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A Simple Definition of Transfer Factors for Unramified Groups

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ABSTRACT. This paper gives a simple characterization of transfer factors for unramified groups which is shown to be equivalent to the definition of Langlands and Shelstad under mild restrictions on the residual field characteristic.

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PART I. PRELIMINARIES

1. Introduction.

Transfer factors have been defined for arbitrary reductive groups in [21]. When the group G and the endoscopic group H are unramified, it is possible to give a simple characterization of the transfer factors when we place mild restriction on the residual characteristic.

The following rules, which are described precisely in the body of this paper, give a complete characterization of the transfer factor for unramified groups on the subgroup $G_{der}(F)Z_G^0(F)$ of finite index in $G(F)$. The endoscopic group determines a character $\kappa : H^1(\mathrm{Gal}(\overline{F}/F), T(\overline{F})) \to \mathbb{C}^\times$.

(1) The transfer factor at a noncompact element g coincides with the product of the transfer factor for a proper Levi subgroup of G containing g with a relative discriminant factor (9.2).
(2) The transfer factor on compact elements is an explicit unramified character on the center of G times the transfer factor on strongly compact elements (11.1), (11.2), (11.3).
(3) The transfer factor on strongly compact elements g coincides with the transfer factor of the connected centralizer of the absolutely semisimple part of g (10.18).
(4) The transfer factor on topologically unipotent elements is obtained by homogeneously extending the transfer factor defined near regular unipotent elements (10.17).
(5) The transfer factor near a regular unipotent element u^g is given by $\kappa_0(\sigma(g)g^{-1})$ times a relative discriminant factor for an explicit character κ_0 on $H^1(\mathrm{Gal}(\overline{F}/F), Z_G)$ provided u is an element which is regular modulo p and $\sigma(g)g^{-1} \in Z_G$, for $\sigma \in \mathrm{Gal}(\overline{F}/F)$ (Section 14). κ_0 is the restriction of κ to the center of G. This character is trivial, for instance, if G is an adjoint group.
(6) The transfer factor at γ^g, with γ strongly regular semisimple and $g \in (T\backslash G)(F)$, is given by $\kappa(\sigma(g)g^{-1})$ times the transfer factor at γ.

Rules 5 and 6 may be replaced by the cohomological description of the Shalika germs for the regular unipotent conjugacy class obtained by Shelstad [24].

Rules 5,6 give the transfer factor near the identity, rule 4 extends it to all topologically unipotent elements, rule 3 extends it to strongly compact elements, and rule 2 extends it to compact elements. Finally, rule 1 gives the transfer factor in terms of its values on compact elements.

The rules 1–6 are given in a form that I hope will be amenable to the study of the fundamental lemma, for parallel to each of these rules is an established fact about the orbital integrals of the unit element of the Hecke algebra:

(1) The orbital integral of a non-elliptic element g coincides with the orbital integral on a proper Levi subgroup of G containing g.
(2) The orbital integral of suitable compact elements is an explicit unrami-

fied character of the center of G times the orbital integral on a strongly compact element.

(3) The orbital integral of a strongly compact element g coincides with the orbital integral of the centralizer of the absolutely semisimple part of g.

(4) The Shalika germs of orbital integrals are homogeneous functions sufficiently close to the identity element.

(5) the κ-orbital integral of a regular unipotent class coincides with the stable orbital integral on the regular unipotent class of the endoscopic group.

Statement 1 is known as Harish-Chandra descent, statement 2 is trivial. The character appearing in statement 2 is the same as the character in rule 2 above. Statement 3 is a lemma of Kazhdan/Kottwitz; statement 4 is due to Harish-Chandra, and statement 5 is formulated precisely and proved in this paper (14.2). To prove the fundamental lemma it would suffice to prove a stronger version of statements 4 and 5.

The main missing ingredient that prevents us from proving the fundamental lemma much more generally is the homogeneity of Shalika germs for all topologically unipotent elements for functions in the Hecke algebra. Assuming homogeneity, to establish the fundamental lemma for the identity of the Hecke algebra it would be sufficient to prove the fundamental lemma either for elements sufficiently close to the identity or for elements at the other extreme: nearly absolutely semisimple. Thus the fundamental lemma would follow from homogeneity together with either a knowledge of the Shalika germs or an improved version of a lemma of Kazhdan **[15]**. These are the two approaches taken in **[9,25]** for the group $SL(n)$ (where homogeneity of germs is known). In both approaches the simple formulation of the transfer factor provided here plays an important role. For example, the results of this paper imply that for $SL(n)$ the transfer factor on strongly compact elements is simply the quotient of the discriminant factors for $SL(n)$ and H, when the endoscopic group H splits over an unramified extension of odd degree.

An approach based on unipotent orbital integrals, the trace formula, or local character identities **[1,6,18,25]**, might make it possible to deduce the fundamental lemma from the matching of the unit element of the Hecke algebra.

The transfer factors were designed with the fundamental lemma and a number of other constraints in mind. It is evidence of the power and foresight of the definition of Langlands and Shelstad that a factor designed for many purposes should serve so well its purpose in this context.

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Assumptions and Notation. Let F be a p-adic field of characteristic zero. We assume res. char. $F \neq 2$. Let O_F denote the ring of integers of F. Let G be a connected reductive group over O_F. We assume that $G(O_F)$ is a hyperspecial maximal compact subgroup of $G(F)$ (modulo the center) and that G is quasi-split and splits over some unramified extension E of F. Let π be a prime element of O_F. Let q be the cardinality of the residue field of F and let p denote the rational prime dividing q. We let $\tilde{}: G(O_F) \to G(\mathbb{F}_q)$, $g \mapsto \tilde{g}$, denote reduction modulo the maximal ideal (π). Let ${}^LG = \widehat{G} \rtimes \mathrm{Gal}(E/F)$ be the dual group of G. Let F^u denote a maximal unramified extension of F, and let $\overline{F}$ be an algebraic closure of F. Let $\mathbb{T}_G$ be a maximally split Cartan subgroup of G, and let $\mathbb{B}$ be a Borel subgroup over F containing $\mathbb{T}_G$. Define $\mathbb{T}_H$ similarly for an endoscopic group H of G. If H splits over an unramified extension then $\mathbb{T}_H$ is also unramified. We often identify $\mathbb{T}_H$ with an isomorphic Cartan subgroup of G.

2. Compact Elements.

We say that $g \in G(F)$ is *strongly compact* if the following equivalent conditions hold:

(1) g lies in a compact subgroup of $G(F)$.
(2) The eigenvalues of $\rho(g)$ are units in $\overline{F}$ for some faithful finite-dimensional rational representation $\rho: G \to \mathrm{GL}(V)$ defined over $\overline{F}$.

We say that $g \in G(F)$ is *compact* if its image in $G_{\mathrm{adj}}(F)$ is a strongly compact element of $G_{\mathrm{adj}}(F)$. This definition of compact coincides with the usage of [**4,5,7**]. Every strongly compact element is compact.

If an element is not compact, then it is not elliptic, and we may select a Levi factor M of G such that it is elliptic in M.

3. Topological Jordan Decomposition.

This section reviews some properties of the topological Jordan decomposition [**15,26**]. If K is a profinite group with a normal pro-p-group L of finite index, then the prime-to-p part of the order of K/L is an integer independent of the choice of L which we denote by c_K. If G is a reductive p-adic group, then it has only finitely many maximal compact subgroups up to conjugacy $K_1, \ldots, K_f$, and we let $c = c_G$ be the least common multiple of the c_{K_i}.

We define an absolutely semisimple element in G to be a semisimple element γ such that $\gamma^c = 1$. Such elements are strongly compact. We define a topologically unipotent element γ of $G(F)$ to be an element such that $\lim_{m\to\infty} \gamma^{q^m} = 1$. Such elements are also strongly compact. For each strongly compact element γ we define a decomposition, called the topological Jordan decomposition, $\gamma = \gamma_s\gamma_u = \gamma_u\gamma_s$ into an absolutely semisimple element γ_s and topologically unipotent element γ_u as follows. Let ℓ be a positive integer such that $q^\ell \equiv 1 \mod c$. Set $\gamma_s = \lim_{m\to\infty} \gamma^{q^{\ell m}}$ and $\gamma_u = \gamma\gamma_s^{-1}$. It is readily verified that this limit exists [**14**]. One can verify without difficulty that the following properties hold:

(1) $\gamma_s \in G(F)$ is absolutely semisimple.
(2) γ_u is topologically unipotent.
(3) If $\gamma = \gamma'_s\gamma'_u = \gamma_s\gamma_u$ are two decompositions into commuting absolutely semisimple and topologically unipotent elements, then $\gamma_s = \gamma'_s$ and $\gamma_u = \gamma'_u$.

(4) If $\rho : G \to G'$ is a morphism of reductive groups defined over a finite extension of F, then $\rho(\gamma)_s = \rho(\gamma_s)$, $\rho(\gamma)_u = \rho(\gamma_u)$. In particular :
   (a) If E is a finite extension of F, then the topological Jordan decomposition is the same for strongly compact elements of $G(F)$ whether viewed as elements of $G(F)$ or $G(E)$.
   (b) $(\gamma^g)_u = (\gamma_u)^g$, $(\gamma^g)_s = (\gamma_s)^g$.

(5) If $G = \mathrm{GL}(1)$, then $\gamma \mapsto \gamma_s$, for $\gamma \in O_F^\times$, is the Teichmüller character of $O_F^\times$ [14].

(6) Let H be an endoscopic group satisfying the same conditions as G (that is, defined over O_F, unramified, etc). Suppose that $\gamma \in G(O_F)$ is an image of some strongly G-regular semisimple element $\gamma_H \in H(O_F)$ then γ_s is a $C_G(\gamma)$-image of $\gamma_{H,s}$ (for terminology see [21]).

(7) If $\gamma \in G(O_F)$, then $\tilde\gamma = \tilde\gamma_s\tilde\gamma_u$ is the Jordan decomposition of $\tilde\gamma \in G(\mathbb{F}_q)$.

If $F/\mathbb{Q}_p$ is a finite extension, then define e_F to be the ramification degree of F over $\mathbb{Q}_p$. If T is a Cartan subgroup, define e_T to be the minimum of e_E over all fields E splitting T. Finally if G is a reductive group, define e_G to be the maximum of e_T as T ranges over all Cartan subgroups defined over the ground field. Also write $U(F)$ for the topologically unipotent elements of $F^\times$.

LEMMA 3.1. *Suppose $p > e_F + 1$ and x is topologically unipotent in $F^\times$. If $Q = p^\ell$, $\ell > 0$, then $(x^Q - 1)/Q(x-1)$ is topologically unipotent.*

PROOF. By the binomial theorem, $(1+y)^Q - 1 = yQ(1 + \sum_{i=2}^{Q} \frac{1}{Q}\binom{Q}{i} y^{i-1})$, where $x = 1 + y$. For $1 \le k < Q$, $(Q-k)/(-k)$ is topologically unipotent. So $\frac{1}{Q}\binom{Q}{i} = u_i(-1)^{i-1}/i$ with u_i topologically unipotent. The lemma will follow if $|y^{i-1}/i| < 1$ for $i \ge 2$. The worst case is $y = \pi$, $(i) = (p^r) = (\pi^{er})$. Then y^{i-1}/i behaves as $\pi^{p^r - 1 - er}$. The worst case is $r = 1$. So we need $|\pi^{p-1-e}| < 1$ or $p > e_F + 1$. □

REMARK 3.2. The condition for the exponential power series to converge on $U(F) - 1$ is $p > e_F + 1$. Also the logarithmic power series converges on $U(F)$ to an element of $U(F) - 1$ if $p > e_F$.

