5/2016
THALES

Pentagons
May 15, 2016 notes.
(ℓ,θ), (ℓ,θ') coordinates.

SINGLE EDGE
COORDINATES.

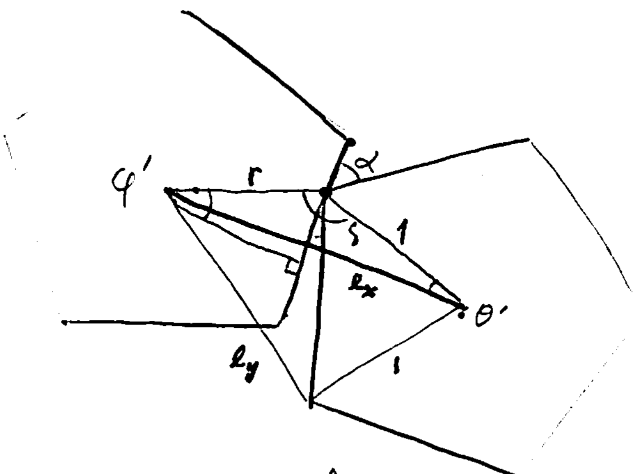
BANANAS

MONOTONICITY.

EXTENDED VARIABLES $(\ell,\theta,\theta')(x_\alpha,\alpha)$.

5/2016.

Given (l, θ') find θ



$\theta' > 0$ in figure.

$l_x = l.$ (l)

$l_y = \text{loc } 1, l_x\ (2\pi/5 - \theta').$

$l_y^2 = l_x^2 + 1 - 2 l_x \cos\left(\frac{2\pi}{5} - \theta'\right)$ (l_y)

$r = \text{loc } 1\ l_x\ \theta'$ (r)

$r^2 = h^2 + K^2$ (h)

h up to a sign (sign_h)

both signs occur. Use both.

$x_\alpha = h + r$ (x_α)

$\delta = \text{arc } 1\ r\ l_x \begin{cases} \cancel{l_x = \text{loc } r\ 1\ \delta} \quad \text{~~two solutions~~ } \cancel{\delta, 2\pi - \delta}\ \text{arc.} \\ \cancel{l_y = \text{loc } r\ (2\sigma)\ (\delta - \tfrac{3\pi}{10})} \quad \left[(\delta - \tfrac{3\pi}{10}) = \text{arc } r\ (2\sigma)\ l_y.\right] \end{cases}$ (δ)

(use second formula if $l_x \approx 1 + r$, $\delta \approx \pi$.)

Always use this if $h \geq 0$.

two solutions δ, $\delta > \pi \iff \theta' < 0.$

$\sin \varphi' = h/r.$ (φ')

$\delta = \frac{6\pi}{5} - \alpha - \varphi'$ (α)

$\theta + \theta' = \alpha \pmod{2\pi/5}$ (θ)

(on this page θ is only defined modulo $2\pi/5$).

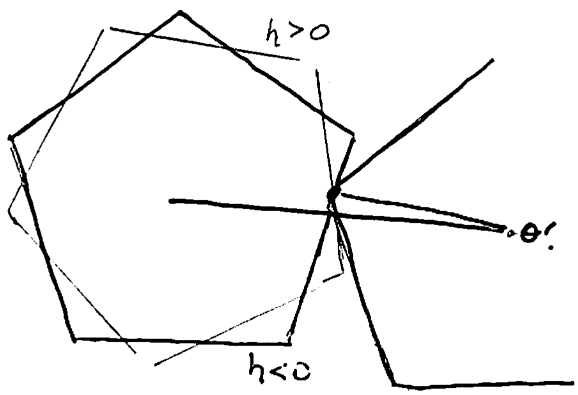
Note if $(l, \theta') \to (l, -\theta')$. Then a symmetry.

5/2016.

Given (l, θ') find θ

supplementary notes.

- Both $\pm h$ occur.



- if we reverse the orientation of the edge

$(l, \theta') \longmapsto (l, -\theta')$

$r \longmapsto r.$

$h \longmapsto -h$

$\theta \longmapsto -\theta \bmod 2\pi/5.$

So by a symmetry, we can arrange $h \geq 0$.

- $|\theta| \geq |\theta'|$. ~~Suppose ...~~ (when $\theta \in [-\pi/5, \pi/5]$). continuity of θ

Look at domain $h \geq 0$.

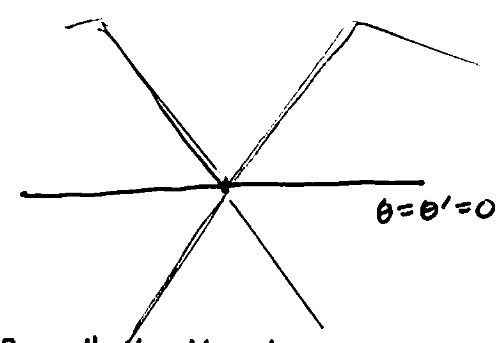
Normalize so that $\theta \in [0, 2\pi/5]$

There is a jump in θ at endpoints.

There $\exists!$ configuration s.t. $\theta \equiv 0 \equiv 2\pi/5 \bmod 2\pi/5$.

$h \geq 0$ implies $h \approx r$ $\theta \in [0, 2\pi/5]$ is near 0.

Hence $\theta \in [0, 2\pi/5]$ is continuous on $\underline{h \geq 0}$.



- In code throw unstable if it jumps across $0 - 2\pi$. It shouldn't happen in practice.

5/2016

Given (ℓ, θ'). Supplementary notes.

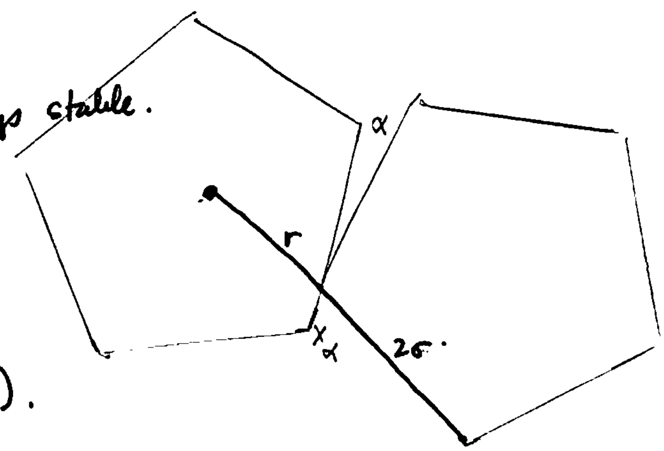
- Which formula for δ to use?

(*) $\left(\delta - \frac{3\pi}{10}\right) = \text{arc } r\ (2\sigma)\ l_y$, is <u>unstable</u>

when triangle degenerates:

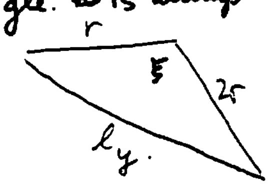
Only get degeneracies when $h \leq 0$.

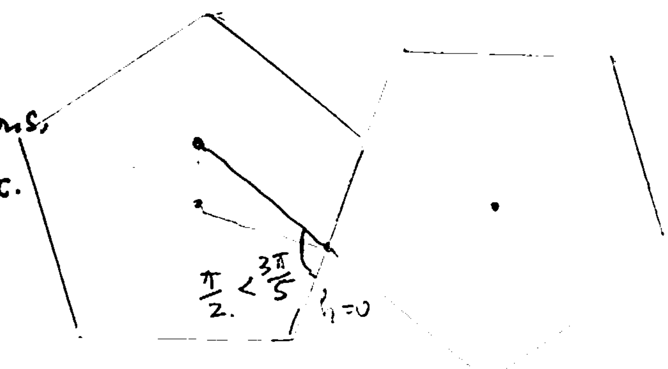
Assuming $h \geq 0$.
Then (*) is always stable.
Always use it.



(When $h \leq 0$ use symmetries).

- Assume $h \geq 0$.
By similar considerations,
the angle ξ is always $< \pi$.





So we always should use the branch of $(\text{arc } r\ (2\sigma)\ l_y)$
that is $< \pi$.

$\delta = \frac{3\pi}{10} + (\text{arc } r\ (2\sigma)\ l_y)$ always works.
for $h \geq 0$.

Supplementary Notes -

<u>Claim</u>. Formula for $\theta = \theta(l, \theta')$ is always in range $[0, 2\pi/5]$.
$h \geq 0$.

We can drop "mod $2\pi/5$".

Can use analytic continuity, if we show analytic throughout region.

l ✓, l_y ✓, r ✓, $\underline{\hat{h}\text{ ok if } h>0}$, x_α ✓, δ ✓,
trouble if $h=0$. (by comments)
Can use continuity.

φ' (between $[-\pi/5, \pi/5]$) ✓, α ✓, θ ✓.

formulas all continuous.

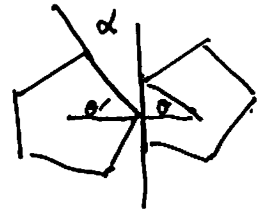
When $\theta' = 0$, $l = 1+k$, $h = 0$,
$\delta = \pi$; $\varphi' = \cancel{\pi/5}\ 0$, $\alpha = \frac{6\pi}{5} - \pi - \varphi' = \frac{\pi}{5}$, $\theta = \pi/5 \in [0, 2\pi/5]$.
By continuity anything we join a path too stays positive,
unless we come to $0, 2\pi/5$, which only occurs at 1 pt.
Region still connected w/o that point ✓.

5/2016

Values of $\theta \in [0, 2\pi/5]$

Geometric Interpretation

(l, θ') coordinates $h \geq 0$.



