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Practical Lab Numerical Computing

Computational Finance

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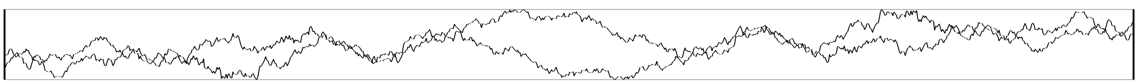

Worksheet 2

Complete your work and mail the results (PDF-Document + Code)
to financelab@ins.uni-bonn.de by November, 24th, 2019

1 Introduction

After we learned how to simulate stock-prices that are modeled by a geometric Brownian motion (GBM) on the first sheet, we will learn about certain derivatives that are defined on stock prices and allow bets on many different behaviours of these underlying prices on this sheet. Moreover, we will discuss several approaches to compute expected values for options and finally derive a formula for a *fair price*.

2 Options

An option is the right, but not the obligation to buy (*call-option*) or sell (*put-option*) a certain amount of an underlying asset for a certain price (the *strike price*) within a certain period of time (the *expiration date*).

The price of an option derives from the difference between the reference price and the value of the underlying asset (commonly a stock, a bond, a currency or a futures contract) plus a premium based on the time remaining until the expiration of the option. Other types of options exist, and options can in principle be created for any type of valuable asset, like stock indices, currencies, interest rates, etc.

Usually options are emitted from a bank, which sets the conditions on the strike-price, the quantity or the expiration date, until which the buyer can choose to either exercice or lapse the option.

If the option is of *European* style it can only be exercised at the expiration date T. In contrast, *American* options can be exercised at any time before. Most options traded are of American style, but since they are more difficult to price we will focus on European options for now.

So let V be the value of an European option on some underlying asset S. The value of V depends on the price of the underlying $S(t)$, which varies in time $0 \leq t \leq T$, thus we write $V(S,t)$. Let K be the strike-price, then the value of a call-option at the expiration date $t = T$ is given by

$$V_{\text{call}}(S,T) = \max\{S(T)-K,0\} = (S(T)-K)^+. \tag{1}$$

In case of a put-option we have

$$V_{\text{put}}(S,T) = \max\{K-S(T),0\} = (K-S(T))^+. \tag{2}$$

These functions $V(S,T)$ are called *payoff functions* and define the value of an option at the expiration-date. On later sheets we will learn about more complicated payoff functions, but here we will stick to the simple case of European calls and puts.

Now the question is: What is a fair value for an option at time $t < T$ or even $t = 0$?

3 Option-pricing by simulation

On the first sheet you learned how to simulate stock prices, which follow the dynamics of a GBM.

It is straightforward to extend this principle of simulation to functions which are defined on these prices. To do so, you simply simulate the path of a geometric Brownian motion (which models the stock price of your underlying) and then apply the payoff function (1) or (2), respectively, to the realization of a path.

Task 1. *Simulate* $N = 1000$ *geometric Brownian pathes with parameters* $S(0) = 10$*,* $\mu = 0.1$*,* $\sigma = 0.2$*,* $T = 2$*,* $\Delta t = 0.2$*, apply the payoff function* $V_{call}(S,T)$ *with strike* $K = 10$ *to each path and compute the mean of the resulting values.*

Now repeat the procedure for different $\sigma = 0.0, 0.2, 0.4, 0.6, 0.8$*. Plot your mean-estimates against the value of* σ*.*

If the payoff function only depends on the value of $S(T)$ and not on $S(t), t < T$, there is a particular feature of the GBM, which allows us to simulate only one single timstep $\Delta t = T$. Validate this by doing the following task.

Task 2. *Simulate* $N = 1000$ *geometric Brownian pathes with parameters* $S(0) = 10$*,* $\mu = 0.1$*,* $\sigma = 0.2$*,* $T = 2$*,* $\Delta t = 0.2$ *again, apply* $V_{call}(S,T)$ *with* $K = 10$ *to each of them and compute the mean and variance. Now repeat the procedure for* $\Delta t = 0.4, 1.0, 2.0$*. Does the variance depend on* Δt*? Interpret your results.*

4 Representation as an integral

From the law of large numbers we know that for i.i.d. random variables $x_i \sim \mathcal{N}(0,1)$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\frac{t^2}{2}}\, dt = \lim_{n\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

Thus, it is natural to interpret our simulation approach as a discretized integral.

