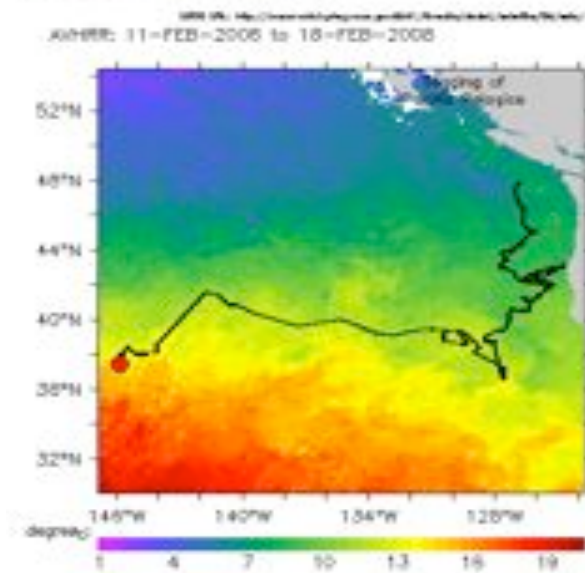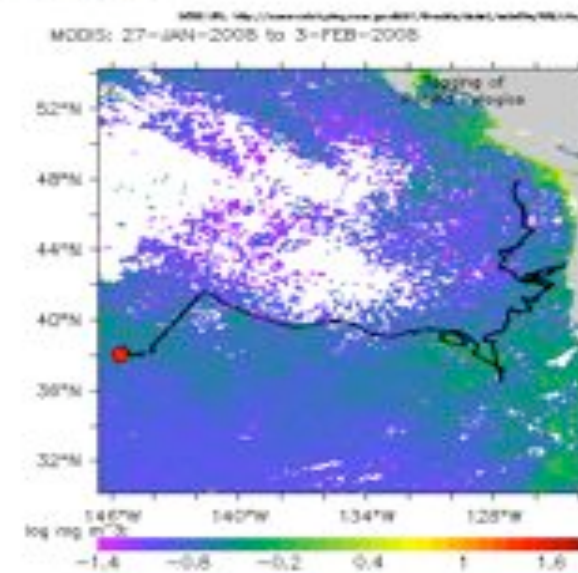# Lecture 13

Drunk Man Walks

H. Risken, The Fokker-Planck Equation (Springer-Verlag Berlin 1989)
C. W. Gardiner, Handbook of Stochastic Methods (Springer Berlin 2004)

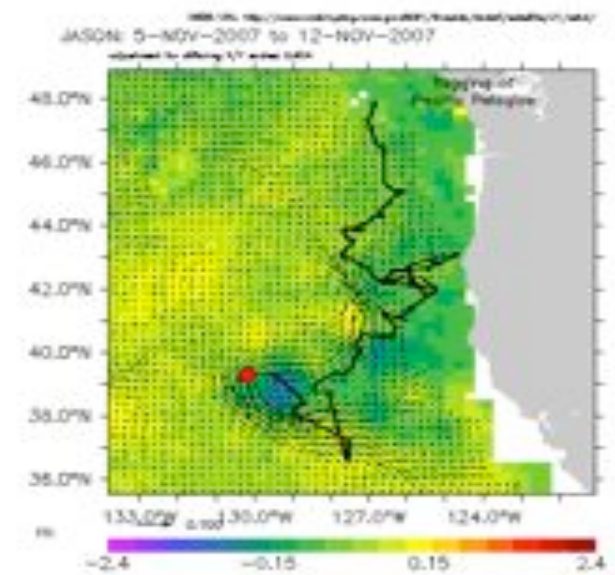# Blue Shark 68493 1607005 25 Jul 2007 to 15 Feb 2008

Sea Surface Temperature
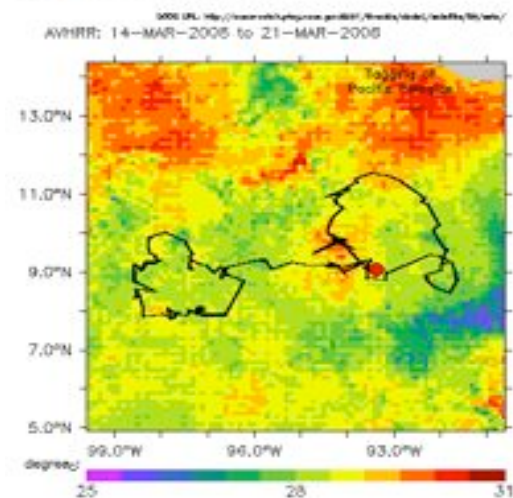
Chlorophyll-a

Sea Surface Height Deviation and Geostrophic Current

Blue Whale 01388 2208010 12 Jan 2008 to 22 Mar 2008

Sea Surface Temperature

Chlorophyll-a
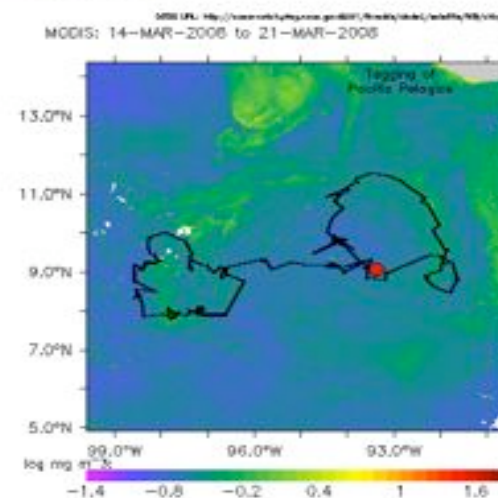
http://topp.org/
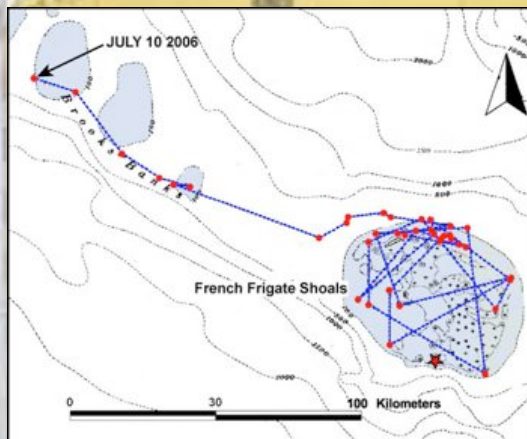
http://topp.org/species/mako_shark

Great White Shark swims 12,400 miles, shocks scientists
WCS release
October 6, 2005

Random Walks and Levy flights

Random Walk

The term "random walk" was first used by Karl Pearson in 1905. He proposed a simple model for *mosquito infestation in a forest*: at each time step, a single mosquito moves a fixed length at a randomly chosen angle. Pearson wanted to know the mosquitos' distribution after many steps.

The paper (a letter to Nature) was answered by Lord Rayleigh, who had already solved the problem in a more general form in the context of *sound waves in heterogeneous materials*.

Actually, the the theory of random walks was developed a few years before (1900) in the PhD thesis of a young economist: Louis Bachelier. He proposed the random walk as the fundamental *model for financial time series*. Bachelier was also the first to draw a connection between discrete random walks and the continuous diffusion equation.

Curiously, in the same year of the paper of Pearson (1905) Albert Einstein published his paper on *Brownian motion* which he modeled as a random walk, driven by collisions with gas molecules. Einstein did not seems to be aware of the related work of Rayleigh and Bachelier.

Smoluchowski in 1906 also published very similar ideas.