$\theta' + \theta = \alpha$

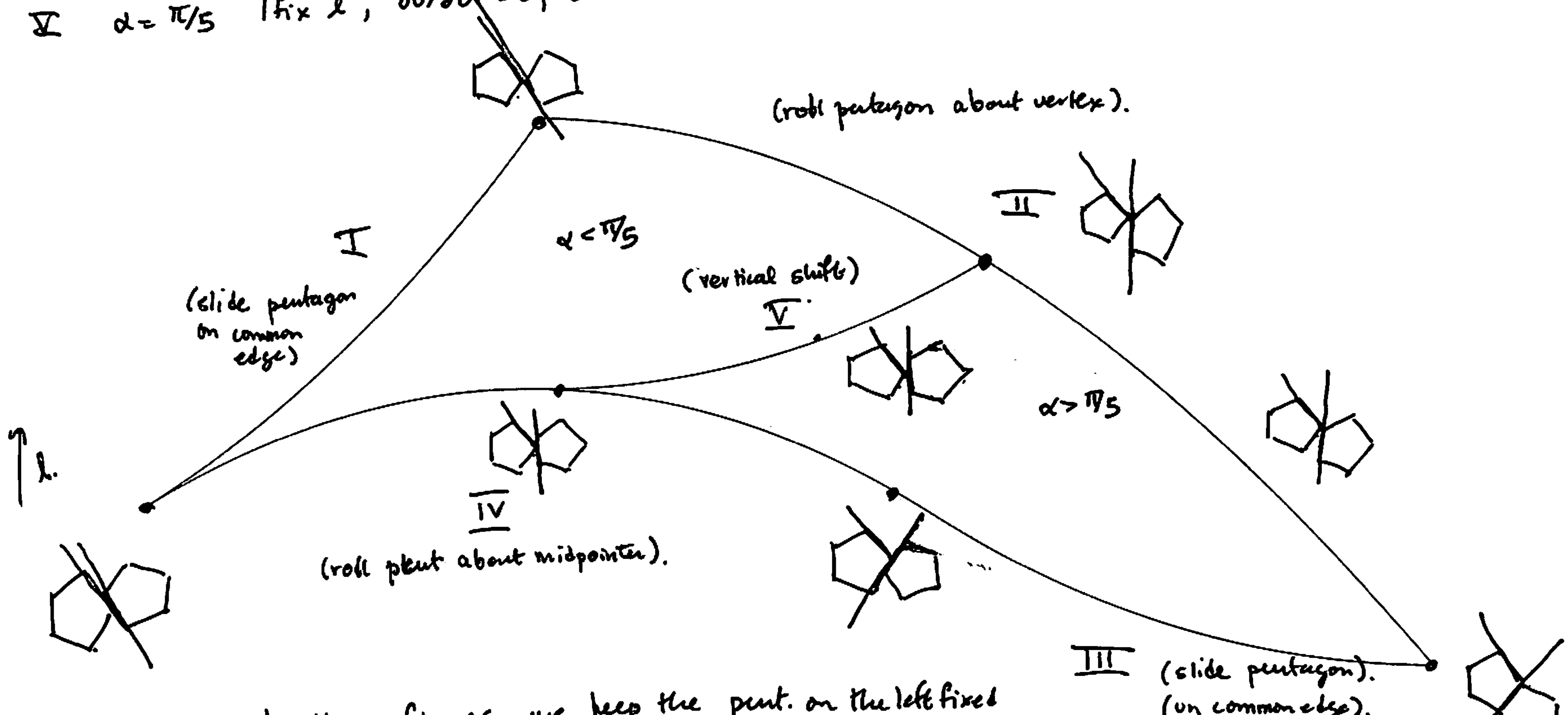
I $\alpha = 0$ (constraint $\alpha \geq 0$)

II $h = \sigma$ $x_\alpha = 2\sigma$.

III $\alpha = \frac{2\pi}{5}$ (constraint $\alpha \leq 2\pi/5$).

IV $h = 0$ $x_\alpha = 0$ midpointer. (constraint $h \geq 0$).

V $\alpha = \pi/5$ (fix l, $\partial\theta/\partial\theta' = 0$, clearly $\alpha = \pi/5$).



In these figures, we keep the pent. on the left fixed move the pent on the right.

$\boxed{\text{Boundary Curves}}$ - explicit equations.

$(\ell, \theta, h \geq 0)$. $\quad ks = \sqrt{(2K)^2 + \sigma^2} \approx 1.72$.

I : $(\arccos(2K/\ell) - \pi/5, \ell)$ $\quad \ell \in [ks, 2]$

II : $(\operatorname{acos}(\ell/2), \ell)$ $\quad \ell \in [2K, 2]$

III : $(\pi/5 - \operatorname{acos}(2K/\ell), \ell)$ $\quad \ell \in [2K, ks]$

IV : $(\pm \operatorname{acos}\left(\frac{1+\ell^2-K^2}{2\ell}\right), \ell)$ $\quad \ell \in [ks, 1+K]$

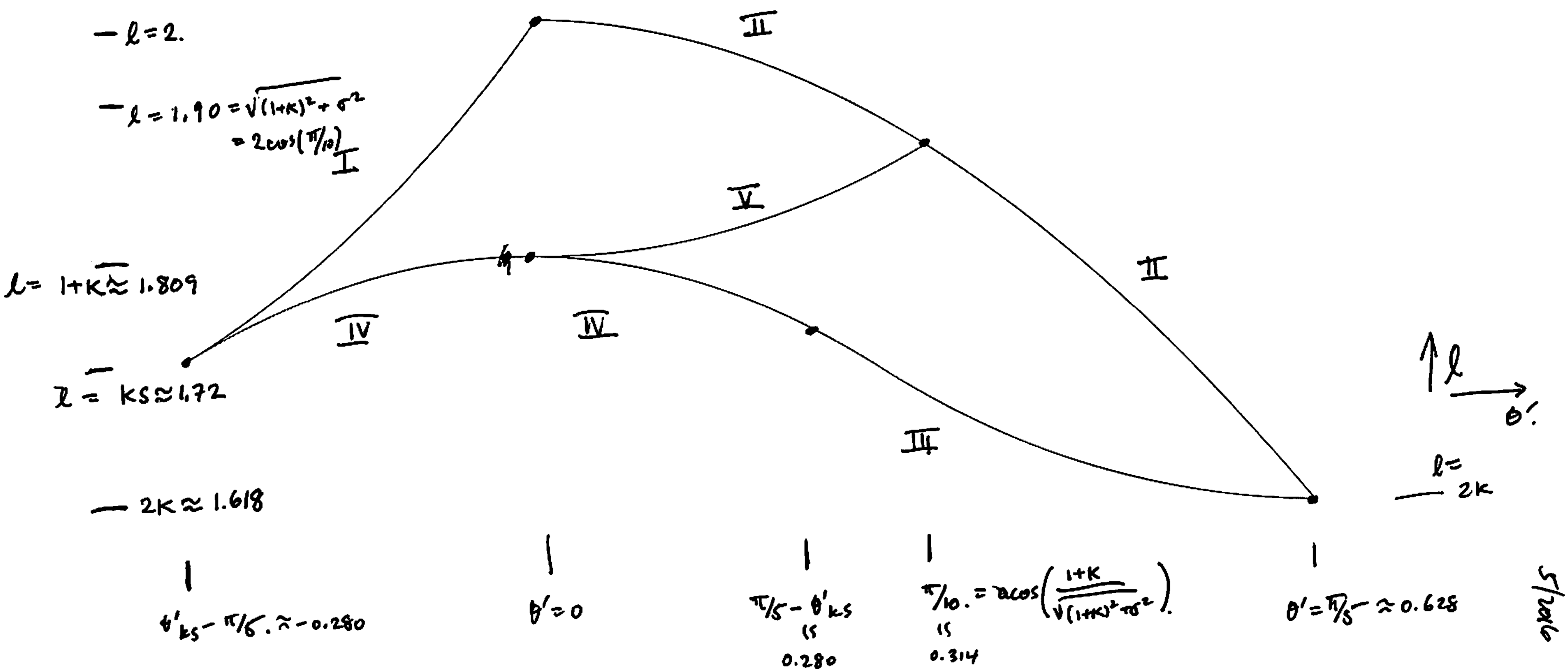
V : $(\operatorname{acos}\left(\frac{1+K}{\ell}\right), \ell)$ $\quad \ell \in \left[1+K, \sqrt{(1+K)^2+\sigma^2}\right]$, $\sqrt{(1+K)^2+\sigma^2} = \sqrt{2(1+K)}$

$\theta'_{ks} = \operatorname{acos}\left(\frac{2K}{\sqrt{(2K)^2+\sigma^2}}\right) \approx 0.348.$

<u>inverse</u>:

I	$(\theta', 2K/\cos(\theta'+\pi/5))$	$\theta' \in (\theta'_{ks} - \pi/5, 0]$.
II	$(\theta', 2\cos\theta')$	$\theta' \in [0, \pi/5]$
III	$(\theta', 2K/\cos(\pi/5 - \theta'))$	$\theta' \in [\pi/5 - \theta'_{ks}, \pi/5]$
IV	$(\theta', \cos\theta' + \sqrt{\cos^2\theta' + K^2 - 1})$	$\theta' \in [\theta'_{ks} - \pi/5, \pi/5 - \theta'_{ks}]$
V	$(\theta', (1+K)/\cos\theta')$	$\theta' \in [0, \pi/10]$

$\left\{ \begin{array}{l} \text{all curves monotonic} \\ \text{except IV.} \end{array} \right\}$.



