Københavns Universitet
LinAlgDat - Project A

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Indhold

1 Opgave

1.a

Vi omskriver ligningssystemet til totalmatrix-form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & a \\ a & a & 4a & a \\ 2 & 2 & 2a^2 & 0 \end{array}\right]$$

Vi benytter Gauss-Jordan elimination til at omskrive totalmatrix'en til en reduceret ræk-keeechelonform.

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & a \\ a & a & 4a & a \\ 2 & 2 & 2a^2 & 0 \end{array}\right] \begin{array}{l} \cdot 2a \\ \cdot 2 \\ \cdot a \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2a & 4a & 16a & 2a^2 \\ 2a & 2a & 8a & 2a \\ 2a & 2a & 2a^3 & 0 \end{array}\right] \begin{array}{l} \\ -r_1 \\ -r_1 \end{array} \rightsquigarrow$$
$$\left[\begin{array}{ccc|c} 2a & 4a & 16a & 2a^2 \\ 0 & -2a & -8a & 2a-2a^2 \\ 0 & -2a & 2a^3-16a & -2a^2 \end{array}\right] \begin{array}{l} \\ \\ -r_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2a & 4a & 16a & 2a^2 \\ 0 & -2a & -8a & 2a-2a^2 \\ 0 & 0 & 2a^3-8a & -2a \end{array}\right] \begin{array}{l} +2r_2 \\ \\ \\ \end{array} \rightsquigarrow$$
$$\left[\begin{array}{ccc|c} 2a & 0 & 0 & 4a-2a^2 \\ 0 & -2a & -8a & 2a-2a^2 \\ 0 & 0 & 2a^3-8a & -2a \end{array}\right] \begin{array}{l} \\ \cdot\frac{1}{-8a} \\ \cdot\frac{1}{2a^3-8a} \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2a & 0 & 0 & 4a-2a^2 \\ 0 & \frac{1}{4} & 1 & \frac{2a-2a^2}{-8a} \\ 0 & 0 & 1 & -\frac{2a}{2a^3-8a} \end{array}\right] \begin{array}{l} \\ -r_3 \\ \\ \end{array} \rightsquigarrow$$
$$\left[\begin{array}{ccc|c} 2a & 0 & 0 & 4a-2a^2 \\ 0 & \frac{1}{4} & 0 & \frac{a^3-a^2-4a+8}{4a^2-16} \\ 0 & 0 & 1 & -\frac{2a}{2a^3-8a} \end{array}\right] \begin{array}{l} \cdot\frac{1}{2a} \\ \cdot 4 \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4a-2a^2}{2a} \\ 0 & 1 & 0 & \frac{4a^3-4a^2-16a+32}{4a^2-16} \\ 0 & 0 & 1 & -\frac{2a}{2a^3-8a} \end{array}\right] \mathit{reducer} \rightsquigarrow$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2-a \\ 0 & 1 & 0 & \frac{a^3-a^2-4a+8}{a^2-4} \\ 0 & 0 & 1 & \frac{1}{4-a^2} \end{array}\right]$$

Har nu reduceret til echelonform, så vi er dermed færdige.

1.b

Vi opskriver igen vores ligningssystem som en totalmatrix, og erstatter denne gang a med 0:

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & 0 \\ 0 & 0 & 4\cdot 0 & 0 \\ 2 & 2 & 2\cdot 0^2 & 0 \end{array}\right] \mathit{reducer} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right] \; r_2 \leftrightarrow r_3 \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 8 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ -2r_1 \\ \\ \end{array} \rightsquigarrow$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & 0 \\ 0 & -2 & -16 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cdot\left(-\frac{1}{2}\right) \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 8 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -2r_2 \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -8 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Vi kan nu aflæse løsningerne til:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 8 \\ -8 \\ 1 \end{bmatrix}$$

Vi ser nu, hvad vi får, hvis vi bruger den rækkereducerede totalmatrix fra tidligere, når vi erstatter a med 0:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2-a \\ 0 & 1 & 0 & \frac{(a^3-a^2-4a+8)}{(a^2-4)} \\ 0 & 0 & 1 & \frac{1}{(4-a^2)} \end{array}\right] a = 0 \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2-0 \\ 0 & 1 & 0 & \frac{(0^3-0^2-4\cdot0+8)}{(0^2-4)} \\ 0 & 0 & 1 & \frac{1}{(4-0^2)} \end{array}\right] reducer \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right]$$

Denne matrix antyder en unik løsning, hvilket ikke afspejler hvad vi fandt lige før, hvor vi fandt uendelig mange løsninger. Dette skete sandstnligvis fordi at den rækkereducerede totalmatrix er lavet ud fra antagelsen, at $a \neq 0$.