The simplest possible problem: 1 dimensional Random Walk

-4 -3 -2 -1 0 1 2 3
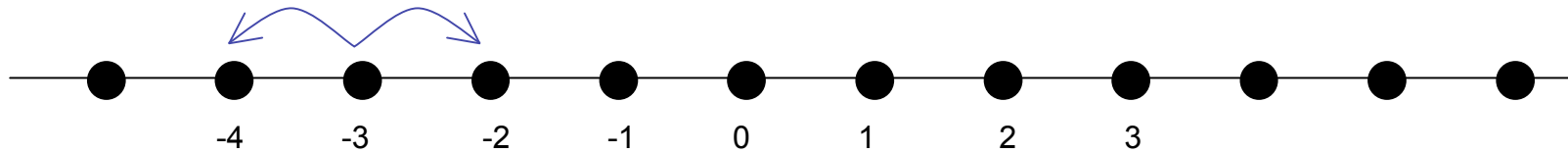
The "walker" starts at position *x = 0* at the step *t = 0;* at each time-step the walker can go either forward or backward of one position with equal probabilities *1/2*.
We ask the probability *P(x,t)* to find the walker at the position *x* a the time step *t*.

probability of any given sequence of *t* steps

Number of sequences that take to x in *t* steps

$$p(x,t) = \frac{t!}{((t+x)/2)!((t-x)/2)!}\left(\frac{1}{2}\right)^t \underset{t>>1}{\sim} \sqrt{\frac{2}{\pi\,t}}\,e^{\frac{-x^2}{2t}}$$

(t+x)/2 steps taken in the positive direction

(t-x)/2 steps taken in the negative direction

Stochastic process - normal noise (finite variance) 1 dimension

$$x(t + 1) = x(t) + \eta(t)$$
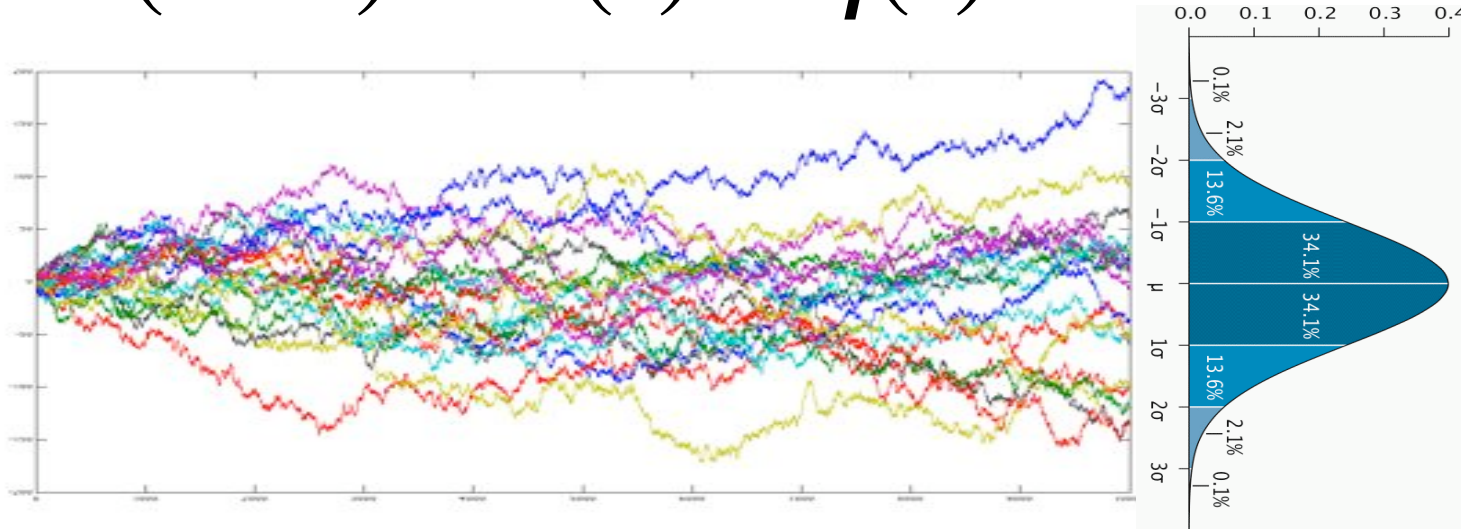
Central limit theorem (lecture 2)

The sum of independent identically-distributed variables with finite variance will tend to be normally distributed. [lecture 2]

$$x(t) = \sum_{\tau=1}^{t} \eta(\tau)$$

$$p(x,t) \underset{t \gg 1}{\sim} \frac{1}{\sqrt{2\pi \ t\langle\eta^2\rangle}} \exp\left(-\frac{x^2}{2t\langle\eta^2\rangle}\right) \qquad (\langle\eta\rangle = 0)$$
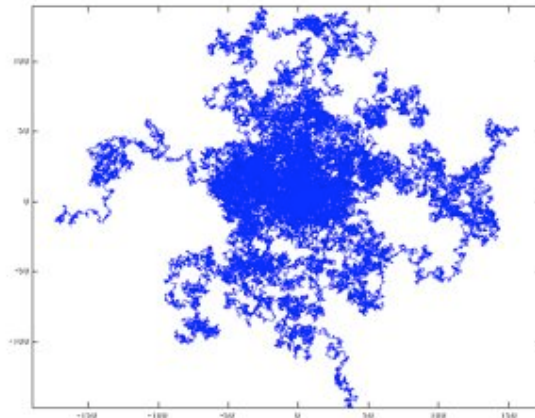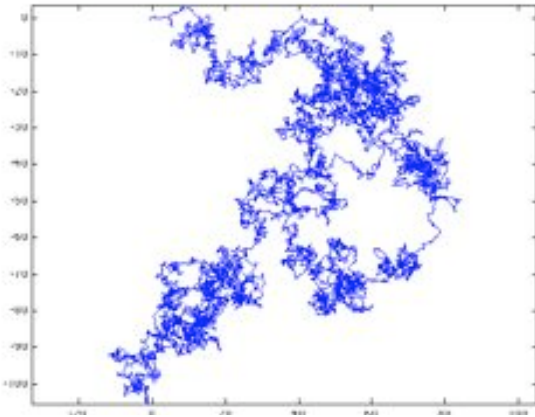
Average traveled distance:

$$\langle x \rangle_t = \langle \eta \rangle = 0 \qquad \sqrt{\langle x^2 \rangle_t - \langle x \rangle_t^2} = \sqrt{\langle\eta^2\rangle t} \propto \sqrt{t}$$

Stochastic process - normal noise (finite variance) D dimensions

$$\vec{r}(t+1) = \vec{r}(t) + \vec{\eta}(t)$$

iid D-dimesional vectors
isotropic (uniformly distributed in the direction)
with moduli distributed accordingly to a given
probability density function $p(|\eta|)$.

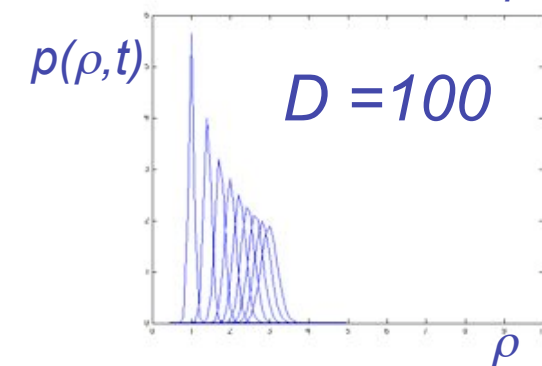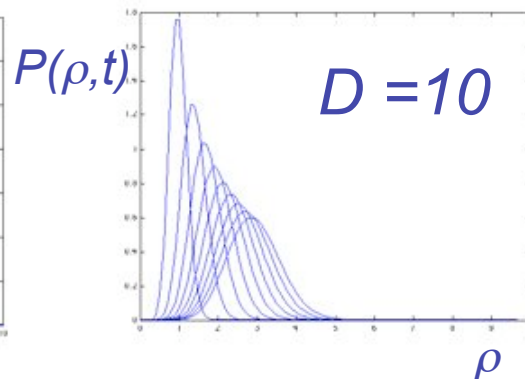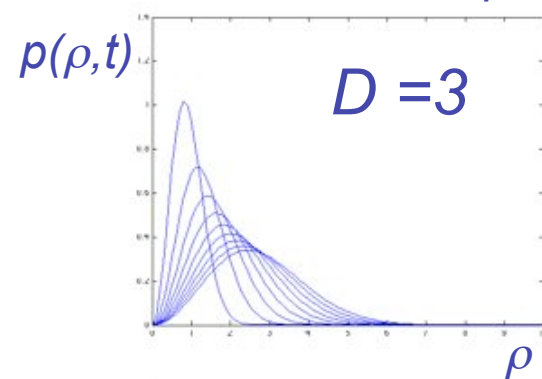$$p(\vec{r}, t+1) = \int_0^\infty p(\vec{\eta}) p(\vec{r} - \vec{\eta}, t) d^D\vec{\eta}$$

$$p(\vec{r}, t+1) = \int_0^\infty p(\vec{\eta}) \left[p(\vec{r}, t) - \vec{\eta}\vec{\nabla}p(\vec{r}, t) + \frac{1}{2}\vec{\eta}\vec{\nabla}\vec{\nabla}p(\vec{r}, t)\vec{\eta} + ...\right] d^D\vec{\eta}$$

$$p(\vec{r}, t+1) - p(\vec{r}, t) = \frac{\langle\eta^2\rangle}{2D}\nabla^2 p(\vec{r}, t) + ...$$

$$\frac{\partial p}{\partial t} = \frac{\langle\eta^2\rangle}{2D}\nabla^2 p$$

$$p(\rho, t) = \frac{2D^{D/2}\rho^{D-1}}{\Gamma(D/2)\left(2\langle\eta^2\rangle t\right)^{D/2}} \exp\left(\frac{-D\rho^2}{2\langle\eta^2\rangle t}\right) \qquad \rho = |\vec{r}|$$

