Københavns Universitet
Introduktion til diskret matematik og algoritmer - Problem set 3

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Contents

1 Recall that a standard deck of cards has 52 cards partitioned into four suits (hearts, spades, clubs, and diamonds) with 13 ranks each (2-10 plus jack, queen, king, and ace). In this problem, we assume that you are dealt 5 cards from a perfectly shuffled deck of cards.

1.a What is the probability that you get a flush, i.e., 5 cards of the same suit but not all in sequence with respect to rank? (Because five cards of the same suit in sequential rank would be a straight flush.)

Først skal vi finde ud af, hvor mange mulige hænder vi kan trække. Siden vi får 5 tilfældige kort uden tilbagelægning, og rækkefølgen ikke betyder noget, kan vi beskrive dette som en binomialkoefficient:

$$_{52}C_5 = \binom{52}{5} =$$

$$\frac{52!}{(52-5)! \cdot 5!} =$$

$$\frac{52!}{47! \cdot 5!} = 2.598.960$$

Nu skal vi bestemme mængden af hænder, der laver en flush. Der er 4 forskellige kulører, og vi skal bruge 5 af samme kulør:

$$_{13}C_5 = \binom{13}{5} =$$

$$\frac{13!}{(13-5)! \cdot 5!} =$$

$$\frac{13!}{8! \cdot 5!} = 1287$$

Vi skal dog også huske, at der er 4 forskellige kulører, så vi vælger 1 af de 4:

$$_4C_1 \cdot 1287 = \binom{4}{1} \cdot 1287 =$$

$$\frac{4!}{(4-1)! \cdot 1!} \cdot 1287 =$$

$$\frac{4!}{3!} \cdot 1287 =$$

$$4 \cdot 1287 = 5148$$

Der er dog også mulighed for at trække en straight flush, som vi skal trække fra:

$$_{10}C_1 \cdot _4 C1 = \binom{10}{1} \cdot \binom{4}{1} =$$

$$\frac{10!}{(10-1)! \cdot 1!} \cdot \frac{4!}{(4-1)! \cdot 1!} =$$

$$\frac{10!}{9!} \cdot \frac{4!}{3!} =$$

$$10 \cdot 4 = 40$$

Så den samlede mængde af hænder, der laver en flush uden en straight flush, er:

$$5148 - 40 = 5108$$

Så sandsynligheden for at trække en flush er:

$$\frac{5108}{2598960} \approx 0.00197$$

1.b What is the probability that you get a straight, i.e., 5 cards of sequential rank but not all of the same suit? (Because if the latter condition also held, we would again have a straight flush.)

Vi bestemmer mængden af hænder, der laver en straight. Den laveste straight vi kan lave er $A, 2, 3, 4, 5$, og den højeste er $10, J, Q, K, A$. Fra A til 10 er der 10 måder at lave en straight, hvis vi bare tæller fra og med laveste kort tal af straight'en op til de 10.

$$_{10}C_1 = \binom{10}{1} =$$

$$\frac{10!}{(10-1)! \cdot 5!} =$$

$$\frac{10!}{9! \cdot 1!} = 10$$

Vi skal dog også huske, at der er 4 forskellige kulører, så vi vælger 1 af de 4, for hvert af de 5 kort.

$$_4C_1 \cdot_4 C_1 \cdot_4 C_1 \cdot_4 C_1 \cdot_4 C_1 \cdot 10 = \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot 10 =$$

$$\frac{4!}{(4-1)! \cdot 1!} \cdot \frac{4!}{(4-1)! \cdot 1!} \cdot \frac{4!}{(4-1)! \cdot 1!} \cdot \frac{4!}{(4-1)! \cdot 1!} \cdot \frac{4!}{(4-1)! \cdot 1!} \cdot 10 =$$

$$\frac{4!}{3!} \cdot \frac{4!}{3!} \cdot \frac{4!}{3!} \cdot \frac{4!}{3!} \cdot \frac{4!}{3!} \cdot 10 =$$

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 10 = 10240$$

Der er dog også mulighed for at trække en straight flush, som vi skal trække fra:

$$10240 - 40 = 10200$$

Så sandsynligheden for at trække en straight er:

$$\frac{10200}{2598960} \approx 0.00393$$

2 Prove mathematically that among all numbers on the form 11...100...0, i.e., numbers consisting of m ones followed by n zeros for some $m, n \in N^+$ (sometimes notation like 1^m0^n is used to describe text strings constructed in such a way), there is some number that is divisible by 2025. Hint: Look at all numbers $1^m = 11...1$ and consider what their remainders can be modulo 2025.