5/2016

$\boxed{\text{Values of } \theta \quad \text{Contour Plots}}$

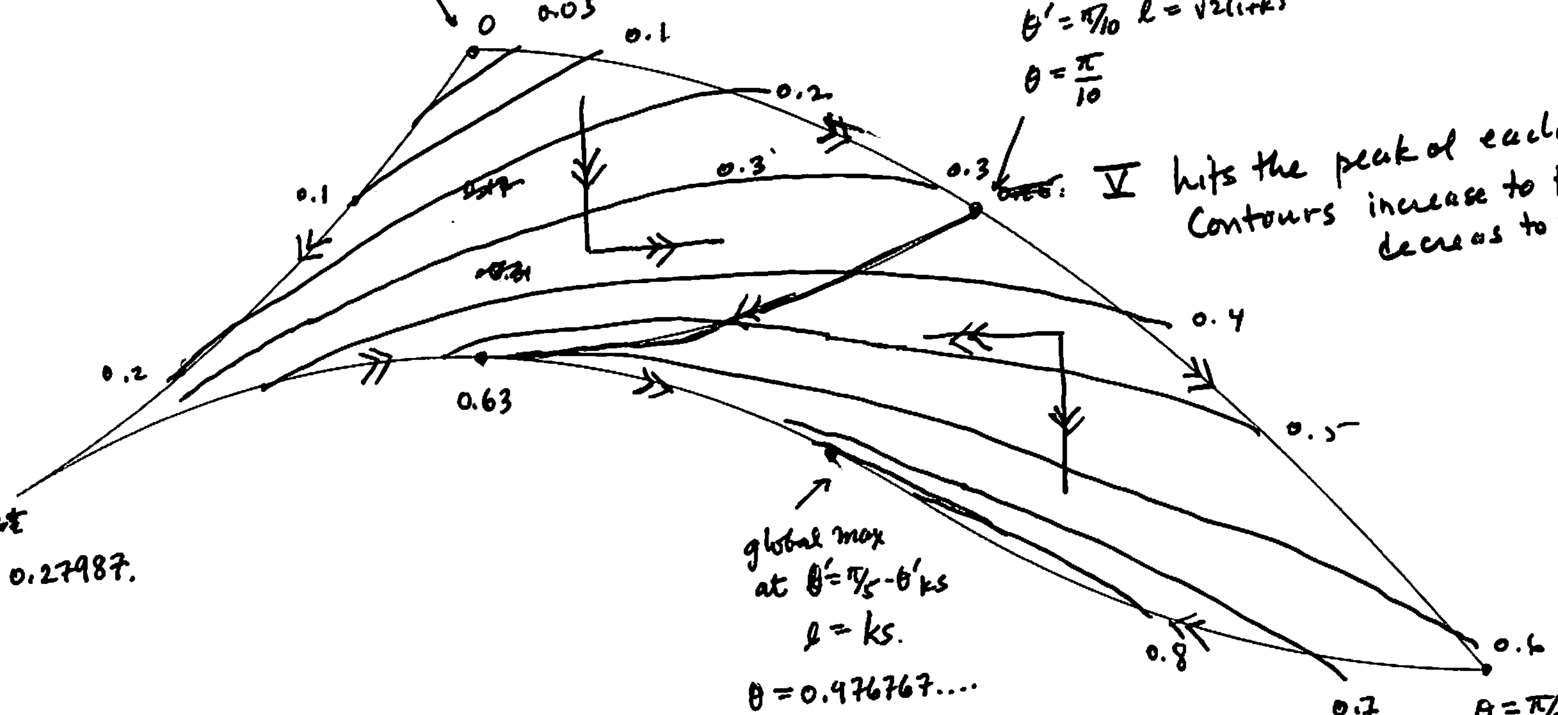
Monotonicities all have an obvious geometric interpretation.

(ℓ, θ') coords $h \geq 0$.

Generated Mathematica
ContourPlot[$\theta == c, \ldots$].

Arrows point to increasing θ.
$\twoheadrightarrow$

global min at
$\theta' = 0,\ \ell = 2,\ \theta = 0.$



$\theta' = \pi/10 \quad \ell = \sqrt{2(1+k)}$
$\theta = \frac{\pi}{10}$

~~0.3~~: V hits the peak of each contour.
Contours increase to the left of V
decrease to the right of V.

~~edge~~
$\theta = 0.27987.$

global max
at $\theta' = \pi/5 - \theta'_{ks}$
$\ell = ks.$
$\theta = 0.976767\ldots$

$\theta = \pi/5 = 0.628\ldots$

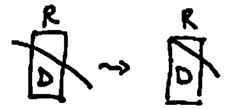
5/2016

(l, θ') coordinates, $h \geq 0$.

Partition domain D into rectangles $B_1, \dots, B_K$. $D \subseteq B_1 \cup \dots \cup B_K \Rightarrow D_i$ drop i.

Want to maximize/minimize θ over a rectangle R ~~...~~.

Partition R s.t. wlog $R \subseteq B_i$ some i.



Truncate R s.t. an $\exists$ upper corner p of R $p \in D$.

Truncate R s.t. $\exists$ lower corner $p \in R \cap D$.

Same $\exists$ left corner $p \in R \cap D$, $\exists$ right corner $p \in R \cap D$.

So every edge of R has a corner in $R \cap D$.

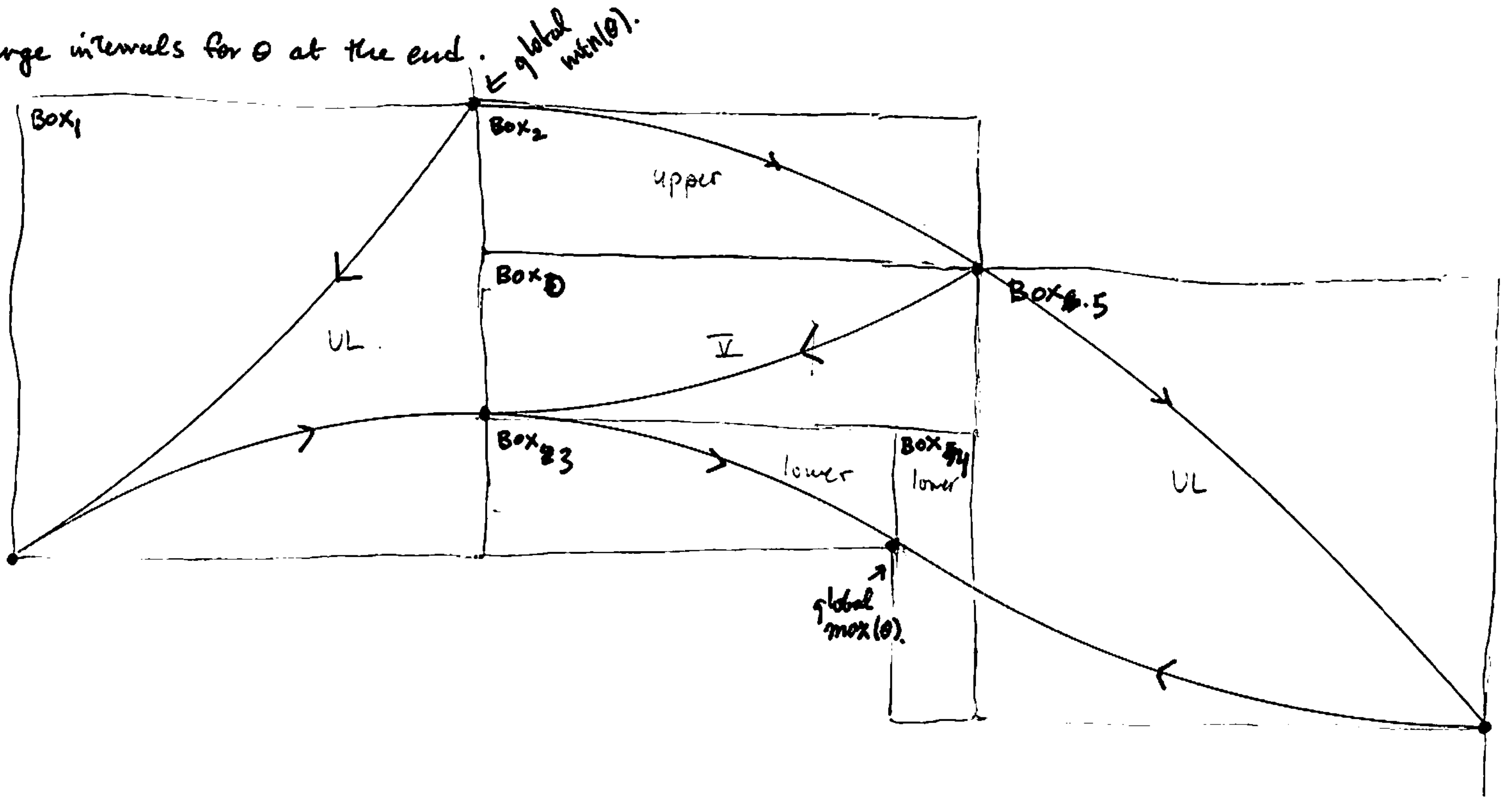
~~Except~~ Even on V, max is on ∂R, at a corner or $\partial R \cap C_{\mathrm{V}}$ (case V) or $\partial R \cap \partial D$

Always min is on ∂R, at a corner or $\partial R \cap \partial D$.

On V. can assume all corners of $R \subseteq D$.

- min at ~~lower~~ upper edge corner
- max at lower edge corner or if lower edge crosses V then on V.

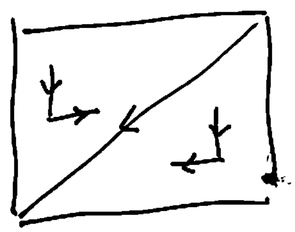
Remerge intervals for θ at the end.



