Computational Discrete Geometry

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Abstract. In recent years, computers have been used regularly to solve major problems in discrete geometry. The talk at ICMS 2010 will give a survey of the computational methods. The extended abstract that is provided below mentions a few of the problems that will be discussed.

Newton-Gregory Problem

In a famous discussion, Isaac Newton claimed that at most twelve nonoverlapping congruent balls in Euclidean three space can touch one further ball at the center of them all. Gregory thought that it might be possible for thirteen balls to touch the one at the center. It was only in 1953 that Newton was finally proved correct. Earlier this year, Musin and Tarasov announced that they have finally determined the optimal arrangement of thirteen balls [**?**]. These thirteen balls do not touch the one at the center, but they come as close as possible. Their proof involves an analysis of more than 94 million planar graphs, which have been generated with the program *plantri* [**?**]. Linear programming methods are used to exclude all but the one optimal graph.

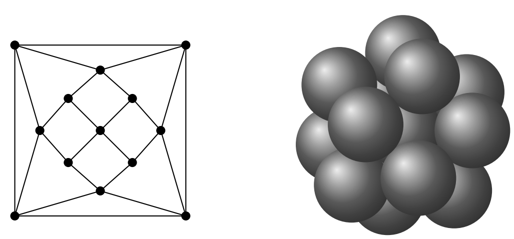

Fig. 1. Musin and Tarasov recently proved that this arrangement of thirteen congruent is optimal. Each node of the graph represents one of the thirteen balls and each edge represents a pair of touching balls. The node at the center of the graph corresponds to the uppermost ball in the second frame.

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Hilbert's Eighteenth Problem

In 1900, in his famous list of problems, Hilbert asked, "How can one arrange most densely in space an infinite number of equal solids of given form, e. g., spheres with given radii or regular tetrahedra with given edges (or in prescribed position), that is, how can one so fit them together that the ratio of the filled to the unfilled space may be as great as possible" [**?**]?

Dense Sphere Packings The solution to the sphere-packing problem was published in [**?**]. It is now the subject of a large scale formal-proof project, Flyspeck, in the HOL Light proof assistant. The talk will describe the current status of this project.

Tetrahedra Aristotle erroneously believed that the regular tetrahedron tiles three dimensional space: "It is agreed that there are only three plane figures which can fill a space, the triangle, the square, and the hexagon, and only two solids, the pyramid and the cube" [**?**]. In fact, the tetrahedron cannot tile because its dihedral angle is about 70.5°, which falls short of the angle $72 = 360/5$ that would be required of a space-filling tile.

Attention has turned to the tetrahedron-packing problem, which has come under intensive investigation over the past few years [**?**], [**?**]. As a result of Monte Carlo simulations by Chen et al., the optimal packing is now given by an explicit conjecture.

Dense Lattice Packings of Spheres in High Dimensions

Lagrange found that the denset packing of congruent disks in the plane, among all lattice packings, is the hexagonal packing [**?**]. See Figure **??**. Gauss solved the analogous problem in three dimensions [**?**]. During the early decades of the twentieth century, the problem of determining the densest lattice packing of balls was solved in dimensions up to eight. Cohn and Kumar, in a computer assisted proof, have solved the problem in dimension 24 [**?**]. Their proof relies on a variety of computations and mathematical methods, including the Poisson summation formula and spherical harmonics.

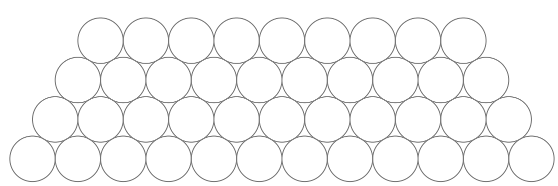

Fig. 2. Lagrange proved that this is the densest of all lattice packings in two dimensions.

Other problems

This abstract has mentioned just a few of a large number of problems in discrete geometry that have been or that are apt to be solved by computer. Others include Fejes Toth's contact conjecture, the Kelvin problem, circle packing problems, the strong dodecahedral conjecture, the Reinhardt conjecture, and the covering problem. Discrete geometry depends on the development of software to assist in the solution to these problems.

References

1. Aristotle, *On the heaven*, translated by J.L. Stocks, `http://classics.mit.edu/Aristotle/heavens.html`, 350BC.
2. G. Brinkmann and B. D. McKay, *Fast generation of planar graphs (expanded edition)*, `http://cs.anu.edu.au/~bdm/papers/plantri-full.pdf`, 2007.
3. Beth Chen, Michael Engel, and Sharon C. Glotzer, *Dense crystalline dimer packings of regular tetrahedra*, `http://arxiv.org/abs/1001.0586`, 2010.
4. H. Cohn and A. Kumar, *The densest lattice in twenty-four dimensions*, Electronic Research Annoucements of the American Mathematical Society **10** (2004), 58–67, `math.MG/0408174`.
5. C. F. Gauss, *Untersuchungen über die Eigenscahften der positiven ternären quadratischen Formen von Ludwig August Seber*, Göttingische gelehrte Anzeigen (1831), also published in *J. reine angew. Math.* 20 (1840), 312–320, and *Werke*, vol. 2, Königliche Gesellschaft der Wissenschaften, Göttingen, 1876, 188–196.
6. Thomas C. Hales and Samuel P. Ferguson, *The Kepler conjecture*, Discrete and Computational Geometry **36** (2006), no. 1, 1–269.
7. D. Hilbert, *Mathematische probleme*, Archiv Math. Physik **1** (1901), 44–63, (also in Proc. Sym. Pure Math. 28, 1976, 1–34).
8. J. L. Lagrange, *Recherches d'arithmétique*, Mem. Acad. Roy. Sc. Bell Lettres Berlin **3** (1773), 693–758, volume and pages refer to Œuvres.
9. Oleg R. Musin and Alexey S. Tarasov, *The strong thirteen spheres problem*, preprint `http://arxiv.org/abs/1002.1439`, February 2010.
10. S. Torquato and Y. Jiao, *Exact constructions of a family of dense periodic packings of tetrahedra*, Physical Review E **81** (2010), 041310–1–11, `http://cherrypit.princeton.edu/papers.html`.