Københavns Universitet
Introduktion til diskret matematik og algoritmer - Problem set 3

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Contents

1 Recall that a standard deck of cards has 52 cards partitioned into four suits (hearts, spades, clubs, and diamonds) with 13 ranks each (2-10 plus jack, queen, king, and ace). In this problem, we assume that you are dealt 5 cards from a perfectly shuffled deck of cards.

1.a What is the probability that you get a flush, i.e., 5 cards of the same suit but not all in sequence with respect to rank? (Because five cards of the same suit in sequential rank would be a straight flush.)

Først skal vi finde ud af, hvor mange mulige hænder vi kan trække. Siden vi får 5 tilfældige kort uden tilbagelægning, og rækkefølgen ikke betyder noget, kan vi beskrive dette som en binomialkoefficient:

$$_{52}C_5 = \binom{52}{5} =$$
$$\frac{52!}{(52-5)! \cdot 5!} =$$
$$\frac{52!}{47! \cdot 5!} = 2.598.960$$

Nu skal vi bestemme mængden af hænder, der laver en flush. Der er 4 forskellige kulører, og vi skal bruge 5 af samme kulør:

$$_{13}C_5 = \binom{13}{5} =$$
$$\frac{13!}{(13-5)! \cdot 5!} =$$
$$\frac{13!}{8! \cdot 5!} = 1287$$

Vi skal dog også huske, at der er 4 forskellige kulører, så vi vælger 1 af de 4:

$$_4C_1 \cdot 1287 = \binom{4}{1} \cdot 1287 =$$
$$\frac{4!}{(4-1)! \cdot 1!} \cdot 1287 =$$
$$\frac{4!}{3!} \cdot 1287 =$$
$$4 \cdot 1287 = 5148$$

Der er dog også mulighed for at trække en straight flush, som vi skal trække fra:

$$_{10}C_1 \cdot_4 C1 = \binom{10}{1} \cdot \binom{4}{1} =$$
$$\frac{10!}{(10-1)! \cdot 1!} \cdot \frac{4!}{(4-1)! \cdot 1!} =$$
$$\frac{10!}{9!} \cdot \frac{4!}{3!} =$$

$$10 \cdot 4 = 40$$

Så den samlede mængde af hænder, der laver en flush uden en straight flush, er:

$$5148 - 40 = 5108$$

Så sandsynligheden for at trække en flush er:

$$\frac{5108}{2598960} \approx 0.00197$$

1.b What is the probability that you get a straight, i.e., 5 cards of sequential rank but not all of the same suit? (Because if the latter condition also held, we would again have a straight flush.)

Vi bestemmer mængden af hænder, der laver en straight. Den laveste straight vi kan lave er $A, 2, 3, 4, 5$, og den højeste er $10, J, Q, K, A$. Fra A til 10 er der 10 måder at lave en straight, hvis vi bare tæller fra og med laveste kort tal af straight'en op til de 10.

$${}_{10}C_1 = \binom{10}{1} =$$

$$\frac{10!}{(10-1)! \cdot 5!} =$$

$$\frac{10!}{9! \cdot 1!} = 10$$

Vi skal dog også huske, at der er 4 forskellige kulører, så vi vælger 1 af de 4, for hvert af de 5 kort.

$${}_4C_1 \cdot_4 C_1 \cdot_4 C_1 \cdot_4 C_1 \cdot_4 C_1 \cdot 10 = \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot 10 =$$

$$\frac{4!}{(4-1)! \cdot 1!} \cdot \frac{4!}{(4-1)! \cdot 1!} \cdot \frac{4!}{(4-1)! \cdot 1!} \cdot \frac{4!}{(4-1)! \cdot 1!} \cdot \frac{4!}{(4-1)! \cdot 1!} \cdot 10 =$$

$$\frac{4!}{3!} \cdot \frac{4!}{3!} \cdot \frac{4!}{3!} \cdot \frac{4!}{3!} \cdot \frac{4!}{3!} \cdot 10 =$$

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 10 = 10240$$

Der er dog også mulighed for at trække en straight flush, som vi skal trække fra:

$$10240 - 40 = 10200$$

Så sandsynligheden for at trække en straight er:

$$\frac{10200}{2598960} \approx 0.00393$$

2 Prove mathematically that among all numbers on the form 11...100...0, i.e., numbers consisting of m ones followed by n zeros for some $m, n \in N^+$ (sometimes notation like $1^m 0^n$ is used to describe text strings constructed in such a way), there is some number that is divisible by 2025. Hint: Look at all numbers $1^m = 11...1$ and consider what their remainders can be modulo 2025.