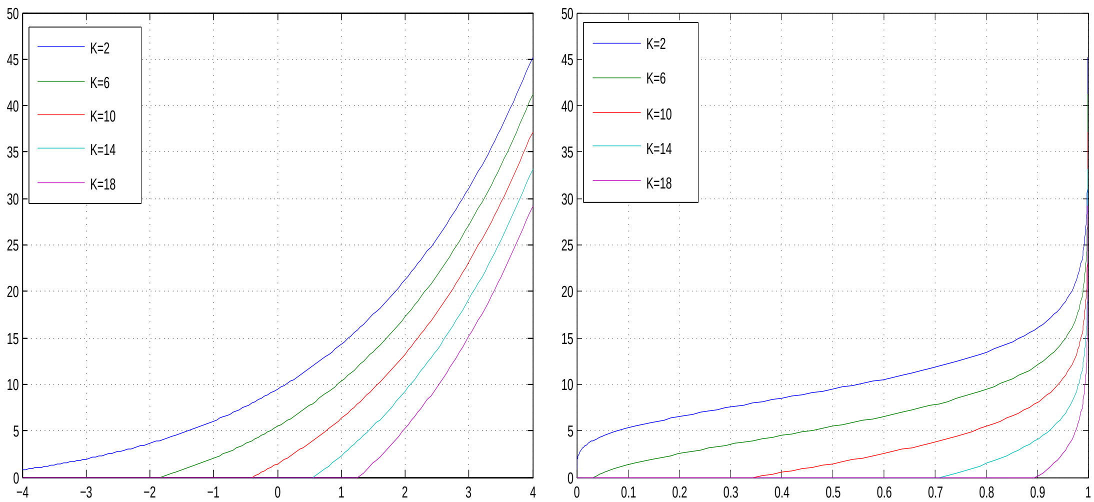

Figure 1: Plots of the integrands $f_{\text{call}}(s)$ versus s for an European call with different strike-prices K. Left: Original integrand. Right: Integrand after transformation with the inverse normal c.d.f.

4.1 Derivation of the integrand

$$\begin{aligned}
\mathbb{E}[V_{\text{call}}(S_T, 0)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(S_T) e^{-\frac{s^2}{2}}\, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (S(T) - K)^+ \, e^{-\frac{s^2}{2}}\, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S(0) \exp\left((\mu - \frac{1}{2}\sigma^2) \cdot T + \sigma\sqrt{T} s \right) - K \right)^+ e^{-\frac{s^2}{2}}\, ds
\end{aligned}$$

Thus, the integrand for a European call-option is given by

$$f_{\text{call}}(s) := \left(S(0) \exp\left((\mu - \frac{1}{2}\sigma^2) \cdot T + \sigma\sqrt{T} s \right) - K \right)^+ . \tag{3}$$

In Figure 1 you can see $f_{\text{call}}(s)$ plotted versus s for different values of K and $S(0) = 10$, $\mu = 0.1$, $\sigma = 0.2$, $T = 2$, $\Delta t = 2$.

4.2 Computing the root of f_{call}

As we will see later on, it is adavantegous to know the root of f_{call} if we neglected the outermost $^+$ function. Therefore, we compute:

$$
\begin{aligned}
& f_{\text{call}}(\chi) = 0 \\
\Leftrightarrow \quad & (\mu - \sigma^2/2)T + \sigma\sqrt{T}\chi = \log\left(\frac{K}{S_0}\right) \\
\Leftrightarrow \quad & \chi = \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{K}{S_0}\right) - (\mu - \sigma^2/2)T\right)
\end{aligned}
$$

Now we can restrict the integration problem to the part of the domain on which f_{call} does not vanish.

