

Københavns Universitet

LinAlgDat - Project B

Victor Vangkilde Jørgensen - kft410
kft410@alumni.ku.dk
Hold 13 Mach

10. maj 2025

Indhold

1	Opgave	3
1.a	3
1.b	3
1.c	3
1.d	4
1.e	4
2	Opgave	4
2.a	4
2.b	6
2.c	6
2.d	6
2.e	6

1 Opgave

1.a

Vi kan aflæse M_a til:

$$\begin{bmatrix} a & -1 & -1 \\ 0 & (a-1) & -1 \\ 0 & 2 & (a+2) \end{bmatrix}$$

1.b

$$\begin{aligned} & \left[\begin{array}{ccc|c} a & -1 & -1 & 0 \\ 0 & a-1 & -1 & 0 \\ 0 & 2 & a+2 & 0 \end{array} \right] \cdot \frac{1}{a-1} \rightsquigarrow \left[\begin{array}{ccc|c} a & -1 & -1 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 \\ 0 & 2 & a+2 & 0 \end{array} \right] \xrightarrow{-2r_2} \left[\begin{array}{ccc|c} a & -1 & -1 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 \\ 0 & 0 & \frac{a^2+a}{a-1} & 0 \end{array} \right] \\ & \xrightarrow{\cdot \frac{a-1}{a^2+a}} \left[\begin{array}{ccc|c} a & -1 & -1 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{+r_3} \left[\begin{array}{ccc|c} a & -1 & -1 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{+r_3 \cdot \frac{1}{a-1}} \left[\begin{array}{ccc|c} a & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \\ & \xrightarrow{+r_2} \left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{\cdot \frac{1}{a}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \end{aligned}$$

T_a er altså injektiv.

T_a er surjektiv, da vi har 3 vektorer. T_a er dermed bijektiv, da den både er injektiv og surjektiv.

Vi bestemmer nu T_a^{-1} :

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} a & -1 & -1 & 1 & 0 & 0 \\ 0 & a-1 & -1 & 0 & 1 & 0 \\ 0 & 2 & a+2 & 0 & 0 & 1 \end{array} \right] \cdot \frac{1}{a-1} \rightsquigarrow \left[\begin{array}{ccc|ccc} a & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 & \frac{1}{a-1} & 0 \\ 0 & 2 & a+2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{-2r_2} \left[\begin{array}{ccc|ccc} a & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 & \frac{1}{a-1} & 0 \\ 0 & 0 & \frac{a^2+a}{a-1} & 0 & -\frac{2}{a-1} & 1 \end{array} \right] \\ & \xrightarrow{\cdot \frac{a-1}{a^2+a}} \left[\begin{array}{ccc|ccc} a & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 & \frac{1}{a-1} & 0 \\ 0 & 0 & 1 & 0 & -\frac{2}{a-1} & \frac{a-1}{a^2+a} \end{array} \right] \xrightarrow{+r_2} \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & \frac{1}{a-1} & \frac{1}{a-1} \\ 0 & 1 & 0 & 0 & \frac{1}{a-1} & \frac{1}{a-1} \\ 0 & 0 & 1 & 0 & -\frac{2}{a-1} & \frac{a-1}{a^2+a} \end{array} \right] \xrightarrow{\cdot \frac{1}{a}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & \frac{1}{a^2+a} & \frac{1}{a^2+a} \\ 0 & 1 & 0 & 0 & \frac{1}{a^2+a} & \frac{1}{a^2+a} \\ 0 & 0 & 1 & 0 & -\frac{2}{a^2+a} & \frac{a-1}{a^2+a} \end{array} \right] \end{aligned}$$

1.c

Vi opstill igen T_a , hvor $a = -1$:

$$\begin{aligned}
& \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 2 & -1+2 & 0 \end{array} \right] \text{ reducer } \rightsquigarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{array} \right] \rightsquigarrow -r_2 \\
& \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \begin{array}{l} +r_3 \\ +r_3 \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \begin{array}{l} \cdot(-1) \\ \cdot(-\frac{1}{2}) \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \begin{array}{l} -r_2 \\ \\ \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]
\end{aligned}$$

