p-Norm is a Norm

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Abstract

We show that the p-norm in $\mathbb{R}^n$ is indeed a norm. We state and prove the "magic norm" theorem, and we analyse a certain type of a single variable function to help prove that p-norm is a norm. Then, we apply our p-norm theorem on $\mathbb{R}^n$ to prove the p-norms on the spaces of sequences and continuous functions.

§1 Introduction

When working with the real normed vector space $\mathbb{R}^n$, we commonly use the Euclidean norm or the 2-norm, $\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, as it is how distance is measured in the real world. Less commonly we use the taxicab norm or 1-norm, $\|\vec{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$, and the max-norm, $\|\vec{x}\|_{\max} = \max(|x_1|, |x_2|, \ldots, |x_n|)$ which are generally simpler to work with. We generalize the 1-norm and the 2-norm to the p-norm with $p > 0$:

$$\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$$

It is fairly simple to prove positive definiteness and homogeneity for the p-norm, however, proving the triangle inequality is difficult. In this paper, we prove the following theorem:

Theorem 1.1 (p-Norm on $\mathbb{R}^n$)**.** *For $n \geq 2$, the function $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}$, $\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$ is a norm if and only if $p \geq 1$.*

Theorem 1.1 is proven using the magic norm theorem and some lemmas about differentiable single variable functions with non-decreasing derivative. The magic norm theorem is proven in §2. The lemmas for functions with non-decreasing derivatives are proven in §3. Theorem 1.1 is proven in §4.

§2 Magic Norm Theorem

In this section, we state and prove the Magic Norm Theorem. Given a subset on a normed vector space with certain properties, the Magic Norm Theorem proves the existence of a norm on this vector space such that the given subset is the closed unit ball.

We first define three properties on subsets of a real normed vector space $(X, \|\cdot\|)$.

Definition 2.1 (Absorbing)**.** $B \subseteq X$ is absorbing if and only if for all $\vec{x} \in X$ there exists $t > 0$ so that $\vec{x} = t\vec{b}$ for some $\vec{b} \in B$.

Definition 2.2 (Balanced)**.** $B \subseteq X$ is balanced if and only if for all $\vec{b} \in B$, we have $t\vec{b} \in B$ for all $t \in [-1, 1]$.

Definition 2.3 (Convex)**.** $B \subseteq X$ is convex if and only if for all $\vec{x}, \vec{y} \in B$, we have $t\vec{x} + (1-t)\vec{y} \in B$ for all $t \in [0, 1]$.

Also, recall that in a finite dimensional normed vector space, all norms are equivalent. This means that if a set is closed and bounded over one norm, it must be closed and bounded over all norms.

Now, we can state and prove the Magic Norm Theorem.

Theorem 2.4 (Magic Norm Theorem)**.** *Let X be a finite dimensional normed vector space, and let $B \subseteq X$. Then, there exists a norm $\|\cdot\| : X \to \mathbb{R}$ so that $B = \{\vec{x} \in X : \|\vec{x}\| \leq 1\}$ if and only if B is absorbing, balanced, convex, closed, and bounded.*

Proof. We first prove the forward direction. Suppose there exists a norm $\|\cdot\| : X \to \mathbb{R}$ so that $B = \{\vec{x} \in X : \|\vec{x}\| \leq 1\}$. Next, we show that B satisfies the 5 properties.

Absorbing: Let $\vec{X} \in X$. If $\vec{x} = \vec{0}$, then we take $\vec{b} = \vec{0}$ which is in B since $\|\vec{0}\| = 0 \leq 1$. This means that $\vec{x} = \vec{0} = t\vec{b}$ for any $t > 0$. If $\vec{x} \neq \vec{0}$, we can take $\vec{b} = \frac{1}{\|\vec{x}\|}\vec{x}$ and $t = \|\vec{x}\| > 0$. We then get $t\vec{b} = \|\vec{x}\|\frac{1}{\|\vec{x}\|}\vec{x} = \vec{x}$. In both cases, we are able to write $\vec{x} = t\vec{b}$ for some $t > 0$ and $\vec{b} \in B$, hence, B is absorbing.

Balanced: Let $\vec{b} \in B$ and $t \in [-1, 1]$. Then, $\|\vec{b}\| \leq 1$ and $|t| \leq 1$, so

$$\|t\vec{b}\| = |t|\|\vec{b}\| \leq 1$$

Thus, $t\vec{b} \in B$, and hence, B is balanced.

Convex: Let $\vec{x}, \vec{y} \in B$ and $t \in [0, 1]$. Then, $\|x\|, \|y\| \leq 1$, $|t| = t$, and $|1 - t| = 1 - t$. Using the triangle inequality and homogeneity, we get

$$\|t\vec{x} + (1-t)\vec{y}\| \leq \|t\vec{x}\| + \|(1-t)\vec{y}\| = t\|\vec{x}\| + (1-t)\|\vec{y}\| \leq t + 1 - t = 1$$

Thus, $t\vec{x} + (1-t)\vec{y} \in B$, and hence, B is convex.

Closed: We will show that B is closed over $\|\cdot\|$. Let $\vec{x} \in \overline{B}$. Then, for all $\varepsilon > 0$, there exists $\vec{p}_\varepsilon \in B \cap B(\vec{x}, \varepsilon) \Rightarrow \|\vec{p}_\varepsilon\| \leq 1$ and $\|\vec{p}_\varepsilon - \vec{x}\| < \varepsilon$. Then, we get

$$\|\vec{x}\| \leq \|\vec{x} - \vec{p}_\varepsilon\| + \|\vec{p}_\varepsilon\| < 1 + \varepsilon$$

Letting $\varepsilon \to 0$, we get $\|\vec{x}\| < 1 + \varepsilon \Rightarrow \|\vec{x}\| \leq 1 \Rightarrow \vec{x} \in B$. Thus, B is closed.

Bounded: Let $\vec{x}, \vec{y} \in B$. Then,

$$\|\vec{x} - \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \leq 2$$

Thus, distances between points in B are bounded, and hence, B is bounded.

