Københavns Universitet
LinAlgDat - Project A

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Indhold

1 Opgave

1.a

Vi omskriver ligningssystemet til totalmatrix-form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & a \\ a & a & 4a & a \\ 2 & 2 & 2a^2 & 0 \end{array}\right]$$

Vi benytter Gauss-Jordan elimination til at omskrive totalmatrix'en til en reduceret række-keeechelonform.

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & a \\ a & a & 4a & a \\ 2 & 2 & 2a^2 & 0 \end{array}\right] \begin{array}{l} \cdot 2a \\ \cdot 2 \\ \cdot a \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2a & 4a & 16a & 2a^2 \\ 2a & 2a & 8a & 2a \\ 2a & 2a & 2a^3 & 0 \end{array}\right] \begin{array}{l} \\ -r_1 \\ -r_1 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2a & 4a & 16a & 2a^2 \\ 0 & -2a & -8a & 2a - 2a^2 \\ 0 & -2a & 2a^3 - 16a & -2a^2 \end{array}\right] \begin{array}{l} \\ \\ -r_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2a & 4a & 16a & 2a^2 \\ 0 & -2a & -8a & 2a - 2a^2 \\ 0 & 0 & 2a^3 - 8a & -2a \end{array}\right] \begin{array}{l} +2r_2 \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2a & 0 & 0 & 4a - 2a^2 \\ 0 & -2a & -8a & 2a - 2a^2 \\ 0 & 0 & 2a^3 - 8a & -2a \end{array}\right] \begin{array}{l} \\ \cdot \frac{1}{-8a} \\ \cdot \frac{1}{2a^3-8a} \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2a & 0 & 0 & 4a - 2a^2 \\ 0 & \frac{1}{4} & 1 & \frac{2a-2a^2}{-8a} \\ 0 & 0 & 1 & -\frac{2a}{2a^3-8a} \end{array}\right] \begin{array}{l} \\ -r_3 \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2a & 0 & 0 & 4a - 2a^2 \\ 0 & \frac{1}{4} & 0 & \frac{a^3-a^2-4a+8}{4a^2-16} \\ 0 & 0 & 1 & -\frac{2a}{2a^3-8a} \end{array}\right] \begin{array}{l} \cdot \frac{1}{2a} \\ \cdot 4 \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4a-2a^2}{2a} \\ 0 & 1 & 0 & \frac{4a^3-4a^2-16a+32}{4a^2-16} \\ 0 & 0 & 1 & -\frac{2a}{2a^3-8a} \end{array}\right] \mathit{reducer} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 - a \\ 0 & 1 & 0 & \frac{a^3-a^2-4a+8}{a^2-4} \\ 0 & 0 & 1 & \frac{1}{4-a^2} \end{array}\right]$$

Har nu reduceret til echelonform, så vi er dermed færdige.

1.b

Vi opskriver igen vores ligningssystem som en totalmatrix, og erstatter denne gang a med 0:

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & 0 \\ 0 & 0 & 4 \cdot 0 & 0 \\ 2 & 2 & 2 \cdot 0^2 & 0 \end{array}\right] \mathit{reducer} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right] > \mathit{byttes} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 8 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] -2r_1 \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & 0 \\ 0 & -2 & -16 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cdot (-\tfrac{1}{2}) \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 8 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -2r_1 \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -8 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Vi kan nu aflæse løsningerne til:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 8 \\ -8 \\ 1 \end{bmatrix}$$

Vi ser nu, hvad vi får, når vi bruger den rækkereducerede totalmatrix fra tidligere, når vi erstatter a med 0:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2-a \\ 0 & 1 & 0 & \frac{(a^3-a^2-4a+8)}{(a^2-4)} \\ 0 & 0 & 1 & \frac{1}{(4-a^2)} \end{array}\right] a=0 \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2-0 \\ 0 & 1 & 0 & \frac{(0^3-0^2-4\cdot 0+8)}{(0^2-4)} \\ 0 & 0 & 1 & \frac{1}{(4-0^2)} \end{array}\right] reducer \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right]$$

Denne matrix antyder en unik løsning, hvilket ikke afspejler hvad vi fandt lige før, hvor vi fandt uendelig mange løsninger. Dette skete sandstnligvis fordi at den rækkereducerede totalmatrix er lavet ud fra antagelsen, at $a \neq 0$.