1.c

Vi opskriver igen vores ligningssystem som en totalmatrix, og erstatter denne gang a med 2:

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & 2 \\ 0 & 0 & 4\cdot2 & 2 \\ 2 & 2 & 2\cdot2^2 & 0 \end{array}\right] reducer \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 8 & 2 \\ 0 & 0 & 8 & 2 \\ 2 & 2 & 8 & 0 \end{array}\right] \quad r_2 \leftrightarrow r_3 \quad \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 8 & 2 \\ 2 & 2 & 8 & 0 \\ 0 & 0 & 8 & 2 \end{array}\right] \quad -2r_1 \quad \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & 2 \\ 0 & -2 & -8 & -4 \\ 0 & 0 & 8 & 2 \end{array}\right] \begin{array}{c} -r_3 \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -2 & -8 & -4 \\ 0 & 0 & 8 & 2 \end{array}\right] \begin{array}{c} \\ +r_3 \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 8 & 2 \end{array}\right] \begin{array}{c} +r_2 \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 8 & 2 \end{array}\right] \begin{array}{c} \\ \cdot(-\frac{1}{2}) \\ \cdot\frac{1}{8} \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right]$$

Vi kan nu aflæse løsningen til:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ \frac{1}{4} \end{bmatrix}$$

Lad os nu se, hvad vi får, hvis vi bruger den rækkereducerede totalmatrix fra tidligere, når vi erstatter a med 2:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2-2 \\ 0 & 1 & 0 & \frac{(2^3-2^2-4\cdot2+8)}{(2^2-4)} \\ 0 & 0 & 1 & \frac{1}{(4-2^2)} \end{array}\right] \quad reducer \quad \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{0} \\ 0 & 0 & 1 & \frac{1}{0} \end{array}\right]$$

Vi ser nu, at vi får division med 0, hvilket må betyde, at der ikke er nogen løsninger, når $a = 2$. Dette må være grunden til, at totalmatricen var lavet under antagelsen, at $a \neq 2$.

1.d

Vi omskriver ligningssystemet til koefficientmatrice-form på venstre side, hvor $a = 1$, og sætter identitetmatricen på højre side. Herefter reducerer vi med Gauss-Jordan indtil, at vi har en reduceret rækkeeechelonform:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 1 & 1 & 4\cdot1 & 0 & 1 & 0 \\ 2 & 2 & 2\cdot1^2 & 0 & 0 & 1 \end{array}\right] reducer \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \quad -1r_1 \quad \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 0 & -1 & -4 & -1 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \\ -2r_1 \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 0 & -1 & -4 & -1 & 1 & 0 \\ 0 & -2 & -14 & -2 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \\ -2r_2 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 0 & -1 & -4 & -1 & 1 & 0 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} \cdot 3 \\ \cdot(-3) \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 3 & 6 & 24 & 3 & 0 & 0 \\ 0 & 3 & 12 & 3 & -3 & 0 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} +4r_3 \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 0 & 3 & -8 & 4 \\ 0 & 3 & 12 & 3 & -3 & 0 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} \\ +2r_3 \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 3 & 6 & 0 & 3 & -8 & 4 \\ 0 & 3 & 0 & 3 & -7 & 2 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} -2r_2 \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & -3 & 6 & 0 \\ 0 & 3 & 0 & 3 & -7 & 2 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} \cdot\frac{1}{3} \\ \cdot\frac{1}{3} \\ \cdot(-\frac{1}{6}) \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -\frac{7}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{6} \end{array}\right]$$

Vi har nu en rduceret rækkeeechelonform, så vi aflæser den inverse matrix af A til:

$$A^{-1} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -\frac{7}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$$

Følgende er på formen $Ax = b$:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

og kan omskrives til $A^{-1}b = x$:

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & -\frac{7}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{5}{3} \\ \frac{1}{6} \end{bmatrix}$$

