Controlling traffic congestions

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Abstract—Traffic congestion is a serious problem in large cities. This project explores the use of dynamic speed limits as a solution, reproducing the results of previous works. Specifically, the Lighthill-Whitham-Richards/Greenshields model is used for traffic flow. The Linear Quadratic Regulator (LQR) method is adapted for this application by linearizing the nonlinear traffic model around equilibrium points. Numerical results demonstrate that dynamic speed control effectively stabilizes traffic density and improves flow conditions.

1 Introduction

Traffic congestion is a growing problem in modern cities, as it leads to an increase in commuting times, air pollution and stress on the population [9]. In the latest years, many efforts have been made to produce accurate models of traffic with the goal of understanding and solving this problem. From a practical point of view, control can be obtained by acting on different aspects of traffic. Some proposals act directly on the topology of the traffic infrastructure, by studying how to optimally add new roads to reduce the car density [10]. Other common proposals focus on active forms of control, such as dynamically adjusting traffic lights [6] or traffic-regulating self driving vehicles [8]. In this report, I will focus on *speed limits* and I will show how they can be used to reduce congestion. In particular, I will try and reproduce the results of [2]. The main idea behind this proposal is to take a stretch of road and implement a way to adjust speed limits along its length. Then, vary the speed limits using some control law that aims to maximize the traffic stability according to some mathematical model. In the real world, this can be easily achieved by using cameras to detect traffic parameters and the speed-limit displays that are already found in modern freeways. I will now start with a brief theoretical description of the tools needed to understand my work; then, I will derive the optimal control strategy for the chosen traffic model and I will conclude the discussion by showing some numerical integrations of the results.

2 Theoretical background

2.1 Optimal control: the Linear Quadratic Regulator

Consider a system whose time evolution can be described by an equation in the form $\dot{x}(t) = f(x(t), u)$. Here, x represents the state of the system and u represents the state of the controls; that is, u is a set of parameters that we can dynamically change to alter the evolution of the system. Now, we would like to find the optimal control law $u = u(t)$ such that out system converges to a chosen equilibrium point x_0. This defines a *control problem*. There exist different techniques to find a solution for such problem [11]; here we will focus on the *Linear Quatratic Regulator*, LQR. It is a form of *optimal* control, as it is formulated with the minimization of a (quadratic) cost functional, and it was originally developed for working with linear, finite-dimensional systems.

LQR formulation

Consider a linear system written in state space representation:

$$\dot{x} = Ax + Bu \tag{1}$$

where A, B are real finite-dimensional matrices and u, x are exactly as above. LQR consists in finding the control law $u(t)$ that minimizes the cost functional

$$J(x, u) = \int_0^{+\infty} x^T Q x + u^T R u \, dt, \tag{2}$$

where Q and R are real matrices that quantify the *cost* of being away from the equilibrium point and of the actuation of the controls. The solution is found by imposing the variational principle $\delta J = 0$ which, in practice, results in having to solve the Euler-Lagrange equations applied to the integrand of (2).

Working in infinite-dimensional systems

There exists a generalization of LQR for infinite-dimensional systems [1]. For the purposes of this work, it is sufficient to say that the system needs to be written in the same form as of (1), but now the mathematical objects are slightly different:

- The state space is now an Hillbert space: $x \in \mathcal{H} = L^2(X)$, $u \in L^2(X)$, where X is the spatial region over which we want to apply the control.
- A, B are now operators: $A : D(A) \to \mathcal{H}$, $B : D(B) \to \mathcal{H}$.
- A can be written as: $A = V\frac{\partial}{\partial z}$ with V being a real symmetric matrix and $B = B_0\mathbb{I}$ with B_0 being a real matrix.

The cost functional now takes the form:

$$J(x,u) = \int_0^{+\infty} \langle x, Qx\rangle + \langle u, Ru\rangle dt$$

with $< f, g >= \int_X fg\, d\mu$ being the standard inner product in L^2 and $Q = Q_0\mathbb{I}$, $R = R_0\mathbb{I}$ being operators parametrized by the scalars $Q_0, R_0 \in \mathbb{R}$.

It can be shown [1, 4] that if we impose $\delta J = 0$, the solution for the optimal control law has the form

$$u_{opt}(t) = K_0 x(t) \tag{3}$$

where $K_0 = -R^{-1}B^*P$ and $P \in L^2(X)$ is the solution to the following Riccati equation:

$$\left[A^*P + PA + C^*QC - PBR^{-1}B^*P\right] x = 0.$$

In our case, it can be shown [1] that the solution for P is given by:

$$\begin{aligned} P &= \Phi(z) \\ \Phi(l) &= 0 \\ V\frac{d\Phi}{dz} &= Q_0 - \Phi B_0 R_0^{-1} B_0^* \Phi \end{aligned} \tag{4}$$

Working with nonlinear systems

LQR cannot be used *directly* to control a nonlinear system. In this work, the traffic model which we will be using is nonlinear and we will need to adapt it to work in conjuction with LQR. The most common way to do so is:

1) linearize the model around an equilibrium point,
2) compute the control law using the linearized system, and
3) apply that to the full nonlinear system and check if the results are still valid.