The topological Jordan decomposition leads to the following decomposition of a field. If E is any finite extension of F, then $E^\times \xrightarrow{\sim} \langle\pi_E\rangle \times A_E \times U(E)$, with π_E a uniformizer, A_E absolutely semisimple, and $U(E)$ topologically unipotent. If the ramification degree of E over F is e, then we take $\pi = \pi_E^e$ as a uniformizer of F. Then $F^\times \xrightarrow{\sim} \langle\pi\rangle \times A \times U(F)$, where $A \subseteq A_E$ absolutely semisimple, $\langle\pi\rangle \subseteq \langle\pi_E\rangle$, and $U(F) \subseteq U(E)$ is topologically unipotent.

DEFINITION. *We say a character θ on $F^\times$ is tamely ramified if $\theta(U(F)) = 1$.*

LEMMA 3.3. *Let θ be a tamely ramified character of $F^\times$ and let E be a finite extension of F. Then there is a tamely ramified character on $E^\times$ extending θ.*

PROOF. Take the extension to be trivial on $U(E)$ and arbitrary extensions on A_E, $\langle\pi_E\rangle$. □

4. Centralizers of Semisimple Elements.

In this section, let G be a complex semisimple group. Let T be a maximal torus of G with Lie algebra Lie(T). We let $G_{\rm adj}$ and $G_{\rm sc}$ be the corresponding adjoint and simply-connected groups. Let $\tilde{\Delta} = \Delta \cup \{-r_0\}$ be the extended system of simple roots for T. Let $\exp : \mathrm{Lie}(T) \to T$ be the exponential map, and let X_* be the kernel of this map. X_* acts on Lie(T) by translation $\mu.Y = \mu + Y$. We also set $\exp_{\rm sc} : \mathrm{Lie}(T) \to T_{\rm sc} \subseteq G_{\rm sc}$, $X_{*,\rm sc} = \ker(\exp_{\rm sc} : \mathrm{Lie}(T) \to T_{\rm sc})$. Let W be the Weyl group for T. Let $W_{\rm aff} = X_* \rtimes W$ be the affine Weyl group, and $W_{\rm aff,sc} = X_{*,\rm sc} \rtimes W$. Two elements $Z, Y \in \mathrm{Lie}(T)$ have conjugate images $\exp(Z)^g = \exp(Y)$, $g \in G$, if and only if $w.Z = Y$ for some $w \in W_{\rm aff}$. Let $\mathcal{C}$ be a fundamental chamber for the action of $W_{\rm aff,sc}$ on Lie(T). For $Y \in \mathcal{C}$ let $\mathcal{C}_Y$ be the smallest facet of $\mathcal{C}$ containing Y. Let $\Omega = \{w \in W_{\rm aff} \mid w\mathcal{C} = \mathcal{C}\}$, then by [13]

$$W_{\rm aff,sc} \rtimes \Omega \xrightarrow{\sim} W_{\rm aff}.$$

For the next few lemmas set $s = \exp(Y)$, $Y \in \mathcal{C}$.

LEMMA 4.1. *The inclusion* $C_{N_G(T)}(s) \subseteq C_G(s)$ *induces an isomorphism between* $C_{N_G(T)}(s)/(C_{N_G(T)}(s) \cap C_G(s)^0)$ *and* $C_G(s)/C_G(s)^0$.

PROOF. [3] □

LEMMA 4.2. $C_{N_G(T)}(s) \cap C_G(s)^0/T \xrightarrow{\sim} C_{W_{\rm aff,sc}}(Y)$.

PROOF. If $n \in C_G(s)^0 \cap C_{N_G(T)}(s)$, then lift n to $\hat{n} \in G_{\rm sc}$, and s to $\hat{s}$. Then $\hat{n}\hat{s}\hat{n}^{-1} = \hat{s}$. So $\hat{n} \in C_{G_{sc}}(\hat{s}) = C_{G_{sc}}(\hat{s})^0$ [3]. Thus

$$C_G(s)^0 \cap C_{N_G(T)}(s) = \mathrm{image}\left(C_{N_{G_{\rm sc}}(T_{\rm sc})}(\hat{s}) \to G\right).$$

If $n \in C_{N_{G_{\rm sc}}(T_{\rm sc})}(\hat{s})$, then its image w in W acts on Y by $wY = \mu Y$, $\mu \in X_{*,\rm sc}$. So replacing w by $w\mu^{-1}$ we obtain a surjection $C_{N_{G_{\rm sc}}(T_{\rm sc})}(\hat{s}) \to C_{W_{\rm aff,sc}}(Y)$. The isomorphism follows easily. □

Similarly,

LEMMA 4.3. $C_{N_G(T)}(s)/T \xrightarrow{\sim} C_{W_{\rm aff}}(Y)$.

Thus,

COROLLARY 4.4. $C_G(s)/C_G(s)^0 \xrightarrow{\sim} C_{W_{\rm aff}}(Y)/C_{W_{\rm aff,sc}}(Y)$.

PROOF. Lemmas 4.2 and 4.3. □

In the disconnected case, set $G' = G \rtimes I$ and $G'_{\rm sc} = G_{\rm sc} \rtimes I$, where I is a finite group of outer automorphisms of G and $G_{\rm sc}$ fixing T and $T_{\rm sc}$. Set $W' = W \rtimes I$, $W'_{\rm aff} = X_* \rtimes W'$, $\Omega' = \{w \in W'_{\rm aff} \mid w\mathcal{C} = \mathcal{C}\}$. Set $\Omega'_Y = \{w \cdot Y = Y\}$. Assume that I acts in such a way to stabilize $\mathcal{C}$, so that $I \subseteq \Omega'$. We obtain as in the connected case

LEMMA 4.5. $C_{G'}(s)/C_{G'}(s)^0 \xrightarrow{\sim} C_{W'_{\rm aff}}(Y)/C_{W_{\rm aff,sc}}(Y) \xrightarrow{\sim} \Omega'_Y \subseteq \Omega'$.

LEMMA 4.6. *The following exact sequence splits.*

$$1 \to C_{W_{\mathrm{aff,sc}}}(Y) \to C_{W'_{\mathrm{aff}}}(Y) \to \Omega'_Y \to 1.$$

PROOF. $C_{W_{\mathrm{aff,sc}}}(Y)$ is the Weyl group of reflections stabilizing $\mathcal{C}_Y$ the smallest facet of $\mathcal{C}$ containing Y. It acts transitively on the set of chambers containing $\mathcal{C}_Y$. Thus if $w \in C_{W_{\mathrm{aff,sc}}}(Y)$ then $w \cdot Y = Y$, and $w \cdot \mathcal{C}_Y = \mathcal{C}_Y$. Select $w' \in C_{W_{\mathrm{aff,sc}}}(Y)$ such that $w'w \cdot \mathcal{C} = \mathcal{C}$; then $w'w \in \Omega'$ gives the splitting. □

For each $\omega \in \Omega'$, let $n_\omega \in N_G(T) \rtimes I$ be an element such that

$$n_\omega \mod T \quad = \quad \omega \mod X_* \in W \rtimes I.$$

Combining all of the previous results we obtain the following theorem. It tells us that the components of $C_{G'}(s)$ are represented by elements n_ω independent of s.

THEOREM 4.7. *Let $s = \exp(Y)$, $Y \in \mathcal{C}$, be semisimple. Then*

$$C_{G'}(s) = C_G(s)^0\{n_\omega\}_{\omega \in \Omega'_Y}.$$

5. Abelian Class Field Theory.

In this section, we recall Labesse's version [19] of the isomorphism between

$$\mathrm{Hom}_c(T(F), \mathbb{C}^\times) \quad \text{and} \quad H^1_c(W_{E/F}, \widehat{T}(\mathbb{C})) \qquad (c = \text{ continuous})$$

where E is a Galois extension which splits T. Set $\Gamma = \mathrm{Gal}(E/F)$, $X_* =$ cocharacters of $T(\overline{F})$ viewed as a Γ-module, $W_{E/F}$ the Weil group, $\widehat{T}$ the complex dual of T. We start with the surjective corestriction map:

$$H^1_c(E^\times, \widehat{T}) \xrightarrow{\mathrm{Cor}} H^1_c(W_{E/F}, \widehat{T}) \tag{1}$$

corresponding to the short exact sequence $1 \longrightarrow E^\times \longrightarrow W_{E/F} \longrightarrow \Gamma \longrightarrow 1$. If $\{w_\sigma\}_{\sigma\in\Gamma}$ are representatives in $W_{E/F}$ of the elements of Γ, then $\mathrm{Cor}\varphi(w) \stackrel{\mathrm{def}}{=} \sum_{\sigma\in\Gamma} w_\tau \varphi(w_\tau^{-1} w w_\sigma)$ where $\tau = \tau(\sigma) \in \Gamma$ is defined by $w_\tau^{-1} w w_\sigma \in E^\times$. We have

$$H^1_c(E^\times, \widehat{T}) = \mathrm{Hom}_c(E^\times, \widehat{T}) \tag{2}$$
$$= \mathrm{Hom}_c(E^\times, \mathrm{Hom}(X_*, \mathbb{C}^\times)) \tag{3}$$
$$= \mathrm{Hom}_c(E^\times \otimes X_*, \mathbb{C}^\times) \tag{4}$$
$$= \mathrm{Hom}_c(T(E), \mathbb{C}^\times) \tag{5}$$
$$\mathrm{Hom}_c(T(E), \mathbb{C}^\times) \twoheadrightarrow \mathrm{Hom}_c(T(F), \mathbb{C}^\times). \tag{6}$$

If we now replace each of the groups A in lines 2–5 by $H_0(\Gamma, A) \stackrel{\mathrm{def}}{=} A/I_\Gamma A = A/\{\sigma a - a \mid a \in A, \sigma \in \Gamma\}$ then the maps in lines 1 and 5 descend to the quotient in homology and become isomorphisms. This gives the desired isomorphism between $H^1_c(W_{E/F}, \widehat{T})$ and $\mathrm{Hom}_c(T(F), \mathbb{C}^\times)$. For proofs see [19]. By the explicit form of the isomorphism in lines 2–6 we see

LEMMA 5.1. *Suppose that $\varphi \in Z_c^1(E^\times, \widehat{T}) = H_c^1(E^\times, \widehat{T}) = \mathrm{Hom}_c(E^\times, \widehat{T})$ is tamely ramified. Then the corresponding homomorphism in* $\mathrm{Hom}_c(T(F), \mathbb{C}^\times)$ *(obtained by the maps in lines 2-6) is also tamely ramified.*

6. Embeddings of L-Groups.

Let $(H, \mathcal{H}, s, \xi)$ be endoscopic data for G (see [21]). We say that $(H, \mathcal{H}, s, \xi)$ is *unramified* endoscopic data for G if

(1) G is defined over $O_F, G(O_F)$ is hyperspecial, and G splits over an unramified extension of F.
(2) H is defined over $O_F, H(O_F)$ is hyperspecial, and H splits over an unramified extension of F.
(3) $\mathcal{H} = {}^LH$ is the L-group of H.
(4) $(H, \mathcal{H}, s, \xi)$ is endoscopic data for G.
(5) The embedding ξ descends to some finite unramified extension E/F. In other words we have a commutative diagram

$$\begin{array}{ccc} {}^LH & \xrightarrow{\xi} & {}^LG \\ {\scriptstyle\varphi_H}\downarrow & & {\scriptstyle\varphi_G}\downarrow \\ \widehat{H} \rtimes \mathrm{Gal}(E/F) & \xrightarrow{\xi_0} & \widehat{G} \rtimes \mathrm{Gal}(E/F). \end{array}$$

The maps φ_H, φ_G are induced by the canonical projection $\varphi : W_F \to \mathrm{Gal}(E/F)$.

