Monte Carlo Variance Scaling

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$E_{X\sim P}\left[\frac{f(X)}{p(X)}\right] = \int_a^b \frac{f(x)}{p(x)}p(x)dx = \int_a^b f(x)\,dx$ given that P has the support $[a,b]$. We use the estimator $\hat{I} = \langle\frac{f(x)}{p(x)}\rangle_{x\sim P} = \frac{1}{n}\sum_{i=1}^n \frac{f(x_i)}{p(x_i)}$ where x_i are drawn from the P distribution.

The expected value of the estimator is

$$\begin{aligned} E\left[\hat{I}\right] &= E\left[\frac{1}{n}\sum_{i=1}^n \frac{f(x_i)}{p(x_i)}\right] \\ &= \frac{1}{n}\sum_{i=1}^n E\left[\frac{f(x_i)}{p(x_i)}\right] \\ &= \frac{1}{n}nE\left[\frac{f(X)}{p(X)}\right] \\ &= E_{X\sim P}\left[\frac{f(X)}{p(X)}\right] \end{aligned}$$

The variance of the estimator is

$$\begin{aligned} Var\left[\hat{I}\right] &= Var\left[\frac{1}{n}\sum_{i=1}^n \frac{f(x_i)}{p(x_i)}\right] \\ &= \frac{1}{n^2}\sum_{i=1}^n Var\left[\frac{f(x_i)}{p(x_i)}\right] \\ &= \frac{1}{n^2}nVar\left[\frac{f(X)}{p(X)}\right] \\ &= \frac{1}{n}Var\left[\frac{f(X)}{p(X)}\right] \end{aligned}$$

which we can in turn estimate as

$$S^2\left[\hat{I}\right] = \frac{1}{n}S^2\left[\frac{f(x_i)}{p(x_i)}\right]$$

The variance of the estimation points are

$$Var\left[\frac{f(X)}{p(X)}\right]$$

which we estimate as

$$S^2\left[\frac{f(x_i)}{p(x_i)}\right]$$

Now, if we scale our estimator by a constant C, that is, $\hat{I}' = \frac{C}{n}\sum_{i=1}^{n}\frac{f(x_i)}{p(x_i)}$, its variance becomes

$$Var\left[\hat{I}'\right] = Var\left[C\hat{I}\right] = C^2 Var\left[\hat{I}\right]$$

and its estimator

$$S^2\left[\hat{I}'\right] = C^2 S^2\left[\hat{I}\right] = \frac{C^2}{n} S^2\left[\frac{f(x_i)}{p(x_i)}\right]$$

If we consider the estimation points also scaled by C, that is, $\hat{I}' = \frac{1}{n}\sum_{i=1}^{n}\frac{Cf(x_i)}{p(x_i)}$, the variance of the estimation points becomes

$$Var\left[C\frac{f(X)}{p(X)}\right] = C^2 Var\left[\frac{f(X)}{p(X)}\right]$$

and its estimator

$$C^2 S^2\left[\frac{f(x_i)}{p(x_i)}\right]$$