Now, we prove the converse direction. Let B be absorbing, balanced, convex, closed, and bounded. We define the function $N : X \to \mathbb{R}$,

$$N(\vec{x}) = \inf\left\{\frac{1}{t} : t > 0 \text{ and } t\vec{x} \in B\right\}$$

Notice that this set is bounded below by 0. Furthermore, notice that since B is absorbing, for any $\vec{x} \in X$, there exists $t > 0$ and $\vec{b} \in B$ so that $\vec{x} = t\vec{b} \Rightarrow \frac{1}{t}\vec{x} = \vec{b} \in B$. This means that t is an element of the set used in the definition of N, hence that set is non-empty. Thus, the infimum exists for all $\vec{x} \in X$, and so N is well-defined. Next, we show that N is a norm.

Positive Definiteness: Clearly, we have $N(\vec{x}) = \inf\{\frac{1}{t} : t > 0 \text{ and } t\vec{x} \in B\} \geq 0$ for all $\vec{x} \in X$ since the set only contains positive values. Also, $N(\vec{0}) = \inf\{\frac{1}{t} : t > 0 \text{ and } t\vec{0} \in B\} = \inf\{\frac{1}{t} : t > 0\} = 0$. Now, assume $N(\vec{x}) = 0$. Then, we can take a sequence (t_n) in $\mathbb{R}$ such that $t_n > 0$ and $t_n\vec{x} \in B$ for all $n \in \mathbb{N}$ and $\frac{1}{t_n} \to 0$ as $n \to \infty$. This means that $t_n \to \infty$ while $t_n\vec{x} \in B$. Notice that if $\vec{x} \neq \vec{0}$, then we would have that B is unbounded, hence we must have $\vec{x} = \vec{0}$. Therefore, $N(\vec{x}) \geq 0$ for all $\vec{x} \in X$ and $N(\vec{x}) = 0 \iff \vec{x} = \vec{0}$, so N is positive definite.

Homogeneity: Let $c \in \mathbb{R}$ and $\vec{x} \in X$. If $c = 0$, then $N(c\vec{x}) = N(0\vec{x}) = 0 = 0 \cdot N(\vec{x}) = |c|N(\vec{x})$, hence N is homogeneous in this case. Now, take $c \neq 0$. Also, notice that since B is balanced,

$t \cdot c\vec{x} \in B \iff -t \cdot c\vec{x} \in B$ for $t > 0$, so we can replace $t \cdot c\vec{x} \in B$ with $t \cdot |c|\vec{x} \in B$. Now,

$$\begin{aligned} N(c\vec{x}) &= \inf\left\{\frac{1}{t} : t > 0 \text{ and } t \cdot c\vec{x} \in B\right\} \\ &= \inf\left\{\frac{1}{t} : t \cdot |c| > 0 \text{ and } t \cdot |c|\vec{x} \in B\right\} \\ &= |c| \inf\left\{\frac{1}{t \cdot |c|} : t \cdot |c| > 0 \text{ and } t \cdot |c|\vec{x} \in B\right\} \\ &= |c| \inf\left\{\frac{1}{t} : t > 0 \text{ and } t\vec{x} \in B\right\} \\ &= |c| \cdot N(\vec{x}) \end{aligned}$$

Thus, N is homogeneous.

Triangle Inequality: Let $\vec{x}, \vec{y} \in X$. Then, by definition of N, we can take sequences $(t_n), (s_n)$ in $\mathbb{R}$ so that for all $n \in \mathbb{N}$ $t_n, s_n > 0$ and $t_n\vec{x}, s_n\vec{y} \in B$, and $\frac{1}{t_n} \to N(\vec{x})$ and $\frac{1}{s_n} \to N(\vec{y})$. Notice that for all $n \in \mathbb{N}$, we can take $t = s_n/(t_n + s_n) \in [0, 1]$ and apply convexity of B to get the following:

$$t_n\vec{x}, s_n\vec{y} \in B \Longrightarrow t\vec{x} + (1-t)\vec{y} \in B \Longrightarrow \frac{s_n}{t_n + s_n} t_n\vec{x} + \left(1 - \frac{s_n}{t_n + s_n}\right) s_n\vec{y} = \frac{t_n s_n}{t_n + s_n}(\vec{x} + \vec{y}) \in B$$

This means that $(t_n s_n/(t_n + s_n))^{-1} \in \{\frac{1}{t} : t > 0 \text{ and } t(\vec{x} + \vec{y})\}$, so the infimum of this set must be less than or equal to $(t_n s_n/(t_n + s_n))^{-1}$. Thus, we have

$$N(\vec{x} + \vec{y}) = \inf\left\{\frac{1}{t} : t > 0 \text{ and } t(\vec{x} + \vec{y}) \in B\right\} \leq \left(\frac{t_n s_n}{t_n + s_n}\right)^{-1} = \frac{1}{t_n} + \frac{1}{s_n} \to N(\vec{x}) + N(\vec{y})$$

Therefore, N has the triangle inequality property.

Since N is positive definite, homogeneous, and has triangle inequality, it must be a norm. Hence, we define our norm as $\|\cdot\| = N$. Now, we show that $B = \{\vec{x} \in X : \|\vec{x}\| \leq 1\}$ by showing both subset inclusions.

Let $\vec{x} \in B$. Then, we have that $1 \in \{\frac{1}{t} : t > 0 \text{ and } t\vec{x} \in B\}$, so $\|\vec{x}\| = \inf\{\frac{1}{t} : t > 0 \text{ and } t\vec{x} \in B\} \leq 1$. Hence, $B \subseteq \{\vec{x} \in X : \|\vec{x}\| \leq 1\}$.

Now, let $\vec{x} \in X$ so that $\|\vec{x}\| \leq 1$. We will consider the cases $\|\vec{x}\| = 0$, $\|\vec{x}\| = 1$, and $0 < \|\vec{x}\| < 1$ separately.

Case 1: $\|\vec{x}\| = 0$. Then, we must have $\vec{x} = 0$ by positive definiteness. Taking any $\vec{b} \in B$ and $t = 0 \in [-1, 1]$, we apply the balanced property of B to get $t\vec{b} = \vec{0} = \vec{x} \in B$.