5/2016

5/2016

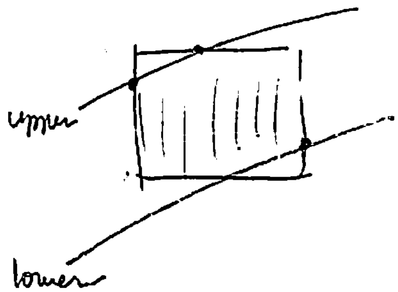
~~Region~~ Curve 5 ~~[illegible]~~ optimization

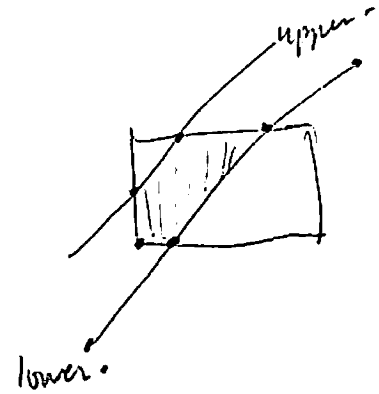


$R \subseteq B$.

<u>Max</u> on bottom edge λ.
If edge meets ridge C then $C \cap \lambda$
else closest endpoint.

<u>Min</u> ~~if edge~~ on upper edge μ
If edge meets ridge C, then one of the two endpoints of μ
else farthest endpoint.





4 new points

compute $upper_{LHS}$ $upper_{RHS}$

$lower_{LHS}$ $lower_{RHS}$.

<u>add</u> those that are in R.

<u>Drop</u> corners that are outside domi

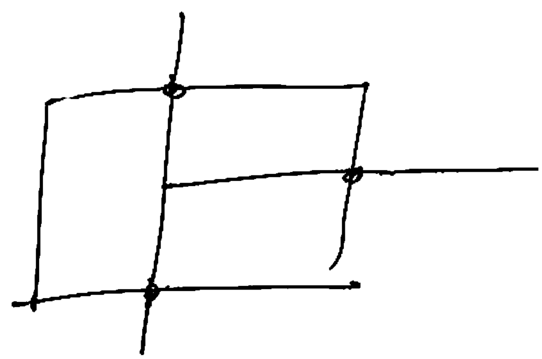
<u>add</u> if merge $upper_{LHS}$ $upper_{RHS}$ meets horizontal edge, add its pt.

<u>add</u> similar for merge $lower_{LHS}$ $lower_{RHS}$.

These are the critical points to be considered.

Final filter: { Every critical point must meet. an edge of the original box. or global max. or global min.

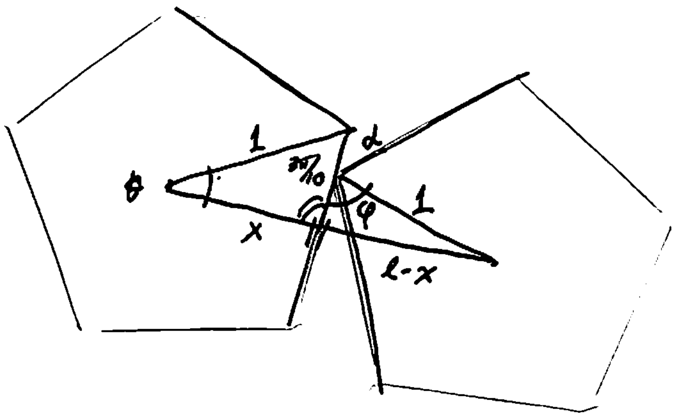
Could also filter out critical points coming from $\partial B \cap \partial R$.



5/2016

Given (l, θ), find θ'!

$\theta \in [0, 2\pi/5]$, $\theta' \in [-\pi/5, \pi/5]$ mod $\frac{2\pi}{5}$.



$$\frac{x}{\sin\left(\frac{3\pi}{10}\right)} = \frac{1}{\sin\left(\pi - \theta - \frac{3\pi}{10}\right)} \qquad (x)$$

$$\frac{l-x}{\sin\varphi} = \frac{1}{\sin\left(\theta + \frac{3\pi}{10}\right)} \qquad (\varphi)$$

two solutions in φ $\quad \varphi, \pi - \varphi$

use both. (primarily acute φ)

$$\varphi = \alpha' + \frac{3\pi}{10} \qquad (\alpha')$$

$$\alpha + \alpha' = \frac{2\pi}{5} \qquad (\alpha).$$

$$\alpha = \theta + \theta' \cancel{\text{mod}\left(\frac{2\pi}{5}\right)} \qquad (\theta')$$

$\left(\frac{2\pi}{5} - \alpha' - \theta\right) = \theta'$.

<u>Note</u>: φ obtuse $\iff \varphi \geq \pi/2 \iff \alpha' + \frac{3\pi}{10} \geq \frac{\pi}{2} \iff \alpha' \geq \frac{\pi}{5}$

φ acute $\iff \alpha' < \pi/5$.

- If we switch the orientation. So switching orientation if necessary, we can assume φ is $\leq \pi/2$.

$l \to l$

$\theta \to 2\pi/5 - \theta$.

$\alpha \to 2\pi/5 - \alpha$

$\varphi \to \pi - \varphi$.

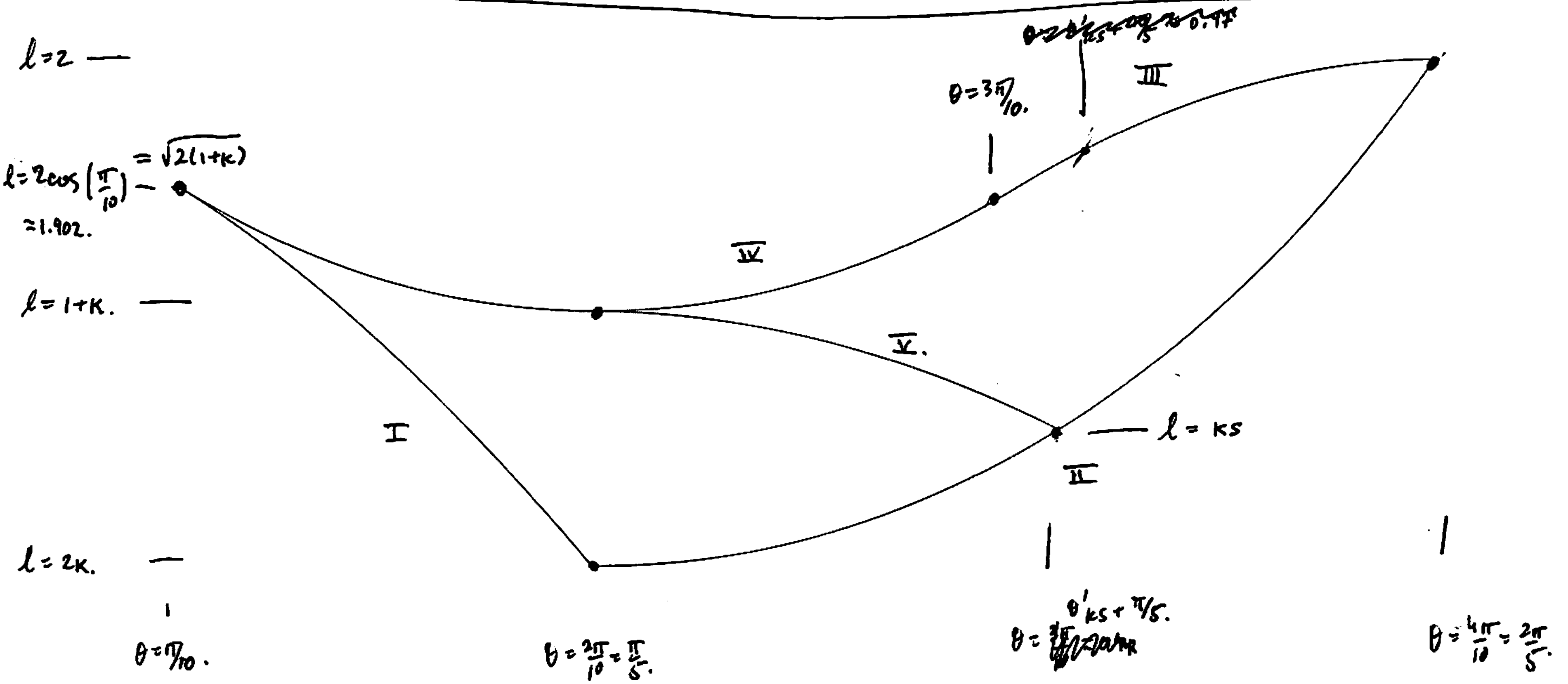
$\theta' \to -\theta'$.

(ℓ, θ, φ acute)   Boundary Curves

Explicit Coordinates.

("curve" in code).