For at omformulere opgaven, skal vi vise:

$$\exists m, n \in \mathbb{N}^+ \quad 2025 \ \mid 1^m0^n$$

3 Let $a \in R^+$ be any positive real number. Show that for any integer $n \geq 2$ there is a rational number $\frac{c}{d}, c, d \in \mathbb{Z}, d \leq n$, that approximates a to within error $\frac{1}{dn}$, ie., $|a - \frac{c}{d}| \leq \frac{1}{dn}$. Hint: Consider the numbers $a, 2a, \ldots, n \cdot a$ and show that one of these numbers is at distance at most $\frac{1}{n}$ from some integer.

4 In this problem we focus on relations. Suppose that $A = \{e_0, e_1, \ldots, e_5\}$ is a set of 6 elements and consider the relation R on A represented by the matrix

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(where element e_i corresponds to the row and column $i + 1$).

4.a Let us write S to denote the symmetric closure of the relation R. What is the matrix representation of S? Can you explain in words what the relation S is by describing how it can be interpreted?

Vi får givet en matrice med relationen R på A. Vi ser, at hvis vi repræsenterer matricen som en graf, ender vi med en directed graf, da alle vertices har en directed edge, og ingen af disse går tilbage igen. Grafisk er dette 2 'lukkede' cykler. Den symetriske closure af en matrix består af dens relation R og dens converse relation R^{-1}. For en relation R er dette defineret som:

$$R^{-1} = (b, a) : (a, b) \in R$$

som når anvendt på en matrix beskrives som:

$$M_{R^{-1}} = M_R^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Vi bemærker, at dette er grafisk repræsenteret som de samme 2 lukkede cykler som før, dog er retningerne af edges'ne omvendt.

$$M_S = M_R \ \vee M_R^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Resultatet er, at de 2 lukkede cykler nu både har udgående edges fra M_R og M_R^T. Edges'ne går nu bidirectional for alle edges i grafen, eksempelvis har e_1 en edge til e_3 i M_R, og e_3 har en edge til e_1 i M_R^T. Da dett gøres for alle edges, er grafen for M_S bidirectional.

4.b Now let T be the transitive closure of the relation S. What is the matrix representation of T? Can you explain in words what the relation T is by describing how it can be interpreted?

Vi husker, at for en relation R er dens transitive bestemt som:

$$((a,b) \wedge (b,c)) \Rightarrow (a,c) : a,b,c \in R$$

For vores relation S på matrix'en M_S kan vi beskrive dette som:

$$M_T = M_S \vee M_S^2 \vee M_S^3 \vee \cdots \vee M_S^n$$

eller

$$M_T = \bigvee_{i=1}^{n} M_S^i$$

Rent sprogligt kan vi fortolke dette som at vi har alle de edges M_T, som er i M_S, og alle de edges, der kan nås fra M_S i n steps. Dette betyder, at hvis vi har en edge fra e_1 til e_2 i M_S, og en edge fra e_2 til e_3 i M_S, så vil der også være en edge fra e_1 til e_3 i M_T.
Vi kender i forvejen M_S, så vi går videre til M_S^2

$$M_S^2 = M_S \vee M_S = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Når vi undersøger M_S^2, ser vi at elementer i cykel 1 (e_0, e_2, e_4) har 1 på positioner svarende til cykel 1, og elementer i cykel 2 (e_1, e_3, e_5) har 1 på positioner svarende til cykel 2. Der er altså 1 på hele diagonalen, hvilket betyder, at hvert element kan nå sig selv på 2 skridt i en cykel med 3 elementer. Vi kan verificere at $M_S^3 = M_S^2$, hvilket betyder at vi har nået et fast punkt, og yderligere matrix potenser vil ikke ændre resultatet.

$$M_T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Vi er nu færdige. Kort sagt kan vi fortolke grafen for M_T som to bidirectional cykler baseret på om deres indeks er lige eller ulige, og hvor vertex danner en edge med sig selv.