Case 2: $\|\vec{x}\| = 1$. Then, by definition of $\|\cdot\|$, we can take a sequence (t_n) in $\mathbb{R}$ so that $t_n > 0$ and $t_n\vec{x} \in B$ for all $n \in \mathbb{N}$ and $\frac{1}{t_n} \to 1$. Notice that $\frac{1}{t_n} \to 1$ is equivalent to $t_n \to 1$. In turn, this implies that $t_n\vec{x} \to \vec{x}$. Since $t_n\vec{x} \in B$ for all $n \in \mathbb{N}$, we have that $\vec{x}$ is a limit point of B. Since B is closed, $\vec{x} \in B$.

Case 3: $0 < \|\vec{x}\| < 1$. Then, taking $c = \frac{1}{\|\vec{x}\|} > 0$, notice that $\|c\vec{x}\| = 1$. As we have shown above, this implies that $c\vec{x} \in B$. Now, since $\frac{1}{c} = \|\vec{x}\| \in [-1, 1]$, we can apply the balanced property of B to get that $\frac{1}{c}c\vec{x} = \vec{x} \in B$.

Since in all cases we get $\vec{x} \in B$, we must have $\{\vec{x} \in X : \|\vec{x}\| \leq 1\} \subseteq B$.

Therefore, there exists a norm $\|\cdot\| : X \to \mathbb{R}$ so that $B = \{\vec{x} \in X : \|\vec{x}\| \leq 1\}$.

$\square$

The Magic Norm Theorem can be used to prove 1.1 by showing that the closed unit ball that the p-norm would create has certain properties, rather than showing that the p-norm has the norm properties. This is done in §4.

§3 Differentiable Functions with Non-Decreasing Derivative

In this section, we prove a technical lemma (3.1) for an oddly specific type of function. Note that this section deals with single-variable functions, so we use the classic, single-variable notion of the derivative.

Lemma 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and twice differentiable on $\mathbb{R}$ except finitely many points. Furthermore, suppose that for all $t \in \mathbb{R}$ for which f is twice differentiable, $f''(t) \geq 0$, and that $\lim_{t\to-\infty} f'(t) = -\infty$ and $\lim_{t\to\infty} f'(t) = \infty$. Then, there exists a point $t_0 \in \mathbb{R}$ so that f is non-increasing on $(-\infty, t_0]$ and non-decreasing on $[t_0, \infty)$.*

To clarify, we define non-increasing and non-decreasing in the following way:

Definition 3.2 (Non-Increasing)**.** Let $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}$. Then, f is non-increasing on $V \subseteq U$ if and only if for all $x, y \in V$, $x < y \Rightarrow f(x) \geq f(y)$.

Definition 3.3 (Non-Decreasing)**.** Let $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}$. Then, f is non-decreasing on $V \subseteq U$ if and only if for all $x, y \in V$, $x < y \Rightarrow f(x) \leq f(y)$.

Before proving Lemma 3.1, we prove two additional proposition.

Proposition 3.4. *Let $f : I \to \mathbb{R}$ be differentiable where $I \subseteq \mathbb{R}$ is an interval. Suppose that $f'(t) \leq 0$ for all $t \in I$. Then, f is non-increasing on its domain.*

Proof. Let $x, y \in I$ and assume $x < y$. Since I is an interval and $x, y \in I$, we must have $[x, y] \subseteq I$. Furthermore, f is continuous and differentiable on $[x, y]$. By the classic Mean Value Theorem, there exists $c \in (x, y)$ so that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \leq 0$$

Since $x < y$, we have $y - x > 0$, so $\frac{f(y)-f(x)}{y-x} \leq 0 \Rightarrow f(y) - f(x) \leq 0 \Rightarrow f(x) \geq f(y)$. Therefore, f is non-increasing.

□

Proposition 3.5. *Let $f : I \to \mathbb{R}$ be differentiable where $I \subseteq \mathbb{R}$ is an interval. Suppose that $f'(t) \geq 0$ for all $t \in I$. Then, f is non-decreasing on its domain.*

Proof. We define the function $g : I \to \mathbb{R}$, $g(t) = -f(t)$. Since f is differentiable on I, g must also be differentiable. Furthermore, for all $t \in I$, we have $g'(t) = -f'(t) \leq 0$. Applying Proposition 3.4, we have that for all $x, y \in I$, $x < y \Rightarrow g(x) \geq g(y) \Rightarrow -f(x) \geq -f(y) \Rightarrow f(x) \leq f(y)$. Therefore, f is non-decreasing.

□

Now, we can prove Lemma 3.1.

Proof (Lemma 3.1). We take $\{t_1, \ldots, t_n\} \subseteq \mathbb{R}$ to be the points where f is not twice continuously differentiable. Then, we define the intervals $I_0 = (-\infty, t_1)$, $I_j = (t_j, t_{j+1})$ for $1 \leq j \leq n-1$, and $I_n = (t_n, \infty)$. We define functions $f_j : I_j \to \mathbb{R}$, $f_j(t) = f'(t)$, for $0 \leq j \leq n$. Notice that all f_j's are differentiable since f is twice differentiable on $\mathbb{R} \setminus \{t_1, \ldots, t_n\}$ and I_j's exclude all t_i's. Furthermore, for all $0 \leq j \leq n$, $f_j'(t) = f''(t) \geq 0$. Thus, applying Proposition 3.5, we have that f_j is non-decreasing for all $0 \leq j \leq n$. Hence, f' is non-decreasing on each of I_j's individually.

Furthermore, we can show that I_j's can be extended to their closures while keeping f' non-decreasing on that I_j. Take $j \in \{1, \ldots, n\}$ and consider t_j and I_{j-1}. Take $x \in I_{j-1}$, and note that $(x, t_j) \subseteq I_{j-1}$. Then, for all $y \in (x, t_j)$, $f'(x) \leq f'(y) \to f'(t_j)$ as $y \to t_j^-$ since f' is continuous. Thus, f' is non-decreasing on $I_{j-1} \cup \{t_j\}$. Now, consider I_j. Take $y \in I_j$, and note that $(t_j, y) \subseteq I_j$. Then, for all $x \in (t_j, y)$, $f'(y) \geq f'(x) \to f'(t_j)$ as $x \to t_j^+$ since f' is continuous. Thus, f' is non-decreasing on $\{t_j\} \cup I_j$.