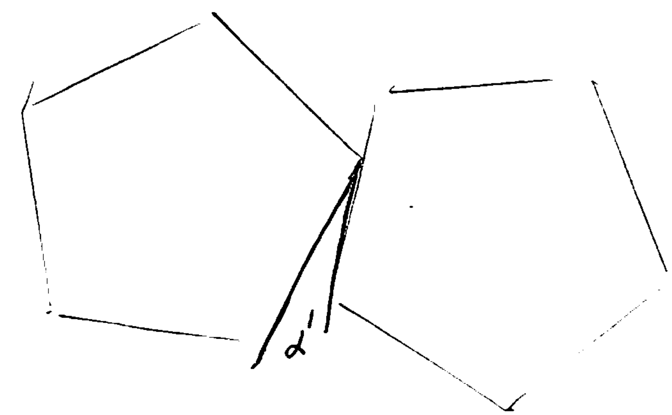
I	$\left(\operatorname{acos}(\ell/2), \ell\right)$	$\ell \in [2K, \sqrt{2(1+K)}]$	$(\theta, 2\cos\theta)$	$\theta \in [\pi/10, \pi/5]$
II	$\left(\cancel{4}\pi/5 \; \cancel{-}\!+ \operatorname{acos}\left(\frac{2K}{\ell}\right), \ell\right)$ $\pi/5 + $ "	$\ell \in [2K, 2]$	$(\theta, 2K/\cos(\theta - \cancel{3\pi/5}))$ $(\theta - \pi/5)$	$\theta \in [\pi/5, 2\pi/5]$
III	$\left(\frac{2\pi}{5} \cancel{+}- \operatorname{acos}(\ell/2), \ell\right)$	$\ell \in [\sqrt{2(1+K)}, 2]$	$(\theta, 2\cos(\theta - 2\pi/5))$	$\theta \in \left[\frac{3\pi}{10}, \frac{2\pi}{5}\right]$
IV	$\left(\pi/5 \pm \operatorname{acos}\left(\frac{1+K}{\ell}\right), \ell\right)$	$\ell \in [1+K, \sqrt{2(1+K)}]$	$(\theta, (1+K)/\cos(\pi/5 - \theta))$	$\theta \in \left[\frac{\pi}{10}, \frac{3\pi}{10}\right]$
V	$\left(\operatorname{acos}\left(\frac{K^2+\ell^2-1}{2\ell K}\right) + \frac{\pi}{5}, \ell\right)$	$\ell \in [K_S, 1+K]$	$\left(\theta, K\cos(\theta - \pi/5) + \sqrt{K^2\cos^2(\theta - \pi/5) - K^2 + 1}\right)$	$\theta \in \left[\frac{\pi}{5}, \frac{\pi}{5} + \theta'_{ks}\right]$



5/2016

$(l, \theta, \varphi$ acute$)$ coordinates $\quad \alpha' \leq \pi/5.$

Geometric Interpretation.



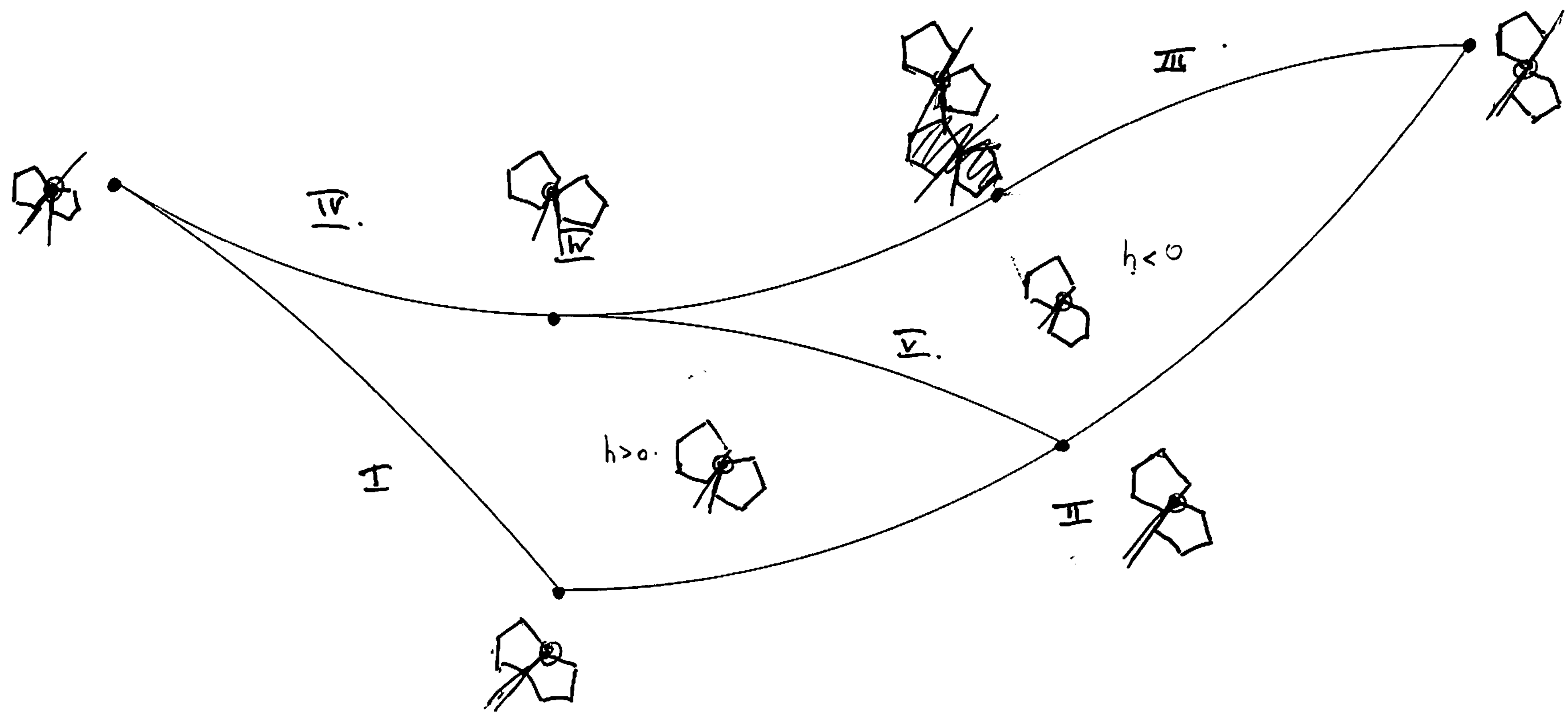
$\underline{\mathrm{I}}$ tip-to-tip. (rotate pentagon about vertex)

$\underline{\mathrm{II}}$ $\alpha'=0$ edge-to-edge. (slide along edge)

$\underline{\mathrm{III}}$ tip-to-tip. (rotate pentagon about vertex).

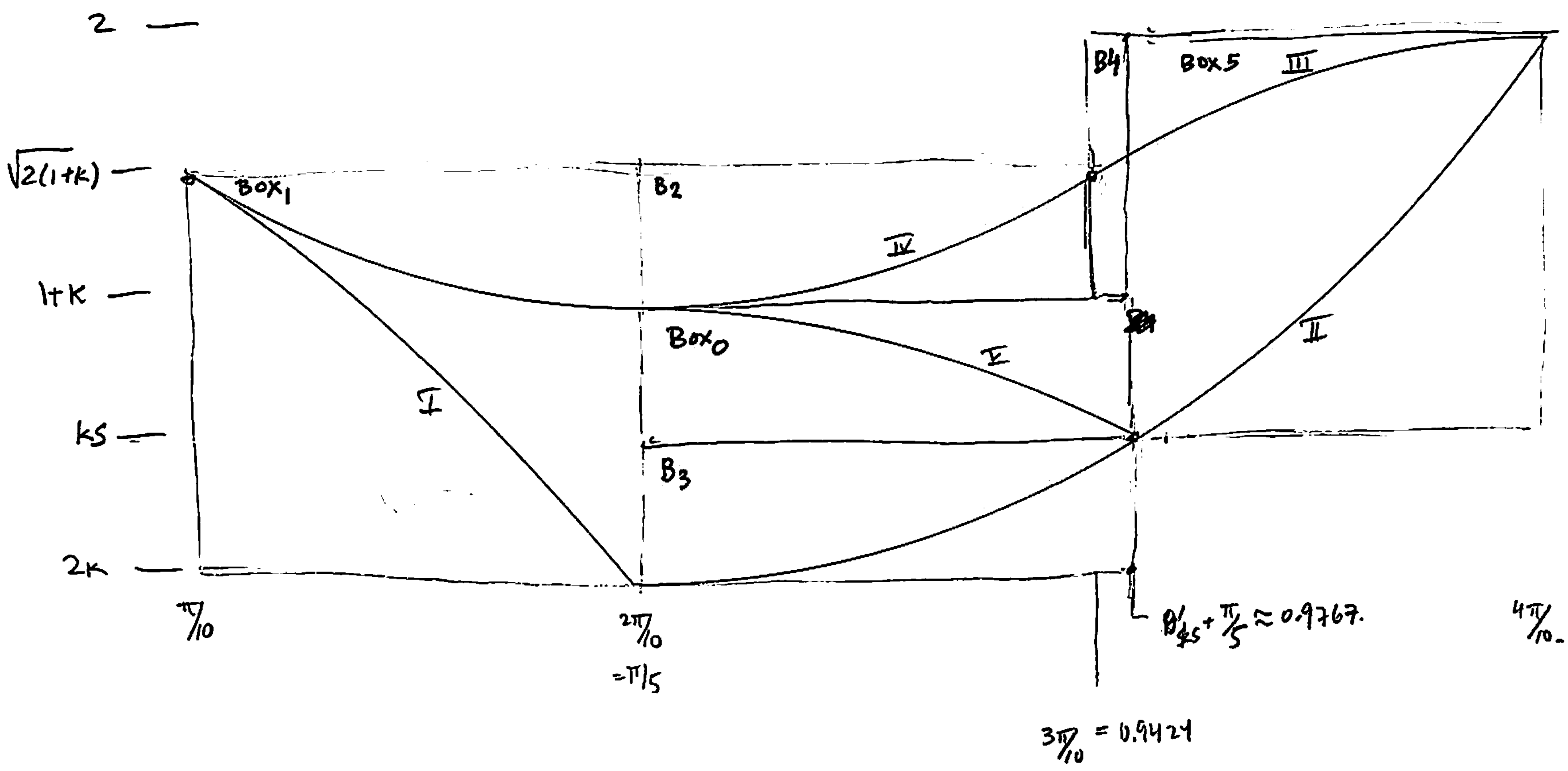
$\underline{\mathrm{IV}}$ $\alpha' = \pi/5.$

$\underline{\mathrm{V}}$ midpoints.



5/2016.

(l, θ) coordinate of acute.



5/2016.