$$
\begin{aligned}
\mathbb{E}[V_{\text{call}}(S_T, 0)] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_{\text{call}}(s) e^{-\frac{s^2}{2}}\, ds && (4)\\
=& \frac{1}{\sqrt{2\pi}} \int_{\chi}^{\infty} \left(S(0)\exp\left((\mu - \frac{1}{2}\sigma^2)\cdot T + \sigma\sqrt{T}s\right) - K\right) e^{-\frac{s^2}{2}}\, ds && (5)
\end{aligned}
$$

4.3 Closed-form solution

Task 3. *Using the formula for χ, prove that*

$$
\mathbb{E}[V_{call}(S_T, 0)] = S(0)e^{\mu T}\Phi(\sigma\sqrt{T} - \chi) - K\Phi(-\chi). \tag{6}
$$

Remark 1.

$$
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt
$$

denotes the normal c.d.f. and Φ^{-1} is its inverse.

Now we can use the exact solution (6) to create a convergence-plot.

Task 4. *Using the parameters $S(0) = 10$, $K = 10$, $\mu = 0.1$, $\sigma = 0.2$, $T = 1$, $\Delta t = T = 1$ compute an approximation to $\mathbb{E}[V_{call}(S_T, 0)]$ by simulation, as you have done in tasks 1 and 2 before, i.e. calculate*

$$
\hat{m}_N = \frac{1}{N}\sum_{i=1}^{N} V(S_i, T),
$$

where S_i is the i-th path from a simulation run.

Plot N against the error of $\hat{m}_N$ in double-logarithmic axes with $N = 1, 10, 10^2, 10^3, \ldots, 10^6$. Repeat this procedure 5 times and put all 5 plots in the same picture. What can you say about the rate of convergence?

Even though we know by now that our problem has a closed-form solution (6), we will use it in the following as a *toy-problem* to compare and explain several approaches, that might also be used for more complicated problems.

4.4 Transformation to the unit-interval

At the moment, our integration problem

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{-\frac{s^2}{2}}\,ds$$

is defined on the whole real axis $(-\infty,\infty)$, which is an unbounded domain and thus not suited for many popular numerical integration methods. For this reason we use a change-of-variable $s = \Phi^{-1}(t)$, where $\Phi^{-1} : (0,1) \to (-\infty,\infty)$ is the inverse of the cumulative distribution function Φ (see Sheet #1) to obtain

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{-\frac{s^2}{2}}\,ds = \int_0^1 f\left(\Phi^{-1}(t)\right)\,dt. \tag{7}$$

Task 5. *Prove formula* (7)*.*

Remark 2. *The transformation from an unbounded domain onto the open intervall* $(0,1)$ *might introduce boundary singularities into the transformed integrand because* $\Phi^{-1}(0) = -\infty$ *and* $\Phi^{-1}(1) = \infty$*. This should be kept in mind when applying numerical quadrature-rules to* $f \circ \Phi^{-1}$*.*

5 Numerical integration in one dimension

In this section we will deal with one-dimensional quadrature-rules. The most general applicable method is Monte Carlo (MC), which only requires finite variance of the integrand to yield a convergence rate of $\mathcal{O}(N^{-1/2})$. But there are other methods, which are able to benefit from higher-order smoothness of the integrand.

To compare the various methods, we have a look at the integral

$$\int_{-\infty}^{\infty} f_{\text{call}}(s)\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds$$

of which we already know a closed-form solution. On later sheets we will learn about more complicated integrands, which do not allow a closed solution and thus can only be computed by numerical approximation.