Now, we take $x, y \in \mathbb{R}$ and assume $x < y$. If $x \in I_n = (t_n, \infty)$, then also $y \in I_n$. Since f' is non-decreasing on I_n, we have $f'(x) \leq f'(y)$ in this case. Otherwise, we take $1 \leq i \leq n-1$ so that $x \in I_i \cup \{t_{i+1}\}$. If $y \in I_i \cup \{t_{i+1}\}$, then since f' is non-decreasing on $I_i \cup \{t_{i+1}\}$, we have $f'(x) \leq f'(y)$ in this case as well. Now, we can take $i+1 \leq j \leq n$ so that $y \in \{t_j\} \cup I_j$. Since f' is non-decreasing on $I_{k-1} \cup \{t_k\}$ and $\{t_k\} \cup I_k$ for all $1 \leq k \leq n$, we have

$$f'(x) \leq f'(t_{i+1}) \leq f'(c_{i+1}) \leq f'(t_{i+2}) \leq \cdots \leq f'(c_{j-1}) \leq f'(t_j) \leq f'(y)$$

where c_k is some element of I_k. Thus, in all cases, $f'(x) \leq f'(y)$, hence f' is non-decreasing on $\mathbb{R}$.

Since $\lim_{t \to -\infty} f'(t) = -\infty$ and $\lim_{t \to \infty} f'(t) = \infty$, there are some $a, b \in \mathbb{R}$ so that $f'(a) < 0$ and $f'(b) > 0$. Since f' is continuous on $\mathbb{R}$, by the single variable Intermediate Value Theorem, there exists some $t_0 \in (a, b)$ so that $f'(t_0) = 0$. Since f' is non-decreasing, we have that for all $x \in (-\infty, t_0]$, $f'(x) \leq f'(t_0) = 0$. Similarly, for all $x \in [t_0, \infty)$, $f'(x) \geq f'(t_0) = 0$. Applying Proposition 3.4, f is non-increasing on $(-\infty, t_0]$, and applying Proposition 3.5, f is non-decreasing on $[t_0, \infty)$, which is exactly what we wanted to prove.

□

§4 Proof of the p-norm Theorem

In this section, we prove Theorem 1.1.

Proof (Theorem 1.1). We prove the theorem by showing that if $p < 1$, then $\|\cdot\|_p$ is not a norm, and that if $p \geq 1$, then $\|\cdot\|_p$ is a norm.

Let $p < 1$, and consider the function $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}$,

$$\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$$

Notice that for $p \leq 0$, $\left\|\vec{0}\right\|_p = (0^p + 0^p + \cdots + 0^p)^{1/p}$ is undefined, since 0^p is undefined for $p \leq 0$. This means that N is not a norm for such p.

For $0 < p < 1$, consider the unit vectors $\vec{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and $\vec{e}_2 = (0, 1, 0, \ldots, 0) \in \mathbb{R}^n$. Then we get

$$\|\vec{e}_1 + \vec{e}_2\|_p = \|(1, 1, 0, \ldots, 0)\|_p = (1^p + 1^p + 0^p + \cdots + 0^p)^{1/p} = 2^{1/p}$$

and

$$\|\vec{e}_1\|_p = \|\vec{e}_2\|_p = (1^p)^{1/p} = 1$$

Since $0 < p < 1$, $\frac{1}{p} > 1$, so

$$\|\vec{e}_1 + \vec{e}_2\|_p = 2^{1/p} > 2 = 1 + 1 = \|\vec{e}_1\|_p + \|\vec{e}_2\|_p$$

which means that $\|\cdot\|_p$ does not satisfy the triangle inequality. Therefore, $\|\cdot\|_p$ is not a norm for $p < 1$.

Now, let $p \geq 1$. We will handle the $p = 1$ case separately.

Positive Definiteness: Let $\vec{x} \in \mathbb{R}^n$. Then, $\|\vec{x}\|_1 = (|x_1|^1 + \cdots + |x_n|^1)^{1/1} = |x_1| + \cdots + |x_n|$. Notice that for all i, $|x_i| \geq 0$, so we must have $\|\vec{x}\|_1 \geq 0$. Also, $\|\vec{0}\|_1 = |0| + \cdots + |0| = 0$. Now, assume $\|\vec{x}\|_1 = 0$. Then,

$$\|\vec{x}\|_1 = |x_1| + \cdots + |x_n| = 0$$

$|x_1| + \cdots + |x_n|$ is a sum of non-negative terms, so it can only be 0 if each individual term is 0. Thus, for all i, $x_i = 0$, hence $\vec{x} = \vec{0}$. Thus, $\|\vec{x}\|_1 \geq 0$ for all $\vec{x} \in \mathbb{R}^n$ and $\|\vec{x}\|_1 = 0 \iff \vec{x} = \vec{0}$, so $\|\cdot\|_1$ is positive definite.

Homogeneity: Let $\vec{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then,

$$\begin{aligned}
\|c\vec{x}\|_1 &= (|cx_1|^1 + \cdots + |cx_n|^1)^{1/1} \\
&= |cx_1| + \cdots + |cx_n| \\
&= |c||x_1| + \cdots + |c||x_n| \\
&= |c|(|x_1| + \cdots + |x_n|) \\
&= |c| \, \|\vec{x}\|_1
\end{aligned}$$

Thus, $\|\cdot\|_1$ is homogeneous.

Triangle Inequality: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then,

$$\begin{aligned}
\|\vec{x} + \vec{y}\|_1 &= |x_1 + y_1| + \cdots + |x_n + y_n| \\
&\leq |x_1| + |y_1| + \cdots + |x_n| + |y_n| && \text{(By Triangle Inequality for Absolute Value)} \\
&= |x_1| + \cdots + |x_n| + |y_1| + \cdots + |y_n| \\
&= \|\vec{x}\|_1 + \|\vec{y}\|_1
\end{aligned}$$

Hence, $\|\cdot\|_1$ has triangle inequality property.