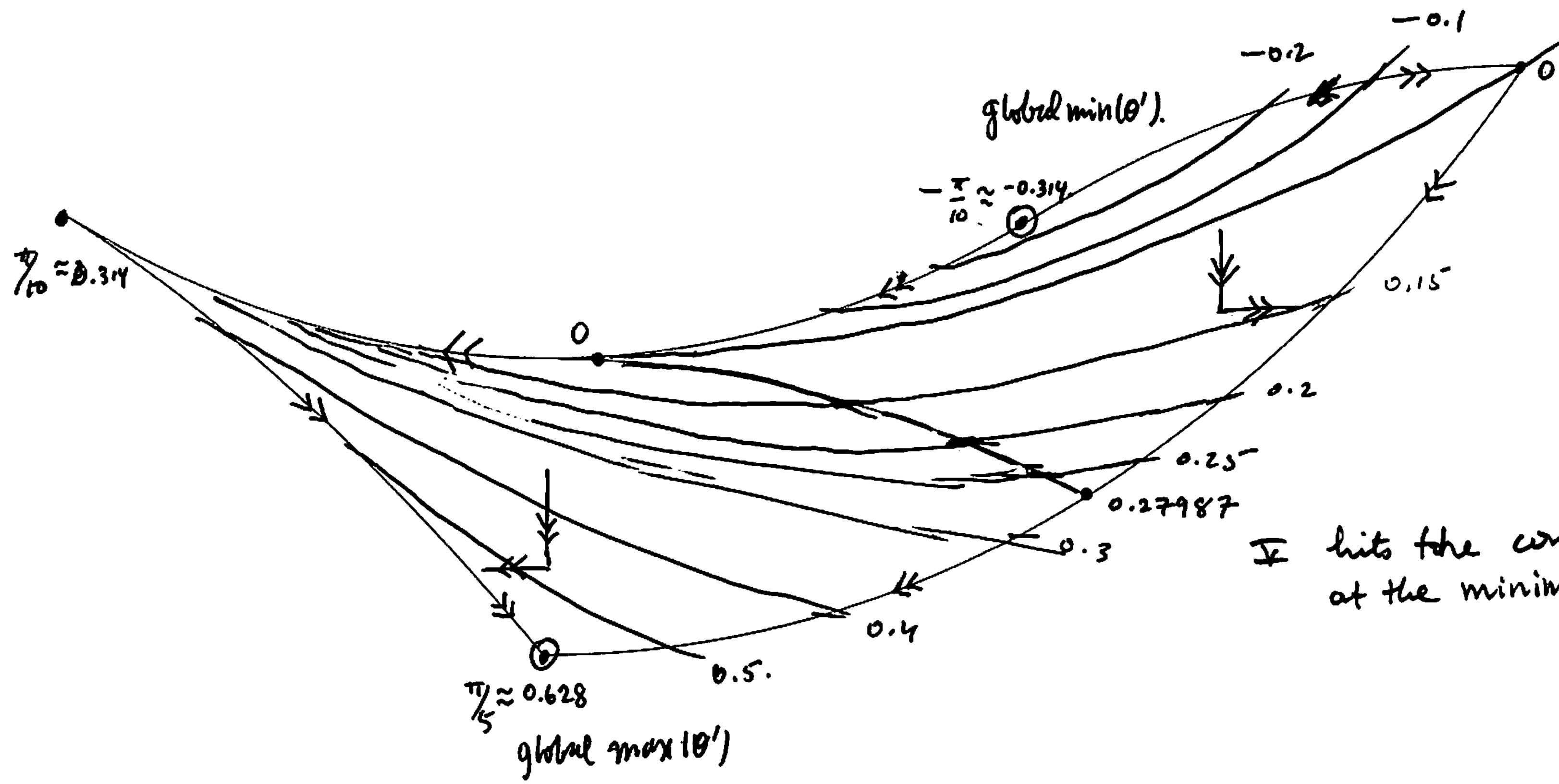
$(l, \theta, \alpha' \le \pi/5)$ coordinates.

Contour Plot.

arrows point to increasing θ'.

$\theta' \in [-\pi/5, \pi/5]$ (already) without taking mod $2\pi/5$.

Monotonicity has clear geometric interpretation.



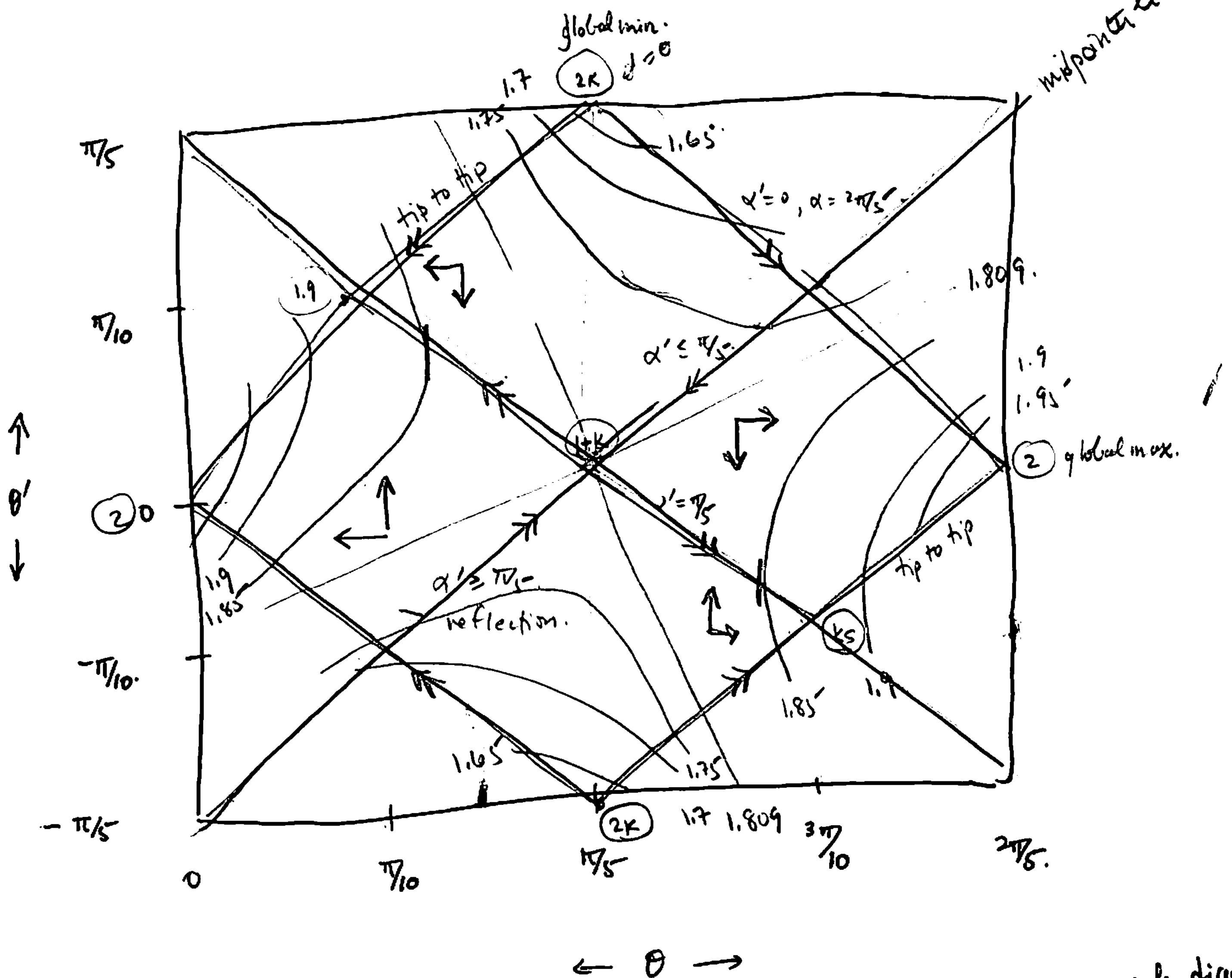
I hits the contour curves at the minimum of the graph

5/2016

5/2016.

(θ, θ') coords.
calculated l.

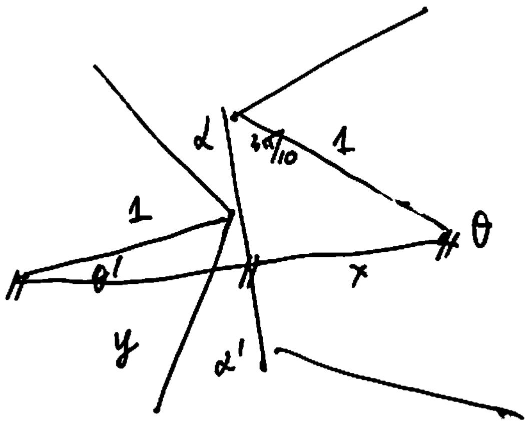
min l. given (θ, θ').



l Monotonic on each diamond.
extrema at corners or
where at domain boundary ∂D
$\cap \partial R$

or on diagonals $\cap \partial R$.