Random Walk - any dimension

$p(\rho,t)$ D =1

$P(\rho,t)$ D =2

ρ ρ

$p(\rho,t)$ D =3

$P(\rho,t)$ D =10

ρ ρ

$p(\rho,t)$ D =100

$P(\rho)$ at t = 0,1,..,9
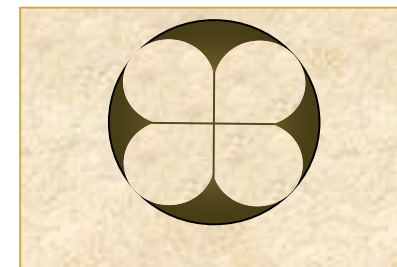for $\langle \eta \rangle = 0$ and $\langle \eta^2 \rangle = 1$

ρ
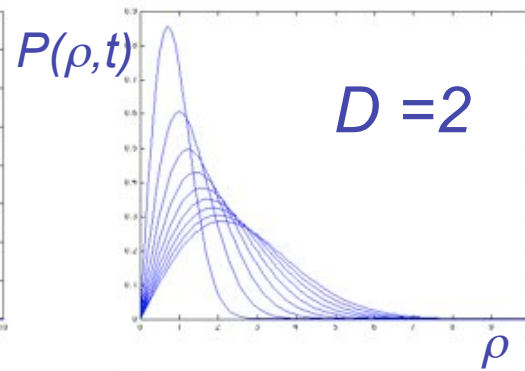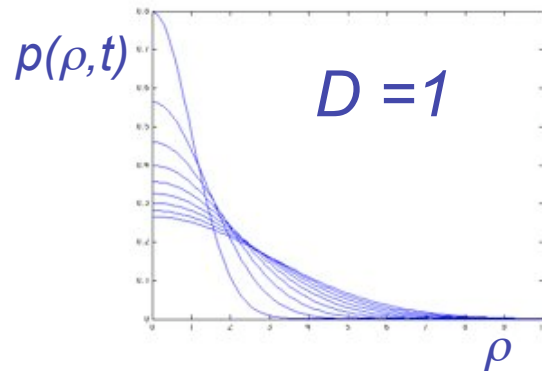
$$\langle \rho \rangle = \frac{\Gamma\left(\frac{(D+1)}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} \sqrt{\frac{2\langle \eta^2 \rangle}{D}}\, t$$

$$\langle \rho^2 \rangle = \langle \eta^2 \rangle t$$
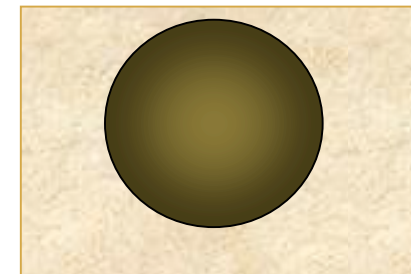
This is not a sphere.

Is this a sphere?

...further jumps…

Stochastic process with large noise fluctuations (non-finite variance)

$$x(t+1) = x(t) + \eta(t)$$

We already discussed the 1 dimensional case in the context of the central limit theorem and stable distributions [Lecture 3].

$$f(x,\alpha,\beta,c,\mu) = \int\limits_{-\infty}^{+\infty} dk\, e^{ikx} \exp\left(ik\mu - |ck|^{\alpha-1}\left(1 - i\beta\frac{k}{|k|}\Phi(\alpha-1,k)\right)\right)$$

$$p(\eta) \propto \frac{1}{|\eta|^\alpha}$$

$$p(x,t) \propto \frac{a(t)}{|x|^\alpha}$$

α= 3.0
α= 2.5
α= 2.0
α= 1.5
β=0
c=1
μ=0
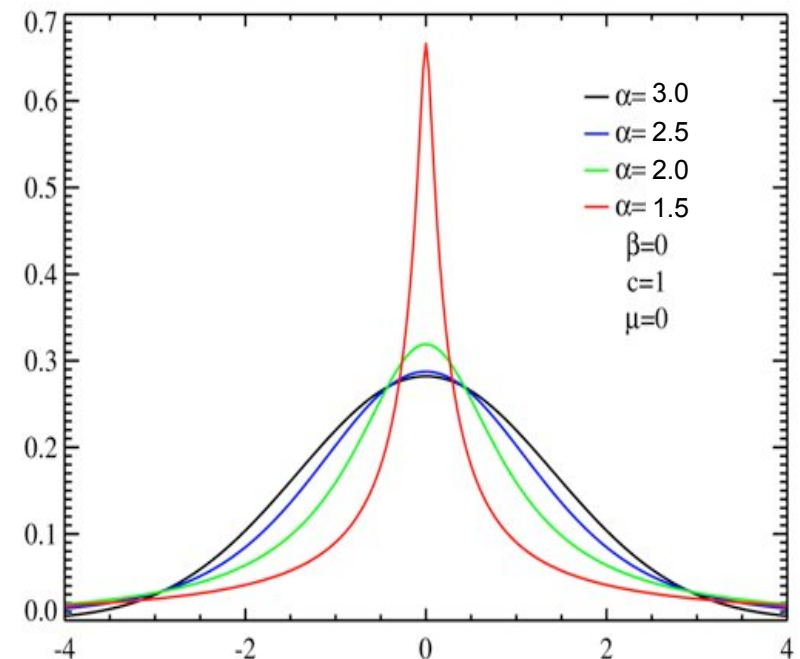
Super diffusive behavior

Probability of a jump larger or equal to L_{max}

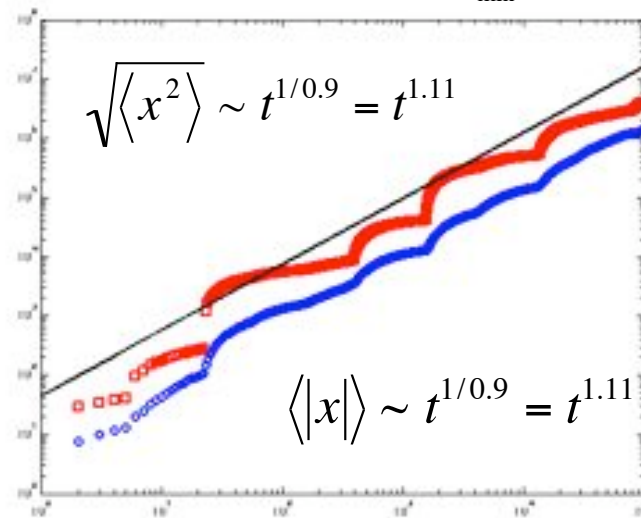$$P_r(L \geq L_{max}) = \int_{L_{max}}^{\infty} p(\eta)d\eta \sim \frac{1}{L_{max}^{\alpha-1}} \qquad \alpha > 1$$

in t steps I have a finite probability of a jump equal to L_{max}
if: $\qquad t \; P_r(L \geq L_{max}) \sim 1$

$$\boxed{L_{max} \sim t^{1/\alpha - 1}}$$

Mean Square Displacement: $\boxed{\langle x^2 \rangle = t \langle \eta^2 \rangle} = t \int_{L_{min}}^{L_{max}} \eta^2 p(\eta)d\eta \sim tL_{max}^{3-\alpha} \sim t^{2/(\alpha-1)}$

$$1 < \alpha < 3$$

$$x(t+1) = x(t) + \eta(t)$$
$$p(\eta) \propto \frac{1}{|\eta|^{\alpha}} \qquad \alpha = 1.9$$

$$\sqrt{\langle x^2 \rangle} \sim t^{1/0.9} = t^{1.11}$$

$$\langle |x| \rangle \sim t^{1/0.9} = t^{1.11}$$
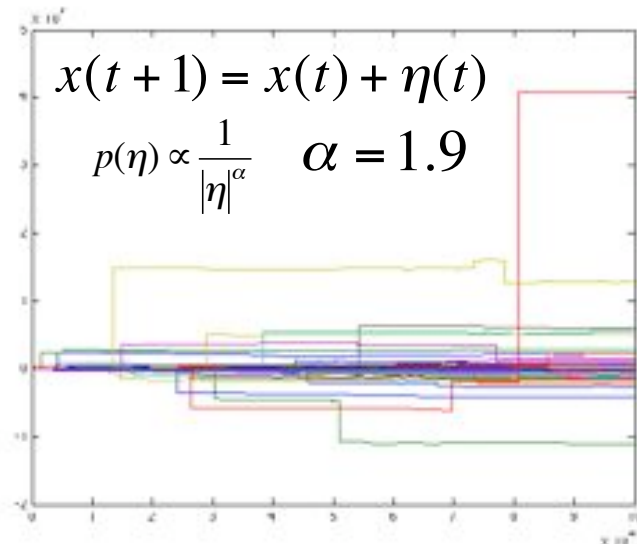
Large jumps
dominate the
behavior