For the next lemma we make a series of assumptions. Suppose H is an unramified reductive group defined over F. Suppose that $\widehat{H} = C_{\widehat{G}}(s)$ for some semisimple element $s \in \widehat{G}$. Suppose also that for every $\sigma \in \mathrm{Gal}(E/F)$ there is an element $\omega(\sigma) \rtimes \theta$ in ${}^LG$ such that the action of σ as an outer automorphism of $\widehat{H}$ coincides with the outer automorphism defined by conjugation by $\omega(\sigma) \rtimes \theta$.

LEMMA 6.1. *Under the assumptions above, there exists an embedding ξ : ${}^LH \to {}^LG$ that makes $(H, {}^LH, s, \xi)$ into unramified endoscopic data.*

REMARK. *This lemma is not true without the assumption that H is unramified. In fact the purpose of* [20] *is to circumvent this lemma in the ramified case.*

PROOF. The group ${}^LH$ is defined by some $\widehat{H} \rtimes \mathrm{Gal}(E/F)$, E/F unramified. Let σ be a generator of $\mathrm{Gal}(E/F)$. Let $\omega(\sigma)$ be in the Weyl group such that σ acts on $\widehat{H}$ by $\omega(\sigma) \rtimes \theta$ with θ outer in $\widehat{G}$. We may assume $\omega(\sigma)$, σ and θ act as automorphisms of extended Dynkin diagrams and stabilize $\widehat{B}_H$, $\widehat{T}_H$ (Borel and Cartan for $\widehat{H}$). Then $w = n(\omega(\sigma)) \rtimes \theta$ also stabilizes $\widehat{B}_H$, $\widehat{T}_H$. Here $n(\cdot)$ is a representative in the normalizer defined in [21]. By construction w is of finite order k. In fact, $w^{[E:F]}$ has order 2 [21]. Let $O_1, \cdots, O_r$ be the orbits of the positive simple roots of H under $\langle w \rangle$. For each orbit O_i pick $t_i \in \widehat{T}_H$ of finite order such that $\alpha(t_i) = 1$ for $\alpha \notin O_i$ and for which there exist root vectors $\{X_\alpha\}$, $\alpha \in O_i$ stable under $t_i w$. Then $w' = (\prod t_i)w$ is of finite order k' and stabilizes the set of root vectors for simple roots of $\widehat{H}$. Let E' be the unramified extension of F of degree k', and σ' a generator of $\mathrm{Gal}(E'/F)$ over $\sigma \in \mathrm{Gal}(E/F)$. Then define $\xi_0 : \widehat{H} \rtimes \mathrm{Gal}(E'/F) \to \widehat{G} \rtimes \mathrm{Gal}(E'/F)$ by $\xi_0(\sigma') = w'$. This gives ξ. $\square$

PART II. TRANSFER FACTORS

7. Canonical Normalization of Transfer Factors.

By our assumptions on G we may assume $G = G^*$ (the quasi-split inner form) and that the inner twist $\psi : G \to G^*$ to the quasi-split form is the identity. For each G and set of unramified endoscopic data $(H, {}^LH, s, \xi)$ we define a canonical transfer factor $\Delta_0(\gamma_H, \gamma_G)$. Recall that the transfer factor Δ_0 of **[21]** depends on a choice of splitting. We will use the subgroup $G(O_F)$ to determine a splitting.

DEFINITION 7.1. *We say that a splitting $(B, T, \{X_\alpha\})$ is admissible if the following conditions hold:*

(1) *B and T are Cartan subgroups over F.*
(2) *If u_α is the homomorphism $u_\alpha : \mathbb{G}_a \to G$ determined by X_α, then u_α is defined over O_{F^u} and $\tilde{u}_\alpha(1)$ is not the identity element in $G(\overline{\mathbb{F}}_q)$.*

We define $\Delta_0(\gamma_H, \gamma_G)$ to be the transfer factor $\Delta_0(\gamma_H, \gamma_G)$ of **[21]** defined relative to an admissible splitting of G.

LEMMA 7.2. *$\Delta_0(\gamma_H, \gamma_G)$ is independent of the admissible splitting of G.*

PROOF. $\Delta_I(\gamma_H, \gamma_G)$ is the only term of the transfer factor depending on a splitting and $\Delta_I(\gamma_H, \gamma_G)/\Delta_I(\overline{\gamma}_H, \overline{\gamma}_G)$ is independent of the splitting of G. Thus it is enough to prove the independence of the admissible splitting for a pair γ_H, γ_G of our choice. We take γ_H to lie in the unramified maximal split torus $\mathbb{T}_H$ of H. Identify $\mathbb{T}_H$ with an isomorphic Cartan subgroup of G. With an appropriate choice we may identify $\mathbb{T}_H(O_F) = H(O_F) \cap \mathbb{T}_H(F)$ with $G(O_F) \cap \mathbb{T}_H(F)$. With an unramified Cartan subgroup we are allowed to take all our a-data to be units in an unramified extension of F. Since u_α is defined over O_{F^u} and $\tilde{u}_\alpha(1) \neq 1$, the coroots $\alpha^\vee : \mathbb{G}_m \to G$ and negative roots $u_{-\alpha} : \mathbb{G}_a \to G$ are also defined over O_{F^u}. Hence **[21]**

$$x(\sigma_T) \stackrel{\mathrm{def}}{=} \prod_{1,\sigma}^{p} a_\alpha^{\alpha^\vee} \in G(O_{F^u})$$

$$n(\alpha) \stackrel{\mathrm{def}}{=} \exp(X_\alpha)\exp(-X_{-\alpha})\exp(X_\alpha) \in G(O_{F^u}).$$

Similarly $n(\omega)$ and $m(\sigma_T) = x(\sigma_T)n(\omega_T(\sigma))$ belong to $G(O_{F^u})$.

Since the pair $(B, \mathbb{T}_H)$ is isomorphic to $(\mathbb{B}, \mathbb{T}_G)$ over an unramified extension, and since our integral structures have been chosen compatibly, we may select $h \in G(O_{F^u})$ such that $(B, \mathbb{T}_H)^h = (\mathbb{B}, \mathbb{T}_G)$. Then the cocycle $\sigma \mapsto hm(\sigma_T)\sigma(h^{-1})$ in $\mathbb{T}_H(\overline{F})$ actually lands in $H^1(F^u, \mathbb{T}_H(O_{F^u})) = \{1\}$. This cocycle defines the element $\lambda(\mathbb{T}_H) = \lambda_{\{a_\alpha\}}(\mathbb{T}_H)$ so that $\lambda(\mathbb{T}_H) = 1$. Hence

$\Delta_I(\gamma_H, \gamma_G) \stackrel{\text{def}}{=} \langle \lambda(\mathbb{T}_{H,sc}), s_T\rangle = 1$. In particular, it is independent of our choice of admissible splitting. $\square$

8. Coherence of Canonical Normalization.

Fix $\gamma_H \in H(O_F)$. We will assume that this element is strongly compact and strongly G-regular. Set $\varepsilon_H = \gamma_{H,s}$ and $\varepsilon_G = \varepsilon = \gamma_{G,s}$. Let $\Delta_{\varepsilon,0}$ be the canonical transfer factor associated to $(G_\varepsilon, H_\varepsilon)$ as in **[22]**. In **[22]** it is shown that $\Delta_0(\gamma_H, \gamma_G) = c_\varepsilon \Delta_{\varepsilon,0}(\gamma_H, \gamma_G)$ for some c_ε independent of the elements γ_H, γ_G and the Cartan subgroups T_H containing γ_H for γ_H near ε_H, γ_G near ε_G. We need the refinement:

Lemma 8.1. *$\Delta_0(\gamma_H, \gamma_G) = \Delta_{\varepsilon,0}(\gamma_H, \gamma_G)$ for γ_H near ε_H, γ_G near ε_G.*

Proof. Since the constant c_ε is independent of the choice of γ_H, γ_G we are free to choose any convenient γ_H, γ_G. We choose $\gamma_H \in H(O_F)$, and $\gamma_G \in G(O_F)$. We take γ_H to lie in an unramified Cartan subgroup T_H of H. The proof of the previous lemma shows $\Delta_I = 1$ for (G, H) and $(G_\varepsilon, H_\varepsilon)$ when the a-data a_α are chosen to be units, provided the integral structures are chosen compatibly for T_H on G and H. $\square$

We pick the χ-data for an unramified torus to consist of unramified characters. We select the same χ-data and a-data for both (G, H) and $(G_\varepsilon, H_\varepsilon)$ for those roots common to both. Then

$$\chi_\alpha\left(\frac{\alpha(\gamma)-1}{a_\alpha}\right) = \begin{cases} \chi_\alpha\left(\frac{\alpha(\gamma_u)-1}{a_\alpha}\right) & \text{if } \alpha(\gamma_s) = 1 \\ \chi_\alpha\left(\frac{\alpha(\gamma_s)-1}{a_\alpha}\right) & \text{if } \alpha(\gamma_s) \neq 1. \end{cases}$$

But $\chi_\alpha\left(\frac{\alpha(\gamma_s)-1}{a_\alpha}\right) = 1$ if $\alpha(\gamma_s) \neq 1$, for both $\alpha(\gamma_s) - 1$ and a_α are units. The roots of T_H in G outside H such that $\alpha(\gamma_s) = 1$ are the same as the roots of T_H in G_ε outside H_ε. So Δ_{II} is the same for both (G, H) and $(G_\varepsilon, H_\varepsilon)$ on the unramified pair (γ_H, γ_G).

Now turn to Δ_{III_1}. We take the element γ to lie in $G(O_F)$. So $\varepsilon_G = \gamma_s \in G_\varepsilon(O_F), G(O_F)$. By **[16]** if $h\gamma_G h^{-1} = \gamma$, then $h\varepsilon_G h^{-1} = \varepsilon$ and also $h \in G(O_F)G_\varepsilon$. We may adjust h by an element in $G(O_F) \subseteq G(F)$ to obtain $h \in G_\varepsilon(\overline{F})$ which represents the same class. Thus $\mathrm{inv}(\gamma_H, \gamma_G)$ is the same for both (G, H) and $(G_\varepsilon, H_\varepsilon)$. So Δ_{III_1} is also the same for both.