(θ, θ') coords.
calculate l.



$$\theta + \theta' = \alpha. \qquad (\alpha).$$

$$\alpha + \alpha' = 2\pi/5. \qquad (\alpha')$$

$$\frac{x}{\sin(\frac{3\pi}{10})} = \frac{1}{\sin(\theta + \frac{3\pi}{10})} \qquad (x)$$

$$\frac{y}{\sin(\alpha' + \frac{3\pi}{10})} = \frac{1}{\sin(\theta' + \alpha' + \frac{3\pi}{10})} \qquad (y)$$

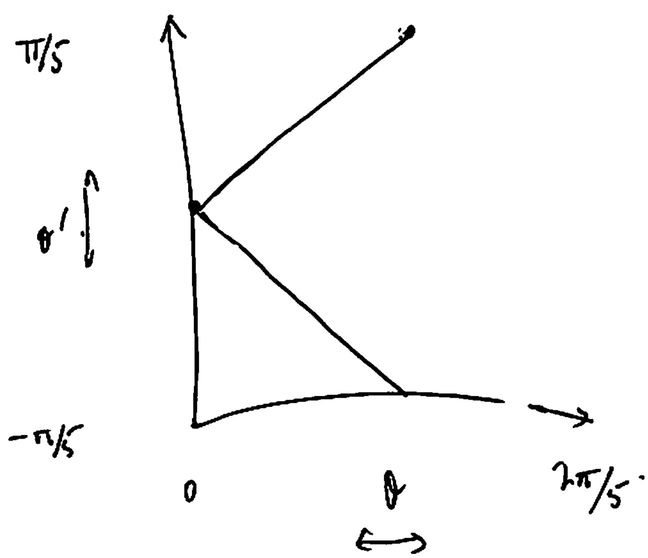
$$l = x + y. \qquad (l)$$

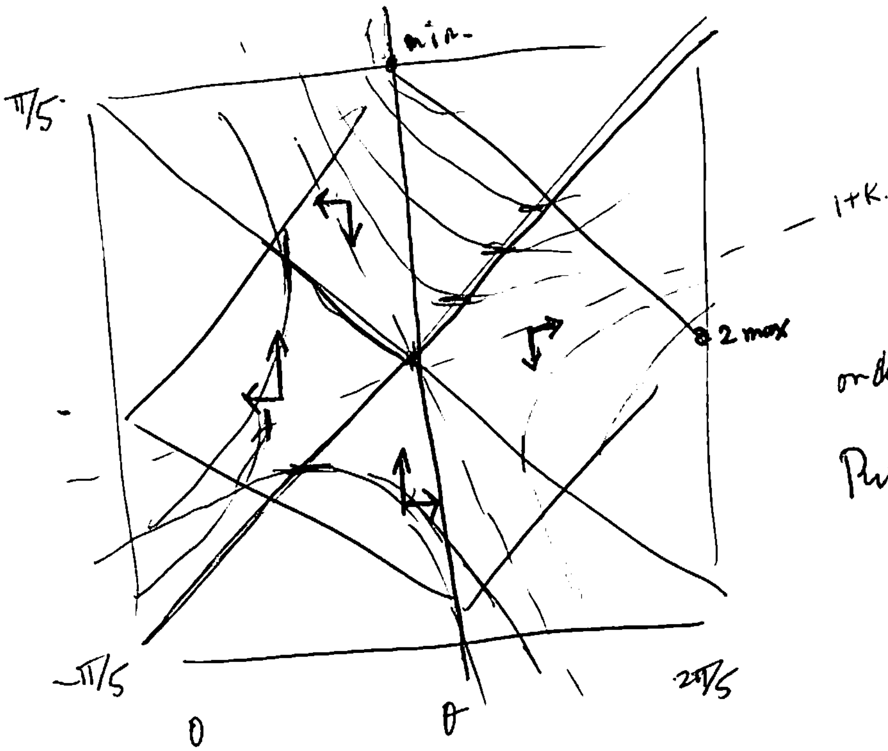
$\frac{3\pi}{5} \leq \theta + \frac{3\pi}{10} \leq \frac{\pi}{2}$, $\sin(\ ) \neq 1$ ok.

$\theta \in [0, 2\pi/5]$

$\theta' \in [-\pi/5, \pi/5]$

$\alpha \in [-\pi/5, 3\pi/5]$





order so $\theta, \theta' \in [-\pi/5, \pi/5]$
order $|\theta'| \le |\theta|$
negate so $\theta \ge 0$.

Put $\theta \in [0, 2\pi/5]$ so $[0, \pi/5]$

$\theta' \in [-\pi/5, \pi/5]$

∘ X take all cases.

∘ check θ, θ'.

$0 \le \theta + \theta' \le 2\pi/5$.

$|\theta - \theta'| \le \pi/5$.

$\theta' \le \theta$.

~~So t~~

Construction of pet

puts $\theta' \in [0, \pi/5]$

$\theta \in [0, 2\pi/5]$

if $\theta' \le \theta$

$\theta' \le 2\pi/5 - \theta$.

$$l \overset{?}{\in} \left[\frac{\sin\left(\frac{3\pi}{10}\right)}{\sin\left(\theta + \frac{3\pi}{10}\right)} + \frac{\sin\left(\alpha' + \frac{3\pi}{10}\right)}{\sin\left(\alpha' + \theta' + \frac{3\pi}{10}\right)} \right]$$

Use monotonicity if.

Box does not meet lines.

$\cancel{\theta' = \theta - \pi/5 \quad \pi/5 \le \theta - \theta'}$

$\theta' = \cancel{-\theta + \pi/5 \quad \pi/5 \le \theta + \theta'}$

$$\alpha' + \theta' + \frac{3\pi}{10} = \frac{2\pi}{5} - \alpha + \theta' + \frac{3\pi}{10}$$

$$= \frac{7\pi}{10} - \theta$$

$$\alpha' + \frac{3\pi}{10} = \frac{2\pi}{5} - \alpha + \frac{3\pi}{10}$$

$$= \frac{7\pi}{10} - (\theta + \theta').$$

$(\theta + \theta' = \alpha.$
$\alpha + \alpha' = 2\pi/5.)$

6/2016

Continuous Extension of Coordinates

(l_x, θ, θ') as a function of (x_α, α)

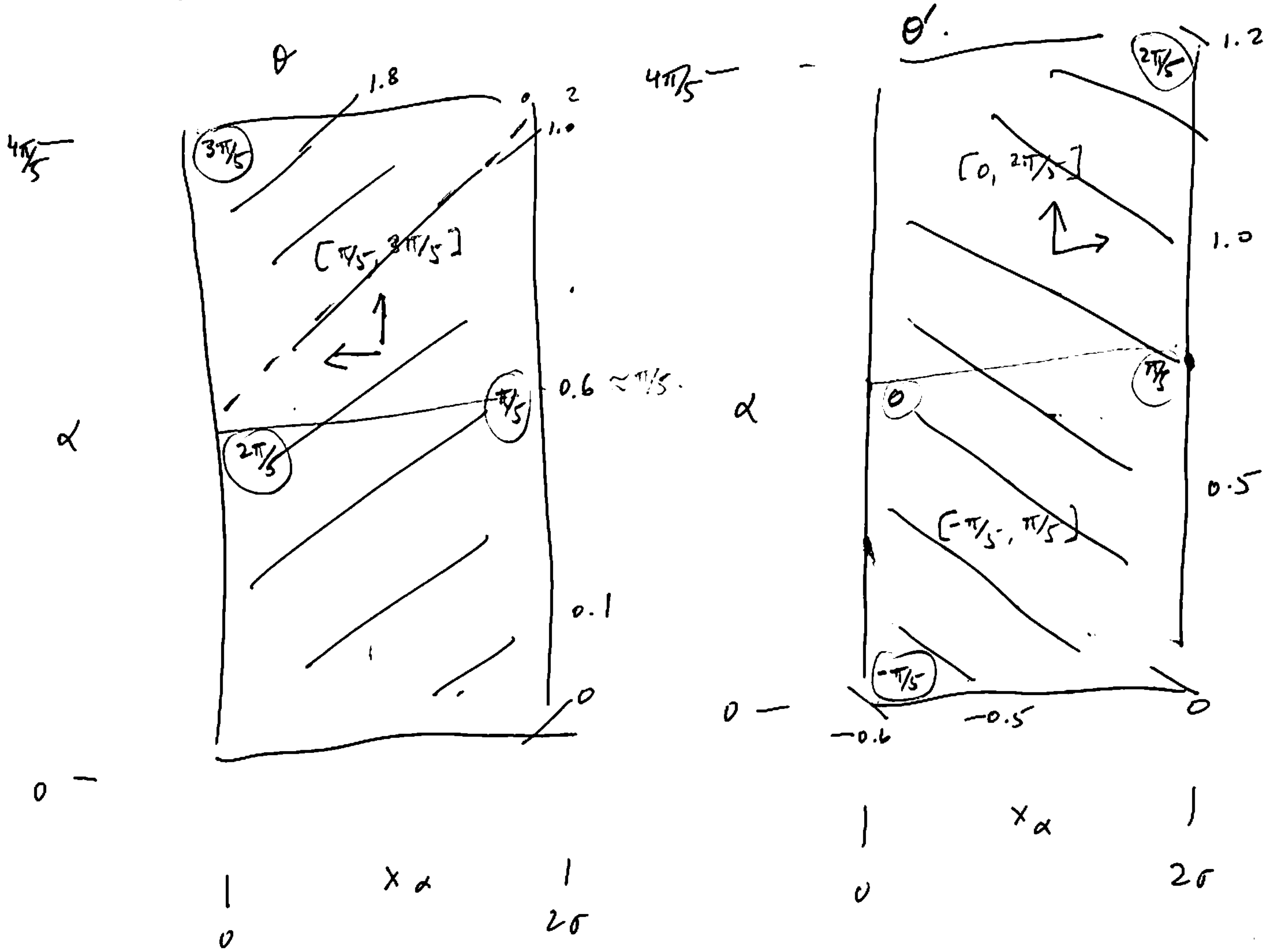
$0 \le x_\alpha \le 2\sigma$ $0 \le \alpha \le 4\pi/5$ (was $(0 \le \alpha \le 2\pi/5)$)

if $\alpha > 2\pi/5$, $\alpha_1 = \alpha - 2\pi/5$ $x^*_{\alpha.1} = 2\sigma - x_\alpha$.

$(l_x, \theta, \theta')(x_\alpha, \alpha) = (l_x, \theta' + 2\pi/5, \theta)(\cancel{\alpha - 2\pi/5, 2\sigma})$ $(2\sigma - x_\alpha, \alpha - 2\pi/5)$

Note that θ and θ' get swapped, so extended coordinates unify A pointer to B with B pointer to A.