If the model deviates too much from equilibrium, then there are many nonlinear control techniques that can be used, but for now they do not concern us. This approach works fairly well in real world applications. As a matter of fact, I wrote my Bachelor's thesis on this, and in there I have shown how the prototypical *inverted-pendulum-on-a-cart* system can be controlled with a combination of different techniques [3].

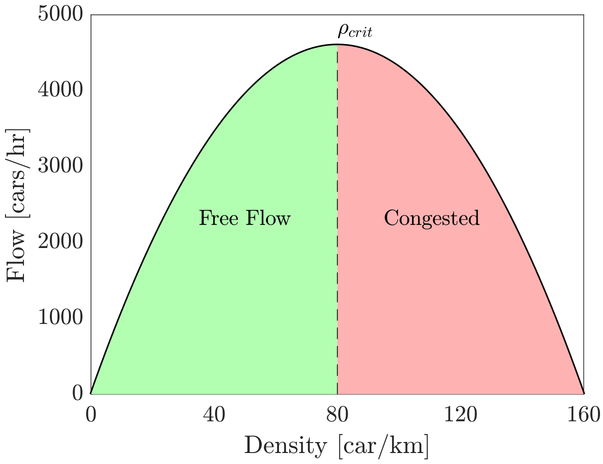


Figure 1: Fundamental diagram of Greenshields model. You can see that this model predicts two different phases of traffic flow: if the car density is low enough, we are in the free flow regime. If we increase the density, the throughput of the road increases up to a point in which the cars are forced to slow down. After this point, we are in the congested regime: the high density forces drivers to reduce their velocity and the car throughput drops.

2.2 The LWR-Greenshield traffic model

The Lighthill-Whitham-Richards, LWR, model [7] is a macroscopic model defined by the conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial z} = 0. \tag{5}$$

Here $\rho = \rho(t,z)$ is the traffic density and $q = q(\rho)$ is the *fundamental diagram* which defines the relationship between traffic density and velocity: $q = \rho v(\rho)$. For this study, we will be using the Greenshields model (Figure 1) [5]:

$$v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right). \tag{6}$$

3 Control law derivation

First, we start by adding controls to the model. We want to alter the speed limit in the road so a good choice is altering (6) by adding a control parameter b that changes the $v_{\max}$ constant:

$$v_{ctrl}(\rho) = bv_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{7}$$

Now we can linearize the model. We consider small deviations in the equilibrium velocity and in the control parameter

$$\begin{aligned} \rho &= \rho_0 + \Delta\rho \\ b &= b_0 + \Delta b \end{aligned} \tag{8}$$

and by inserting (7) and (8) in the model (5), we get the linearized form:

$$\frac{\partial \Delta\rho}{\partial t} + b_0 v_{\max}\left(1 - 2\frac{\rho_0}{\rho_{\max}}\right)\frac{\partial \Delta\rho}{\partial z} + \\ + \rho_0 v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)\frac{\partial \Delta b}{\partial z} = 0 \quad (9)$$

(note that $\partial\Delta\rho = \partial\rho$, $\partial\Delta b = \partial b$). Now, we rewrite this model in the state-space representation. If we define

$$u = \frac{\partial b}{\partial z}$$
$$B = -\rho_0 v_{\max}\left(1 - \frac{\rho_0}{\rho_{\max}}\right)$$
$$V = -b_0 v_{\max}\left(1 - 2\frac{\rho_0}{\rho_{\max}}\right)$$
$$A = V\frac{d}{dz}$$

we get the equation

$$\dot\rho(t, z) = A\rho(t, z) + Bu(t, z) \quad (10)$$

and we can check (10) and (9) are equal. Before moving on, please note that the control parameter that we have here is $u = db/dz$, and not b. This change is needed just so we can apply LQR to the linearized system; in practice, we can get back b by just integrating: $b(t, z) = \int_0^z u(t, z')dz' + b_0$.