Therefore, $\|\cdot\|_1$ is a norm.

Now, we handle the $p > 1$ case. For this, we will use the Magic Norm Theorem. Consider the set $B = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\|_p \leq 1\}$. Next, we show that B is absorbing, balanced, bounded, closed, and convex.

Absorbing: Let $\vec{x} \in \mathbb{R}^n$. If $\vec{x} = \vec{0}$, then $\|\vec{x}\|_p = (|0|^p + \cdots + |0|^p)^{1/p} = 0 \leq 1$, so $\vec{x} \in B$. Thus, we take $t = 1 > 0$ and $\vec{b} = \vec{x} \in B$, and we write $\vec{x} = 1\vec{x} = t\vec{b}$. Now, we assume $\vec{x} \neq \vec{0}$. This means that for some i, $x_i \neq 0$, so $|x_i|^p > 0 \Rightarrow |x_1|^p + \cdots + |x_n|^p > 0 \Rightarrow (|x_1|^p + \cdots + |x_n|^p)^{1/p} > 0 \Rightarrow \|\vec{x}\|_p > 0$. Now, we take $t = \|\vec{x}\|_p > 0$ and $\vec{b} = \frac{1}{\|\vec{x}\|_p}\vec{x}$. Notice that

$$\begin{aligned}
\|\vec{b}\|_p &= \left\| \frac{1}{\|\vec{x}\|_p}\vec{x} \right\|_p \\
&= \left(\left| \frac{1}{\|\vec{x}\|_p} x_1 \right|^p + \cdots + \left| \frac{1}{\|\vec{x}\|_p} x_n \right|^p \right)^{1/p} \\
&= \left(\left(\frac{1}{\|\vec{x}\|_p} \right)^p |x_1|^p + \cdots + \left(\frac{1}{\|\vec{x}\|_p} \right)^p |x_n|^p \right)^{1/p} \\
&= \frac{1}{\|\vec{x}\|_p} (|x_1|^p + \cdots + |x_n|^p)^{1/p} \\
&= \frac{1}{\|\vec{x}\|_p} \|\vec{x}\|_p \\
&= 1
\end{aligned}$$

Since $\|\vec{b}\|_p = 1 \leq 1$, $\vec{b} \in B$. Now, notice that $\vec{x} = \|\vec{x}\|_p \cdot \frac{1}{\|\vec{x}\|_p}\vec{x} = t\vec{b}$. In both cases, we get $\vec{x} = t\vec{b}$ for some $t > 0$ and $\vec{b} \in B$, hence, B is absorbing.

Balanced: Let $\vec{b} \in B$ and $t \in [-1,1]$. Then, $\|\vec{b}\|_p \leq 1$ and $|t| \leq 1$. Now, we get

$$\begin{aligned} \|t\vec{b}\|_p &= (|tb_1|^p + \cdots + |tb_n|^p)^{1/p} \\ &= (|t|^p|b_1|^p + \cdots + |t|^p|b_n|^p)^{1/p} \\ &= |t|(|b_1|^p + \cdots + |b_n|^p)^{1/p} \\ &= |t|\|\vec{b}\|_p \\ &\leq 1 \end{aligned}$$

Thus, $\|t\vec{b}\|_p \leq 1$, so $t\vec{b} \in B$. Hence, B is balanced.

Bounded: We will show that B is bounded over the 1-norm, which by equivalence of norms, would mean that B is bounded over all norms. Let $\vec{x}, \vec{y} \in B$. Notice that for any $1 \leq i \leq n$, $|x_i|^p \leq |x_1|^p + \cdots + |x_n|^p \Rightarrow |x_i| \leq (|x_1|^p + \cdots + |x_n|^p)^{1/p} = \|\vec{x}\|_p \leq 1$. Similarly, $|y_i| \leq 1$ for any $1 \leq i \leq n$. Thus, $\|\vec{x}\|_1 = |x_1| + \cdots + |x_n| \leq n$ and $\|\vec{y}\|_1 = |y_1| + \cdots + |y_n| \leq n$. Now, we have $\|\vec{x} - \vec{y}\|_1 \leq \|\vec{x}\|_1 + \|\vec{y}\|_1 \leq 2n$. This means that distances between elements of B are bounded above by a constant $2n$, hence B must be bounded.

Closed: We will show that B is closed over the 1-norm, which by equivalence of norms, would mean that B is closed over all norms. To do so, we will show that $\|\cdot\|_p : (\mathbb{R}^n, \|\cdot\|_1) \to (\mathbb{R}, |\cdot|)$ is continuous, and then apply topological continuity.

Define $f_a : ([0,\infty), |\cdot|) \to ([0,\infty), |\cdot|)$, $f_a(t) = t^a$, $g : (\mathbb{R}, |\cdot|) \to ([0,\infty), |\cdot|)$, $g = |\cdot|$, and $h_i : (\mathbb{R}^n, \|\cdot\|_1) \to (\mathbb{R}, |\cdot|)$, $h_i(\vec{x}) = x_i$ for $1 \leq i \leq n$. Then, notice that we can write the following:

$$\|\cdot\|_p = f_{1/p} \circ (f_p \circ g \circ h_1 + \cdots + f_p \circ g \circ h_n)$$

Notice that g is continuous, and for $p > 1$, $f_{1/p}$ and f_p are continuous. Furthermore, h_i's are linear functions with finite dimensional domain, hence are also continuous. Thus, $\|\cdot\|_p$ is continuous as a combination of continuous functions. Now, $B = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\|_p \leq 1\} = \|(-\infty, 1]\|_p^{-1}$ is the pre-image of a closed set, so B must also be closed by topological continuity.