In Section 11 we will prove that the unramified χ-data and unramified endoscopic data determine an unramified character $\langle a, \cdot\rangle$ of the unramified torus T_H. Hence $\langle a, \cdot\rangle$ is trivial on units. Thus $\Delta_2 = 1$ for both (G, H) and $(G_\varepsilon, H_\varepsilon)$. Finally

$$|\alpha(\gamma) - 1| = \begin{cases} |\alpha(\gamma_u) - 1| & \text{if } \alpha(\gamma_s) = 1 \\ 1 & \text{if } \alpha(\gamma_s) \neq 1. \end{cases}$$

So the product of $|\alpha(\gamma) - 1|$ over roots in G outside H coincides with the product over roots in G_ε outside H_ε.

9. Descent for Levi Factors.

LEMMA 9.1. *Suppose that M is a Levi factor of a parabolic subgroup P of G. Then M is quasisplit and $M \cap G(O_F)$ is hyperspecial.*

PROOF. See [16]. □

Let Δ_0^M and Δ_0^G denote the canonical transfer factors for M, G. Let H_M be the endoscopic group associated to M by descent [22].

LEMMA 9.2. *If $\gamma_G \in M(F)$ and $\gamma_H \in H_M(F)$, then*

$$\Delta_0^G(\gamma_H, \gamma_G) = \Delta_0^M(\gamma_H, \gamma_G)|\prod_\alpha(\alpha(\gamma) - 1)|^{1/2}$$

(product over roots of G outside M and H).

PROOF. We may pick the same a-data and χ-data for M and G. We may choose the a-data and χ-data to be trivial on the roots of the unipotent radical of P. We pick the element h in the definition of Δ_{III_1} to lie in $M(\overline{F})$. With these choices it is clear from the definitions that $\Delta_{\mathrm{I}}^M = \Delta_{\mathrm{I}}^G$, $\Delta_{\mathrm{II}}^M = \Delta_{\mathrm{II}}^G$, $\Delta_{\mathrm{III}_1}^M = \Delta_{\mathrm{III}_1}^G$, $\Delta_{\mathrm{IV}}^G = |\prod_\alpha(\alpha(\gamma) - 1)|^{1/2}\Delta_{\mathrm{IV}}^M$ (product over roots in G outside M and H). We have a commutative diagram (identifying ${}^LT$ and ${}^LT_H$).

$$\begin{tikzcd}
& {}^LG & \\
{}^LH \arrow[ur, "{[2]}"] & {}^LT \arrow[u, "\xi_T"] \arrow[l, "\xi_{T_H}"'] \arrow[r, "\xi_T^M"] \arrow[d, "\xi_{T_H}^M"] & {}^LM \arrow[ul, "{[3]}"'] \\
& {}^LH_M \arrow[ul, "{[1]}"] \arrow[ur, "{[4]}"'] &
\end{tikzcd}$$

The triangles [1] and [3] as well as the outside square are commutative. The triangles [2] and [4] fail to commute by the classes a^G and a^M respectively. By the commutativity of [1], [3] and the outside square we have $a^G = a^M$ so that $\Delta_{\mathrm{III}_2}^M = \Delta_{\mathrm{III}_2}^G$. □

10. Homogeneity and General Descent.

We fix in this section an integer Q satisfying $Q = p^\ell$, $\ell > 0$, $Q \equiv 1 \mod c$, $c = c_G$. We wish to compare the constants $\Delta_0(\gamma_H, \gamma_G)$ and $\Delta_0(\gamma_H^Q, \gamma_G^Q)$. We assume in this section that (γ_H, γ_G) are matching strongly G-regular, strongly compact elements such that $\gamma_H \in H(O_F)$, $\gamma_G \in G(O_F)$. We also assume that the same a-data, χ-data, splitting, etc. are used for (γ_H, γ_G) and (γ_H^Q, γ_G^Q).

First of all, $\Delta_{\mathrm{I}}(\gamma_H, \gamma_G) = \Delta_{\mathrm{I}}(\gamma_H^Q, \gamma_G^Q)$ because this term depends on the above data and Cartan subgroups T_H and T and not directly on the elements (γ_H, γ_G) and (γ_H^Q, γ_G^Q). Also $\Delta_{\mathrm{III}_1}(\gamma_H, \gamma_G) = \Delta_{\mathrm{III}_1}(\gamma_H^Q, \gamma_G^Q)$ if we choose the element h, which enters into the definition of this term, to be the same for (γ_H, γ_G) and (γ_H^Q, γ_G^Q).

Write $\prod_{G/H}$ (resp. $\prod^{\circ}_{G/H}$) for the product of roots (resp. orbits of roots under the T-action of $\mathrm{Gal}(\overline{F}/F)$) in G but not H. Define $\prod^{\varepsilon}_{G/H}$, and $\prod^{\varepsilon,\circ}_{G/H}$ similarly with respect to the roots in G_ε but not H_ε.

LEMMA 10.1. *If $p > e_G + 1$,*

$$\Delta_{\mathrm{IV}}(\gamma_H^Q, \gamma_G^Q) = |Q|^{m/2}\Delta_{\mathrm{IV}}(\gamma_H, \gamma_G) =$$
$$\Delta_{\mathrm{IV},\varepsilon}(\gamma_H^Q, \gamma_G^Q) = |Q|^{m/2}\Delta_{\mathrm{IV},\varepsilon}(\gamma_H, \gamma_G)$$

where m is the number of roots in G_ε but not in H_ε and ε is the absolutely semisimple part of γ_G.

PROOF. $\Delta_{\mathrm{IV}}(\gamma_H, \gamma_G) = |\prod_{G/H}(\alpha(\gamma) - 1)|^{1/2}$. If $\gamma = \gamma_s\gamma_u$ is the topological Jordan decomposition,

$$|\prod_{\alpha}(\alpha(\gamma) - 1)|^{1/2} = |\prod_{\alpha(\gamma_s)\neq 1}(\alpha(\gamma_s) - 1)|^{1/2} \quad |\prod_{\alpha(\gamma_s)=1}(\alpha(\gamma) - 1)|^{1/2}$$
$$= |\prod^{\varepsilon}_{G/H}(\alpha(\gamma) - 1)|^{1/2}.$$

Similarly, $\Delta_{\mathrm{IV}}(\gamma_H^Q, \gamma_G^Q) = \Delta_{IV,\varepsilon}(\gamma_H^Q, \gamma_G^Q)$. If $\alpha(\gamma_s) = 1$, then by (3.1), we have $|\alpha(\gamma)^Q - 1| = |Q||\alpha(\gamma) - 1|$. So $\Delta_{IV,\varepsilon}(\gamma_H^Q, \gamma_G^Q) = |Q|^{m/2}\Delta_{\mathrm{IV},\varepsilon}(\gamma_H, \gamma_G)$. □

By (3.3), we may assume the χ-data are selected to be tamely ramified characters.

LEMMA 10.2. *If $p > e_G + 1$, then $\Delta_{\mathrm{II}}(\gamma_H^Q, \gamma_G^Q) = \Delta_{\mathrm{II}}(\gamma_H, \gamma_G)\prod^{\varepsilon,\circ}_{G/H}\chi_\alpha(Q)$.*

COROLLARY 10.3. *Under the same conditions, $\Delta_{\mathrm{II}}(\gamma_H^{Q^2}, \gamma_G^{Q^2}) = \Delta_{\mathrm{II}}(\gamma_H, \gamma_G)$.*

PROOF. χ_α on $F_\pm \ni Q$ has order 2. □

PROOF. (Lemma).

$$\frac{\Delta_{\mathrm{II}}(\gamma_H^Q, \gamma_G^Q)}{\Delta_{\mathrm{II}}(\gamma_H, \gamma_G)} = \prod^{\circ}_{\alpha}\chi_\alpha\left(\frac{\alpha(\gamma)^Q - 1}{\alpha(\gamma) - 1}\right).$$

If $\alpha(\gamma_s) \neq 1$ then

$$\frac{\alpha(\gamma)^Q - 1}{\alpha(\gamma) - 1} = \frac{1 + x_s'(x_u^Q - 1)}{1 + x_s'(x_u - 1)}$$

is topologically unipotent, where $\alpha(\gamma) = x_s x_u$ is the topological Jordan decomposition of $\alpha(\gamma)$, and $x_s' = x_s/(x_s - 1)$. So $\chi_\alpha\left(\frac{\alpha(\gamma)^Q-1}{\alpha(\gamma)-1}\right) = 1$ if $\alpha(\gamma_s) \neq 1$. So the product can be taken over roots in G_ε outside H_ε. If $\alpha(\gamma_s) = 1$ we have by (3.1), $\chi_\alpha\left(\frac{\alpha(\gamma)^Q-1}{\alpha(\gamma)-1}\right) = \chi_\alpha(Q)$, and the result follows. □

Before turning to the factor Δ_{III_2} we state a few preparatory lemmas.

LEMMA 10.4. *Suppose $p > e_G + 1$. If a^{2^r} is a tamely ramified character on $T(F)$ for some $r > 0$, then a is a tamely ramified character.*

PROOF. If res. char. $\neq 2$ every topologically unipotent element is a square, and hence a 2^r-power. So if γ is topologically unipotent, then $\gamma = \delta^{2^r}$ for some δ and $\langle a, \gamma\rangle = \langle a^{2^r}, \delta\rangle = 1$. □

LEMMA 10.5. *r_q^2 is independent of the gauge q.*

PROOF. $r_q = s_{q/p} r_p$ and $s_{q/p}^2 = 1$. [21, 2.4]. □

LEMMA 10.6. *r_q^2 is a cocycle.*

PROOF. The coboundary of r_q is t_q and $t_q^2 = 1$ [21,2.1.B,2.5.A]. □

Next we give an expression for the cocycle $a(w) \in H^1(W_{E/F}, \widehat{T}(\mathbb{C}))$. We add subscripts G, H as needed to distinguish between data for G and H. We also distinguish data by adding a bar to data on H, e.g. $\bar{r}_p$. We use notation from [21].

$$\begin{aligned}
\xi_T(w) &= r_p(w) n_G(\omega_T(\sigma)) \times w, && w \mapsto \sigma \in \Gamma, \quad \xi_T : {}^LT \to {}^LG,\\
\xi_{T_H}(w) &= \bar{r}_p(w) n_H(\omega_{T_H}(\sigma)) \times w, && \xi_{T_H} : {}^LT_H \to {}^LH,\\
\xi(w) &= m_0(\sigma) \times w, && \xi : {}^LH \to {}^LG.
\end{aligned}$$

By definition, a is given by

$$\begin{aligned}
\xi \cdot \xi_{T_H}(w) = \xi(\bar{r}_p(w) n_H(\omega_{T_H}(\sigma)) \times w) &= \bar{r}_p(w) n_H(\omega_{T_H}(\sigma)) m_0(\sigma) \times w\\
&= a(w) r_p(w) n_G(\omega_T(\sigma))) \times w = a \cdot \xi_T(w).
\end{aligned}$$

Define λ_0 by $m_0(\sigma) = {}^x\lambda_0(\sigma) n_G(\omega_H(\sigma))$, $x \overset{\text{def}}{=} \omega_{T_H}(\sigma)^{-1}$ where ω_H is defined by $\omega_{T_H}(\sigma)\omega_H(\sigma) = \omega_T(\sigma)$. Then

$$\begin{aligned}
\bar{r}_p(w) n_H(\omega_{T_H}(\sigma)) \cdot {}^x\lambda_0(\sigma) &= a(w) r_p(w) n_G(\omega_T(\sigma)) n_G(\omega_H(\sigma))^{-1},\\
n_G(\omega_T(\sigma)) = n_G(\omega_{T_H}(\sigma)\omega_H(\sigma)) &= t_G(\omega_{T_H}(\sigma), \omega_H(\sigma))^{-1} n_G(\omega_{T_H}(\sigma)) n_G(\omega_H(\sigma)).
\end{aligned}$$

So

$$\bar{r}_p(w)\lambda_0(\sigma) n_H(\omega_{T_H}(\sigma)) = a(w) r_p(w) t_G(\omega_{T_H}(\sigma), \omega_H(\sigma))^{-1} n_G(\omega_{T_H}(\sigma)).$$

(10.7)

$$a(w) = \underbrace{\bar{r}_p(w)}_{(1)} \underbrace{\lambda_0(\sigma)}_{(2)} \underbrace{[n_H(\omega_{T_H}(\sigma)) n_G^{-1}(\omega_{T_H}(\sigma))]}_{(3)} \underbrace{t_G(\omega_{T_H}(\sigma), \omega_H(\sigma))}_{(4)} \underbrace{r_p(w)^{-1}}_{(5)}.$$

Each of the bracketed terms lies in $\widehat{T}(\mathbb{C})$.

