Københavns Universitet
LinAlgDat - Project B

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Hold 13 Mach

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Indhold

1 Opgave

1.a

Vi kan aflæse M_a til:

$$\begin{bmatrix} a & -1 & -1 \\ 0 & (a-1) & -1 \\ 0 & 2 & (a+2) \end{bmatrix}$$

1.b

$$\left[\begin{array}{ccc|c} a & -1 & -1 & 0 \\ 0 & a-1 & -1 & 0 \\ 0 & 2 & a+2 & 0 \end{array}\right] \begin{matrix} \\ \cdot\frac{1}{a-1} \\ \\ \end{matrix} \rightsquigarrow \left[\begin{array}{ccc|c} a & -1 & -1 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 \\ 0 & 2 & a+2 & 0 \end{array}\right] \begin{matrix} \\ \\ -2r_2 \end{matrix} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} a & -1 & -1 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 \\ 0 & 0 & \frac{a^2+a}{a-1} & 0 \end{array}\right] \begin{matrix} \\ \\ \cdot\frac{a-1}{a^2+a} \end{matrix} \rightsquigarrow \left[\begin{array}{ccc|c} a & -1 & -1 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{matrix} +r_3 \\ +r_3\cdot\frac{1}{a-1} \\ \\ \end{matrix} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} a & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{matrix} +r_2 \\ \\ \\ \end{matrix} \rightsquigarrow \left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{matrix} \cdot\frac{1}{a} \\ \\ \\ \end{matrix} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

T_a er altså injektiv.
T_a er surjektiv, da vi har 3 vektoerer. T_a er dermed bijektiv, da den både er injektiv og surjektiv.

Vi bestemmer nu T_a^{-1}:

$$\left[\begin{array}{ccc|ccc} a & -1 & -1 & 1 & 0 & 0 \\ 0 & a-1 & -1 & 0 & 1 & 0 \\ 0 & 2 & a+2 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \cdot\frac{1}{a-1} \\ \\ \end{matrix} \rightsquigarrow \left[\begin{array}{ccc|ccc} a & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 & \frac{1}{a-1} & 0 \\ 0 & 2 & a+2 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ -2r_2 \end{matrix} \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} a & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 & \frac{1}{a-1} & 0 \\ 0 & 0 & \frac{a^2+a}{a-1} & 0 & -\frac{2}{a-1} & 1 \end{array}\right] \begin{matrix} \\ \\ \cdot\frac{a-1}{a^2+a} \end{matrix} \rightsquigarrow \left[\begin{array}{ccc|ccc} a & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{a-1} & 0 & \frac{1}{a-1} & 0 \\ 0 & 0 & 1 & 0 & -\frac{2}{a^2+a} & \frac{a-1}{a^2+a} \end{array}\right] \begin{matrix} +r_3 \\ +r_3\cdot\frac{1}{a-1} \\ \\ \end{matrix} \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} a & -1 & 0 & 1 & -\frac{2}{a^2+a} & \frac{a-1}{a^2+a} \\ 0 & 1 & 0 & 0 & \frac{a+2}{a^2+a} & \frac{1}{a^2+a} \\ 0 & 0 & 1 & 0 & -\frac{2}{a^2+a} & \frac{a-1}{a^2+a} \end{array}\right] \begin{matrix} +r_2 \\ \\ \\ \end{matrix} \rightsquigarrow \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & \frac{1}{a+1} & \frac{1}{a+1} \\ 0 & 1 & 0 & 0 & \frac{a+2}{a^2+a} & \frac{1}{a^2+a} \\ 0 & 0 & 1 & 0 & -\frac{2}{a^2+a} & \frac{a-1}{a^2+a} \end{array}\right] \begin{matrix} \cdot\frac{1}{a} \\ \\ \\ \end{matrix} \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & \frac{1}{a^2+a} & \frac{1}{a^2+a} \\ 0 & 1 & 0 & 0 & \frac{a+2}{a^2+a} & \frac{1}{a^2+a} \\ 0 & 0 & 1 & 0 & -\frac{2}{a^2+a} & \frac{a-1}{a^2+a} \end{array}\right]$$

1.c

Vi opstill igen T_a, hvor $a = -1$:

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -1-1 & -1 & 0 \\ 0 & 2 & -1+2 & 0 \end{array}\right] \; reducer \rightsquigarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \\ -r_2 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \begin{array}{c} \\ \\ \cdot\frac{1}{2} \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} +r_3 \\ +r_3 \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} \cdot(-1) \\ \cdot(-\frac{1}{2}) \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} -r_2 \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

1.d

1.e

2 Opgave

2.a

Vi opstiller et ligningssystem i form af en totalmatrix, hvor vi sætter u_1, u_2, u_3 lig hhv. v_1, v_2, v_3, og finder løsningerne til disse, ved brug af Gauss-Jordan elimination.