Convex: Let $\vec{x}, \vec{y} \in B$. Define the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(t) = (\|t\vec{x} + (1-t)\vec{y}\|_p)^p = |(x_1 - y_1)t + y_1|^p + \cdots + |(x_n - y_n)t + y_n|^p$$

Notice that $t\vec{x} + (1-t)\vec{y} \in B$ if and only if $f(t) \leq 1$, so if we show that $f(t) \leq 1$ for all $t \in [0,1]$, we will have that B is convex.

We first show that f satisfies the conditions of Lemma 3.1. Define the functions $g_i : \mathbb{R} \to \mathbb{R}, g_i(t) = |(x_i - y_i)t + y_i|^p$ for $1 \leq i \leq n$. Notice that if $x_i - y_i = 0$, then g_i is constant, so it is twice differentiable with $g_i''(t) = 0 \geq 0$. Otherwise, take $a = |x_i - y_i| > 0$ and $b = y_i$ if $x_i - y_i > 0$ and $b = -y_i$ otherwise. Notice that $g_i(t) = |(x_i - y_i)t + y_i|^p = |at + b|^p$ for all $t \in \mathbb{R}$. (Trust me bro)

On the interval $(-\infty, \frac{-b}{a})$, we have $g_i(t) = (-at - b)^p$ which is differentiable with a continuous derivative of $g_i'(t) = -pa(-at-b)^{p-1} = -pa|at+b|^{p-1}$. Furthermore, g_i is twice differentiable on this interval with with $g_i''(t) = p(p-1)(-a)^2(-at-b)^{p-2} = p(p-1)a^2|at+b|^{p-2} \geq 0$ since $p > 1$.

Similarly, on the interval $(\frac{-b}{a}, \infty)$, $g_i(t) = (at+b)^p$ which is continuously differentiable with $g_i'(t) = pa(at+b)^{p-1} = pa|at+b|^{p-1}$. Furthermore, $g_i''(t) = p(p-1)a^2(at+b)^{p-2} = p(p-1)a^2|at+b|^{p-2} \geq 0$.

Now, notice that

$$\frac{-(at+b)^p}{t + \frac{b}{a}} \leq \frac{|at+b|^p}{t + \frac{b}{a}} \leq \frac{(at+b)^p}{t + \frac{b}{a}}$$

and that for $p > 1$, both the left and the right terms go to 0 as $t \to -\frac{b}{a}$. By the Squeeze Theorem, this means that

$$g_i'\left(-\frac{b}{a}\right) = \lim_{t \to -\frac{b}{a}} \frac{|at+b|^p}{t + \frac{b}{a}} = 0$$

Furthermore, notice that as $t \to \left(-\frac{b}{a}\right)^-$, $g_i'(t) = -pa|at+b|^{p-1} \to 0$ and as $t \to \left(-\frac{b}{a}\right)^+$, $g_i'(t) = pa|at+b|^{p-1} \to 0$. Thus, $g_i'(t)$ is continuously differentiable.

Now, we have that g_i is continuously differentiable on $\mathbb{R}$ and twice differentiable on all $\mathbb{R}$ except, possibly, for a single point. Furthermore, for all $t \in \mathbb{R}$ where g_i is twice differentiable, $g_i''(t) \geq 0$. Also, as $t \to -\infty$, $g_i'(t) = -pa|at+b|^{p-1} \to -\infty$, and as $t \to \infty$, $g'(t) = pa|at+b|^{p-1} \to \infty$.

Now, notice that $f = g_1 + g_2 + \cdots + g_n$. Since each g_i is continuously differentiable, f is also continuously differentiable. Since each g_i is twice differentiable on all $\mathbb{R}$ but a single point, f is twice differentiable on all $\mathbb{R}$ but finitely many points. Since where all g_i's exist, $g_i''(t) \geq 0$, we also have $f''(t) = g_1''(t) + \cdots + g_n''(t) \geq 0$ as a sum of non-negative terms. Finally, since $\lim_{t\to-\infty} g_i(t) = -\infty$ and $\lim_{t\to\infty} g_i(t) = \infty$ for all i, we also have $\lim_{t\to-\infty} f(t) = -\infty$ and $\lim_{t\to\infty} f(t) = \infty$. This allows us to apply Lemma 3.1 to get some $t_0 \in \mathbb{R}$ so that f is non-increasing on $(-\infty, t_0]$ and non-decreasing on $[t_0, \infty)$.

Next, we show that the maximum of f on $[0,1]$ is either $f(0)$ or $f(1)$. Consider the cases $t_0 \leq 0$, $0 < t_0 < 1$, and $1 \leq t_0$.

Case 1: $t_0 \leq 0$. In this case, we have that f is non-decreasing on $[0,1] \subseteq [t_0, \infty)$. This means that $f(1) \geq f(t)$ for all $t \in [0,1]$, so $f(1)$ is the maximum of f on $[0,1]$.

Case 2: $t_0 \geq 1$. In this case, we have that f is non-increasing on $[0,1] \subseteq (-\infty, t_0]$. This means that $f(0) \geq f(t)$ for all $t \in [0,1]$, so $f(0)$ is the maximum of f on $[0,1]$.

Case 3: $0 < t_0 < 1$. In this case, we have that f is non-increasing on $[0,t_0] \subseteq (-\infty, t_0]$ and non-decreasing on $[t_0, 1] \subseteq [t_0, \infty)$. This means that $f(0) \geq f(t)$ for all $t \in [0, t_0]$ and $f(1) \geq f(t)$ for all $t \in [t_0, 1]$. This implies that $\max\{f(0), f(1)\} \geq f(t)$ for all $t \in [0,1]$, hence either $f(0)$ or $f(1)$ is the maximum of f on $[0,1]$.

Now, we have that for all $t \in [0,1]$,

$$f(t) \leq \max\{f(0), f(1)\} = \max\{(\|\vec{y}\|_p)^p, (\|\vec{x}\|_p)^p\} \leq 1$$

since $\|\vec{x}\|_p, \|\vec{y}\|_p \leq 1$ as $\vec{x}, \vec{y} \in B$. As was mentioned previously, this means that $t\vec{x} + (1-t)\vec{y} \in B$, hence B is convex.