Contour Plots. Monotonicity and Continuity are clear.

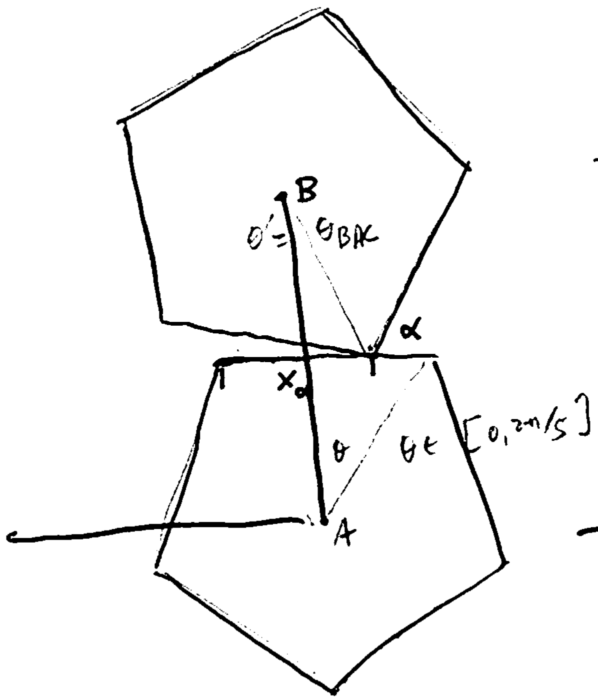


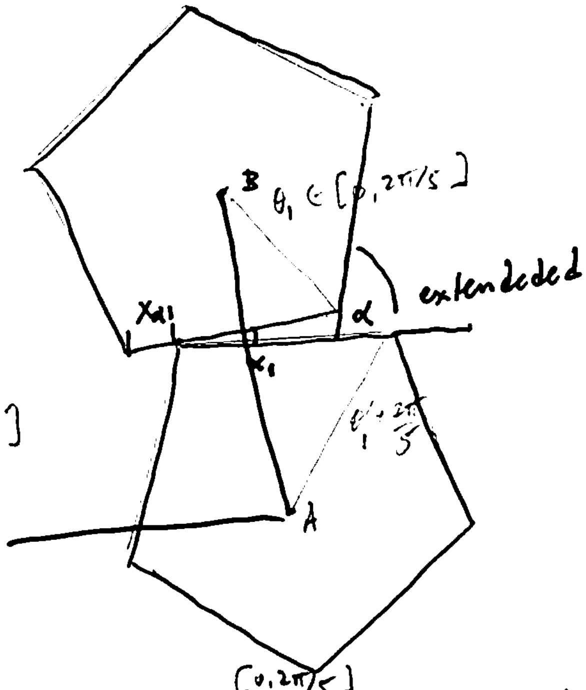
Check continuous extension of. φ_A

(l_x, θ, θ')

$0 \leq x_\alpha \leq 2\sigma$

$0 \leq \alpha \leq 4\pi/5$.





$\alpha > 2\pi/5$.

$\alpha_1 = \alpha - \frac{2\pi}{5}$

$X_{\alpha 1} = \cancel{x_\alpha}\ 2\sigma - x_\alpha$.

$[0, 2\pi/5]$

$$(l_x, \theta, \theta')(x_\alpha, \alpha) = (l_x, \theta' + \frac{2\pi}{5}, \theta)(\underbrace{2\sigma - x_\alpha}_{x_{\alpha,1}}, \underbrace{\alpha - 2\pi/5}_{\alpha_1})$$

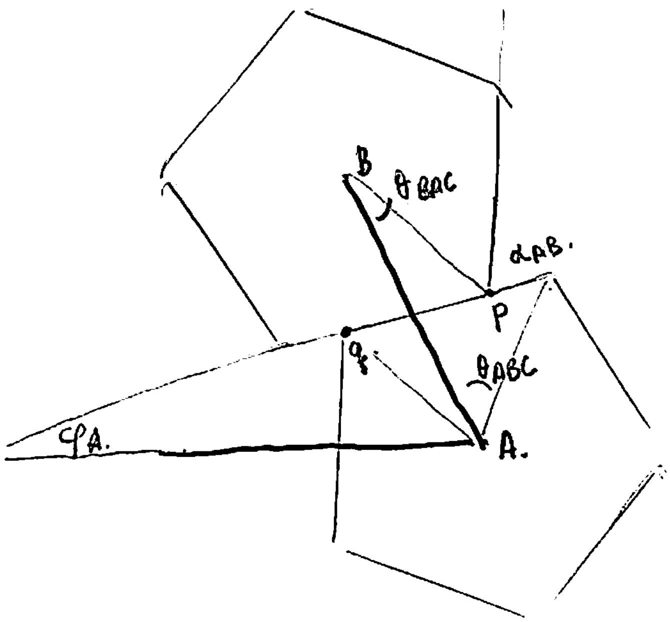
$[\pi/5, 3\pi/5]$ $\quad [0, 2\sigma]$ $\quad [\frac{2\pi}{5}, \frac{4\pi}{5}]$

relation with mk2Ce(true, true)

$(dAB, \theta_{ABC}, \theta_{BAC})$ ~~Bac~~
pointer

$(dBC, \theta_{CBA}, \theta_{BCA})$
pointer.

6/2016



Using $B \to A$ at p

$\alpha'_{AB} = 0$

$\alpha_{AB} = 2\pi/5$.

Use $\tilde{\theta}$ for variables on $\alpha \in [2\pi/5, 4\pi/5]$

$\tilde{\alpha}_{AB} = 2\pi/5$

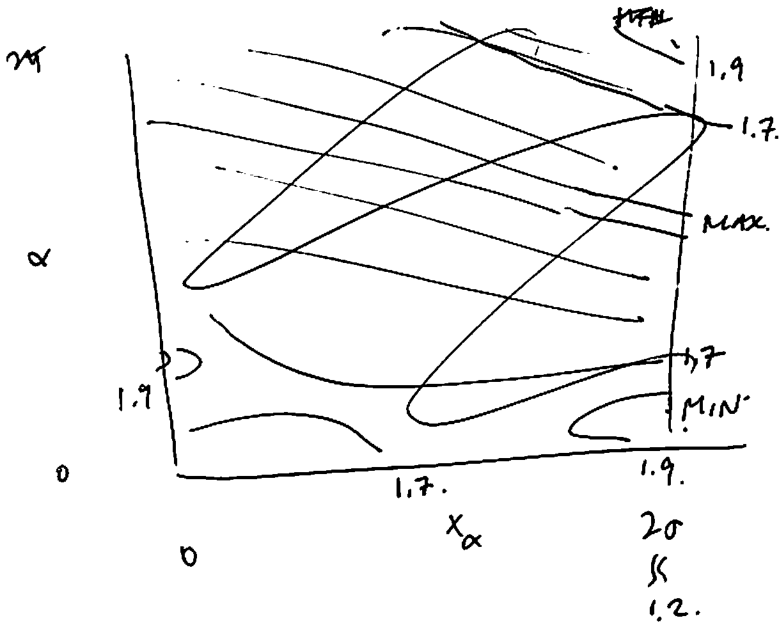
Extended variables on

$(l, \theta, \theta')(x_\alpha, \alpha)$

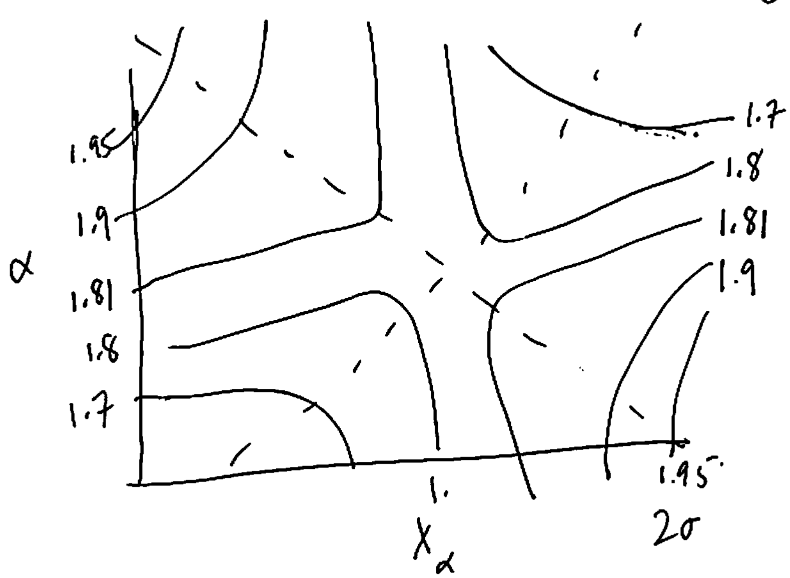
Picture where $\alpha = 2\pi/5$ where they meet

6/2016.

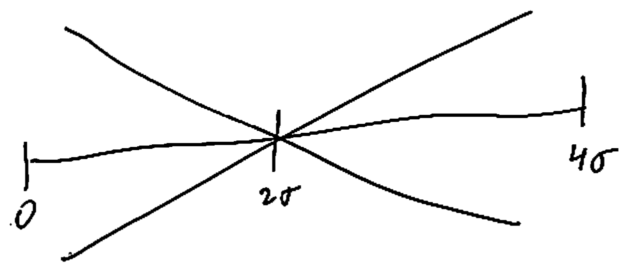
Contour Plot. $l_x(x_\alpha, \alpha)$



Contour Plot $l_x(x_\alpha, \alpha)$.



Extend slider coordinates
from $x_\alpha \in [0, 2\sigma]$ to $x_\alpha \in [0, 4\sigma]$



Allows swapping of θ, θ' coordinates.