Now we proceed by finding the solution v_{opt}. We start by integrating (4), from which we get

$$\frac{V}{2B_0\sqrt{Q_0}} \ln\frac{|B_0\Phi + \sqrt{Q_0}|}{|B_0\Phi - \sqrt{Q_0}|} = z + const.$$

The absolute values lead to two possible solutions, but by applying the condition $\Phi(l) = 0$ we are left with only one:

$$\Phi = \frac{\sqrt{Q_0}}{B_0}\tanh\left[\frac{2B_0\sqrt{Q_0}}{V}(z - L)\right].$$

To get the optimal control law, we combine this last result with (3) and after setting $R_0 = 1$ we get

$$u_{opt} = -\Delta\rho\sqrt{Q_0}\tanh\left[\frac{2B_0\sqrt{Q_0}}{V}(z - L)\right].$$

Setting $R_0 = 1$ is done so that the only tuning parameter is Q_0.

4 Results and Conclusions

To reproduce the results of [2], I performed a numerical integration of the model. The parameters I have used are summarized in Table 1. I did not perform any parameter tuning on Q_0, but I chose a value that would yield good results. I had a hard time reproducing the exact results of [2] since they specified the parameters they used using the wrong dimensions. The initial conditions I used are representative of a stretch of road situated right after an entrance ramp on a highway: there is an increasing and oscillating flow of cars coming in from $z = 0$ and there is already some variance in the density at $t = 0$:

$$\rho(0, z) = \rho_0\left(1 + \frac{1}{10}\sin\frac{\pi z}{l}\right)$$
$$\rho(t, 0) = \rho_0\left[1 + \frac{1}{2}\left(1 - e^{-\frac{t}{100}}\right) + \frac{1}{10}\sin\frac{\pi t}{50}\right]$$

The behavior of $\rho(t, 0)$ is shown in Figure 2.

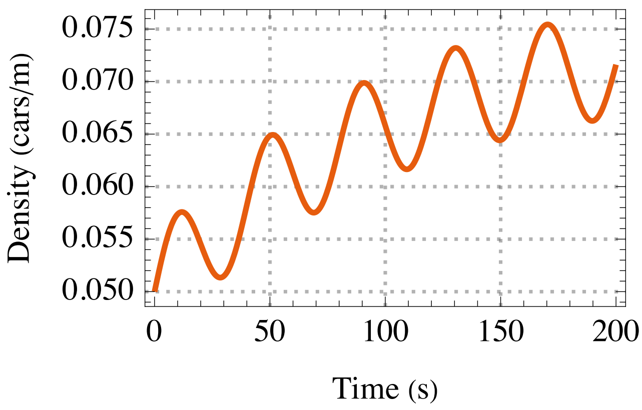


Figure 2: Initial condition for $z = 0$. The flow of cars is increasing and oscillating. This initial condition is different with respect to the one in [2], as theirs was written in a rather messy way. The idea behind their choice was the same as mine.

Parameter		Value	unit
Default speed control	b_0	1	/
Maximum density	$\rho_{\max}$	.16	$cars/m$
Average density	ρ_0	.05	$cars/m$
Road length	l	2000	m
Maximum speed	$v_{\max}$	30	m/s
Simulation time	T	200	s
Controller gain	Q_0	.05	/

Table 1: Parameters used for the integrations. These are mostly the same as in [2]

4.1 Results

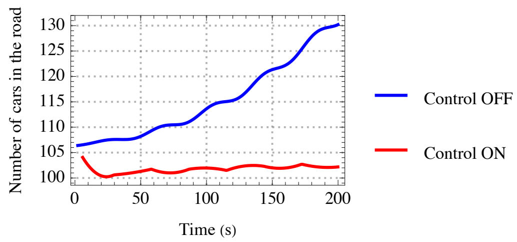
First, I integrated the linear system with and without LQR (Figure 4, Figure 5). The results show that by adding the control, the density of cars gets brought down to a stable value. Moreover, in the uncontrolled system we see that the density gets close to the critical value of $\rho_{\max}/2$; this does not happen in the controlled system.

LQR also manages to stabilize the full nonlinear system (Figure 6). In this last integration, I used an approximation and treated b as constant along z: $b(t) = \int_0^l db(t,z)/dz\, dz$. I did this to speed up the computations[1] and also because it appears it is the same thing they did in [2].

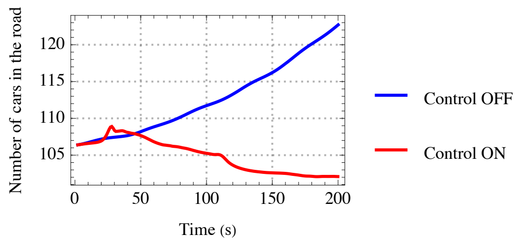
These results qualitatively match the ones in [2].

4.2 Conclusions

In this work, I have shown how it is possible to reduce traffic congestion in a stretch of road by adding dynamic speed limits which adapt to the traffic density according to an optimal control law. My work took inspiration from a recent paper [2] and I was able to fully reproduce their results. The final picture of my results is summarized in Figure 3 where it is shown that, by applying this control strategy, traffic would flow faster and more uniformly. Concerning real-world applications, I expect these results to be valid for scenarios in which the LWR/Greenshields model is valid.