1.d

1.e

2 Opgave

2.a

Vi opstiller et ligningssystem i form af en totalmatrix, hvor vi sætter u_1, u_2, u_3 lig hhv. v_1, v_2, v_3 , og finder løsningerne til disse, ved brug af Gauss-Jordan elimination.

$$u_1 + u_2 + u_3 = v_1 \Leftrightarrow$$

$$\begin{aligned}
& \left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 1 & 1 & -1 & -2 \\ -1 & -1 & 2 & 3 \\ 1 & 1 & 2 & 1 \end{array} \right] \begin{array}{l} \cdot 2 \\ \cdot(-2) \\ \cdot 2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 2 & 2 & -2 & -4 \\ 2 & 2 & -4 & -6 \\ 2 & 2 & 4 & 2 \end{array} \right] \begin{array}{l} -r_1 \\ -r_1 \\ -r_1 \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 0 & 2 & -3 & -11 \\ 0 & 2 & -5 & -13 \\ 0 & 2 & 3 & -5 \end{array} \right] \begin{array}{l} \\ -r_2 \\ -r_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 0 & 2 & -3 & -11 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 6 & 6 \end{array} \right] \begin{array}{l} \\ \\ +3r_3 \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 0 & 2 & -3 & -11 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \\ \cdot(-\frac{1}{2}) \\ \cdot(-\frac{1}{2}) \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 0 & 2 & -3 & -11 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \\ -1r_3 \\ +3r_3 \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & 2 & 0 & -8 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \cdot\frac{1}{2} \\ \cdot\frac{1}{2} \\ \cdot\frac{1}{2} \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]
\end{aligned}$$

Vores første kolonne i $P_{B \leftarrow C}$ er dermed: $\begin{bmatrix} 3 \\ -4 \\ 1 \\ 0 \end{bmatrix}$

$$u_1 + u_2 + u_3 = v_2 \Leftrightarrow$$

$$\begin{aligned}
& \left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 1 & 1 & 2 & -3 \end{array} \right] \begin{array}{l} \cdot 2 \\ \cdot (-2) \\ \cdot 2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 2 & 2 & -2 & 0 \\ 2 & 2 & -4 & 2 \\ 2 & 2 & 4 & -6 \end{array} \right] \begin{array}{l} -r_1 \\ -r_1 \\ -r_1 \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 2 & -5 & 3 \\ 0 & 2 & 3 & -5 \end{array} \right] \begin{array}{l} -r_2 \\ -r_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 6 & -6 \end{array} \right] \begin{array}{l} +3r_3 \\ -1r_3 \\ +3r_3 \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \cdot (-\tfrac{1}{2}) \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} -1r_3 \\ +3r_3 \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \cdot \frac{1}{2} \\ \cdot \frac{1}{2} \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \\
& \text{Vores anden kolonne i } P_{B \leftarrow C} \text{ er dermed: } \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}
\end{aligned}$$

$$u_1 + u_2 + u_3 = v_3 \Leftrightarrow$$

$$\begin{aligned}
& \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{array} \right] \begin{array}{l} \cdot 2 \\ \cdot (-2) \\ \cdot 2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 2 & 2 & -2 & -2 \\ 2 & 2 & -4 & -4 \\ 2 & 2 & 4 & 4 \end{array} \right] \begin{array}{l} -r_1 \\ -r_1 \\ -r_1 \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -3 & -5 \\ 0 & 2 & -5 & -7 \\ 0 & 2 & 3 & 1 \end{array} \right] \begin{array}{l} -r_2 \\ -r_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -3 & -5 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 6 & 6 \end{array} \right] \begin{array}{l} +3r_3 \\ -1r_3 \\ +3r_3 \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -3 & -5 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right] \cdot (-\tfrac{1}{2}) \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -3 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} -1r_3 \\ +3r_3 \end{array} \rightsquigarrow \\
& \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \cdot \frac{1}{2} \\ \cdot \frac{1}{2} \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \\
& \text{Vores sidste kolonne i } P_{B \leftarrow C} \text{ er dermed: } \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

Sammensætter vi nu vores tre kolonner til en matrix, får vi:

$$P_{B \leftarrow C} = \begin{bmatrix} 3 & 0 & 1 \\ -4 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

2.b

2.c

2.d

2.e