1.c

Vi opskriver igen vores ligningssystem som en totalmatrix, og erstatter denne gang a med 2:

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & 2 \\ 0 & 0 & 4\cdot 2 & 2 \\ 2 & 2 & 2\cdot 2^2 & 0 \end{array}\right] reducer \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 8 & 2 \\ 0 & 0 & 8 & 2 \\ 2 & 2 & 8 & 0 \end{array}\right] \begin{array}{c} \\ \\ +r_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 8 & 2 \\ 2 & 2 & 8 & 0 \\ 0 & 0 & 8 & 2 \end{array}\right] -2r_1 \rightsquigarrow$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & 2 \\ 0 & -2 & -8 & -4 \\ 0 & 0 & 8 & 2 \end{array}\right] \begin{array}{c} -r_3 \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -2 & -8 & -4 \\ 0 & 0 & 8 & 2 \end{array}\right] +r_3 \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 8 & 2 \end{array}\right] \begin{array}{c} +r_2 \\ \\ \\ \end{array} \rightsquigarrow$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 8 & 2 \end{array}\right] \cdot(-\tfrac{1}{2}) \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 8 & 2 \end{array}\right] \cdot\tfrac{1}{8} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right]$$

Vi kan nu aflæse løsningen til:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ \frac{1}{4} \end{bmatrix}$$

Lad os nu se, hvad vi får, når vi bruger den rækkereducerede totalmatrix fra tidligere, når vi erstatter a med 2:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2-2 \\ 0 & 1 & 0 & \frac{(2^3-2^2-4\cdot 2+8)}{(2^2-4)} \\ 0 & 0 & 1 & \frac{1}{(4-2^2)} \end{array}\right] reducer \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{0} \\ 0 & 0 & 1 & \frac{1}{0} \end{array}\right]$$

Vi ser nu, at vi får division med 0, hvilket må betyde, at der ikke er nogen løsninger, når $a = 2$.

1.d

Vi omskriver ligningssystemet til koefficientmatrice-form på venstre side, hvor $a = 1$, og sættr identitetmatricen på højre side:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 1 & 1 & 4\cdot 1 & 0 & 1 & 0 \\ 2 & 2 & 2\cdot 1^2 & 0 & 0 & 1 \end{array}\right] reducer \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right] -1r_1 \rightsquigarrow$$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 0 & -1 & -4 & -1 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \\ -2r_1 \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 0 & -1 & -4 & -1 & 1 & 0 \\ 0 & -2 & -14 & -2 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \\ -2r_2 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 0 & -1 & -4 & -1 & 1 & 0 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} \cdot 3 \\ \cdot(-3) \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 3 & 6 & 24 & 3 & 0 & 0 \\ 0 & 3 & 12 & 3 & -3 & 0 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} +4r_3 \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 0 & 3 & -8 & 4 \\ 0 & 3 & 12 & 3 & -3 & 0 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} \\ +2r_3 \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 3 & 6 & 0 & 3 & -8 & 4 \\ 0 & 3 & 0 & 3 & -7 & 2 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} -2r_2 \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & -3 & 6 & 0 \\ 0 & 3 & 0 & 3 & -7 & 2 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} \cdot\frac{1}{3} \\ \cdot\frac{1}{3} \\ \cdot(-\frac{1}{6}) \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -\frac{7}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{6} \end{array}\right]$$

Vi aflæser løsningen til:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x = A^{-1}b = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -\frac{7}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{5}{3} \\ \frac{1}{6} \end{bmatrix}$$