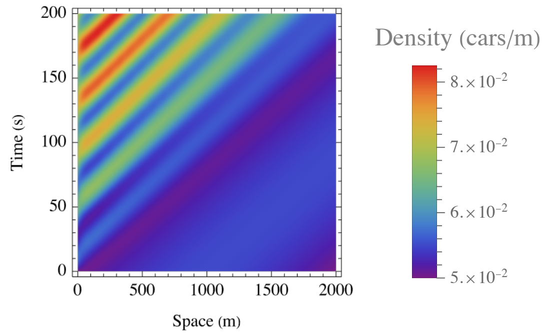


(a) Linear model

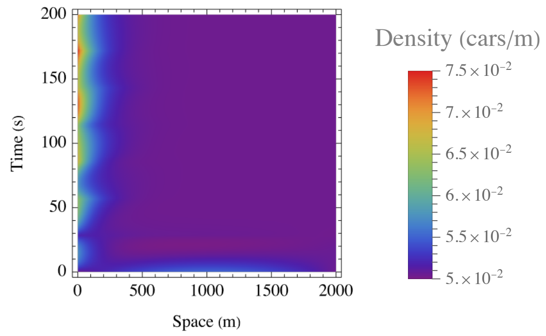


(b) Nonlinear model

Figure 3: Number of cars stuck travelling through the simulated stretch of road for both the linear (a) and nonlinear (b) model. The results show that by applying the control strategy described in this paper, this number gets brought down to a constant value, thus increasing flow and lowering the density.

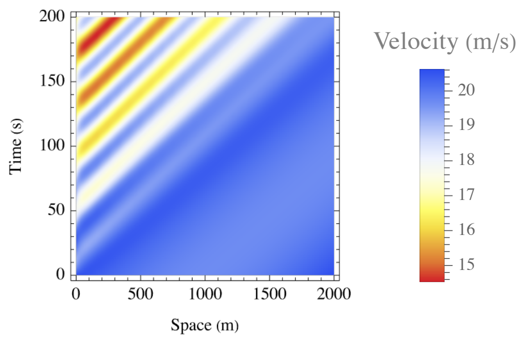


(a) Control OFF

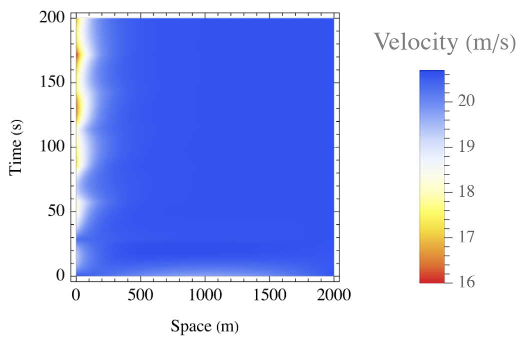


(b) Control ON

Figure 4: Integration results for the density of the linear model. When control is turned ON, the car density is lower on average and the flow is constant.



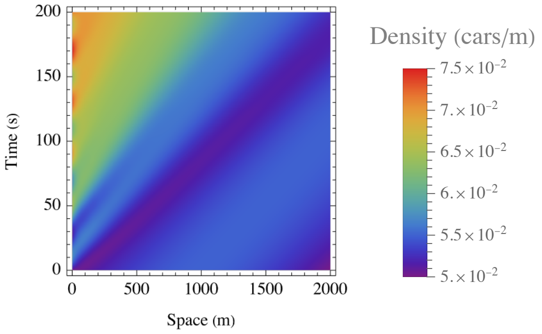
(a) Control OFF



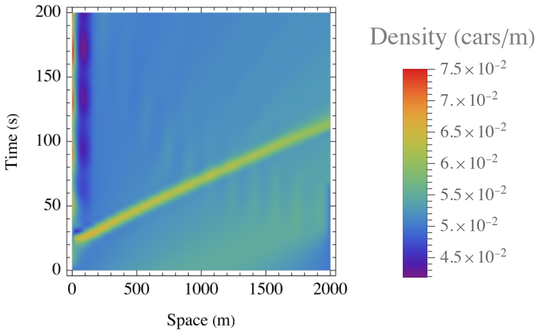
(b) Control ON

Figure 5: Integration results for the velocity of the linear model. When control is turned ON, cars move faster on average and the flow is constant.

1. I had some trouble with Mathematica not liking the equations with control. I would have had to re-do the integration with another software but I did not have enough time for that.



(a) Control OFF



(b) Control ON

Figure 6: Integration results for the density of the nonlinear model. The control computed using the linear model appears to work well even in this case. It brings down the density to a constant low value and reduces oscillations.

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