We will check that by raising a to the fourth power, the 3rd and 4th bracketed terms will become 1. Also the remaining three terms $\bar{r}_p(w)$, $\lambda_0(\sigma)$, $r_p(w)^{-1}$ will each be cocycles when raised to the fourth power.

LEMMA 10.8.

$$n_G(\omega_1\omega_2\cdots\omega_r) = t\, n_G(\omega_1)\cdots n_G(\omega_r)$$

for some $t = t(\omega_1, \dots, \omega_r)$ satisfying $t^2 = 1$.

PROOF. Clear from [21,2.1.A]. □

By [8] we may assume that the simple roots of $\widehat{H}$ are a subset of the extended set of simple roots $\Delta \cup \{-r_0\} = \widetilde{\Delta}$ of $\widehat{G}$.

LEMMA 10.9. *There is a choice of root vectors X^H_α, X^G_α such that*

(i) $n_G(\omega_\alpha) = n_H(\omega_\alpha)$ *if α is a simple root of H in Δ.*

(ii) $n_G(\omega_{-r_0}) = t_0 n_H(\omega_{-r_0})$ *where $t_0^4 = 1$.*

PROOF. The first statement is obvious by definition if the root vectors are chosen to be the same: $X^H_\alpha = X^G_\alpha$, $\alpha \in \Delta$. Define t_0 by $n_G(\omega_{-r_0}) = t_0 n_H(\omega_{-r_0})$. Set $\omega = \omega_{-r_0}$.

$$n_G(\omega)^2 = t_G(\omega,\omega) = t_0^\omega t_0 n_H^2(\omega_{-r_0}) = t_0^\omega t_0 t_H(\omega,\omega).$$

We are still allowed to pick X_-, the simple root vector associated to $-r_0$. Allowing a momentary conflict in the meaning of H, we let $\varphi : SL_2 \to \widehat{H}$ be attached to the Lie triple $\{X_{-r_0} = X_-, X_+$ and coroot $H\}$, and set $T_0 = \{x \in T \mid \alpha_{-r_0}(x) = 1\}$. Let M_0 be the Levi factor such that $M_0 = T_0\varphi(SL_2)$. We select X_- so that $n_G(\omega_{-r_0})n_H(\omega_{-r_0})^{-1} = t_0 \in T_0$. This is always possible by the 2×2 calculation:

$$\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/x & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -t/x & 0 \\ 0 & -x/t \end{pmatrix}.$$

Then $t_0^\omega t_0 = t_0^2 = t_G(\omega,\omega)/t_H(\omega,\omega)$, which has order 2. □

COROLLARY 10.10. $a^4 = \overline{r}_p^4\lambda_0^4/r_p^4$.

PROOF. By (10.8) and (10.9), the 4th powers of the terms

$$n_H(\omega_{T_H}(\sigma))n_G^{-1}(\omega_{T_H}(\sigma)) \qquad \text{and} \qquad t(\omega_{T_H}(\sigma), \omega_H(\sigma))$$

in (10.7) are 1. □

COROLLARY 10.11. *λ_0^4 is a cocycle.*

PROOF. The other terms of (10.10) are cocycles. □

REMARK 10.12. $(\overline{r}_p/r_p)^2$ is independent of the choice of $\xi : {}^LH \to {}^LG$.

LEMMA 10.13. *λ_0^4 depends on T_H only through H.*

PROOF. Write $\lambda_0^4 = \lambda_{T_H}$ to make the possible dependence on T_H explicit.

$$\lambda_{T_H}(\sigma) = {}^{\omega_{T_H}(\sigma)}\left[m_0(\sigma)n_G(\omega_H(\sigma))\right]^4.$$

Since λ_{T_H} is a cocycle, $\lambda_{T_H}(\sigma)^{-1} = \sigma(\lambda_{T_H}(\sigma^{-1}))$. Hence

$$\lambda_{T_H}(\sigma)^{-1} = {}^{(\omega_T(\sigma)\times\sigma)\omega_{T_H}(\sigma^{-1})}\left[m_0(\sigma^{-1})n_G(\omega_H(\sigma^{-1}))\right]^4.$$

But since $\Gamma \to \Omega \rtimes \Gamma$, $\sigma \mapsto \omega_T(\sigma) \rtimes \sigma$ is a homomorphism we have

$$\begin{aligned}(\omega_T(\sigma) \rtimes \sigma)\omega_{T_H}(\sigma^{-1}) &= (\omega_T(\sigma) \rtimes \sigma)\omega_T(\sigma^{-1})\omega_H(\sigma^{-1})^{-1} \\ &= (\omega_T(\sigma) \rtimes \sigma)(\omega_T(\sigma^{-1}) \rtimes \sigma^{-1})(1 \rtimes \sigma)\omega_H(\sigma^{-1})^{-1} \\ &= (1 \rtimes \sigma)(\omega_H(\sigma^{-1})^{-1})\end{aligned}$$

which is independent of T_H. Hence λ_{T_H} is independent of T_H. □

COROLLARY 10.14. *λ_0^4 gives an unramified character on the maximally split Cartan subgroup $\mathbb{T}_H(F)$ of H.*

PROOF. The cocycle λ_0^4 of the Weil group is clearly the inflation of a cocycle on $\mathrm{Gal}(E/F)$ for an appropriate unramified extension E/F because both $\omega_H(\sigma)$ and $m_0(\sigma)$ factor through an unramified extension. By [19], such cocycles correspond to unramified characters on the $\mathbb{T}_H(F)$. □

PROPOSITION 10.15. *If the χ-data is tamely ramified then the character $\langle a, \gamma\rangle$ given in the term Δ_{III_2} is also tamely ramified. (As always* res. char. $\neq 2$.*)*

PROOF. By the construction of Labesse, it is enough to show that $a^4 \in H^1(W_{E/F}, \widehat{T}(\mathbb{C}))$ is the corestriction of a tamely ramified character

$$\varphi \in \mathrm{Hom}_c(E^\times, \widehat{T}(\mathbb{C})).$$

This follows immediately from the following lemma and (10.10), (10.14). □

Extend each character χ_λ to a tamely ramified character $\chi_{\lambda,E}$ on $E^\times$. Then define $\varphi_\lambda \in \mathrm{Hom}_c(E^\times, \widehat{T})$ by $\varphi_\lambda(u) = \chi_{\lambda,E}(u)^\lambda$.

LEMMA 10.16. $\mathrm{Cor}\left[\prod^\circ_{G/H} \varphi_\lambda\right]$ *is cohomologous to $r_p^2/\bar{r}_p^2$.*

PROOF. r_p itself is a product of terms r_p^λ indexed by orbits of roots λ. If χ_λ is a character on $F_+^\times$ we may view it as a character $\widehat{\chi}_\lambda$ on W_{E/F_+} by local class field theory. By Section 5 applied to the split rank-one torus, this identification is given by

$$\chi_\lambda \mapsto \widehat{\chi}_\lambda = \mathrm{Cor}'_{E/F_+} \chi_{\lambda,E}$$

where Cor'_{E/F_+} is the corestriction from $\mathrm{Hom}_c(E^\times, \mathbb{C}^\times)$ to $H^1_c(W_{E/F_+}, \mathbb{C}^\times)$. Let Cor_+ be the corestriction map associated to $W_{E/F_+} \to W_{E/F}$. Since $\mathrm{Gal}(E/F_+)$ acts trivially on λ, we have $\mathrm{Cor}_{E/F}\,\varphi_\lambda = \mathrm{Cor}_+\left[\mathrm{Cor}'_{E/F_+} \chi_{\lambda,E}{}^\lambda\right] = \mathrm{Cor}_+(\widehat{\chi}_\lambda^\lambda)$ which we will now show to be equal to $r_p^\lambda(w)^2$.

We recall the definition of $r_p^\lambda(w)$. Fix representatives $\{\tau_p\}$ in Γ for $\Gamma/\Gamma_\pm$. Fix a, b in $\Gamma_\pm \subseteq \Gamma$ trivial and nontrivial respectively in $\Gamma_\pm/\Gamma_+$. Set $R^+ = R_p^+ = \{a\tau_p\}$, $R^- = R_p^- = \{b\tau_p\}$. Then fixing appropriate representatives w_τ in W for elements of R^+, R^-,

$$r_p^\lambda(w) \stackrel{\mathrm{def}}{=} \prod_{\tau \in R^+} \widehat{\chi}_\lambda(w_\tau w w_\sigma^{-1})^{w_\tau^{-1}\lambda}$$

where $\sigma \in R^+ \cup R^-$ is defined by $w_\tau w w_\sigma^{-1} \in W_{E/F_+}$. Now we change the set of representatives to $\tau_q = a^{-1}b\tau_p$. So $R_q^+ = R_p^-$, $R_q^- = R_p^+$. The gauge for this data is $q = -p$. This gives

$$r_{-p}^\lambda(w) = \prod_{R_q^+} \widehat{\chi}_\lambda(w_\tau w w_\sigma^{-1})^{w_\tau^{-1}\lambda} = \prod_{R_p^-} \widehat{\chi}_\lambda(w_\tau w w_\sigma^{-1})^{w_\tau^{-1}\lambda}.$$

Now

$$r_p^\lambda r_{-p}^\lambda = \prod_{R^+\cup R^-} (\cdots) = \mathrm{Cor}_+ \widehat{\chi}_\lambda^\lambda = \mathrm{Cor}_{E/F}\,\varphi_\lambda.$$

Also $r_{-p} = s_{-p/p} r_p$ and $s_{-p/p} = 1$ ([21, 2.4]). The result follows. $\square$

Now finally, we turn to $\Delta_{III_2}(\gamma_H, \gamma_G)$ and the completion of the results of this section.