2 Opgave

2.a

Vi opskriver vores rækkeoperationer som elementærmatricer:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & \frac{1}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Når man udfører rækkeoperationerne i rækkefølgen $E_4 \cdot (E_3 \cdot (E_2 \cdot (E_1 \cdot A)))$ svarer det til at gange med elementærmatricerne fra venstre:

$$E_4 \cdot (E_3 \cdot (E_2 \cdot (E_1 \cdot A))) = \begin{bmatrix} 1 & 0 & \frac{1}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 & 0 \\ -2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \right) \right) \right)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ifølge theorem 2.7 op slides'ne i forelæsning 3, er det at lave rækkeoperationer der samme som at gange med elementærmatricerne fra venstre.
Det må betyde, at en mulig opskrivning er:

$$E_4E_3E_2E_1A = I$$

da vi laver rækkeoperationer i rækkefølgen E_1, E_2, E_3, E_4.

Ifølge definition 2.7 på side 75 i Lineær Algebra i Datalogi, gælder der for en square matrix:

$$AX = I,\ XA = I$$

Det må dermed gælde, at:

$$E_4E_3E_2E_1A = I = AE_4E_3E_2E_1$$

2.b

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2a & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] +r_1 \cdot 2a \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2a & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \cdot\frac{1}{5} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right] \Rightarrow E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] r_1 \leftrightarrow r_3 \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \Rightarrow E_3^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{a^2} & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] -r_3 \cdot \frac{1}{a^2} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\frac{1}{a^2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow E_4^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2a \\ \frac{1}{5} & 0 & -\frac{1}{5a^2} \end{bmatrix}$$

2.c

$$E_1E_2E_3E_4 = \begin{bmatrix} 1 & 0 & 0 \\ -2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{a^2} & 0 & 5 \\ -2a & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

X er givet som række 1 i X = række 1 i $E_1E_2E_3E_4$, og række 2 i X er = række 3 i $E_1E_2E_3E_4$:

$$X = \begin{bmatrix} \frac{1}{a^2} & 0 & 5 \\ 1 & 0 & 0 \end{bmatrix}$$

Vi skal nu bestemme, om X: er venstreinvers til V:

$$XV = \begin{bmatrix} \frac{1}{a^2} & 0 & 5 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2a \\ \frac{1}{5} & -\frac{1}{5a^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Da XV giver enhedsmatricen, er X venstreinvers til V.

Vi bestemmer nu alle venstreinverser til V, ved at lave et ligningssystem:
$x_3 = 1$
$x_1 + 2ax_2 - \frac{1}{5a^2}x_3 = 0$
$\frac{1}{5}x_6 = 0$
$x_4 + 2ax_5 - \frac{1}{5a^2}x_6 = 1$

Vi opskriver vores ligningssystem som en totalmatrix:

$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2a & -\frac{1}{5a^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 2a & -\frac{1}{5a^2} & 1 \end{array}\right] \quad r_1 \leftrightarrow r_2 \quad \rightsquigarrow \left[\begin{array}{cccccc|c} 1 & 2a & -\frac{1}{5a^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 2a & -\frac{1}{5a^2} & 1 \end{array}\right] \quad r_3 \leftrightarrow r_4 \quad \rightsquigarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 2a & -\frac{1}{5a^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2a & -\frac{1}{5a^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \end{array}\right] \begin{array}{c} \\ \\ \\ \cdot 5 \end{array} \quad \rightsquigarrow \left[\begin{array}{cccccc|c} 1 & 2a & -\frac{1}{5a^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2a & -\frac{1}{5a^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} +r_2 \cdot \frac{1}{5a^2} \\ \\ +r_4 \cdot \frac{1}{5a^2} \\ \\ \end{array} \quad \rightsquigarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 2a & 0 & 0 & 0 & 0 & \frac{1}{5a^2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2a & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Vi kan nu aflæse løsningerne til:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{5a^2} \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2a \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2a \\ 1 \\ 0 \end{bmatrix}$$

3 Opgave

3.a

Vi sætter et 1-tal i hver e_{uv}-element i matricen for hver edge $E(u, v)$ i grafen G_N:

$$N = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For at finde antallet af veje fra knude 4 til knude 1 med netop længde 8, skal vi kigge på element $(4, 1)$ i N^8. Vi kan finde $(N^8)_{4,1}$ ved:

$$69 \cdot 0 + 45 \cdot 1 + 45 \cdot 1 + 18 \cdot 1 + 29 \cdot 1 = 137$$

unikke veje fra knude 4 til knude 1 med længde 8.