$u_1 + u_2 + u_3 = v_1 \Leftrightarrow$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 1 & 1 & -1 & -2 \\ -1 & -1 & 2 & 3 \\ 1 & 1 & 2 & 1 \end{array}\right] \begin{array}{c} \\ \cdot 2 \\ \cdot(-2) \\ \cdot 2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 2 & 2 & -2 & -4 \\ 2 & 2 & -4 & -6 \\ 2 & 2 & 4 & 2 \end{array}\right] \begin{array}{c} \\ -r_1 \\ -r_1 \\ -r_1 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 0 & 2 & -3 & -11 \\ 0 & 2 & -5 & -13 \\ 0 & 2 & 3 & -5 \end{array}\right] \begin{array}{c} \\ \\ -r_2 \\ -r_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 0 & 2 & -3 & -11 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 6 & 6 \end{array}\right] \begin{array}{c} \\ \\ \\ +3r_3 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 0 & 2 & -3 & -11 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \\ \cdot(-\frac{1}{2}) \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 7 \\ 0 & 2 & -3 & -11 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} -1r_3 \\ +3r_3 \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & 2 & 0 & -8 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \cdot\frac{1}{2} \\ \cdot\frac{1}{2} \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Vores første kolonne i $P_{B \leftarrow C}$ er dermed: $\begin{bmatrix} 3 \\ -4 \\ 1 \\ 0 \end{bmatrix}$

$u_1 + u_2 + u_3 = v_2 \Leftrightarrow$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 1 & 1 & 2 & -3 \end{array}\right] \begin{array}{c} \\ \cdot 2 \\ \cdot(-2) \\ \cdot 2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 2 & 2 & -2 & 0 \\ 2 & 2 & -4 & 2 \\ 2 & 2 & 4 & -6 \end{array}\right] \begin{array}{c} \\ -r_1 \\ -r_1 \\ -r_1 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 2 & -5 & 3 \\ 0 & 2 & 3 & -5 \end{array}\right] \begin{array}{c} \\ \\ -r_2 \\ -r_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 6 & -6 \end{array}\right] \begin{array}{c} \\ \\ \\ +3r_3 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \\ \cdot(-\frac{1}{2}) \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} -1r_3 \\ +3r_3 \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \cdot\frac{1}{2} \\ \cdot\frac{1}{2} \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Vores anden kolonne i $P_{B\leftarrow C}$ er dermed: $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

$u_1 + u_2 + u_3 = v_3 \Leftrightarrow$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{array}\right] \begin{array}{c} \\ \cdot 2 \\ \cdot(-2) \\ \cdot 2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 2 & 2 & -2 & -2 \\ 2 & 2 & -4 & -4 \\ 2 & 2 & 4 & 4 \end{array}\right] \begin{array}{c} \\ -r_1 \\ -r_1 \\ -r_1 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -3 & -5 \\ 0 & 2 & -5 & -7 \\ 0 & 2 & 3 & 1 \end{array}\right] \begin{array}{c} \\ \\ -r_2 \\ -r_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -3 & -5 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 6 & 6 \end{array}\right] \begin{array}{c} \\ \\ \\ +3r_3 \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -3 & -5 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \\ \cdot(-\frac{1}{2}) \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -3 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} -1r_3 \\ +3r_3 \\ \\ \\ \end{array} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \cdot\frac{1}{2} \\ \cdot\frac{1}{2} \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Vores sidste kolonne i $P_{B\leftarrow C}$ er dermed: $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

Sammensætter vi nu vores tre kolonner til en matrix, får vi:

$$P_{B\leftarrow C} = \begin{bmatrix} 3 & 0 & 1 \\ -4 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

2.b

$$x = \begin{bmatrix} 7 \\ -2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ 4 \\ 0 \end{bmatrix}$$

Da konstanterne foran v i hvert led er 1, og $v_1, v_2, v_3 \in \mathcal{C}$, er koordinaterne for x med henhold til $\mathcal{C}$:

$$[x]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Vi benytter vores basisskriftmatrice til at transformere vores koordinater til basen $\mathcal{B}$ fra $\mathcal{C}$:

$$[x]_{\mathcal{B}} = \begin{bmatrix} 3 & 0 & 1 \\ -4 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 1 \end{bmatrix}$$

2.c

Vi ganger kolonne 2 i vores basisskriftmatrice på u_1 og u_2:

$$-1 \cdot u_1 + (-1) \cdot u_2 = -1 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ -3 \end{bmatrix}$$

Vi får v_2, så v_2 må dermed række spannet af $u2, u3$.

Mangler at lave resten af opgaven

2.d

2.e