1 will be the ghost points in the left and right boundaries respectively. Thus, you need to consider total 203 grid points. Typical grids to be used is shown in the Figure 2

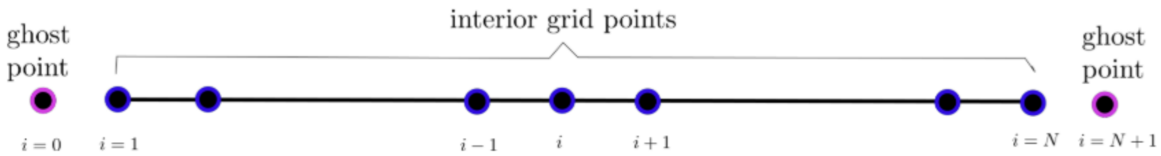


Figure 2: Typical equi-spaced grid

Governing Equation:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Where

$$U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}$$

$$F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix}$$

These equations can be written in a split flux form as:

$$\frac{\partial U}{\partial t} + \frac{\partial F^+}{\partial x} + \frac{\partial F^-}{\partial x} = 0$$

Implemented Schemes: (Applying first-order upwind scheme)

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(F_i^{+n} - F_{i-1}^{+n}) - \frac{\Delta t}{\Delta x}(F_{i+1}^{-n} - F_i^{-n})$$

Split Fluxes for van Leer Method:

For $M \leq -1$

$$\boldsymbol{F}_{VL}^{+} = 0 \qquad \text{and} \qquad \boldsymbol{F}_{VL}^{-} = \boldsymbol{F}$$

For $-1 < M < 1$

$$\boldsymbol{F}_{VL}^{+} = \frac{1}{4}\rho a(M+1)^2 \begin{bmatrix} 1 \\ \frac{2a}{\gamma}\left(1+\frac{\gamma-1}{2}M\right) \\ \frac{2a^2}{\gamma^2-1}\left(1+\frac{\gamma-1}{2}M\right)^2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{F}_{VL}^{-} = \boldsymbol{F} - \boldsymbol{F}_{VL}^{+}$$

For $1 < M$

$$\boldsymbol{F}_{VL}^{+} = \boldsymbol{F} \qquad \text{and} \qquad \boldsymbol{F}_{VL}^{-} = \boldsymbol{0}$$

Initial Conditions:

Temperature and velocity at all grid points are set to 300K and zero respectively. Whereas, pressure is set to 5 atm at the points located in the range $0 \leq x < 0.5$ and 1 atm in the range $0.5 \leq x \leq 1$, because the diaphragm is located at x=0.5. Above mentioned conditions are explained in Figure 1**.**

Boundary Conditions:

Since the waves will not reach the boundaries in the time 0.75×10^{-3} seconds, following boundary conditions can be specified.

At Left Boundary

$$p[0] = p[1]$$

$$T[0] = T[1]$$

$$u[0] = u[1]$$

At Right Boundary

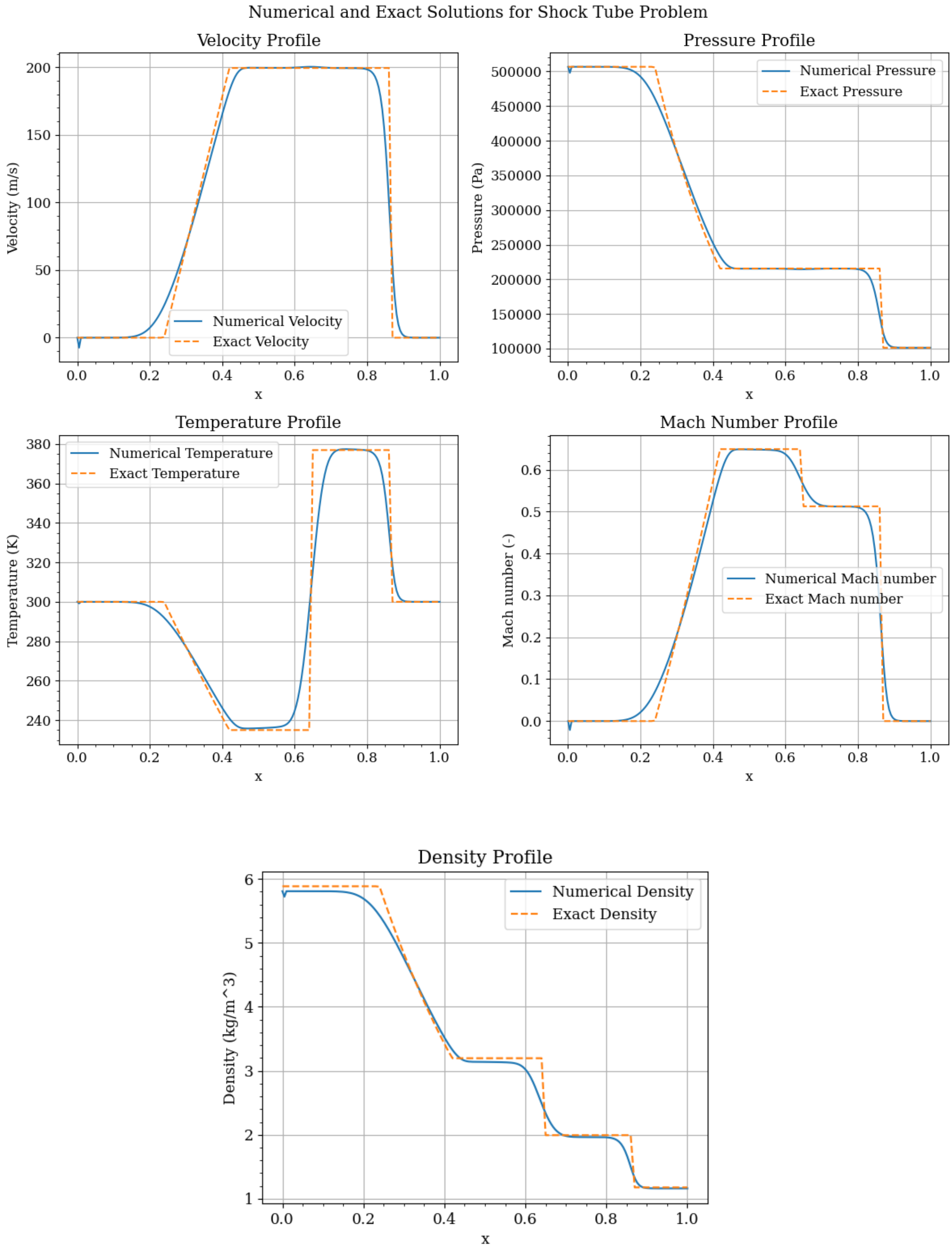
$$\mathrm{p[N+1] = p[N]}$$

$$\mathrm{T[N+1] = T[N]}$$

$$\mathrm{u[N+1] = u[N]}$$

Results:

Comparison between Exact and Numerical solutions



The numerical solution has been obtained for a CFL value of 0.1

Conclusions:

- The shock wave can be found at x = 0.87 m (approximately).
- The rarefaction wave can be found at x = 0.3 m (approximately).
- The contact wave can be found at x = 0.65 m (approximately). Pressure and velocity remain constant across the contact wave.
- In the exact solution the change across the state variables is more sudden across the shock and contact wave as compared to the rarefaction wave.
- The numerical solution captures the change across shock wave more accurately as compared to the other two waves. The change across the contact wave is the least accurate amongst all in the numerical solution. It is the most sluggish in nature.

Note:

Flowchart, zip file of code repository and README.txt has been provided with instructions on running the code.