The general form of an univariate quadrature-rule for integration with respect to a measure μ over a domain $\Omega \subset \mathbb{R}$ is

$$Q^N(f) := \sum_{j=1}^{N} w_j f(x_j) \approx \int_\Omega f(x)\,d\mu(x), \tag{8}$$

where the $w_j \in \mathbb{R}$ are called *quadrature-weights* and the $x_i \in \Omega$ are called *quadrature-nodes.*

The *absolute error* of a quadrature rule is

$$R^N(f) = \left|\int_\Omega f(x)\,d\mu(x) - Q^N(f)\right|$$

and the *relative error* is

$$\hat{R}^N(f) = \frac{R^N(f)}{\left|\int_\Omega f(x)\,d\mu(x)\right|}.$$

5.1 Monte Carlo integration

Monte Carlo integration is equivalent to the simulation approach we used in section 3. It relies on simulating several (thousands) of scenarios and compute the mean over all the outcomes:

$$Q_{\mathrm{mc}}^{N}(f) = \frac{1}{N}\sum_{i=1}^{N} f(x_i), \quad x_i \in \Omega \text{ are i.i.d.}$$

This approach is justified by the *law of large numbers* and the *central limit theorem* which yields the (probabilistic) rate of convergence

$$\mathbb{E}\left[R_{\mathrm{mc}}^{N}\right] \leq \frac{\sigma(f)}{\sqrt{N}}.$$

5.2 Trapezoidal rule

The compound trapezoidal rule on $\Omega = [0,1]$ uses equidistant nodes and equal weights. If the two outer-most points (at 0 and 1) are left out (open trapezoidal rule), the two second outer-most points are weighted with a factor of $\frac{3}{2}$.

We assume that the number of points fulfills

$$N_l = 2^l - 1, \tag{9}$$

where $l = 1, 2, \ldots$ is called the *level*. Then, the open compound trapezoidal rule can be written as

$$Q_{\mathrm{Tr}}^{N_l}(f) = \frac{1}{N_l+1}\left(\frac{3}{2}\cdot f\left(\frac{1}{N_l+1}\right) + \sum_{i=2}^{N_l-1} f\left(\frac{i}{N_l+1}\right) + \frac{3}{2}\cdot f\left(\frac{N_l}{N_l+1}\right)\right). \tag{10}$$

The rate of convergence of the trapezoidal-rule with N_l nodes is for two times continuously differentiable functions $f \in C^2((0,1))$

$$R_{\mathrm{Tr}}^{N_l} = O(N_l^{-2}). \tag{11}$$

The trapezoidal-rule is exact for piecewise linear functions.

Task 6. *Implement the open compound trapezoidal-rule on* $(0,1)$ *as a function of* l*, which returns* N_l *nodes and weights.*

What can you say about the nodes that are contained in level l *and* $(l+1)$ *? Is there a relationship?*

5.3 Gaussian quadrature

Gaussian quadrature rules are optimal in the sense, that they yield the maximum degree of exactness for polynomials. This means, that all polynomials up to degree $2N_l - 1$ are integrated exactly.

Task 7. *Implement the Gauss Legendre - rule on* $(0,1)$ *as a function of* l*, which returns* N_l *nodes and weights. You can use a library of your choice (e.g. the* `GSL`*), but validate your implementation, e.g. by checking if it really integrates polynomials of degree* $2N_l - 1$ *exactly with respect to the uniform distribution and be prepared to explain how the computation of Gaussian quadrature rules work, at least in theory.*

What can you say about the nodes that are contained in level l *and* $(l+1)$ *? Is there a relationship?*

5.4 Clenshaw Curtis

By now you should have seen that the computation of Gaussian quadrature rules is rather complicated. An alternative, which is easier to implement is *Clenshaw-Curtis* quadrature, which uses the extrema of the Tschebyscheff-polynomials as nodes.

Again we fix the number of points to be N_l, then the nodes and weights for level l are given by

$$x_i = \frac{1}{2}\left(1 - \cos\frac{\pi i}{N_l + 1}\right) \tag{12}$$

and

$$w_i = \frac{2}{N_l + 1}\sin\left(\frac{\pi i}{N_l + 1}\right)\sum_{j=1}^{(N_l+1)/2}\frac{1}{2j-1}\sin\left(\frac{(2j-1)\pi i}{N_l + 1}\right) \tag{13}$$

for $i = 1, \ldots, N_l$. These rules are sometimes also refered to as *Fillipi-formulae.* The degree of polynomial exactness of the Clenshaw–Curtis formulae is N_l and for functions $f \in C^r((0,1))$, we get an error-estimate

$$R_{\mathrm{CC}}^{N_l} = O(N_l^{-r}). \tag{14}$$

Task 8. *Implement the Clenshaw Curtis-rule on* $(0,1)$ *as a function of* l*, which returns* N_l *nodes and weights. What can you say about the nodes that are contained in level* l *and* $(l+1)$ *? Is there a relationship?*