For at omformulere opgaven, skal vi vise:

$$\exists m, n \in \mathbb{N}^+ \quad 2025 \mid 1^m 0^n$$

3 Let $a \in R^+$ be any positive real number. Show that for any integer $n \geq 2$ there is a rational number $\frac{c}{d}, c, d \in \mathbb{Z}, d \leq n$, that approximates a to within error $\frac{1}{dn}$, ie., $|a - \frac{c}{d}| \leq \frac{1}{dn}$. Hint: Consider the numbers $a, 2a, \ldots, n \cdot a$ and show that one of these numbers is at distance at most $\frac{1}{n}$ from some integer.

4 In this problem we focus on relations. Suppose that $A = \{e_0, e_1, \ldots, e_5\}$ is a set of 6 elements and consider the relation R on A represented by the matrix

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(where element e_i corresponds to the row and column $i + 1$).

4.a Let us write S to denote the symmetric closure of the relation R. What is the matrix representation of S? Can you explain in words what the relation S is by describing how it can be interpreted?

Vi får givet en matrice med relationen R på A. Vi ser, at hvis vi repræsenterer matricen som en graf, ender vi med en directed graf, da der ikke er nogle

$$M_R^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_S = M_R \vee M_R^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

4.b **Now let T be the transitive closure of the relation S. What is the matrix representation of T? Can you explain in words what the relation T is by describing how it can be interpreted?**

4.c **Suppose that we instead let T' be the transitive closure of the relation R, and then let S' be the symmetric closure of T'. Are S' and T the same relation? If they are not the same, show some way in which they differ. If they are the same, is it true that S' and T constructed in this way from some relation R on a set A will always be the same? Please make sure to motivate your answers clearly.**

5 **Recall that an undirected graph $G = (V, E)$ consists of a set of vertices V connected by edges E, where every edge is an unordered pair of vertices. If there is an edge (u, v) between two vertices u and v, then we say that u and v are the endpoints of the edge, and the two vertices are said to be neighbours. We say that a sequence of edges $(v_1, v_2), (v_2, v_3), (v_3, v_4), \ldots, (v_{k-1}, v_k)$, in E is a path from v_1 to v_k. In this problem, we wish to express properties of graphs in both natural language and pred- icate logic, and to translate between the two forms. We do this as follows:**

- **The universe is the set of vertices V of G.**
- **The binary predicate $E(u, v)$ holds if and only if there is an edge between u and v in G.**
- **The unary predicate $S(v)$ is used to identify a subset of vertices $S = \{v \mid v \in V, S(v) \text{ is true}\}$ for which some property might or might not hold.**

For example, we can write the natural language statement "S is a set containing exactly k distinct elements" as a formula

$$\textbf{setsize}(S, k) := \exists u_1 \cdots \exists u_k \left(\bigwedge_{i=1}^{k-1} \bigwedge_{j=i+1}^{k} (u_i \neq u_j) \wedge \bigwedge_{i=1}^{k} S(u_i) \right)$$
$$\wedge \neg \exists u_1 \cdots \exists u_{k+1} \left(\bigwedge_{i=1}^{k} \bigwedge_{j=i+1}^{k+1} (u_i \neq u_j) \wedge \bigwedge_{i=1}^{k} S(u_i) \right) \qquad (1)$$

where $\bigwedge_{i=1}^{k} \phi_i$ is shorthand for $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_k$, and where we use standard notation $\neg$ for logical negation. In natural language, this formula can be read as: "There exist k elements u_1 to u_k such that (i) for every pair $(u_i, u_j), i \neq j$, the elements themselves are also distinct $(u_i \neq u_j)$; and (ii) all the u_i for $i = 1, \ldots, k$ are members of S, but no such set of $k+1$ elements u_1 to u_{k+1} exists." Below you find six graph properties defined in natural language and six graph properties written as predicate logic formulas. Most of the natural language definitions have equivalent predicate logic formulas, but not all. Your task is to translate each predicate logic formula (a), ..., (f) into a natural language description, and argue which—if any—of the natural language definitions (1), ..., (6) it matches.