$$\langle x^2 \rangle \sim L_{max}^2$$
$$\langle |x| \rangle \sim L_{max}$$

Diffusive behavior *('jumps' with finite variance)*
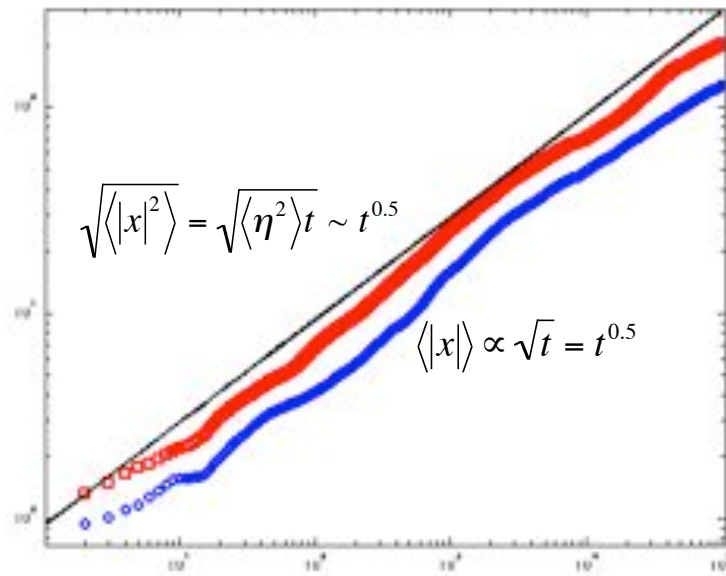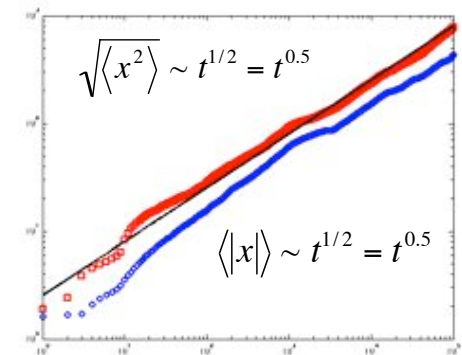

$\alpha = 3$

$$\sqrt{\langle x^2 \rangle} \sim t^{1/2} = t^{0.5}$$

$$\langle |x| \rangle \sim t^{1/2} = t^{0.5}$$

Super-diffusive behavior

$\alpha = 2.5$

$$\sqrt{\langle x^2 \rangle} \sim t^{1/1.5} = t^{0.66}$$

$$\langle |x| \rangle \sim t^{1/1.5} = t^{0.66}$$

$\alpha = 1.9$

$$\sqrt{\langle x^2 \rangle} \sim t^{1/0.9} = t^{1.11}$$

$$\langle |x| \rangle \sim t^{1/0.9} = t^{1.11}$$

$$\sqrt{\langle |x|^2 \rangle} = \sqrt{\langle \eta^2 \rangle t} \sim t^{0.5}$$

$$\langle |x| \rangle \propto \sqrt{t} = t^{0.5}$$
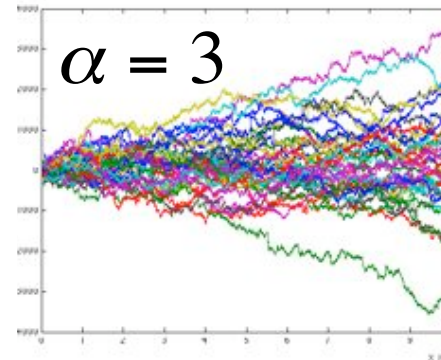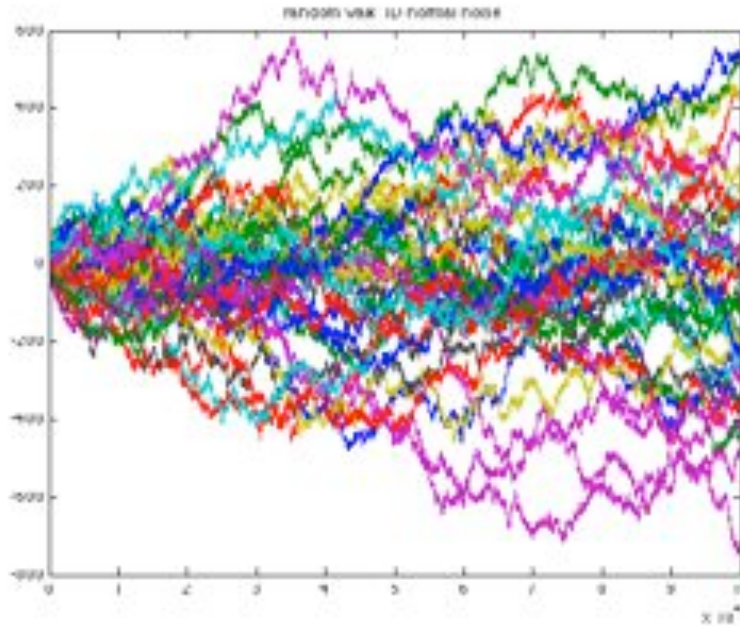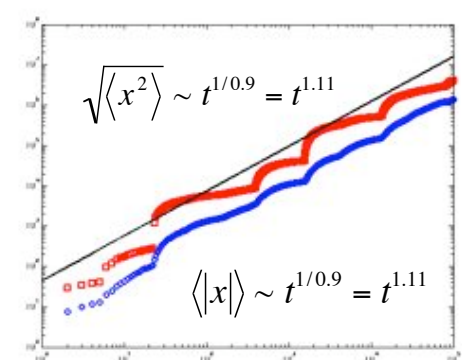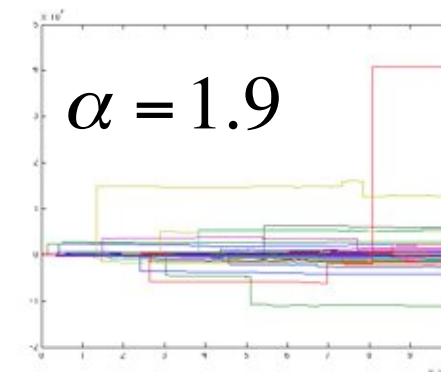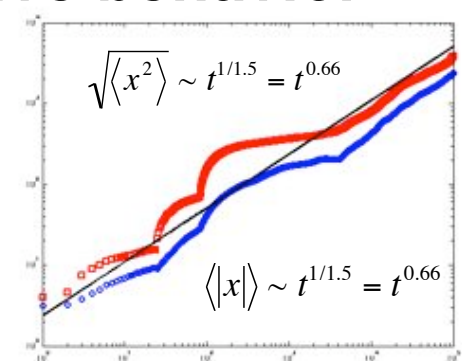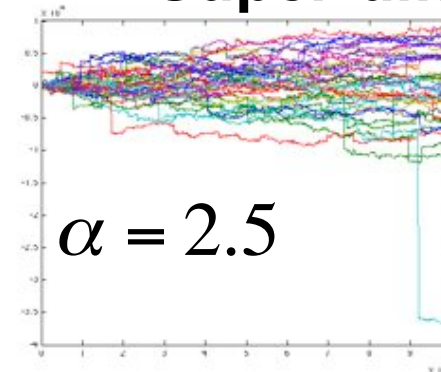
Scale free and Self similarity