4.c Suppose that we instead let T' be the transitive closure of the relation R, and then let S' be the symmetric closure of T'. Are S' and T the same relation? If they are not the same, show some way in which they differ. If they are the same, is it true that S' and T constructed in this way from some relation R on a set A will always be the same? Please make sure to motivate your answers clearly.

Vi udregner R transitive closure på samme måde vi gjorde for S tidligere. Vi kender M_R, så lad os udregne M_R^2:

$$M_R^2 = M_R \vee M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Vi bestemmer nu næste matrix potens, M_R^3:

$$M_R^3 = M_R^2 \vee M_R = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Vi får en identity matrix for M_R^3, hvilket betyder at vi har nået et punkt, hvor yderligere matrix potenser vil ikke ændre resultatet.

$$M_{T'} = M_R^3 \vee M_R^2 \vee M_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Derefter finder vi S' som den symmetriske lukning af T'. Da T' allerede er symmetrisk (vi kan se dette fordi hver row i T' allerede er identisk med den tilsvarende column), vil S' være præcis den samme som T'.

$$M_{S'} = M_{T'} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

hvilket også er den samme relation som den givet i M_T. Relationerne på $M_{S'}$ og på M_T er altså ens.

$$aRb \Rightarrow bRa \quad bRc \Rightarrow aRc$$

$$aRb \quad bRc \Rightarrow aRc \Rightarrow cRa$$

5 Recall that an undirected graph $G = (V, E)$ consists of a set of vertices V connected by edges E, where every edge is an unordered pair of vertices. If there is an edge (u, v) between two vertices u and v, then we say that u and v are the endpoints of the edge, and the two vertices are said to be neighbours. We say that a sequence of edges $(v_1, v_2), (v_2, v_3), (v_3, v_4), \ldots, (v_{k-1}, v_k)$, in E is a path from v_1 to v_k. In this problem, we wish to express properties of graphs in both natural language and pred- icate logic, and to translate between the two forms. We do this as follows:

- **The universe is the set of vertices V of G.**
- **The binary predicate $E(u, v)$ holds if and only if there is an edge between u and v in G.**
- **The unary predicate $S(v)$ is used to identify a subset of vertices $S = \{v \mid v \in V, S(v)$ is true$\}$ for which some property might or might not hold.**

For example, we can write the natural language statement "S is a set containing exactly k distinct elements" as a formula

$$\textbf{setsize}(S, k) := \exists u_1 \cdots \exists u_k \left(\bigwedge_{i=1}^{k-1} \bigwedge_{j=i+1}^{k} (u_i \neq u_j) \wedge \bigwedge_{i=1}^{k} S(u_i) \right)$$

$$\wedge \neg \exists u_1 \cdots \exists u_{k+1} \left(\bigwedge_{i=1}^{k} \bigwedge_{j=i+1}^{k+1} (u_i \neq u_j) \wedge \bigwedge_{i=1}^{k} S(u_i) \right) \tag{1}$$

where $\bigwedge_{i=1}^{k} \phi_i$ is shorthand for $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_k$, and where we use standard notation $\neg$ for logical negation. In natural language, this formula can be read as: "There exist k elements u_1 to u_k such that (i) for every pair $(u_i, u_j), i \neq j$, the elements themselves are also distinct ($u_i \neq u_j$); and (ii) all the u_i for $i = 1, \ldots, k$ are members of S, but no such set of $k+1$ elements u_1 to u_{k+1} exists." Below you find six graph properties defined in natural language and six graph properties written as predicate logic formulas. Most of the natural language definitions have equivalent predicate logic formulas, but not all. Your task is to translate each predicate logic formula (a), ..., (f) into a natural language description, and argue which—if any—of the natural language definitions (1), ..., (6) it matches.