THEOREM 10.17. *For $p > e_G + 1$ and γ_H, γ_G strongly compact*

$$\Delta_0(\gamma_H^{Q^2}, \gamma_G^{Q^2}) = |Q|^m \Delta_0(\gamma_H, \gamma_G)$$

where m is the number of roots outside H_ε but in G_ε.

PROOF. By the preceding results, it is enough to check that $\Delta_{\mathrm{III}_2}(\gamma_H^Q, \gamma_G^Q) = \Delta_{\mathrm{III}_2}(\gamma_H, \gamma_G)$. But

$$\begin{aligned}\Delta_{\mathrm{III}_2}(\gamma_H, \gamma_G) &\overset{\mathrm{def}}{=} \langle a, \gamma\rangle = \langle a, \gamma_s\rangle\langle a, \gamma_u\rangle = \langle a, \gamma_s\rangle \\ &= \langle a, \gamma_s^Q\rangle = \langle a, \gamma_s^Q\rangle\langle a, \gamma_u^Q\rangle = \langle a, \gamma^Q\rangle \\ &= \Delta_{\mathrm{III}_2}(\gamma_H^Q, \gamma_G^Q)\end{aligned}$$

since $\gamma_s^Q = \gamma_s$ and a is tamely ramified by (10.15). $\square$

THEOREM 10.18. *For $p > e_G + 1$, $\Delta_0(\gamma_H, \gamma_G) = \Delta_{0,\varepsilon}(\gamma_H, \gamma_G)$.*

PROOF. $\Delta_0(\gamma_H, \gamma_G) = \Delta_0(\gamma_H^{Q^2}, \gamma_G^{Q^2})/|Q|^m$. As $Q \to \infty$, $\gamma_H^Q \to \gamma_{H,s}$, $\gamma_G^Q \to \gamma_{G,s}$ and we have seen $\Delta_0(\gamma_H^Q, \gamma_G^Q) = \Delta_{0,\varepsilon}(\gamma_H^Q, \gamma_G^Q)$ near $\gamma_{H,s}$, $\gamma_{G,s}$. Also

$$\Delta_{0,\varepsilon}(\gamma_H, \gamma_G) = \Delta_{0,\varepsilon}(\gamma_H^{Q^2}, \gamma_G^{Q^2})/|Q|^m$$

which equals $\Delta_0(\gamma_H^{Q^2}, \gamma_G^{Q^2})/|Q|^m = \Delta_0(\gamma_H, \gamma_G)$. Here ε is the absolute semisimple part of γ_G (or γ_H). $\square$

11. Central Characters.

We identify $Z^0 = Z_G^0$ canonically with a subgroup of Z_H.

LEMMA 11.1. *For $z \in Z^0$, we have $\Delta(z\gamma_H, z\gamma_G) = \langle a, z\rangle\Delta(\gamma_H, \gamma_G)$, where $a \in H^1(W_F, \widehat{T}(\mathbb{C}))$ is computed relative to any $\gamma_H, T_H \to T$, etc.*

PROOF. [21]. $\square$

LEMMA 11.2. *$\langle a,\cdot\rangle$ is an unramified character on every unramified Cartan subgroup T_H of H.*

PROOF. Referring to the definition of the cocycle a, and using [21, 2.1.A], together with $\omega_{T_H}(\sigma)=1$, the cocycle a becomes

$$a(w)=\overline{r}_p(w)m_0(\sigma)n_G(\omega_H(\sigma))^{-1}r_p(w)^{-1}.$$

Select the χ-data to be unramified. Let E/F be an unramified extension splitting T_H. By the criterion of Labesse [19,6.3,5.9], $\langle a,\cdot\rangle$ is unramified on $T_H(F)$ if a is represented by a cocycle in $Z^1_c(W_{E/F},\widehat{T}_H)$ which is trivial on the units of $E^\times\subset W_{E/F}$. On $E^\times$, $a(x)$ reduces further to

$$a(x)=\overline{r}_p(x)r_p(x)^{-1}.$$

By Langlands and Shelstad [21,p237], the restriction of $r_p=\prod r_p^\lambda$ to $E^\times$ is given by

$$r_p^\lambda(x)=\prod_{i=1}^n\chi_\lambda(\mathrm{Nm}^E_{F_+}\sigma_i x)^{\sigma_i^{-1}\lambda},\quad x\in E^\times,$$

and a similar formula holds for $\overline{r}_p$. Since χ_λ is unramified, it is now clear that $a(x)$ restricted to units of $E^\times$ is trivial. □

To identify the unramified character $\langle a,z\rangle$ in terms suitable for application to endoscopy we recall some well-known constructions found in [2],[23]. Adopt the notation of [2], adding subscripts G and H to distinguish data on $\widehat{G}$ and $\widehat{H}$. There is a canonical surjection $\widehat{T}_G\to\widehat{Y}_G$ inducing bijections

$$\widehat{Y}_G/_kW_G\overset{\text{bij}}{\leftrightarrow}(\widehat{T}_G\rtimes\sigma)/\mathrm{int}\,{}_k\widehat{N}\overset{\text{bij}}{\leftrightarrow}(\widehat{G}\rtimes\sigma)_{ss}/\mathrm{int}\,\widehat{G}.$$

Similar bijections hold with G replaced by H. Using $\xi:{}^LH\to{}^LG$ and the bijections above, functions on $\widehat{Y}_G/_kW_G$ pull back by ξ^* to functions on $\widehat{Y}_H/_kW_H$. Let s_H and s_G be matching elements in $\widehat{Y}_H$ and $\widehat{Y}_G$ by these bijections, and let t_H and t_G be corresponding lifts to $\widehat{T}_H$ and $\widehat{T}_G$. By construction $\xi(t_H\rtimes\sigma)=t_Hm_0(\sigma)\rtimes\sigma\in{}^LG$ is $\widehat{G}$-conjugate to $t_G\rtimes\sigma\in\widehat{T}_G\rtimes\sigma$. The spherical measures $\widehat{\omega}_s$ attached to $s\in\widehat{Y}_G$ identifies the Hecke algebra $H(G)$ on G with coordinate functions on $\widehat{Y}_G/_kW_G$ by

$$f:s\to\widehat{\omega}_s(f).$$

By these identifications ξ^* induces a map from the Hecke algebra of G to that of H, which we denote $\mathcal{H}:H(G)\to H(H)$.

Let $\mathbb{T}_d$ denote the maximal split subtorus of $\mathbb{T}_G$. Then $\widehat{Y}_G$ is canonically identified with the group of unramified characters on $\mathbb{T}_G$ giving a pairing $\widehat{Y}_G\times\mathbb{T}_G(F)\to^{\langle,\rangle}\mathbb{C}^\times$. Pick $z\in\mathbb{T}_d(F)\cap Z^0_G$. Define R^G_z for $z\in Z_G(F)$ by $(R^G_zf)(x)=f(xz)$, and define R^H_z analogously on $C^\infty_c(H)$.

LEMMA 11.3. *For $z\in\mathbb{T}_d(F)\cap Z^0_G$, we have $R^H_z\mathcal{H}(f)=\langle a,z\rangle\mathcal{H}(R^G_zf)$.*

PROOF. Tracing the identifications through using [23] one easily checks that $\widehat{\omega}^G_s(R^G_zf)=\langle s,z^{-1}\rangle\widehat{\omega}^G_s(f)$, $s\in\widehat{Y}_G$, $f\in H(G)$. A similar formula holds on the Hecke algebra of H.

Now if s_H and s_G in $\widehat{Y}_H$ and $\widehat{Y}_G$ correspond by ξ, and if $s \in Z_G^0 \cap \mathbb{T}_d(F)$, then

$$\widehat{\omega}^H_{s_H}(\mathcal{H}R^G_z f) = \widehat{\omega}^G_{s_G}(R^G_z f) = \langle s_G, z^{-1}\rangle \widehat{\omega}^G_{s_G}(f) = \langle s_G, z^{-1}\rangle \widehat{\omega}^H_{s_H}(\mathcal{H}f),$$

$$\widehat{\omega}^H_{s_H}(R^H_z \mathcal{H}f) = \langle s_H, z^{-1}\rangle \widehat{\omega}^H_{s_H}(\mathcal{H}f).$$

Hence

$$\langle s_G, z\rangle \widehat{\omega}^H_{s_H}(\mathcal{H}R^G_z f) = \langle s_H, z\rangle \omega^H_{s_H}(R^H_z \mathcal{H}f).$$

Now we will show below that (*) $\langle s_G, z\rangle\langle s_H, z^{-1}\rangle = \langle a, z\rangle$. Hence

$$\widehat{\omega}^H_{s_H}(\langle a, z\rangle \mathcal{H}R^G_z f) = \widehat{\omega}^H_{s_H}(R^H_z \mathcal{H}f)$$

for all $s_H \in \widehat{Y}_H$. Thus $\langle a, z\rangle \mathcal{H}(R^G_z f) = R^H_z \mathcal{H}(f)$.

Finally we return to the identity (*). Place a subscript "un" to denote unramified classes in H^1. Let $a_G, a_H \in H^1_{un}(W_{E/F}, \widehat{T})$ represent $t_G \rtimes \sigma, t_H \rtimes \sigma$ respectively. We must show $a_G a_H^{-1} a^{-1}$ has trivial image in $H^1_{un}(W_{E/F}, \widehat{Z}_G)$. Let $\widehat{G}_{der}$ denote the derived subgroup of $\widehat{G}$, and set $\widehat{T}_{der} = \widehat{T} \cap \widehat{G}_{der}$. It is easily checked that the composite

$$H^1_{un}(W_{E/F}, \widehat{T}_{der}) \to H^1_{un}(W_{E/F}, \widehat{T}) \to H^1_{un}(W_{E/F}, \widehat{Z}_G)$$

is zero. Hence $a_G a_H^{-1} a^{-1}$ has trivial image in $H^1_{un}(W_{E/F}, \widehat{Z}_G)$ if its image in $H^1_{un}(W_{E/F}, \widehat{T}/\widehat{T}_{der}) = H^1_{un}(W_{E/F}, \widehat{G}/\widehat{G}_{der})$ is trivial. Now by definition t_G is σ-conjugate to $t_H m_0(\sigma)$, $(t_G = x^{-1} t_H m_0(\sigma)\sigma(x))$ so that t_G and $t_H m_0(\sigma)$ lead to the same class in $H^1_{un}(W_{E/F}, \widehat{G}/\widehat{G}_{der})$. But the product of t_H with $a(w) = \bar{r}_p(w) m_0(\sigma) n_G(\omega_H(\sigma))^{-1} r_p(w)^{-1}$ is clearly equal to $t_H m_0(\sigma)$ modulo $\widehat{G}_{der}$ (which contains $\bar{r}_p$, r_p, n_G). Hence $a_G = a_H a$ in $H^1_{un}(W_{E/F}, \widehat{Z}_G)$. □

PART III. ORBITAL INTEGRALS

12. Harish-Chandra Descent.

Suppose we have a commutative diagram

$$\begin{tikzcd} {}^L H \arrow[r, "\xi"] & {}^L G \\ {}^L H_M \arrow[u] \arrow[r, "\xi_M"] & {}^L M \arrow[u] \end{tikzcd}$$

where M is a Levi factor of a parabolic subgroup of G. Let $\mathcal{H}^A_B$ be the map from the Hecke algebra of A to the Hecke algebra of B corresponding to ${}^L A \to {}^L B$. Then $\mathcal{H}^M_{H_M} \mathcal{H}^G_M = \mathcal{H}^H_{H_M} \mathcal{H}^G_H$. By Harish-Chandra descent and induction on groups of smaller rank, for suitable elements $(\gamma_H, \gamma_G) \in H_M \times M$ (by using 9.2)

$$\begin{aligned}
&\sum \Delta_0(\gamma_H, \gamma_G)\Phi_{T,G}(\gamma_G, f) \\
&\qquad = \sum \Delta_0^M(\gamma_H, \gamma_G) \prod_{H/H_M} |1-\alpha(\gamma)|^{-1/2}\Phi_{T,M}(\gamma_G, \mathcal{H}_M^G(f)) \\
&\qquad = \prod_{H/H_M} |1-\alpha(\gamma)|^{-1/2}\Phi_{T,H_M}^{st}(\gamma_G, \mathcal{H}_{H_M}^M \mathcal{H}_M^G(f)) \\
&\qquad = \prod_{H/H_M} |1-\alpha(\gamma)|^{-1/2}\Phi_{T,H_M}^{st}(\gamma_G, \mathcal{H}_{H_M}^H \mathcal{H}_H^G(f)) \\
&\qquad = \Phi_{T,H}^{st}(\gamma_G, \mathcal{H}_H^G(f)).
\end{aligned}$$

So the fundamental lemma in general follows from the fundamental lemma for compact elements. So it is enough to consider compact elements.