$$\Delta x \sim (\Delta t)^2$$

$$\Delta x \sim (\Delta t)^{1/\alpha}$$

$$\alpha = 0.5$$
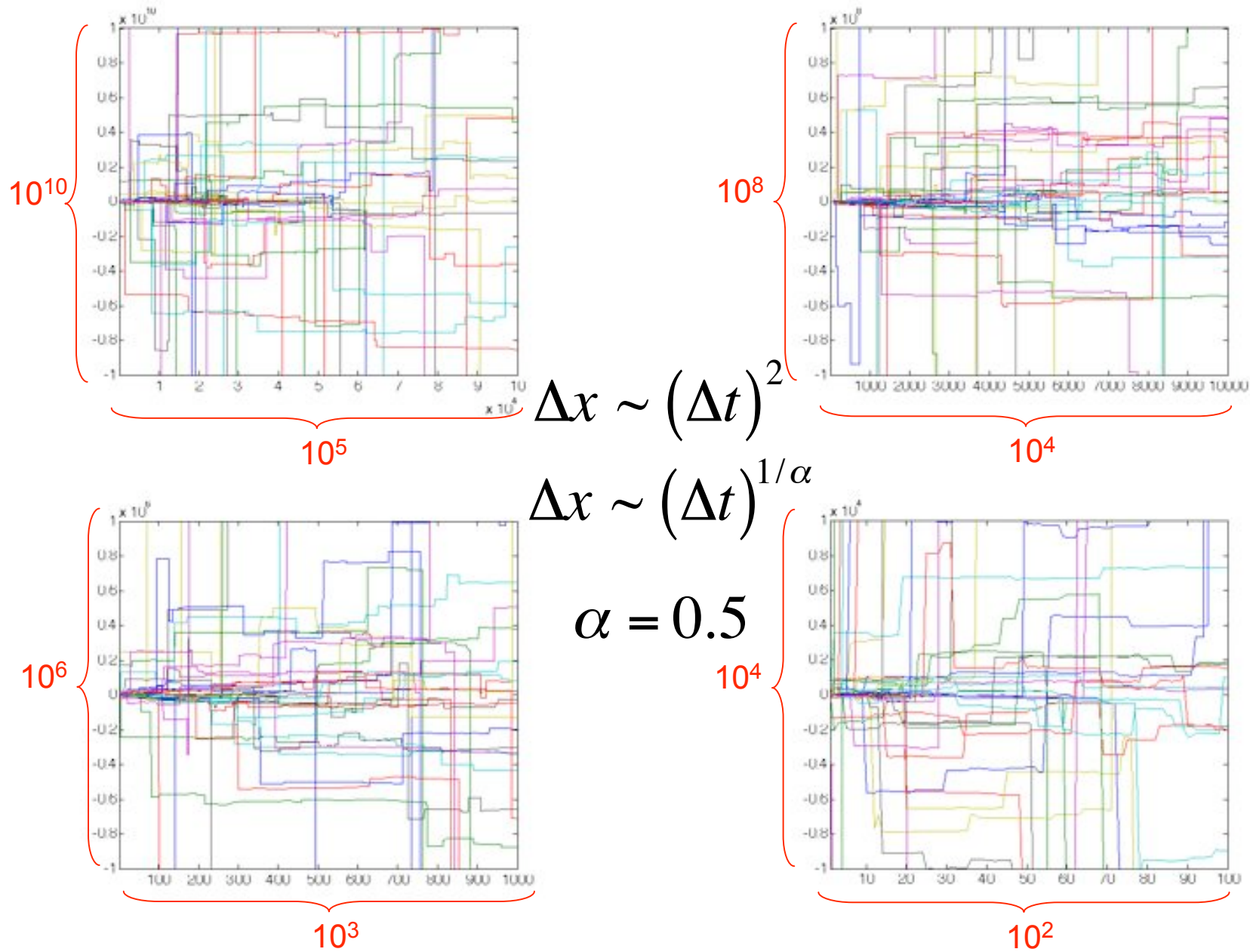
Higher dimensions: Levi Flights

$$\vec{r}(t+1) = \vec{r}(t) + \vec{\eta}(t)$$

$$\alpha = 1.1$$

$$\hat{p}(\vec{k}) \sim \exp(-c|k|^{\alpha})$$

$$p(\vec{\eta}) \sim \frac{1}{|\eta|^{d+\alpha}}$$
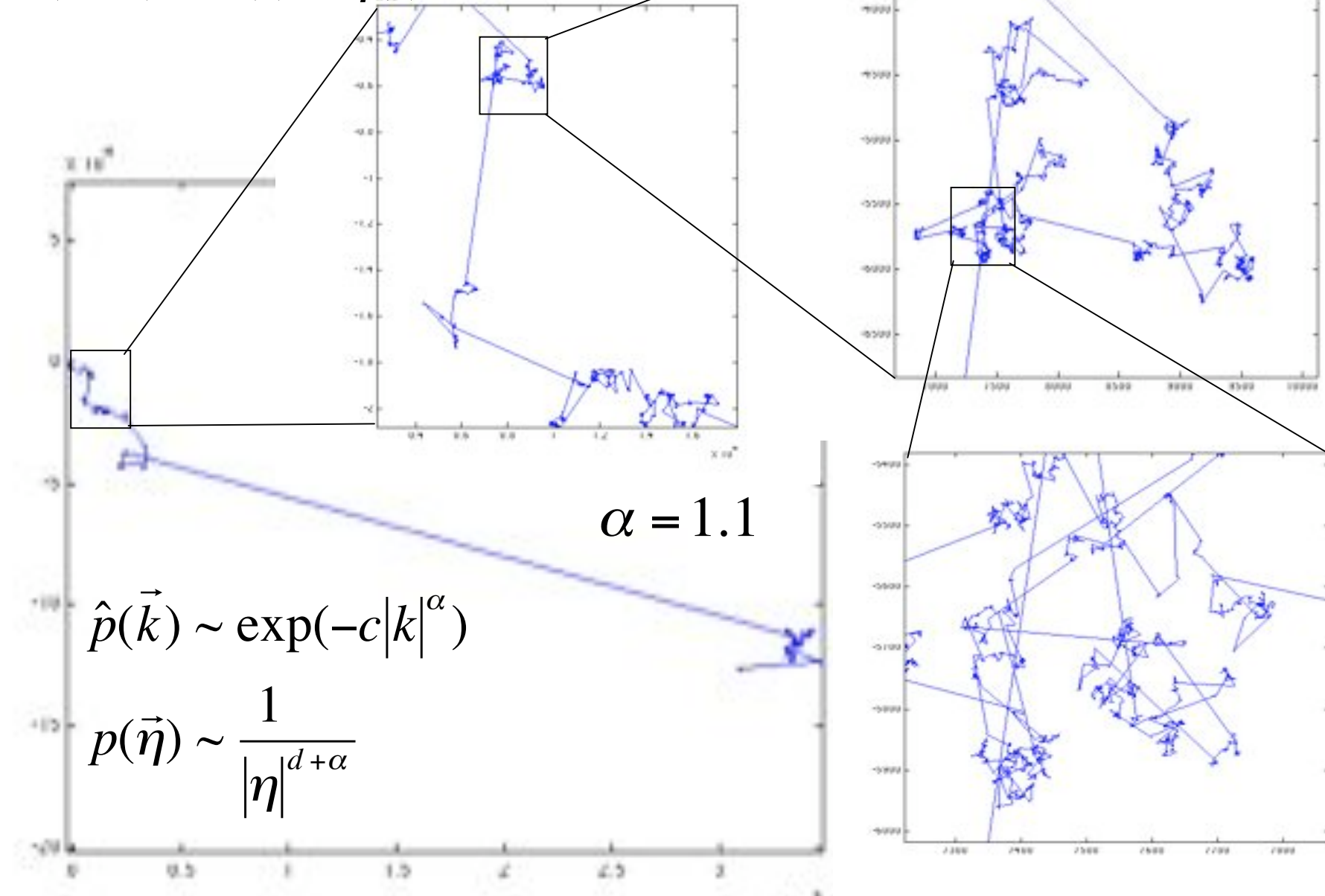
Sub-diffusive behaviors

$$x(t + \tau) = x(t) + \eta(t)$$ with $\langle \eta^2 \rangle$ finite, but

with ***power law distributed waiting times***: $p(\tau) \sim \dfrac{1}{\tau^\alpha}$

In n steps the mean square displacement will grow as

$$\langle x^2 \rangle = n \langle \eta^2 \rangle$$ the total time elapsed is $t \sim n \langle \tau \rangle$

Case $\alpha > 2$ $\Rightarrow$ $t \sim n$ $\langle x^2 \rangle \sim t$ diffusive

Case $1 < \alpha < 2$ $\Rightarrow$ $t \sim \tau_{\max} \sim n^{1/(\alpha-1)}$ (same reasoning as for L_{max} in previous slide)

$$\langle x^2 \rangle \sim t^{\alpha-1}$$ sub-diffusive!