Natural Language Definitions:

(1) A ***dominating set*** of size k for a graph $G = (V, E)$ is a set S of k distinct vertices such that every vertex v in the graph either is in S or is a neighbour of a vertex in S.

(2) A ***clique*** S of size k in a graph $G = (V, E)$ is a set S of k distinct vertices such that all vertices in S are neighbours with each other.

(3) A ***disconnected vertex set*** of size k in a graph $G = (V, E)$ is a set S of k distinct vertices such that there are no edges from any $u \in S$ to any $v \in V \setminus S$. [Here $V \setminus S$ denotes set subtraction, so that $V \setminus S = \{v \mid v \in V \text{ and } v \notin S\}$.]

(4) A ***vertex cover*** of size k of a graph $G = (V, E)$ is a set S of k distinct vertices such that for every edge $(u, v) \in E$ it holds that at least one of the endpoints is in S.

(5) A graph $G = (V, E)$ is ***bipartite***, with one of the parts in the bipartition having size k, if there a set S of k distinct vertices such that all edges in the graph go between S and $V \setminus S$.

(6) A ***connected component*** S of size k in a graph $G = (V, E)$ is a set S of k distinct vertices such that for every pair of distinct vertices u and v in S there is a path from u to v consisting of vertices in S.

Predicate Logic Formulas:

(a) $setsize(S, k) \wedge \forall v \forall w \, (E(v, w) \rightarrow S(v) \vee S(w))$

(b) $setsize(S, k) \wedge \forall v \, (S(v) \vee \exists w (S(w) \wedge E(v, w)))$

(c) $setsize(S, k) \wedge \forall u \forall w \, ((u \neq w \wedge S(u) \wedge S(w)) \rightarrow \exists v (E(u, v) \wedge E(v, w)))$

(d) $setsize(S, k) \wedge \forall v \forall w \, (E(v, w) \rightarrow ((S(v) \wedge \neg S(w)) \vee (\neg S(v) \wedge S(w))))$

(e) $setsize(S, k) \land \forall v\, (S(v) \to \exists w(\neg S(w) \land E(v, w)))$

(f) $setsize(S, k) \land \forall v \forall w\, ((v \neq w \land S(v) \land S(w)) \to E(v, w))$

For every predicate logic formula that matches a natural language description, explain clearly why there is a match. For formulas that do not match a description, write a natural language definition along the lines of (1), . . . , (6) that describes the property that the formula encodes.

5.a Predicate Logic Formulas

Vi oversætter først formlerne, hvorefter vi finder den bedst passende beskrivelse til dem.

(a) Et sæt S af en graf $G = (V, E)$ bestående af k unikke vertices, hvor mindst en af alle de vetices der danner en edge med i S.

(b) Et sæt S af en graf $G = (V, E)$ bestående af k unikke vertices, hvor enhver vertex i grafen enten selv er i S eller er nabo til mindst en vertex i S.

(c) Et sæt S af en graf $G = (V, E)$ bestående af k unikke vertices, hvor for ethvert par af forskellige vertices $u, w \in S$ findes der en vertex v i grafen, som er forbundet direkte til både u og w.

(d) Et sæt S af en graf $G = (V, E)$ bestående af k unikke vertices, hvor for enhver edge $(v, w) \in E$ gælder det, at præcis et af vertices'ne er i S (altså enten $v \in S$ og $w \notin S$, eller $v \notin S$ og $w \in S$). Dette beskriver en bipartition graf, hvor S udgør den ene del af opdelingen.

(e) Et sæt S af en graf $G = (V, E)$ bestående af k unikke vertices, hvor enhver vertex $v \in S$ har mindst en edge med en vertex w som ikke er i S.

(f) Et sæt S af en graf $G = (V, E)$ bestående af k unikke vertices, hvor alle vetices har en edge med en vertex, som ikke er sig selv.

5.b Natural Language Definitions

(a) = ?
(b) = beskrivelse 1