Vi ender altså med løsningen:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{5}{3} \\ \frac{1}{6} \end{bmatrix}$$

2 Opgave

2.a

Vi opskriver vores rækkeoperationer som elementærmatricer:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & \frac{1}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Når man udfører rækkeoperationerne i rækkefølgen $E_4 \cdot (E_3 \cdot (E_2 \cdot (E_1 \cdot A)))$ svarer det til at gange med elementærmatricerne fra venstre:

$$E_4 \cdot (E_3 \cdot (E_2 \cdot (E_1 \cdot A))) = \begin{bmatrix} 1 & 0 & \frac{1}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 & 0 \\ -2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \right) \right) \right)$$

Det må betyde, at en mulig opskrivning er:

$$E_4 E_3 E_2 E_1 A = I$$

da vi laver rækkeoperationer på A i rækkefølgen E_1, E_2, E_3, E_4, og at dette er givet i opgaven, som lig enhedsmatricen.

Ifølge definition 2.7 på side 75 i Lineær Algebra i Datalogi, gælder der for en square matrix:

$$XA = I \Rightarrow AX = I$$

Det må dermed gælde, at:

$$E_4 E_3 E_2 E_1 A = I \Rightarrow A E_4 E_3 E_2 E_1 = I$$

2.b

Vi bestemmer $E_1^{-1}, E_2^{-1}, E_3^{-1}, E_4^{-1}$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2a & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] +r_1 \cdot 2a \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2a & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \cdot \frac{1}{5} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right] \Rightarrow E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] r_1 \leftrightarrow r_3 \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \Rightarrow E_3^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{a^2} & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] -r_3 \cdot \frac{1}{a^2} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\frac{1}{a^2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow E_4^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A er givet som produktet af de inverse elementærmatricer:

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2a \\ \frac{1}{5} & 0 & -\frac{1}{5a^2} \end{bmatrix}$$

2.c

X er givet som række 1 i X = række 1 i $E_1E_2E_3E_4$, og række 2 i X er = række 3 i $E_1E_2E_3E_4$:

$$E_1E_2E_3E_4 = \begin{bmatrix} 1 & 0 & 0 \\ -2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{a^2} & 0 & 5 \\ -2a & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{1}{a^2} & 0 & 5 \\ 1 & 0 & 0 \end{bmatrix}$$

Vi skal nu bestemme, om X er venstreinvers til V:

$$XV = \begin{bmatrix} \frac{1}{a^2} & 0 & 5 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2a \\ \frac{1}{5} & -\frac{1}{5a^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Da XV giver enhedsmatricen, er X venstreinvers til V.

Vi bestemmer nu alle venstreinverser til V, ved at lave et ligningssystem:
$x_3 = 1$
$x_1 + 2ax_2 - \frac{1}{5a^2}x_3 = 0$
$\frac{1}{5}x_6 = 0$
$x_4 + 2ax_5 - \frac{1}{5a^2}x_6 = 1$

Vi opskriver vores ligningssystem som en totalmatrix:

$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2a & -\frac{1}{5a^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 2a & -\frac{1}{5a^2} & 1 \end{array}\right] r_1 \leftrightarrow r_2 \rightsquigarrow \left[\begin{array}{cccccc|c} 1 & 2a & -\frac{1}{5a^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 2a & -\frac{1}{5a^2} & 1 \end{array}\right] r_3 \leftrightarrow r_4 \rightsquigarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 2a & -\frac{1}{5a^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2a & -\frac{1}{5a^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \end{array}\right] \begin{array}{c} \\ \\ \\ \cdot 5 \end{array} \rightsquigarrow \left[\begin{array}{cccccc|c} 1 & 2a & -\frac{1}{5a^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2a & -\frac{1}{5a^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} +r_2 \cdot \frac{1}{5a^2} \\ \\ +r_4 \cdot \frac{1}{5a^2} \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 2a & 0 & 0 & 0 & 0 & \frac{1}{5a^2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2a & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Vi kan nu aflæse løsningerne til:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{5a^2} \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2a \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2a \\ 1 \\ 0 \end{bmatrix}$$

For at se, om V har en højre invers, skal VX være lig med enhedsmatricen:

$$VX = \begin{bmatrix} 0 & 1 \\ 0 & 2a \\ \frac{1}{5} & -\frac{1}{5a^2} \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & 0 & 5 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Da vi ikke får enhedsmatricen, har V ikke en højre invers.