Random walk on graphs

Which is the probability to find the walker at distance *j* after *t* steps?
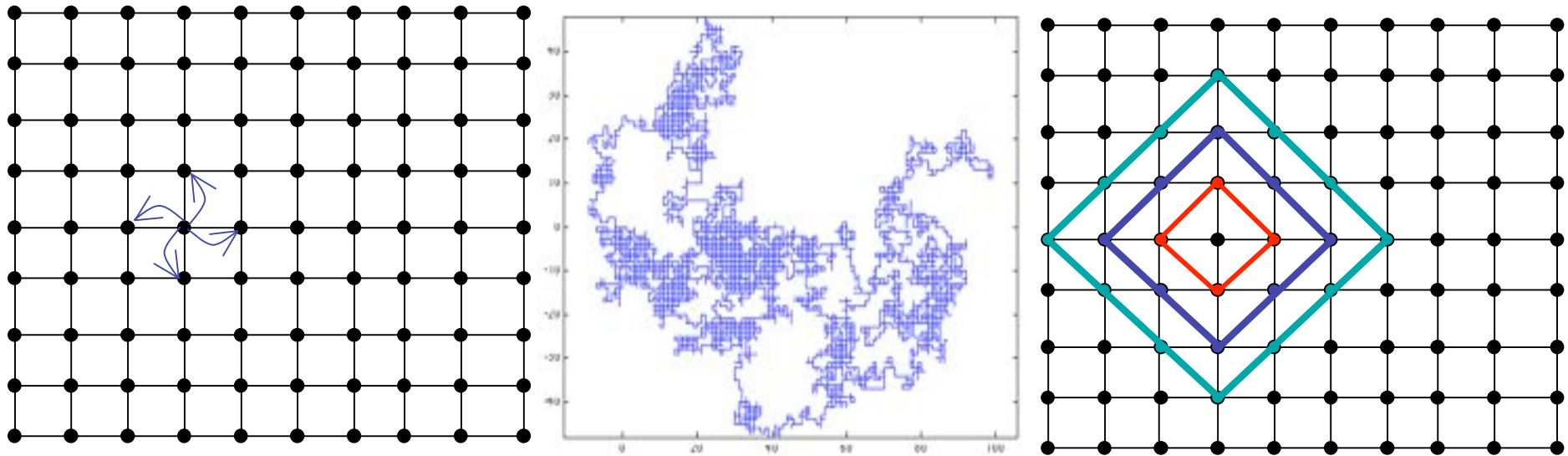
The probability to be at time *t+1* at a given geodesic distance *j* form the starting point is given by the probability that at time *t* the walker is

probability to be at distance *j* and stay there

$$P(j,t+1) = \overbrace{p_{stay}(j)P(j,t)} + \underbrace{p_{out}(j-1)P(j-1,t)} + \underbrace{p_{in}(j+1)P(j+1,t)}$$

probability to be at distance *j-1* and move forward   probability to be at distance *j+1* and move inward

with $P(0,t+1) = p_{in}(1)P(1,t)$

and $P(j,0) = \delta_{j,0}$

Random walk on graphs - continuous limit

$$P(j,t+1) = p_{stay}(j)P(j,t) + p_{out}(j-1)P(j-1,t) + p_{in}(j+1)P(j+1,t)$$

the equation for *P(j,t)* can be re-written as

$$P(j,t+1) - P(j,t) = \frac{1}{2}\left[p_{in}(j+1) + p_{out}(j+1)\right]P(j+1,t)$$

$$+ \frac{1}{2}\left[p_{in}(j-1) + p_{out}(j-1)\right]P(j-1,t)$$

$$+ \left[p_{in}(j) + p_{out}(j)\right]P(j,t)$$

$$+ \frac{1}{2}\left[p_{in}(j+1) - p_{out}(j+1)\right]P(j+1,t)$$

$$- \frac{1}{2}\left[p_{in}(j-1) - p_{out}(j-1)\right]P(j-1,t)$$

the continuous limit (*j* →*ρ; t* → *τ*) gives (Fokker-Plank Equation)

$$\frac{\partial P(\rho,\tau)}{\partial t} = \frac{\partial^2}{\partial \rho^2}\left[\frac{p_{out}(\rho) + p_{in}(\rho)}{2}\right]P(\rho,\tau) - \frac{\partial}{\partial \rho}\left[p_{out}(\rho) - p_{in}(\rho)\right]P(\rho,\tau)$$

Random walk on graphs - shell map

Each one of the probabilities $p_{in}(j)$ $p_{out}(j)$ $p_{stay}(j)$
is proportional to the relative number of paths that take the
walker inwards, outwards or within the "shell" j

square lattice ($j > 1$): - inward $4(j - 1);$
                          - outward $4(j+1);$
                          - stay 0
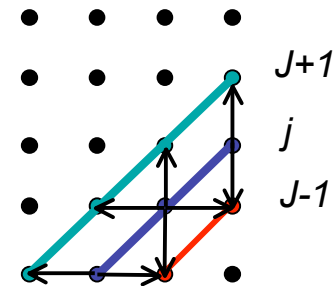
For a broad class of graphs holds ($j >> 1$):
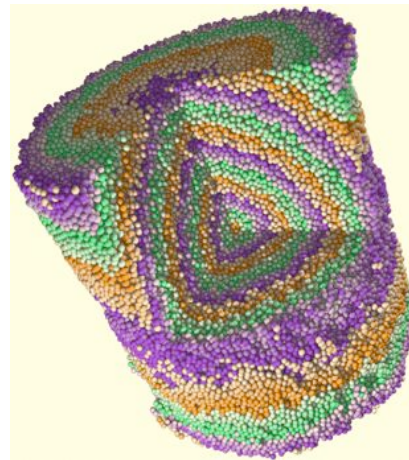
$$p_{out}(j) + p_{in}(j) \sim Const$$

$$p_{out}(j) - p_{in}(j) \propto \frac{V(j+1) - V(j)}{V(j)}$$

Random walk on graphs - continuous asymptotic solutions

$$p_{out}(\rho) + p_{in}(\rho) \sim Const = \frac{\sigma^2}{D}$$

$$p_{out}(\rho) - p_{in}(\rho) = \frac{\sigma^2}{DV(\rho)}\frac{\partial V(\rho)}{\partial \rho}$$

Finite dimensions: $\quad V(\rho) \sim \rho^{D-1} \quad p_{out}(\rho) - p_{in}(\rho) \sim \frac{(D-1)\sigma^2}{D\rho^{D-1}}\rho^{D-2} \sim \frac{1}{\rho}$

$$P(\rho,t) = \frac{2D^{D/2}\rho^{D-1}}{\Gamma(D/2)(2\sigma^2 t)^{D/2}}\exp\left(\frac{-D\rho^2}{2\sigma^2 t}\right)$$

Hyperbolic spaces: $\quad V(\rho) \sim \exp(\rho) \qquad p_{out}(\rho) - p_{in}(\rho) \sim Const$

which leads to an equation of the form

$$\frac{\partial P(\rho,\tau)}{\partial t} = -D_1\frac{\partial}{\partial \rho}P(\rho,\tau) + D_2\frac{\partial^2}{\partial \rho^2}P(\rho,\tau)$$

and the solution for large times is a density wave which
moves ballistically outwards:

$$<\rho> \sim t, \quad <\rho^2> \sim t^2 \quad and \quad <\rho^2> - <\rho>^2 \sim t$$

T. Aste, "Random walks on disordered networks", *Phys. Rev. E* 55 (1997), p.6233-6236.

Will the walker ever return to the origin?

Will the walker ever return to the origin?

The mean time spent in the origin is:

$$\sum_{t=0}^{\infty} P(0,t)$$

The probability to return to the origin is (Polya theorem)

$$\Pi = 1 - \frac{1}{\sum_{t=0}^{\infty} P(0,t)}$$

Which can be computed using the characteristic function

$$P(0,t) = \frac{1}{(2\pi)^D} \int_{-\infty}^{+\infty} \hat{P}(\vec{k},t) d^D\vec{k} \qquad \hat{P}(\vec{k},t) = \left[\hat{p}(\vec{k})\right]^t$$