Suppose $z \in Z_G^0(F)$, and suppose the matching is known for γ_H. Then using (11.1) and (11.3)

$$\begin{aligned}
\sum_{\gamma_G} \Delta(z\gamma_H, z\gamma_G)\Phi_T(z\gamma_G, f) &= \sum_{\gamma_G} \langle a, z\rangle \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, R_z^G f) \\
&= \langle a, z\rangle \Phi_{T,H}^{st}(\gamma_H, \mathcal{H}(R_z^G f)) \\
&= \Phi_{T,H}^{st}(\gamma_H, \langle a, z\rangle \mathcal{H}(R_z^G f)) \\
&= \Phi_{T,H}^{st}(\gamma_H, R_z^H \mathcal{H}(f)) \\
&= \Phi_{T,H}^{st}(z\gamma_H, \mathcal{H}(f)).
\end{aligned}$$

So the fundamental lemma follows for $z\gamma_H$ from the fundamental lemma for γ_H.

13. Descent for Absolutely Semisimple Elements.

Assume in this section that G_{der} is simply connected. We will apply descent arguments to $\gamma = \gamma_s\gamma_u$ noting that every element is either topologically unipotent ($\gamma_s = 1$) or contained in the centralizer $C_G(\gamma_s)^0$ of a nontrivial, absolutely semisimple element. The following version of Kazhdan's lemma, found in Kottwitz [**16**], is essential.

LEMMA 13.1. *Let $\varepsilon \in G(O_F)$ be absolutely semisimple. Then G_ε is quasi-split and splits over an unramified extension. Also G_ε is naturally defined over O_F as well as the inclusion $G_\varepsilon \subseteq G$. $G_\varepsilon(O_F)$ is hyperspecial. Finally, if $\varepsilon^h \in G(O_F)$ then $h \in G_\varepsilon(\overline{F})G(O_F)$.*

Let $\gamma \in G(O_F)$ be a strongly regular element with absolutely semisimple part ε. We consider κ-orbital integrals:

$$\Phi_{T,G}^{\kappa}(\gamma, f) = \int_{(T\backslash G)(F)} \kappa(\sigma(g)g^{-1})f(g^{-1}\gamma g)d\dot{g}.$$

Suppose that dt and dg are invariant measures on G (arising as usual from an invariant form) and that $dt\,d\dot{g} = dg$. Let dh be an invariant measure on G_ε, and suppose that $dt\,d\dot{h} = dh$ (same dt as above). Set

$$f_1 = \frac{\operatorname{char} G(O_F)}{\operatorname{vol}(G(O_F), dg)} \quad \text{and} \quad f_2 = \frac{\operatorname{char} G_\varepsilon(O_F)}{\operatorname{vol}(G_\varepsilon(O_F), dh)}.$$

Let κ_ε be the restriction of κ to G_ε.

LEMMA 13.2. *Under the assumptions given above,*

$$\Phi^{\kappa}_{T,G}(\gamma, f_1) = \Phi^{\kappa_\varepsilon}_{T,G_\varepsilon}(\gamma, f_2).$$

PROOF. Same as [9], replacing Kazhdan's lemma by (13.1). □

Combined with (10.18), this shows that the fundamental lemma for the identity of the Hecke algebra follows from the fundamental lemma for topologically unipotent elements.

14. Regular Unipotent Orbital Integrals.

It can be easily seen that the canonical normalization of transfer factors is the correct normalization for the fundamental lemma by comparing the orbital integrals of strongly regular absolutely semisimple elements. This section shows that a much stronger compatibility condition holds for the transfer factors: the first term of the asymptotic expansion (Shalika germ expansion) for the κ-orbital integral of G with transfer factor, coincides with the first term of the first term of the asymptotic expansion on H. By results of Shelstad [24] the germ is either 1 or 0. So to check that the first term of the expansions coincide it is enough to compute and compare the unipotent regular orbital integrals for G and H. This computation is carried out in this section. When G and H split over the same unramified extension, descent properties for the maximally split Cartan subgroup of G give the comparison by trivial arguments. When G and H split over different extensions more work is required. We make use of the results of Section 4 to reduce to a small number of cases.

Let κ be the character of $H^1(\mathrm{Gal}(E/F), \mathbb{T}_H)$ defining the endoscopic group H of G. Then κ defines [11,VI-2] for each simple root α a character $\kappa^\alpha : F_\alpha \to \mathbb{C}^\times$ where F_α is the field extension of F over which the root α becomes fixed by the Galois group. We recall the definition. We have a sequence

$$\begin{aligned} F^\alpha &\xrightarrow{\varphi_\alpha} \mathbb{T}_{G,\mathrm{adj}}(F) \to \mathbb{T}_{G,\mathrm{adj}}(F)/\mathbb{T}_G(F) \\ &\xrightarrow{\delta} H^1(\mathrm{Gal}(E/F), Z_G) \to H^1(\mathrm{Gal}(E/F), \mathbb{T}_H) \xrightarrow{\kappa} \mathbb{C}^\times. \end{aligned}$$

The composite of these maps is κ^α. Here φ_α is defined by

$$\beta(\varphi_\alpha(x)) = \begin{cases} \sigma x & \text{if } \beta = \sigma\alpha \quad \sigma \in \mathrm{Gal}(F_\alpha/F) \\ 1 & \text{otherwise.} \end{cases} \quad \text{for all roots } \beta \text{ of } \mathbb{T}_G.$$

It follows from the work of Langlands and Shelstad that the orbital integral on the stable class of regular unipotent elements that arises as the coefficient of the regular germ of a κ-orbital integral is

$$f \mapsto \mu^{\kappa}_{reg}(f) = \int_{\overline{N}} \int_{N} m_{\kappa}(n) f(n^{\overline{n}}) dn d\overline{n}$$

where $f \in C_c^{\infty}(G)$, N is the unipotent radical of $\mathbb{B}$, and $\overline{N}$ is the unipotent radical of the Borel subgroup opposite $\mathbb{B}$ through $\mathbb{T}_G$. Also $m_{\kappa}(n) \stackrel{\text{def}}{=} \prod_S \kappa^{\alpha}(x_{\alpha})$ where $n = \prod_{\beta>0} \exp(x_{\beta} X_{\beta})$ and the product $\prod_S$ runs over the set S of $\mathrm{Gal}(\overline{F}/F)$-orbits of simple roots. Finally, dn, and $d\overline{n}$ are Haar measures on $N, \overline{N}$ obtained as a product of Haar measures dx_{β} (β roots) normalized so that $\int_{O_{F_{\beta}}} dx_{\beta} = 1$. Another expression for this integral is [12]

$$f \to \mu^{\kappa}_{reg}(f) = c \int_{K} \int_{N} f(n^{k}) m_{\kappa}(n) dn\, dk$$

for some constant c independent of κ and f. It is this latter integral that we will compute on the characteristic function of $K \stackrel{\text{def}}{=} G(O_F)$ to obtain

$$\mu^{\kappa}_{reg}(f) = c \int_{N \cap K} m_{\kappa}(n) dn \operatorname{vol}(K, dk) = c\{\prod_S \int_{O_{F_{\alpha}}} \kappa^{\alpha}(x_{\alpha}) dx_{\alpha}\} \operatorname{vol}(K, dk).$$

Since $\mathbb{T}_H$ is unramified, a prime π of O_F is also prime in $O_{F_{\alpha}}$. If we set $\kappa^{\alpha}(\pi) = \zeta_{\alpha}$, and set $q_{\alpha} = q^{[F_{\alpha}:F]}$ then

$$\begin{aligned} \mu^{\kappa}_{\text{reg}}(\text{char}(K))/\operatorname{vol}(K, dk) &= c \prod_S \left(1 - \frac{1}{q_{\alpha}}\right) \left(1 + \frac{\zeta_{\alpha}}{q_{\alpha}} + \frac{\zeta_{\alpha}^2}{q_{\alpha}^2} + \cdots\right) \\ &= c \prod_S \frac{(1 - \frac{1}{q_{\alpha}})}{(1 - \frac{\zeta_{\alpha}}{q_{\alpha}})}. \end{aligned}$$

To compute c we set $\kappa = 1$. Then $\mu^{st}_{\text{reg}}(f) = c \cdot \iint f(n^k) dn dk$. Set f equal to the characteristic function of $\{n^{\overline{n}} \mid n \in N \cap K, \overline{n} \in \overline{N} \cap K\}$. For any unramified group G over O_F set $[G] \stackrel{\text{def}}{=} |G(\mathbb{F}_q)|/q^{\dim G}$. Then integrating f as in [9,10] we obtain:

LEMMA 14.1.

$$\mu^{\kappa}_{\text{reg}}(\text{char}(K)) = \frac{|\tilde{G}|}{|\tilde{B}||\tilde{N}|} \prod_S \frac{(1 - \frac{1}{q_{\alpha}})}{(1 - \frac{\zeta_{\alpha}}{q_{\alpha}})} = \frac{[G]}{[\mathbb{T}_G]} \prod_S \frac{(q_{\alpha} - 1)}{(q_{\alpha} - \zeta_{\alpha})}.$$

Set $1_G = \text{char}(G(O_F))/[G]$ for any unramified group G over O_F. The main result of this section is

PROPOSITION 14.2. $\mu^{\kappa}_{\mathrm{reg}}(1_G) = [\mathbb{T}_H]^{-1} = \mu^{st}_{\mathrm{reg}}(1_H)$.