3 Opgave

3.a

Vi sætter et 1-tal i hver e_{uv}-element i matricen for hver edge $E(u, v)$ i grafen G_N:

$$N = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For at finde antallet af veje fra knude 4 til knude 1 med netop længde 8, skal vi kigge på element $(4, 1)$ i N^8. Vi kan finde $(N^8)_{4,1}$ ved:

$$69 \cdot 0 + 45 \cdot 1 + 45 \cdot 1 + 18 \cdot 1 + 29 \cdot 1 = 137$$

unikke veje fra knude 4 til knude 1 med længde 8.

3.b

Linkmatricen A findes ved at tage N^T og dividere hvert element med antallet af udgående edges i dens column.

$$N^T = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \end{bmatrix}$$

3.c

Vi kan lave et ligningssystem ud fra N, hvor alle elementer nu divideres med mængden af udgående edges den vertex i kolonnen er connected med:

$x_1 = \frac{1}{2}x_2 + x_3$
$x_2 = \frac{1}{2}x_1 + \frac{1}{2}x_5$
$x_3 = \frac{1}{2}x_1$
$x_4 = \frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3 + \frac{1}{2}x_5$
$x_5 = \frac{1}{2}x_1 + \frac{1}{4}x_4$

Vi opskriver vores ligningssystem som en totalmatrix:

$$\left[\begin{array}{ccccc|c} -1 & \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & -1 & 0 \end{array}\right] \begin{array}{l} r_1 \cdot 2 \\ r_2 \cdot 4 \\ r_3 \cdot 4 \\ r_4 \cdot 4 \\ r_5 \cdot 4 \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 & 2 & 0 \\ 2 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & 4 & -4 & 2 & 0 \\ 2 & 0 & 0 & 1 & -4 & 0 \end{array}\right] \begin{array}{l} \\ +r_1 \\ +r_1 \\ +r_1 \\ +r_1 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 3 & 6 & -4 & 2 & 0 \\ 0 & 1 & 2 & 1 & -4 & 0 \end{array}\right] \begin{array}{l} \\ \\ r_3 \cdot 3 \\ \\ r_5 \cdot 3 \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 3 & -6 & 0 & 0 & 0 \\ 0 & 3 & 6 & -4 & 2 & 0 \\ 0 & 3 & 6 & 3 & -12 & 0 \end{array}\right] \begin{array}{l} \\ \\ +r_2 \\ +r_2 \\ +r_2 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 0 & -4 & 0 & 2 & 0 \\ 0 & 0 & 8 & -4 & 4 & 0 \\ 0 & 0 & 8 & 3 & -10 & 0 \end{array}\right] \begin{array}{l} \\ \\ \cdot 2 \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 8 & -4 & 4 & 0 \\ 0 & 0 & 8 & 3 & -10 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ +r_3 \\ +r_3 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 & 8 & 0 \\ 0 & 0 & 0 & 3 & -6 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \cdot 3 \\ \cdot 4 \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 12 & -24 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ +r_4 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \cdot 4 \\ \cdot 4 \\ \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -8 & 4 & 8 & 0 & 0 & 0 \\ 0 & -12 & 8 & 0 & 8 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} +r_3 \\ +r_3 \\ \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -8 & 4 & 0 & 0 & 4 & 0 \\ 0 & -12 & 0 & 0 & 12 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \cdot 3 \\ \\ \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} -24 & 12 & 0 & 0 & 12 & 0 \\ 0 & -12 & 0 & 0 & 12 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} +r_2 \\ \\ \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} -24 & 0 & 0 & 0 & 24 & 0 \\ 0 & -12 & 0 & 0 & 12 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \cdot(-\frac{1}{24}) \\ \cdot(-\frac{1}{12}) \\ \cdot(-\frac{1}{8}) \\ \cdot(-\frac{1}{12}) \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Vi kan nu aflæse løsningerne til:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \\ 2 \\ 1 \end{bmatrix}$$

Vi ser nu, at side 4 er vigtigst, 1, 2, 5 næst mest vigtige, og 3 er mindst vigtig.

4 Opgave

Se vedhæftede python-fil.