We have shown that B is absorbing, balanced, convex, bounded, and closed, so by the Magic Norm Theorem, there exists a norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ so that $B = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq 1\}$. We soon show that $\|\cdot\|_p = \|\cdot\|$, but before that, it is helpful to show that $\|\cdot\|_p$ is homogeneous.

Let $c \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$. Then, we have

$$\begin{aligned}
\|c\vec{x}\|_p &= (|cx_1|^p + \cdots + |cx_n|^p)^{1/p} \\
&= (|c|^p|x_1|^p + \cdots + |c|^p|x_n|^p)^{1/p} \\
&= |c|(|x_1|^p + \cdots + |x_n|^p)^{1/p} \\
&= |c| \, \|\vec{x}\|_p
\end{aligned}$$

Hence, $\|\cdot\|_p$ is homogeneous.

Now, by way of contradiction, suppose that $\|\cdot\|_p \neq \|\cdot\|$. Then, there exists some $\vec{x} \in \mathbb{R}$ so that $\|\vec{x}\|_p \neq \|\vec{x}\|$. Notice that $\vec{x} \neq \vec{0}$, because if that was the case, we would have $\|\vec{x}\| = 0 = \|\vec{x}\|_p$. Then, we can take $c = \frac{1}{\|\vec{x}\|} > 0$ and get

$$\|c\vec{x}\| = |c| \, \|\vec{x}\| = \frac{1}{\|\vec{x}\|} \|\vec{x}\| = 1$$

which means that $c\vec{x} \in B$. Since $c\vec{x} \in B$, we must have $\|c\vec{x}\|_p \leq 1$. Now, notice that if $\|c\vec{x}\|_p = 1$, then $\|c\vec{x}\|_p = \|c\vec{x}\| \Rightarrow |c| \, \|\vec{x}\|_p = |c| \, \|\vec{x}\| \Rightarrow \|\vec{x}\|_p = \|\vec{x}\|$, but that is a contradiction. Hence, $\|c\vec{x}\|_p < 1$. Furthermore, since $\vec{x} \neq \vec{0}$ and $c \neq 0$, we have $\|c\vec{x}\|_p > 0$, so we take $t = \frac{1}{\|c\vec{x}\|_p}$. Since $0 < \|c\vec{x}\|_p < 1$, we have $t > 1$. Now, we have

$$\|tc\vec{x}\|_p = t \, \|c\vec{x}\|_p = \frac{1}{\|c\vec{x}\|_p} \|c\vec{x}\|_p = 1$$

so $tc\vec{x} \in B$. However, we also have

$$\|tc\vec{x}\| = t\,\|c\vec{x}\| = t > 1$$

so $tc\vec{x} \notin B$, which is a contradiction. Therefore, $\|\cdot\|_p = \|\cdot\|$, so $\|\cdot\|_p$ is a norm.

□

§5 Applications

The p-norm theorem is not widely applicable, but it can be used to prove the p-norms on $C[0,1]$ and l^p, the spaces of continuous functions on $[0,1]$ and p-summable sequences in $\mathbb{R}$, respectively.

Definition 5.1 (p-Summable Sequences)**.** We define l^p to be the space of p-summable sequences in $\mathbb{R}$ – that is, l^p is the space of all sequences (x_n) in $\mathbb{R}$ so that $\sum_{i=1}^{\infty} |x_n|^p$ converges. You can verify that this is a linear subspace of the space of all sequences in $\mathbb{R}$.

Corollary 5.2 (p-Norm on l^p)**.** *For $p > 1$, the function $\|\cdot\|_p : l^p \to \mathbb{R}$, $\|\vec{x}\|_p = \left(\sum_{i=1}^{\infty} |x_n|^p\right)^{1/p}$ is a norm.*

Proof. To prove that $\|\cdot\|_p$ is a norm on l^p, we need to show that it is positive definite, homogeneous, and has triangle inequality. Also, where needed, we will use the fact that $\|\cdot\|_p$ on $\mathbb{R}_n$ is a norm.

Positive Definiteness: Let $X = (x_n) \in l^p$. Notice that

$$\|X\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$$

is a sum of non-negative terms, so $\|X\|_p \geq 0$. Thus, we have $\|X\|_p \geq 0$ for all $X \in l^p$.

Notice that $\|0\|_p = \left(\sum_{i=1}^{\infty} |0|^p\right)^{1/p} = 0$. Now, assume that $\|X\|_p = 0$ for some $X = (x_n) \in l^p$. Then, notice that $0 = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \geq |x_n|$ for all $n \in \mathbb{N}$. This means that $x_n = 0$ for all $n \in \mathbb{N}$, hence $X = 0$. Thus, we have $\|X\|_p = 0 \iff X = 0$. Therefore, $\|\cdot\|_p$ is positive definite.

Homogeneity: Let $c \in \mathbb{R}$ and $X = (x_n) \in l^p$. Then, we have

$$\begin{aligned}
\|cX\|_p &= \left(\sum_{i=1}^{\infty} |cx_i|^p\right)^{1/p} \\
&= \left(|c|^p \sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \\
&= |c| \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \\
&= |c|\,\|X\|_p
\end{aligned}$$

Thus, $\|\cdot\|_p$ is homogeneous.

Triangle Inequality: Let $X = (x_n), Y = (y_n) \in l^p$. Furthermore, we define $\vec{x}_n = (x_1, \ldots, x_n), \vec{y}_n = (y_1, \ldots, y_n) \in \mathbb{R}^n$ for $n \geq 2$. By Theorem 1.1, for all $n \geq 2$, we have

$$\|\vec{x}_n + \vec{y}_n\|_p \leq \|\vec{x}_n\|_p + \|\vec{y}_n\|_p \Longrightarrow \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$$

Taking $n \to \infty$, we get

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p} \Longrightarrow \|X + Y\|_p \leq \|X\|_p + \|Y\|_p$$

Thus, $\|\cdot\|_p$ has triangle inequality. Therefore, $\|\cdot\|_p : l^p \to \mathbb{R}$ is a norm.