$$\sum_{t=0}^{\infty} P(0,t) = \frac{1}{(2\pi)^D} \int_{-\infty}^{+\infty} \sum_{t=0}^{\infty} \left[\hat{p}(\vec{k})\right]^t d^D\vec{k} = \frac{1}{(2\pi)^D} \int_{-\infty}^{+\infty} \frac{1}{1-\hat{p}(\vec{k})} d^D\vec{k}$$
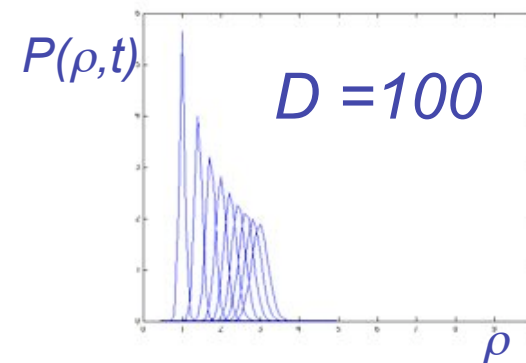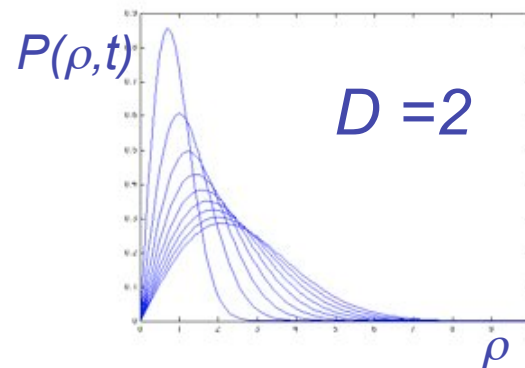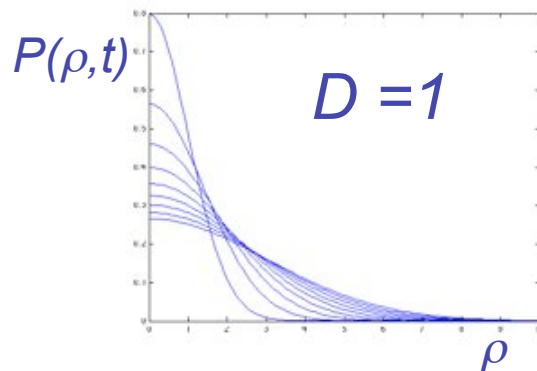
He might never get back home…

$$p(\vec{k}) \sim \exp(-c|k|^{\alpha}) \sim 1 - c|k|^{\alpha}$$

$$\int_{-\infty}^{+\infty} \frac{1}{1-p(\vec{k})} d^D\vec{k} \sim \int_{-\infty}^{+\infty} \frac{1}{|k|^{\alpha}} d^D\vec{k} \sim \int_{-\infty}^{+\infty} \frac{|k|^{D-1}}{|k|^{\alpha}} d|k|$$

$$\Pi = \begin{cases} \begin{cases} 1 & for \quad D \leq 2 \\ <1 & for \quad D > 2 \end{cases} \\ \begin{cases} 1 & for \quad D \leq \alpha \\ <1 & for \quad D > \alpha \end{cases} \end{cases}$$

Finite variance

Power Law, **Non defined variance**

$\alpha < 2$

Recurrent $\Pi = 1$

*(he **always** get back)*

Transient $\Pi < 1$

*(he might **never** get back)*

$P(\rho,t)$ D =1

$P(\rho,t)$ D =2

$P(\rho,t)$ D =100

$P(\rho)$ at t = 0,1,..,9 for $<\eta> = 0$ and $<\eta^2> = 1$

Are sharks intelligent mathematicians?

G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphy, P. A. Prince, H. E. Stanley, Levy flight search patterns of wandering albatrosses, *Nature 381 (1996)* 413 - 415.
A.M. Reynolds, Cooperative random Levy flight searches and the flight patterns of honeybees, Physics Letters A 354 (2005) 384-388.

"maximally searchable graphs" (lecture 9)

We want to be able to navigate efficiently with local information only (*greedy algorithm*).

Let us put the vertices on a lattice and choose shortcuts between lattice vertices with probability

$$p_s(i,j) \propto d_{i,j}^{-\gamma}$$

with $d_{i,j}$ the distance between the vertices on the lattice.

It results that *any target vertex from any random starting vertex is found within a time ~ (Log V)²* **only when**:
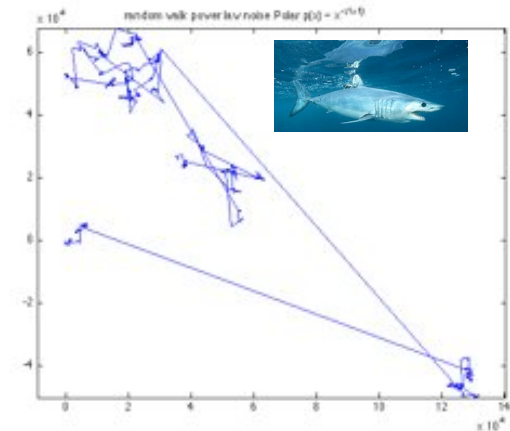
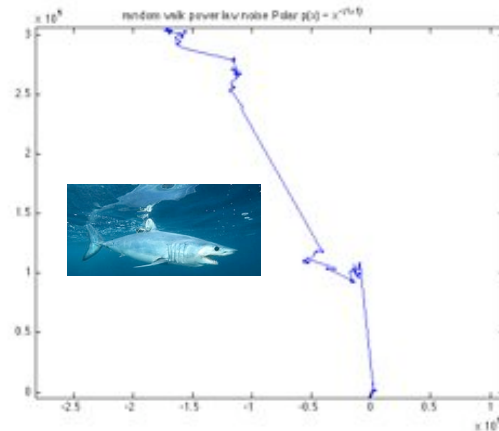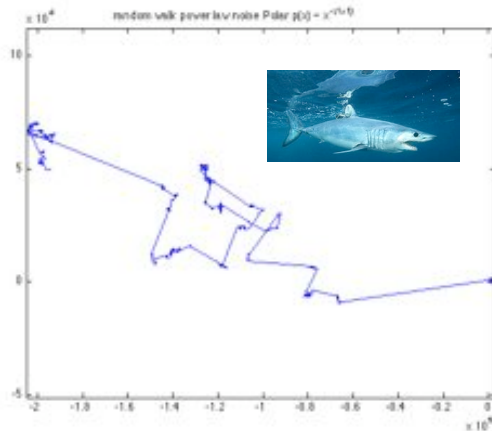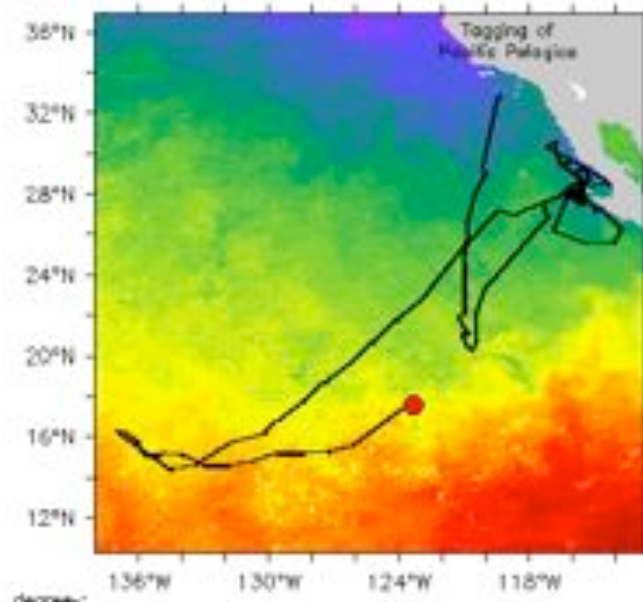$$\gamma = \text{space dimension}$$
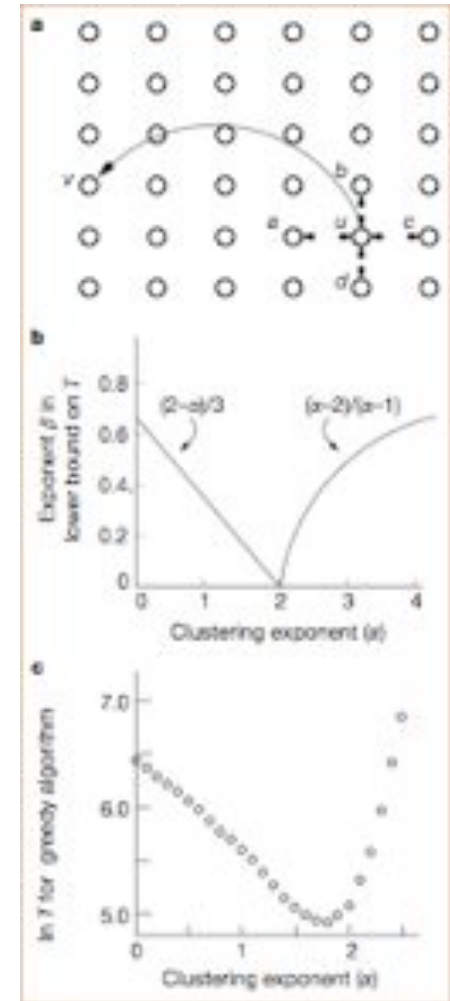
J. M. Kleinberg, Navigation in a small world, Nature, 406 (2000), p. 845.
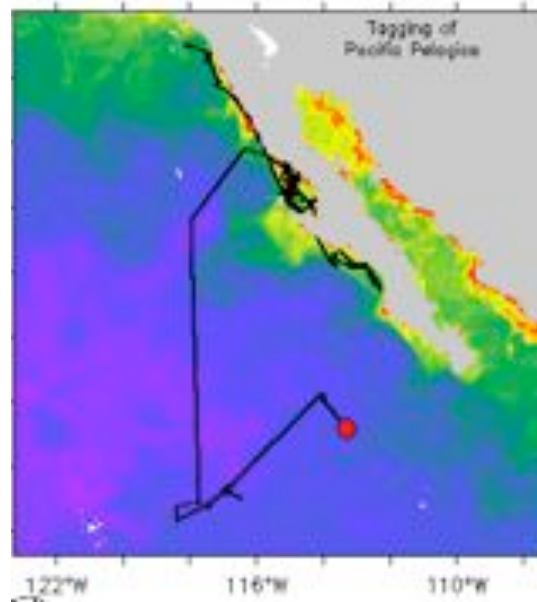
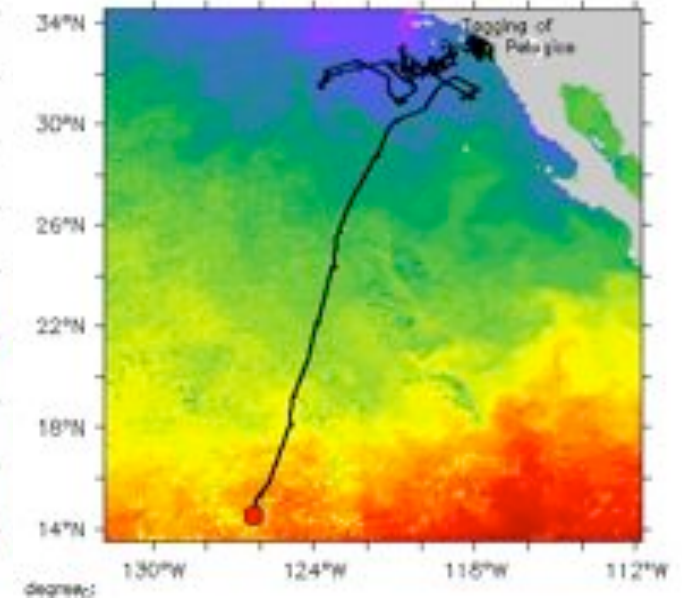Two dimensions: $$p_s(r) \propto r^{-2}$$