3.b

Linkmatricen A findes ved at tage N^T og dividere hvert element med antallet af udgående edges i dens column.

$$N^T = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \end{bmatrix}$$

3.c

Vi kan lave et ligningssystem ud fra N, hvor alle elementer nu divideres med mængden af udgående edges den vertex i kolonnen er connected med:

$x_1 = \frac{1}{2}x_2 + x_3$
$x_2 = \frac{1}{2}x_1 + \frac{1}{2}x_5$
$x_3 = \frac{1}{2}x_1$
$x_4 = \frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3 + \frac{1}{2}x_5$
$x_5 = \frac{1}{2}x_1 + \frac{1}{4}x_4$

Vi opskriver vores ligningssystem som en totalmatrix:

$$\left[\begin{array}{ccccc|c} -1 & \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & -1 & 0 \end{array}\right] \begin{array}{l} r_1 \cdot 2 \\ r_2 \cdot 4 \\ r_3 \cdot 4 \\ r_4 \cdot 4 \\ r_5 \cdot 4 \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 & 2 & 0 \\ 2 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & 4 & -4 & 2 & 0 \\ 2 & 0 & 0 & 1 & -4 & 0 \end{array}\right] \begin{array}{l} \\ +r_1 \\ +r_1 \\ +r_1 \\ +r_1 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 3 & 6 & -4 & 2 & 0 \\ 0 & 1 & 2 & 1 & -4 & 0 \end{array}\right] \begin{array}{l} \\ \\ r_3 \cdot 3 \\ \\ r_5 \cdot 3 \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 3 & -6 & 0 & 0 & 0 \\ 0 & 3 & 6 & -4 & 2 & 0 \\ 0 & 3 & 6 & 3 & -12 & 0 \end{array}\right] \begin{array}{l} \\ \\ +r_2 \\ +r_2 \\ +r_2 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 0 & -4 & 0 & 2 & 0 \\ 0 & 0 & 8 & -4 & 4 & 0 \\ 0 & 0 & 8 & 3 & -10 & 0 \end{array}\right] \begin{array}{l} \\ \\ \cdot 2 \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 8 & -4 & 4 & 0 \\ 0 & 0 & 8 & 3 & -10 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ +r_3 \\ +r_3 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 & 8 & 0 \\ 0 & 0 & 0 & 3 & -6 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \cdot 3 \\ \cdot 4 \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 12 & -24 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ +r_4 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \cdot 4 \\ \cdot 4 \\ \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -8 & 4 & 8 & 0 & 0 & 0 \\ 0 & -12 & 8 & 0 & 8 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} +r_3 \\ +r_3 \\ \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -8 & 4 & 0 & 0 & 4 & 0 \\ 0 & -12 & 0 & 0 & 12 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \cdot 3 \\ \\ \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -24 & 12 & 0 & 0 & 12 & 0 \\ 0 & -12 & 0 & 0 & 12 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} +r_2 \\ \\ \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -24 & 0 & 0 & 0 & 24 & 0 \\ 0 & -12 & 0 & 0 & 12 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \cdot(-\frac{1}{24}) \\ \cdot(-\frac{1}{12}) \\ \cdot(-\frac{1}{8}) \\ \cdot(-\frac{1}{12}) \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Vi kan nu aflæse løsningerne til:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \\ 2 \\ 1 \end{bmatrix}$$

Vi ser nu, at side 4 er vigtigst, 1, 2, 5 næst mest vigtige, og 3 er mindst vigtig.

4 Opgave

Se vedhæftede python-fil.