PROOF. If $\kappa = 1$, $H = G$ then by (14.1), $\mu^{\kappa}_{\mathrm{reg}}(1_G) = [\mathbb{T}_G]^{-1} = \mu^{st}_{\mathrm{reg}}(1_G)$ and the proposition follows in this case. Also by (14.1)

$$\mu^{\kappa}_{\mathrm{reg}}(1_G)/\mu^{st}_{\mathrm{reg}}(1_G) = \prod_S \frac{(q_\alpha - 1)}{(q_\alpha - \zeta_\alpha)}.$$

Thus it is enough to show that

$$[\mathbb{T}_G]/[\mathbb{T}_H] = \prod_S (q_\alpha - 1) / \prod_S (q_\alpha - \zeta_\alpha). \tag{14.3}$$

The possibilities for F_α, κ^α were calculated in **[11,VI-2,3]**. We begin with a special case:

LEMMA 14.4. *If the connected dual $\widehat{G}$ of G is simply connected, then $[\mathbb{T}_G] = [\mathbb{T}_H]$, and $\kappa^\alpha = 1$ for all α.*

PROOF. In this case $\mathbb{T}_G = \mathbb{T}_H$. By the formula defining κ^α, we have $\zeta_\alpha = 1$ for all α. □

We let Fr be the Frobenius map $G \to G$ for G a reductive group over $\mathbb{F}_q$. According to **[3,3.3.5]**:

LEMMA 14.5. *Let T be an Fr-stable maximal torus of $G/\mathbb{F}_q$ obtained from the maximally split torus T_0 by twisting with the Weyl group element ω. Then the order of T^{Fr} is given by*

$$|T^{Fr}| = |\det_{Y_0 \otimes \mathbb{R}}(\omega^{-1} \cdot Fr - 1)|$$

($Y_0 =$ character group of T_0 and ${}^\omega\chi(t) = \chi(t^\omega)$, ${}^{Fr_0}(\chi)(t) = \chi(Fr_0(t))$, $\chi \in Y$.) Moreover, if we write $Fr = qFr_0$ where $q > 1$ and Fr_0 has finite order then $|T^{Fr}| = \chi(q)$ where χ is the characteristic polynomial of $Fr_0^{-1}\omega$ on $Y_0 \otimes \mathbb{R}$.

Now to prove (14.3) we may work with adjoint groups.

LEMMA 14.6. $[\mathbb{T}_G] = \prod_S (q_\alpha - 1)$ *for adjoint groups G.*

PROOF. In the general case $[\mathbb{T}_G] = \prod_\alpha (q - \lambda_\alpha)$ where λ_α are the eigenvalues of Frobenius. If the orbit of α is $\{\alpha_1, \dots, \alpha_r\}$, then Fr_0 permutes these roots cyclically. Hence $\{\lambda_{\alpha_1}, \dots, \lambda_{\alpha_r}\} = \{1, \zeta, \dots, \zeta^{r-1}\}$, ζ a primitive r-th root of unity. So $\prod_{i=1}^r (q - \lambda_{\alpha_i}) = (q_\alpha - 1)$. The result follows. □

We now assume that ω acts on $\widehat{H}$ by an element of the extended Dynkin diagram as in (4.7). We may identify Y_0 with the subspace of codimension 1 in the geometric realization $V = \oplus \mathbb{R} e_\alpha$, α a simple root of the extended Dynkin diagram. The eigenvalues of ω on V are 1 (for the complement of Y_0) together with the eigenvalues of ω on Y_0. Thus we must check the roots of unity $\{\zeta_\alpha\}$ defined above together with 1 coincide with the eigenvalues of ω on V. If $q_\alpha = q^{m_\alpha}$ and if ζ_α is the label, then the eigenvalues of ω should be the m_α distinct roots $\zeta_\alpha^{1/m_\alpha}$. These facts are clear by inspection of the relevant Dynkin diagrams and the extended diagrams of the dual. These diagrams are described below completing the proof of (14.2). □ □

Split Groups

Type A_n. The extended Dynkin diagram of the Langlands dual to A_n is an $n+1$-gon. The automorphism ω acts by rotation of the $n+1$-gon. The characteristic polynomial is $X^{n+1}-1$. The vertices of the Dynkin diagram of A_n are labelled $\zeta, \zeta^2, \ldots, \zeta^n$. This gives the polynomial

$$\prod_{i=1}^{n}(X-\zeta^i)=(X^{n+1}-1)/(X-1).$$

Type B_n. The extended Dynkin diagram of the dual to B_n is C_n^{aff}. This extended dual diagram has an automorphism of order two interchanging the long roots of the diagram. The characteristic polynomial is $(X^2-1)^{(n+1)/2}$ if n is odd, and $(X^2-1)^{n/2}(X-1)$ if n is even. The vertices of the Dynkin diagram of B_n are labelled $\zeta, \zeta^2, \ldots, \zeta^n$, $\zeta=-1$ (ζ^n being the label on the short root). This gives the polynomial

$$\prod_{i=1}^{n}(X-\zeta^i)=\begin{cases}(X^2-1)^{(n+1)/2}/(X-1) & \text{if } n \text{ is odd}\\ (X^2-1)^{n/2} & \text{if } n \text{ is even.}\end{cases}$$

Type C_n. The extended Dynkin diagram of the dual to C_n is B_n^{aff}. The extended dual diagram has an automorphism of order 2 interchanging two vertices of the D_k subdiagram of B_n^{aff} and fixing the other vertices. This gives the characteristic polynomial $(X^2-1)(X-1)^{n-1}$. The short vertices of the Dynkin diagram of C_n are all labelled 1. The long vertex is labelled $\zeta=-1$. This gives the polynomial $(X^2-1)(X-1)^{n-2}$.

Type D_n. The extended Dynkin diagram of the dual to D_n is D_n^{aff}. Here there are several choices of automorphisms of the dual diagram. Not all possibilities are realizable for the split group and we refer the reader below to Type ${}^2D_n$ for those not listed here. Here we give three possibilities. The extended dual diagram D_n^{aff} has two subdiagrams of type D_k, and we may distinguish the automorphisms by whether they interchange the two D_k subdiagrams, and by whether they act as outer automorphisms on the individual D_k subdiagrams.

Type D_n, Automorphism I. The extended diagram D_n^{aff} has an automorphism of order two preserving each D_k subdiagram but acting as an outer automorphism on each factor. This gives the characteristic polynomial $(X^2-1)^2(X-1)^{n-3}$. The vertices of the subdiagram of D_n of type A_{n-2} are all labelled 1. The remaining two vertices are labelled $\zeta=-1$. This gives the polynomial $(X^2-1)^2(X-1)^{n-3}/(X-1)$.

Type D_n, Automorphism II, ($n = 2\ell+1$ is odd). The extended diagram D_n^{aff} has an automorphism of order four interchanging the D_k subdiagrams; the square of the automorphism is the automorphism described in the previous case. This gives the characteristic polynomial $(X^4-1)(X^2-1)^{\ell-1}$. The vertices of the subdiagram of D_n of type A_{n-2} are labelled ζ^i, $\zeta = -1$. The remaining two vertices are labelled $\zeta = i, -i$. This gives the polynomial $(X^4-1)(X^2-1)^{\ell-1}/(X-1)$.

Type D_n, Automorphism III, ($n = 2\ell$ is even). The extended diagram D_n^{aff} has an automorphism of order two interchanging the D_k subdiagrams. This gives the characteristic polynomial $(X^2-1)^{\ell}(X-1)$. The vertices of the subdiagram of D_n of type A_{n-2} are labelled ζ^i, $\zeta = -1$. The remaining two vertices are labelled $\zeta = 1, -1$. This gives the polynomial $(X^2-1)^{\ell}$.

Type E_6. The extended dual diagram E_6^{aff} has an automorphism of order 3, fixing one vertex. This gives the characteristic polynomial $(X^3-1)^2(X-1)$. The vertices of the Dynkin diagram of E_6 are labelled $\zeta, \zeta^2, 1, \zeta, \zeta^2, 1$ where ζ is a cube root of unity. This gives the polynomial $(X^3-1)^2$.

Type E_7. The extended dual diagram has an automorphism of order two. This gives the characteristic polynomial $(X^2-1)^3(X-1)^2$. The vertices of the Dynkin diagram of E_7 are labelled $\zeta, 1, \zeta, 1, 1, 1, \zeta$. This gives the polynomial $(X^2-1)^3(X-1)$.

Types E_8, F_4, G_2 These groups have no center, and their extended dual diagrams have no automorphisms.

Quasisplit Groups

Type ${}^2A_{2n}$. The extended dual diagram is a $2n+1$-gon and has an automorphism acting by reflection. This gives the characteristic polynomial $(X^2-1)^n(X-1)$. The diagram for ${}^2A_{2n}$ has $2n$ vertices arranged into n orbits of two vertices each. The labels on all the vertices are 1. This gives the polynomial $(X^2-1)^n$.

Type ${}^2A_{2n+1}$. The extended Dynkin diagram of the dual to A_{2n+1} is a $2n+2$-gon. There are two possibilities.

Type ${}^2A_{2n+1}$, Automorphism I. The extended dual diagram has an automorphism of order two acting by reflection whose axis passes through two opposite vertices. This gives the characteristic polynomial $(X^2-1)^n(X-1)^2$. The diagram for ${}^2A_{2n+1}$ has $2n+1$ vertices. Of these, $2n$ are arranged into n orbits of two vertices each, and a final vertex belongs to an orbit of its own. The labels on all the vertices are 1. This gives the polynomial $(X^2-1)^n(X-1)$.

Type ${}^2A_{2n+1}$ Automorphism II. The extended dual diagram has an automorphism of order two acting by a reflection whose axis passes through the midpoints of two opposite edges. This gives the characteristic polynomial $(X^2 -$

$1)^{n+1}$. The diagram for ${}^2A_{2n+1}$ is as in the previous case, except that the label on the solitary vertex is $\zeta = -1$. This gives the polynomial $(X^2-1)^{n+1}/(X-1)$

Type ${}^2D_{2n}$. The extended dual diagram has an automorphism of order four similar to that in Type D_n, Automorphism II described above. This gives the characteristic polynomial $(X^4-1)(X^2-1)^{n-2}(X-1)$. The Dynkin diagram is made up of $2n$ vertices. Two of the the vertices group together into a single orbit; the remaining vertices are in an orbit of their own. The labels on the solitary vertices are $-1, 1, -1, \ldots$. The label on the orbit with two vertices is -1. This gives the polynomial $(X^4-1)(X^2-1)^{n-2}$.

Type ${}^2D_{2n+1}$. The extended dual diagram has an automorphism of order two described above in Type D_n, Automorphism III. This gives the characteristic polynomial $(X^2-1)^{n+1}$. The Dynkin diagram is the same as in the previous case, except that the label on the orbit of two vertices is $\zeta = 1$. This gives the polynomial $(X^2-1)^{n+1}/(X-1)$.

Type ${}^3D_4$. The automorphism of order three on the extended dual diagram D_4^{aff} fixes two vertices and permutes the remaining three. This gives the characteristic polynomial $(X^3-1)(X-1)^2$. The Dynkin diagram of ${}^3D_4$ has 4 vertices arranged into two orbits of one and three vertices. The labels are all one. This gives the polynomial $(X^3-1)(X-1)$.

Type ${}^2E_6$. The automorphism of order two on the extended dual diagram E_6^{aff} has characteristic polynomial $(X^2-1)^2(X-1)^3$. The diagram ${}^2E_6$ has six vertices broken into four orbits of one, one, two and two vertices. The labels are all one. This gives the polynomial $(X^2-1)^2(X-1)^2$.

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