http://topp.org/species/mako_shark

Shortfin Mako Shark 53802
1507012 26 Jul 2007 to 23 Mar
2008

Shortfin Mako Shark 68509 1507007
13 Jul 2007 to 9 Mar 2008

Shortfin Mako Shark 53802 1507012 26
Jul 2007 to 23 Mar 2008

Supplemetary material

Weierstrass Random Walk
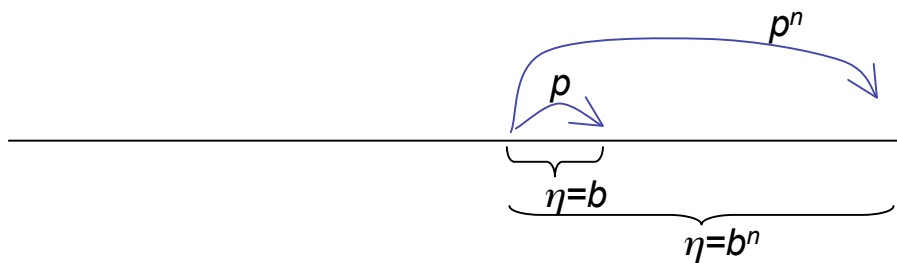
The "walker" starts at position *x = 0* at step *t = 0*; at each time-step the walker can go either forward or backward b^n positions with probability $\sim p^n$.

$$p(n) \sim p^n \qquad \eta = \pm b^n \qquad n = \frac{\log(|\eta|)}{b} \qquad p(\eta) \sim \frac{1}{|\eta|^{1+\alpha}} \qquad \alpha = -\frac{\log(p)}{\log(b)}$$

$$p(\eta) = \frac{(1-p)}{2} \sum_{m=0}^{\infty} p^n \left[\delta(\eta - b^n) + \delta(\eta + b^n) \right]$$

Characteristic function

$$\hat{p}(k) = \int_{-\infty}^{+\infty} p(\eta) e^{ik\eta} = (1-p) \sum_{m=0}^{\infty} p^m \cos(kb^m) \underset{k \to \infty}{\sim} \exp(-c(\alpha)|k|^\alpha)$$

$$p(\eta) \sim \frac{1}{|\eta|^{1+\alpha}} \qquad \text{Levy stable} \qquad P(x,t) \sim \frac{1}{|x|^{1+\alpha}}$$

$$p(x,t) = \int_{-\infty}^{+\infty} d\eta_1 ... \int_{-\infty}^{+\infty} d\eta_t \, p(\eta_1)...p(\eta_t)\delta(\eta_1 + ... + \eta_t - x)$$

$$\hat{p}(k,t) = \int_{-\infty}^{+\infty} dx \, e^{ikx} \int_{-\infty}^{+\infty} d\eta_1 ... \int_{-\infty}^{+\infty} d\eta_t \, p(\eta_1)...p(\eta_t)\delta(\eta_1 + ... + \eta_t - x)$$

solution $\qquad \hat{p}(k,t) = \left[\hat{p}(k)\right]^t \sim \exp\left(-D_{eff} t |k|^\alpha\right)$

<u>Non-equally distributed time-steps</u> can lead to the same
kind of fat-tailed jump probabilities.

$$\boxed{\frac{\partial \hat{P}(k,t)}{\partial t} = -D_{eff} |k|^\alpha \hat{P}(k,t)}$$

Diffusive behavior $\alpha = 2$

<u>Fractional diffusion behavior</u>

$$\frac{\partial P(x,t)}{\partial t} = D_{eff} \frac{\partial^2 \hat{P}(k,t)}{\partial x^2}$$

$$\frac{\partial P(x,t)}{\partial t} = D_{eff} \frac{\partial^\alpha \hat{P}(k,t)}{\partial x^\alpha}$$

$$\boxed{\frac{\partial^\alpha}{\partial x^\alpha} := -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk |k|^\alpha e^{-ikx}}$$
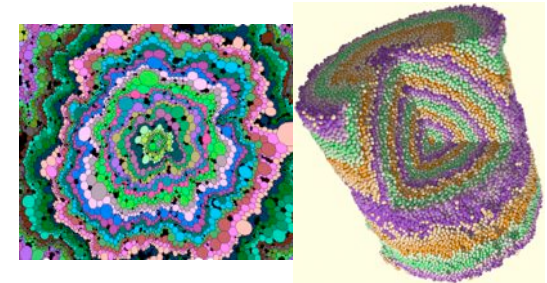
<u>Fractional calculus</u>

Probabulity to return to the origin, on graphs…

$$P(j,t+1) = p_{stay}(j)P(j,t) + p_{out}(j-1)P(j-1,t) + p_{in}(j+1)P(j+1,t)$$

From the above relation one can calculate the mean time spent in the origin:

$$\sum_{t=0}^{\infty} P(j,t) = \sum_{j=0}^{\infty} \frac{1}{\# \text{ of paths between shell } j \text{ and shell } j+1}$$

$$(\# \text{ of paths between shell } j \text{ and shell } j+1 \sim \frac{\langle k^2 \rangle_j}{\langle k \rangle_j} V(j) \sim \begin{cases} V(j) & \text{for } \langle k^2 \rangle \text{ finite} \\ [V(j)]^{\frac{2}{\alpha-1}} & \text{scale free with } 1 < \alpha < 2 \\ [V(j)]^{\frac{\alpha}{\alpha-1}} & \text{scale free with } 2 \le \alpha < 3 \end{cases}$$

$$V(j) \sim j^{D-1}$$

$$\Pi = 1 - \frac{1}{\sum\limits_{t=0}^{\infty} P(0,t)} \sim \begin{cases} 1 & \text{for } D \le 2 \text{ and finite } \langle k^2 \rangle \\ 1 & \text{for } scale \text{ free with } 2 \le \alpha < 3 \text{ and } D < 2 - \frac{(3-\alpha)}{2} \\ 1 & \text{for } scale \text{ free with } 1 < \alpha < 2 \text{ and } D < 2 - \frac{1}{\alpha} \\ < 1 & \text{otherwhise} \end{cases}$$

T. Aste, "Random walks on disordered networks", *Phys. Rev. E* 55 (1997), p.6233-6236.

Brownian motion is a fractal with dimension 4/3

G.F. Lawler, W. Werner (1999), Intersection exponents for planar Brownian motion, Ann. Probab. 27, 1601-1642.