Remark 3. *If an integrand is* C^∞ *or even analytic, the error-analysis is usually more difficult, but one can expect an exponential rate of convergence.*

5.5 Comparison of the methods

Now we are going to compare the different integration methods by applying them to test-functions.

Task 9. *Let* $f_\gamma(x) = 1 + \gamma\exp(\frac{1}{2}x)$*. Compute the reference value* $\int_0^1 f_\gamma(x)\,dx$ *and make a convergence plot for* $\gamma = 1$ *for Monte-Carlo, Trapezoidal rule, Clenshaw-Curtis and Gauss-Legendre in one picture. (Plot* ***relative errors****, i.e. divide the absolute errors by the reference value.)*

Task 10. *Make two convergence plots in which you compare Clenshaw-Curtis, Gauss-Legendre, Monte Carlo and the Trapezoidal rule for the original call-option integrand* $f_{call} \circ \Phi^{-1} : (0,1) \to \mathbb{R}$ *from* (3)*. Use the parameters from task 4 – once with* $K = 10$ *and once with* $K = 0$*.*

Use the exact solution from formula (6) *as a* reference value *and plot relative errors.*

6 Equivalent martingale principle

In this section we will justify the rather practial approach of this sheet by the theory of the Black-Scholes model and the concept of the equivalent martingale measure.

6.1 Martingale approach

The martingale-approach is an important principle in option pricing. It says, that the fair value of an option (without early exercice) is the discounted expected value of the payoff function with respect to the risk-free probability measure. This means, that

$$V(S,0) = e^{-rT}E^*(V(S,T)), \tag{15}$$

where E^* is the expected value with respect to the so-called equivalent martingale measure. In case of the Black-Scholes model (see next section) the drift μ is replaced by the risk-free interest rate r.
The factor e^{-rT} is due to continuous compound interest.

6.2 Black-Scholes formula

The Black-Scholes (or BlackScholesMerton) model was first published in 1973 and makes the following explicit assumptions:

- There is no arbitrage opportunity (i.e., there is no way to make a riskless profit).
- It is possible to borrow and lend cash at a known constant risk-free interest rate.
- It is possible to buy and sell any amount, even fractional, of stock (this includes short selling).
- The above transactions do not incur any fees or costs (i.e., frictionless market).

The model assumes, that the stock price follows a geometric Brownian motion with constant drift and volatility, i.e. it is a solution to the stoachstic differential equation

$$dS(t) = \mu S(t)\ dt + \sigma S(t)\ dW(t). \tag{16}$$

It can be shown, that within the assumptions of the BS-model the drift-term μ can be replaced by the risk-free interest rate r. Then the closed form solution to this equation is

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)}, \tag{17}$$

which should be well known by now.
Usually the Black-Scholes formula is stated in the following form, which is equivalent to the formula we derived in Task 3, without the compound interest factor e^{-rT}:

$$V_{\text{call}}(S,0) = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{18}$$

and

$$V_{\text{Put}}(S,0) = Ke^{-rT}\Phi(-d_2) - S(0)\Phi(-d_1), \tag{19}$$

where

$$d_1 = \frac{\ln(S(0)/K) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{20}$$

and

$$d_2 = d_1 - \sigma\sqrt{T} \tag{21}$$

hold.

7 Summary

By now you should have some basic knowledge of

- How to tackle an integration problem: Seek a closed-form solution first.
- If no easy solution is available, find a form of the integrand that is smooth and apply a suitable quadrature-rule.
- If no smooth integrand is available: Use simulation. “If nothing works: Monte Carlo works.”