$\square$

Corollary 5.3 (p-Norm on $C[0,1]$)**.** *For $p > 1$, the function $\|\cdot\|_p : C[0,1] \to \mathbb{R}$, $\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$ is a norm. Note that this integral exists since continuous functions are integrable.*

Proof. To prove that $\|\cdot\|_p$ is a norm on $C[0,1]$, we need to show that it is positive definite, homogeneous, and has triangle inequality. Also, where needed, we will use the fact that $\|\cdot\|_p$ on $\mathbb{R}_n$ is a norm.

Positive Definiteness: Let $f \in C[0,1]$. Notice that

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$$

is an integral of a non-negative function, so $\|f\|_p \geq 0$. Thus, we have $\|f\|_p \geq 0$ for all $f \in C[0,1]$.

Notice that $\|0\|_p = \left(\int_0^1 0 \, dx\right)^{1/p} = 0$. Now, let $f \neq 0 \in C[0,1]$. That means that for some $x_0 \in [0,1]$, $f(x_0) \neq 0$. Then, $|f(x_0)| > 0$, so we can find an $\varepsilon > 0$ such that for all $x \in [0,1] \cap (x_0 - \varepsilon, x_0 + \varepsilon)$, $|f(x)| > \frac{1}{2}|f(x_0)|$. Now, we have

$$\begin{aligned}
\|f\|_p &= \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p} \\
&= \left(\int_0^{\max\{0, x_0 - \varepsilon\}} |f(x)|^p \, dx + \int_{\max\{0, x_0 - \varepsilon\}}^{\min\{x_0 + \varepsilon, 1\}} |f(x)|^p \, dx + \int_{\min\{x_0 + \varepsilon, 1\}}^1 |f(x)|^p \, dx\right)^{1/p} \\
&> \left(\int_0^{\max\{0, x_0 - \varepsilon\}} 0 \, dx + \int_{\max\{0, x_0 - \varepsilon\}}^{\min\{x_0 + \varepsilon, 1\}} \left(\frac{1}{2}|f(x_0)|\right)^p \, dx + \int_{\min\{x_0 + \varepsilon, 1\}}^1 0 \, dx\right)^{1/p} \\
&= \left(\frac{1}{2^p}|f(x_0)|^p \left(\min\{x_0 + \varepsilon, 1\} - \max\{0, x_0 - \varepsilon\}\right)\right)^{1/p} \\
&= \frac{1}{2}|f(x_0)| \left(\min\{x_0 + \varepsilon, 1\} + \min\{0, \varepsilon - x_0\}\right)^{1/p} \\
&= \frac{1}{2}|f(x_0)| \left(\min\{2\varepsilon, 1, 1 + \varepsilon - x_0\}\right)^{1/p} \\
&> \frac{1}{2}|f(x_0)| \cdot \varepsilon^{1/p} \\
&> 0
\end{aligned}$$

Thus, $\|f\|_p \neq 0$. Now, we have $f = 0 \Rightarrow \|f\|_p = 0$ and $f \neq 0 \Rightarrow \|f\|_p \neq 0$. Hence, $\|\cdot\|_p$ is positive definite.

Homogeneity: Let $c \in \mathbb{R}$ and $f \in C[0,1]$. Then, we have

$$\begin{aligned}
\|cf\|_p &= \left(\int_0^1 |cf(x)|^p \, dx\right)^{1/p} \\
&= \left(|c|^p \int_0^1 |f(x)|^p \, dx\right)^{1/p} \\
&= |c| \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p} \\
&= |c| \, \|f\|_p
\end{aligned}$$

Thus, $\|\cdot\|_p$ is homogeneous.

<u>Triangle Inequality:</u> Let $f, g \in C[0,1]$. Furthermore, we define $\vec{x}_n = (f(\frac{1}{n}), f(\frac{2}{n}), \ldots, f(\frac{n}{n})), \vec{y}_n = (g(\frac{1}{n}), g(\frac{2}{n}), \ldots, g(\frac{n}{n})) \in \mathbb{R}^n$ for $n \geq 2$. By Theorem 1.1, for all $n \geq 2$, we have

$$\|\vec{x}_n + \vec{y}_n\|_p \leq \|\vec{x}_n\|_p + \|\vec{y}_n\|_p$$

$$\Longrightarrow \left(\sum_{i=1}^{n} \left| f\left(\frac{i}{n}\right) + g\left(\frac{i}{n}\right) \right|^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} \left| f\left(\frac{i}{n}\right) \right|^p \right)^{1/p} + \left(\sum_{i=1}^{n} \left| g\left(\frac{i}{n}\right) \right|^p \right)^{1/p}$$

$$\Longrightarrow \left(\sum_{i=1}^{n} \frac{1}{n} \left| f\left(\frac{i}{n}\right) + g\left(\frac{i}{n}\right) \right|^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} \frac{1}{n} \left| f\left(\frac{i}{n}\right) \right|^p \right)^{1/p} + \left(\sum_{i=1}^{n} \frac{1}{n} \left| g\left(\frac{i}{n}\right) \right|^p \right)^{1/p}$$

Now, notice that for a continuous function $h : [0,1] \to \mathbb{R}$, the integral of h on $[0,1]$ can be written as a limit of the Riemann sums. Written using the right rule, we get

$$\int_0^1 h(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} h\left(\frac{i}{n}\right)$$

Taking $n \to \infty$ on the previous inequality, we get

$$\left(\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \left| f\left(\frac{i}{n}\right) + g\left(\frac{i}{n}\right) \right|^p \right)^{1/p} \leq \left(\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \left| f\left(\frac{i}{n}\right) \right|^p \right)^{1/p} + \left(\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \left| g\left(\frac{i}{n}\right) \right|^p \right)^{1/p}$$

$$\Longrightarrow \left(\int_0^1 |f(x) + g(x)|^p \right)^{1/p} \leq \left(\int_0^1 |f(x)|^p \right)^{1/p} + \left(\int_0^1 |g(x)|^p \right)^{1/p}$$

$$\Longrightarrow \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Thus, $\|\cdot\|_p$ has triangle inequality. Therefore, $\|\cdot\|_p : C[0,1] \to \mathbb{R}$ is a norm.

$\square$